B2-1

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B2-1.pdf

B2-1: B-field energy

When a dipole of moment $\vec{m}$ is held at an angle θ to a uniform magnetic field $\vec{B}$, the dipole will experience a torque $\vec{N}$, equal to

$$\vec{N} = \vec{m} \times \vec{B} \Rightarrow N = mB\sin\theta$$

The torque will rotate the dipole till $\vec{m}$ & $\vec{B}$ are alligned.

Work is required to rotate the dipole some angle θ. So we can write torque as

$$N = \frac{dW}{d\theta} \Rightarrow dW = N\,d\theta = mB\sin\theta\,d\theta$$

The total work between angle θ_1 & angle θ_2 is

$$W = \int_{\theta_1}^{\theta_2} mB\sin\theta\,d\theta = -mB\left(\cos\theta_2 - \cos\theta_1\right)$$

If $\theta_1 = 90°$ & $\theta_2 = \theta$ (this is equivalent to going from $U_f \to U_i$)

$$W = -mB\left(\cos\theta - \cos(90°)\right) = -mB\cos\theta$$

$$W = -mB\cos\theta = -\vec{m}\cdot\vec{B} = U$$

The Hamiltonian is defined as (quantum mechanical Hamiltonian)

$$\hat{H} = \hat{T} + \hat{U}$$

T - kinetic energy

U - potential energy

Because there is no translational motion $T=0$. So the Hamiltonian is just equal to the potential energy.

$$\hat{H} = 0 + \hat{U} = -\vec{m}\cdot\vec{B}$$

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Alternate B2-1 (from Jackson, 5.6)

If the external magnetic induction varies slowly over the region of current, a Taylor series expansion can be used to find the dominant in the force & torque.

(centered at origin)

$$\vec{B}_i(x) = \vec{B}_i(0) + x \cdot \vec{\nabla} B(0) + \dots$$

From Jackson,

$$\vec{F} = \frac{1}{c}\int \vec{J} \times \vec{B}\, d^3x' \equiv -\frac{1}{c}\vec{B}(0) \times \int J(x')\, d^3x' + \frac{1}{c}\int J(x') \times [(x' \cdot \nabla) B(0)]\, d^3x' + \dots$$

$$\left[\begin{array}{l} \int J\, d^3x \to 0 \quad \text{for steady-state current} \\ J \times [(x' \cdot \nabla) B] = J \times \nabla (x' \cdot B) = -\nabla \times [J (x' \cdot B)] \end{array}\right]$$

$$\vec{F} = -\frac{1}{c}\vec{\nabla} \times \int \vec{J} (\vec{x}' \cdot \vec{B})\, d^3x' + \dots$$

$$\approx -\frac{1}{c}\vec{\nabla} \times -\frac{1}{2}\vec{B} \times \int (\vec{x}' \times \vec{J})\, d^3x' = \frac{1}{2c}\vec{\nabla} \times \vec{B} \times \int (\vec{x}' \times \vec{J})\, d^3x'$$

$$\vec{F} = \frac{1}{2c}\vec{\nabla} \times \vec{B} \times 2c\vec{m} = \vec{\nabla} \times \vec{B} \times \vec{m} \qquad \vec{m} = \frac{1}{2c}\int (\vec{x}' \times \vec{J})\, d^3x'$$

$$= (\vec{m} \cdot \vec{\nabla})\vec{B} = \vec{\nabla}(\vec{m} \cdot \vec{B})$$

$$\vec{F} = -\vec{\nabla} U = \vec{\nabla}(\vec{m} \cdot \vec{B}) \Longrightarrow \boxed{U = -\vec{m} \cdot \vec{B}}$$

The current density of a number of charged particles with charge q_i and mass M_i in motion with velocities $\vec{v}_i$, the magnetic moment can be expressed in terms of the orbital angular momentum.

$$\vec{J} = \sum_i q_i \vec{v}_i \, \delta(\vec{x} - \vec{x}_i)$$

$$\downarrow$$

$$\vec{m} = \frac{1}{2c} \sum_i \vec{x}_i \times q_i \vec{v}_i = \frac{1}{2c} \sum_i q_i (\vec{x}_i \times \vec{v}_i)$$

$$\left(L_i = M_i (\vec{x}_i \times \vec{v}_i) \right)$$

$$\downarrow$$

$$\vec{m} = \sum_i \frac{q_i}{2M_i c} \vec{L}_i$$

If the ratio q_i/M_i is the same for all particles, $\vec{m}$ can be written as.

$$\vec{m} = \frac{e}{2Mc} \sum_i \vec{L}_i = \frac{e}{2Mc} \vec{L}_{tot}$$

For elementary particles, the intrinsic moment is off by a factor called the "g" factor. For electrons, the g-factor is nearly equal to 2.

Additionally, L (the total angular momentum) is replaced by the spin, S. The intrinsic moment is then written as.

$$\vec{m} = \frac{ge}{2Mc} \vec{S}$$

We can update the Hamiltonian

$$\mathcal{H} = -\vec{m}\cdot\vec{B} = -\frac{ge}{2mc}\vec{S}\cdot\vec{B}$$