A spd $\quad Ax = b \overset{?}{\iff} \frac{1}{2} x^T A x - b^T x \to \min$

$A \in \mathbb{R}^{m \times m}$

$$\frac{\partial}{\partial x_k} \sum_{i=1}^{m} \sum_{j=1}^{m} x_i A_{ij} x_j = 2 \underbrace{A_{kk} x_k}_{k=i=j} + \overbrace{\underbrace{A_{ij} x_j}_{k=i\neq j} + \underbrace{A_{ij}}_{A_{ji}} x_i}^{\text{all } j}$$

$$\frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} x_i A_{ij} x_j - \underbrace{\sum_{i=1}^{m} b_i x_i}$$

$$\frac{\partial}{\partial x_k} \sum_{i=1}^{m} b_i x_i = \frac{\partial}{\partial x_k} \left(b_1 x_1 + \dots + b_k x_k + \dots + b_m x_m \right) = b_k$$

$$\nabla x^T A x = 2 A x$$

$$\frac{d}{dx}\left(\frac{1}{2} a x^2 - b x\right) = a x - b = 0$$

$a > 0$

$a x - b = 0$

$$\boxed{\frac{1}{2} a x^2 - b x \to \min}$$

$k \neq 1 \quad \frac{\partial b_1 x_1}{\partial x_k} = 0 \quad \frac{\partial b_k x_k}{\partial x_k} = b_k$

$$\frac{\partial}{\partial x_k} \sum_{i=1}^{m} \sum_{j=1}^{m} x_i A_{ij} x_j$$

$$\frac{\partial}{\partial x_k} x_i A_{ij} x_j = \begin{cases} 0 & \text{if } k \neq i, k \neq j \\ A_{ij} x_j & k = i \neq j \\ x_i A_{ij} & k = j \neq i \\ 2 A_{kk} x_k & k = i = j \end{cases}$$

$$\begin{bmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_m} \end{bmatrix} x^T A x = 2 A x$$

$$Ax = b \iff \min_x \underbrace{\tfrac{1}{2} x^T A x - b^T x}_{J(x)}$$

descent methods: direction d $\quad x \leftarrow x + td \quad$ find t so that $\min_t J(x+td)$

$$J(x+td) = \frac{1}{2}(x+td)^T A (x+td) = \frac{1}{2} x^T A x + \frac{1}{2} t d^T A x + \frac{1}{2} t x^T A d + \frac{1}{2} t^2 d^T A d \iff \frac{d}{dt} \frac{1}{2}(x+td)^T A (x+td) - b^T(x+t\ldots$$

$-b^T(x+dt)$

$$\frac{d}{dt} J(x+td) = \frac{d}{dt} \frac{1}{2} x^T A x + t d^T A x + \frac{1}{2} t^2 d^T A d = 0 + d^T A x + t d^T A d - \underbrace{b^T d}_{d^T b} = 0$$

$$\frac{d}{dt}\left(-b^T(x+dt)\right)$$

$$d^T(Ax-b) + t d^T A d = 0$$

$$t = -\frac{d^T(Ax-b)}{d^T A d}$$

at solution $Ax - b = 0$

$d) \longrightarrow \min_t \quad t \in \mathbb{R}$

d – descent direction

$x + td$

1 step of GS in direction x_i

$- \dfrac{\sum_{j=1}^{m} a_{ij} x_j}{a_{ii}}$

$$x_i \Leftarrow x_i + \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{m} a_{ij} x_j\right)$$

GS: for $i = 1, \dots, m$

sweep

$$x_i \Leftarrow x_i + \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{m} a_{ij} x_j\right)$$

Gradient descent choose $d = \nabla J(x) = \nabla\left(\frac{1}{2} x^T A x - x^T b\right) = Ax - b$

for $k = 1, \dots$ $d = Ax - b$ residual

$$t = \frac{d^T(Ax-b)}{d^T A d} = -\frac{d^T r}{d^T A d}$$

$$x \Leftarrow x + td$$