Standard 8

Use the Sackur-Tetrode equation to show that a adiabatic compression of an ideal gas is reversable.

$$\frac{S}{k_B N} = \ln\left[\frac{V}{N}\left(\frac{4\pi m U}{3h^2 N}\right)^{3/2}\right] + \frac{5}{2}$$

(**Sackur-Tetrode equation**: expresses the entropy of a monatomic ideal gas in terms of its volume (V), energy (U), and number of particles (N).

$$pV = Nk_B T$$

(Ideal gas equation)

The compression of an ideal gas is said to be adiabatic if the process is so fast that no heat flows out of (or into) the gas. In other words, an adiabatic process is one where the change in energy is equal to the change in the work.

$$dU = W$$

$$\frac{f}{2} N k_B dT = -p\,dV$$

(where f is the degrees of freedom)

Using the ideal gas law to solve for pressure:

$$\frac{f}{2} N k_B\, dT = -\left(\frac{Nk_B T}{V}\right) dV$$

$$\int_{T_i}^{T_f} \frac{f}{2}\frac{dT}{T} = -\int_{V_i}^{V_f} \frac{dV}{V}$$

$$\frac{f}{2}\ln\left(\frac{T_f}{T_i}\right) = \ln\left(\frac{V_i}{V_f}\right)$$

$$\frac{T_f^{f/2}}{T_i^{f/2}} = \frac{V_i}{V_f} \Rightarrow T_f^{f/2} V_f = T_i^{f/2} V_i = \text{constant} \quad (1.)$$

$$\left(\frac{p_f V_f}{\ldots}\right)^{f/2} V_f = \left(\frac{p_i V_i}{Nk\ldots}\right)^{f/2} V_i$$

$$\left(\frac{P_f V_f}{N k_B}\right)^{f/2} V_f = \left(\frac{P_i V_i}{N k_B}\right)^{f/2} V_i$$

$$\left(\frac{P_f}{P_i}\right)^{\frac{f}{2}\cdot\frac{2}{f}} = \left(\frac{V_i}{V_f}\right)^{\frac{f+2}{2}\cdot\frac{2}{f}} \Rightarrow \frac{P_f}{P_i} = \frac{V_i^{\frac{f+2}{f}}}{V_f^{\frac{f+2}{f}}}$$

$$P_f V_f^{\frac{f+2}{f}} = P_i V_i^{\frac{f+2}{f}} = \text{constant} \quad (2.)$$

(1) and (2) are both part of the definition of an adiabatic process.

A process is said to be reversible if the total entropy of the universe unchanged.

$$\Delta S = S_f - S_i = 0$$

If we take the definition of entropy below:

$$\Delta S = \frac{Q}{T}$$

We should be able to see how an adiabatic compression of an ideal gas like we assumed above is reversible.

$$Q = 0$$

$$\Delta S = \frac{0}{T} = 0$$

Assuming a reversible compression with the Sackur-Tetrode equation we get the following process.

$$\Delta S = S_f - S_i = \cancel{N}k_B\left[\ln\left(\frac{V_f}{N}\left(\frac{4\pi m U_f}{3h^2 N}\right)^{3/2}\right) + \cancel{\frac{5}{2}}\right]$$

$$- \cancel{N}k_B\left[\ln\left(\frac{V_i}{N}\left(\frac{4\pi m U_i}{3h^2 N}\right)^{3/2}\right) + \cancel{\frac{5}{2}}\right] = 0$$

$$\Rightarrow \ln\left(\frac{V_f}{\cancel{N}}\right) + \frac{3}{2}\ln\left(\frac{\cancel{4\pi m}\, U_f}{\cancel{3h^2 N}}\right) - \ln\left(\frac{V_i}{\cancel{N}}\right) - \frac{3}{2}\ln\left(\frac{\cancel{4\pi m}\, U_i}{\cancel{3h^2 N}}\right) = 0$$

$$\Rightarrow \ln\left(\frac{V_f}{V_i}\right) + \frac{3}{2}\ln\left(\frac{U_f}{U_i}\right) = 0$$

$$\Rightarrow \left(\frac{U_f}{U_i}\right)^{3/2} = \frac{V_i}{V_f}$$

$$\Rightarrow \left(\frac{\cancel{\frac{3}{2}}\cancel{N}\cancel{k_B}\,T_f}{\cancel{\frac{3}{2}}\cancel{N}\cancel{k_B}\,T_i}\right)^{3/2} = \frac{V_i}{V_f}$$

$$\Rightarrow V_f T_f^{3/2} = V_i T_i^{3/2}$$

(This is the same as (1) above which we got from the definition of adiabatic compression. In this case f = 3, which is the number of degrees of freedom of a monatomic ideal gas.)

Then is also follows:

$$\Rightarrow V_f\left(\frac{P_f V_f}{N k_B}\right)^{3/2} = V_i\left(\frac{P_i V_i}{N k_B}\right)^{3/2}$$

$$P_f V_f^{5/2} = P_i V_i^{5/2}$$

$$\frac{3+2}{2} = \frac{5}{2} \checkmark$$

By assuming that our process was reversible, we were able to trace back towards to show that it must have been an adiabatic compression.