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Method of Mathematical Physics

Dirichlet vs Fejer Kernel Fourier Series

Introduction

So long as a function satisfies the Dircihlet conditions, its Fourier series approximation is guaranteed to converge. That is, for a periodic function $f(x)$ that is single-valued over a period T that has a finite number of discontinuities and extrema, as well as a finite total variation over T, then its Fourier series S_N converges to $f(x)$ at all points where $f(x)$ is continuous; at jump discontinuities, S_N converges to the midpoint of the jump [1]. If we consider the final part of the previous statement, specifically for the case of a finite Fourier series, we quickly run into a non-negligible source of error known as the Gibbs phenomenon.

Simply, the Gibbs phenomenon is a consequence of the slow convergence of the Fourier coefficients near discontinuities. As the number of terms in the series increases, the Fourier series oscillations intensify, causing the overshoot to become more pronounced. Interestingly, and maybe somewhat counter-intuitively, the overshoot does not diminish with additional terms but instead transforms into a narrower and taller spike, with a height around $\sim 9\%$ of the total height of the jump [2].

While the Gibbs phenomenon does not affect the overall convergence of the Fourier series to the original function away from the points of discontinuity, it does highlight a limitation of Fourier series approximations when dealing with functions featuring sudden changes [3]. However, it is possible to employ corrective fixes, such as the use of alternate kernels like the Fejér kernel, to reduce the impact of the Gibbs phenomenon and enhance the accuracy of Fourier series approximations [4].

In this paper, we are going to explore the use of an alternate kernel, namely the Fejér kernel, as it helps to reduce the impact of the Gibb phenomenon. First, we will discuss the Dirichlet kernel and the role it plays in the common Fourier series. Then, we will provide a brief overview of the Fejér kernel, as well as provide a brief discussion of its properties. Finally, we will do a comparison of the two kernels in the approximation of a square wave in order to demonstrate the reduction of the Gibbs phenomenon.

Fourier Series and the Dirichlet Kernel

The Fourier is a powerful mathematical tool which allows us to describe a periodic function in terms of a linear combination of orthogonal sinusoidal functions. First, we should recall that the complex form of the Fourier series is

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \text{ [1].}$$

We can rewrite this as an N-th partial sum

$$S_N f(x) = \sum_{k=-N}^{N} c_k e^{ikx} \text{ [4] .}$$

We should also remember that for some function that is absolutely integrable over the interval $[-\pi, \pi]$, the definition of its coefficients is

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}dt \text{ [4].}$$

Combining this definition with the N-th partial sum, we get

$$S_N f(x) = \sum_{k=-N}^{N} (\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt)e^{inx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) \sum_{k=-N}^{N} e^{in(x-t)}dt$$

From the above relation, we refer to the term in the final summation term the Dirichlet kernel, written more generally as

$$D_N(x) = \sum_{k=-N}^{N} e^{ikx} \text{ [4].}$$

From this we can see that the Dirichlet kernel is responsible for introducing the oscillating components of the Fourier series (contained within the complex exponential term).

Additionally, if we form a partial Fourier sum (that is over some interval of iterations $[-N, N]$) we get

$$S_N f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) \sum_{k=-N}^{N} e^{in(x-t)}dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x-t)f(t)dt = \frac{1}{2\pi}D_N * f(x) \text{ [5].}$$

Note how we are able to obtain the definition of convolution when we use this identity. This shows us that the convolution of the Dirichlet kernel with any function $f(x)$ of some period is equal to the Fourier series approximation of $f(x)$. The convolution in this process effectively means that the Dirichlet kernel is moved along the function $f(t)$ over the period, where it is multiplied pointwise and then integrated over the interval. This has the effect of superimposing the oscillations from the Dirichlet kernel onto the original function.

It’s easy to see that the definition of the Dirichlet kernel is a geometric series. Meaning that we can use the formula for a geometric series $\sum_{k=0}^{n} ar^k = a\frac{(1-r^{n+1})}{(1-r)}$ to solve for it.

$$\sum_{k=-N}^{N} r^k = r^{-N}\frac{1-r^{2N+1}}{1-r} = \frac{r^{-N-1/2}}{r^{-1/2}}\frac{1-r^{2N+1}}{1-r} = \frac{r^{-(N+1/2)}-r^{N+1/2}}{r^{-1/2}-r^{1/2}}$$

In this case, $r = e^{ix}$. Plugging that value in gives us

$$\sum_{k=-N}^{N} e^{ikx} = \frac{e^{-(N+\frac{1}{2})ix}-e^{(N+\frac{1}{2})ix}}{e^{-\frac{ix}{2}}-e^{\frac{ix}{2}}} == \frac{cos((N+\frac{1}{2})x)-isin((N+\frac{1}{2})x)-cos((N+\frac{1}{2})x)-isin((N+\frac{1}{2})x)}{cos(\frac{x}{2})-isin(\frac{x}{2})-cos(\frac{x}{2})-isin(\frac{x}{2})} = \frac{-2isin((N+\frac{1}{2})x)}{-2isin(\frac{x}{2})}$$

$$\sum_{k=-N}^{N} e^{ikx} = \frac{-2isin((N+\frac{1}{2})x)}{-2isin(\frac{x}{2})} == \frac{sin((N+\frac{1}{2})x)}{sin(\frac{x}{2})}$$ [6]

This finally gives us the equation

$$D_N(x) = \frac{sin((N+\frac{1}{2})x)}{sin(\frac{x}{2})}$$

This form will work for most values of x. However, we can also see that this is undefined for $x = 0$. In order to find the Dirichlet kernel in this case, we take the limit and apply l'Hôpital's rule.

$$\lim_{x \to 0} \frac{d}{dx}\left(\frac{sin((N+\frac{1}{2})x)}{sin(\frac{x}{2})}\right) = \lim_{x \to 0} \frac{(N+\frac{1}{2})cos((N+\frac{1}{2})x)}{\frac{1}{2}cos(\frac{x}{2})} = \frac{N+\frac{1}{2}}{\frac{1}{2}} = 2N + 1$$

With this value, we can now write the Dirichlet kernel as

$$D_N(x) = \left\{\frac{2N+1}{2\pi},\ x = 0\ :\ \frac{1}{2\pi}\frac{sin((N+\frac{1}{2})x)}{sin(\frac{x}{2})}\ \ otherwise\right\}$$

It’s worth noting the the Dirichlet kernel above has an additional constant term $\frac{1}{2\pi}$included. So far, this has come from the definition of the Fourier series, but we will find it advantageous to include it in the definition of the Dirichlet kernel from here on out.

Cesàro summation and the Fejér kernel

We should now introduce the Cesàro mean. As a general technique, a Cesàro mean is simply the arithmetic mean and is used to enhance the convergence properties of a series. This is particularly useful when the original series exhibits oscillations (such as we see in the Gibbs phenomenon) or does not converge. Taking the arithmetic mean of the first N partial sum of a series.

$$\sigma_N(x) = \frac{1}{N+1}\sum_{k=0}^{N} S_k \text{ (}N\text{-th Cesàro mean for a given series } S_k\text{)}$$

Applying our definition of a partial Fourier series in terms of the convolution of the Dirichlet kernel and some periodic function, we find that

$$\sigma_N f(x) = \frac{1}{N+1}\sum_{k=0}^{N} S_k f(x) = \int_{-\pi}^{\pi} \frac{1}{N+1}\sum_{k=0}^{N} D_k(x - t)f(t)dt.$$

From this we get a new kernel, which we define as the Fejér kernel.

$$K_N(x) = \frac{1}{N+1}\sum_{k=0}^{N} D_k(x) \text{ [4]}$$

Using this, we can define the Fejér kernel more explicitly.

$$K_N = \frac{1}{2\pi N}\sum_{k=0}^{N-1}\frac{sin((k+\frac{1}{2})x)}{sin\frac{x}{2}} = \frac{1}{2\pi N sin^2(\frac{x}{2})}\sum_{k=0}^{N-1} sin((k + \frac{1}{2})x)sin(\frac{x}{2})$$

$$= \frac{1}{2\pi N sin^2(\frac{x}{2})}\sum_{k=0}^{N-1}\frac{1}{2}(cos(kx) - cos((k + 1)x)$$

$$K_N = \frac{1}{2\pi N}\frac{1}{sin^2(\frac{x}{2})}\frac{1-cos(Nx)}{2} = \frac{1}{2\pi N}\frac{1}{sin^2(\frac{x}{2})}sin^2(\frac{Nx}{2}) = \frac{1}{2\pi N}\left(\frac{sin(\frac{Nx}{2})}{sin\ (\frac{x}{2})}\right)^2 \text{[4]}$$

This definition holds for almost any value of x, except for $x = 0$. We can find this by using a combination of trigonometric identities and l'Hopital's rule. When we do this, we find that

$$K_N(x = 0) = \frac{N+1}{2\pi} \text{ [4].}$$

This allows us to write the Fejér kernel as

$$K_N(x) = \left\{\frac{N+1}{2\pi},\ x = 0\ :\ \frac{1}{2\pi N}\left(\frac{sin(\frac{Nx}{2})}{sin\ (\frac{x}{2})}\right)^2\ otherwise\right\}.$$

If we are to trace back and apply the Fejér kernel in our definition of the Cesàro mean, we get

$$\sigma_N f(x) = \frac{1}{N+1}\sum_{k=0}^{N} S_k f(x) = \int_{-\pi}^{\pi}\frac{1}{N+1}\sum_{k=0}^{N} D_k(x - t)f(t)dt = \int_{-\pi}^{\pi} K_N(x - t)f(t)dt = K_N * f$$

showing that the convolution of the Fejér kernel with any periodic function returns the Cesàro mean of the Dirichlet kernels.

The Fejér kernel has a few notable traits. Most notably, it is non-negative and has an average value of 1. The non-negative aspect turns out to make a substantial difference as it prevents the introduction of opposite signed oscillations, which contribute to the overshooting effect we get with the Dirichlet kernel. Additionally, with an average value of 1 results in the Fejér kernel behaving similarly to the Dirac delta function. Specifically, since we know

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)dx = 1 \text{ (average value)}$$

As well the definition of the Cesàro-Fourier series we obtained above, we can reason that

$$\sigma_N f(x) = \int_{-\pi}^{\pi} K_N(x - t)f(t)dt \approx f(x)\int_{-\pi}^{\pi} K_N(x - t)dt = f(x) \cdot 1 \text{ [4].}$$

The reason we can do this is that K_N is sharply peaked around $t = x$, so as the N becomes greater and greater, the main contribution to the integral comes from $f(x) \approx f(t)$ so long as f is a well-behaved function (i.e. continuous) [4]. What this means is that for a continuous function, the Cesàro-Fourier series converges uniformly– this is essentially Fejér's theorem.

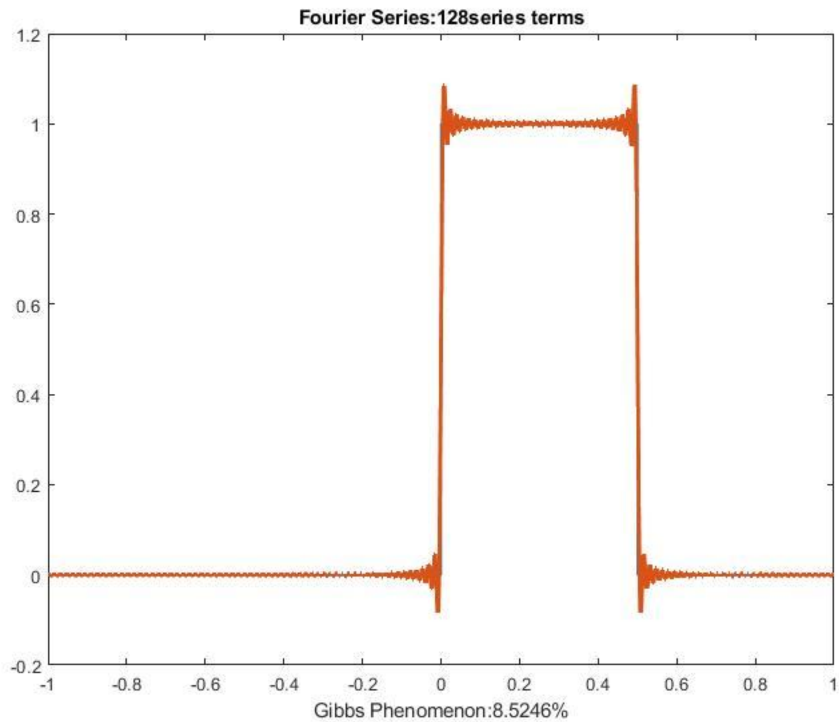
On the following three pages we have attached an example made in MATLAB, comparing the use of the Fourier Series and the Cesàro-Fourier series in approximating a square wave.

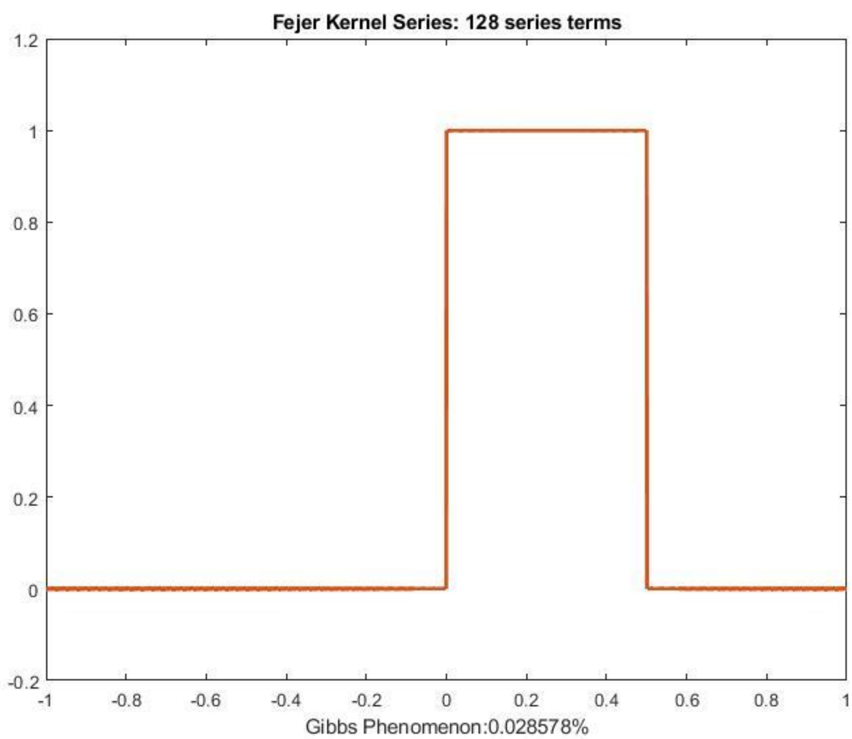
Fejér Kernel code example

```
%-------------------------------------------------------------%
% Physics Math Methods
% PHYS 3120
% Term Paper - Dirichlet vs Fejer Kernel Fourier Series
%
% Zach Eggeman, James Amidei, Steven Bann
%-------------------------------------------------------------%

clc
clear
close all

%-------------------------------------------------------------%
%Constants
%-------------------------------------------------------------%

N = 128; %number of series terms

L = 1; %upper and lower limit
n = 1024;
dx = 2*L/(n-1);
x = -L:dx:L;

A = zeros(N);
B = zeros(N);
sum_fej = zeros(N);

%-------------------------------------------------------------%
%Original Function
%-------------------------------------------------------------%

f =0*x;
for i=1:numel(x)
    if (x(i)>0) && (x(i)<0.5)
        f(i) = 1;
    end
end

%-------------------------------------------------------------%
%Fourier Series
%-------------------------------------------------------------%
```

```
A0 = sum(f.*ones(size(x)))*dx/L;
fFS = A0/2;
for k=1:N
    A(k) = sum(f.*cos(pi*k*x/L))*dx/L;
    B(k) = sum(f.*sin(pi*k*x/L))*dx/L;
    fFS = fFS + A(k)*cos(k*pi*x/L) + B(k)*sin(k*pi*x/L);
end
fFS_max = max(fFS);
Gibbs = ((fFS_max - L)/L)*100;

%-------------------------------------------------------------%
%Fourier Series - Fejer
%-------------------------------------------------------------%

A0_fej = sum(f.*ones(size(x)))*dx/L;
fFS_fej = A0_fej/2;
for k=1:N
    sum_fej(k) = sum(f.*((1/(2*pi*(k+1)))*(sin(((k+1)/2)*x)/(sin(x/2)).^2)));
    fFS_fej = fFS_fej + abs(sum_fej(k)*exp(1i*k*pi*x/L));
end

fFS_max_fej = max(fFS_fej);
Gibbs_fej = ((fFS_max_fej - L)/L)*100;

%-------------------------------------------------------------%
%Plotting
%-------------------------------------------------------------%
```

```
%Fourier
figure(1)
plot(x,f,'LineWidth',1.5)
hold on
plot(x,fFS,'LineWidth',2)
title_fourier = strcat('Fourier Series: ',num2str(N),'series terms');
title(title_fourier)
x_lbl = strcat('Gibbs Phenomenon: ',num2str(Gibbs),'%');
xlabel(x_lbl)

%Fourier Fejer
figure(2)
plot(x,f,'LineWidth',1.5)
hold on
plot(x,fFS_fej,'LineWidth',2)
title_fourier_fej = strcat('Fejer Kernel Series: 128 series terms');
title(title_fourier_fej)
x_lbl = strcat('Gibbs Phenomenon: ',num2str(Gibbs_fej),'%');
xlabel(x_lbl)
```





Here we can see that the Fejer kernel series converges to the desired waveform quicker and with significantly less overshoot than the standard fourier series.

References

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