Two body central force problem

Bohr model using forces

Now we must correct for the fact that the nucleus is not infinitely massive. Thus we start by writing the dynamics for both the electron of mass m acted upon by force F_{Mm} ("nucleus-on-electron") and the nucleus of mass M acted upon by force F_{mM} ("electron-on-nucleus"):

$$m\frac{d^2\mathbf{r}_m}{dt^2} = \mathbf{F}_{Mm}.$$

$$M\frac{d^2\mathbf{r}_M}{dt^2} = \mathbf{F}_{mM}.$$

EXERCISE 14

Show that by subtracting a suitable multiple of the second equation from a suitable multiple of the first equation, and using Newton's Third Law that $\mathbf{F}_{mM} =- \mathbf{F}_{Mm}$, one can get the following equation for the second derivative of the relative position vector $\mathbf{r}_{rel} = \mathbf{r}_m - \mathbf{r}_M$:

$$\frac{Mm}{m+M}\frac{d^2\mathbf{r}_{rel}}{dt^2} = \mathbf{F}_{Mm}.$$

ANSWER

Multiply the first equation by M and subtract the second equation multiplied by m:

$$Mm\frac{d^2\mathbf{r}_m}{dt^2} - mM\frac{d^2\mathbf{r}_M}{dt^2} = M\mathbf{F}_{Mm} - m\mathbf{F}_{mM}.$$

Grouping terms on the left and use $\mathbf{F}_{mM} =- \mathbf{F}_{Mm}$,

$$Mm\frac{d^2(\mathbf{r}_m - \mathbf{r}_M)}{dt^2} = M\mathbf{F}_{Mm} - m(-\mathbf{F}_{Mm}).$$

Then

$$Mm\frac{d^2(\mathbf{r}_m - \mathbf{r}_M)}{dt^2} = (M+m)\mathbf{F}_{Mm}.$$

Finally

$$\frac{Mm}{M+m}\frac{d^2(\mathbf{r}_m-\mathbf{r}_M)}{dt^2}=\mathbf{F}_{Mm}$$

or, defining $\mathbf{r}_{rel} \equiv \mathbf{r}_m - \mathbf{r}_M$,

$$\frac{Mm}{M+m}\frac{d^2\mathbf{r}_{rel}}{dt^2}=-\frac{e^2}{4\pi\epsilon_0 r_{rel}^2}\hat{\mathbf{r}}_{rel}.$$

We define the reduced mass μ by

$$\mu \equiv \frac{Mm}{M+m}=\frac{1}{1+\frac{m}{M}}m.$$

Recalling that the force of the nucleus on the electron $\mathbf{F}_{Ne}$ is just the Coulomb force, we write the dynamics as:

$$\mu\frac{d^2\mathbf{r}}{dt^2}=-\frac{e^2}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}.$$

Thus all of the results for the central force Bohr model must be corrected by replacing the mass m of the electron with the reduced mass μ. Most importantly, the energy states are given by:

$$E_n=-\frac{1}{2}\frac{\mu e^4}{(4\pi\epsilon_0)^2\hbar^2}\frac{1}{n^2}=-\frac{1}{2}\frac{\mu}{m}\frac{me^4}{(4\pi\epsilon_0)^2\hbar^2}\frac{1}{n^2}=-\frac{\mu}{m}\frac{E_0}{2}\frac{1}{n^2}.$$

where the Hartree energy E_0 is

$$E_0 \equiv \frac{me^4}{(4\pi\epsilon_0)^2\hbar^2}.$$

Two-Body Schrödinger equation

The two-body Schrödinger equation is:

$$-\frac{\hbar^2}{2m}\nabla_m^2\,\psi(\vec{r}_m,\vec{r}_M)-\frac{\hbar^2}{2M}\nabla_M^2\,\psi(\vec{r}_m,\vec{r}_M)+V(|\vec{r}_m-\vec{r}_M|)=E\psi(\vec{r}_m,\vec{r}_M)$$

where

$$V(|\vec{r}_m-\vec{r}_M|)=-\frac{e^2}{4\pi\epsilon_0|\vec{r}_m-\vec{r}_M|}.$$

Make the coordinate transformation

$$\vec{r}_{cm}=\frac{m}{M+m}\vec{r}_m+\frac{M}{M+m}\vec{r}_M.$$

$$\vec{r}_{rel} = \vec{r}_m - \vec{r}_M.$$

EXERCISE (a)

Show that the inverse transformation is:

$$\vec{r}_m = \vec{r}_{cm} + \frac{M}{M+m}\vec{r}_{rel}.$$

$$\vec{r}_M = \vec{r}_{cm} - \frac{m}{M+m}\vec{r}_{rel}.$$

EXERCISE (b)

Show that through a change of coordinates, the two-body Schrodinger equation becomes:

$$-\frac{\hbar^2}{2\mu}\nabla^2_{rel}\,\psi(\vec{r}_{rel},\vec{r}_{cm}) - \frac{\hbar^2}{2M_{tot}}\nabla^2_{cm}\,\psi(\vec{r}_{rel},\vec{r}_{cm}) + V(|\vec{r}_{rel}|) = E\psi(\vec{r}_{rel},\vec{r}_{cm})$$

where

$$M_{tot} \equiv M + m$$

and

$$\mu \equiv \frac{Mm}{M+m}.$$

EXERCISE (c)

Show that we can use separation of variables

$$\psi(\vec{r}_{rel},\vec{r}_{cm}) = \psi_{rel}(\vec{r}_{rel})\psi_{cm}(\vec{r}_{cm})$$

to obtain the system of equations:

$$-\frac{\hbar^2}{2M_{tot}}\nabla^2_{cm}\,\psi_{cm}(\vec{r}_{cm}) = E_{cm}\psi_{cm}(\vec{r}_{cm})$$

$$-\frac{\hbar^2}{2\mu}\nabla^2_{rel}\,\psi_{rel}(\vec{r}_{rel}) + V(|\vec{r}_{rel}|) = (E - E_{cm})\psi_{rel}(\vec{r}_{rel})$$

where E_{cm} has served as a separation constant and

$$V(|\vec{r}_{rel}|) = -\frac{e^2}{4\pi\epsilon_0|\vec{r}_{rel}|}.$$

Thus the second equation is like Schrodinger's equation for an electron in a Couloumb potential but now with the electron mass m replaced by the reduced mass μ.

Formal solution of this equation will again yield energy eigenvalues E_n for $E - E_{cm}$

$$E_n = -\frac{1}{2}\frac{\mu e^4}{(4\pi\epsilon_0)^2\hbar^2}\frac{1}{n^2} = -\frac{1}{2}\frac{\mu}{m}\frac{me^4}{(4\pi\epsilon_0)^2\hbar^2}\frac{1}{n^2} = -\frac{\mu}{m}\frac{E_0}{2}\frac{1}{n^2}.$$

where again the Hartree energy E_0 is

$$E_0 \equiv \frac{me^4}{(4\pi\epsilon_0)^2\hbar^2}.$$