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Assignment #2

3.13

$$\left(-\frac{\hbar^2}{2m_0}\nabla^2 + V\right)\psi = i\hbar\frac{\partial}{\partial t}\psi \quad (\text{T.D.S.E.})$$

where $\alpha = \frac{4\hbar\pi^2}{2m_0 L_z^2}$ & $\beta = \frac{2\pi}{L_z}$

ii) $e^{i\left(\frac{4\hbar\pi^2}{2m_0L_z^2}\right)t}\sin\left(\frac{2\pi}{L_z}z\right) = \psi(z,t) = e^{i\alpha t}\sin(\beta z)$

Sign on time component is wrong. $\rightarrow$ Because of this, it will not be a solution.

$$\frac{\partial\psi}{\partial t} = i\alpha e^{i\alpha t}\sin(\beta z) \qquad \nabla\psi = e^{i\alpha t}\beta\cos(\beta z) \rightarrow \nabla^2\psi = e^{i\alpha t}(-\beta^2\sin(\beta z))$$
$$= -e^{i\alpha t}\beta^2\sin(\beta z)$$

$$-\frac{\hbar^2}{2m_0}\left(-e^{-i\alpha t}\sin(\beta z)\beta^2\right) + Ve^{i\alpha t}\sin(\beta z) = i\hbar\left(i\alpha e^{i\alpha t}\sin(\beta z)\right)$$

$$\frac{\hbar^2}{2m_0}\beta^2\cancel{e^{i\alpha t}}\cancel{\sin(\beta z)} + V\cancel{e^{i\alpha t}}\cancel{\sin(\beta z)} = -\hbar\alpha\cancel{e^{i\alpha t}}\cancel{\sin(\beta z)}$$

$$\frac{\hbar^2}{2m_0}\left(\frac{2\pi}{L_z}\right)^2 + V = -\frac{4\hbar^2\pi^2}{2m_0L_z^2}$$

– when $V=0$ –

$$\frac{4\hbar^2\pi^2}{2m_0L_z^2} = -\frac{4\hbar^2\pi^2}{2m_0L_z^2}$$

$1 = -1$ Not a solution

iv)

$$\psi(z,t) = 2e^{-i\left(\frac{\hbar\pi^2}{2m_0L_z^2}\right)t}\sin\left(\frac{\pi}{L_z}z\right) - ie^{-i\left(\frac{9\hbar\pi^2}{2m_0L_z^2}\right)t}\sin\left(\frac{3\pi}{L_z}z\right)$$

Linear combination of two solutions, so also a solution.

$$\alpha = \frac{\hbar\pi^2}{2m_0L_z^2}, \quad \beta = \frac{\pi}{L_z}$$

$$\psi(z,t) = 2e^{-i\alpha t}\sin(\beta z) - ie^{-i9\alpha t}\sin(3\beta z)$$

$$\frac{\partial\psi}{\partial t} = -2i\alpha e^{-i\alpha t}\sin(\beta z) - 9\alpha e^{-i9\alpha t}\sin(3\beta z)$$

$$\nabla\psi = 2e^{-i\alpha t}\beta\cos(\beta z) - ie^{-i9\alpha t}3\beta\cos(3\beta z)$$

$$\nabla^2\psi = -2e^{-i\alpha t}\beta^2\sin(\beta z) + ie^{-i9\alpha t}3^2\beta^2\sin(3\beta z)$$

$$-\frac{\hbar^2}{2m_0}\left(-2e^{-i\alpha t}\beta^2\sin(\beta z) + i9e^{-i9\alpha t}\beta^2\sin(3\beta z)\right) + V\left(-2e^{-i\alpha t}\beta^2\sin(\beta z) + i9e^{-i9\alpha t}\beta^2\sin(3\beta z)\right)$$

$$= i\hbar\left(-2i\alpha e^{-i\alpha t}\sin(\beta z) - 9\alpha e^{-i9\alpha t}\sin(3\beta z)\right)$$

$$\frac{\hbar}{2m_0}\beta^2\left(\cancel{2e^{-i\alpha t}\sin(\beta z) - i9e^{-i9\alpha t}\sin(3\beta z)}\right) - V\beta^2\left(\cancel{2e^{-i\alpha t}\sin(\beta z) - i9e^{-i9\alpha t}\sin(3\beta z)}\right)$$

$$= \hbar\alpha\left(\cancel{2e^{-i\alpha t}\sin(\beta z) - i9e^{-i9\alpha t}\sin(3\beta z)}\right)$$

$$\frac{\hbar^2}{2m_0}\beta^2 - V\beta^2 = \hbar\alpha$$

$$- V = 0 -$$

$$\frac{\hbar^2}{2m_0}\left(\frac{\pi}{L_z}\right)^2 = \hbar\left(\frac{\hbar\pi^2}{2m_0L_z^2}\right)$$

$$\frac{\hbar^2\pi^2}{2m_0L_z^2} = \frac{\hbar^2\pi^2}{2m_0L_z^2} \quad \text{is a solution}$$

$$1 = 1$$

3.6.1

From page 62:

"... if we make a linear superposition of two energy eigenstates with energies E_a & E_b, the resulting probability distribution will oscillate at the frequency $\omega_{ab} = |E_a - E_b|/\hbar$."

From page 52:

$$E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{L_z}\right)^2, \quad n = 1, 2, \ldots$$

radians per second

Units

$$\left\{\frac{J\cdot s}{kg\cdot m^2}\right\} \equiv \left\{\frac{kg\cdot m^2}{s^2}\,\frac{s}{kg\cdot m^2}\right\} = \left\{\frac{1}{s}\right\}$$

$$L_z = 4\,\mathring{A} = 4\times10^{-10}\,m$$

$n_a = 1$

$n_b = 3$

$|1 - 9| = 8$

$$\omega_{13} = \frac{|E_1 - E_2|}{\hbar} = \frac{\hbar^2\pi^2}{2mL_z^2}|n_a^2 - n_b^2|\left(\frac{1}{\hbar}\right) = \frac{\hbar\pi^2}{2mL_z^2}|n_a^2 - n_b^2|$$

$$= \frac{8\hbar\pi^2}{2mL_z^2} = \frac{4\hbar\pi^2}{mL_z^2} = \frac{4(1.055\times10^{-34})(3.142)^2}{(9.109\times10^{-31})(4\times10^{-10})^2}$$

$$\boxed{\omega_{13} = 2.858\times10^{16}\ s^{-1}}$$ ← in radians per sec

3.6.3

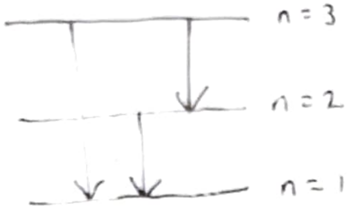
$L_z = 1\,nm = 10^{-9}\,m$

$$\psi = c_1 e^{-iE_1t/\hbar} \sin\left(\frac{\pi}{L_z} z\right) + c_2 e^{-iE_2t/\hbar} \sin\left(\frac{2\pi}{L_z} z\right) + c_3 e^{-iE_3t/\hbar} \sin\left(\frac{3\pi}{L_z} z\right)$$

$$E_n = \frac{\hbar^2}{2m}\left(\frac{\pi n}{L_z}\right)^2 = \frac{\hbar^2\pi^2 n^2}{2mL_z^2}$$

$\hbar = 1.055 \times 10^{-34}\ J\cdot s$

$m_e = 9.11 \times 10^{-31}\ kg$



$3 \to 1$ $\quad \Delta E_{31} = \hbar(\omega_3 - \omega_1)$

$3 \to 2$ $\quad \Delta E_{32} = \hbar(\omega_3 - \omega_2)$

$2 \to 1$ $\quad \Delta E_{21} = \hbar(\omega_2 - \omega_1)$

$$\omega_{31} = \frac{\Delta E_{31}}{\hbar} = \omega_3 - \omega_1 = \frac{\hbar\pi^2}{2mL_z^2}(3^2 - 1^2) = 8\frac{\hbar\pi^2}{2mL_z^2}$$

$$\omega_{32} = \frac{\Delta E_{32}}{\hbar} = \omega_3 - \omega_2 = \frac{\hbar\pi^2}{2mL_z^2}(3^2 - \underset{2^2}{4}) = 5\frac{\hbar\pi^2}{2mL_z^2}$$

$$\omega_{21} = \frac{\Delta E_{21}}{\hbar} = \omega_2 - \omega_1 = \frac{\hbar\pi^2}{2mL_z^2}(2^2 - 1^2) = 3\frac{\hbar\pi^2}{2mL_z^2}$$

$$\frac{\hbar\pi^2}{2mL_z^2} = 5.71 \times 10^{14}\ rads/s \cdot \frac{1}{2\pi}$$

$$= 9.10 \times 10^{13}\ Hz$$

$\omega_{31} = 8(9.10 \times 10^{13}\ Hz) = 7.28 \times 10^{14}\ Hz$

$\omega_{32} = 5(9.10 \times 10^{13}\ Hz) = 4.55 \times 10^{14}\ Hz$

$\omega_{21} = 3(9.10 \times 10^{13}\ Hz) = 2.73 \times 10^{14}\ Hz$

3.12.1

Page 62 & 52

i)

$$\omega_{21} = \frac{E_2 - E_1}{\hbar} = \frac{1}{\hbar}\left(\frac{\hbar^2 \pi^2 n_a^2}{2mL_z^2} - \frac{\hbar^2 \pi^2 n_b^2}{2mL_z^2}\right) = \frac{\hbar\pi^2}{2mL_z^2}\left(n_a^2 - n_b^2\right)$$

$$2^2 - 1^2 = 3$$

$n_a = 2$

$n_b = 1$

$$\omega_{21} = 3\frac{\hbar\pi^2}{2mL_z^2}$$

ii)

$$\hat{p} = -i\hbar\nabla$$

Page 52

$$\psi_n(z) = \sqrt{\frac{2}{L_z}}\sin\left(\frac{n\pi z}{L_z}\right)$$

$$\langle\hat{p}\rangle = \int_0^{L_z}\psi^*(-i\hbar\nabla)\psi\,dz$$

Page 60

$$\Psi_{12} = c_1 e^{-iE_1t/\hbar}\psi_1(r) + c_2 e^{-iE_2t/\hbar}\psi_2(r)$$

Page 61

$$= \frac{1}{\sqrt{L_z}}\left(e^{-iE_1t/\hbar}\sin\left(\frac{\pi}{L_z}z\right) + e^{-iE_2t/\hbar}\sin\left(\frac{2\pi}{L_z}z\right)\right)$$

$$-i\hbar\nabla\psi = -i\hbar\frac{1}{\sqrt{L_z}}\nabla\left(e^{-iE_1t/\hbar}\sin\left(\frac{\pi}{L_z}z\right) + e^{-iE_2t/\hbar}\sin\left(\frac{2\pi}{L_z}z\right)\right)$$

$$= \frac{-i\hbar}{\sqrt{L_z}}\left(\frac{\pi}{L_z}e^{-iE_1t/\hbar}\cos\left(\frac{\pi}{L_z}z\right) + \frac{2\pi}{L_z}e^{-iE_2t/\hbar}\cos\left(\frac{2\pi}{L_z}z\right)\right)$$

$$= -\frac{i\hbar\pi}{L_z^{3/2}}\left(e^{-iE_1t/\hbar}\cos\left(\frac{\pi}{L_z}z\right) + 2e^{-iE_2t/\hbar}\cos\left(\frac{2\pi}{L_z}z\right)\right)$$

$$\psi^*(-i\hbar\nabla\psi) = \left[\frac{1}{\sqrt{L_z}}\left(e^{iE_1t/\hbar}\sin\left(\frac{\pi}{L_z}z\right) + e^{iE_2t/\hbar}\sin\left(\frac{2\pi}{L_z}z\right)\right)\right]$$
$$\times\left[-\frac{i\hbar\pi}{L_z^{3/2}}\left(e^{-iE_1t/\hbar}\cos\left(\frac{\pi}{L_z}z\right) + 2e^{-iE_2t/\hbar}\cos\left(\frac{2\pi}{L_z}z\right)\right)\right]$$

$$= -\frac{i\hbar\pi}{L_z}\left(\sin\left(\frac{\pi}{L_z}z\right)\cos\left(\frac{\pi}{L_z}z\right) + 2\sin\left(\frac{\pi}{L_z}z\right)\cos\left(\frac{2\pi}{L_z}z\right)e^{i(E_1-E_2)t/\hbar}\right.$$
$$\left. + \sin\left(\frac{2\pi}{L_z}z\right)\cos\left(\frac{\pi}{L_z}z\right)e^{i(E_2-E_1)t/\hbar} + 2\sin\left(\frac{2\pi}{L_z}z\right)\cos\left(\frac{2\pi}{L_z}z\right)\right)$$

$$\langle\hat{p}\rangle = -\frac{i\hbar\pi}{L_z}\int_0^{L_z} \overset{0}{\cancel{\sin\left(\frac{\pi}{L_z}z\right)\cos\left(\frac{\pi}{L_z}z\right)}}\,dz - \frac{i\hbar\pi}{L_z}\int_0^{L_z} \overset{2\left(-\frac{2L}{3\pi}\right)}{2\sin\left(\frac{\pi}{L_z}z\right)\cos\left(\frac{2\pi}{L_z}z\right)} e^{i(E_1-E_2)t/\hbar}\,dz$$

$$-\frac{i\hbar\pi}{L_z}\int_0^{L_z} \overset{\frac{4L_z}{3\pi}}{\sin\left(\frac{2\pi}{L_z}z\right)\cos\left(\frac{\pi}{L_z}z\right)} e^{i(E_2-E_1)t/\hbar}\,dz - \frac{i\hbar\pi}{L_z}\int_0^{L_z} \overset{0}{\cancel{2\sin\left(\frac{2\pi}{L_z}z\right)\cos\left(\frac{2\pi}{L_z}z\right)}}\,dz$$

$$\frac{4}{3}\frac{L_z}{\pi}$$

$$= -\frac{i\hbar\pi}{L_z}\int_0^{L_z} 2\sin\left(\frac{\pi}{L_z}z\right)\cos\left(\frac{2\pi}{L_z}z\right) e^{i(E_1-E_2)t/\hbar}\,dz$$

$$\int_0^{L_z} \sin\left(\left(\frac{\pi}{L_z}+\frac{2\pi}{L_z}\right)z\right) + \sin\left(\left(\frac{\pi}{L_z}-\frac{2\pi}{L_z}\right)z\right)dz$$

$$\int_0^{L_z} \sin\left(\frac{3\pi}{L_z}z\right) - \sin\left(\frac{\pi}{L_z}z\right)dz = -\frac{L_z}{3\pi}\cos\left(\frac{3\pi}{L_z}z\right) + \frac{L_z}{\pi}\cos\left(\frac{\pi}{L_z}z\right)\Bigg|_0^{L_z}$$

$$= -\frac{L_z}{3\pi}\cos(3\pi) + \frac{L_z}{\pi}\cos(\pi) + \frac{L_z}{3\pi}\cos(0) - \frac{L_z}{\pi}\cos(0)$$

$$-\frac{i\hbar\pi}{L_z}\int_0^{L_z}\sin\left(\frac{2\pi}{L_z}z\right)\cos\left(\frac{\pi}{L_z}z\right) e^{\frac{i(E_2-E_1)t}{\hbar}}\,dz$$

$$= \frac{L_z}{3\pi} + -\frac{L_z}{\pi} + \frac{L_z}{3\pi} - \frac{L_z}{\pi} = \frac{2L_z}{3\pi} - \frac{2L_z}{\pi}$$

$$= \frac{2L_z}{3\pi} - \frac{6L_z}{3\pi}$$

$$= -\frac{4L_z}{3\pi}$$

$$\int_0^{L_z}\frac{1}{2}\left(\sin\left(\left(\frac{2\pi}{L_z}+\frac{\pi}{L_z}\right)z\right) + \sin\left(\left(\frac{2\pi}{L_z}-\frac{\pi}{L_z}\right)z\right)\right)dz$$

$$\int_0^{L_z}\frac{1}{2}\left(\sin\left(\frac{3\pi}{L_z}z\right) + \sin\left(\frac{\pi}{L_z}\right)\right)dz = \frac{1}{2}\left(-\frac{L_z}{3\pi}\cos\left(\frac{3\pi}{L_z}z\right) - \frac{L_z}{\pi}\cos\left(\frac{\pi}{L_z}z\right)\right)\Bigg|_0^{L_z}$$

$$= \frac{1}{2}\left(-\frac{L_z}{3\pi}\cos(3\pi) - \frac{L_z}{\pi}\cos(\pi) + \frac{L_z}{3\pi}\cos(0) + \frac{L_z}{\pi}\cos(0)\right)$$

$$= \frac{1}{2}\left(\frac{L_z}{3\pi} + \frac{L_z}{\pi} + \frac{L_z}{3\pi} + \frac{L_z}{\pi}\right) = \frac{1}{2}\left(\frac{2L_z}{3\pi} + \frac{2L_z}{\pi}\right) = \frac{1}{2}\left(\frac{2L_z}{3\pi} + \frac{6L_z}{3\pi}\right)$$

$$= \frac{1}{2}\left(\frac{8L_z}{3\pi}\right) = \frac{4L_z}{3\pi}$$

$$\cancel{\langle \hat{p} \rangle = -\frac{i\hbar\pi}{L_z}\left(-\frac{4}{3}\frac{L_z}{\pi} + \frac{4}{3}\frac{L_z}{\pi}\right)}$$

$$\cancel{\langle \hat{p} \rangle = -\frac{i\hbar\pi}{L_z}\left(-\frac{4L_z}{3\pi} e^{i(E_1+E}\right.}$$

$$\langle \hat{p} \rangle = -\frac{i\hbar\pi}{L_z}\left(-\frac{4L_z}{3\pi} e^{i(E_1-E_2)t/\hbar} + \frac{4L_z}{3\pi} e^{i(E_2-E_1)t/\hbar}\right)$$

$$= \frac{4}{3}\hbar i\left(e^{i(E_1-E_2)t/\hbar} - e^{i(E_2-E_1)t/\hbar}\right)$$

$$= \frac{4}{3}\hbar i\left(e^{i(E_1-E_2)t/\hbar} - e^{-i(E_1-E_2)t/\hbar}\right) = \frac{4}{3}\hbar i\left(2\sin\left(\frac{E_1-E_2}{\hbar}t\right)\right)$$

$$\langle \hat{p} \rangle = \frac{8\hbar}{3} i \sin\left(\frac{E_1-E_2}{\hbar}t\right)$$

iii) Yes.

$$\langle \hat{p} \rangle \propto \int_0^{L_z} \cancelto{0}{\sin\left(\frac{\pi}{L_z}z\right)\cos\left(\frac{\pi}{L_z}z\right)}\,dz + \int_0^{L_z} 3\sin\left(\frac{\pi}{L_z}z\right)\cos\left(\frac{3\pi}{L_z}z\right)dz$$

$$+ \int_0^{L_z} \sin\left(\frac{3\pi}{L_z}z\right)\cos\left(\frac{\pi}{L_z}z\right)dz + \int_0^{L_z} \cancelto{0}{3\sin\left(\frac{3\pi}{L_z}z\right)\cos\left(\frac{3\pi}{L_z}z\right)}\,dz$$

$$\propto \int_0^{L_z} 3\sin\left(\frac{\pi}{L_z}z\right)\cos\left(\frac{3\pi}{L_z}z\right)dz + \int_0^{L_z} \sin\left(\frac{3\pi}{L_z}z\right)\cos\left(\frac{\pi}{L_z}z\right)dz$$

$$\downarrow$$

$$= \int_0^{L_z} \frac{3}{2}\left(\sin\left(\left(\frac{\pi}{L_z}+\frac{3\pi}{L_z}\right)z\right) + \sin\left(\left(\frac{\pi}{L_z}-\frac{3\pi}{L_z}\right)z\right)\right)dz + \int_0^{L_z} \frac{1}{2}\left(\sin\left(\left(\frac{3\pi}{L_z}+\frac{\pi}{L_z}\right)z\right) + \sin\left(\left(\frac{3\pi}{L_z}-\frac{\pi}{L_z}\right)z\right)\right)dz$$

$$= \frac{3}{2}\int_0^{L_z} \sin\left(\frac{4\pi}{L_z}z\right) + \sin\left(-\frac{2\pi}{L_z}z\right) + \frac{1}{2}\int_0^{L_z} \sin\left(\frac{4\pi}{L_z}z\right) + \sin\left(\frac{2\pi}{L_z}z\right)dz$$

$$= \frac{3}{2}\left(-\frac{L_z}{4\pi}\cos\left(\frac{4\pi}{L_z}z\right) + \frac{L_z}{2\pi}\cos\left(\frac{2\pi}{L_z}z\right)\right) + \frac{1}{2}\left(-\frac{L_z}{4\pi}\cos\left(\frac{4\pi}{L_z}z\right) - \frac{L_z}{2\pi}\cos\left(\frac{2\pi}{L_z}z\right)\right)\Bigg|_0^{L_z}$$

$$= \frac{3}{2}\left(-\frac{L_z}{4\pi}\cancelto{1}{\cos(4\pi)} + \frac{L_z}{2\pi}\cancelto{1}{\cos(2\pi)}\right) + \frac{1}{2}\left(-\frac{L_z}{4\pi}\cancelto{1}{\cos(4\pi)} - \frac{L_z}{2\pi}\cancelto{1}{\cos(2\pi)}\right)$$

$$- \frac{3}{2}\left(-\frac{L_z}{4\pi}\cancelto{1}{\cos(0)} + \frac{L_z}{2\pi}\cancelto{1}{\cos(0)}\right) - \frac{1}{2}\left(-\frac{L_z}{4\pi}\cancelto{1}{\cos(0)} - \frac{L_z}{2\pi}\cancelto{1}{\cos(0)}\right)$$

$$= -\cancel{\frac{3}{2}\frac{L_z}{4\pi}} + \cancel{\frac{3}{2}\frac{L_z}{2\pi}} - \cancel{\frac{L_z}{4\pi}\frac{1}{2}} - \cancel{\frac{1}{2}\frac{L_z}{2\pi}} + \cancel{\frac{3}{2}\frac{L_z}{4\pi}} - \frac{3}{2}\frac{L_z}{2\pi} + \cancel{\frac{1}{2}\frac{L_z}{4\pi}} + \cancel{\frac{1}{2}\frac{L_z}{2\pi}}$$

$$= 0$$