Linear convergence of Newton's method

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

converge quadratic to r, $f(r)=0$, if $f'(r) \neq 0$

example: $f(x) = x^2$ $\quad f(x)=0$ $\quad r=0$ $\quad f'(x)=2x$ $\quad x_i = \frac{1}{2^i}$

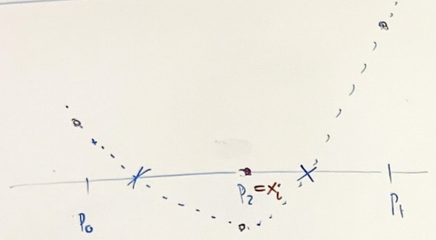
choose $x_0 = 1$

i	x_i	$f(x_i)$	$f'(x_i)$	$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$
0	1	1	2	$1 - \frac{1}{2} = \frac{1}{2}$
1	$\frac{1}{2}$	$\frac{1}{4}$	1	$\frac{1}{2} - \frac{1/4}{1} = \frac{1}{4}$
2	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{2}$	$\frac{1}{4} - \frac{1/16}{1/2} = \frac{1}{8}$
3	$\frac{1}{8}$	...		

modified Newton

x_i	$f(x_i)$	$f'(x_i)$	$f(x_i)/f'(x_i)$	$x_{i+1} = x_i - 2\frac{f(x_i)}{f'(x_i)}$
1	1	2	$\frac{1}{2}$	0

Muller's method



interpolate known $f(p_0)$ $f(p_1)$ $f(p_2)$ by a polynomial

Lagrange interpolation polynomial

want $p(x) = ax^2 + bx + c$ s.t $p(p_0) = f(p_0)$
$p(p_1) = f(p_1)$
$p(p_2) = f(p_3)$

$$p(x) = f(p_2)\underbrace{\frac{(x-p_0)(x-p_1)}{(p_2-p_0)(p_2-p_1)}}_{\substack{=0 \text{ at } p_0\ p_1 \\ =1 \text{ at } p_2}} + f(p_1)\frac{(x-p_2)(x-p_0)}{(p_1-p_2)(p_1-p_0)} + f(p_0)\frac{(x-p_2)(x-p_1)}{(p_0-p_2)(p_0-p_1)}$$

$$\boxed{\begin{aligned} p(p_2) &= f(p_2) \\ p(p_0) &= 0 \\ p(p_1) &= 0 \end{aligned}}$$

$\Rightarrow$ $p(x) = ax^2 + bx + c$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

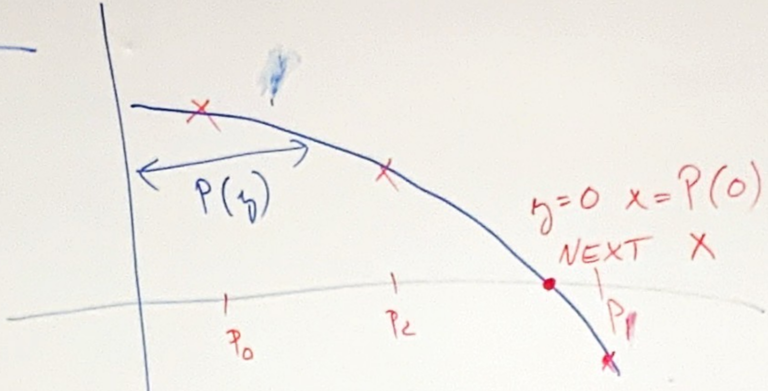
choose the closer to the previous x_i as x_{i+1}

Brent's method

$p_0 \quad f(p_0)$

$p_1 \quad f(p_1)$

$p_2 \quad f(p_2)$



Find $P(y)$

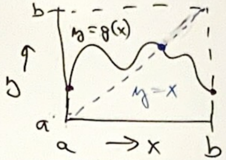
$$P(f(p_0)) = p_0$$

$$P(f(p_1)) = p_1$$

$$P(f(p_2)) = p_2$$

Fixed-point iterations – further look

Theorem! $g:[a,b]\to[a,b]$ is continuous. Then exists $x\in[a,b]$: $x=g(x)$

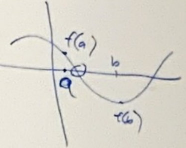


$y=g(x)$
$y=x$
$x=g(x)$

INTERMEDIATE VALUE THM:

$f(x)$ continuous $[a,b]$, & $f(a)f(b)<0$

then $\exists\, x_0\in[a,b]$ s.t. $f(x_0)=0$



define $f(x)=g(x)-x$ case: $g(a)=a$, done. otherwise $f(a)=g(a)-a>0$

case $g(b)=b$ done. otherwise $f(b)=g(b)-b<0$

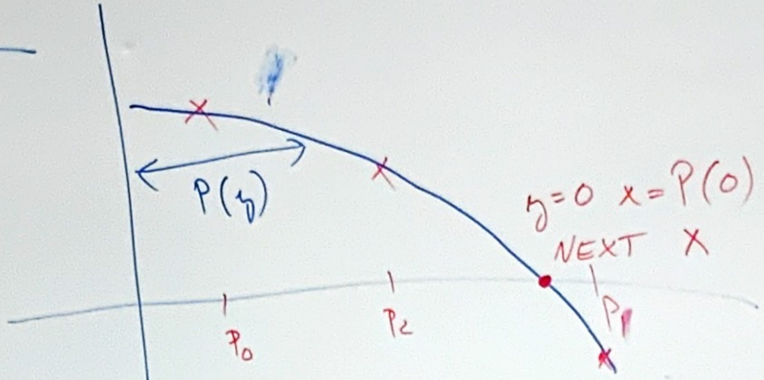
remaining: $f(a)f(b)<0$, $\exists\, x\in[a,b]$: $f(x)=0$, i.e., $x=g(x)$

Brent's method

p_0 $f(p_0)$

p_1 $f(p_1)$

p_2 $f(p_2)$



Find $P(y)$

$$P(f(p_0)) = p_0$$
$$P(f(p_1)) = p_1$$
$$P(f(p_2)) = p_2$$