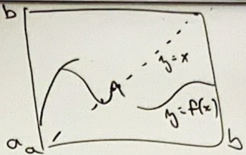
9.1 If $f:[a,b]\to[a,b]$ is continuous, then $\exists x: f(x)=x$



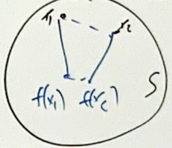
More general: Map of ball B in $\mathbb{R}^n$ to itself

not on doughnut

9.2 If $f:[a,b]\to[a,b]$ and has a fixed point, and $\forall x,y\in[a,b]: \underbrace{|f(x)-f(y)|\le\lambda|x-y|}_{\text{assumption}}$ with constant $\lambda<1$,
then the fixed point is unique.

Proof Let $f(x_1)=x_1$ $f(x_2)=x_2$ $x_1\ne x_2$ then $\underbrace{|x_1-x_2|}_{\ne 0} = |f(x_1)-f(x_2)| \le \underbrace{\lambda}_{<1}\underbrace{|x_1-x_2|}_{\ne 0}$ contradiction

Definition $f: S\subset\mathbb{R}^n\to S$ s.t. $\forall x_1,x_2\in S: \|f(x_1)-f(x_2)\|\le\lambda\|x_1-x_2\|$ for constant $\lambda<1$ is called contraction



<u>Theorem</u> (contraction theorem) $S \subset \mathbb{R}^n$, closed (contains the limits of all sequences $\{x_n\} \subset S$ which have a limit)

$f: S \to S$, is contraction. Then $\exists$ unique fixed point of f in S $(x \in S,\ x = f(x))$, and for any $x_0 \in S$, $x_{n+1} = f(x_n)$, it holds that $\lim_{n\to\infty} x_n = x$

estimates of $x - x_n$ in terms of $x_0 - x_1$:

<u>Theorem 10</u> $\underline{g: [a,b] \to [a,b]}$, $\exists \lambda < 1\ \forall x, y \in [a,b]$: $\underline{|g(x) - g(y)| \le \lambda |x-y|}$ then the fixed point iteration $x_{n+1} = g(x_n)$ converges to the unique fixed point of g in $[a,b]$.

constant

<u>Proof</u> (i) <u>g is continuous</u> Let $x_n \to x$ in $[a,b]$ i.e. $|x_n - x| \to 0$ as $n \to \infty$. Then $|g(x_n) - g(x)| \le \lambda \underbrace{|x_n - x|}_{\to 0} \to \lambda \cdot 0 = 0$

So $g(x_n) \to g(x)$

contraction assumption

Thm. 9 $\Rightarrow \exists\, p \in [a,b] : p = g(p)$ +

iterations $\longrightarrow$ $x_{n+1} = g(x_n)$ −

assumption, $\lambda < 1$

$$|p - x_{n+1}| = |g(p) - g(x_n)| \le \lambda\, |p - x_n|$$

proven: $|p - x_{n+1}| \le \lambda |p - x_n|$

$0 \le \lambda < 1 \Rightarrow \lim_{n\to\infty} \lambda^n = 0$

induction:

$|p - x_1| \le \lambda |p - x_0|$

$|p - x_2| \le \lambda |p - x_1|$ $\qquad |p - x_2| \le \lambda^2 |p - x_0|$

$|p - x_3| \le \lambda |p - x_2|$ $\qquad |p - x_3| \le \lambda^3 |p - x_0|$

⋮

$$|p - x_n| \le \lambda^n |p - x_0| \longrightarrow 0 \text{ as } n \to \infty$$

got estimate:

$|p - x_n| \le \lambda^n |p - x_0|$

know λ, do not know p

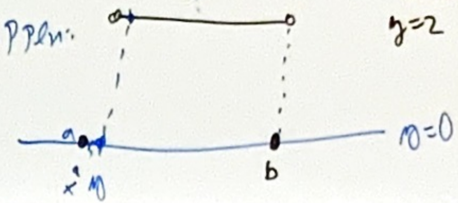
Remark ? $|g'| \le \lambda$ on (a,b)

g continuous on $[a,b]$

$\Rightarrow \forall x,y \in [a,b] : |g(x) - g(y)| \le \lambda |x - y|$

if g not assumed continuous on $[a,b]$

can happen:



got estimate:

$$|p - x_n| \le \lambda^n |p - x_0|$$

know λ, do not know p

Remark ? $|g'| \le \lambda$ on (a,b)

g continuous on $[a,b]$

$\Rightarrow \forall x,y \in [a,b]: |g(x) - g(y)| \le \lambda |x - y|$

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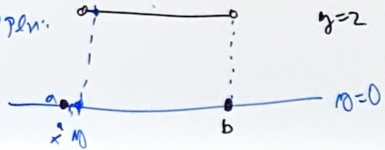
can happen:



$n \to \infty$

use $|x_1 - x_0| = |g(x_0) - x_0|$?

Corollary 3.1 If $g:[a,b] \to [a,b]$, $\exists \lambda < 1 \; \forall x,y \in [a,b]: |g(x) - g(y)| \le \lambda |x - y|$

then $|p - x_n| \le \dfrac{\lambda^n}{1-\lambda} |x_1 - x_0|$

Exercise: given g, verify $g:[a,b] \to [a,b]$
estimate $|g'| \le \lambda$
given $\varepsilon > 0$ find n
$|p - x_n| < \varepsilon$

Triangle inequality

$$|p - x_0| \le \underbrace{|p - x_1|}_{\le \lambda |p - x_0|} + |x_1 - x_0| \qquad |p - x_1| \le \lambda |p - x_0|$$

known $x_1 = g(x_0)$

$$|p - x_0| \le \lambda |p - x_0| + |x_1 - x_0|$$

$$\underbrace{|p - x_0| - \lambda |p - x_0|}_{\underbrace{(1-\lambda)}_{>0} |p - x_0|} \le |x_1 - x_0| \Rightarrow |p - x_0| \le \frac{|x_1 - x_0|}{1 - \lambda}$$

$$|p - x_0| \le \frac{|x_1 - x_0|}{1-\lambda}$$

know: $|p - x_n| \le \lambda^n |p - x_0|$

$$\boxed{|p - x_n| \le \frac{\lambda^n}{1-\lambda}|x_1 - x_0|}$$

$\alpha = 2$

In Newton: $g(x) = x - \frac{f(x)}{f'(x)}$

$f(p) = 0 \Rightarrow g(p) = x$

$f'(p) \neq 0 \Rightarrow g'(p) = 0$

$g''(p) \neq 0$ in general

$$g''(p) = \frac{2f''(p)}{f'(p)}$$

2.7 High order fixed point iterations

Thm 12 $g \in C^\infty(\bar{I})$ interval I - open, $p \in I$, $p = g(p)$

$g'(p) = g''(p) = \dots = g^{(\alpha-1)}(p) = 0$, $g^{(\alpha)}(p) \neq 0$

then $\lim_{n\to\infty} \frac{x_{n+1} - p}{(x_n - p)^\alpha} = \frac{g^{(\alpha)}(p)}{\alpha!} \neq 0$ for x_0 close to p.

$x_{n+1} = g(x_n)$

$g(x) = p + (x-p)\underbrace{g'(p)}_{=0} + \dots + \frac{(x-p)^{\alpha-1}}{\alpha-1!}\underbrace{g^{(\alpha-1)}(p)}_{=0} + \frac{(x-p)^\alpha}{\alpha!} g^{(\alpha)}(\xi_n)$ $\quad \xi_n$ between p and x

$g(x_n) - p = \frac{(x_n - p)^\alpha}{\alpha!} g^{(\alpha)}(\xi_n)$ $\quad \xi_n = \xi_n(x_n) \to p$ $\quad$ already know $x_n \to p$

$g^{(\alpha)}(\xi_n) \to g^{(\alpha)}(p)$

$$\boxed{\frac{x_{n+1} - p}{(x_n - p)^\alpha} \to \frac{g^{(\alpha)}}{\alpha!}}$$