Hydrogen Deuterium Spectroscopy Idea Calculations

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Excersize 1: Momentum to Coulomb Expression for Electrostatic Force

$p = mv$, $v = \frac{dr}{dt}$, $F = \frac{dp}{dt}$

=>

$$v = \frac{1}{m}p$$

=>

$$\frac{dr}{dt} = \frac{1}{m}p$$

=>

$$\frac{dp}{dt} = \frac{1}{4\pi\epsilon_0}\frac{qq}{r^2}\hat{r}$$

=>

$$\frac{dp}{dt} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2}\hat{r}$$

Excersize 2: Relationship of Scales in Terms of Electrostatic Force

$r = r_0 r'$, $p = p_0 p'$, $t = t_0 t'$, Atomic Number is 1

=>

$$\frac{r_0}{t_0}\frac{dr'}{dt'} = \frac{1}{m}p_0 p'$$

=>

$$\frac{dr'}{dt'} = \frac{t_0}{r_0}\frac{p_0}{m}p'$$

=>

$$\frac{p_0}{t_0}\frac{dp'}{dt'} = -\frac{1}{r_0^2}\frac{e^2}{4\pi\epsilon_0 r'^2}\hat{r} = -\frac{t_0}{r_0^2}\frac{p_0 e^2}{4\pi\epsilon_0 r'^2}\hat{r}$$

=>

$$\frac{t_0}{r_0}\frac{p_0}{m} = \frac{e^2}{4\pi\epsilon_0}\frac{t_0 p_0}{r_0^2}$$

$\frac{dr'}{dt'} = p'$ and $\frac{dp'}{dt'} = -\frac{1}{r'^2}\hat{r}$

=>

$$m = \frac{p_0 t_0}{r_0}$$

and

$$\frac{t_0}{p_0 r_0^2} = \frac{4\pi\epsilon_0}{e^2}$$

Excersize 3: Atomic Length, Time, and Momentum Scales

$r_0 = a_0$, $a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2}$, $\hbar = r_0 p_0$

a)

$$\frac{p_0}{r_0}t_0 = \frac{4\pi\epsilon_0 r_0^2 p_0}{e^2}\frac{p_0}{r_0}$$

=>

$$m = \frac{4\pi\epsilon_0\hbar^2}{e^2 r_0}$$

=>

$$r_0 = \frac{4\pi\epsilon_0\hbar^2}{e^2 r_0} = a_0$$

b)

$$t_0 = \frac{mr_0}{p_0} = \frac{mr_0^2}{\hbar} = \frac{m}{\hbar}\left(\frac{4\pi\epsilon_0\hbar^2}{e^2 m}\right)^2$$

C)

$$p_0 = \frac{\hbar}{r_0}$$

=>

$$p_0 = \frac{e^2 m\hbar}{4\pi\epsilon_0}$$

Excersize 4: Atomic Scale for Velocity

$v_0 = \frac{p_0}{m}$, $\beta_0 = \frac{v_0}{c}$

=>

$$v_0 = \frac{1}{m}\frac{me^2}{4\pi\epsilon_0\hbar} = \frac{e^2}{4\pi\epsilon_0\hbar}$$

=>

$$\beta_0 = \frac{e^2}{4\pi\epsilon_0\hbar(c)}$$

Excersize 5: Atomic Scale for Energy

a)

$$E_0 = \frac{p_0}{t_0}r_0 = \frac{me^4}{(4\pi\epsilon_0)^2\hbar^2}$$

b)

$$\frac{1}{a_0} = \frac{me^2}{4\pi\epsilon_0\hbar^2}$$

=>

$$E = \frac{1}{a_0}\frac{e^2}{4\pi\epsilon_0}$$

Excersize 6: Relationship Between Energy and Time Scales

$$E_0 t_0 = \left(\frac{me^4}{(4\pi\epsilon_0)^2\hbar^2}\right)\left(\frac{(4\pi\epsilon_0)^2\hbar^3}{me^4}\right) = \hbar$$

Excersize 7: Bohr Radius, Hartree, and Fine Structure Constant Relationship

a)

=>

$$\frac{1}{\alpha}\frac{hc}{2\pi(mc^2)}$$

=>

$$(137)\frac{1240eVnm}{(2\pi)(0.511x10^6eV)} = 0.0529nm$$

b)

$$E = \frac{me^4}{(4\pi\epsilon_0)^2\hbar^2} = mc^2\frac{e^4}{(4\pi\epsilon_0)^2\hbar(c)^2} = \alpha^2mc^2$$

=>

$$\frac{1}{(137)^2}(0.511x10^6eV) = 27.2eV$$

c)

$$\frac{v_0}{c} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{\hbar}\frac{1}{c} = \alpha = \frac{1}{137}$$

Excersize 8: Time Scale and Momentum Scale

a)

$$t_0 = \frac{(4\pi\epsilon_0)^2\hbar^3}{me^4}$$

=>

$$\frac{(4\pi\epsilon_0)^2\hbar^2c^2}{e^4}\frac{hc}{2\pi}\frac{1}{mc^2}\frac{1}{c}$$

=>

$$\frac{1}{2\pi}\alpha^{-2}\hbar(c)(mc^2)^{-1}c^{-1}$$

=>

$$t_0 = \frac{1}{2\pi}(137)^2(1240eVnm)(0.511x10^6)^{-1}(3x10^{17})^{-1} = 2.42x10^{-17}s$$

b)

$$p_0 = \frac{me^2}{4\pi\epsilon_0\hbar} = \frac{e^2}{4\pi\epsilon_0\hbar(c)}mc^2$$

=>

$$\alpha(mc^2) = (137)^{-1}(0.511x10^6 eV) = 3730 eV$$

Excersize 9: Newton's Second Law for Circular Motion

=>

$$F_{e^-} = -\frac{e^2}{4\pi\epsilon_0 r^2} = -m\frac{v^2}{r} = m\alpha$$

Excersize 10: Kinetic and Potential Energy

=>

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}\frac{e^2}{4\pi\epsilon_0 r} = -\frac{1}{2}V(r)$$

=>

$$V(r) = PE$$

=>

$$E_{total} = KE + V(r) = -\frac{1}{2}V(r) + V(r) = \frac{1}{2}V(r) = -\frac{1}{2}\frac{e^2}{4\pi\epsilon_0 r}$$

Excersize 11: Quantization of Orbital Angular Momentum

$$L = mvr = mv\frac{n\lambda}{2\pi} = mv\frac{1}{2\pi}n\frac{h}{2\pi} = n\hbar$$

Excersize 12: Quantization of Angular Momentum with Newton's Law

$r_n = a_0 n^2 and E_0 = -\frac{E_0}{2}\frac{1}{n^2}$

=>

$$-m\frac{v^2}{r} = -\frac{e^2}{4\pi\epsilon_0 r^2}$$

=>

$$L^2 = (mvr)^2 = m^2v^2r^2 = mr^3\left(m\frac{v^2}{r}\right) = mr^3\left(\frac{e^2}{4\pi\epsilon_0 r^2}\right)$$

=>

$$L^2 = \frac{me^2}{4\pi\epsilon_0} r = n^2 \hbar^2$$

=>

$$r = \frac{n^2 \hbar (4\pi\epsilon_0)}{me^2} = n^2 a_0$$

=>

$$E = -\frac{e^2}{4\pi\epsilon_0 r} = -\frac{1}{2}\frac{e^2}{4\pi\epsilon_0 n^2 a_0} = -\frac{1}{2n^2} E_0$$

Excersize 13: Wavelengths of Light Emitted by Excited Hydrogen Atoms

a)

$$\frac{1}{\lambda} = R_\infty (n_f^{-2} - n_i^{-2}) = \frac{\Delta(E)}{hc} = \frac{1}{hc}(E_i - E_f)$$

=>

$$\frac{1}{hc}\frac{E}{2}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

b)

$$E_0 = \alpha^2 mc^2$$

=>

$$R_\infty = \frac{1}{hc}\frac{E_0}{2} = \frac{1}{2}\frac{1}{hc}\alpha^2 mc^2$$

=>

$$R_\infty = \frac{1}{2}\frac{1}{1240eVnm}\frac{1}{137^2}(0.511x10^6 eV) = 0.01097nm^{-1}$$

Excersize 14: Newton's Third Law for Relative Position Vector

$$F_{eN} = -F_{Ne}, r = r_e - r_N = \frac{d^2 r}{dt^2} = \frac{d^2 r_e}{dt^2} - \frac{d^2 r_N}{dt^2}, \frac{M_m}{m+M}\frac{d^2 r}{dt^2} = F_{Ne}$$

=>

$$M_m \frac{d^2 r}{dt^2} - mM\frac{d^2 r}{dt^2} = MF_{Ne} - mF_{Ne} - MF_{Ne} - m(-F_{Ne}) = (M+m)F_{Ne}$$

=>

$$Mm\frac{d^2 r}{dt^2} = Mm\frac{d^2(r_e - r_N)}{dt^2} = (M+m)F_{Ne}$$

=>

$$\frac{Mm}{(M+m)}\frac{d^2 r}{dt^2} = F_{Ne} = -\frac{e^2}{4\pi\epsilon_0 r^2}\hat{r}$$

Excersize 15: Reduced Mass of an Electron

=>

$$\mu = \frac{Mm}{M+m} = \frac{M}{M}\frac{m}{1+\frac{m}{M}} = \frac{1}{1+\frac{m}{M}}m = \left(1 - \frac{m}{M}\right)m$$

=>

$$(1+x)^n = 1 + nx$$

Excersize 16: Rydberg Constants for Hydrogen and Deuterium

=>

$$Deuterium : \frac{\mu_D}{m} = \frac{1}{1+\frac{m_e}{M_D}} = 0.999728$$

=>

$$Hydrogen : \frac{\mu_H}{m} = \frac{1}{1+\frac{m_e}{M_p}} = 0.999457$$

$$R_\infty = 1.09737x10^7 m^{-1}$$

=>

$$R_D = (0.999728)(1.09737x10^7 m^{-1}) = 1.09707x10^{10} m^{-1}$$

=>

$$R_H = (0.999457)(1.09737x10^7 m^{-1}) = 1.09677x10^{10} m^{-1}$$

Excersize 17: Shift of Wavelength for Hydrogen and Deuterium

=>

$$\frac{\lambda_H - \lambda_D}{\lambda_H} = \frac{m}{M_p}\left(1 - \frac{M_p}{M_D}\right), \frac{1}{\lambda} = \frac{\mu}{m}(n_f^{-2} - n_i^{-2})$$

=>

$$\lambda\left(\frac{1}{\lambda_D} - \frac{1}{\lambda_H}\right) = \frac{m}{M_D}\left(\frac{M_D}{m} - \frac{M_H}{m}\right) = 1 - \frac{\lambda_D}{\lambda_H} = 1 - \frac{\mu_H}{\mu_D}$$

=>

$$\frac{\lambda_H - \lambda_D}{\lambda_H} = \frac{1 + \frac{m}{M_D}}{1 + \frac{m}{M_p}} = \frac{m}{M_p} - \frac{m}{M_D} = \frac{m}{M_p}\left(1 - \frac{M_p}{M_D}\right)$$

=>

$$\frac{m}{M_p}\left(1 - \frac{M_p}{M_D}\right) = (0.0005446170215)\left(1 - \frac{1}{1.999}\right) = 0.0002722$$

Excersize 18: Spectrometer Detection of Hydrogen-Deuterium Line Shift

=>

$$R = \frac{\lambda}{\Delta\lambda} = \frac{\lambda_H}{\lambda_H - \lambda_D} = \frac{M_p}{m}\left(1 - \frac{M_p}{M_D}\right)^{-1}$$

=>

$$(1836.15)\left(1 - \frac{1}{1.999}\right)^{-1} = 3674$$

Excersize 20: Ratio of Electron Mass to Proton Mass

=>

$$\frac{m}{M_p} = \frac{\frac{R_D}{R_H} - 1}{1 - \frac{M_p}{M_D}}, \frac{R_D}{R_H} = 1 - \frac{m}{M_D} + \frac{m}{M_p} - \frac{m^2}{M_p M_D}$$

=>

$$\frac{R_D}{R_H} = 1 + \frac{m}{M_p}\left(1 - \frac{M_p}{M_D}\right)$$

=>

$$\frac{\frac{R_D}{R_H} - 1}{1 - \frac{M_p}{M_D}} = \frac{m}{M_p}$$

Excersize 21: Value for Ratio of Proton to Electron Mass

=>

$$\frac{m}{M_p} = (2.001)\left(\frac{1.09707}{1.09677} - 1\right) = 0.0005473$$

=>

$$\frac{M_p}{m} = (0.0005473)^{-1} = 1827$$

In [ ]:

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