Final Review

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This is a live document and will be updated during the final review week.

1. 3.2 Iterative methods

   (a) Formulate the Jacobi iterative method for the system

   $$\begin{aligned} 3x + 2y &= 1 \\ x + 2y &= 0 \end{aligned}$$

   and perform one step starting from $x = 1$, $y = 1$. Do you expect the method to converge?

   (b) Ditto for Gauss-Seidel

   (c) Perform one step of SOR with with $\omega = 1.5$

2. 3.3 Methods for SPD matrices

   (a) Definition of SPD, show that a 2x2 matrix is SPD.

   (b) Find Cholesky Decomposition for the matrix

   $$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 10 \end{bmatrix}$$

   Hint: If you need to derive the recursive formula, use

   $$A = \begin{bmatrix} a_{11} & b^T \\ b & C \end{bmatrix} = R^T R, \quad R = \begin{bmatrix} \alpha & \beta^T \\ 0 & \Gamma \end{bmatrix}$$

   to find α, β, and Γ.

   (c) Find a solution of linear system from Choleski decomposition

   (d) Conjugate Gradients

   Suppose A is SPD.

   (a) Show that $\langle Au, v \rangle = v^T Au$ defines an inner product (i.e., satisfies its axioms)

   (b) Show that if $J(x) = \frac{1}{2} x^T Ax - b^T x$, then $\nabla J(x) = Ax - b$

(c) Define a step of conjugate gradients as the descent from x in direction u, so that $J(x+\alpha u)$ is minimal. Find a formula for α.

(d) For $r^{(k)}$ (the residual in step k) and $u^{(k-1)}$ (the previous search direction), find β such that , $u^{(k)} = r^{(k)} - \beta A u^{(k-1)}$ is A-orthogonal to $u^{(k-1)}$. (We did not cover the justification why $u^{(k)}$ is also A-orthogonal to the previous search directions, $u^{(\ell)}$, $\ell < k-1$, so it won't be on the exam, but you need to know that it is)

3. 4.1 Polynomial interpolation

   (a) Find interpolating polynomial from monomial bases, Lagrange basis, Newton basis (we skip divided differences though)

4. 4.2 High degree polynomial interpolation

   (a) Estimate the maximum of the polynomial in the remainder $(x-x_0)\ldots(x-x_n)$ and why/when its value can get large

5. 4.3 Hermite interpolation

   (a) Find Hermite interpolating polynomial in a simple example such as $f(0)=0$ $f'(0)=1$ $f(1)=0$ $f'(1)=1$

6. 4.4 Piecewise polynomial interpolation

   (a) Find and draw piecewise linear interpolation, such as $f(0)=1$, $f(1)=0$, $f(2)=3$ what is $f(1.8)$?

   (b) Find piecewise Hermite interpolation on 3 points: $f(1)=1$ $f'(1)=0$ $f(2)=1$ $f'(2)=1$ $f(3)=1$ $f'(3)=1$

   (c) Find the natural cubic spline for the same $f(1)$ $f(2)$ $f(3)$

   (d) Find a clamped cubic spline passing through the points $(0,0)$ $(1,1)$ $(2,0)$ with $f'(0) = f'(2) = 0$

7. 6.1 Discrete least squares

   (a) Find the best least squares approximation of data points (x_i, y_i) by a line $y = a + bx$ For example $(1,1)$ $(2,0)$ $(3,1)$

   (b) Know the normal equation for the least squares solution of $Ax = b$. Solve a system, for

   (c) Given QR decomposition of A, use it to solve $Ax = b$ in the sense of least squares

8. 7.1 Golden section method for minimization in 1D

   (a) Decide if function $f(x) = x^2$ is unimodal on the interval $[0,1]$.

   (b) Suppose the golden section method for minimization of function f starts from $[a,b] = [-1,1]$.What will be the probe points x_1 and x_2? What will be the next $[a,b]$ when $f(x_1) = 3$ and $f(x_2) = 4$?

9. 7.2 Newton's method for minimization and for systems

   (a) Set up the Newton's method for the minimization of $f(x, y) = x^2 + y^2 + \cos(xy)$

   (b) Set up the multivariate Newton's method for the system

$$
\begin{aligned}
x^2 + y^2 &= 1 \\
x - y &= 0
\end{aligned}
$$

10. 7.2 Descent Methods

    (a) Perform one step of gradient descent for $f(x, y) = x^2 + y^2 + \cos(xy)$ starting from $x = 1$ $y = 0$

11. 7.2 Conjugate gradients for minimization