EM5-1: Static Electric Field

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I can identify the integral and differential forms of Maxwell's equations that describe the electric field in materials and explain the terms in those equations.

For a conductor, the charged particles within are all considered free charges, and will accumulate on the surface when an electric field is incident upon it. For a dielectric, the charged particles are all attached to an atom or molecule, and are thus considered bound charges, only able to move within their associated atom or molecule. In the latter case, the material itself will generate an internal electric field due to polarization within. In this case, we may find it easier to describe the electric field with modified version of Maxwell's equations.

Definitions: (from pp 172 and 174 of Griffiths)

$\sigma_b \equiv \vec{P}\cdot\hat{n}$ (surface charge density of a polarized object {C/m^2})

$\rho_b \equiv -\vec{\nabla}\cdot\vec{P}$ (volume charge density of a polarized object {C/m^3})

$\vec{P} \equiv$ dipole moment per unit volume

(polarization => dipole moment per unit volume)

Gauss's Law

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$$

The total field in a polarized object is going to the result of the field produced by the bound charge plus the field due to the free charges (in this case meaning anything that is not the result of polarization). Within a dielectric, we can write the total charge density as

$$\rho = \underbrace{\rho_b}_{\text{bound}} + \underbrace{\rho_f}_{\text{free}}$$

Where the subscript b signifies the bound charge density and the subscript f signifies the free charge density.

Plugging this in to Gauss's law above and using the definition of the volume density of a polarized object for the bound charge density

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}(\rho_b + \rho_f) = \frac{1}{\varepsilon_0}\left(-\vec{\nabla}\cdot\vec{P} + \rho_f\right)$$

We can then consolidate the two divergence terms and define a new quantity called the electric displacement

$$\varepsilon_0\vec{\nabla}\cdot\vec{E} + \vec{\nabla}\cdot\vec{P} = \rho_f$$

(electric displacement $\vec{D} = \varepsilon_0\vec{E} + \vec{P}$)

$$\vec{\nabla}\cdot(\varepsilon_0\vec{E} + \vec{P}) = \rho_f$$

$$\vec{\nabla}\cdot\vec{D} = \rho_f$$

This gives us a form of Gauss's law that specifically makes reference to only free charges. This is useful when describing dielectrics because we are often know (or can easily determine) the free charge density, but not the bound charge density. Additionally, when free charges are put into place, a certain polarization necessarily follows. "... free charge is the stuff we control. Bound charge comes along for the ride... " (pp 182, Griffiths).

Differential	Integral	Gauss's Law
$\vec{\nabla}\cdot\vec{D} = \rho_f$	$\oint\vec{D}\cdot dA = Q_{f\,enc}$	

This is the only modification of Maxwell's equation if we are strictly talking about the electric field. There is a similar modification of the Ampere's law that can be made for magnetic field. I wasn't sure if I should include that in this writeup as well.

Magnetic field in matter

$$\vec{J} = \underbrace{\vec{J}_f}_{\text{free}} + \underbrace{\vec{J}_b}_{\text{bound}} + \underbrace{\vec{J}_p}_{\text{polarized}}$$

ε_0 — vacuum permittivity

$$\left\{\frac{s^2\cdot C^2}{kg\cdot m^2} \equiv F/m\right\}$$

μ_0 — vacuum permeability

$$\left\{\frac{kg\cdot m}{s^2 A}\right\}$$

Maxwell's Equations in matter

$$\vec{\nabla}\cdot\vec{D} = \rho_f$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t}$$

$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$ (electric displacement) $\{C/m\}$

$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$ (Auxiliary field) $\{A/m\}$

$\vec{P} = \varepsilon_0\chi_e\vec{E}$ $\quad \vec{D} = \varepsilon E$

$\vec{M} = \chi_m\vec{H}$ $\quad H = \frac{1}{\mu}B$

$$\oint\vec{D}\cdot d\vec{A} = Q_{f,enc}$$

$$\oint\vec{E}\cdot d\vec{l} = \int_S B\cdot d\vec{A}$$

$$\oint\vec{B}\cdot d\vec{A} = 0$$

$$\oint\vec{H}\cdot d\vec{l} = I_{f\,enc} + \frac{\partial\vec{D}}{\partial t}$$

free bound polarized

$$\vec{J}_b = \vec{\nabla} \times \vec{M} \Leftarrow \text{magnetization}$$

$$\vec{J}_p = \frac{\partial \vec{P}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\frac{1}{\mu_0} \vec{\nabla} \times \vec{B} = \vec{J}_f + \vec{J}_b + \vec{J}_p + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$= \vec{J}_f + \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \vec{\nabla} \times \vec{M} = \vec{J}_f + \frac{\partial}{\partial t}\left(\varepsilon_0 \vec{E} + \vec{P}\right)$$

$$\vec{\nabla} \times \left(\frac{1}{\mu_0} \vec{B} - \vec{M}\right) = \vec{J}_f + \frac{\partial}{\partial t}\left(\varepsilon_0 \vec{E} + \vec{P}\right)$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

Ampere's law with Maxwell's addition in matter

$$\int_S (\vec{\nabla} \times \vec{H}) \cdot d\vec{A} = \int_S J_f \cdot d\vec{A} + \frac{\partial}{\partial t} \int_S \vec{D} \cdot d\vec{A}$$

$$\oint \vec{H} \cdot d\vec{l} = \int_S J_f \cdot d\vec{A} + \frac{\partial}{\partial t} \int_S \vec{D} \cdot d\vec{A}$$