Physics 4(5)211 Fall 2024
Homework 02

Total Points: 37 points
All questions are graded for **reasoning** and correctness. Providing *only* the correct answer without clearly showing your work and/or explaining your reasoning is generally worth no more than 25% of the possible credit. Submitting solutions to problem sets is about growing your skills both in problem solving and in formal, scientific communication.

After each homework, you should review both your answers and the solutions to make sure that you understand all the questions and how to answer them correctly. Going forward, on each homework you will receive credit for reviewing your previous homework. Please review your homework submission and the solution set and reach out to me if you have any questions or concerns.

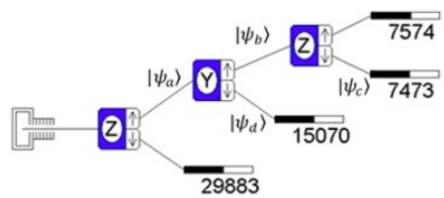
1. **[1.5 points]** Going forward, part of each homework will be submitting a correction from the previous week. Please select a problem from the prior homework for which you had the wrong answer and:

i. Identify the question number you are correcting
ii. State/copy your original wrong answer
iii. Explain where your original reasoning was incorrect, the correct reasoning for the problem, and how it leads to the right answer.

If you got all the answers correct, that's great! Please review your work anyway and discuss which problem you found the most interesting, enjoyable, and/or helpful and why.

2. **[1.5 points]** Select another problem from the prior homework for which you had the wrong answer and correct it as above. If you got all the answers correct (or have nothing left to correct) that's great! Please review your work anyway and discuss which problem you found the *least* interesting, enjoyable, and/or helpful and why.

3. **[4 points]** Consider the chained Stern-Gerlach setup shown at right. The thermal source creates a beam of spin-1/2 particles. Three students are discussing the states following some of the measurements:



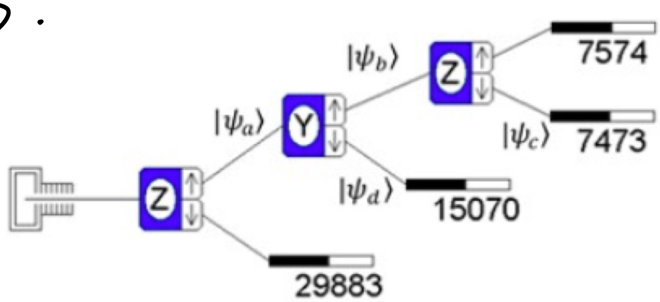
Student A: "The y-analyzer (Analyzer 2) changes its incoming particles from being all plus-z to an output $|\psi_b\rangle$, which is a mixture of half of the particles being in the plus-z state and half being in the minus-z state. That's why the farthest right z-analyzer (Analyzer 3) measures half up and half down."

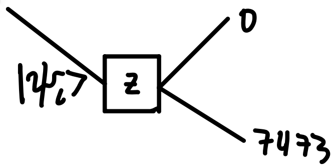
Student B: "No, that's not right. All the particles entering Analyzer 3 are in the same state: $|\psi_b\rangle$ is a specific superposition of up-z and down-z states so the particles are in both the up-z and down-z states at the same time."

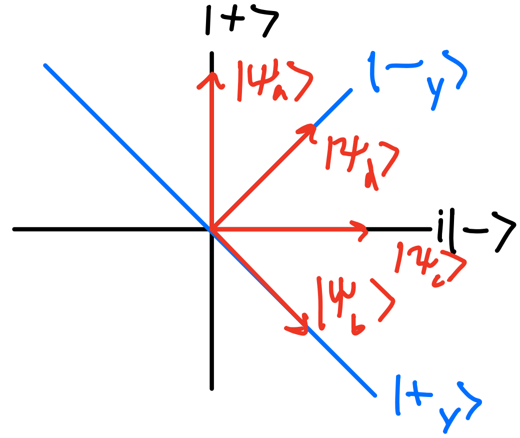
Student C: "There's no way to tell the difference between those arguments, both say that the last analyzer measures 50-50, so this discussion is more about philosophy than physics."

Describe a simple experiment (or experiments) whose results show that student B is correct. Explain clearly how the experimental results not only show that student B is correct, but that students A and C are *incorrect*. Sketch SG-experiment diagrams (like the one above) **and** Hilbert Space diagrams as part of your explanation.

3.







Student B is correct b/c the thing we are measuring is the state of the particles when they change directions. It has nothing to do w/ the analyzer itself, but the direction it measures. If we add another z-analyzer to our SG-experiment after the second z analyzer, it will measure the same state again and yield 0/100 rather than 50/50. There is no 'hidden variable' that determines what particles are in which orientation before the state is measured (unless it's known).

For questions 4-6, please use **Dirac notation** (bras and kets, not matrices).

=)

4. **[6 points total]** Consider the following vectors:

$$|\psi_1\rangle = \frac{\sqrt{3}}{2}|+\rangle - \frac{i}{2}|-\rangle \qquad and \qquad |\psi_2\rangle = \frac{1}{3}|+\rangle + \frac{2}{3}e^{-i\frac{3\pi}{4}}|-\rangle$$

a) **[1 point]** Could these two vectors be valid descriptions of physical states? Explain.

b) **[3 points]** For *each* state, find a normalized ket $|\phi_n\rangle$ that is orthogonal to it. Use the convention discussed in class, that the coefficient of the $|+\rangle$ ket is real and positive.

c) **[2 points]** Calculate the inner products $\langle\psi_1|\psi_2\rangle$ and $\langle\psi_2|\psi_1\rangle$. How are these numbers related? Is this consistent with the idea of an inner product being a "dot" product? Explain.

5. **[5 points total]** Consider the following vectors:

$$a{:}\,1\,|+\rangle - 2e^{i\frac{\pi}{4}}|-\rangle \qquad and \qquad b{:}\,1\,|+\rangle - ie^{-i\pi}|-\rangle$$

a) **[1 point]** Normalize each expression to create states $|\psi_a\rangle$ and $|\psi_b\rangle$. Follow the standard convention for the coefficients of kets.

b) **[1 point]** For **each** state, find the probability of measuring $S_z = -\hbar/2$.

c) **[1.5 points]** For **only** $|\psi_a\rangle$, find the probability of measuring $S_x = -\hbar/2$.

d) **[1.5 points]** For **only** $|\psi_b\rangle$, find the probability of measuring $S_y = -\hbar/2$.

6. **[3 points]** Consider a general state $|\psi\rangle$. Show that the probability of obtaining a particular spin measurement (in *any* direction) is unaffected by changing the state to $|\phi\rangle = e^{i\beta}|\psi\rangle$. Discuss the physical significance of this result, and how it is related to our standard convention for the coefficient of the $|+\rangle$ ket.

7. **[10 points total]** Matrix representations:

a) **[3 points]** Given:

$$\hat{S}_y|\pm\rangle_y = \pm\frac{\hbar}{2}|\pm\rangle_y \qquad (Experimental\ SG\ results)$$

$$|\pm\rangle_y \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm i\end{pmatrix} \qquad$$ ~~(derived in question 6)~~

derive the **matrix** representation of $\hat{S}_y$ (in the z-basis). As always, show your work clearly.

While we normally work in the z-basis, this is a convention - there is nothing truly special about the z direction. So instead, **let's work in the *y*-basis.**

b) **[1 point]** How do you represent the states $|\pm\rangle_y$ and the operator $\hat{S}_y$ in the y-basis (using matrix notation)? Explain.

c) **[1.5 points]** How do you represent the state $|+\rangle$ (meaning $|+\rangle_z$) in the y-basis? Explain.

d) **[2.5 points]** How do you represent the state $|+\rangle_x$ in the y-basis? Explain. *Hint:* Consider your answer to question 6 regarding overall phase.

e) **[2 points]** Calculate the bracket ${}_x\langle+|+\rangle$ using matrices in the y-basis. Show that this is the same result you would get using the z-basis.

4)

a. $|\psi_1\rangle = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{i}{2}\right)^2 = \frac{3}{4} + \frac{1}{4} = 1$ yes, normalized

$|\psi_2\rangle = \left(\frac{1}{3}\right)^2 + \left|\frac{2}{3}\right)^2 = \frac{1}{9} + \frac{4}{9} = \frac{\sqrt{5}}{3}$ no, not normalized

b. $|\psi_1\rangle = \begin{pmatrix} \sqrt{3}/2 \\ -i/2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

$|\psi_1\rangle_\perp = \begin{pmatrix} c \\ d \end{pmatrix}$ $\quad a^* \cdot c + b^* \cdot d = 0$ ← orthogonality

$\Rightarrow \frac{\sqrt{3}}{2} c + \frac{i}{2} d = 0$, $d = 1$

$c = \frac{-i}{\sqrt{3}}$

$\Rightarrow |\psi_1\rangle_\perp = \begin{pmatrix} -i/\sqrt{3} \\ 1 \end{pmatrix} = \sqrt{-i/\sqrt{3}^2 + 1^2} = \sqrt{1/3 + 1} = \frac{2}{\sqrt{3}}$

$\Rightarrow |\psi_1\rangle_\perp = \frac{\sqrt{3}}{2} \cdot \begin{pmatrix} -i/\sqrt{3} \\ 1 \end{pmatrix} = \begin{pmatrix} -i/2 \\ \sqrt{3}/2 \end{pmatrix}$

$|\psi_2\rangle_{norm} = \frac{3}{\sqrt{5}} \cdot \left(\frac{1}{3} |+\rangle + \frac{2}{3} e^{-i\frac{3\pi}{4}} |-\rangle \right) = \frac{1}{\sqrt{5}} |+\rangle + \frac{2}{\sqrt{5}} e^{\cdots} |-\rangle$

$|\psi_2\rangle = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$ $\quad \frac{1}{\sqrt{5}} \cdot c + \frac{2}{\sqrt{5}} d = 0$, $d = 1$

$c = -2$

$|\psi_2\rangle_\perp = \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \sqrt{1+4} = \sqrt{5}$

$\Rightarrow |\psi_2\rangle_\perp = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$ including $e \ldots$ $\rightarrow = \begin{pmatrix} -2/\sqrt{5} \\ \frac{1}{\sqrt{5}} e^{-i\frac{3\pi}{4}} \end{pmatrix}$

c. $\langle \psi_1 | \psi_2 \rangle = \left(\frac{\sqrt{3}}{2}|+\rangle - \frac{i}{2}|-\rangle\right)\left(\frac{1}{3}|+\rangle + \frac{2}{3}e^{\cdots}|-\rangle\right)$

mixed brakets will $= 0$

$$\Rightarrow \frac{\sqrt{3}}{6}\langle + | + \rangle - \frac{2ie^{\cdots}}{6}\langle - | - \rangle = \frac{\sqrt{3} - 2ie^{\cdots}}{6}$$

scalar multiplication is commutative $\therefore$

$\langle \psi_2 | \psi_1 \rangle = \langle \psi_1 | \psi_2 \rangle$

consistent w/ dot product

5)

a. $|\psi_1\rangle = 1 \cdot |+\rangle - 2e^{i\pi/4}|-\rangle = \sqrt{1^2 + -2^2} = \sqrt{5}$

$$\Rightarrow |\hat{\psi}_1\rangle = \frac{1}{\sqrt{5}}|+\rangle - \frac{2}{\sqrt{5}}e^{i\pi/4}|-\rangle$$

$|\psi_2\rangle = 1 \cdot |+\rangle - ie^{-i\pi}|-\rangle = \sqrt{1^2 + -i^2} = \sqrt{2}$

$$|\hat{\psi}_2\rangle = \frac{1}{\sqrt{2}}|+\rangle - \frac{i}{\sqrt{2}}e^{-i\pi}|-\rangle$$

b. $|\psi_1\rangle$: $P_+ = \left(\frac{1}{\sqrt{5}}\right)^2 = \frac{1}{5}$, $P_- = \left(-\frac{2}{5}\right)^2 = \frac{4}{5}$

$|\psi_2\rangle$: $P_+ = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$, $P_- = \left(-\frac{i}{2}\right)^2 = \frac{1}{2}$

c. $\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \hat{S}_x|\hat{\psi}_1\rangle = \frac{-2}{\sqrt{5}}e^{i\pi/4}|+\rangle + \frac{1}{\sqrt{5}}|-\rangle$

$P_- = \left(\frac{1}{\sqrt{5}}\right)^2 = \frac{1}{5}$

d. $\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \Rightarrow \hat{S}_y|\hat{\psi}_1\rangle = \frac{i}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}e^{-i\pi}|-\rangle$, $P_- = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

6) $$P = \left|{}_x\langle -|\psi\rangle\right|^2 = \left|\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}\right|^2 = \frac{1}{\sqrt{2}^2}a^*a - \frac{1}{\sqrt{2}^2}b^*b$$

$$a\cdot e^{iB} = A, \quad b\cdot e^{iB} = B$$

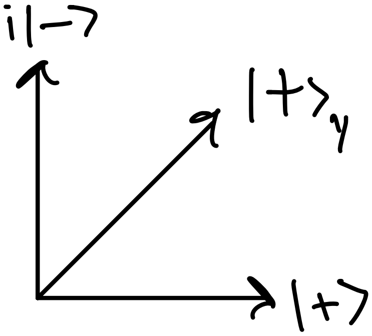
$$P = \left|{}_x\langle -|\phi\right|^2 = \left|\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}A\\ B\end{pmatrix}\right|^2 = \frac{1}{\sqrt{2}^2}A^*A - \frac{1}{\sqrt{2}^2}B^*B$$

$$e^{iB}\times e^{-iB} = 1$$

Taking the magnitude squared of an unknown component involves taking the complex conjugate of a/b. This makes it so anything in the $|+\rangle$ will <u>always</u> be real + positive.

7)

a.



$$\hat{S}_y = \begin{pmatrix}a & b\\ c & d\end{pmatrix}_z \qquad |+\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$$

$$|-\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}$$

$$\hat{S}_y|+\rangle_y = \frac{\hbar}{2}|+\rangle_y$$

$$\Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}a & b\\ c & d\end{pmatrix}_z\begin{pmatrix}1\\ i\end{pmatrix} = \frac{\hbar}{2}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}a+bi\\ c+di\end{pmatrix} = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$$

$$a+bi = \frac{\hbar}{2}$$
$$c+di = \frac{\hbar}{2}i$$

$$\hat{S}_y|-\rangle_y = \frac{\hbar}{2}|-\rangle_y$$

$$\Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}a & b\\ c & d\end{pmatrix}_z\begin{pmatrix}1\\ -i\end{pmatrix} = \frac{-\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}a-bi\\ c-di\end{pmatrix} = \frac{-\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}$$

$$a-bi = \frac{-\hbar}{2}$$
$$c-di = \frac{\hbar}{2}i$$

$$\begin{array}{r} a+bi = \frac{\hbar}{2} \\ + \quad a-bi = -\frac{\hbar}{2} \\ \hline 2a = 0 \end{array}
\qquad
\begin{array}{r} c+di = \frac{\hbar}{2}i \\ - \quad c-di = \frac{\hbar}{2}i \\ \hline -2di = 0 \\ d = 0 \end{array}$$

$$0 + bi = \frac{\hbar}{2} \qquad c + 0 = \frac{\hbar}{2}i$$

$$b = \frac{\hbar}{2i} = -\frac{\hbar}{2}i \qquad c = \frac{\hbar}{2}i$$

$$\Rightarrow \hat{S}_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\hbar/2i \\ \hbar/2i & 0 \end{pmatrix}_z$$

b. There is nothing special about z, so writing y in the y basis would look just like z in the z.

$$\Rightarrow \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

c. $$|\psi\rangle_y = \begin{pmatrix} {}_y\langle +|\psi\rangle \\ {}_y\langle -|\psi\rangle \end{pmatrix}_y \qquad {}_y\langle +| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}$$

8. **[6 points total]** Suppose we act the $\hat{S}_z$ operator on the general state $|\psi\rangle = a\,|+\rangle + b\,|-\rangle$.

a) **[1 point]** What kind of *object* (bra, ket, operator, number, etc.) is $\hat{S}_z\,|\psi\rangle$?

b) **[1.5 points]** Determine an expression for $\hat{S}_z\,|\psi\rangle$ using *matrix notation*. Is your result an *eigenequation*?

c) **[2 points]** Consider the following discussion between three students:

Student A: "I think this formula tells you the experimental results of measuring the z-component of spin on some general input state."

Student B: "I disagree. I think it tells you what the resulting quantum **state** is after you measure the z-component of spin. That's different from an experimental result, which would be a number (with dimensions), not a state."

Student C: "I think you're both over-interpreting this expression. It's a mathematical equation, it doesn't mean anything physically."

With which student, if any, do you agree? Discuss *both* what you agree with and what you don't.

d) **[1.5 points]** Based on your answers above, can you explain why I **strongly** disagree with the claim that $\hat{O}|\psi\rangle$ means "a measurement of $\hat{O}$ on the state $|\psi\rangle$"?

a) $\hat{S}_z|\psi\rangle = |\beta\rangle$, a ket

b) $|\psi\rangle \doteq \begin{pmatrix} a \\ b \end{pmatrix}$, $\hat{S}_z = \frac{\hbar}{2}|+\rangle - \frac{\hbar}{2}|-\rangle \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\therefore \quad \hat{S}_z|\psi\rangle = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$

No, either both or neither vector component has to be changed in the same way. Can't be flipped either. Since B is negative & A is not, this violates the rule.

c) I agree w/ student B because applying an operator on a state gives us another state, rather than a specific eigenvalue. If our operator was S_z that isolated either up or down spin, then we could get an experimental result.

d) The expression represents how the operator changes the state, not the outcome/measurement, which would be a number.