Lect. 1 ex. 1.18 $\sqrt{9.01} - 3 = 3\sqrt{1+\frac{0.01}{9}} - 3 = 3\left(\sqrt{1+\frac{0.01}{9}} - 1\right) \approx 3\left(1 + \frac{0.01}{2\cdot 9} - 1\right) = \frac{0.01}{6}$

Taylor theorem: $\sqrt{1+x} = f(x)$ $\quad f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$

$f(0) = 1$

$f'(x) = (1+x)^{\frac{1}{2}} = \frac{1}{2}(1+x)^{-\frac{1}{2}}$ $\quad f'(0) = \frac{1}{2}$

$\sqrt{1+x} \approx 1 + \frac{x}{2}$

GET RID OF SUBTRACTING CLOSE NUMBER!

better:

$$\frac{(\sqrt{9.01} - 3)(\sqrt{9.01} + 3)}{\sqrt{9.01} + 3} = \frac{9.01 - 9}{\sqrt{9.01} + 3} = \frac{0.01}{\sqrt{9.01} + 3}$$

$\sqrt{9.0000001} - 3 \approx \frac{0.0000001}{6}$ Taylor

$\sqrt{9.0000001} - 3 = 1.766666671\ldots \cdot 10^{-8}$

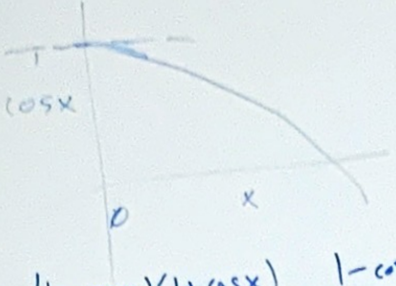
Scientific notation

$1.0 * 10^{-6}$ computer $\boxed{1.0e-6}$

example: division by a small number

compute $\frac{1-\cos x}{\sin x}$ for small x

$$\lim_{x\to 0} \frac{1}{x}\,\frac{1-\cos x}{\sin x} = \frac{1}{2}$$



$$\left.\begin{aligned}\cos x &= 1-\frac{x^2}{2}+\dots \\ \sin x &= x+\dots\end{aligned}\right\}\text{Taylor}$$

$$\frac{1}{x}\,\frac{x-\left(x-\frac{x^2}{2}+\dots\right)}{x+\dots} \to \frac{1}{2}$$

better: $\frac{1-\cos x}{\sin x} = \frac{(1-\cos x)(1+\cos x)}{\sin x(1+\cos x)} = \frac{1-\cos^2 x}{\sin x(1+\cos x)}$

$= \frac{\sin^2 x}{\cancel{\sin x}(1+\cos x)} = \frac{\sin x}{1+\cos x}$ sound for $x \approx 0$

$$\frac{1}{x}\,\frac{1-\cos x}{\sin x} = \frac{\sin x}{x}\,\frac{1}{1+\cos x}$$

computer add: $fl(x + y)$

↑ rounding

exact

approximation

(absolute) relative error of x is $\frac{|x - x^*|}{|x|}$

book: first $\frac{eps}{2}$ instead of eps

Ch. 1.3 remark 2

in computer arithmetic, $x^* = fl(x)$

relative error of floating point representation of x is $\frac{|x - fl(x)|}{|x|}$

$|x - fl(x)| \le |x| \frac{eps}{2}$

eps is smallest $x > 0$
$1.0 + x > 1.0$

relative error of sum: given inputs x, y add $fl(x+y)$

assume $x = fl(x)$ $y = fl(y)$

relative error $\frac{|x + y - fl(x+y)|}{|x+y|}$

magnification of error when $x + y \approx 0$

x, y from outside
convert to $fl(x), fl(y)$

$$\frac{|x + y - fl(fl(x) + fl(y))|}{|x+y|}$$

evaluate polynomial

potentially large

example $P(x) = 2x^4 + 3x^3 - 3x^2 + 5x - 1$

x^n expensive

$|x| > 1$ x^n large

$|x| < 1$ x^n small

nested evaluation:

$$P(x) = x(\underbrace{2x^3 + 3x^2 - 3x + 5}_{\text{have } x}) - 1$$

$$= x(x(2x^2 + 3x - 3) + 5) - 1$$

$$= x(x(\underbrace{x(\underbrace{2x+3}_{t}) - 3}_{t}) + 5) - 1$$

program:

$t = 2x + 3$

$t = xt - 3$

$t = xt + 5$

$t = xt - 1$

4 multiply-add operations