# Activity 3 - James Amidei

**Part I: Quantum Mice**
Consider a quantum object, let's call it a *quantum mouse*, with several physical properties we can measure ("observables"). We'll consider two specific properties, *eye size* (specifically pupil-size) and *mood*. Since these are observable properties, they must *each* be associated with a Hermitian operator. Let's call them $\hat{S}$ (eye size) and $\hat{M}$ (mood) respectively.

**1.** When we measure the eye size of any quantum mouse, we find **only** two values: *tiny-eyes* with 1 *mm* pupils and *wide-eyes* with 2 *mm* pupils. These are quantum states, so let's define kets to label these two eye size states: *tiny-eyes* $= |*\rangle$ and *wide-eyes* $= |\odot\rangle$. Being either tiny-eyed or wide-eyed is totally *normal.* In fact, let's assume it's *ortho*normal (and complete).

a) Write two *eigen-equations* relating the operator $\hat{S}$ to its eigenkets and eigenvalues. Explain your reasoning.

$$\hat{S}\,|*\rangle = (1\,mm)\,|*\rangle$$

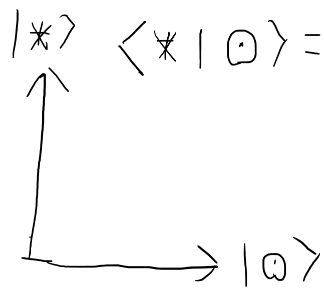
$$\hat{S}\,|\odot\rangle = (2\,mm)\,|\odot\rangle$$

Both the operator and the state vector are in the same basis, which allows us to just pull out the state's associated eigenvalue.

b) What can you say about the numerical value of $\langle * | \odot \rangle$? How do you know?

Both the tiny-eyes state vector and wide-eyes state vector form an orthonormal eye-size basis. Because they are orthonormal, their inner product will be 0.

$$\langle * | \odot \rangle = 0$$




**2.** To measure the mood of a quantum mouse, we look at its expression. When we do, we only *ever* observe either a smile (mood $= +1$) or a frown (mood $= -1$). Being happy or sad is also orthonormal and complete, with associated kets $|\text{☺}\rangle$ and $|\text{☹}\rangle$.

a) Write the two eigen-equations for the mood operator $\hat{M}$. Briefly explain.

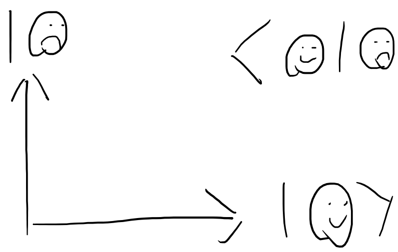
$$\hat{M}\,|\text{☺}\rangle = +1\,|\text{☺}\rangle$$

$$\hat{M}\,|\text{☹}\rangle = -1\,|\text{☹}\rangle$$

Both the operator and the state vector are in the same basis, which allows us to just pull out the state's associated eigenvalue.

b) What can you say about the numerical value of $\langle \text{☺} | \text{☹} \rangle$? How do you know?

Both the happy state vector and sad state vector form an orthonormal mood basis.
Because they are orthonormal, their inner product will be 0.




$\langle ☺ | ☹ \rangle = 0$

c) What, if anything, can you say about the numerical value of ⟨∗|☺⟩? Explain.

In order to say anything about the numerical value of the inner product of the happy state vector and the tiny eyes state vector, we would need to know how to express either on in terms of the other's basis. This is something we can only find through experimentation, so as of now, we can't say anything about the result of this inner product.

**Part II: The Mood Basis**

After many (**many**) measurements we find that $|⊙\rangle = \frac{1}{\sqrt{5}}|☺\rangle + \frac{2}{\sqrt{5}}|☹\rangle$.

**1.** Describe the experiment(s) that were conducted on our quantum mice to determine this. Make sure to discuss which mice we experimented on, what we measured, and what the outcomes were.

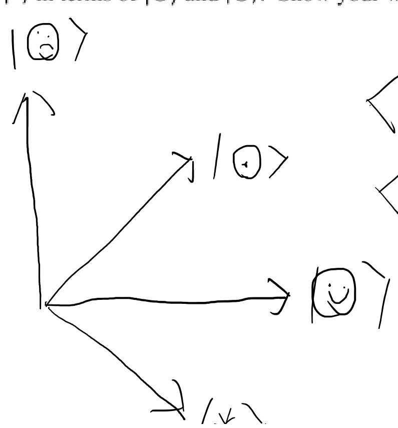
After measuring the mood of many wide-eyed mice, we found that 1/5 == 20% of them were happy, wile 4/5 == 80% of them were sad.

**2.** The expression above suggests that wide-eyed mice are rather *stressed.*

a) Briefly explain why I might say that. Does this statement apply to *individual mice, many mice* (a mischief!), or *both*? Explain.

It seems reasonable to say that the wide-eyed mice are stressed because a sizeable majority the wide-eyed mice whose emotional states were measured were found to be sad. This doesn't completely preclude the possibility of a single mouse being found in the happy state, this just speaks to the distribution of emotional states found across the group. A single mouse could be measured to be in either emotional state, but just that they are more likely to be sad than to be happy.

b) Using orthonormality and completeness, write the tiny-eyed eigenstate in the mood basis. That is, write |∗⟩ in terms of |☺⟩ and |☹⟩. Show your work and explain.

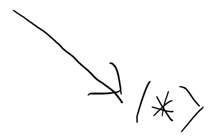



$$\langle ⊙ | * \rangle = 0 = \frac{1}{\sqrt{5}} a + \frac{2}{\sqrt{5}} b$$

$$\langle * | * \rangle = 1 = a^* a + b^* b$$

$$= |a|^2 + |b|^2$$

We can just swap the components from the wide-eye state expressed in the mood basis and change the sign



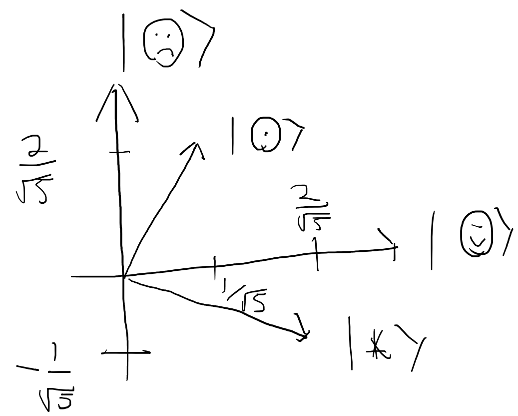

We can just swap the components from the wide-eye state expressed in the mood basis and change the sign to one.

$$|*\rangle = \frac{2}{\sqrt{5}}|☺\rangle - \frac{1}{\sqrt{5}}|☹\rangle$$

$$\langle \odot | * \rangle = \frac{1}{\sqrt{5}}\left(\frac{2}{\sqrt{5}}\right) + \frac{2}{\sqrt{5}}\left(-\frac{1}{\sqrt{5}}\right) = 0$$

$$\langle * | * \rangle = \frac{4}{5} + \frac{1}{5} = 1$$

**3.** Make a sketch representing the Hilbert Space of the "Mood Basis."




a) Indicate the mouse states $|\odot\rangle$ and $|*\rangle$ on the plot above. Describe how you could have determined an expression for the state $|*\rangle$ from this plot. Be explicit about any additional information/conventions you would use, and how this relates to your work in question II.2.b.

Since we know that the wide-eye and tiny-eye states have to be orthogonal to one another and also be normalized, we could find the tiny-eye state vector in the mood basis by taking the wide-eye state vector in the mood basis and rotating it 90-degrees in some direction (either clockwise or counter-clockwise). If we rotate the wide-eye vector 90-degree clockwise, we end up swapping which constant goes with the two basis vectors and flipping the sign on the sad component.

**4.** Suppose I give you a quantum mouse. When you measure its eye size ($\hat{S}$) you find a value of 1 $mm$.

a) What is the state of the mouse now? Is there any ambiguity about this state? Explain.

The state of the mouse is in the tiny-eyes state, which corresponds to one of the eigenstates in the S/eye-size basis. Because this is an eigenstate of the basis, there is no ambiguity in your measurement.

b) If you measure $\hat{S}$ again, what results might you find? With what probabilities? After this second measurement what state is the mouse in?

If you were to measure the eye-size again, you'd have 100% certainty of measuring the mouse to be in the tiny-eyed state. This is analogous to preparing a beam of articles in the "up-z" direction, then feeding that beam through another z-analyzer. You will measure 100% of the particles to be in the "up-z".

c) You now measure the mood of the mouse. What is the probability that you find this mouse to be unhappy? Show your work and explain.

$$|*\rangle = \frac{2}{\sqrt{5}}|\text{☺}\rangle - \frac{1}{\sqrt{5}}|\text{☹}\rangle$$

$$P_s = |\langle \text{☹}|*\rangle|^2 = \frac{1}{5} \Rightarrow 20\%$$

Given a tiny-eyed mouse, you will have a probability of 20% that it will be in the sad state after measuring its mood. This follows from the definition of the tiny-eyed state in the mood basis we found above.

**5.** I said above that "wide-eyed mice are stressed." In the same spirit, how might you describe small-eyed mice? Does this apply to individual mice, a mischief, or both? Explain.

$$|\langle \text{☺}|*\rangle|^2 = \frac{2^2}{\sqrt{5}^2} = \frac{4}{5} \Rightarrow 80\%$$

In the same way that we said the wide-eyed mice seemed stressed, the tiny-eyed mice can be seen as largely being happy. There is still a possibility that we will find a tiny-eyed mouse that is sad, so the statement that "tiny-eyed mice seem to be happy" does not describe the emotional state of a specific mouse, but it does describe the general trend of the mischief whose emotional states have been measured.

**6.** Consider a new mouse. You measure $\hat{M}$ and find a value of $-1$.

a) What is the state of the mouse now? Is there any ambiguity about this state? Explain.

$$M|\text{☹}\rangle = -1|\text{☹}\rangle$$

The state of the mouse is in the sad-eyes state, which corresponds to one of the eigenstates in the M/mood basis. Because this is an eigenstate of the basis, there is no ambiguity in your measurement.

b) What is the probability that you will measure this mouse to be *tiny-eyed*? Show your work and explain.

$$|\langle *|\text{☹}\rangle|^2 = \frac{1}{5} \Rightarrow 20\%$$

Given the input of a mouse in the sad mood state, we will have 1 20% probability of it being tiny-eyed, following from the definition of the tiny-eyed state in the mood basis we found above.

c) Say that you *do* measure this (originally unhappy) mouse to be tiny-eyed. If you measure the mood of this mouse again, what will you find? Is this what you would expect for a "classical" mouse? Why?

For a quantum mouse, once we measure its eye-size state, we destroy any information we previously had about its mood/emotional-state. This means if we were to measure its mood-state and find it to be sad, measure its eye-size state and find it to be tiny-eyed, and hen measure its mood state again, it would have a 80% probability of being happy or a 20% probability of being sad after this second measurement.

This is different than a classical mouse in which we'd expect its emotional state to be consistent through the experiment and not change based on which measurements were made before hand, i.e. a sad mouse with tiny eyes would remain a sad mouse with tiny eyes.

**Part III: Matrix Representations in the mood basis**

**1.** Represent each of the quantum mouse states ( |☺⟩, |☹⟩, |∗⟩, |⊙⟩ ) as *matrices* in the mood basis. Briefly explain.

$$|☺\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |*\rangle = \begin{pmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{pmatrix}$$

$$|☹\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |\odot\rangle = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$$

**2.** Use the mood eigen-equations (I.2.a.) to determine a matrix representation of the mood operator (in the mood basis). *Hint:* Written generally, the Mood operator is $\hat{M} \doteq \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Since we are in the mood basis, we could say that M is diagonal with the entries along its main diagonal being the eigen values of the different mood measurements.

$$\hat{M} = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}$$

**3.** **Challenge** (NOT required for full credit on this activity, but this will be on the HW!): Use the eye-size eigen-equations (I.1.a.) to determine the matrix representation of the $\hat{S}$ operator **in the mood basis**.

$$\hat{S} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\hat{S}|*\rangle = (1\,\mathrm{mm})|*\rangle$$

$$\begin{pmatrix} a & b \\ & \end{pmatrix} \begin{pmatrix} 2/\sqrt{5} \\ \end{pmatrix} = (1\,\mathrm{mm}) \begin{pmatrix} 2/\sqrt{5} \\ \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{pmatrix} = (1\,mm) \begin{pmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{pmatrix}$$

$$(1) \quad \frac{2}{\sqrt{5}} a - \frac{1}{\sqrt{5}} b = \frac{2}{\sqrt{5}} \Rightarrow 2a - b = 2$$

$$(2) \quad \frac{2}{\sqrt{5}} c - \frac{1}{\sqrt{5}} d = -\frac{1}{\sqrt{5}} \Rightarrow 2c - d = -1$$

$$\hat{S} |\ominus\rangle = (2\,mm) |\ominus\rangle$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} = (2\,mm) \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$$

$$(3) \quad \frac{1}{\sqrt{5}} a + \frac{2}{\sqrt{5}} b = \frac{2}{\sqrt{5}} \Rightarrow a + 2b = 2$$

$$a = 2 - 2b$$

$$(4) \quad \frac{1}{\sqrt{5}} c + \frac{2}{\sqrt{5}} d = \frac{4}{\sqrt{5}} \Rightarrow c + 2d = 4$$

$$c = 4 - 2d$$

(1) \$ (3)

$$2a - b = 2 \qquad a = 2 - 2b$$

$$4 - 4b - b = 2$$

$$-5b = -2$$

$$b = 2/5$$

$$a = 2 - 2\left(\frac{2}{5}\right)$$

$$a = 2 - \frac{4}{5} = \frac{6}{5}$$

(2) \$ (4)

$$2c - d = -1 \qquad c = 4 - 2d$$

$2c - d = -1 \qquad c = 4 - 2d$

$8 - 4d - d = -1$

$-5d = -9$

$d = 9/5$

$c = 4 - 2\left(\frac{9}{5}\right) = 4 - \frac{18}{5} = \frac{2}{5}$

$$\hat{S} = \begin{pmatrix} 6/5 & 2/5 \\ 2/5 & 9/5 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix}$$