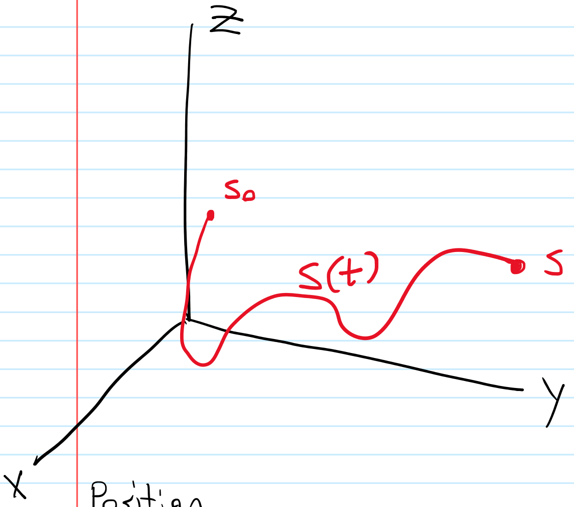
Standard 3 - 3D Path x(t)



A particle moves along the path s(t) from point s_0 to point s_f over some interval of time in cartesian coordinates.

Position

$$\vec{S} = x\hat{i} + y\hat{j} + z\hat{k}$$

Infinitesimal displacement

$$d\vec{S} = dx\hat{i} + y\hat{j} + z\hat{k}$$

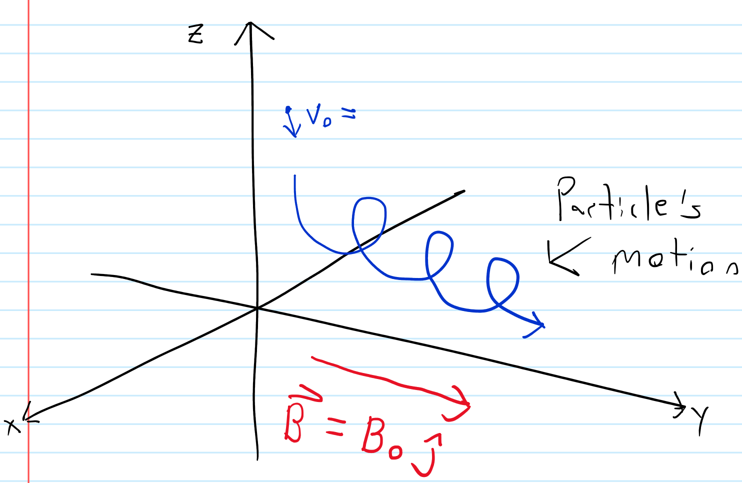
Velocity

$$\vec{v} = \frac{d\vec{S}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

Acceleration

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{S}}{dt^2} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt}\hat{k}$$

From page 73: "... Consider a charged particle entering a region of uniform magnetic field B-for example, the earth's field-{as shown below}. Determine its subsequent motion."



$$\sum F = m\vec{a} = q\vec{v} \times \vec{B}$$

$$m\left(\frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k}\right) = q\left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right) \times \left(B_0 \hat{j}\right)$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ q\frac{dx}{dt} & q\frac{dy}{dt} & q\frac{dz}{dt} \\ 0 & B_0 & 0 \end{vmatrix} = \hat{i}\left(q\frac{dy}{dt}\cdot 0 - B_0 q\frac{dz}{dt}\right) - \hat{j}\left(q\frac{dz}{dt}\cdot 0 - 0\cdot q\frac{dx}{dt}\right)$$

$$+ \hat{k}\left(q\frac{dx}{dt}B_0 - 0\cdot q\frac{dy}{dt}\right)$$

$$m\left(\frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\right) = qB_0\left(-\frac{dz}{dt}\hat{i} + \frac{dx}{dt}\hat{k}\right)$$

$$m\frac{d^2x}{dt^2} = -qB_0\frac{dz}{dt}$$

$$m\frac{d^2y}{dt^2} = 0$$

$$m\frac{d^2z}{dt^2} = qB_0\frac{dx}{dt}$$

$$\boxed{\frac{d^2y}{dt^2} = 0 \qquad \frac{dy}{dt} = \frac{dy_0}{dt} \qquad y = \frac{dy_0}{dt}t + y_0}$$

$$m\frac{d^2x}{dt^2} = -qB_0\frac{dz}{dt} \quad \& \quad m\frac{d^2z}{dt^2} = qB_0\frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} = -\frac{q}{m}B_0\frac{dz}{dt} \qquad \frac{d^2z}{dt^2} = \frac{q}{m}B_0\frac{dx}{dt}$$

$$\alpha = \frac{q}{m} B_0$$

$$\frac{d^2x}{dt^2} = -\alpha \frac{dz}{dt} \qquad \frac{d^2z}{dt^2} = \alpha \frac{dx}{dt}$$

$$\frac{d^3x}{dt^3} = -\alpha \frac{d^2z}{dt^2} \qquad \frac{d^3z}{dt^3} = \alpha \frac{d^2x}{dt^2}$$

$$\frac{d^3x}{dt^3} = -\alpha^2 \frac{dx}{dt} \qquad \frac{d^3z}{dt^3} = -\alpha^2 \frac{dz}{dt}$$

$$\frac{d^3x}{dt^3} + \alpha^2 \frac{dx}{dt} = 0 \qquad \frac{d^3z}{dt^3} + \alpha^2 \frac{dz}{dt} = 0$$

$x = e^{rt}$ $\qquad$ $z = e^{st}$

$\frac{dx}{dt} = re^{rt}$ $\qquad$ $\frac{dz}{dt} = se^{st}$

$\frac{d^2x}{dt^2} = r^2e^{rt}$ $\qquad$ $\frac{d^2z}{dt} = s^2e^{st}$

$\frac{d^3x}{dt^3} = r^3e^{rt}$ $\qquad$ $\frac{d^3z}{dt^3} = s^3e^{st}$

$r^3e^{rt} + \alpha^2 re^{rt} = 0$ $\qquad$ $s^3e^{st} + \alpha^2 se^{st} = 0$

$r^2 + \alpha^2 = 0$ $\qquad$ $s^2 + \alpha^2 = 0$

$s = \pm \alpha i$

$$r = \frac{-0 \pm \sqrt{0^2 - 4\alpha^2}}{2}$$

$s = \pm \alpha i$

$r = \pm \alpha i$

$x = c_1 e^{\alpha i t} + c_2 e^{-\alpha i t}$

$x = C_1 (\cos(\alpha t) + i\sin(\alpha t))$
$\quad + C_2(\cos(\alpha t) - i\sin(\alpha t))$

$z = (d_1 + d_2)\cos(\alpha t)$
$\quad + (d_1 - d_2)\, i \sin(\alpha t)$

$d_1 + d_2 = C$

$$+c_2(\cos(\alpha t) - i\sin(\alpha t))$$

$$x = (c_1 + c_2)\cos(\alpha t)$$

$$+ (c_1 - c_2) i \sin(\alpha t)$$

$$d_1 + d_2 = C$$

$$(d_1 - d_2) i = D$$

$$c_1 + c_2 = A$$

$$(c_1 - c_2) i = B$$

$$x = A\cos(\alpha t) + B\sin(\alpha t) + x_0 \qquad z = C\cos(\alpha t) + D\sin(\alpha t) + z_0$$

$$\frac{dx}{dt} = -\alpha A\sin(\alpha t) + \alpha B\cos(\alpha t) \qquad \frac{dz}{dt} = -\alpha C\sin(\alpha t) + \alpha D\cos(\alpha t)$$

$$\frac{d^2x}{dt^2} = -\alpha^2 A\cos(\alpha t) - \alpha^2 B\sin(\alpha t) \qquad \frac{d^2z}{dt^2} = -\alpha^2 C\cos(\alpha t) - \alpha^2 D\sin(\alpha t)$$

$$\frac{d^2x}{dt^2} = -\alpha \frac{dz}{dt} \qquad \frac{d^2z}{dt^2} = \alpha \frac{dx}{dt}$$

$$-\alpha^2(A\cos(\alpha t) + B\sin(\alpha t)) = -\alpha(-\alpha C\sin(\alpha t) + \alpha D\cos(\alpha t))$$

$$A\cos(\alpha t) + B\sin(\alpha t) = -C\sin(\alpha t) + D\cos(\alpha t)$$

$\underline{t = 0}$

$$A = D$$

$$t = \frac{\pi}{\alpha 2}$$

$$B = -C$$

$$\boxed{\begin{aligned} x &= A\cos(\alpha t) + B\sin(\alpha t) + x_0 \\ z &= -B\cos(\alpha t) + A\sin(\alpha t) + z_0 \end{aligned}}$$

$$z = -B\cos(\alpha t) + A\sin(\alpha t) + z_0$$

$$y = \frac{dy_0}{dt} t + y_0$$

At $t = 0$, $\frac{dz}{dt} = \frac{dz_0}{dt}$ & $\frac{dx}{dt} = 0$

$$\frac{dx}{dt}(t=0) = -\alpha A\sin(\alpha(0)) + \alpha B\cos(\alpha(0))$$

$$= \alpha B = 0$$

$$B = 0$$

$$\frac{dz}{dt}(t=0) = \alpha B\sin(\alpha(0)) + \alpha A\cos(\alpha(0)) = \frac{dz_0}{dt}$$

$$= \alpha A = \frac{dz_0}{dt}$$

$$A = \frac{1}{\alpha}\frac{dz_0}{dt}$$

$$x = \left(\frac{m}{qB_0}\frac{dz_0}{dt}\right)\cos\left(\frac{qB_0}{m}t\right) + x_0$$

$$y = \frac{dy_0}{dt}t + y_0$$

$$z = \left(\frac{m}{qB_0}\frac{dz_0}{dt}\right)\sin\left(\frac{qB_0}{m}t\right) + z_0$$