The effects of nuclear mass on the wavelengths of emitted electromagnetic radiation from H2 and D2 atoms

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Abstract

In the early 20th century, the Bohr model of the hydrogen atom was developed to address many failures in the classical model. This described the atom in terms of central force problem, mediated through the Coulomb attractive force, in which a negatively charged electron orbits a positively charged nucleus with quantized angular momenta. This helped to give a theoretical basis to the Rydberg equation, which described the observed spectral lines of electromagnetic radiation emitted from the hydrogen atom. Additionally, this model helped to show how the Rydberg equation would change depending on the reduced mass of an atomic electron in response to nuclear mass. Using the Bohr model as the theoretical basis, we perform a spectroscopy analysis of electromagnetic radiation emitted from a narrow, low pressure tube of excited hydrogen H2 and deuterium D2 gas. The radiation beam is processed by a device known as a grating monochromator spectrometer, which then feeds the processed signal into a photomultiplier detector and is digitized. The data are then taken and plotted using a linearized version of the Rydberg equation where a fit function is applied in order to obtain the value of each atom's respective Rydberg constant. Using this value, we then attempt to derive the reduced mass of each atomic electron and compare this result with a theoretically calculated value.

I. Introduction

The existence of atoms has only be accepted relatively recently in the history of physics. This is despite the idea of atoms as a fundamental building block of all matter going back almost as far as what we now conceptualize as western thought itself [1]. This was due in no small part to the lack of technologies that allowed us to observe particles and molecules at the atomic scale [2]. In the modern day we are lucky to have a variety of techniques and devices that allow for the observation of atoms and atomic behaviors [2]. With these, we have the ability to study atomic and molecular structure and interactions with a level of precision nearly unthinkable just about 100 years ago.

In this experiment, we examine the simplest atom, hydrogen H2, made of just a single nuclear proton and orbiting electron. While the simplicity of the hydrogen atoms makes for relatively easy calculations and modeling, we should not mistake this for shallowness. From this hydrogen model, we are able to derive many phenomena that are easily able to be scaled up to more complicated atomic structures. The ability to do this has led to the hydrogen atom's reputation as a kind of fundamental building block itself for the rest of atomic physics [2].

In order to demonstrate the ways in which atomic phenomena change with respect to atomic structures, we include hydrogen's closest isotope, deuterium D2, whose nucleus contains a neutron in addition to the single proton found in the hydrogen H2 nucleus. Intuitively, the D2 nucleus will have a greater nuclear mass, resulting in small, yet measurable changes to some important atomic behaviors. As we will see in section II, these changes will make it important to consider the role of the reduced electron mass. This is obtained through the Bohr model, which described the interaction between the nucleus and orbiting electron as a central force problem, determined by the Coulomb attraction between opposite charges [2].

The specific behavior that we are attempting to observe and measure is the spectral lines of electromagnetic radiation from the two atoms as their orbital electrons relax from some excited state back down to some lower energy state. In particular, we are most interested in the first four wavelengths given by the Balmer series, which corresponds to the visible light photons emitted as the Hydrogen atom relaxes from its 3rd through 6th excited energy states back down to its 2nd. Each of these photons will have an energy equal to the difference between the energy states the electron moves between, which is inversely proportional to each photon's respective wavelength. In addition to the energy state, these wavelengths may also be used to gain information on the orbiting electron's reduced mass, which itself contains information about the nuclear mass and composition. Therefore, the measurement of these wavelengths proves to be valuable tool when attempting to determine characteristics of a particular atom.

In order to measure wavelengths, we will make use of a rather sophisticated device known as a grating monochromator spectrometer. This device allows us to focus a radiation signal, split into its various wavelengths, and measure those with the highest intensity. More details about this will be given in section III.

The model that allows us to model the wavelengths of the lines of the emitted electromagnetic radiation is the Rydberg equation [3]. Roughly, this describes the wavelength as a function of the difference in energy state numbers—an integer value which corresponds to the distance from the ground energy state—times some constant of proportionality known as the Rydberg constant. Importantly, the Rydberg constant will change by a factor of the reduced mass ratio, i.e. the ratio of the electron's reduced mass and the accepted, non-reduced electron mass [2] [3]. Upon measuring the wavelengths of the emitted electromagnetic radiation, we will use a linearized version of the Rydberg equation to determine this ratio as the slope of a fit function. We are then able to compare our measured value with a previous determined, calculated value to gauge the quality of our measurements.

II. Theory

The Bohr model of the atom was developed in 1913 in order to address multiple shortcomings of the then dominant Rutherford model. Maxwell's equations tell us that an accelerating charged particle will emit electromagnetic (EM) radiation and as a result lose energy. Since the Rutherford model involved electrons orbiting a nucleus in a circular orbit, these electrons would experience an acceleration everywhere along their orbital path, emitting a continuous spectrum of EM radiation with ever shortening wavelengths, losing all of their energy till they crash into the atomic nucleus. The derivation is a bit beyond the scope of this report, but for an electron orbiting a hydrogen atom at a Bohr radius, this collapse would happen in $\sim 10^{-11}s$ [4].

These predictions were obviously incorrect since they couldn't explain the existence of stable atoms which for the basis for most matter, and EM radiation had been observed as discrete spectral lines in experimentation. Therefore, it was obvious that the Rutherford model was incorrect on some fundamental level. The Rydberg equation, which had been around since the late 19th century, was able to describe the observed spectral lines, but lacked a theoretical basis. It was the development of the Bohr model which ultimately provided the first solution to all of the above problems [5].

The Bohr model puts forth that electrons must be able to move around a nucleus in stable orbits without radiating any energy at specific, discrete distances. Each one of these stable orbits has a corresponding angular momentum which is an integer multiple n of the reduced Planck's constant. For a hydrogen atom, the smallest integer multiple $n = 1$ gives us the Bohr radius, the smallest radial distance an electron can be from a nucleus; $\sim 0.0529nm$. Following from the discrete angular momenta of each discrete orbit, there is also a corresponding energy which is constant while the electron occupies that specific orbit. An electron will only lose or gain energy by moving between these stable energy states, leading to the conclusion that an electron can only lose or gain specific amounts of energy determined by the difference between the corresponding energy states. For an electron that absorbs energy from an incident photon, or loses energy by emitting electromagnetic radiation in the form of a photon, the energy difference is given by the formula below, where h is the unreduced Planck's constant, ν is the frequency of the photon, c is the speed of light, and λ is the wavelength of the photon.

$$\Delta E = E_2 - E_1 = h\nu = \frac{hc}{\lambda}$$

This can be compared with the Rydberg equation below, where n_i and n_f are the integer values that correspond to the initial and final energy states respectively.

$$\frac{1}{\lambda} = R\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

It's important to note that in the above the value of the Rydberg constant R is for an infinitely massive nucleus and is a theoretical value. Common sense will lead one to conclude that there is, after all, no such thing as an infinitely massive nucleus. In the case of hydrogen, where the nuclear mass is as small as it could ever be and this value is a decent approximation, we will see a small yet substantial difference in the value of R. Thus, it seems worthwhile to go over how we arrive at a correction for this value.

Starting with the Bohr model, we can assume three postulates: (1) the radial distance of the electron has some typical scale given by r_0, (2) the momentum has some typical scale of p_0, and the time for any significant change in position and momentum has a typical scale t_0. Here we write the following equations with the dimensionless terms r', p', and t' [2].

$$r = r_0 r' \qquad p = p_0 p' \qquad t = t_0 t'$$

These postulates seem reasonable since we already have some basis as to the Bohr atomic model following from some natural scale. By separating our equations into normalized dimensional terms and dimensionless scalar terms, we set ourselves up to determine any other constants which may arise in our system.

Using these, we write the momentum as

$$p = m\frac{dr}{dt} \rightarrow p_0 p' = m\frac{r_0}{t_0}\frac{dr'}{dt'}$$

$$\frac{1}{m}p_0 p' = \frac{r_0}{t_0}\frac{dr'}{dt'}$$

$$F = \frac{dp}{dt} = \frac{p_0}{t_0}\frac{dp'}{dt'} = \frac{1}{r_0^2}\frac{e^2\hat{r}}{4\pi\epsilon_0 r'^2}$$

.

Grouping the dimensionless terms on one side and the dimension bestowed terms on the other:

$$\frac{dr'}{dt'} = \frac{t_0}{r_0}\frac{1}{m}p_0 p' \qquad \frac{dp'}{dt'} = \frac{t_0}{p_0}\frac{e^2}{4\pi\epsilon_0 r_0^2}\frac{\hat{r}}{r'^2}$$

.

We can now group all the constant scale terms together and set them equal to one, since they must result in some unitless relation.

$$\frac{t_0}{r_0}\frac{p_0}{m} = \frac{e^2}{4\pi\epsilon_0}\frac{t_0}{p_0 r_0^2} = 1$$

We also take our dynamics equations in terms of a natural scale like so.

$$\frac{dr'}{dt'} = p' \qquad \frac{dp'}{dt'} = \frac{\hat{r}}{r'^2}$$

Lastly, we can rearrange our constant scale relationships to the following.

$$\frac{p_0 t_0}{r_0} = m \qquad \frac{t_0}{p_0 r_0^2} = \frac{4\pi\epsilon_0}{e^2}$$

With our natural scale constants, we are able to refer back to one of the postulates of the Bohr model. That is that the angular momentum at each possible electron orbit is equal to an integer multiple of the reduced Planck's constant. Assuming $n = 1$, we can now write $r_0 p_0 = \hbar$.

Starting with the definitions above $\frac{p_0 t_0}{r_0} = m$ and $\frac{t_0}{p_0 r_0^2} = \frac{4\pi\epsilon_0}{e^2}$ we can write the natural scale mass and radius as follows.

$$\frac{4\pi\epsilon_0}{e^2}\frac{p_0^2 r_0^2}{r_0} = \frac{4\pi\epsilon_0}{e^2}\frac{\hbar^2}{r_0} = m$$

$$r_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2}$$

Following from the definition of r_0 as some natural scale for the distance at the atomic scale, we now define $r_0 \equiv a_0$, which will be the smallest discrete distance between an electron and nucleus, which we now refer to it as the Bohr radius. This value is already well established, being equal to $a_0 = 0.0529$ nm for a H2 atom. We are able to determine this value below.

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2}$$

From the definition of the natural scale radius, we can go on to obtain a natural scale for the momentum, and from the momentum, the energy.

$$p_0 = \frac{me^2}{4\pi\epsilon_0\hbar}$$

.

$$E_0 = \frac{p_0}{t_0}r_0 = \frac{\frac{me^2}{4\pi\epsilon_0\hbar}}{\frac{(4\pi\epsilon_0)^2\hbar^3}{me^4}}\frac{4\pi\epsilon_0\hbar^2}{me^2} = \frac{me^4}{(4\pi\epsilon_0)^2\hbar^2}$$

And E_0 at a_0 is given as

$$\frac{1}{a_0} = \frac{me^2}{4\pi\epsilon_0\hbar^2}$$

$$\frac{e^2}{4\pi\epsilon_0}\frac{1}{a_0} = \frac{e^2}{4\pi\epsilon_0}\frac{me^2}{4\pi\epsilon_0\hbar^2}$$

$$E_0 = \frac{me^4}{(4\pi\epsilon_0)^2\hbar^2} = \frac{e^2}{4\pi\epsilon_0 a_0}$$

With a natural scale for the radius, momentum, and energy, we seemingly have everything we will need to describe the system. As a final point, we ought to briefly look to the force. Since the electron will experience and acceleration towards the center from a force that is the result of a positively charged proton nucleus, we can write the force as follows.

$$F = -m\frac{v^2}{r} = -\frac{e^2}{4\pi\epsilon_0 r^2}$$

With a bit of manipulation, we are able to relate this definition to the angular momentum like so.

$$(mvr)^2 = m\left(m\frac{v^2}{r}\right)r^3 = m\frac{e^2}{4\pi\epsilon_0 r^2}r^3 = \frac{me^2}{4\pi\epsilon_0}r$$

From the Bohr model postulate about the quantization of angular momentum, we can utilize the definition below.

$$n^2\hbar^2 = \frac{me^2}{4\pi\epsilon_0}r$$

$$r = n^2 \frac{4\pi\epsilon_0}{me^2} = n^2 \frac{4\pi\epsilon_0 \hbar^2}{me^2} = n^2 a_0$$

Taking this definition of the orbital radii, we can then come back to the total energy through the force.

$$-m\frac{v^2}{r} = -\frac{e^2}{4\pi\epsilon_0 r^2}$$

Solving for the velocity squared, we obtain the following from the classical definition of the kinetic energy.

$$v^2 = \frac{e^2}{4\pi\epsilon_0 m r}$$

$$K = \frac{1}{2}m\left(\frac{e^2}{4\pi\epsilon_0 m r}\right) = -\frac{1}{2}\frac{e^2}{4\pi\epsilon_0 r}$$

The kinetic energy is equal to half of the electrical potential energy, given by the the identity $V(r) = -\int F(r)dr$.

From here we are able to write the total energy.

$$E = K(r) + V(r) = \frac{1}{2}\frac{e^2}{4\pi\epsilon_0 r} - \frac{e^2}{4\pi\epsilon_0 r} = -\frac{1}{2}\frac{e^2}{4\pi\epsilon_0 r}$$

Worth noting, this is strikingly similar to the relationship between the kinetic, potential, and total energies in the case of a circular orbit under gravitational forces. This makes sense because Bohr model of the hydrogen atom is based upon an analogous picture.

We now take the definition of the orbital radii in terms of the product of an integer n and the Bohr radius, we are able to write the energy as follows.

$$E = -\frac{1}{2}\frac{e^2}{4\pi\epsilon_0 r} = -\frac{1}{2}\frac{e^2}{4\pi\epsilon_0 n^2 a_0} = -\frac{1}{n^2}\frac{1}{2}\frac{e^2}{4\pi\epsilon_0 a_0} = -\frac{1}{n^2}\frac{E_0}{2}$$

Lastly, we go back to the Rydberg equation and the change of energy as a function of the photon wavelength laid out above towards the beginning of this section.

$$\frac{1}{\lambda} = \frac{\Delta E}{hc} = \frac{E_i - E_f}{hc} = \frac{1}{hc}\left(\frac{1}{n_f^2}\frac{E_0}{2} - \frac{1}{n_i^2}\frac{E_0}{2}\right) = \frac{1}{hc}\frac{E_0}{hc}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

Setting this equal to the Rydberg equation, we are able to write the Rydberg constant for an infinite mass in terms of the natural scale energy.

$$R\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) = \frac{1}{hc}\frac{E_0}{2}(\frac{1}{n_f^2} - \frac{1}{n_i^2}\Big)$$

$$R = \frac{1}{hc}\frac{E_0}{2} = 0.01097\ \mathrm{nm}^{-1}$$

As touched upon above, this value is in need of a correction. In order to do this, we consider forces of the negatively charged electron and the positively charged nucleus on each other. This are written below as $\vec{F}_{+-}$ and $\vec{F}_{-+}$, as the force of the proton nucleus on the electron and the electron on the proton nucleus respectively.

$$\vec{F}_{+-} = m\frac{d^2\vec{r}_m}{dt^2} \qquad \vec{F}_{-+} = M\frac{d^2\vec{r}_M}{dt^2}$$

Assuming the two forces are in equilibrium, we can say that $\vec{F}_{+-} = -\vec{F}_{-+}$, which gives us the following, after multiplying the first force equation by M and the second by m and subtracting the two.

$$M\vec{F}_{+-} - m\vec{F}_{-+} = Mm\frac{d^2\vec{r}_m}{dt^2} - mM\frac{d^2\vec{r}_M}{dt^2}$$

$$M\vec{F}_{+-} + m\vec{F}_{+-} = Mm\frac{d^2(\vec{r}_m - \vec{r}_M)}{dt^2}$$

$$(M+m)\vec{F}_{+-} = Mm\frac{d^2(\vec{r}_m - \vec{r}_M)}{dt^2}$$

$$\vec{F}_{+-} = \frac{Mm}{M+m}\frac{d^2(\vec{r}_m - \vec{r}_M)}{dt^2}$$

We then define $\vec{r}_{rel} \equiv \vec{r}_m - \vec{r}_M$ and set $\mu = \frac{Mm}{M+m}$, which gives us the reduced mass of the electron. This finally gives us the force from the proton nucleus on the electron as

$$\vec{F}_{+-} = \mu\frac{d^2\vec{r}_{rel}}{dt^2}$$

Which may also be written in terms of the Coulomb force.

$$\mu\frac{d^2\vec{r}_{rel}}{dt^2} = -\frac{1}{4\pi\epsilon_0}\frac{\mu e^2}{r_{rel}^2}$$

In order to correct the equations that we've obtained thus far, we simply multiply by the ratio of the electron reduced mass and mass of a free electron.

$$\frac{\mu}{m_e}E_0 = \frac{\mu}{m_e}\frac{m_e e^4}{(4\pi\epsilon_0)^2\hbar^2} = \frac{\mu e^4}{(4\pi\epsilon_0)^2\hbar^2}$$

$$\frac{\mu}{m}R\left(\frac{1}{n_f^2}-\frac{1}{n_i^2}\right)=\frac{1}{hc}\frac{\mu}{m}\frac{E_0}{2}(\frac{1}{n_f^2}-\frac{1}{n_i^2}\right)$$

The values for the reduced mass ratios for hydrogen and deuterium can easily be calculated analytically and are given below.

$$\frac{\mu_H}{m_e}=0.999457 \qquad \frac{\mu_D}{m_e}=0.999728$$

As we can see, the reduced mass ratio will return a reduced value of the Rydberg equation and reduced change in energy between energy states. Since these both have an inverse relationship with the wavelength, we will then expect to observe longer wavelengths than in the case of the Rydberg constant for an infinitely massive nucleus. From the calculated values above, we will then expect hydrogen H2 to have longer wavelengths that are longer than deuterium D2. Additionally, the difference of these wavelengths will be small, since the difference of the ratios is rather small. Indeed, in the following section we will see that for each line in the Balmer series, we will see two wavelengths, very close by each other. The shorter of the two will correspond to D2 and the longer will correspond to H2. This theoretical basis is what allows us to determine which wavelength is emitted by each atom and sets up the basis of all of our further analysis.

III. Experimental Methods

In order to measure the photon wavelengths, a narrow vertical discharge lamp, filled with a low pressure gas, was used as a source of EM radiation, i.e. as a light source. This was done by running an electrical current through the gas in order to disassociate the molecules and to excite the electrons of each atom. As the electrons relax, they emit photons, which are measured as waves of light. The resulting beam of light is then passed through a convex lens onto the entrance slit of a Jobin-Yvon 750s grating monochromator spectrometer. The aperture of the entrance slit is adjustable over a range of $3-3000\mu m$. The geometry of the monochromator is such that the entrance slit is at the focal point of a a concave spherical mirror, which collimates and directs the light onto a 110x100 mm diffraction grating with a series of small grooves—1800 per mm. The diffracted light is then refocused by another spherical concave mirror onto the exit slit. After passing through the exit slit, the light signal is then collected into a photomultiplier tube (PMT) detector [6].

The data collection process was controlled from a software interface called SynerJY by Horiba [7]. The high voltage for the PMT was set to 750 V, and both the entrance an exit slits were set were set at 0.022 mm.

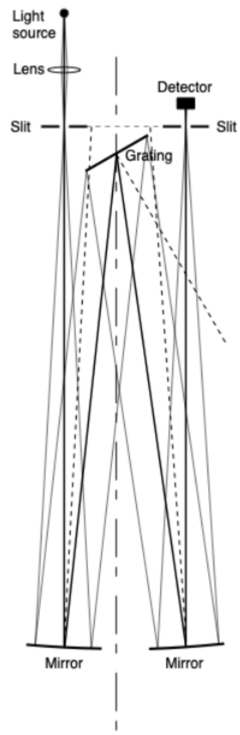
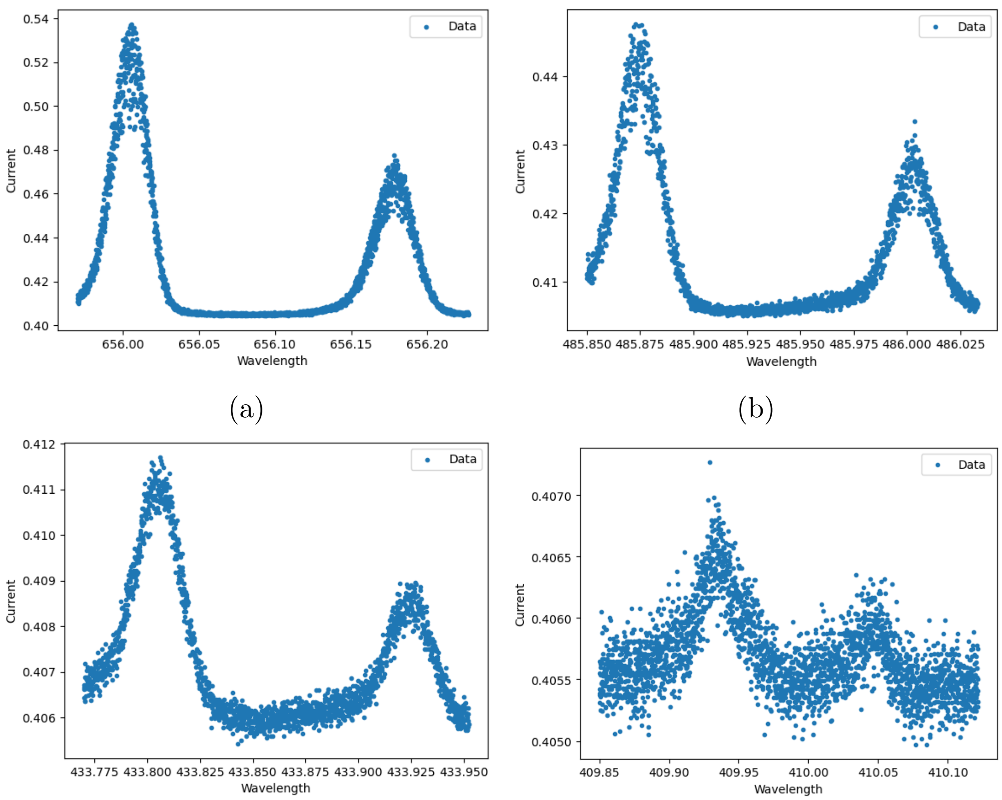


Figure 1: Diagram of the optical system with special emphasis on the inside of the monochromator. A light source is placed before a lens which focuses a beam of light onto the entrance slit. The beam is then reflected off a concave mirror towards the diffraction grating, which reflects the scattered light towards another concave mirror, which then directs it towards the exit slit. After leaving through the exit slit, the light is then picked up by a PMT detector [6]

The experiment was initially calibrated using a mercury Hg discharge lamp. The Hg lamp was turned on and a sweep between 546.1 nm – 546.3 nm was run with an integration time of 0.01s and a increment time of 0.001 nm. The vacuum wavelength that mercury exhibits around this range is 546.0735 nm [8]. Taking this, we then measured the air temperature, atmospheric pressure, and air humidity in order to calculate the expected wavelength in ambient air as well as the refractive index for the air. We then used the calculated wavelength in the ambient air to set the center position in SynerJY, so that our measurements would compensate for any modification of the wavelengths from the expected value due to the ambient refraction.

After calibration, the Hg discharge tube was replaced by an H2D2 source. The integration time was adjusted to 0.05 s and a series of sweeps were performed, starting around the highest wavelength in the Balmer series. This was repeated for the second, third, and fourth highest wavelengths in the series, i.e. 656.279 nm, 486.135 nm, 434.0472 nm, and 410.1734 nm [9]. For each sweep, we initially set a wide range of roughly $\pm$50nm in order to compensate for any modification of the hydrogen H2 and deuterium D2 wavelengths in ambient air. For each wavelength, we recorded information about the air temperature, pressure, and humidity so as to make note of any substantial changes from our calibration values. Over our data collection, no substantial changes were recorded.

IV. Results



(a) (b) (c) (d)

Figure 2: Peaks corresponding to the first four wavelengths in the Balmer series emitted from both hydrogen and deuterium. Value of wavelengths for each peak from left to right: (a) 656.004nm and 656.179nm; (b) 485.874nm and 486.002nm; (c) 433.805nm and 433.924nm; (d) 409.935nm and 410.043nm

One the data were collected, each peak was fit with a Gaussian function in order to determine precise values for each wavelength. As expected, there were two nearby peaks near each value of the Balmer series, corresponding to the electromagnetic radiation emitted by both the H2 and D2 atoms. From our theoretical analysis done, we knew to expect D2 to emit radiation with smaller wavelengths than H2, since it had a greater reduced mass and thus a greater adjusted Rydberg constant. We used this to determine which peak corresponded to which atom and group the data accordingly.

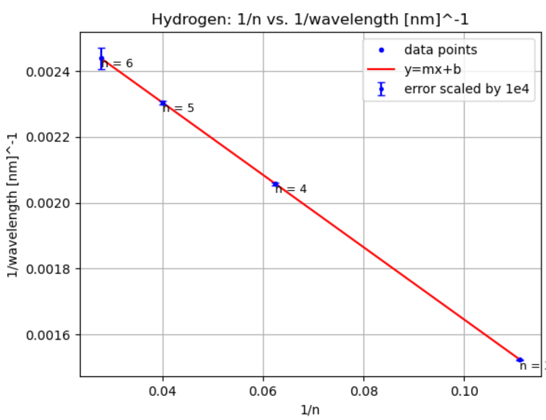
V. Analysis

For each line in the Balmer series, two wavelengths were measured. In accordance with our theoretical prediction, the shorter wavelength of the two was grouped as deuterium D2 and the longer was grouped as hydrogen H2. Once the identity of each wavelength was determined, they were fit using a using Gaussian fit function. This returned each peaks measured value, as well as an error. One each peaks value was determined with our fit function, the inverse of the wavelengths were plotted with respect to the inverse of there energy states number in order to linearize the Rydberg equation.

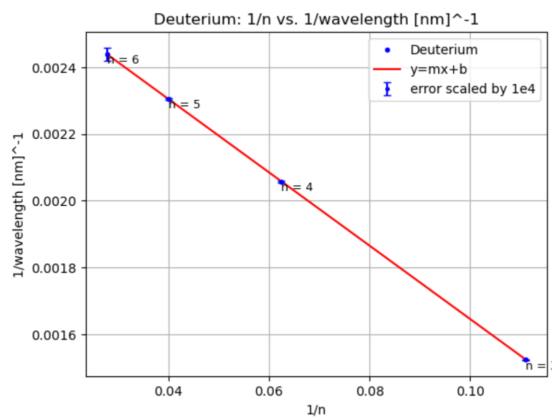
$$\frac{1}{\lambda} = R\left(\frac{1}{2^2} - \frac{1}{n^2}\right) \rightarrow \frac{1}{\lambda} = \frac{R}{2^2} - \frac{R}{n^2}$$

$$y = \frac{R}{2^2} - Rx$$

The equation above was used to make two linear plots (Figure 3a, 3b.). The pre-made Linregress function from the SciPy.stats library for Python was used to determine an equation for the line-of-best-fit, with the slope directly corresponding to the reduced Rydberg constant and the y-intercept equally a quarter of the Rydberg constant.



(a)



(b)

Figure 3: Plots made with the linearized version of Rydberg's equation. (a) $y = -(1.097464 \times 10^{-2}\ \mathrm{nm}^{-1} \pm 1.0 \times 10^{-6})\ \mathrm{nm}^{-1} + (2.74306 \times 10^{-3}\ \mathrm{nm}^{-1} \pm 7.0 \times 10^{-8}\ \mathrm{nm}^{-1}$; (b) $y = -(1.097763 \times 10^{-2}\ \mathrm{nm}^{-1} \pm 8.7 \times 10^{-7}\ \mathrm{nm}^{-1}) + (2.7438 \times 10^{-3}\ \mathrm{nm}^{-1} \pm 6.0 \times 10^{-8})\ \mathrm{nm}^{-1}$

From the linear regression, we determined that the values of the Rydberg constants for hydrogen H2 was $1.097464 \times 10^{-2}\ \mathrm{nm}^{-1} \pm 1.0 \times 10^{-6}\ \mathrm{nm}^{-1}$ and deuterium D2 was $1.097763 \times 10^{-2}\ \mathrm{nm}^{-1} \pm 8.7 \times 10^{-7}\ \mathrm{nm}^{-1}$. From here, we can see that the difference between the two values is minute, yet still large enough to be experimentally observed.

With the value of the slope, we can divide by the determined value for the Rydberg constant for an infinitely massive nucleus, and obtain the reduced mass ratio for each atom's orbiting electron.

$$R_\infty = 1.09737 \times 10^{-2}\ \text{nm}^{-1}$$

$$R_H = \frac{\mu_H}{m_e} R_\infty \qquad R_D = \frac{\mu_D}{m_e} R_\infty$$
$$\frac{R_H}{R_\infty} = \frac{\mu_H}{m_e} \qquad \frac{R_D}{R_\infty} = \frac{\mu_H}{m_e}$$

However, with just a cursory glance, the ratios of the two respective Rydberg constants and R_∞ shows that this ratio will be greater than unity.

$$\frac{\mu_H}{m_e} = \frac{1.097464}{1.09737} = 1.00009 \qquad \frac{\mu_H}{m_e} = \frac{1.097763}{1.09737} = 1.00036$$

This is certainly a problematic result, since it directly follows that the reduced electron mass would be greater than the regular non-reduced electron mass. This is of course scientifically and linguistically incorrect.

We can compare these measured values with the values calculated using our model with the following values: the mass of an H2 nucleus 1.67262×10^{-27} kg, the mass of a D2 nucleus is 3.34449×10^{-27} kg, and the mass of an electron is 9.10938×10^{-31} kg [10].

$$R_H = \frac{\mu_H}{m_e} R_\infty = \frac{\frac{M_H m_e}{M_H + m_e}}{m_e} R_\infty = \frac{1}{1 + \frac{m_e}{M_H}} R_\infty = 1.09677 \times 10^{-2}\ \text{nm}^{-1}$$

$$R_D = \frac{\mu_D}{m_e} R_\infty = \frac{\frac{M_D m_e}{M_D + m_e}}{m_e} R_\infty = \frac{1}{1 + \frac{m_e}{M_D}} R_\infty = 1.09707 \times 10^{-2}\ \text{nm}^{-1}$$

$$\frac{\mu_H}{m_e} = \frac{1.09677}{1.09737} = 0.99945 \qquad \frac{\mu_H}{m_e} = \frac{1.09707}{1.09737} = 0.99973$$

This gives us a relative error in our measured values of

$$R_{D_{err}} \approx 6.4035 \times 10^{-2}\ \% \qquad R_{D_{err}} \approx 6.3017 \times 10^{-2}\ \%$$

.

This is an extremely small error, yet at the atomic scale, it is large enough to return incorrect reduced mass ratios. Interesting, when we look at the ratio of the measured and calculated ratios, we get the following.

$$(\text{measured})\ \frac{\mu_H}{\mu_D} = 0.99973 \qquad (\text{calculated})\ \frac{\mu_H}{\mu_D} = 0.99973$$

This seems to suggest that the values from our measurement are at least consistent with each other within the model we laid out. Therefore, in order to explain why our reduced mass ratio is so obviously incorrect, we need to consider sources of error.

Since our measurements seem to be theoretically consistent, as well as the fact that the propagated error we calculated was incredibly small, having to be scaled by a factor of $10,000$ to even be visible on our linear plot, the most probably source of error is likely to be in our calibration or some calculation error. In either case, this may warrant a further analysis where additional data would need to be taken.

VI. Conclusion

While we did not arrive at correct values for the reduced electron masses for both types of atom, our data and calculations seemed to have an internal consistency which allows us to see that the model laid done in section II holds. This is supported by the two peaks per Balmer wavelength that we were able to measure, as well as the near equality of the ratio of the measured and calculated reduced masses. Additionally, while the calculated values are off, they're only off by a slight amount. This is a testament to how minute and exquisite phenomena at the atomic scale truly are, as well as the level of precision needed to be able to measure and observe them.

References

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