$x \in \mathbb{R}^n$ $\quad \|x\|_\infty = \max_i |x_i|$ $\quad \le \|x\|_\infty$

$\|y\|_\infty$ of $y = Ax$

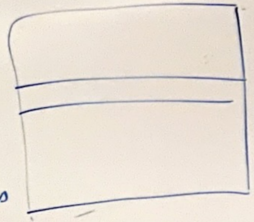
$A = [a_{ij}]$

$$|y_i| = \left|\sum_{j=1}^m a_{ij}x_j\right| \le \sum_{j=1}^m |a_{ij}x_j| = \sum_{j=1}^n |a_{ij}||x_j| \le \left(\sum_{j=1}^n |a_{ij}|\right)\|x\|_\infty$$

$$\|Ax\|_\infty = \max_i |y_i| \le \underbrace{\left(\max_i \sum_{j=1}^\infty |a_{ij}|\right)}_{\|A\|_\infty}\|x\|_\infty$$

$$\|A\|_\infty = \max_i \sum_j |a_{ij}|$$

max row sum of absolute values



$$\|Ax\|_\infty \le \|A\|_\infty \|x\|_\infty$$

$$\|A\|_\infty = \max_{x \ne 0} \frac{\|Ax\|_\infty}{\|x\|_\infty}$$

$$\max_x \frac{\|Ax\|_\infty}{\|x\|_\infty} \le \|A\|_\infty$$ know already

show $\ge$?

$\exists x \ne 0 \quad \max_i \left|\sum_j a_{ij}x_j\right| = \left(\max_i \sum_j |a_{ij}|\right) \cancel{\|x\|_\infty}^{=1}$ $\quad$ take $x_i = \pm 1$

$\boxed{\text{take } i}$ where the max is attained

find x $\quad \sum_j a_{ij}x_j = \sum_j |a_{ij}|$ $\quad \boxed{\begin{array}{l} x_j = 1 \text{ if } a_{ij} \ge 0 \\ x_j = -1 \text{ if } a_{ij} < 0 \end{array}}$

found x s.t.

$$\|Ax\|_\infty = \|A\|_\infty \|x\|_\infty$$

Def $A = [a_{ij}]$, $n \times n$, is strictly diagonally dominant if $\boxed{\sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|} < |a_{ii}|$ for all i

Thm If A is strictly diagonally dominant, then Jacobi method converges.

Proof Know $\|x^{(k+1)} - x^*\|_\infty \leq \|I - D^{-1}A\|_\infty \|x^{(k)} - x^*\|_\infty$

m.t.s. $\|I - D^{-1}A\|_\infty < 1$ i.e.

$$I - D^{-1}A = \begin{bmatrix} 1 & 0 & \\ 0 & 1 & \\ & & 1 \end{bmatrix} - \begin{bmatrix} 1/a_{11} & & \\ & 1/a_{22} & \\ & & \ddots \\ & & & 1/a_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & & \\ a_{n1} & & & a_{nn} \end{bmatrix} = \begin{bmatrix} \cancel{1}\overset{0}{-1} & -\frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} & \cdots & -\frac{a_{1n}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & & \frac{a_{23}}{a_{22}} & \cdots & \\ & & & & \end{bmatrix}$$

$$\|I - D^{-1}A\|_\infty = \max_i \sum_{\substack{j=1 \\ j \neq i}}^{n} \left|\frac{a_{ij}}{a_{ii}}\right| = \max_i \frac{1}{|a_{ii}|} \underbrace{\boxed{\sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|}}_{< |a_{ii}|} < 1$$

Jacobi for $\sum_{j=1}^{n} a_{ij} x_j = b_i$ $\quad a_{i1}x_1 + a_{i2}x_2 + \dots a_{ii}x_i \dots + a_{in}x_n = b_i \quad i=1,\dots,n$

Jacobi: $x_i = \frac{1}{a_{ii}}(b_i - (a_{i1}x_1 + a_{i2}x_2 \dots + a_{i\,i-1}x_{i-1} + a_{i\,i+1}x_{i+1} + \dots a_{in}x_n)$

$$\boxed{x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij} x_j^{(k)}\right)} \quad i=1,2,\dots,n$$

Jacobi

Gauss-Seidel

$$\boxed{\text{for } i=1,\dots,n \quad x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}\right)}$$

code: for $i=1,\dots,n$ $\quad x_i \leftarrow \frac{1}{a_{ii}}\left(b_i - \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij} x_j\right)$

Example: $4x_1 - 3x_2 = -1$

$2x_1 + 5x_2 = 19$

Jacobi

$x_1^{(k+1)} = \frac{1}{4}(-1 \quad + 3x_2^{(k)})$

$x_2^{(k+1)} = \frac{1}{5}(19 - 2x_1^{(k)} \quad)$

Gauss-Seidel

$x_1^{(k+1)} = \frac{1}{4}(-1 \quad + 3x_2^{(k)})$

$x_2^{(k+1)} = \frac{1}{5}(19 - 2x_1^{(k+1)} \quad)$

Implementation:

$x_1 \leftarrow \frac{1}{4}(-1 + 3x_2)$

$x_2 \leftarrow \frac{1}{5}(19 - 2x_1)$

Def $A = [a_{ij}]$, $n \times n$, is strictly diagonally dominant if $\boxed{\sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| < |a_{ii}|}$ for all i

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Proof Know $\|x^{(k+1)} - x^*\|_\infty \le \|I - D^{-1}A\|_\infty \|x^{(k)} - x^*\|_\infty$

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$$\|I - D^{-1}A\|_\infty = \max_i \sum_{\substack{j=1 \\ j \neq i}}^{n} \left| \frac{a_{ij}}{a_{ii}} \right| = \max_i \frac{1}{|a_{ii}|} \underbrace{\sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|}_{< |a_{ii}|} < 1$$

Jacobi for $\sum_{j=1}^{n} a_{ij} x_j = b_i$ $\quad a_{i1}x_1 + a_{i2}x_2 + \cdots a_{ii}x_i \cdots + a_{in}x_n = b_i \quad i=1,\dots,n$

Jacobi: $x_i = \frac{1}{a_{ii}}\left(b_i - (a_{i1}x_1 + a_{i2}x_2 \cdots + a_{i\,i-1}x_{i-1} + a_{i\,i+1}x_{i+1} + \cdots a_{in}x_n)\right)$

$$\boxed{x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij} x_j^{(k)}\right)} \qquad i=1,2,\dots,n$$

Jacobi

Example: $4x_1 - 3x_2 = -1$

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Jacobi

$x_1^{(k+1)} = \frac{1}{4}(-1 \quad + 3x_2^{(k)})$

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Gauss-Seidel

$x_1^{(k+1)} = \frac{1}{4}(-1 \quad + 3x_2^{(k)})$

$x_2^{(k+1)} = \frac{1}{5}(19 - 2x_1^{(k+1)} \quad)$

in book

Implementation:

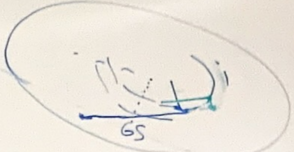
$x_1 \leftarrow \frac{1}{4}(-1 + 3 x_2)$

$x_2 \leftarrow \frac{1}{5}(19 - 2x_1)$

$\boxed{\text{gauss-seidel}}$

for $i=1,\dots,n$ $\quad x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}\right)$

code: for $i=1,\dots,n$ $\quad x_i \leftarrow \frac{1}{a_{ii}}\left(b_i - \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij} x_j\right)$



Successive over-relaxation

Gauss Seidel $x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right)$ for $i=1,\dots,n$

$$x_i^{(k+1)} = x_i^{(k)} + \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} \underbrace{- a_{ii}x_i^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}}\right)$$

SOR

$x_i^{(k+1)} = x_i^{(k)} + \omega\delta_i$

relaxation parameter

$\omega \in (0,2)$

$$\underset{\text{new}}{x_i^{(k+1)}} = \underset{\text{old}}{x_i^{(k)}} + \underbrace{\frac{\omega}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1\;(<i)} a_{ij}x_j^{(k+1)} - \sum_{j=i}^{n} a_{ij}x_j^{(k)}\right)}_{\text{increment}}$$

δ_i

$\omega = 0$ $x^{(k)}$ constant

$\omega = 1$ Gauss-Seidel

DAVID YOUNG JR

$\boxed{\text{MR SOR}}$
TEXAS