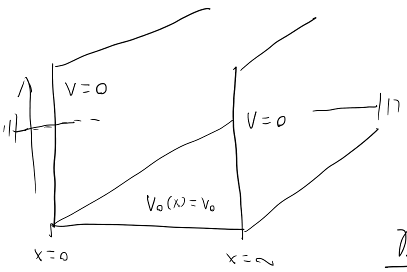
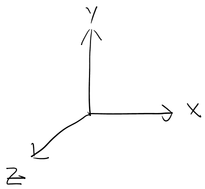
Poisson 2





$x \in [0, a]$
$y \in [0, \infty)$
$z \in (-\infty, \infty)$

B.C.s

$V(x=0, y>0) = 0$

$V(x=a, y>0) = 0$

$V(x, y\to\infty) = 0$

$V(x, y=0) = V_0(x) = V_0 = \text{constant}$

Since the voltage is defined as the distance in the y direction from the origin on the xz plane, we can treat this as a function of only x and y. z will not matter in this case due to symmetry.

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$V(x,y) = X(x)\,Y(y)$$

$$\frac{\partial^2 V}{\partial x^2} = Y X'' \qquad \frac{\partial^2 V}{\partial y^2} = X Y''$$

$$\nabla^2 V = Y X'' + X Y'' = 0 \qquad \left(\text{both } \frac{X''}{X} \,\&\, \frac{Y''}{Y} \text{ must be constant}\right)$$

$$\frac{1}{XY}\nabla^2 V = \frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -k^2$$

$$X'' - k^2 X = 0 \qquad Y'' + k^2 Y = 0$$

$$X'' + k^2 X = 0 \qquad Y'' - k^2 Y = 0$$

S.H.M. equation

Has exponential solutions

$$X = \begin{cases} \sin(kx) \\ \cos(kx) \end{cases} \qquad Y = \begin{cases} e^{ky} \\ e^{-ky} \end{cases}$$

$$V(x=0, y>0) = 0 \qquad V(x, y\to\infty) = 0$$

$$X = \begin{Bmatrix} \sin(k(0)) \\ \cancel{\cos(k(0))} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \cancel{1} \end{Bmatrix} = 0 \qquad Y = \lim_{y\to\infty} \begin{Bmatrix} \cancel{e^{ky}} \\ e^{-ky} \end{Bmatrix} = \begin{Bmatrix} \cancel{\infty} \\ 0 \end{Bmatrix} = 0$$

$$\begin{pmatrix} \cos(\cancel{k}(0)) \end{pmatrix} \quad (\cancel{\ })$$

$$\begin{pmatrix} e^{-ky} \end{pmatrix} \quad (0)$$

$$X = A\sin(kx)$$

$$V(x=a, y>0) = 0$$

$$X = A\sin(ka) = 0$$

$$\sin(ka) = 0$$

$$ka = n\pi$$

(where $n = 1, 2, 3 \ldots$)

$$k = \frac{n\pi}{a}$$

$$Y(y) = Be^{-ky}$$

$$Y(y) = Be^{-\frac{n\pi}{a}y}$$

$$X(x) = A\sin\left(\frac{n\pi}{a}x\right)$$

$$AB = C \quad \text{(consolidating constants)}$$

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n}{a}\pi y} \sin\left(\frac{n\pi}{a}x\right)$$

We write this as a summation because we have really found a family of solutions for this particular voltage configuration. The idea is similar to writing vectors in a given vector space as a linear combination of some constants multiplied by some set of orthonormal basis vectors which span the entirety of said vector space, i.e. a set of eigenvalues times a set of eigenvectors. Using thisformula, we are able to describe any possible vector in a given vector space, so long as we know its orthonormal basis vectors.

Here, we similarly have a given mathematical space, except in this case, we could think of it as a voltage space rather thana vector space. Similarly to how we defined the above definition of vectors, we are now describing our voltage as a linear combinationof infinitely many linearly independent solutions. These linearly independent solutions are act similarly in that they are someconstant, time some combination of orthogonal and normalized root functions; i.e. eigenfunctions, in this case an exponential times a sinusoid. Since we know the form of our basis set, we can now exploit the orthogonality of this basis set to find the pertinent constants by use of the definition of orthogonality off to the side.

$\vec{A}$ & $\vec{B}$ are orthogonal if they scalar product is 0,

$$\sum_i A_i B_i = 0$$

$\vec{A}$ & $\vec{B}$ are not orthogonal if

$$\sum_i A_i B_i \neq 0$$

By analogy, we can say that the functions $A(x)$ & $B(x)$ are orthogonal if

$$\int_a^b A(x)B(x)\,dx = 0$$

and not orthogonal when

$$\int^b A(x)B(x) \neq 0$$

Before using the orthogonality to find the constants, we can simplify our expression further using the final boundary condition. In this case, V_0(x) = constant because the material is a conductor, this would not hold if the material were not a conductor.

$$V(x, y=0) = V_0$$

$$= \sum_{n=1}^{\infty} C_n e^{-0} \sin\left(\frac{n\pi}{a}x\right)$$

$$= \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{a}x\right) = V_0$$

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$$\int_a^b A(x)B(x) \neq 0$$

$$\sum_{m=1}^{\infty} V_0 \sin\left(\frac{m\pi}{a}x\right) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)$$

$$\Rightarrow \int_{x=0}^{x=a} V_0 \sin\left(\frac{m\pi}{a}x\right)dx = C_{n,m}\int_{x=0}^{x=a} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)dx$$

$$\int_0^a \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)dx = \int_0^a \frac{1}{2}\left[\cos\left((n-m)\frac{\pi}{a}x\right) - \cos\left((n+m)\frac{\pi}{a}x\right)\right]dx$$

$$= \frac{1}{2}\left[\frac{\sin\left((n-m)\frac{\pi}{a}x\right)}{(n-m)\frac{\pi}{a}} - \frac{\sin\left((n+m)\frac{\pi}{a}x\right)}{(n+m)\frac{\pi}{a}}\right]\Bigg|_{x=0}^{x=a}$$

$$= \frac{a}{2\pi}\left[\frac{\sin((n-m)\pi)}{(n-m)} - \frac{\sin((n+m)\pi)}{(n+m)}\right] = \begin{cases} n=m, & \text{indeterminant} \\ n\neq m, & 0 \end{cases}$$

When n does not equal m, each of the sine function will be for some integer multiple of \pi, which is always zero. This confirms the idea that each basis function we are using will be orthogonal when their indices are different. We should now look at the special case in which the indices are equal, i.e. where n=m.

$$\int_0^a \sin^2\left(\frac{n\pi}{a}x\right)dx = \int_0^a \frac{1-\cos\left(\frac{2n\pi}{a}x\right)dx}{2}$$

$$= \frac{1}{2}x - \frac{a}{2^2 n\pi}\sin\left(\frac{2n\pi}{a}x\right)\Bigg|_{x=0}^{x=a}$$

$$= \frac{1}{2}a - \frac{a}{2^2 n\pi}\sin(2n\pi) - 0 = \frac{a}{2}$$

The sine term will be an integer multiple of 2\pi everywhere, and thus be equal to zero. This leaves us with the solution above, which confirms that when the indices are the same, the basis functions are identical.

$$\int^a \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)dx = \begin{cases} 0, & n\neq m \end{cases}$$

$$\int_0^a \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)dx = \begin{cases} 0, & n \neq m \\ \frac{a}{2}, & n = m \end{cases}$$

$$\int_{x=0}^{x=a} V_0 \sin\left(\frac{n\pi}{a}x\right)dx = \frac{a}{2}C_n$$

$$C_n = \frac{2}{a}\int_{x=0}^{x=a} V_0 \sin\left(\frac{n\pi}{a}x\right)dx$$

V_0 is constant, so we can pull it out of the integral.

$$C_n = \frac{2V_0}{a}\int_{x=0}^{x=a} \sin\left(\frac{n\pi}{a}x\right)dx = \frac{2V_0}{a}\cdot\frac{a}{n\pi}\left(-\cos\left(\frac{n\pi}{a}x\right)\right)\Big|_0^a$$

$$C_n = \frac{2V_0}{n\pi}\left(-\cos(n\pi)+1\right)$$

$$= \frac{2V_0}{n\pi}\left(1-\cos(n\pi)\right)$$

when n is an even number

$$C_n = \frac{2V_0}{n\pi}(1-1) = 0$$

when n is an odd number

$$C_n = \frac{2V_0}{n\pi}(1+1) = \frac{4V_0}{n\pi}$$

So,

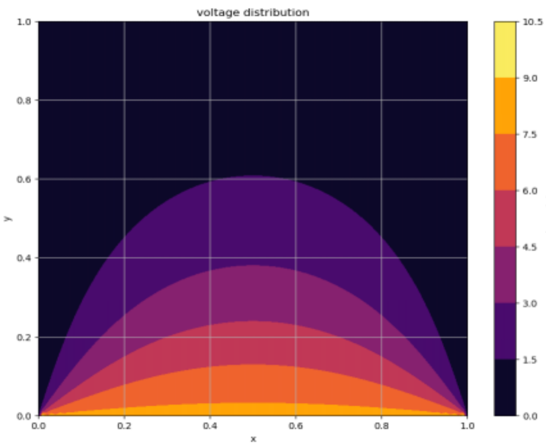
$$C_n = \begin{cases} 0, & n \text{ is even} \\ \frac{4V_0}{n\pi}, & n \text{ is odd} \end{cases}$$

Our complete solution is then

Our complete solution is then

$$\boxed{V(x,y) = \frac{4V_0}{\pi} \sum_{n=\text{odd}} \frac{1}{n} e^{-\frac{n\pi}{a}y} \sin\left(\frac{n\pi}{a}x\right)}$$

Plot of the voltage distribution



(code for plot)

```
import numpy as np
import matplotlib.pyplot as plt

def Vfunc(x, y, V0, a, n):
    sol = 0 # initializes solution
    for k in range(1, n, 2):  # starts at 1 and counts by 2; k = 1,3,5,...,n
        sol += (1/k)*np.exp(-k*np.pi*y/a)*np.sin(k*np.pi*x/a)
    sol *= 4*V0 / np.pi
    return sol

n = 100 # number of steps
V0 = 8.0 # volts
a = 1.0 # mm (around the scale of the width of a capacitor)

x = np.linspace(0, a, 100)
y = np.linspace(0, a, 100)
X, Y = np.meshgrid(x, y)

Z = Vfunc(X, Y, V0, a, n)

plt.figure(figsize=(10, 8))
plt.contourf(X, Y, Z, cmap='inferno')
plt.colorbar(label='voltage (V)')
plt.title('voltage distribution')
plt.xlabel('x')
plt.ylabel('y')
plt.grid(True)
plt.show()
```