HW (4)

6.35

$y = c_1 e^{x} + c_2 e^{-x} + \frac{x}{2} \cosh x$

6.41

$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum^{\infty} \frac{1}{n^2} \cos nx$ $\Rightarrow$ $|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum \frac{e^{inx} + e^{-inx}}{2n^2}$

$n \to$ odd

$D = -1 \pm i$

$y_h = e^{-x} (A\cos x + B \sin x)$

$Y_{p_1} = \frac{\pi}{4}$, $y_{p_k} = c e^{ikx}$ $\to$ $\cancel{c = \frac{2}{\pi k (-k^2 + 2ik + 2)}}$

$y'' + 2y' + 2y = |x|$

$(-k^2 + 2ik + 2) c e^{ik} = -\frac{2}{\pi k} e^{ikx}$

$c = \frac{+2}{\pi k [k^2 + 2ik - 2]}$

(10-15)

$y'' + 3y' - 4y = e^{3t}$

$Y = \mathcal{L} (\frac{e^{t} - e^{-4t}}{5}) \mathcal{L} e^{3t} = G(p) H(p)$

$y = \frac{e^{t}}{-16} + \frac{e^{-4t}}{35} + \frac{1}{14} e^{3t}$

12-7

$$y'' - a^2 y = e^{-t}$$

$$G(t,t') = \begin{cases} 0 \\ \frac{1}{a} \sinh a(t-t') \end{cases}$$

$$y = \int_0^t G(t,t')\, dt'$$

$$= \int \frac{1}{a} \sin a(t-t')\, e^{-t'}\, dt'$$

5) Show $\frac{n}{2\cosh^2(nx)}$ is a $\delta_n(x)$ function.

6) Solve the following integral equation,

$$\int_{-\infty}^{\infty} y(u)y(x-u)du = e^{-x^2}$$

we define a Sequence

If $\delta_n(x) = \frac{n}{2\cosh^2 nx}$, show that

$$\int_{-\infty}^{\infty} \delta_n(x)\,dx = 1 \quad \text{independent of } n.$$

$\int \tanh^2 x\,dx = x - \tanh x$

$$\int \frac{n}{2\cosh^2 nx}\,dx = \frac{n}{2}\int \frac{1}{\cosh^2 nx}\,dx = \frac{n}{2}\int \left[1 - \tanh^2 nx\right]dx$$

$$= \frac{n}{2}\left[x - \frac{(x-\tanh x)}{n}\right] = \frac{n}{2}\left[\cancel{2x} + \tanh x\right] =$$

$$\int_{-\infty}^{\infty} \delta_n(x)\,dx = \int_{-\infty}^{\infty} \frac{n}{2\cosh^2 nx}\,dx = \frac{n}{2}\left[\tanh x\right]_{-\infty}^{\infty} =$$

$$\int \frac{n}{2\cosh^2 nx}\,dx = \frac{\cancel{n}}{2}\int \frac{1}{\cosh^2 u}\,\frac{du}{\cancel{n}} = \frac{1}{2}\int \left[1 - \tanh^2 u\right]du$$

$nx = u$

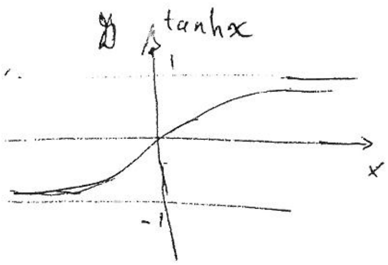
$dx = \frac{du}{n}$

$$= \frac{1}{2}\left[u - \left[u - \tanh u\right]\right]$$

$$= \frac{1}{2}\tanh u = \frac{1}{2}\left[\tanh nx\right]$$

$$\int_{-\infty}^{\infty} \delta_n(x)\,dx = \int_{-\infty}^{\infty} \frac{n}{2\cosh^2 nx}\,dx = \frac{1}{2}\left[\tanh nx\right]_{-\infty}^{\infty}$$

$$= \frac{1}{2}\left[1 - (-1)\right] = \frac{1}{2}(2) = 1$$



(2)

Solve the following integral equation,

$$\int_{-\infty}^{\infty} y(u)\,y(x-u)\,du = e^{-x^2} \xRightarrow[\text{Convolution}]{\text{From definition of}} y * y = e^{-x^2}$$

$$\frac{1}{2\pi} y * y = \frac{1}{2\pi} e^{-x^2}$$

taking the Fourier transform of both side;

$\frac{1}{2\pi}$ Fourier transform of $y_{(x)} * y_{(x)} = Y(\alpha)Y(\alpha)$

$$2\pi Y(\alpha)Y(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-x^2} e^{-i\alpha x}\,dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-x^2} [\cos\alpha x - i\sin\alpha x]\,dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-x^2}\cos\alpha x\,dx - \underbrace{\frac{i}{2\pi}\int_{-\infty} \underbrace{e^{-x^2}\sin\alpha x}_{\text{odd}}\,dx}_{0}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-x^2}\cos\alpha x\,dx$$

$$= \frac{2}{2\pi}\int_{0}^{\infty} e^{-x^2}\cos\alpha x\,dx$$

$$2\pi Y(\alpha)Y(\alpha) = \frac{\sqrt{\pi}}{2\sqrt{\pi}}\, e^{-\frac{\alpha^2}{4}}$$

$$Y^2(\alpha) = \frac{\sqrt{\pi}}{4\pi^2}\, e^{-\alpha^2/4} \Rightarrow Y(\alpha) = \frac{1}{2}\sqrt{\frac{1}{\pi^{3/2}}}\; e^{-\frac{\alpha^2}{8}}$$

$\pi^{3/2}$

Inverse Fourier Transform,

$$y(x) = \int_{-\infty}^{\infty} Y(\alpha)\, e^{i\alpha x}\,d\alpha \Rightarrow y(x) = \frac{1}{2}\sqrt{\frac{1}{\pi^{3/2}}}\int_{-\infty}^{\infty} e^{-\frac{\alpha^2}{8}} \underbrace{e^{i\alpha x}}_{\cos\alpha x + i\underbrace{\sin\alpha x}_{\text{odd}}}\,d\alpha$$

$$= \frac{1}{2}\sqrt{\frac{1}{\pi^{3/2}}}\int^{\infty} e^{-\alpha^2/8}\cos\alpha x\,d\alpha$$

$$2\sqrt{2\pi}\; e^{-2x^2}$$

$$y(x) = \cancel{\sqrt{\frac{1}{\pi^{3/2}}}}\;\frac{1}{2\pi^{\frac{3}{4}}}\int\left[\cancel{\frac{\pi^{1/2}}{2\times 8^{1/2}}\, e^{-\frac{x^2}{8}}}\right] = \frac{\cancel{2}\sqrt{2}(\pi)^{1/2}}{\cancel{2}\pi^{3/2}}\, e^{-2x^2}$$

$$2\sqrt{2}$$

$$\cancel{\boxed{y(x) = \frac{e^{-\frac{x^2}{8}}}{8\sqrt{2}\,\pi^{1/4}}}}$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}}\, e^{-2x^2}$$

$$\boxed{y(x) = \sqrt{\frac{2}{\pi}}\, e^{-2x^2}}$$

In[1]:= $\int_{-\infty}^{\infty} \mathbf{Exp[-x^2]\ Cos\ [\alpha\, x]\ d x}$

Out[1]= If$\left[\mathrm{Im}[\alpha] == 0,\ e^{-\frac{\alpha^2}{4}} \sqrt{\pi},\ \int_{-\infty}^{\infty} e^{-x^2} \mathrm{Cos}[x\,\alpha]\, dx\right]$

In[2]:= $\int_{-\infty}^{\infty} \mathbf{Exp\left[-\frac{\alpha^2}{8}\right] Cos\ [\alpha\, x]\ d\alpha}$

Out[2]= If$\left[\mathrm{Im}[x] == 0,\ 2\, e^{-2 x^2} \sqrt{2\,\pi},\ \int_{-\infty}^{\infty} e^{-\frac{\alpha^2}{8}} \mathrm{Cos}[x\,\alpha]\, d\alpha\right]$