James Amidei - Homework 4

3. [10 points total] Expectation values. Consider the state $|\psi\rangle = \frac{1}{2}(\sqrt{3}\,|+\rangle + 1|-\rangle)$.

a) **[2 points]** Say that you send *many* particles in this state through a z-analyzer. Sketch a histogram (like McIntyre Fig. 2.8) showing the possible outcomes and their probabilities. Based on this, what is the expectation value of spin in the z-direction ($\langle \hat{S}_z \rangle$)? What does the expectation value tell you in this case - is it the value you are most likely to measure (the one you'd "expect" to get)?

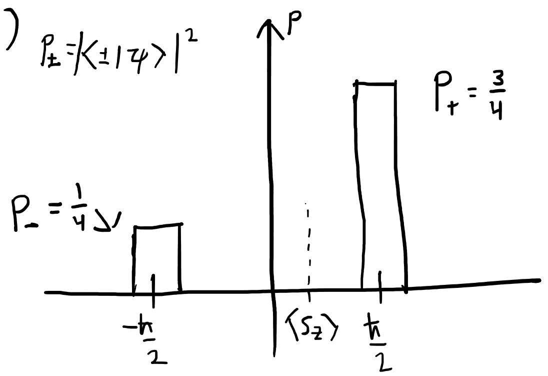
b) **[2 points]** Say instead that you measure the x-component of spin. Determine the expectation value of spin in the x-direction ($\langle \hat{S}_x \rangle$). Sketch a histogram and comment on the result. Again, make sure to discuss what the expectation value tells you.

c) **[2 points]** Determine the expectation value of spin in the y-direction. Sketch a histogram and comment on the result. Again, make sure to discuss what the expectation value tells you.

d) **[2 points]** There are two ways to calculate the expectation value (a matrix/Dirac calculation or a sum of measurement values times their probabilities). Verify your result to part c) using the "other" method. Comment on which method, if either, was (1) easier to compute and (2) easier to understand. Briefly reflect on why you chose the method you used first (there's no correct answer to that, I just want to know what you were thinking!).

e) **[2 points]** In what direction $\hat{n}$ does this state "point"? Sketch the orientation $\hat{n}$ in **cartesian** *(xyz)* space. Discuss how the orientation is (or is not) consistent with your results above.

a.) $P_\pm = |\langle \pm|\psi\rangle|^2$



$$\langle S_z \rangle = \frac{\hbar}{2}\left(\frac{3}{4}\right) + \left(-\frac{\hbar}{2}\right)\left(\frac{1}{4}\right) = \frac{3\hbar}{8} - \frac{\hbar}{8} = \frac{\hbar}{4}$$

The expectation value corresponds to the average measurement we'd expect if we were to measure some large number of samples. This does not have to correspond to an actual value of any given measurement, similarly to how the average family in the US may have something in the ballpark of 1.9 children, but you will never find a family with exactly that number of children.
In this case, since we have a greater probability of measuring an "up-z" state, the expectation value skews towards the "up-z" state's corresponding eigenvalue. However, since there is some probability of measuring a "down-z" state, the expectation value is not perfectly aligned with the "up-z" eigenvalue, and is instead pulled slightly down, between the two measurable eigenvalues.

b.) $$|+\rangle_x = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right) \qquad |-\rangle_x = \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right)$$

$$(|\psi\rangle)_x = \begin{pmatrix} {}_x\langle +|\psi\rangle \\ {}_x\langle -|\psi\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2}\langle +|+\rangle + \frac{1}{\sqrt{2}}\frac{1}{2}\langle -|-\rangle \\ \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2}\langle +|+\rangle - \frac{1}{\sqrt{2}}\frac{1}{2}\langle -|-\rangle \end{pmatrix}$$

$$= \frac{1}{\ldots}\begin{pmatrix} \sqrt{3} + 1 \\ \ldots \end{pmatrix}$$

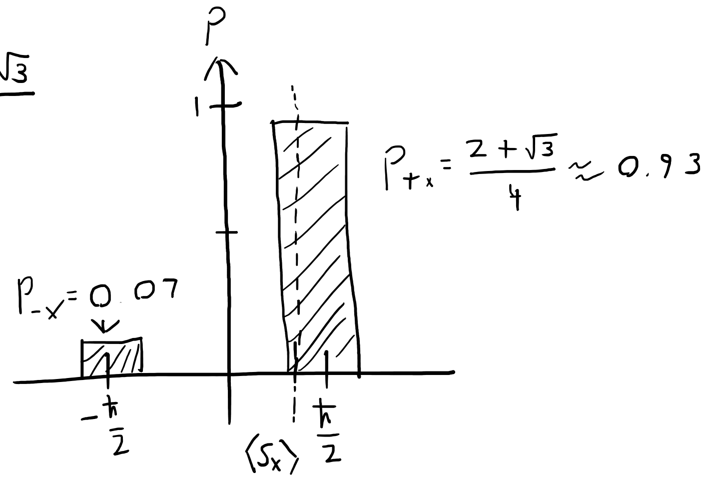
$$= \frac{1}{2\sqrt{2}} \begin{pmatrix} \sqrt{3}+1 \\ \sqrt{3}-1 \end{pmatrix}$$

$$P_{+x} = \left| \frac{\sqrt{3}+1}{2\sqrt{2}} \right|^2$$

$$= \frac{3+2\sqrt{3}+1}{2^3} = \frac{4+2\sqrt{3}}{2^2}$$

$$= \frac{2+\sqrt{3}}{4}$$

$$P_{-x} = 1 - \frac{2+\sqrt{3}}{4}$$



$$\langle S_x \rangle \approx \frac{\hbar}{2}(0.93) + \left(-\frac{\hbar}{2}\right)(0.07)$$

$$= \frac{0.86}{2}\hbar = 0.43\hbar$$

In this case, the expectation value skew heavily towards the "up-x" eigenvalue because of the remarkably high probability of measuring a "up-x" state. It does not perfectly align with the corresponding "up-x" eigenvalue because there is still a small possibility of measuring a "down-x" state.

c.)

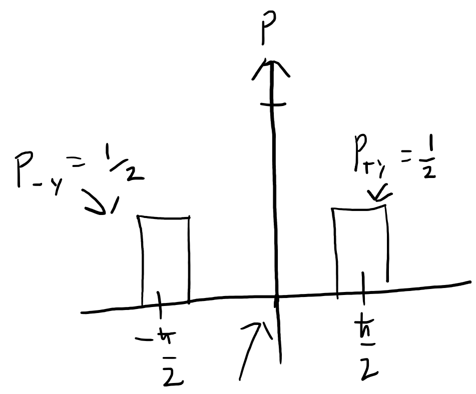
$$(|\psi\rangle)_y = \begin{pmatrix} {}_y\langle +|\psi\rangle \\ {}_y\langle -|\psi\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2}\langle +|+\rangle - \frac{i}{\sqrt{2}}\frac{1}{2}\langle -|-\rangle \\ \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2}\langle +|+\rangle + \frac{i}{\sqrt{2}}\frac{1}{2}\langle -|-\rangle \end{pmatrix}$$

$$= \frac{1}{2\sqrt{2}} \begin{pmatrix} \sqrt{3}-i \\ \sqrt{3}+i \end{pmatrix}$$

$$P_{+y} = \left| \frac{\sqrt{3}-i}{2\sqrt{2}} \right|^2 = \frac{\sqrt{3}+i}{2\sqrt{2}} \frac{\sqrt{3}-i}{2\sqrt{2}} = \frac{3+1}{2^3} = \frac{4}{8} = \frac{1}{2}$$

$$P_{-y} = \left| \frac{\sqrt{3}+i}{2\sqrt{2}} \right|^2 = \frac{\sqrt{3}+i}{2\sqrt{2}} \frac{\sqrt{3}-i}{2\sqrt{2}} = \frac{3+1}{2^3} = \frac{1}{2}$$

$$P_{-y} = \left|\frac{\sqrt{3}+1}{2\sqrt{2}}\right|^2 = \frac{\sqrt{3}+1}{2\sqrt{2}}\,\frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{3+1}{2^3} = \frac{1}{2}$$



$$\langle S_y \rangle = 0$$

$$\langle S_y \rangle = \frac{\hbar}{2}\left(\frac{1}{2}\right) - \frac{\hbar}{2}\left(\frac{1}{2}\right) = 0$$

Because both "up-y" and "down-y" measurements are equally likely, the expectation value equals zero, since it falls in the middle of two equally likely, even spaced measurement values.

d.)

$$\langle \psi | \hat{S}_y | \psi \rangle$$

$$= \frac{1}{2^2}\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$$

$$= \frac{1}{2^2}\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} -i \\ \sqrt{3}\,i \end{pmatrix} = \frac{1}{2^2}\left(-\sqrt{3}\,i + \sqrt{3}\,i\right) = 0$$

Really both methods are as easy as each other, depending on what you are trying to do. In the case where we are making histograms like above, we had to find the probabilities and we already knew the eigenvalues, so it is simplest to just do the summation. However, I think using Dirac notation is simplest when you want to change bases, like we did in the last few parts.

e.)

Based on our results above, we can determine that the n-hat vector will point in some direction with a majority positive z-component, a majority positive x-component, and no y-component (due to both states being equally likely).
Because of the larger contribution of the positive/"up" components along z and x, we can equate the state vector with the "up-n" vector in the z basis.

$$\frac{\sqrt{3}}{2}|+\rangle + \frac{1}{2}|-\rangle = \cos\left(\frac{\theta}{2}\right)|+\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\varphi}|-\rangle$$

$$\frac{\ }{2}|+\rangle + \frac{\ }{2}|-\rangle = \cos\left(\frac{\theta}{2}\right)|+\rangle + \sin\left(\frac{\theta}{2}\right) e^{i\varphi}|-\rangle$$

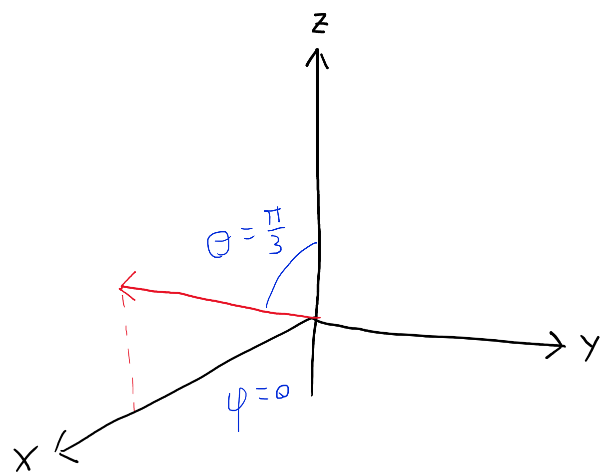
$$\cos\left(\frac{\theta}{2}\right) = \frac{\sqrt{3}}{2} \qquad \sin\left(\frac{\pi}{2\cdot 3}\right) = \frac{1}{2}$$

$$\frac{\theta}{2} = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

$$\theta = \frac{\pi}{3}$$

No imaginary component.
so, $\varphi = 0$

$$\cos\left(\frac{\pi}{2\cdot 3}\right)|+\rangle + \sin\left(\frac{\pi}{2\cdot 3}\right) e^{i(0)}|-\rangle = \frac{\sqrt{3}}{2}|+\rangle + \frac{1}{2}(1)|-\rangle$$



4. **[4 points total]** Quantum Mouse

a) **[2 points]** Please reproduce your work for the last part of the first Quantum Mouse tutorial. That is, determine a matrix representation of the eye-size operator $\hat{S}$, *in the mood basis.* Is your answer "diagonal"? Should it be?

b) **[2 points]** What is the "expected value" $\langle \hat{S} \rangle$ of eye size for a happy mouse? Comment on what this result means about happy mice.

a.) $\hat{S}|*\rangle = (1\,mm)|*\rangle$

$\hat{S}|\odot\rangle = (2\,mm)|\odot\rangle$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{pmatrix} = (1\,mm)\begin{pmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{pmatrix}$$

$$\frac{2}{\sqrt{5}}a - \frac{1}{\sqrt{5}}b = \frac{2}{\sqrt{5}} \qquad \frac{a}{\sqrt{5}} + \frac{2}{\sqrt{5}}b = \frac{2}{\sqrt{5}}$$

$$\frac{2}{\sqrt{5}}c - \frac{1}{\sqrt{5}}d = -\frac{1}{\sqrt{5}} \qquad \frac{c}{\sqrt{5}} + \frac{2}{\sqrt{5}}d = \frac{4}{\sqrt{5}}$$

$$\frac{2}{\sqrt{5}}c - \frac{1}{\sqrt{5}}d = -\frac{1}{\sqrt{5}} \qquad \frac{c}{\sqrt{5}} + \frac{2}{\sqrt{5}}d = \frac{4}{\sqrt{5}}$$

$$2a - b = 2 \qquad a + 2b = 2$$

$$2c - d = -1 \qquad c + 2d = 4$$

$$2c + 1 = d \qquad c + 2(2c+1) = 4$$

$$c + 4c + 2 = 4$$

$$5c = 2$$

$$c = 2/5$$

$$2\left(\frac{2}{5}\right) + 1 = d$$

$$\frac{4}{5} + 1 = \frac{9}{5} = d$$

$$2a - b = 2 \qquad a + 2b = 2$$

$$2a - 2 = b \qquad a + 2(2a-2) = 2$$

$$a + 4a - 4 = 2$$

$$2\left(\frac{6}{5}\right) - 2 = b \qquad 5a = 6$$

$$a = 6/5$$

$$\frac{12}{5} - 2 = \frac{2}{5} = b$$

$$\hat{S} = \begin{pmatrix} 6/5 & 2/5 \\ 2/5 & 9/5 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix}$$

This matrix is not diagonal and it shouldn't be because it is written in another basis, i.e. in the mood basis.

b.) $|\text{☺}\rangle = \begin{pmatrix} +1 \\ 0 \end{pmatrix}$

$$\langle \text{☺} | \hat{S} | \text{☺} \rangle = \begin{pmatrix} +1 & 0 \end{pmatrix} \begin{pmatrix} 6/5 & 2/5 \\ 2/5 & 9/5 \end{pmatrix} \begin{pmatrix} +1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} +1 & 0 \end{pmatrix} \begin{pmatrix} 6/5 \\ 2/5 \end{pmatrix} = \frac{6}{5}$$

For questions 5 and 6: A system has two "observables" (A and B) that can be measured. When you measure A, there are only two possible outcomes, $a_1 = +3\ eV$ and $a_2 = -2\ eV$, where eV is an "electron-volt." When you measure B, the only possible values are $b_1 = +\hbar$ and $b_2 = 0$. We'll choose to label our states (kets) using their corresponding eigenvalues (our "terrible" notation), for example:

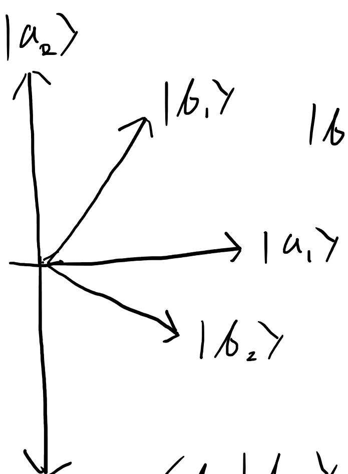
$$\hat{A}|a_1\rangle = +3\ eV\ |a_1\rangle \qquad (1)$$

After *many* experiments, it has been determined that:

$$|b_1\rangle = \frac{3}{5}|a_1\rangle + \frac{4i}{5}|a_2\rangle \qquad (2)$$

5. **[5 points total]** Measurements and Probabilities
   a) **[1.5 points]** Determine the state $|b_2\rangle$ in the A basis – an expression similar to equation (2) above. Make sure that $|b_1\rangle$ and $|b_2\rangle$ form an orthonormal set.
   b) **[1.5 points]** A particle is prepared in the state $|b_1\rangle$. If you send this particle into a device that measures A, what possible values could you measure? With what probabilities? Is it reasonable to sketch a histogram of measurement results for this single particle? Explain.
   c) **[2 points]** Suppose that the result of the prior measurement was $-2\ eV$. If you now measure B for this same particle, what result(s) could you get? With what probabilities? Explain.

5.) a.)



$$|b_2\rangle = \frac{4}{5}|a_1\rangle - \frac{3i}{5}|a_2\rangle$$

$$\langle b_2 | b_1 \rangle = \frac{4}{5}\frac{3}{5} + \frac{3}{5}\frac{4}{5}i^2$$

$$= \frac{4\cdot 3}{5^2} - \frac{4\cdot 3}{5^2} = 0$$

b.)

$$P_+ = |\langle a_1 | b_1 \rangle|^2 = \left| \frac{3}{5}\langle a_1 | a_1 \rangle + \frac{4}{5}i\langle a_1 | a_2 \rangle \right|^2$$

$$P_+ = |\langle a_1|b_1\rangle|^2 = \left|\tfrac{3}{5}\langle a_1|a_1\rangle + \tfrac{4}{5}i\langle a_1|a_2\rangle\right|^2$$

$$= \left|\tfrac{3}{5}\right|^2 = \frac{9}{25}$$

$$P_- = 1 - \frac{9}{25} = \frac{16}{25}$$

I don't know. Call it confirmation.

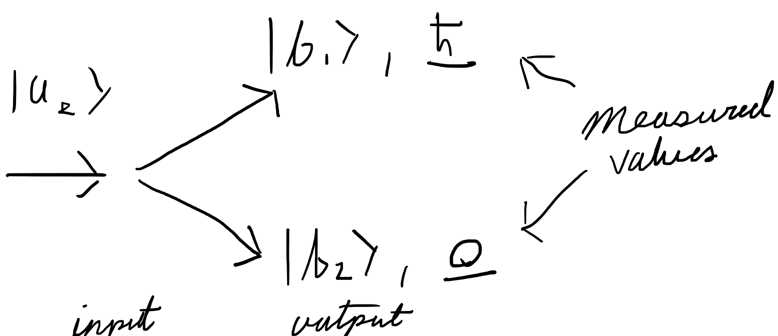
$$= |\langle a_2|b_1\rangle|^2 = \left|\tfrac{3}{5}\langle a_2|a_1\rangle + \tfrac{4i}{5}\langle a_2|a_2\rangle\right|^2$$

$$= \left|\tfrac{4i}{5}\right|^2 = \frac{4^2}{5^2} = \frac{16}{25}$$

Since we are measuring only one particle, it would not make sense to draw a histogram. These probabilities tell us about the odds that the particle will be in one of the two states, so a histogram would show that 100 percent of the particles measured were in whatever state this may be, which is obviously not descriptive of the actual probabilities in the state. We would need to measure many particles in order to draw a histogram that is actually descriptive.

c.)
$$|\langle b_1|a_2\rangle|^2 = \frac{16}{25}$$

$$|\langle b_2|a_2\rangle|^2 = \frac{9}{25}$$



$|a_2\rangle \rightarrow |b_1\rangle$ or $|b_2\rangle$

6. **[5 points total]** Matrices and Statistics
   a) **[1 point]** Write the other three eigenvalue equations – similar to equation (1) above.
   b) **[1 point]** Express A in matrix notation (in the A basis). Briefly explain.

   A particle is prepared in state $|b_1\rangle$.

   c) **[2 points]** Determine the expectation value $\langle A\rangle$ and uncertainty ΔA for this state. Show your work.
   d) **[1 point]** Determine the expectation value $\langle B\rangle$ and uncertainty ΔB for this state. Explain your reasoning *without* doing any calculations.

6.)

a.) $\hat{A}|a_2\rangle = -2\,\text{eV}\,|a_2\rangle$

$\hat{B}|b_1\rangle = +\hbar\,|b_1\rangle$

$\hat{B}|b_2\rangle = 0\,|b_2\rangle$

$$B|b_2\rangle = 0|b_2\rangle$$

b.)

$$\hat{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

eigenvalues
$a_1 = 3\,eV$ & $a_2 = -2\,eV$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} eV$$

Since we are in the A basis, the A matrix can be assumed to be diagonalized, with the entries along the main diagonal corresponding to the A eigenvalues.

c.)

$$\langle b_1|\hat{A}|b_1\rangle$$

$$= \frac{1}{5^2}\begin{pmatrix} 3 & -4i \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 3 \\ 4i \end{pmatrix}$$

$$= \frac{1}{5^2}\begin{pmatrix} 3 & -4i \end{pmatrix}\begin{pmatrix} 3^2 \\ -8i \end{pmatrix} = \frac{1}{5^2}\left(3^3 + 32i^2\right)$$

$$= \frac{1}{25}\left(27 - 32\right) = -\frac{5}{25} = -\frac{1}{5}$$

$$\langle b_1|AA|b_1\rangle = \langle b_1|\hat{A}^2|b_1\rangle$$

$$= \frac{1}{5^2}\begin{pmatrix} 3 & -4i \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 3 \\ 4i \end{pmatrix}$$

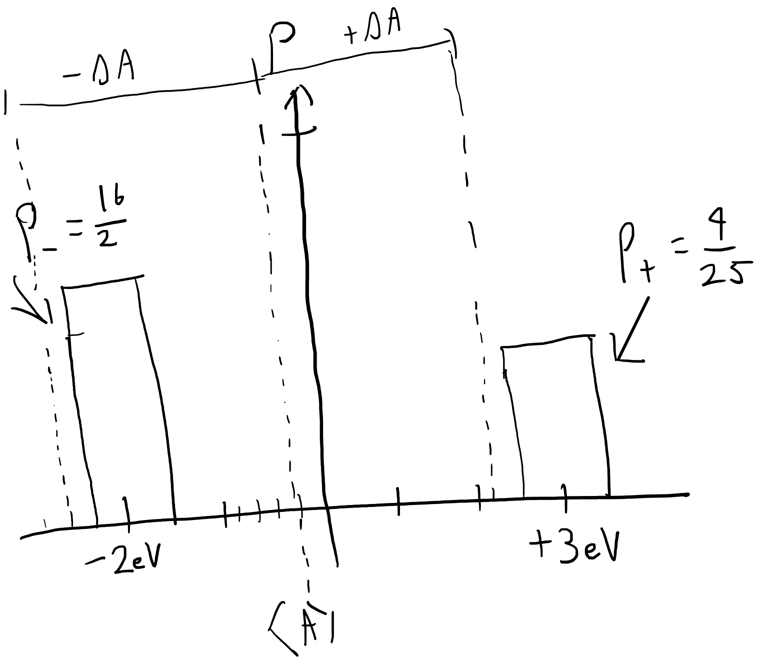
$$1 \qquad \begin{pmatrix} 3^2 & 0 \end{pmatrix} \begin{pmatrix} 3 \end{pmatrix}$$

$$= \frac{1}{5^2}\begin{pmatrix} 3 & -4i \end{pmatrix}\begin{pmatrix} 3^2 & 0 \\ 0 & 2^2 \end{pmatrix}\begin{pmatrix} 3 \\ 4i \end{pmatrix}$$

$$= \frac{1}{5^2}\begin{pmatrix} 3 & -4i \end{pmatrix}\begin{pmatrix} 3^3 \\ 2^4 i \end{pmatrix} = \frac{1}{5^2}\begin{pmatrix} 3^4 & -2^6 i^2 \end{pmatrix}$$

$$= \frac{1}{5^2}\left(81 + 64 \right) = \frac{1}{25}\left(145 \right)$$

$$\Delta A = \sqrt{\frac{145}{25} - \frac{1}{25}} = \sqrt{\frac{144}{25}} = \frac{12}{5} = 2\tfrac{2}{5}$$



d.) $\hat{B}|b_1\rangle = +h|b_1\rangle$ Eigen-equation for B acting on b1 ket.

$\langle b_1|\hat{B}|b_1\rangle = +h$ The expectation value will correspond to the b1 eigenvalue, since there is no other value that can be measured. Basically, the average of a single value is just the value itself.

$\langle b_1|B^2|b_1\rangle = +\hbar^2$ A operator in its own basis will be a diagonal matrix with the possible observable eigenvalues as the entries along the main diagonal. When we square a matrix of this type, we can just square the diagonal entries. In this case, our entries would become h-bar^2 and 0. When we find the expectation value of this squared matrix, since our input state is one of the eigenvectors of the basis, we just pull out the corresponding squared eigenvalue. In this case

$$\langle b_1 | B | b_1 \rangle = +\hbar$$

square the diagonal entries. In this case, our entries would become h-bar^2 and 0. When we find the expectation value of this squared matrix, since our input state is one of the eigenvectors of the basis, we just pull out the corresponding squared eigenvalue. In this case that is h-bar^2.

$$\Delta B = 0$$

Since the two values are the same, they have a difference of zero, which of course has a square root of zero. Another, less mathy way we could approach this is that, since we are certain to get only one value for the reasons stated above, we will have no deviation from the average--which is just the eigenvalue itself. Therefore, the uncertainty/standard deviation will equal zero.

7. **[6 points]** Commutators and Uncertainty

a) **[2 points]** Using the (2D) **matrix** representation of the operators, show that $[S_y, S_z] = i\hbar S_x$

b) **[2 points]** Using the (2D) **matrix** representation of the operators, determine the *minimum* uncertainty in the product $\Delta S_x \cdot \Delta S_z^2$. Does this depend on the initial state $|\psi\rangle$?

c) **[1 point]** When operators commute, they share a common eigen-basis. Are *all* of the eigenstates of S_z^2 also eigenstates of S_x? What about the other way around?

d) **[1 point]** Is the common eigen-basis sufficient to describe *any* 2D (spin-1/2) state? Explain why, or why not.

a.) $$S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[S_y, S_z] = S_y S_z - S_z S_y$$

$$= \frac{\hbar^2}{2^2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{\hbar^2}{2^2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \frac{\hbar^2}{4}\left[\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\right]$$

$$= \frac{\hbar^2}{4}\begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} = i\frac{\hbar^2}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$[S_y, S_z] = i\hbar S_x$$

b.) $$\Delta S_y \cdot \Delta S_z^2 \geq \frac{1}{2}\left|\langle [S_x, S_z^2] \rangle\right|$$

$$[S_x, S_z^2] = S_x S_z^2 - S_z^2 S_x$$

$$= \begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix} \begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix} \begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix}$$

$$- \begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix} \begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix} \begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix} \begin{pmatrix} \hbar^2/2^2 & 0 \\ 0 & \hbar^2/2^2 \end{pmatrix} - \begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix} \begin{pmatrix} 0 & \hbar^2/2^2 \\ -\hbar^2/2^2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \hbar^3/2^3 \\ \hbar^3/2^3 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \hbar^3/2^3 \\ \hbar^3/2^3 & 0 \end{pmatrix} = 0$$

$$\Delta S_y \cdot \Delta S_z^2 \geq 0$$

Or, we could have recognized that Sz^2 would result in the identity operator times some constant term hbar^2/4. Because Sz^2 is proportional to the identity operator, it is compatible with any other operator, of course meaning that it commutes with Sx. Because the two operators are compatible, their eigenvalues can be measured simultaneously, meaning that there is no uncertainty relation between them and that **the initial state does not matter.**

c.) Since Sz^2 is proportional to the identity operator, it acts arbitrarily on upon other operators, assigning the same eigenvalues on each one, and thus must commute with Sx, Sy, and Sz. This also means that all states are eigenstates of the Sz^2 operator, in the same way that all states are eigenstates of the more general S^2 operator. Therefore, **all eigenstates of Sz^2 are eigenstates of Sx, and vice versa.**

d.)

For a 2D (spin-1/2) system, we can choose any two orthonormal basis kets from the eigen states of any operator which span the entirety of the Hilbert space to describe any arbitrary state as a linear combination. In this class, we have predominantly seen this by using the keys |+> and |-> in the z-basis. In other words, we have been able to take any arbitrary state and write it in terms of a|+> + b|->, where a and b are normalized, complex coefficients--i.e., a*a + b*b = |a|^2 + |b|^2 = 1. This is of course not limited to just the z-basis; any set of orthonormal kets which span the 2D Hilbert space are similarly sufficient to describe any other arbitrary state. This is the entire logic that allows us to change basis while retaining the state.

So simply, **so long as we choose a complete set of orthonormal basis kets, in any basis, we are able to describe any other 2D spin state as a linear combination of some normalized complex constants and the relevant kets.**

1.)

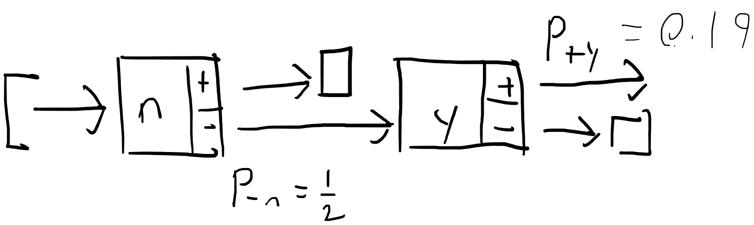
1. **[1.5 points]** Please select a problem from the prior homework for which you had the wrong answer and:

   i. Identify the question number you are correcting
   ii. State/copy your original wrong answer
   iii. Explain where your original reasoning was incorrect, the correct reasoning for the problem, and how it leads to the right answer.

   If you got all the answers correct, that's great! Please review your work anyway and discuss which problem you found the most interesting, enjoyable, and/or helpful and why.

3d.

I calculated the probability of a particle leaving the "up-y" port from the second analyzer, but I did not calculate the full probability, including the probability of a particle leaving the "down-n" port of the first analyzer.
The full correct answer would have been



$$P_{tot} = \frac{1}{2}\,\frac{19}{100} = \frac{19}{200} = 0.095$$

$$\boxed{\Rightarrow 9.5\%}$$

— or —

$$P_{tot} = \frac{1}{2}\left(\frac{1}{2}\left(1 - \frac{\sqrt{6}}{4}\right)\right)$$

$$= \frac{1}{4}\left(1 - \frac{\sqrt{6}}{4}\right) \approx 0.097$$

$$\rightarrow 9.7\%$$

$$\Rightarrow 9.7\%$$

I feel that this is an easy mistake to make, since I think that there have been many times that we have looked at independent, decoupled probabilities. However, when we do this, we are looking at cases where our input is prepared to come into an analyzer ALL in a given state-- e.g., all "up-n". In the future, I will make sure to look out for cases where our input state comes with a specific probability and remember to include that in the total probability at the end.

2. **[1.5 points]** Select another problem from the prior homework for which you had the wrong answer and correct it as above. If you got all the answers correct (or have nothing left to correct) that's great! Please review your work anyway and discuss which problem you found the *least* interesting, enjoyable, and/or helpful and why.

Insofar as the rest of the problems, I don't know if there was one that I found to be *less* interesting or helpful. I'd say all in all, learning about how to write a spin vector in a general state has made the idea of spin click a bit better than it was when we were looking at it in the x, y, and z bases. It makes it far easier to see spin as a physical thing that a particles has, rather than some magical quality which we can only understand through the mathematics. Nonetheless, spin is still very weird. I feel like I see the shape of it, forming somewhere off in the distance, obscured and not completely taking form, with only small parts popping in and out of comprehensibility, like some Lovecraftian monster. But that's quantum physics baby.

Just to fill this space with something more than rambling, I want to walk through the 5000 level question.

5. <u>**[8 points] 5000 Level:**</u> Consider a new operator defined as $\hat{R} \doteq \hbar|+\rangle\langle-|$.

a) **[3 points]** Write this operator as a matrix in the *z-basis* and determine its eigenvalue(s). Is it Hermitian? What does that suggest about the operator?

b) **[2 points]** Explain what this operator *does* to a state. *Hint:* Act this operator on the basis kets $|+\rangle$ *and* $|-\rangle$ and comment on the results.

c) **[1 point]** Is $|-\rangle$ an *eigenvector* of $\hat{R}$? What about $|+\rangle$?

d) **[2 points]** Consider another operator $\hat{L} \doteq \hbar|-\rangle\langle+|$. What does this operator do to a state? How is it related to $\hat{R}$?

a.)

$$\hat{R} = \hbar|+\rangle\langle-|$$

$$\hbar\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \hbar\begin{pmatrix}1\cdot 0 & 1\cdot 1\\ 0\cdot 0 & 0\cdot 1\end{pmatrix}$$

$$\hat{R} = \hbar\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$$

$$\det(\hat{R} - \lambda \mathbb{I})$$

$$\begin{vmatrix} -\lambda & \hbar \\ 0 & -\lambda \end{vmatrix} = \lambda^2 = 0 \Rightarrow \lambda = 0$$

$$\hat{R} \neq \hat{R}^\dagger \Rightarrow \begin{pmatrix} 0 & \hbar \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ \hbar & 0 \end{pmatrix}$$

Because R is not Hermitian, it does not correspond to an observable.

b.)

$$\hat{R}|+\rangle = \hbar|+\rangle\langle -|+\rangle = \hbar|+\rangle \underset{\text{eigenvalue}}{(0)}$$
$$= |0\rangle$$

$$\hat{R}|-\rangle = \hbar|+\rangle\langle -|-\rangle = \underline{\hbar|+\rangle}$$

flips $|-\rangle$ to $|+\rangle$

c.) $|+\rangle$ is an eigenvector of $\hat{R}$

$$\hat{R}|+\rangle = 0|+\rangle$$

d.) $\hat{L} = \hbar|-\rangle\langle +|$ flips $|+\rangle$ to $|-\rangle$

$$\hat{L}|+\rangle = \hbar|-\rangle\langle +|+\rangle = \hbar|-\rangle$$

$$\hat{L}|-\rangle = \hbar|-\rangle\langle +|-\rangle = 0|-\rangle$$

$$\hat{L} = \hbar\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\det(\hat{L} - \lambda \mathbb{I}) = \begin{vmatrix} -\lambda & 0 \\ \hbar & -\lambda \end{vmatrix} = \lambda^2 = 0$$

$$\lambda = 0$$

$|-\rangle$ is an eigenvector of $\hat{L}$

$$\hat{L} = \hat{L}^{+}$$

but

$$\hat{R} = \hat{L}^{+} \Rightarrow \begin{pmatrix} 0 & \hbar \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \hbar & 0 \end{pmatrix}^{+}$$