Homework 1

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Mathematical Methods: Homework 1

4.3) $\frac{1}{i} + \frac{1}{o} = \frac{1}{f}$

$i = 15 \quad o = 10$

$f = \text{const.}$

$i + di = 15 + di$

$o + do = 10.1$

$do = 0.1$

$\lambda = \text{some parameter}$

$$\frac{d}{d\lambda}\left(i^{-1} + o^{-1}\right) = \frac{d}{d\lambda}\left(f^{-1}\right)$$

$$-i^{-2}\frac{di}{d\lambda} - o^{-2}\frac{do}{d\lambda} = 0$$

$$i^{-2}di + o^{-2}do = 0$$

$$di = -\left(\frac{o}{i}\right)^2 do$$

$$= -\left(\frac{15}{10}\right)^2 (0.1)$$

$$= -\frac{9}{4}\left(\frac{1}{10}\right)$$

$$di = -9/40 \approx -0.23$$

$$\boxed{\text{When } o = 10.1, \; i = 15 - 0.23 = 14.77}$$

7.23) <u>Wave Equation</u>

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

$$u = f(x - ct) + g(x + ct)$$

$$\lambda = x - ct \qquad \rho = x + ct$$

$$\frac{\partial}{\partial x} = \frac{\partial \lambda}{\partial x}\frac{\partial}{\partial \lambda} + \frac{\partial \rho}{\partial x}\frac{\partial}{\partial \rho}$$

$$\frac{\partial}{\partial x} = \frac{\partial \lambda}{\partial x}\frac{\partial}{\partial \lambda} + \frac{\partial \rho}{\partial x}\frac{\partial}{\partial \rho}$$

$$\frac{\partial \lambda}{\partial x} = 1 \qquad \frac{\partial \rho}{\partial x} = 1$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \lambda} + \frac{\partial}{\partial \rho} \Rightarrow \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \lambda^2} + 2\frac{\partial^2}{\partial \lambda \partial \rho} + \frac{\partial^2}{\partial \rho^2}$$

$$\frac{\partial}{\partial t} = \frac{\partial \lambda}{\partial t}\frac{\partial}{\partial \lambda} + \frac{\partial \rho}{\partial t}\frac{\partial}{\partial \rho}$$

$$\frac{\partial \lambda}{\partial t} = -c \qquad \frac{\partial \rho}{\partial t} = c$$

$$\frac{\partial}{\partial t} = -c\frac{\partial}{\partial \lambda} + c\frac{\partial}{\partial \rho} \Rightarrow \frac{\partial^2}{\partial t^2} = c^2\frac{\partial^2}{\partial \lambda^2} - 2c^2\frac{\partial^2}{\partial \lambda \partial \rho} + c^2\frac{\partial^2}{\partial \rho^2}$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \lambda^2} + 2\frac{\partial^2}{\partial \lambda \partial \rho} + \frac{\partial^2}{\partial \rho^2} \qquad \frac{\partial^2}{\partial t^2} = c^2\left(\frac{\partial^2}{\partial \lambda^2} - 2\frac{\partial^2}{\partial \lambda \partial \rho} + \frac{\partial^2}{\partial \rho^2}\right)$$

$$\frac{\partial^2}{\partial x^2} - 2\frac{\partial^2}{\partial \lambda \partial \rho} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + 2\frac{\partial^2}{\partial \lambda \partial \rho}$$

$$\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = 4\frac{\partial^2}{\partial \lambda \partial \rho}$$

$$\frac{\partial^2 u}{\partial \lambda \partial \rho} = \frac{\partial}{\partial \lambda}\left(\frac{\partial}{\partial \rho}\left(f(\lambda) + g(\rho)\right)\right) = \frac{\partial}{\partial \lambda}\left(0 + \frac{\partial g}{\partial \rho}\cancelto{1}{\frac{\partial \rho}{\partial \rho}}\right)$$

$$\Rightarrow \frac{\partial}{\partial \lambda}\left(\frac{\partial g}{\partial \rho}\right) = 0$$

$$\frac{\partial^2 u}{\partial \lambda \partial \rho} = \frac{\partial^2 u}{\partial \rho \partial \lambda} = 0$$

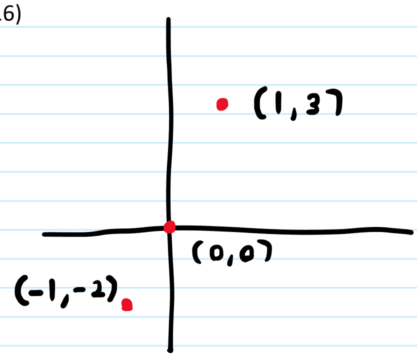
So,

$$\boxed{\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2}{\partial \lambda \partial \rho} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}}$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2}{\partial\lambda\partial\rho} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

when $u = f(\lambda) + g(\rho) = f(x-ct) + g(x+ct)$

8.16)



$y = mx + b$

$(-1,-2) :: -2 = -m + b$

$(0,0) :: 0 = b$

$(1,3) :: 3 = m + b$

$$S(m,b) = (-2+m-b)^2 + b^2 + (3-m-b)^2$$

$$\frac{\partial S}{\partial m} = 2(-2+m-b) - 2(3-m-b)$$

$$= -4 + 2m - 2b - 6 + 2m + 2b$$

$$= -10 + 4m = 0$$

$$m = \frac{5}{2}$$

$$\frac{\partial S}{\partial b} = -2(-2+m-b) + 2b - 2(3-m-b)$$

$$= 4 - 2m + 2b + 2b - 6 + 2m + 2b$$

$$= -2 + 6b = 0$$

$$b = 1/3$$

$$\boxed{y = \tfrac{5}{2}x + \tfrac{1}{3}}$$

– another way –

$$\underbrace{\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ b \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}}_{b}$$

$$Ax = b \Rightarrow x = (A^TA)^{-1}A^Tb$$

$$A^TA = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$(A^TA)^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$$

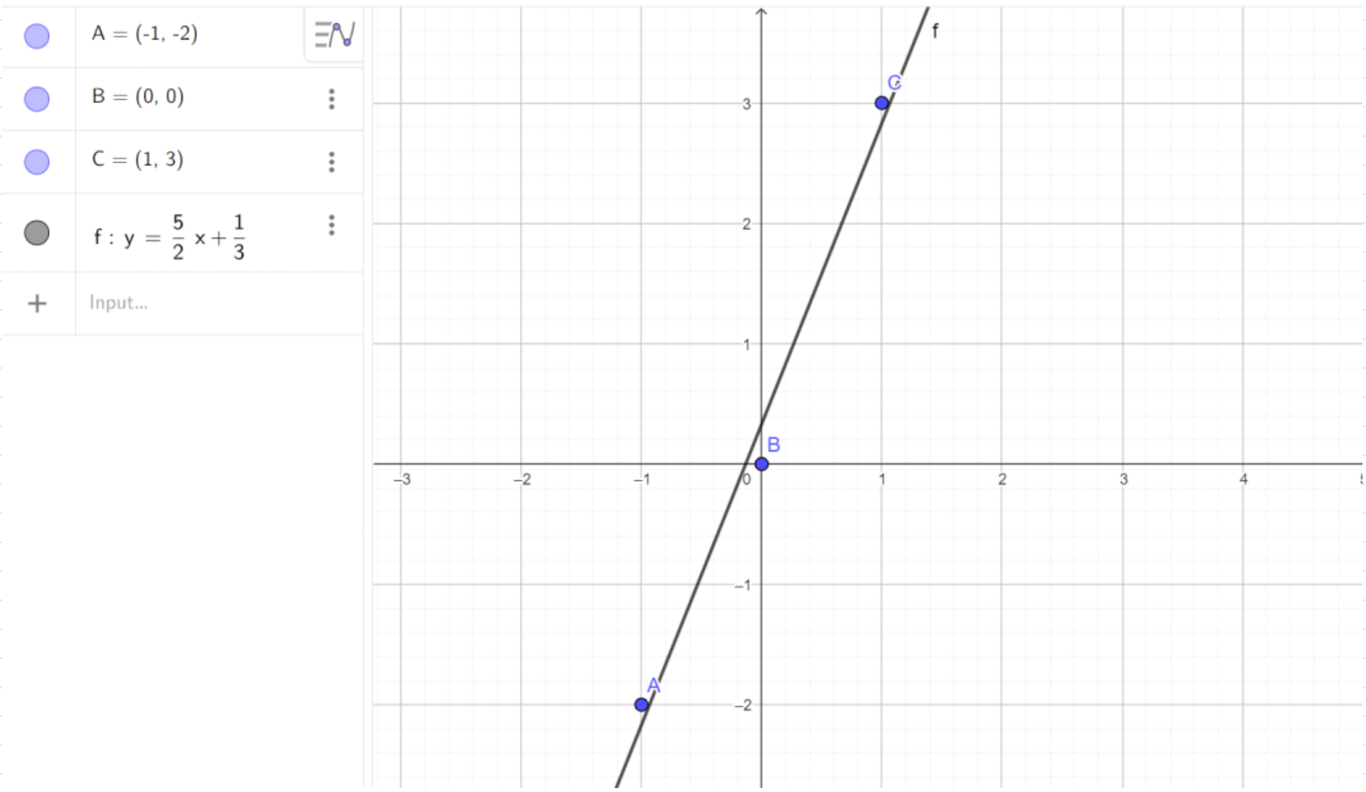
$$(A^TA)^{-1}A^T = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$(A^TA)^{-1}A^T = \begin{bmatrix} -1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 1/3 \end{bmatrix}$$

$m = 5/2$

$b = 1/3$

$$y = \tfrac{5}{2}x + \tfrac{1}{3}$$



11.6)

$$x^2 \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 5y = 0 \quad > \quad \frac{d^2y}{dz^2},\ \frac{dy}{dz},\ y$$

$x = e^z$

$x^2 = e^{2z}$

$dx = e^z dz$

$\frac{dz}{dx} = e^{-z}$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = e^{-z}\frac{dy}{dz}$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dz}\left(\frac{dy}{dx}\right)\frac{dz}{dx}$$

$$= \frac{d}{dz}\left(e^{-z}\frac{dy}{dz}\right)e^{-z}$$

$$= \left(-e^{-z}\frac{dy}{dz} + e^{-z}\frac{d^2y}{dz^2}\right)e^{-z}$$

$$\frac{d^2y}{dx^2} = e^{-2z}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

$$\Rightarrow e^{2z}\left(e^{-2z}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)\right) + 2e^z\left(e^{-z}\frac{dy}{dz}\right) - 5y = 0$$

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} + 2\frac{dy}{dz} - 5y = \boxed{0 = \frac{d^2y}{dz^2} + \frac{dy}{dz} - 5y}$$

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} + 2\frac{dy}{dz} - 5y = \boxed{0 = \frac{d^2y}{dz^2} + \frac{dy}{dz} - 5y}$$

11.11) $L(q,\dot{q})$ $\ddagger$ $dL = \dot{p}\,dq + p\,d\dot{q}$

$H(p,q)$ so that $dH = \dot{q}\,dp - \dot{p}\,dq$

$$dL - d(p\dot{q}) = \dot{p}\,dq + p\,d\dot{q} - \dot{q}\,dp - p\,d\dot{q}$$

$$d(L - p\dot{q}) = \dot{p}\,dq - \dot{q}\,dp = -dH$$

$$dH = \dot{q}\,dp - \dot{p}\,dq = d(p\dot{q} - L)$$

$$H = p\dot{q} - L$$

13.29) $t = 64$, $T = 15.73$

$$T(t) = 100^\circ\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{8/\sqrt{t}} e^{-\tau^2}\,d\tau\right)$$

$$\frac{dT(t)}{d\tau} = \frac{d}{d\tau}\left(100^\circ\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{8/\sqrt{t}} e^{-\tau^2}\,d\tau\right)\right)$$

$$= -2\,\frac{100}{\sqrt{\pi}}\,e^{-\left(\frac{8}{\sqrt{t}}\right)^2} \cdot -\frac{4}{t^{3/2}}\Bigg|_0^{64}$$

$$= \frac{8(100^\circ)}{\sqrt{\pi}}\,e^{-1}\left(\frac{4}{8^3}\right) = \frac{100^\circ}{8^2\sqrt{\pi}\,e}$$

$$\frac{dT}{dt} = \frac{100^\circ}{8^2\sqrt{\pi}\,e} \qquad , dT = 17^\circ - 15.73$$

$$\frac{dT}{dt} = \frac{100^\circ}{8^2\sqrt{\pi}\, e}$$

$$dT = 17^\circ - 15.73 = 1.27^\circ$$

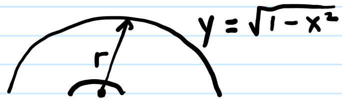
$$(1.27^\circ)\frac{8^2\sqrt{\pi}\, e}{100^\circ} = \underline{dt = 3.9}$$

$$dt = t - 64 = 3.9 \Rightarrow t = 3.9 + 64 = 67.9$$

$$\boxed{\text{When } T = 15^\circ,\ t = 67.9}$$

4.14)

$$I = \int_0^1 dx \int_0^{\sqrt{1-x^2}} e^{-x^2-y^2}\, dy$$



$\theta: 0 \to \pi/2$

$r: 0 \to 1$

$$I = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=1} e^{-r^2} r\, dr\, d\theta$$

$u = -r^2$

$du = -2r\, dr$

$-\frac{1}{2} du = r\, dr$

$r=1,\ u=-1$

$r=0,\ u=0$

$$I = -\frac{1}{2}\int_{\theta=0}^{\theta=\pi/2}\int_{u=0}^{u=-1} e^u\, du\, d\theta = -\frac{1}{2}\int_{\theta=0}^{\theta=\pi/2} e^u \Big|_{u=0}^{u=-1} d\theta = -\frac{1}{2} e^u \theta \Big|_{u=0}^{u=-1}\Big|_{\theta=0}^{\theta=\pi/2}$$

$$I = -\frac{1}{2}(e^{-1} - 1)(\pi/2 - 0) = -\frac{\pi}{4}(e^{-1} - 1) = \frac{\pi}{4}(1 - e^{-1})$$

$$I = \frac{\pi}{4}\left(1 - \frac{1}{e}\right)$$

5.3)

Area of $x^2+y^2=z$ inside the cylinder $x^2+y^2=9$

Area of $x^2+y^2=z$ inside the cylinder $x^2+y^2=9$

$$\iint dA = \iint \sec\gamma \, dx \, dy$$

$$\sec\gamma = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

$$\frac{\partial z}{\partial x} = 2x \qquad \frac{\partial z}{\partial y} = 2y$$

$$\sec\gamma = \sqrt{4x + 4y + 1} = \sqrt{4(x^2+y^2)+1}$$

$$r^2 = x^2 + y^2$$

$$\sec\gamma = \sqrt{1 + 4r^2}$$

$$A = \int_0^{2\pi}\int_0^{r} \sqrt{1+4r^2}\, dr\,(r\,d\theta)$$

$$x^2+y^2 = 3^2 \Rightarrow r^2 = 3^2 \Rightarrow r = 3$$

$$A = \int_0^{2\pi}\int_0^{3} \sqrt{1+4r^2}\, r\,dr\,d\theta \qquad \left| \begin{array}{l} u = 1+4r^2 \\ du = 8r\,dr \\ \frac{1}{8}du = r\,dr \end{array} \right. \qquad \begin{array}{l} r=0,\ u=1 \\ r=3,\ u=37 \end{array}$$

$$= \frac{1}{8}\int_0^{2\pi}\int_{u=1}^{u=37} \sqrt{u}\,du\,d\theta = \frac{1}{8}\,\frac{2}{3}\int_0^{2\pi} u^{3/2}\Big|_{u=1}^{u=37} d\theta$$

$$= \frac{1}{12}\left(u^{3/2}\Big|_{u=1}^{u=37}\right)\left(\theta\Big|_0^{2\pi}\right) = \frac{1}{12}\left(37^{3/2} - 1\right)(2\pi)$$

$$\boxed{A = \frac{\pi}{6}\left(37^{3/2} - 1\right)}$$