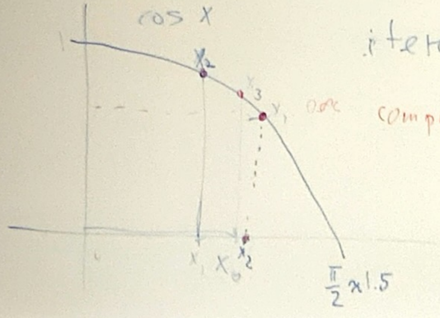
Solving equations

approximations of p $\quad f(p)=0$

approximations p_N $\quad \lim_{N\to\infty} p_N = p$

how fast? how accurate?

examples



iteration: $x_{k+1} = \cos x_k$

compute $x = \cos x$

solving $x = \cos x$

fixed point iteration

solve $x = f(x)$ by 1° choose x_0

2° for $k = 0, 1, \ldots$ $\quad x_{k+1} = f(x_k)$

analysis of fixed point iterations for $x = f(x)$

assume X^* = solution exists

assume exists x^*: $x^* = f(x^*)$

analyze: error $e_k = x_k - x^*$

propagation of error: $x_{k+1} - x^* = f(x_k) - x^* = f(x_k) - f(x^*)$

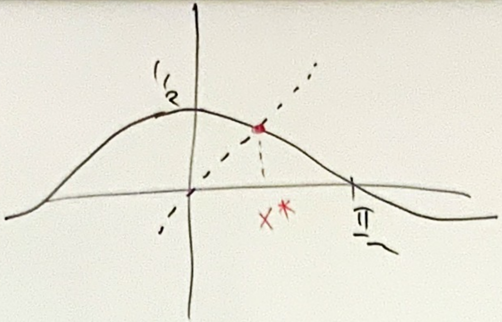
exists ξ_k between x_k and x^*

$f(x_k) - f(x^*) = f'(\xi_k)(x_k - x^*)$

$|f(x_k) - x^*| \leq |f'(\xi_k)|\,|x_k - x^*|$

Example

$$x = \frac{1}{2}\cos x$$



$$x_{k+1} = \frac{1}{2}\cos x_k$$

$$|x_{k+1} - x^*| \le \underbrace{\left|\left(\tfrac{1}{2}\cos x\right)'\Big|_{x=\xi}\right|}_{\underbrace{\left|\frac{1}{2}\sin \xi\right|}_{\le \frac{1}{2}}} \; |x_k - x^*|$$

$\longleftarrow$ for all $\xi \in \mathbb{R}$: $\left|\frac{1}{2}\sin\xi\right| \le \frac{1}{2}$

$$\boxed{|x_{k+1} - x^*| \le \frac{1}{2}|x_k - x^*|}$$

Problem $f(x) = e^x \sin x$

$P_2(x)$ $f'(x)$ $e^x \sin x + e^x \cos x$

$f''(x)$ $-e^x \sin x + e^x \cos x + e^x \cos x + e^x \sin x$

$= 2e^x \cos x$

$f'''(x) = -2e^x \sin x + 2e^x \cos x$

$= 2e^x(\cos x - \sin x)$

$$P_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2$$

$$R_2(x) = \frac{f'''(\xi)}{3!}(x - x_0)^3$$

$$= \boxed{\frac{2e^\xi(\cos\xi - \sin\xi)}{6}} x^3$$

$$\boxed{f(x) = P_2(x) + R_2(x)}$$

interested in $x = 0.4$

to do: find C : $\forall x \in [0, 0.4] : |R_2(x)| \leq C$

find C : $\forall x \in [0, 0.4]\ \forall \xi \in [0, 0.4] : \left|\frac{2e^\xi}{6}(\cos\xi - \sin\xi)\right| \leq C$

$0 < \xi < 0.4$

$$\left|\frac{2e^\xi}{6}(\cos\xi - \sin\xi)\right| \leq \left|\frac{2e^\xi}{6}\right| \left|\cos\xi - \sin\xi\right|$$

$$\leq \underbrace{\left|\frac{2e^\xi}{6}\right|}_{\leq \frac{e^{0.4}}{3}} \left(\underbrace{|\cos\xi|}_{\leq 1} + \underbrace{|\sin\xi|}_{\leq 1}\right) \leq \frac{2}{3} e^{0.4}$$

triangle inequality: $|a + b| \leq |a| + |b|$

$$\int_0^1 f(x)\,dx = \int_0^1 P_2(x)\,dx + \underbrace{\int_0^1 R_2(x)\,dx}$$

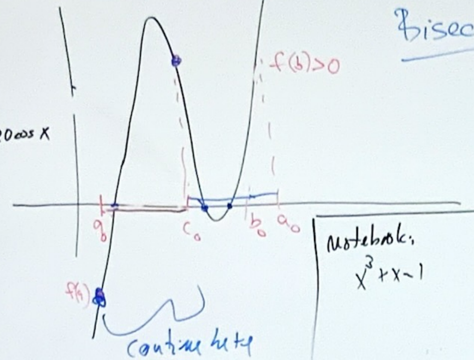
$$\left| \int_0^1 R_2(x)\,dx \right| \leq 1 \cdot \max_{0 \leq x \leq 1} |R_2(x)|$$

Bisection method

Intermediate value thm: If f continuous on $[a,b]$, $f(a)f(b)<0$, then $\exists\, x \in (a,b): f(x)=0$

example:

$f(x) = x - 20\cos x$



notebook:

x^3+x-1

Bisection method

Find $a<b$ s.t. $f(a), f(b)$ have opposite signs

set $a_0=a$, $b_0=b$

for $k=0,1,2,\ldots$

$c_k = \frac{1}{2}(a_k+b_k)$

if $f(c_k)=0$, $x=c_k$, done

if $f(c_k)f(a_k)<0$ then

$b_{k+1}=c_k$, $a_{k+1}=a_k$

else

$a_{k+1}=c_k$, $b_{k+1}=b_k$

Error:

$\exists x: f(x)=0,\ x\in[a_k,b_k]$

$$b_k - a_k = \frac{b-a}{2^k}$$

then

$$\underbrace{\left|x - \frac{a_k+b_k}{2}\right| \le \frac{b-a}{2^k}}_{\text{error estimate}}$$