Cholesky decomposition

A is symmetric positive definite (SPD) if: $\boxed{A=A^T \text{ \& forall } v\neq 0:\ \underbrace{v^TAv>0}_{\sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}v_iv_j>0}}$

example: $A=\begin{bmatrix}4 & -2\\ -2 & 10\end{bmatrix}$

A is spd if for all $v=\begin{bmatrix}v_1\\ v_2\end{bmatrix}\neq 0$: $v^TAv>0$

$$4v_1^2-2v_1v_2-2v_2v_1+10v_2^2\overset{?}{>}0$$

$$4v_1^2-4v_1v_2+10v_2^2\overset{?}{>}0$$

$$(v_1-v_2)^2=v_1^2-2v_1v_2+v_2^2$$

$$2(v_1-v_2)^2=2v_1^2-4v_1v_2+2v_2^2\geq 0$$

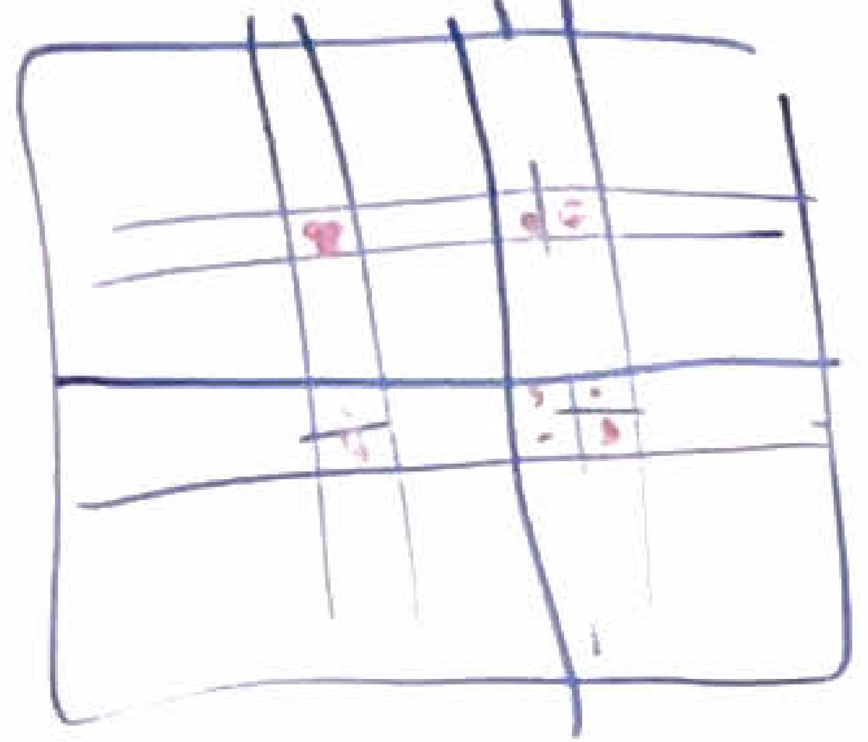
$$2v_1^2 \qquad +8v_2^2>0 \text{ if } v_1\neq 0 \text{ or } v_1\neq 0$$

$$4v_1^2-4v_1v_2+10v_2^2>0 \text{ if } \begin{pmatrix}v_1\\ v_2\end{pmatrix}\neq 0$$

Test

$A=A^T$: A is SPD $\iff$ det A > 0

all principal minors are > 0



principal sub matrix

= some selection rows, columns

minor = determinant

her $\det A = 4\cdot 10-(-2)(-2)=36>0$

$\det[4]=4>0$

$\det[10]=10>0$

example

$$A = \begin{bmatrix} \boxed{3} & \boxed{1} & \boxed{1} \\ 1 & \boxed{4} & 1 \\ \boxed{1} & 1 & \boxed{5} \end{bmatrix}$$

$\det A = \ldots > 0$

$\det \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix} \ldots$

$\det \begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix} > 0$

$\det \begin{bmatrix} 4 & 1 \\ 1 & 5 \end{bmatrix} > 0$

Thm

$A = A^T$ spd $\Leftrightarrow$ all eigenvalues > 0

Proof $\Rightarrow$: $\quad A u = \lambda u \quad u \neq 0$

$$\underbrace{u^T A u}_{\substack{>0 \\ A\text{ SPD}}} = \lambda \underbrace{u^T u}_{\|u\|^2 > 0} \Rightarrow \lambda = \frac{u^T A u}{u^T u} > 0$$

Rayleigh quotient

Properties of SPD matrix A

know: $v \neq 0 \Rightarrow v^T A v > 0$

choose $v = e_k$ $\quad e_k^T A e_k = a_{kk} > 0$

$$A = \begin{array}{c} k\{ \\ n-k\{ \end{array} \underbrace{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}}_{n} \}n \qquad A$$

(columns of width k and $n-k$)

A is SPD $\Rightarrow A_{11}$

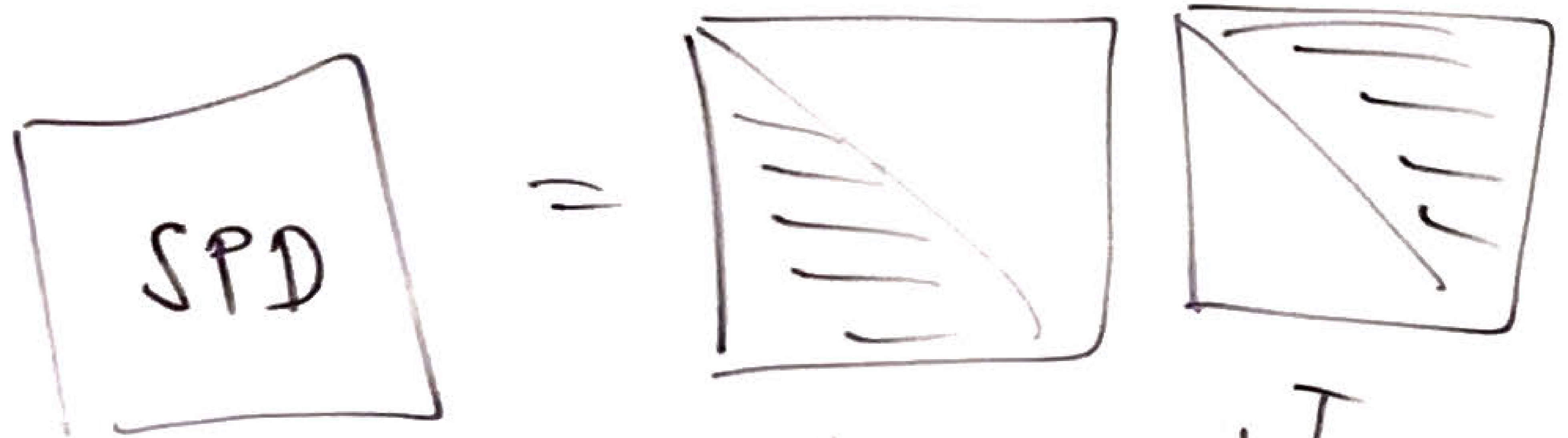
$$\underbrace{[\underset{\neq 0}{v_1^T}, \underset{=0}{v_2^T}]}_{\neq 0} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \begin{array}{l} \neq 0 \\ =0 \end{array} = v_1^T A_{11} v_1 > 0$$

Cholesky decomposition

$A = LU$ $\quad A = A^T$ $\quad LU$ can be made symmetric

solve $Ax = b$

$R \underbrace{R^T x}_{y} = b$



$A = L L^T$

step 1: $Ry = b$

step 2: $R^T x = y$

Book $\boxed{A = R^T R}$

Cholesky method

$$A = \begin{array}{cc} & \begin{array}{cc} 1 & m-1 \end{array} \\ \begin{array}{c} 1 \\ m-1 \end{array} & \begin{bmatrix} a_{11} & b^T \\ b & C \end{bmatrix} \end{array}$$

$$R = \begin{bmatrix} \alpha & \beta \\ 0 & \Gamma \end{bmatrix}$$

Γ upper triangular

$$R^T R = \begin{bmatrix} \alpha & 0 \\ \beta & \Gamma^T \end{bmatrix} \begin{bmatrix} \alpha & \beta^T \\ 0 & \Gamma \end{bmatrix} = \begin{bmatrix} \alpha^2 & \alpha\beta^T \\ \alpha\beta & \beta\beta^T + \Gamma^T\Gamma \end{bmatrix} = \begin{bmatrix} a_{11} & b^T \\ b & C \end{bmatrix}$$

$\alpha^2 = a_{11} \Rightarrow \alpha = \sqrt{a_{11}}$

$\alpha\beta = b \Rightarrow \beta = \frac{1}{\alpha} b = \frac{b}{\sqrt{a_{11}}}$

$\beta\beta^T + \Gamma^T\Gamma = C \Rightarrow \Gamma^T\Gamma = \underbrace{C - \beta\beta^T}_{\text{find cholesky factor } \Gamma} \in \mathbb{R}^{(m-1)\times(m-1)}$