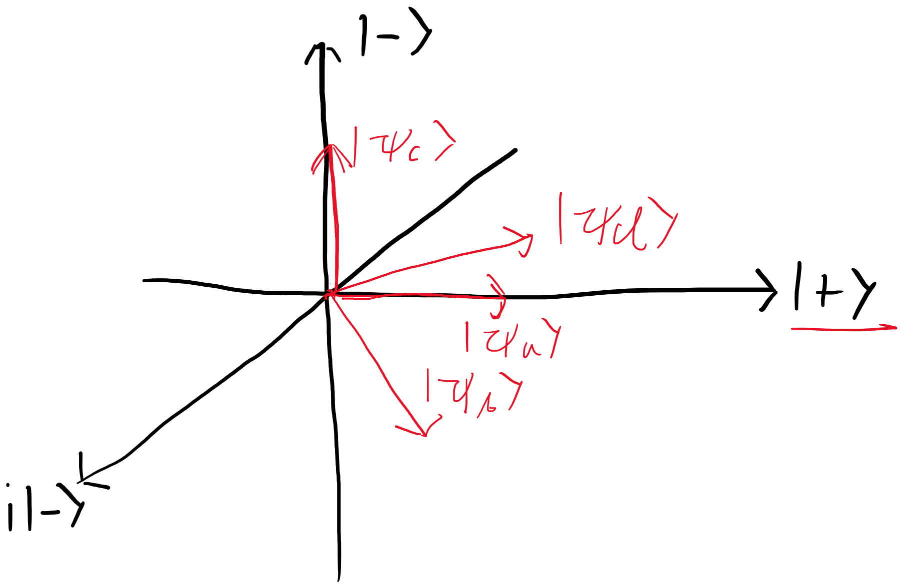
Question 4

$|+\rangle$

$|-\rangle$

$$|+\rangle_y = \frac{1}{\sqrt{2}}\left(|+\rangle + i|-\rangle\right)$$

$$|-\rangle_y = \frac{1}{\sqrt{2}}\left(|+\rangle - i|-\rangle\right)$$



For questions 4-6, please use **Dirac notation** (bras and kets, not matrices).

4. **[6 points total]** Consider the following vectors:

$$|\psi_1\rangle = \frac{\sqrt{3}}{2}|+\rangle - \frac{i}{2}|-\rangle \qquad and \qquad |\psi_2\rangle = \frac{1}{3}|+\rangle + \frac{2}{3}e^{-i\frac{3\pi}{4}}|-\rangle$$

a) **[1 point]** Could these two vectors be valid descriptions of physical states? Explain.

b) **[3 points]** For *each* state, find a normalized ket $|\phi_n\rangle$ that is orthogonal to it. Use the convention discussed in class, that the coefficient of the $|+\rangle$ ket is real and positive.

c) **[2 points]** Calculate the inner products $\langle\psi_1|\psi_2\rangle$ and $\langle\psi_2|\psi_1\rangle$. How are these numbers related? Is this consistent with the idea of an inner product being a "dot" product? Explain.

a.) $|\psi_1\rangle = \frac{\sqrt{3}}{2}|+\rangle - \frac{i}{2}|-\rangle$

$$\langle\psi_1|\psi_1\rangle = \left(\frac{\sqrt{3}}{2}\langle+| + \frac{i}{2}\langle-|\right)\left(\frac{\sqrt{3}}{2}|+\rangle - \frac{i}{2}|-\rangle\right)$$

$$= \frac{3}{4}\langle+|+\rangle - \frac{\sqrt{3}}{2^2}i\langle+|-\rangle$$

$$+\frac{\sqrt{3}}{2^2} i \langle -|+\rangle - \frac{i^2}{4}\langle -|-\rangle$$

$$= \frac{3}{4} + \frac{1}{4} = \frac{4}{4} = 1$$

$$\langle \psi_2|\psi_2\rangle = \left(\frac{1}{3}\langle +| + \frac{2}{3}e^{i\frac{3\pi}{4}}\langle -|\right)\left(\frac{1}{3}|+\rangle + \frac{2}{3}e^{-i\frac{3\pi}{4}}|-\rangle\right)$$

$$= \frac{1}{9} + \frac{4}{9} = 5/9$$

b.) $|\varphi_n\rangle = a|+\rangle + b|-\rangle$

$$\langle \varphi_n|\psi_1\rangle = 0 = \langle \varphi_n|\psi_2\rangle$$

$$\langle \varphi_n|\varphi_n\rangle = 1 = a^*a + b^*b = |a^2| + |b^2|$$

$$\langle \varphi_n|\psi_1\rangle = \begin{pmatrix} a^* & b^* \end{pmatrix}\begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{i}{2} \end{pmatrix} = \frac{\sqrt{3}}{2}a^* - \frac{i}{2}b^* = 0$$

$$\frac{\sqrt{3}}{2}a^* = \frac{i}{2}b^*$$

$$a = -\frac{i}{\sqrt{3}}b \leftarrow a^* = \frac{i}{\sqrt{3}}b^*$$

$$a = \frac{\;}{\sqrt{3}} b$$

$$|\varphi_n\rangle = -\frac{i}{\sqrt{3}} b\,|+\rangle + b\,|-\rangle$$

$$\langle\varphi_n| = \frac{i}{\sqrt{3}} b^* \langle +| + b^* \langle -|$$

$$\langle\varphi_n|\varphi_n\rangle = -\frac{i^2}{3} b^* b + b^* b$$

$$= \left(\frac{1}{3} + 1\right) b^* b = \underbrace{\frac{4}{3} b^* b = 1}$$

$$b^* b = \frac{3}{4}$$

$$a^* a + \frac{3}{4} = 1 \rightarrow a^* a = \frac{1}{4}$$

$$|b|^2 = \frac{3}{4} \qquad |a|^2 = \frac{1}{4}$$

$$|b| = \frac{\sqrt{3}}{2} \qquad |a| = \frac{1}{2}$$

$$\langle\varphi_n|\psi_1\rangle = \begin{pmatrix} a^* & b^* \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{i}{2} \end{pmatrix} = \frac{\sqrt{3}}{2} a^* - \frac{i}{2} b^* = 0$$

$$\langle\varphi_n|\psi_1\rangle = \begin{pmatrix} \frac{1}{2} i & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{i}{2} \end{pmatrix}$$

$$a = -\frac{i}{2}$$

$$= \sqrt{3}\, i \cdot 1 - \sqrt{3} \cdot i$$

$$a = \dots\ \frac{\ }{2}$$

$$b = \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{3}}{2} i \cdot \frac{1}{2} - \frac{\sqrt{3}}{2} \cdot \frac{i}{2}$$

$$\frac{\sqrt{3}}{4} i - \frac{\sqrt{3}\, i}{4} = 0$$

$$\varphi_n = -\frac{i}{2}|+\rangle + \frac{\sqrt{3}}{2}|-\rangle$$

$$\psi_2 = \frac{1}{3}|+\rangle + \frac{2}{3} e^{-i\frac{3\pi}{4}}|-\rangle$$

$$\psi_2^* \psi = 5/9 \qquad \frac{2}{3} \cdot \frac{9}{5} = \frac{2}{5} \cdot 3 = \frac{6}{5}$$

$$|\psi_2\rangle = \frac{1/3}{\sqrt{5}/3}|+\rangle + \frac{2/3}{\sqrt{5}/3} e^{-i\frac{3\pi}{4}}|-\rangle$$

$$= \frac{1}{\sqrt{5}}|+\rangle + \frac{2}{\sqrt{5}} e^{-i\frac{3\pi}{4}}|-\rangle$$

$$\langle \varphi_n | \psi_2 \rangle = \frac{1}{\sqrt{5}} a^* + \frac{2}{\sqrt{5}} b^* e^{-i\frac{3\pi}{4}} = 0$$

$$a^* = 2b^* e^{-i\frac{3\pi}{4}}$$

$$\langle \varphi_n | = 2b^* e^{-i\frac{3\pi}{4}}\langle +| + b^* \langle -|$$

$$\langle \varphi_n | = 2b^* e^{-i\frac{3\pi}{4}} \langle + | + b^* \langle - |$$

$$| \varphi_n \rangle = 2b e^{i\frac{3\pi}{4}} |+\rangle + b|-\rangle$$

$$\langle \varphi_n | \varphi_n \rangle = 2^2 b^* b + b^* b = 1 = a^* a + b^* b$$

$$b^* b (4 + 1) = 1$$

$$b^* b = \frac{1}{5}$$

$$|b| = \frac{1}{\sqrt{5}}$$

$$a^* a = |a|^2 = \frac{4}{5} \Rightarrow |a| = \frac{2}{\sqrt{5}}$$

$$\langle \varphi_n | \psi_2 \rangle = \frac{1}{\sqrt{5}} a^* + \frac{2}{\sqrt{5}} b^* e^{-i\frac{3\pi}{4}} = 0$$

$$\langle \psi_2 | \varphi_n \rangle = 0$$

$$\langle \psi_2 | \varphi_n \rangle = \left(\frac{1}{\sqrt{5}} \langle + | + \frac{2}{\sqrt{5}} e^{i\frac{3\pi}{4}} \langle - | \right) \left(\frac{2}{\sqrt{5}} x |+\rangle + \frac{1}{\sqrt{5}} y |-\rangle \right)$$

$$= \frac{2}{\;} x + \frac{2}{\;} e^{i\frac{3\pi}{4}} y = 0$$

$$= \frac{2}{5}x + \frac{2}{5}e^{i\frac{3\pi}{4}}y = 0$$

$$x = -e^{i\frac{3\pi}{4}} \qquad y = 1$$

$$= -\frac{2}{5}e^{i\frac{3\pi}{4}} + \frac{2}{5}e^{i\frac{3\pi}{4}} = 0$$

$$\boxed{|\varphi_n\rangle = -\frac{2}{\sqrt{5}}e^{i\frac{3\pi}{4}}|+\rangle + \frac{1}{\sqrt{5}}|-\rangle}$$