$$A = QR$$

create example: 2×2

choose $R = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

choose $Q'' = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

verify $Q^TQ = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

GIVEN $A = QR$, $Q = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $R = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

FIND THE LEAST SQUARES SOLUTION OF $Ax = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$A = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}$ $b = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ solve $Ax = b$ in mean square sense

$\|Ax - b\|^2 \to \min_x$

Normal equation: $A^T A x = A^T b$

$$A^T A = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$$

solution: any x_1, x_2 $x_1 + x_2 = -3$

$$x = x(t) = \begin{bmatrix} t \\ -3-t \end{bmatrix} \quad t \in \mathbb{R}$$

$\min_t \|x(t)\|$ "smallest least squares solution"

$QRx = b$

Q size $m \times m$

R size $m \times n$ $\quad n < m$

$\|QRx - b\|^2 \to \min_x$ $\qquad \|Qu\|^2 = u^T Q^T Q u = u^T I u = u^T u = \|u\|^2$

$\|Q^T(QRx - b)\|^2 \to \min_x$ $\qquad Q^T Q = I$

$\|Rx - b\|^2 \to \min_x$

$$\underbrace{\begin{bmatrix} \ddots & & \\ & & \\ \hdashline & & \\ & & \end{bmatrix}}_{R} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} Q^T b \\ \hdashline \\ \end{bmatrix}$$

$$\left\| \begin{bmatrix} R_1 \\ 0 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} - \begin{bmatrix} (Q^T b)_1 \\ (Q^T b)_2 \end{bmatrix} \right\|^2 \to \min$$

$$A = QR \quad Q = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad R = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad A = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\underset{Q}{\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}} \underset{R}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \underset{x}{[x]} = \underset{b}{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}$$

$$\underbrace{\underset{Q^T}{\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}} \underset{Q}{\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}}}_{= I} \underset{R}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} [x] = \underbrace{\underset{Q^T}{\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}} \begin{bmatrix} 2 \\ 3 \end{bmatrix}}_{\begin{bmatrix} -3 \\ -2 \end{bmatrix}}$$

"in least squares sense"

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} [x] = \begin{bmatrix} -3 \\ -2 \end{bmatrix} \qquad \begin{matrix} x = -3 \\ 0 = -2 \end{matrix} \qquad \left\| \begin{matrix} x+3 \\ 2 \end{matrix} \right\|^2 \Rightarrow \min$$

$$\min_x \, (x+3)^2 + 2^2 \Rightarrow x = -3$$

another example $A = QR \quad Ax = b$

$$Q = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad Q^T Q = I$$

$$R = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\overset{Q}{\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \overset{R}{\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}} \overset{x}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \overset{b}{\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}}$$

$$\overset{Q^T}{\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \overset{Q}{\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \overset{R}{\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}} \overset{x}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \overset{Q^T}{\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \overset{b}{\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}}$$

$$\overset{R}{\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}} \overset{x}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \overset{Q^T b}{\begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix}}$$

$$\overbrace{\left\| \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} -2 \\ 4 \end{bmatrix} \right\|^2}^{\text{min } 0}$$

$$\boxed{x_2 = 4}$$

$$\boxed{x_1 = -10 \quad x_2 = 4}$$

$$+ \underbrace{\left\| [0, 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - [3] \right\|^2}_{\text{alway } 3^2} \to \text{min}$$

$$x_1 + 2x_2 = -2$$

$$x_1 = -2 - 2x_2$$

$$= -2 - 2 \cdot (4) = -10$$

$$-10 + 2 \cdot 4 = -2$$

$$f(x,y) = x^2 + y^2 + \cos(xy) \to \min_{(x,y)} \qquad \text{start } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

1° $$\nabla f(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x - y\sin(xy) \\ 2y - x\sin(xy) \end{bmatrix} = g$$

2° $$\min f\left(\begin{bmatrix} x \\ y \end{bmatrix} - \alpha g\right) \qquad \begin{bmatrix} x \\ y \end{bmatrix} - \alpha \underbrace{\begin{bmatrix} 2x - y\sin(x,y) \\ 2y - x\sin(x,y) \end{bmatrix}}_{g = \begin{bmatrix} 2 \\ 0 \end{bmatrix}} \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

solve $\min_{\alpha} f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \alpha \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right)$

$$\min_{\alpha} \underbrace{(1-2\alpha)^2}_{x^2} + \underbrace{0}_{y^2} + \underbrace{\cos((0-\alpha 2)\cdot 0)}_{1} \Rightarrow \alpha = \frac{1}{2} \qquad \text{new} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$f(x,y) = x^2 + y^2 + \cos(xy)$

$\nabla f = 0$

$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x - y\sin(xy) \\ 2y - x\sin(xy) \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} - \underbrace{H_f^{-1}(x,y)\,\nabla f_{(x,y)}}_{u}$

$H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x \partial x} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y \partial y} \end{bmatrix}$

$\begin{bmatrix} 2 - y^2\cos(x,y) & \ldots \\ \ldots & \ldots \end{bmatrix}$

1D $x \leftarrow x - \frac{1}{f''(x)} f'(x)$

$(H_f)^{-1}$ ∇f

1° $H_f(x,y)\,u = \nabla f_{(x,y)}$

solve for increment u

2° $\begin{bmatrix} x \\ y \end{bmatrix} \leftarrow \begin{bmatrix} x \\ y \end{bmatrix} - u$