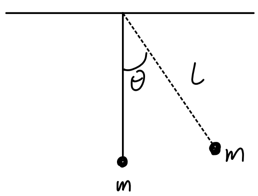
Planar pendulum:



generalized coordinates are any coordinates that are convenient for a particular system.

generalized coordinates here would be θ.

$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$

$U = mgy$

$x = l\sin\theta$ | $\dot{x} = l\dot{\theta}\cos\theta$

$y = -l\cos\theta$ | $\dot{y} = l\dot{\theta}\sin\theta$

, then $T = \frac{1}{2} m l^2 \dot{\theta}^2$

$U = -ml\cos(\theta)$

, I'm doing it in terms of r.

$KE = \frac{1}{2} I\omega^2, \omega = \dot{\theta}$

$KE = \frac{1}{2} mr^2\dot{\theta}^2$

$PE = -mgr\cos(\theta)$

Lagrangian method:

$L(\theta(t)) = T(\theta(t)) - U(\theta(t))$

$L(\theta(t)) = \frac{1}{2} mr^2\dot{\theta}^2 + mgr\cos(\theta)$

depend on time only based on generalized coordinates.

E-L equation:

$\frac{\delta L}{\delta \theta} - \frac{d}{dt}\left(\frac{\delta L}{\partial \dot{\theta}}\right) = 0$

$\frac{\partial L}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\cancel{\frac{1}{2} mr^2\dot{\theta}^2}^{\,0} + mgr\cos\theta\right)$ partial derivative of L w/r to θ

$= -mgr\cos\theta$

$\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2} mr^2\dot{\theta}^2 + \cancel{mgr\cos(\theta)}^{\,0}\right)$ partial derivative of L w/r to $\dot{\theta}$

$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mr^2\ddot{\theta}$ ← time derivative of $\frac{\partial L}{\partial \dot{\theta}}$

So...

$\frac{\delta L}{\delta \theta} - \frac{d}{dt}\left(\frac{\delta L}{\partial \dot{\theta}}\right) = 0$

$-mgr\sin(\theta) - mr^2\ddot{\theta} = 0$

$-\cancel{mgr}\sin(\theta) = \cancel{m}r^2\ddot{\theta}$

$-g\sin(\theta) = r\ddot{\theta}$

$\boxed{\text{Eom: } \ddot{\theta} = -\frac{g}{r}\sin(\theta)}$

↓

$\sin\theta$ is θ when θ is small,

so...

$\ddot{\theta} = -\frac{g}{r}\theta$

Let's solve:

Guess:

$\theta = A\sin(\omega t) + B\cos(\omega t)$

$\dot{\theta} = \omega A\cos(\omega t) - B\omega\sin(\omega t)$

$\ddot{\theta} = -A\omega^2\sin(\omega t) - B\omega^2\cos(\omega t)$

Plug in:

$-A\omega^2\sin(\omega t) - B\omega^2\cos(\omega t) = -\frac{g}{r}\left[A\sin(\omega t) + B\cos(\omega t)\right]$

$-A\omega^2\sin(\omega t) - B\omega^2\cos(\omega t) = -A\frac{g}{r}\sin(\omega t) - \frac{g}{r}B\cos\omega(t)$

Since everything is negative, I just cancelled it out.

$A\omega^2\sin(\omega t) - \frac{g}{r}A\sin(\omega t) = \frac{g}{r}B\cos(\omega t) - B\omega^2\cos(\omega t)$

if $t=0$, then we end up with

$0 = -B\omega^2 + \frac{g}{r}B$

$B\omega^2 = \frac{g}{r}B$

$\omega^2 = \frac{g}{r}$, $\omega = \sqrt{g/r}$

What we found is that this equation satisfies

$\ddot{\theta} = \frac{g}{r}\theta$, as long as we set $\omega = \sqrt{\frac{g}{r}}$.

$\theta(t=0) = \theta_0$

$B\cos(\omega \cdot 0) = B$

$B = \theta_0$

$A = 0$, it has to be zero.

$$\boxed{\theta(t) = \theta_0\cos\left(\sqrt{\frac{g}{r}}\,t\right)}$$