Fixed point methods

- formulate the problem as $x=f(x)$
- solve $x=f(x)$ by

1° choose x_0, $\varepsilon>0$, k_{max}

2° compute $x_1=f(x_0)$

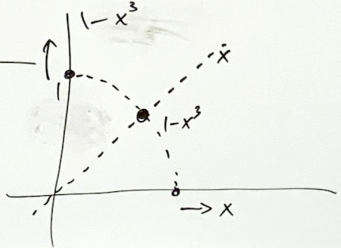
$x_2=f(x_1)$

$\vdots$

$x_{k+1}=f(x_k)$

until $|x_k - f(x_k)| < \varepsilon$ or $k>k_{max}$

x	$1-x^3$
0	1
1	0



Example

$x^3+x-1=0$

reformulate as $x=1-x^3$ choose $x_0=0$

def $f(x)=1-x^3$, compute $x_{k+1}=f(x_k)$

diverges

better try: $x^3+x-1=0$

↑ evaluate from here

$x^3=1-x$

$x=\sqrt[3]{1-x}$

k	x
0	-2
1	$(1+2)^{1/3}=\sqrt[3]{3}\approx 1.4\ldots$
2	$(\underbrace{1-\sqrt[3]{3}}_{<0})^{1/3}$

Lecture2.pdf Def. 1.4

Iterative method $x_{i+1} = g(x_i)$, exact solution $r = g(r)$

Def. 1.4 Def $e_i = |x_i - r|$. If $\lim_{i \to \infty} \frac{e_{i+1}}{e_i} = S$ then method has linear convergence with rate S.

Thm 1.2 Assume g' is continuous, $g(r) = r$, $S \overset{\text{def}}{=} |g'(r)| < 1$. Then for all x_0 sufficiently close to r, iterations $x_{i+1} = g(x_i)$ converge linearly with rate S.

mean value theorem

Proof

$$x_{i+1} = g(x_i)$$
$$r = g(r)$$

subtract $x_{i+1} - r = g(x_i) - g(r) = g'(c_i)(x_i - r)$ for some c_i between x_i and r

$$\underbrace{|x_{i+1} - r|}_{e_{i+1}} = |g'(c_i)|\underbrace{|g(x_i) - g(r)|}_{e_i}$$

$$e_{i+1} = |g'(c_i)|\, e_i$$

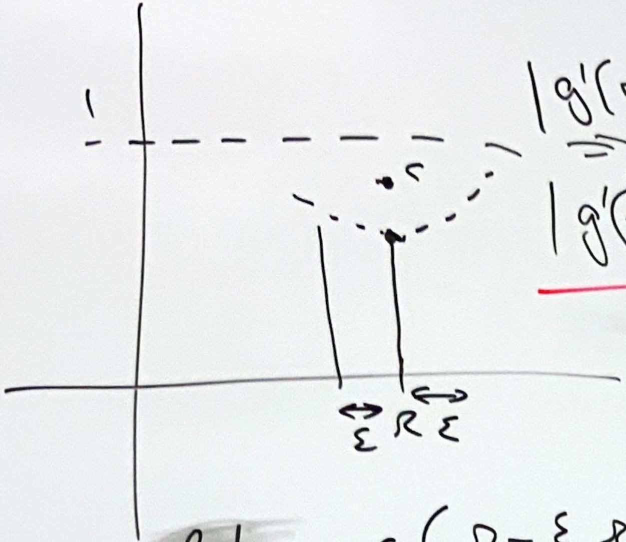
$|g'(r)| < 1 \Rightarrow \boxed{\exists c : |g'(r)| < c < 1}$

assumed $|g'(r)| < 1$

$|g'(x)|$ continuous function of x

$\Rightarrow$ for all x close enough to r, $|g'(x)| < c$ for example $\boxed{c = \frac{|g'(r)| + 1}{2}}$

$0 < c < 1$



$|g'(r)| < 1$

$\Rightarrow$

$|g'(x)| < 1$ on some $(r-\varepsilon, r+\varepsilon)$

Then let $x_0 \in (r-\varepsilon, r+\varepsilon)$ or "close to r"

$$|x_1 - r| = |g(x_0) - r| = \underbrace{|g'(c_0)|}_{<c}\,|x_0 - r| \qquad c_0 \text{ between } x_0 \text{ and } r$$

$$|x_2 - r| = |g(x_1) - r| = |g'(c_1)|\,|x_1 - r| < c\,|x_1 - r| < c^2|x_0 - r|$$

$$\vdots$$

$$|x_n - r| \le c^n |x_0 - r| \to 0 \text{ as } n \to \infty$$

because $0 \le c < 1$

Recall: $x_{i+1} = g(x_i)$, solution $r = g(r)$ $|g'(r)| < 1$ convergence rate $S = \lim_{i \to \infty} \frac{|x_{i+1} - r|}{|x_i - r|} = |g'(r)|$

make $g'(r)$ small by reformulation

to solve $f(x) = 0$ use $\boxed{g(x) = x - \frac{f(x)}{f'(x)}}$ need g' continuous around r, need $f'(r) \neq 0$, f'' continuous $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

compute $g'(r) = \left(x - \frac{f(x)}{f'(x)}\right)' = 1 - \underbrace{\frac{f'(r)f'(r) - f(r)f''(r)}{(f'(r))^2}}_{1} = 0$

at r, $f(r) = 0$

Thm If $f'(r) \neq 0$, f'' continuous in a neighborhood of r, $f(r) = 0$, then Newton's method $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ converges for any x_0 in some neighborhood of r, linearly with rate $S = 0$

$$\lim_{i \to \infty} \frac{e_{i+1}}{e_i} = 0$$