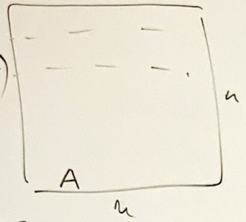
why not gauss elimination?

operations $\mathcal{O}(n^3)$

memory $\mathcal{O}(n^2)$



$n = 3 \cdot 10^6$

$\boxed{\text{ops } \mathcal{O}(9 \cdot 10^{18})}$

memory $\mathcal{O}(9 \cdot 10^{12})$

8 bytes = 1 word

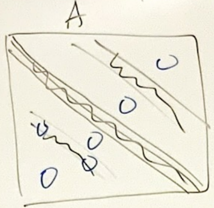
$8 \cdot 9 \cdot 10^{12} = 7.2 \cdot 10^{13}$ bytes

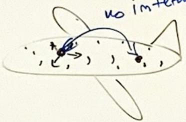
1kB = 1024 byte

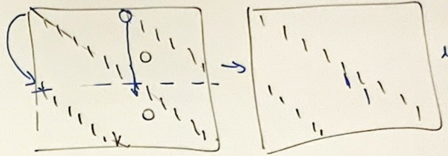
1MB $\approx 10^6$ Byte

$\approx 7.2 \cdot 10^7$ MB = 72TB memory

sparsity

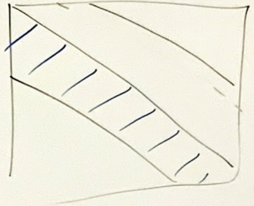






new nonzeros

"fill-in"



sparse methods = LU, but store nonzeros only

(or minimize the zeros stored)

$$4x_1 - 3x_2 = -1 \iff x_1 = \frac{1}{4}(-1 + 3x_2)$$

$$2x_1 + 5x_2 = 19 \iff x_2 = \frac{1}{5}(19 - 2x_1)$$

Iterative method

exact: $Ax^* = b$

for some initial $x_1^{(0)}, x_2^{(0)}$

$$x_1^{(k+1)} = \frac{1}{4}(-1 + 3x_2^{(k)})$$

$$x_2^{(k+1)} = \frac{1}{5}(19 - 2x_1^{(k)})$$

for $k = 0, 1, 2, \dots$

general form:

preconditioner

$$x^{(k+1)} = x^{(k)} + M^{-1}(b - Ax^{(k)}) \quad +$$

$$x^* = x^* + M^{-1}(b - Ax^*) \quad -$$

$$x^{(k+1)} - x^* = (I - M^{-1}A)(x^{(k)} - x^*)$$

error transformation

$M = A \Rightarrow 1$ iteration

$$4x - 10x_2 = 1$$

$$2x_1 + 5x_2 = 19$$

$A - D$

we did: $A = D + (L + U)$

$$b - Ax = b - Dx - (L+U)x = 0$$

$$\boxed{x^{(k+1)} = D^{-1}(b - (L+U)x^{(k)})}$$

$$x^{(k+1)} = x^{(k)} + D^{-1}(b - (D+L+U)x^{(k)})$$

$$x^{(k+1)} = x^{(k)} + D^{-1}(b - Ax^{(k)})$$

equations $Ax = b$ $\qquad A = D + L + U$

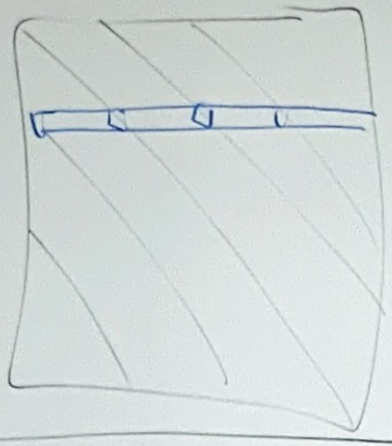
$$b - Ax = b - Dx - \underbrace{(L+U)}_{=A-D}x$$

$$\begin{pmatrix} -1 \\ 19 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 0 & -3 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}^{-1}\left(\begin{pmatrix} -1 \\ 19 \end{pmatrix} - \begin{pmatrix} 0 & -3 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^{(k+1)} = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/5 \end{pmatrix}\begin{pmatrix} -1 + 3x_1^{(k)} \\ 19 - 2x_2^{(k)} \end{pmatrix}$$

sparse matrix – nonzeros only on some diagonals



idea: compute variable x_k from k-th equation

simplest: compute x_k from k-equation for all k at once
keep other variables at previous values

$$\sum_{l=1}^{m} a_{kl} x_l = b_k \qquad a_{kk} x_k = b_k - \sum_{\substack{l=1 \\ l \neq k}}^{m} a_{kl} x_l$$

$$x_k \leftarrow \frac{1}{a_{kk}} \left(b_k - \sum_{\substack{l=1 \\ l \neq k}}^{m} a_{kl} x_l \right)$$

$x^{(k+1)} = x^{(k)} + M^{-1}(b - Ax^{(k)})$ choose $x^{(0)}$ any, $M^{-1} = A^{-1}$

$x^{(1)} = x^{(0)} + A^{-1}(b - Ax^{(0)}) = x^{(0)} + A^{-1}b - x^{(0)} = A^{-1}b = x^*$ exact solution

$$x^{(k+1)} = x^{(k)} + \underbrace{M^{-1}(b - Ax^{(k)})}_{\delta^{(k)}} \qquad x^{(0)} \text{ given}$$

$$\boxed{\begin{aligned} M\delta^{(k)} &= b - Ax^{(k)} \\ x^{(k+1)} &= x^{(k)} + \delta^{(k)} \end{aligned}}$$

$$\delta^{k} = M^{-1}(b - Ax^{(k)})$$

1. residual $r^{(k)} = b - Ax^{(k)}$
2. solve $\boxed{M\delta^{(k)} = r^{(k)}}$
3. insert correction $x^{(k+1)} = x^{(k)} + \delta^{(k)}$

replace solving $Ax = b$
by repeated $M\delta = r$

$0°$ assumption: evaluation of Ax cheap

design of M: $1°$ solving $M\delta = r$ cheap $O(n)$
dense $O(n^2)$

$2°$ $\|I - M^{-1}A\|$ small

$$x^{(k+1)} - x^{*} = (I - M^{-1}A)(x^{(k)} - x^{*})$$

$$\|I - M^{-1}A\|$$

$$\|x^{(k+1)} - x^{*}\| \le \underbrace{\|I - M^{-1}A\|}_{<1}\,\|x^{(k)} - x^{*}\|$$