Standard 1

Saturday, January 27, 2024 12:37 PM

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16. H. B. Callen, *Thermodynamics* (John Wiley & Sons, New York, 1960).
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PROBLEMS

Problem 2.1. Test the following differentials for exactness. For those cases in which the differential is exact, find the function $u(x, y)$.

(a) $du_a = \frac{-y\,dx}{x^2+y^2} + \frac{x\,dy}{x^2+y^2}$.

(b) $du_b = (y - x^2)dx + (x + y^2)dy$.

(c) $du_c = (2y^2 - 3x)dx - 4xy\,dy$.

Problem 2.2. Consider the two differentials (1) $du_1 = (2xy + x^2)dx + x^2dy$ and (2) $du_2 = y(x - 2y)dx - x^2dy$. For both differentials, find the change in $u(x, y)$ between two points, (a, b) and (x, y). Compute the change in two different ways: (a) Integrate along the path $(a, b) \to (x, b) \to (x, y)$, and (b) integrate along the path $(a, b) \to (a, y) \to (x, y)$. Discuss the meaning of your results.

Problem 2.3. Electromagnetic radiation in an evacuated vessel of volume V at equilibrium with the walls at temperature T (black body radiation) behaves like a gas of photons having internal energy $U = aVT^4$ and pressure $P = (1/3)aT^4$, where a is Stefan's constant. (a) Plot the closed curve in the P–V plane for a Carnot cycle using blackbody radiation. (b) Derive *explicitly* the efficiency of a Carnot engine which uses blackbody radiation as its working substance.

Problem 2.4. A Carnot engine uses a paramagnetic substance as its working substance. The equation of state is $M = (nDH/T)$, where M is the magnetization, H is the magnetic field, n is the number of moles, D is a constant determined by the type of substance, and T is the temperature. (a) Show that the internal energy U, and therefore the heat capacity C_M, can only depend on the temperature and not the magnetization. Let us assume that $C_M = C =$ constant. (b) Sketch a typical Carnot cycle in the M–H plane. (c) Compute the total heat absorbed and the total work done by the Carnot engine. (d) Compute the efficiency of the Carnot engine.

2.1) a.) $du_a = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$

$$\frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right) = \frac{-((x^2+y^2)(1)-(2y)(y))}{(x^2+y^2)^2}$$

$$= \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2)(1)-(2x)(x)}{(x^2+y^2)^2}$$

$$= \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\int \frac{-y}{x^2+y^2}dx = -y\left(\frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right)\right)$$

$$\int \frac{x}{x^2+y^2}dy = x\left(\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right)\right)$$

$$\boxed{u_a = \tan^{-1}\left(\frac{y}{x}\right) - \tan\left(\frac{x}{y}\right)}$$

b.) $du_b = (y - x^2)dx + (x + y^2)dy$

$$\frac{\partial}{\partial y}(y - x^2) = 1$$

$$\frac{\partial}{\partial x}(x + y^2) = 1$$

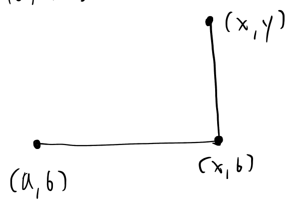
$$\int y - x^2\,dx = yx - \frac{1}{3}x^3$$

$$\int x + y^2\,dy = xy + \frac{1}{3}y^3$$

2.2) $du_1 = (2xy + x^2)dx + x^2dy$

1.) (a,b) & (x,y)

a.) $(a,b) \to (x,b) \to (x,y)$



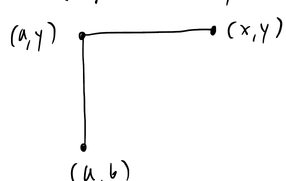
$$\int_a^x (2xy + x^2)dx + \int_b^y x^2dy = x^2y + \frac{1}{3}x^3\Big|_a^x + x^2y\Big|_b^y$$

$$= x^2y + \frac{1}{3}x^3 - a^2y - \frac{1}{3}a^3 + x^2y - x^2b$$

$$= \cancel{x^2b} + \frac{1}{3}x^3 - a^2b - \frac{1}{3}a^3 + x^2y - \cancel{x^2b}$$

$$= \frac{1}{3}x^3 + x^2y - a^2b - \frac{1}{3}a^3$$

b.) $(a,b) \to (a,y) \to (x,y)$



$$\int_b^y x^2dy + \int_a^x (2xy + x^2)dx$$

$$= \cancel{a^2y} - a^2b + x^2y + \frac{1}{3}x^3 - \cancel{a^2y} - \frac{1}{3}a^3$$

$$\int x+y^2\,dy = xy + \frac{1}{3}y^3$$

$$u_b = xy + \frac{1}{3}y^3 - \frac{1}{3}x^3$$

c.) $du_c = (2y^2-3x)dx - 4xy\,dy$

$$\frac{\partial}{\partial y}(2y^2-3x) = 4y \quad \times$$

$$\frac{\partial}{\partial x}(-4xy) = -4y \quad \times$$

$$= \cancel{a^2}y - a^2b + x^2y + \frac{1}{3}x^3 - \cancel{a^2}y - \frac{1}{3}a^3$$

$$= \frac{1}{3}x^3 + x^2y - a^2b - \frac{1}{3}a^3$$

2.) $du_2 = y(x-2y)dx - x^2dy$

b.) $(a,b) \rightarrow (x,b) \rightarrow (x,y)$

$$\int_a^x y(x-2y)\,dx - \int_b^y x^2\,dy$$

$$= \frac{1}{2}yx^2 - 2y^2x \Big|_a^x - x^2y \Big|_b^y$$

$$= \frac{1}{2}yx^2 - 2y^2x - \frac{1}{2}ya^2 + 2y^2a - x^2y + x^2b$$

$$= \frac{1}{2}bx^2 - 2b^2x - \frac{1}{2}ba^2 + 2b^2a - x^2y + x^2b$$

$$= \frac{3}{2}x^2b - x^2y - 2b^2x - \frac{1}{2}ba^2 + 2b^2a$$

b.) $(a,b) \rightarrow (a,y) \rightarrow (x,y)$

$$\int_b^y -x^2\,dy + \int_a^x y(x-2y)\,dx$$

$$= -x^2y \Big|_b^y + \frac{1}{2}yx^2 - 2y^2x \Big|_a^x$$

$$= -x^2y + x^2b + \frac{1}{2}yx^2 - 2y^2x - \frac{1}{2}ya^2 + 2y^2a$$

$$= -a^2y + a^2b + \frac{1}{2}yx^2 - 2y^2x - \frac{1}{2}ya^2 + 2y^2a$$

$$= \frac{3}{2}a^2y + \frac{1}{2}yx^2 - 2y^2x + a^2b + 2y^2a$$

$du_1 = (2xy+x^2)dx + x^2\,dy$

$$\frac{\partial}{\partial y}(2xy+x^2) = 2x$$

$$\frac{\partial}{\partial x}(x^2) = 2x$$

du_1 is exact

$du_2 = y(x-2y)\,dx - x^2dy$

$$\frac{\partial}{\partial y}(y(x-2y)) = x-4y$$

$$\frac{\partial}{\partial x}(-x^2) = -2x$$

du_2 is not exact

The answers above demonstrate how the integrals of exact differentials have path independence. Meaning, the path taken does not affect the answer, all that matters is the end points.

We can also look at this through Stokes' theorem. Given an orthogonal coordinate system and function, we can write the following.

dq_i - coordinates

F - function

$$\nabla F = \frac{\partial f}{\partial q_1}\hat{q}_1 + \frac{\partial f}{\partial q_2}\hat{q}_2 + \ldots + \frac{\partial f}{\partial q_i}\hat{q}_i$$

exact differential

$$\nabla F \cdot dq_i = \frac{\partial f}{\partial q_1}dq_1 + \frac{\partial f}{\partial q_2}dq_2 + \ldots + \frac{\partial f}{\partial q_i}dq_i = dF$$

$$\int_i^f dF = \int_i^f \nabla F \cdot dq_i = F(f) - F(i)$$

Reiterate path independence

Stoke's Theorem

$$\oint_p \nabla F \cdot dq_i = \int_S (\nabla \times \nabla F) \cdot da = 0$$

curl of gradient $= 0$