Waves1-1 (Wave Equation)

Waves1-1: "I can write down a general solution for a time-varying electric and magnetic field and prove that it satisfies Maxwell's equations

Maxwell's Equations

$$\vec{\nabla}\cdot E = \rho/\varepsilon_0 \qquad\qquad \vec{\nabla}\cdot\vec{E} = 0$$

(in a vacuum)

$$\vec{\nabla}\cdot B = 0 \quad \Longrightarrow \quad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad\qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \varepsilon_0\frac{\partial\vec{E}}{\partial t}\right) \qquad\qquad \vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

In a vacuum there is no charge density (\rho) since there is no charged material present and there is no current density (J) since there is no medium through which the electromagnetic wave moves through.

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right) = \vec{\nabla}\left(\cancelto{0}{\vec{\nabla}\cdot\vec{E}}\right) - \vec{\nabla}^2\vec{E}$$

$$= \vec{\nabla}\times -\frac{\partial\vec{B}}{\partial t} = -\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{B}\right)$$

$$\vec{\nabla}^2\vec{E} = \frac{\partial}{\partial t}\left(\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

$$\vec{\nabla}^2\vec{E} = \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$ (Wave equation for an electric field)

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right) = \vec{\nabla}\left(\cancelto{0}{\vec{\nabla}\cdot\vec{B}}\right) - \vec{\nabla}^2\vec{B}$$

$$= \vec{\nabla}\times\left(\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

$$\vec{\nabla}^2\vec{B} = -\mu_0\varepsilon_0\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{E}\right)$$

$$\nabla B = -\mu_0 \varepsilon_0 \frac{\partial}{\partial t} (\nabla \times E)$$

$$\vec{\nabla}^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$ (Wave equation for magnetic field)

$$\left(\vec{\nabla}^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \right) \vec{E} = 0$$

By taking the curl of the curl of each type of field we are able to obtain forms of the wave equation for each type of field.

$$\left(\vec{\nabla}^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \right) \vec{B} = 0$$

Where $\mu_0 \varepsilon_0 = \frac{1}{c^2}$

And c is the speed of light.

General Solution to Wave equation

$$u(\vec{x}, t) = f(\vec{x} + ct) + g(\vec{x} - ct)$$

$$\vec{x} = \langle x, y, z \rangle$$

$$\nabla u = \nabla (f + g) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} + \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial g}{\partial z}$$

$$\nabla^2 u = \nabla^2 (f + g) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} + \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} (f + g) = c \frac{\partial f}{\partial t} - c \frac{\partial g}{\partial t}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial t^2} (f + g) = c^2 \frac{\partial^2 f}{\partial t^2} + c^2 \frac{\partial^2 g}{\partial t^2}$$

$$\nabla^2 u - \mu_0 \varepsilon_0 \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} + \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} - c^2 \frac{\partial^2 f}{\partial t^2} - c^2 \frac{\partial^2 g}{\partial t^2} = 0$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - c^2 \frac{\partial^2}{\partial t^2} \right) (f + g) = 0$$