Homework 2 - Math Methods

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Mathematical Methods: Homework 2

4.18)

$r = r e_r \Rightarrow$ position

$e_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \qquad e_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

$$\frac{dr}{dt} = r\frac{de_r}{dt} + \frac{dr}{dt}e_r$$

$e_\theta \frac{d\theta}{dt}$

$$= r\left(-\sin\theta\frac{d\theta}{dt}\hat{i} + \cos\theta\frac{d\theta}{dt}\hat{j}\right) + \frac{dr}{dt}e_r$$

$$= r\frac{d\theta}{dt}e_\theta + \frac{dr}{dt}e_r \Big)$$ velocity

$-\frac{d\theta}{dt}e_r \qquad e_\theta$

$$\frac{d^2r}{dt} = \frac{dr}{dt}\frac{d\theta}{dt}e_\theta + r\left(\frac{d\theta}{dt}\cancel{\frac{de_\theta}{dt}} + \frac{d^2\theta}{dt^2}e_\theta\right) + \frac{d^2r}{dt^2}e_r + \frac{dr}{dt}\cancel{\frac{de_r}{dt}}$$

$$\frac{de_\theta}{dt} = -\cos\theta\frac{d\theta}{dt}\hat{i} - \sin\theta\frac{d\theta}{dt}\hat{j} = -\frac{d\theta}{dt}e_r$$

$$\frac{d^2r}{dt^2} = \frac{dr}{dt}\frac{d\theta}{dt}e_\theta - r\left(\frac{d\theta}{dt}\right)^2 e_r + r\frac{d^2\theta}{dt^2}e_\theta + \frac{d^2r}{dt^2}e_r + \frac{dr}{dt}\frac{d\theta}{dt}e_\theta$$

$$= \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)e_r + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)e_\theta\Big)$$ acceleration

6.10) $T = xy - x \qquad T = 0, 1, 2, -1, -2$

```
import numpy as np
import matplotlib.pyplot as plt

x = np.linspace(-10, 10, 100)
y = np.linspace(-10, 10, 100)

X, Y = np.meshgrid(x, y)

T_values = X * Y - X

isothermal_T_values = [-2, -1, 0, 1, 2]

colors = ['blue', 'green', 'red', 'purple', 'orange']

for T, color in zip(isothermal_T_values, colors):
    plt.contour(X, Y, T_values, levels=[T], colors=[color])
```

```python
import numpy as np
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X, Y = np.meshgrid(x, y)

T_values = X * Y - X

isothermal_T_values = [-2, -1, 0, 1, 2]

colors = ['blue', 'green', 'red', 'purple', 'orange']

for T, color in zip(isothermal_T_values, colors):
    plt.contour(X, Y, T_values, levels=[T], colors=[color])

plt.xlabel('x')
plt.ylabel('y')
plt.title('Isothermal Curves for T = xy - x')
plt.grid(True)
plt.show()
```
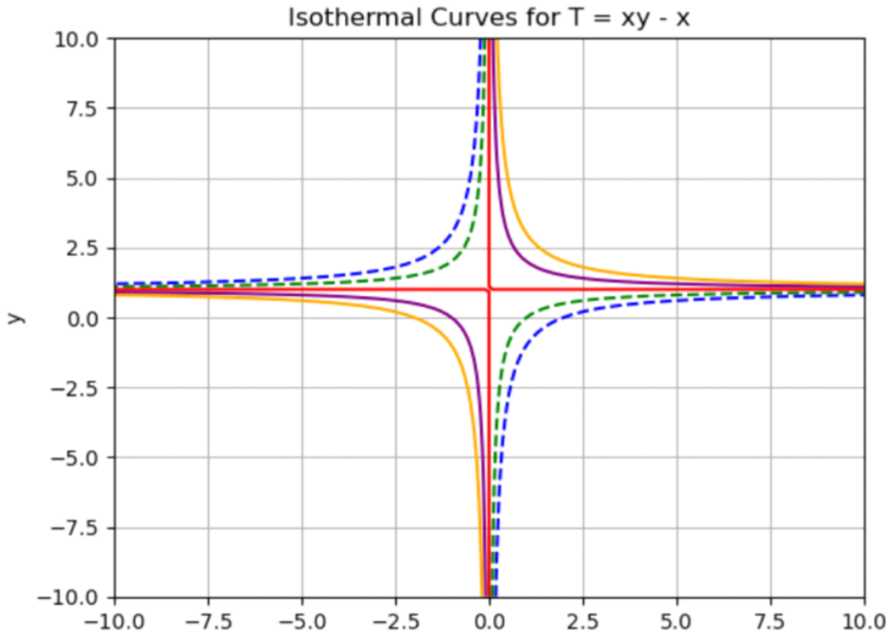



blue = -2
green = -1
red = 0
purple = 1
Orange = 2

$$\nabla T = \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}$$

$$= (y-1)\hat{i} + x\hat{j}$$

$$-\nabla T = (1-y)\hat{i} - x\hat{j}$$

Heat flow

from point $(x_0, y_0) = (1,1)$

$$-\nabla T(x_0, y_0) = -\hat{j} \Rightarrow |-\nabla T| = 1$$

In the direction of $u = 3\hat{i} - 4\hat{j}$

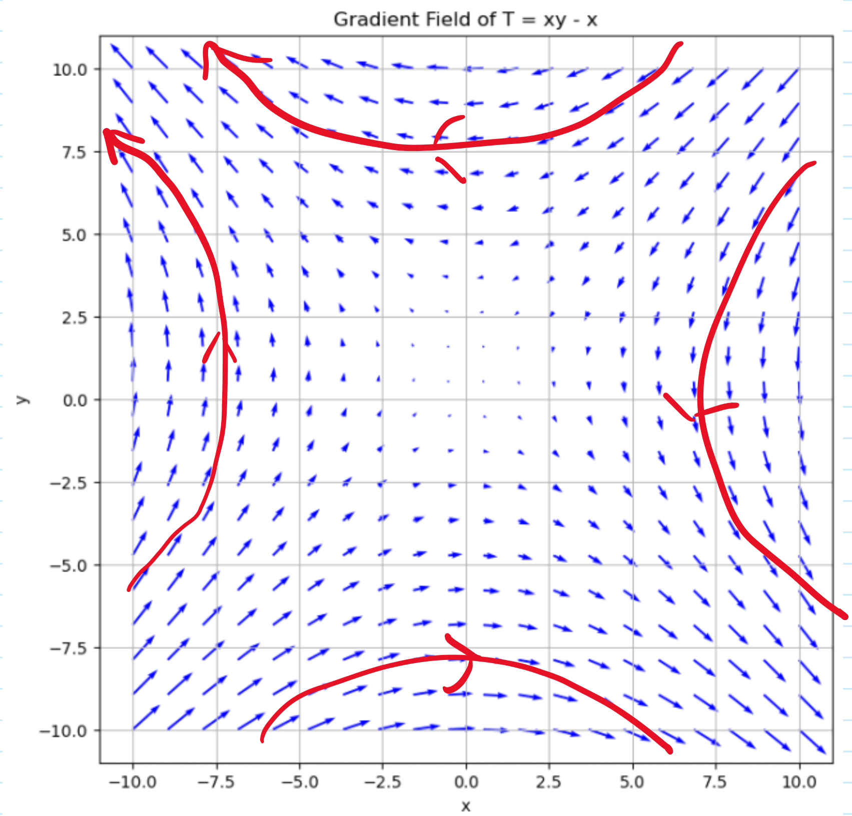
In the direction of $u = 3\hat{i} - 4\hat{j}$

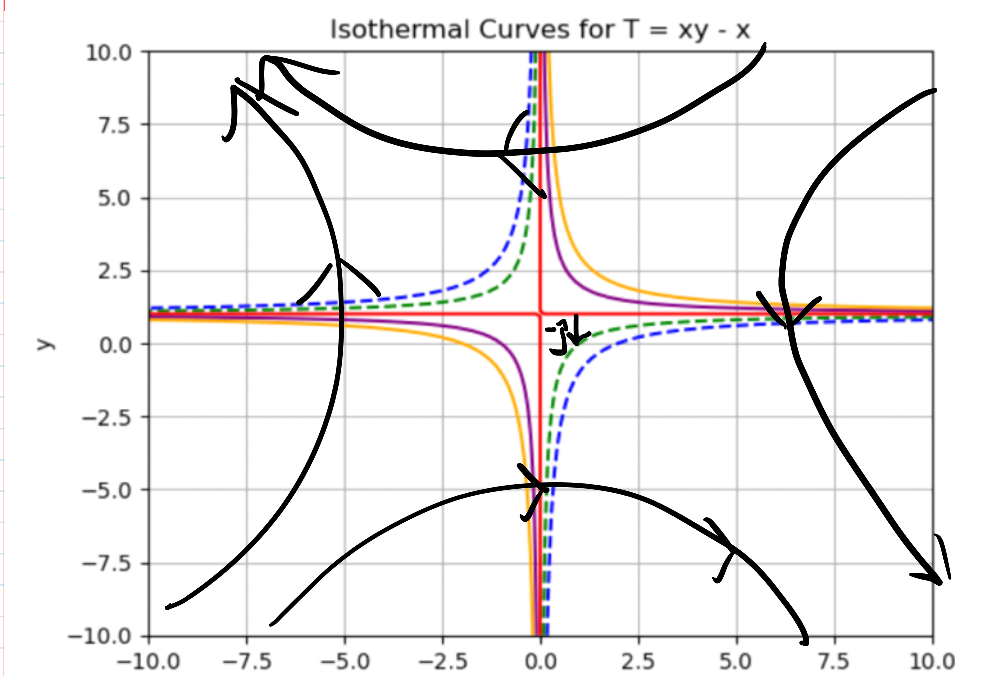
$$\hat{u} = \frac{(3\hat{i} - 4\hat{j})}{\sqrt{3^2 + 4^2}} = \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}$$

$$\frac{dT}{ds} = \vec{\nabla} T \cdot \hat{u} = (0, 1) \cdot \left(\frac{3}{5}, -\frac{4}{5}\right) = -\frac{4}{5}$$

Gradient Field

```python
import numpy as np
import matplotlib.pyplot as plt

x = np.linspace(-10, 10, 20)
y = np.linspace(-10, 10, 20)

X, Y = np.meshgrid(x, y)

dT_dx = (1-Y) * 0.1  # I changed the scale of each vector because it was illegible otherwise.
dT_dy = -X * 0.1

plt.figure(figsize=(8, 8))
plt.quiver(X, Y, dT_dx, dT_dy, color='blue')
plt.xlabel('x')
plt.ylabel('y')
plt.title('Gradient Field of T = xy - x')
plt.grid()
plt.show()
```





black = gradient curves

7.19)

$$\nabla \cdot \left(\frac{r}{|r|}\right) \; ; \; r = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\nabla \cdot \left(\frac{r}{|r|}\right) = \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\nabla\cdot\left(\frac{r}{|r|}\right) = \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)$$

$$+\frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z}}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\sqrt{x^2+y^2+z^2}(1) - x\left(\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-1/2}\cdot\cancel{2}x\right)}{(x^2+y^2+z^2)}$$

$$= \frac{\sqrt{x^2+y^2+z^2} - \frac{x^2}{\sqrt{x^2+y^2+z^2}}}{(x^2+y^2+z^2)} = \frac{\frac{x^2+y^2+z^2 - x^2}{\sqrt{x^2+y^2+z^2}}}{(x^2+y^2+z^2}$$

$$= \frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) = \frac{x^2+z^2}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = \frac{x^2+y^2}{(x^2+y^2+z)^{3/2}}$$

$$\nabla\cdot\left(\frac{r}{|r|}\right) = \frac{y^2+z^2+x^2+z^2+x^2+y^2}{(x^2+y^2+z^2)^{3/2}} = \frac{2(x^2+y^2+z^2)^{2/2}}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{2}{\sqrt{x^2+y^2+z^2}}$$

8.17) $F_1 = -y\hat{\imath} + x\hat{\jmath} + z\hat{k} \quad F_2 = y\hat{\imath} + x\hat{\jmath} + z\hat{k}$

8.17)

$$F_1 = -y\hat{\imath} + x\hat{\jmath} + z\hat{k} \qquad F_2 = y\hat{\imath} + x\hat{\jmath} + z\hat{k}$$

$$\nabla \times F_1 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & z \end{vmatrix} = \hat{\imath}\left(\frac{\partial}{\partial y} z - \frac{\partial}{\partial z} x\right) - \hat{\jmath}\left(\frac{\partial}{\partial z}(-y) - \frac{\partial}{\partial x}(z)\right)$$

$$+\hat{k}\left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right)$$

$F_1 \Rightarrow$ <u>not conservative</u>

$$= \hat{\imath}(0-0) - \hat{\jmath}(0-0) + \hat{k}(1+1) \neq 0$$

$$\nabla \times F_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & z \end{vmatrix} = \hat{\imath}\left(\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(x)\right) - \hat{\jmath}\left(\frac{\partial}{\partial z}(y) - \frac{\partial}{\partial x}(z)\right)$$

$$+\hat{k}\left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(y)\right)$$

$F_2 =$ <u>is conservative</u>

$$= \hat{\imath}(0-0) - \hat{\jmath}(0-0) + \hat{k}(1-1) = 0$$

$$x = \cos(t) \qquad y = \sin(t)$$

$$r = x\hat{\imath} + y\hat{\jmath} \Rightarrow dr = dx\,\hat{\imath} + dy\,\hat{\jmath}$$

$$dx = -\sin(t)\,dt \qquad dy = \cos(t)\,dt$$

$$F_1 \cdot dr = (-y\hat{\imath} + x\hat{\jmath} + z\hat{k}) \cdot (dx\,\hat{\imath} + dy\,\hat{\jmath})$$

$$= -y\,dx + x\,dy$$

$$= -\sin(t)(-\sin(t)\,dt) + \cos(t)\cos(t)\,dt$$

$$= \sin^2(t)\,dt + \cos^2(t)\,dt = dt\,(\underbrace{\sin^2(t) + \cos^2(t)}_{1})$$

$$dW_1 = F_1 \cdot dr = dt$$

$$dW_1 = F_1 \cdot dr = dt$$

$$W_1 = \int_0^{2\pi} dt = 2\pi$$

$$dW_2 = F_2 \cdot dr = (y\hat{\imath} + x\hat{\jmath} + z\hat{k}) \cdot (dx\hat{\imath} + dy\hat{\jmath})$$

$$= y\,dx + x\,dy$$

$$= \sin(t)(-\sin(t)\,dt) + \cos(t)\cos(t)\,dt$$

$$= -\sin^2(t)\,dt + \cos^2(t)\,dt$$

$$= dt(\cos^2(t) - \sin^2(t))$$

$$= dt\cos(2t)$$

$$W_2 = \int_0^{2\pi} \cos(2t)\,dt \quad \left| \begin{array}{l} u = 2t \\ du = 2\,dt \\ \frac{1}{2}du = dt \end{array} \right.$$

$$= \frac{1}{2}\int_{u=0}^{u=4\pi} \cos(u)\,du = \frac{1}{2}\sin(u)\Big|_0^{4\pi} = \frac{1}{2}\left(\overset{0}{\cancel{\sin(4\pi)}} - \sin(0)\right) = 0$$

$$\boxed{W_1 = 2\pi \qquad W_2 = 0}$$

8.21)

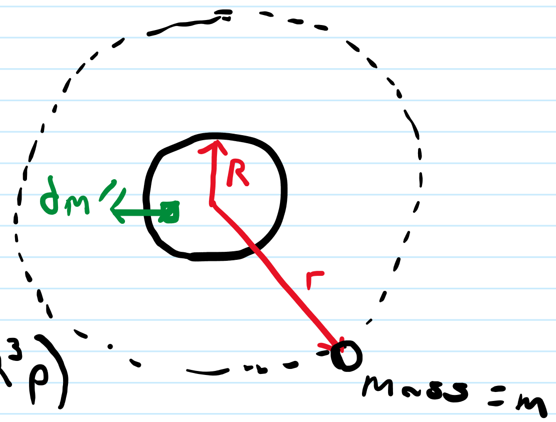
M = total mass of Earth = m'

R = radius of Earth

Const. $r < R$

R = radius of Earth

$$\varphi = \begin{cases} \text{const.} & r < R \\ -\frac{cm'}{r} & r > R \end{cases}$$



For $r > R$

$$\varphi = -\frac{cm'}{r} = -\frac{c}{r}\left(\frac{4}{3}\pi R^3 \rho\right)$$

$$\varphi = -\frac{cm'}{r}$$

$$F = -\nabla\varphi = -\frac{cm'}{r^2} \Rightarrow F = -mg \approx -\frac{cm'}{R^2}$$

$$mgR^2 = cm'$$

$$\boxed{\varphi = -\frac{mgR^2}{r} \qquad F = -\frac{mgR^2}{r^2}}$$

$r < R$

$$\varphi = \underbrace{-\frac{cm'}{r}}_{\text{at } r = R} + \underbrace{c_2}_{\text{inside}}$$

$$\varphi = -\frac{c}{\cancel{r}}\left(\frac{4}{3}\pi r^{\cancel{3}} \rho\right) + c_2 = -\frac{4}{3}\pi r^2 c\rho + c_2$$

$$= -\lambda r^2 + c_2$$

$$-\nabla\varphi = 2\lambda r = F$$

$r = R$

$$2\lambda R = -mg \Rightarrow \lambda = -\frac{mg}{\ldots}$$

$$2\lambda R = -mg \Rightarrow \lambda = -\frac{mg}{2R}$$

$$\boxed{F = -\frac{mgr}{R} e_r}$$

$$\varphi = \frac{mgr^2}{2R} + c_2$$

$r = R$

$$\varphi = \underbrace{\frac{mgR^2}{2R}}_{\text{at surface}} + \underbrace{c_2}_{\text{inside}} = \underbrace{-\frac{mgR^2}{R}}_{\text{outside}}$$

$$\frac{mgR}{2} + c_2 = -mgR$$

$$c_2 = \frac{1}{2}\left(-2mgR - mgR\right)$$

$$c_2 = -\frac{3}{2}mgR$$

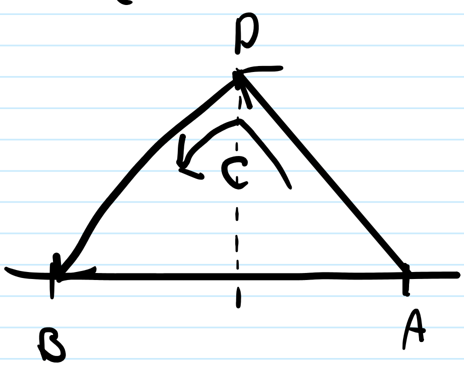
$$\varphi = \frac{mgr^2}{2R} - \frac{3}{2}mgR = \frac{mg}{2}\left(\frac{r^2}{R} - 3R\right)$$

$$\varphi = \frac{mg}{}\left(\frac{r^2}{} - 3\frac{R^2}{}\right) = \frac{mg}{}\left(r^2 - 3R\right)$$

$$\boxed{\varphi = \frac{mg}{2}\left(\frac{r^2}{R} - 3\frac{R^2}{R}\right) = \frac{mg}{2R}\left(r^2 - 3R\right)}$$

9.4)

$$\int_C e^x \cos(y)\,dx - e^x \sin(y)\,dy$$



$C = ADBA$

$P = e^x \cos(y) \qquad Q = -e^x \sin(y)$

$$\frac{\partial P}{\partial y} = -e^x \sin(y) \qquad \frac{\partial Q}{\partial x} = -e^x \sin(y)$$

$$\int_C e^x \cos(y)\,dx - e^x \sin(y)\,dy = \int_C \left(-e^x \sin(y) + e^x \sin(y)\right) dx\,dy$$

$$= 0$$

$$\int_{ADBA} = \int_{ADB} + \int_B^A = 0 \Rightarrow \int_{ADB} = -\int_B^A$$

Over B to A

$x = x \quad dx = dx$

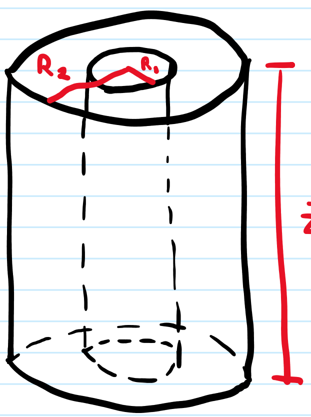
$x = x \quad dx = dx$

$y = 0 \quad dy = 0$

$$= -\int_{-\ln(2)}^{\ln(2)} e^x\left(\cos(0)\,dx - \sin(0)(0)\right) = -\int_{-\ln(2)}^{\ln(2)} e^x\,dx$$

$$= -e^{\ln(2)} + e^{-\ln(2)} = -2 + \frac{1}{2} = -\frac{3}{2}$$

10.12)



$$\left.\begin{aligned} K &= q_1/z \\ -K &= q_2/z \end{aligned}\right\} \text{Linear Charge Density}$$

$$q_{enc}/\varepsilon_0 = \oint E \cdot dA$$

$$q_{enc} = Kz = \oint \vec{D}\cdot\hat{n}\,d\sigma$$

<u>On the surface of the inner cylinder</u>

$$\left.\begin{aligned} \sigma &= 2\pi r h \\ d\sigma &= 2\pi r\,dh \end{aligned}\right\} \text{Surface Area of Cylinder}$$

$$Kz = \oint \vec{D}\cdot\hat{n}\,d\sigma = \int_0^z D(2\pi r)\,dh$$

$$Kz = D(2\pi r z)$$

$$\frac{K}{2\pi r} = D \Rightarrow \underline{E = \frac{K}{2\pi r \varepsilon_0}} \Bigg)\ \vec{E}\text{-field inbetween cylinders} \Bigg)\ \text{at } r = R_1$$

No $\vec{E}$-field when $r < R_1$ because it is a conductor.

Outside of outer cylinder

$$\underbrace{Kz}_{\text{from inner cylinde}} - \underbrace{Kz}_{\text{from outer cylinder}} = \int_0^z D_2(2\pi r)\,dh = D_2(2\pi r z)$$

$$\frac{z(K - K)}{2\pi r z \varepsilon_0} = E_2 = 0$$

$$E = -\nabla\varphi$$

$$\varphi = -\int E\,dr \quad \Bigg)\ E \neq 0 \text{ only when } R_1 \leq r < R_2$$

$$\varphi = -\int_{R_1}^{r} \frac{K}{2\pi r \varepsilon_0}\,dr = -\frac{K}{2\pi\varepsilon_0}\left(\ln(r) - \ln(R_1)\right)$$

$$\varphi = \frac{K}{\ldots}\left(\ln(R_1) - \ln(r)\right) = \frac{K}{\ldots}\ln\left(\frac{R_1}{\ldots}\right), \quad \text{when } R_1 < r < R_2$$

$$\varphi = \frac{k}{2\pi\varepsilon_0}\left(\ln(R_1) - \ln(r)\right) = \frac{k}{2\pi\varepsilon_0}\ln\left(\frac{R_1}{r}\right), \quad \text{when } R_1 < r < R_2$$

$\underline{r \leq R_1}$

No electric potential

$$\varphi = \int 0 \, dr = c_1$$

$$\varphi = \frac{k}{2\pi\varepsilon_0}\ln\left(\frac{R_1}{R_1}\right) = 0$$

$\underline{\text{When } r \geq R_2}$

$$\varphi = \int 0 \, dr = c_2$$

$$\varphi = \frac{k}{2\pi\varepsilon_0}\ln\left(\frac{R_1}{R_2}\right)$$

$$\varphi = \begin{cases} 0, & r < R_1 \\ \frac{k}{2\pi\varepsilon_0}\ln\left(\frac{R_1}{r}\right), & R_1 \leq r \leq R_2 \\ \frac{k}{2\pi\varepsilon_0}\ln\left(\frac{R_1}{R_2}\right), & r \geq R_2 \end{cases}$$

11.20) $V = (x^2 - 2xz)\hat{i} + (y^2 - 2xy)\hat{j} + (z^2 - 2yz + xy)\hat{k}$

$$V = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = \overbrace{\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)}^{V_x}\hat{i} - \overbrace{\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)}^{V_y}\hat{j}$$

$$V = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\quad}{\partial y} - \frac{\quad}{\partial z}\right)\hat{i} - \left(\frac{\quad}{\partial z} - \frac{\quad}{\partial x}\right)\hat{j}$$

$$+ \underbrace{\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)}_{V_z}\hat{k}$$

$$V_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = z e^{zy} + x \sin(zx)$$

$$V_y = \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} = x \cos(xz)$$

$$V_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = -z \sin(zx)$$

$$\frac{\partial A_z}{\partial x} = \frac{\partial A_x}{\partial y} = 0$$

$$\frac{\partial A_x}{\partial z} = -x \cos(xz)$$

$$A_x = -\sin(xz) + f_1(x,y)$$

$$\frac{\partial A_y}{\partial x} = -z \sin(zx) \qquad f_1(x,y) = f_2(x,y) = 0$$

$$A_y = \cos(zx) + f_2(x,y)$$

$$V_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = z e^{zy} + x \sin(zx)$$

$$\frac{\partial A_y}{\partial z} = -x \sin(zx)$$

$$\frac{\partial A_y}{\partial z} = -x\sin(zx)$$

$$\frac{\partial A_z}{\partial y} + x\sin(zx) = ze^{zy} + x\sin(zx)$$

$$\frac{\partial A_z}{\partial y} = ze^{zy}$$

$$A_z = e^{zy}$$

$$\boxed{\vec{A} = -\sin(xz)\,\hat{\imath} + \cos(xz)\,\hat{\jmath} + e^{zy}\,\hat{k}}$$