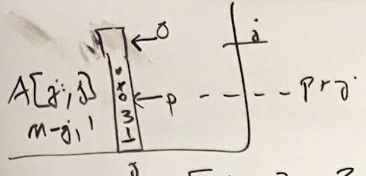


[[1, 2, 3],
[1, 2, 5],
[3, 6, 7]]

$$-1\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 2 & 5 & 2 \\ 3 & 6 & 7 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & 2 & 1 \\ 3 & 6 & 7 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

$R_3 \leftarrow R_3 - 3R_1$

$2x_3 = 1$

$-2x_3 = 0$

no solution

$$-1\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 2 & 5 & 2 \\ 3 & 1 & 7 & 3 \end{array}\right] \overset{-3}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & -5 & -2 & 0 \end{array}\right]$$

$$\underbrace{P}_{\text{permutation}}\ \underbrace{L_2 L_1}_{\text{lower triangular}}[A|b] = \underbrace{[U|\tilde{b}]}_{\text{suitable for back substitution}}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{L_1} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 2 & 5 & 2 \\ 3 & 1 & 7 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 2 & 2 \\ 3 & 1 & 7 & 3 \end{bmatrix}$$

$$\overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}}^{L_2} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 2 & 2 \\ 3 & 1 & 7 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & -5 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -5 & -2 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

U

partial pivoting

$$\begin{array}{c} \text{column } j \\ \left[\begin{matrix} 1 & 2 & 3 & 1 \\ 1 & 2 & 5 & 2 \\ 3 & 1 & 7 & 3 \end{matrix}\right] \end{array} \sim \left[\begin{matrix} 3 & 1 & 7 & 3 \\ 1 & 2 & 5 & 2 \\ 1 & 2 & 3 & 1 \end{matrix}\right] \begin{matrix} -\frac{1}{3} \\ \\ -\frac{1}{3} \end{matrix} \sim \left[\begin{matrix} 3 & 1 & 7 & 3 \\ 0 & \frac{5}{3} & \ddots & \\ 0 & \frac{5}{3} & & \ddots \end{matrix}\right]$$

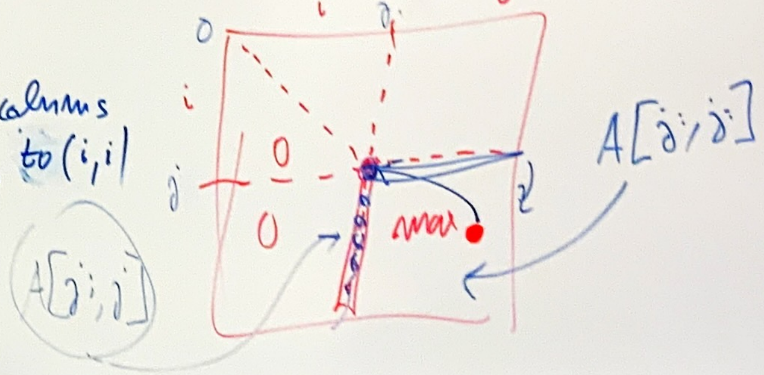
step j: 1° find $\max |a_{ij}|$ $i \geq j$
$= |a_{i_m j}|$

2° swap rows $i_m \leftrightarrow j$

3° eliminate a_{ij}, $i > j$

full pivoting 1° find $\max |a_{i_m j_m}|$ $i_m \geq i = j$, $j_m \geq i = j$

2. swap rows and columns to move $a_{i_m j_m}$ to (i, i)



permutation matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 2 & 2 \\ 1 & -5 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & -5 & -2 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 2 & 2 \\ 1 & -5 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -5 & -2 & 0 \end{bmatrix}$$

pick up 3rd row

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ c \\ b \end{bmatrix}$$

<u>Definition</u> $P = [p_{ij}] \in \mathbb{R}^{m \times m}$ is permutation matrix if each $p_{ij} = 0$ or 1 and each row has exactly one 1 and each column has exactly one 1

Observation about partial pivoting:

We do elimination on matrix PA (A with swapped rows)

end up with $\underbrace{PA}_{} = LU$ multipliers with 1 on diagonal added

A after swapping rows = permuted rows

how accumulate P? $\underbrace{P_m \cdots P_3 P_2 \underbrace{P_1 I}_{}}_{} A$

$= P_m \cdots P_1 I$ same swaps of rows applied to I

$$\begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 0 & 1 & & \\ & & & 1 & 0 & & \\ & & & & & 1 & \\ & & & & & & \ddots \end{bmatrix}$$ swap of 2 rows $= P_m \cdots P_1$

$$PA = LU$$

Full pivoting

$$PAQ = LU$$

Solve $Ax = b$

1. $PAx = Pb$

$$L\underbrace{Ux}_{y} = Pb$$

2. $Ly = Pb$ forward substitution

3 $Ux = y$ back substitution

$$L = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} u_{11} & & & U \\ & u_{22} & & \\ & & \ddots & \\ 0 & & & u_{nn} \end{bmatrix}$$

$b+\delta b$ know rhs to error δb

Solve $A(x+\delta x)=b+\delta b \Rightarrow A\delta x=\delta b$

estimate $\frac{\|\delta x\|}{\|x\|}$

$\delta x = A^{-1}\delta b$

assume $Ax=b$

$\Downarrow$

$\|A\|\|x\| \geq \|b\|$

$\|x\| \geq \frac{\|b\|}{\|A\|}$

$$\boxed{\begin{aligned} \|\delta x\| &\leq \|A^{-1}\|\|\delta b\| \\ \frac{1}{\|x\|} &\leq \|A\| \frac{1}{\|b\|} \end{aligned}}$$

$$\frac{\|\delta x\|}{\|x\|} \leq \underbrace{\|A\|\|A^{-1}\|}_{\text{condition number}} \frac{\|\delta b\|}{\|b\|}$$

customer wants $\frac{\|\delta x\|}{\|x\|} < 10^{-6}$

$\|A\|\|A^{-1}\| \approx 10^{12}$

$\underbrace{A(x+\delta x)}_{\text{numeric solution}} = b + r$ ↑ residual

$$\left[\begin{array}{ccc|c} 1 & 3 & 6 & 7 \end{array}\right] \quad \left[\begin{array}{ccc|c} 3 & 6 & 7 & 3 \end{array}\right] \quad \left[\begin{array}{ccc|c} 0 & 0 & -2 & 0 \end{array}\right] \quad -2x_3=0$$

$R_3 \leftarrow R_3 - 3R_1$

no solution

$$\underbrace{P}_{\text{permutation}} \underbrace{L_2 L_1}_{\text{lower triangular}} [A|b] = \underbrace{[U|\tilde{b}]}_{\text{suitable for back substitution}}$$

L_2

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 2 & 2 \\ 3 & 1 & 7 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & -5 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -5 & -2 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

U