

## 6.3 Kinematics: Third Look

We have now looked at a couple of different ways to find velocities and accelerations. **Any** acceleration or velocity can be found by applying these methods. Once these procedures are known and understood, it is advantageous to look at kinematic expansions for velocity and acceleration. The expansions can be used directly for calculation or can be used as an aid to bring **physical significance** to the equations. The expressions were derived in class and homework.

### 6.3.1 Velocity

To find a velocity,  ${}^f\vec{V}^p$ , using a kinematic expansion, we must still begin with a position vector  $\vec{R}^{qp}$ .  $\vec{R}^{qp}$  begins at some point fixed in frame  $f$ , the base point  $q$ , and terminates at the point  $p$ . As we have done before, we also select an intermediate frame  $f'$ . So far, this is exactly the same procedure used when applying equation (5) from section 6.2.3 (Kinematics, second look). But now introduce a point  $q'$  which is **fixed in frame  $f'$** , and rewrite  $\vec{R}^{qp}$  as the sum of two vectors.

$$\vec{R}^{qp} = \underset{\substack{\text{from } q \\ \text{to } q'}}{\vec{R}^{qq'}} + \underset{\substack{\text{from } q' \\ \text{to } p}}{\vec{R}^{q'p}}$$

Then find the velocity by using

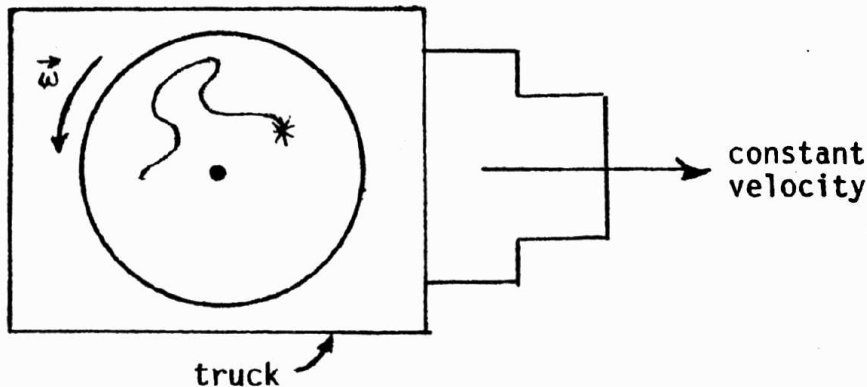
$${}^f\vec{V}^p = {}^{f'}\vec{V}^p + {}^f\vec{V}^{q'} + {}^f\vec{\omega} \times \vec{R}^{q'p} \quad (6)$$

Certain choices of  $\vec{R}^{qq'}$ ,  $\vec{R}^{q'p}$ , and  $f'$  may make the first two terms easy to determine since

$$\begin{aligned} {}^{f'}\vec{V}^p &= \frac{d}{dt} \vec{R}^{q'p} \\ {}^f\vec{V}^{q'} &= \frac{d}{dt} \vec{R}^{qq'} \end{aligned}$$

We might also note that the sum of the 2nd and 3rd terms in Eqn. (6),  ${}^f\vec{V}^{q'} + {}^f\vec{\omega} \times \vec{R}^{q'p}$  is sometimes defined as the **transport velocity**. This is the velocity that the point  $p$  would have if it was fixed in  $f'$ .

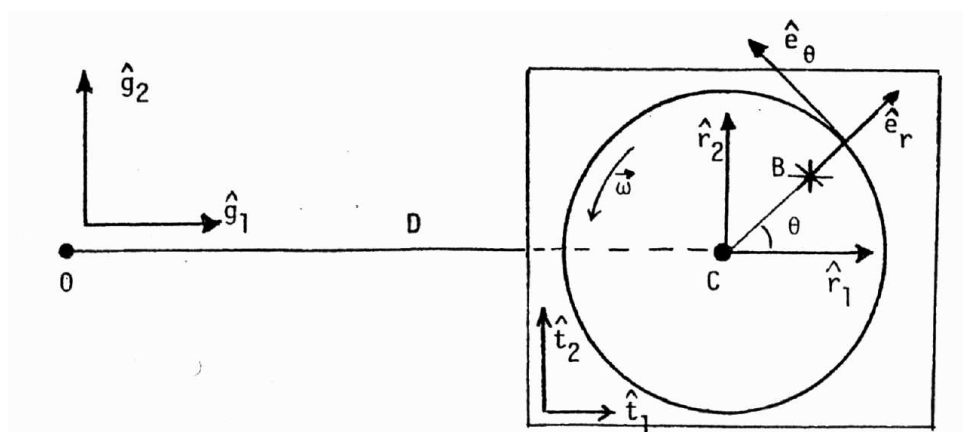
Example, on the problem illustrated below: A bug is walking around on a record which is spinning at a constant angular velocity  $\vec{\omega}$ . The record player is aboard a flatbed truck moving down a straight road at constant velocity  $\vec{v}$ . We are asked to find the velocity of the bug with respect to the ground using Eqn. (6).



Example problem: bug on turntable, carried on a truck.

To solve this, we use the following steps:

1. Redraw the figure with the appropriate unit vectors and define any variables which are not given.



Unit vectors are added to the problem, and a few variables are defined

- frame  $g$  - fixed to the ground
- frame  $t$  - fixed to the truck
- frame  $r$  - fixed to the record
- frame  $e$  - rotating with the bug such that  $\hat{e}_r$  is always directed from  $C$  to  $B$
- point  $0$  - fixed on the ground
- point  $C$  - at the center of the record and therefore fixed in both frame  $r$  and frame  $t$
- point  $B$  - the bug
- angle  $\theta$  - angle made by  $\hat{e}_r$  with  $\hat{r}_1$  (fixed in  $r$ )

2. List the symbols used in Eqn. (6) with those that you have chosen to correspond in the problem

	<u>Equation</u>	<u>Problem</u>
frames	$f$	$g$
	$f'$	$r$
points	$q$	$0$
	$q'$	$C$
	$p$	$B$
vectors	$\vec{R}^{qp}$	$D\hat{t}_1 + r\hat{e}_r$
	$\vec{R}^{qq'}$	$D\hat{t}_1$
	$\vec{R}^{q'p}$	$r\hat{e}_r$

3. Rewrite Eqn. (6) for the problem of interest

$${}^g\vec{V}^B = {}^r\vec{V}^B + {}^g\vec{V}^C + {}^g\vec{\omega}^r \times (r\hat{e}_r)$$

4. Note any angular velocity which may be needed.

$$\begin{aligned} {}^g\vec{\omega}^r &= \omega\hat{r}_3 \\ {}^r\vec{\omega}^e &= \dot{\theta}\hat{r}_3 \end{aligned}$$

5. Solve by finding one term at a time

$$\begin{aligned}
 {}^r\vec{V}^B &= \frac{{}^r d({}^r\hat{e}_r)}{dt} = \frac{{}^e d({}^r\hat{e}_r)}{dt} + {}^r\vec{\omega}^e \times {}^r\hat{e}_r \\
 &= \dot{r}\hat{e}_r + (\dot{\theta}\hat{r}_3) \times (r\hat{e}_r) \\
 &= \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta
 \end{aligned}$$

$${}^s\vec{V}^C = v\hat{t}_1$$

$$\begin{aligned}
 {}^s\vec{\omega}^r \times r\hat{e}_r &= \omega\hat{r}_3 \times r\hat{e}_r \\
 &= r\omega\hat{e}_\theta
 \end{aligned}$$

6. Solution:

$$\begin{aligned}
 {}^s\vec{V}^B &= \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta & + & & v\hat{t}_1 + r\omega\hat{e}_\theta \\
 &\text{velocity of bug with} & & & \text{velocity bug would have if he/she} \\
 &\text{respect to the record} & & & \text{just sat on record}
 \end{aligned}$$

$${}^s\vec{V}^B = v\hat{t}_1 + \dot{r}\hat{e}_r + r(\omega + \dot{\theta})\hat{e}_\theta$$

### 6.3.2 Acceleration

To find an acceleration,  ${}^f\vec{a}^p$ , using a kinematic equation, we define the same frames, points and vectors, then use the following equation

$${}^f\vec{a}^p = {}^{f'}\vec{a}^p + {}^{f'}\vec{a}^{q'} + {}^{f'}\vec{\alpha}^{f'} \times \vec{R}^{q'p} + {}^{f'}\vec{\omega}^{f'} \times ({}^{f'}\vec{\omega}^{f'} \times \vec{R}^{q'p}) + 2 {}^{f'}\vec{\omega}^{f'} \times {}^{f'}\vec{V}^p \quad (7)$$

Two of the terms here have names which are widely used.

$$\begin{aligned}
 {}^{f'}\vec{\omega}^{f'} \times ({}^{f'}\vec{\omega}^{f'} \times \vec{R}^{q'p}) &\implies \textbf{centripetal} \\
 &\text{acceleration} \\
 2 {}^{f'}\vec{\omega}^{f'} \times {}^{f'}\vec{V}^p &\implies \textbf{Coriolis} \\
 &\text{acceleration}
 \end{aligned}$$

Since mass times **centripetal acceleration** is the negative of what is frequently called “centrifugal force,” it is sometimes recognizable in less complicated problems. **Coriolis acceleration** is the origin of a “sideways” force felt, for example, if you try to walk radially outward on a spinning playground merry-go-round.

Example: Find now the acceleration of the bug with respect to the ground. Since we have already redrawn the figure and defined the symbols, move straight to the third step.

3. Rewrite equation (7) for the problem of interest.

$${}^s\vec{a}^B = {}^r\vec{a}^B + {}^s\vec{\alpha}^r \times r\hat{e}_r + {}^s\vec{\omega}^r \times ({}^s\vec{\omega}^r \times r\hat{e}_r) + 2 {}^s\vec{\omega}^r \times {}^r\vec{V}^B$$

4. Note any angular velocities and angular accelerations which may be needed.

$${}^g\vec{\alpha}^r = \frac{{}^g d {}^g \vec{\omega}^r}{dt} = \frac{{}^g d \omega \hat{r}_3}{dt} = 0 \quad \text{since } \omega \text{ is constant}$$

5. Term by term

(a)

$$\begin{aligned} {}^r\vec{a}^B &= \frac{{}^r d {}^r \vec{V}^B}{dt} = \frac{{}^r d}{dt} (\dot{r} \hat{e}_r + r \dot{\theta} \hat{e}_\theta) \\ &= \frac{{}^e d {}^r \vec{V}^B}{dt} + {}^r \vec{\omega}^e \times {}^r \vec{V}^B \\ &= \frac{{}^e d}{dt} (\dot{r} \hat{e}_r + r \dot{\theta} \hat{e}_\theta) + \dot{\theta} \hat{r}_3 \times (\dot{r} \hat{e}_r + r \dot{\theta} \hat{e}_\theta) \\ &= (\dot{r} - r \dot{\theta}^2) \hat{e}_r + (r \ddot{\theta} + 2 \dot{r} \dot{\theta}) \hat{e}_\theta \end{aligned}$$

(b)

$${}^g\vec{a}^C = \frac{{}^g d {}^g \vec{V}^C}{dt} = \frac{{}^g d v \hat{t}_1}{dt} = 0$$

( since  $v$  is constant and the road is straight)

(c)

$${}^g\vec{\alpha}^r \times r \hat{e}_r = 0 \quad \text{since } {}^g\vec{\alpha}^r = 0$$

(d)

$$\begin{aligned} {}^g\vec{\omega}^r \times r \hat{e}_r &= \omega \hat{r}_3 \times r \hat{e}_r = r \omega \hat{e}_\theta \\ {}^g\vec{\omega}^r \times ({}^g\vec{\omega}^r \times r \hat{e}_r) &= \omega \hat{r}_3 \times r \omega \hat{e}_\theta \\ &= -r \omega^2 \hat{e}_r \end{aligned}$$

(e)

$$\begin{aligned} 2 {}^g\vec{\omega}^r \times {}^r \vec{V}^B &= 2 \omega \hat{r}_3 \times (\dot{r} \hat{e}_r + r \dot{\theta} \hat{e}_\theta) \\ &= 2 \dot{r} \omega \hat{e}_\theta - 2 r \omega \dot{\theta} \hat{e}_r \end{aligned}$$

6. Solution:

$${}^g\vec{a}^B = (\dot{r} - r \dot{\theta}^2 - r \omega^2 - 2 r \omega \dot{\theta}) \hat{e}_r + (r \ddot{\theta} + 2 \dot{r} \dot{\theta} + 2 \omega \dot{r}) \hat{e}_\theta$$

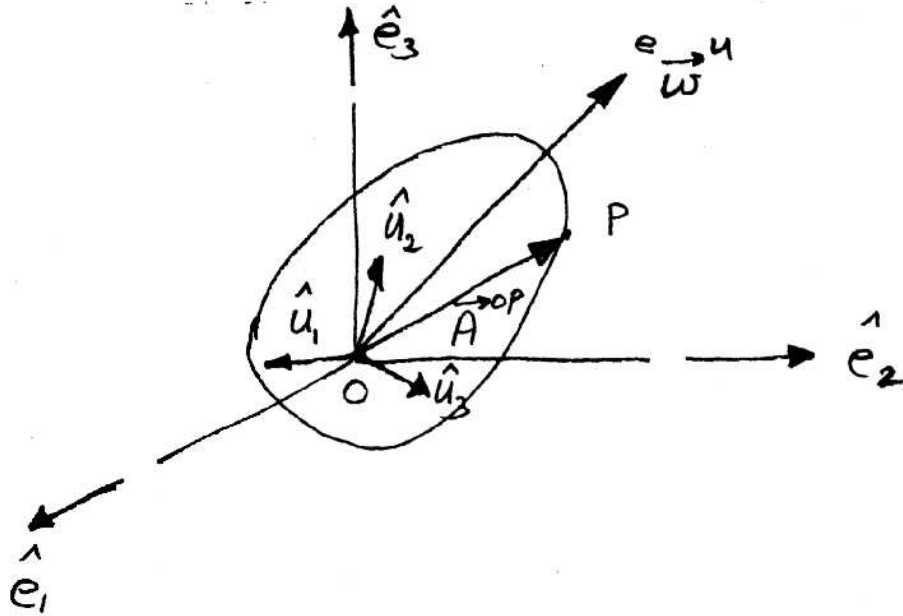
### 6.3.3 Summary

Using kinematic expansions is not difficult to do but requires practice. You also need to have the equations available and understand what the symbols mean. It is probably the most widely used method of calculating velocities and accelerations.

## 6.4 The Kinematics “Cookbook”

We now summarize the solution of kinematics problems in which the BKE (Basic Kinematic Equation) is used:

$$\frac{{}^e d\vec{A}^{op}}{dt} = \frac{{}^u d\vec{A}^{op}}{dt} + {}^e \vec{\omega}^u \times \vec{A}^{op}$$



Two sets of unit vectors on a rigid body, and the angular velocity between them

where

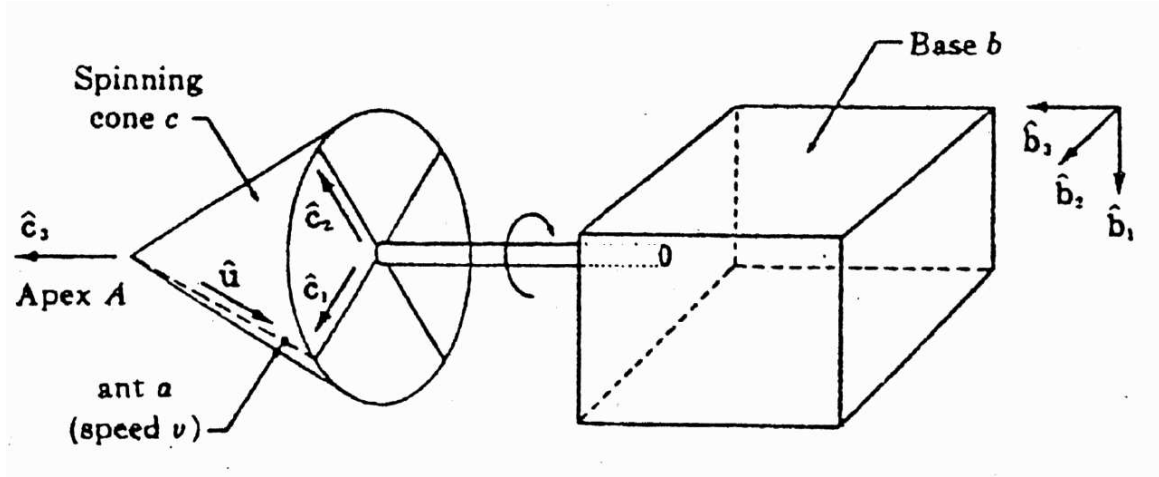
$$\begin{aligned} \frac{{}^e d\vec{A}^{op}}{dt} &= \text{rate of change of } \vec{A} \text{ (of } p \text{ with respect to } 0) \text{ as seen in } e \text{ frame.} \\ \frac{{}^u d\vec{A}^{op}}{dt} &= \text{rate of change of } \vec{A} \text{ (of } p \text{ with respect to } 0) \text{ as seen in } u \text{ frame.} \\ &= 0 \text{ if } p \text{ is fixed in the } u \text{ frame.} \\ {}^e \vec{\omega}^u &= \text{angular velocity of the frame } u \text{ with respect to the frame } e. \\ {}^e \vec{\omega}^u \times \vec{A}^{op} &= \text{rate of change of } \vec{A}^{op} \text{ as seen in the } e \text{ frame due to the rotation rate of the } u \text{ frame (with respect to the } e \text{ frame). If } p \text{ is fixed in the } u \text{ frame, this is the total rate of change as seen in } e. \end{aligned}$$

The advantages of the BKE approach are that the BKE is a simple equation to remember and that it can be easily applied to complex problems involving several rotations (coordinate systems). A disadvantage is that sometimes the BKE method may require a few more steps than the method of using the kinematic expansions. On the other hand, the kinematic expansions are more difficult to recall, involve more notation and are more cumbersome in dealing with several rotations. Both methods are important and have advantages and disadvantages.

In the Kinematic Cookbook, which follows, we will focus on the application of the BKE method. In order to make the method clearer, we will work through a specific example, called the “Ant on a Cone Problem,” which is described by P.W. Likins in *Elements of Engineering Mechanics*, McGraw-Hill Book Company, New York. (Sections of this book are reprinted as part of the class notes.)

## 6.5 The Kinematic Cookbook for the “Ant on a Cone” Problem

An ant, “a,” crawls along a meridian of a spinning cone, “c,” at a constant speed,  $v$ .



The ant on a cone problem, used to illustrate the application of the B.K.E.

The cone is a right circular cone of height 4 inches and base radius 3 inches. It spins about its symmetry axis at the constant rate,  $\Omega$ , with clockwise rotation, as viewed from the base, “b.” The ant begins at the apex, “A,” at time  $t = 0$  when unit vectors  $\hat{c}_1$ ,  $\hat{c}_2$ , and  $\hat{c}_3$  coincide with  $\hat{b}_1$ ,  $\hat{b}_2$ , and  $\hat{b}_3$ , respectively.

The problem is to find  ${}^b\vec{V}^{Aa}$  and  ${}^b\vec{a}^{Aa}$ .

1. Always start from a stationary point, such as the apex A.
2. Specify the position vector with respect to the stationary point:

$$\vec{r}^{Aa} = vt\hat{u}$$

3. Determine the minimum number of reference frames,  $N$ , required by adding the number of rotations,  $n$  (rotating coordinates) to one (for the inertial frame):

$$N = n + 1$$

Note that  $N$  is the minimum. In this example,  $N = n + 1 = 1 + 1 = 2$ , but we have used an extra unit vector  $\hat{u}$ , for convenience.  $\hat{u}$  is along the meridian of the cone and is fixed in the  $c$  reference frame.

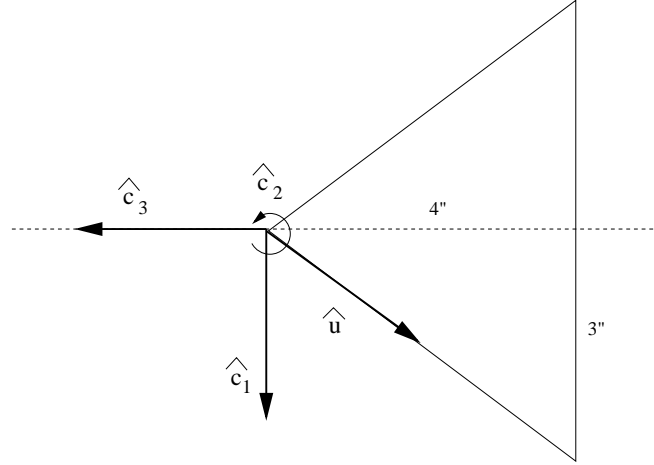
4. Label the reference frames on the appropriate body. Convenient letters such as  $c$  for cone and  $b$  for base provide for easier-to-understand notation.
5. Determine the angular velocities of each frame:

$${}^b\vec{\omega}^c = \Omega\hat{c}_3$$

6. Find the velocity by differentiating the position vector, using the BKE (Basic Kinematic Equation):

$${}^b\vec{V}^{Aa} = \frac{{}^b d}{dt} \vec{r}^{Aa} = \frac{{}^c d}{dt} \vec{r}^{Aa} + {}^b\vec{\omega}^c \times \vec{r}^{Aa}$$

In our particular example we have written  $\vec{r}^{Aa}$  in terms of  $\hat{u}$ . Before applying the BKE we find the components of  $\hat{u}$  in terms of components in the directions of the  $\hat{c}$  unit vectors:



The ant on a cone problem, finding the unit vector  $\hat{u}$

$$\hat{u} = \frac{3}{5}\hat{c}_1 - \frac{4}{5}\hat{c}_3$$

So that

$$\vec{r}^{Aa} = vt(0.6\hat{c}_1 - 0.8\hat{c}_3)$$

and

$$\begin{aligned} {}^b\vec{V}^{Aa} &= \frac{{}^c d}{dt} [vt(0.6\hat{c}_1 - 0.8\hat{c}_3)] + \vec{\Omega}\hat{c}_3 \times [vt(0.6\hat{c}_1 - 0.8\hat{c}_3)] \\ &= 0.6v\hat{c}_1 + 0.6vt\Omega\hat{c}_2 - 0.8v\hat{c}_3 \end{aligned}$$

7. Find the acceleration by differentiating the velocity using the BKE

$${}^b\vec{a}^{Aa} = \frac{{}^b d}{dt} {}^b\vec{V}^{Aa} = \frac{{}^c d}{dt} {}^b\vec{V}^{Aa} + {}^b\vec{\omega}^c \times {}^b\vec{V}^{Aa}$$

In our example, we have:

$$\begin{aligned} {}^b\vec{a}^{Aa} &= \frac{{}^c d}{dt} [0.6v\hat{c}_1 + 0.6vt\Omega\hat{c}_2 - 0.8v\hat{c}_3] \\ &+ \Omega\hat{c}_3 \times [0.6v\hat{c}_1 + 0.6vt\Omega\hat{c}_2 - 0.8v\hat{c}_3] \\ &= -0.6\Omega^2 vt\hat{c}_1 + 1.2v\Omega\hat{c}_2 \end{aligned}$$

Insert Likins pgs 118 to 137 here.

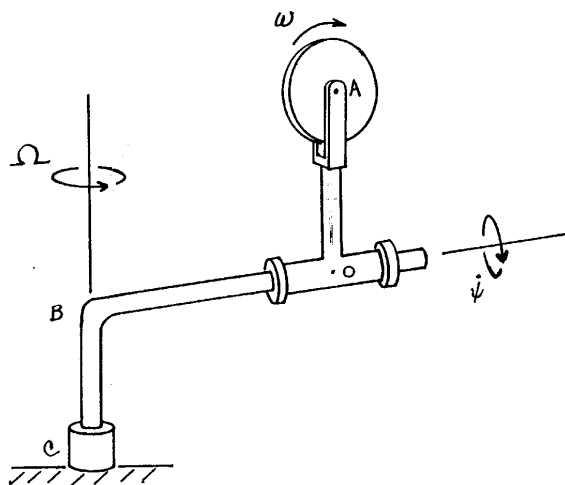


## 6.6 Practice Problems in Kinematics

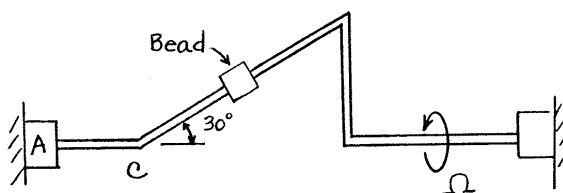
- The disk, which has a radius of  $R$ , spins about a pin at point A with a **constant** angular rate of  $\omega$ . The pin itself turns with shaft OA at a **variable** rate  $\dot{\psi}(t)$ . The arm BO also rotates at a constant rate  $\Omega$ , relative to an inertial observer. Determine a **general** expression for the angular acceleration of the disk relative to the inertial frame:

$$\frac{{}^I d^I \vec{\omega}^D}{dt}$$

in which  $D$  is an observer attached to the disk.

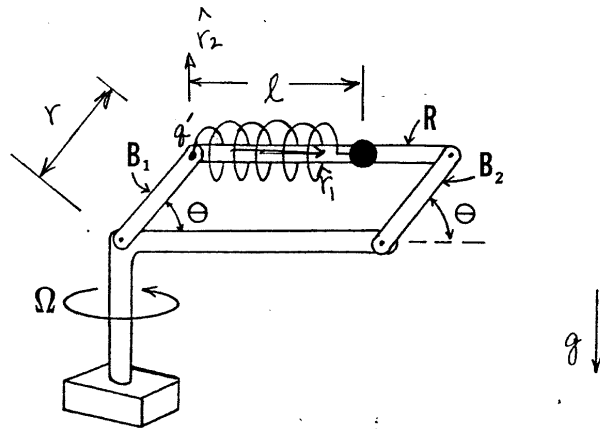


- A bead slides along the bent rod shown below, with a **constant** speed of 1 cm/s. The rod is rotating about the horizontal axis at a **constant** angular rate of  $\omega = 0.5 \text{ rad/sec}$ .



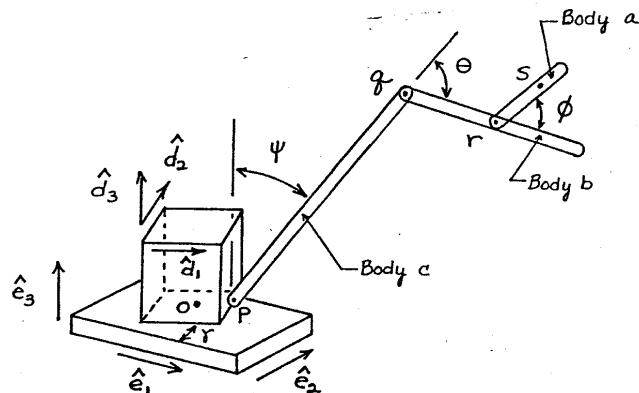
Write a **general** expression for the acceleration of the bead relative to the ground. Find the acceleration (ie., a **numerical value**) when the rod is vertical and the bead is 5 cm from point C.

- A bead of mass  $m$ , connected by a linear spring to point  $q'$  and constrained to slide along rod R as shown. The total length of the spring (time varying) is  $l$ . Rod R forms part of a four-bar mechanism, along with rods  $B_1$  and  $B_2$ , each of which have (fixed) length  $r$ . Pin joints exist at each of the 4 corners of this mechanism, so that as rod  $B_1$  rotates through the variable angle  $\theta(t)$ , rods  $B_1$  and  $B_2$  remain parallel.



Write a **general** expression for the acceleration of the mass  $m$  relative to an inertial reference (assumed for the ground). Define a convenient set of unit vectors to express this acceleration.

4. A power shovel is drawn in the picture below. The unit vectors  $\hat{d}_1, \hat{d}_2, \hat{d}_3$  are attached to the cab, which can rotate through an angle  $\gamma(t)$  relative to observer  $E$ , attached to the ground. The system  $\hat{e}_1, \hat{e}_2, \hat{e}_3$  is attached to observer  $E$ . The motion of jaws  $a$  and  $b$ , relative to the crane body  $c$ , is given by the angles  $\theta(t)$  and  $\phi(t)$ . Observers  $A, B$ , and  $C$  are attached to parts  $a, b$  and  $c$ , respectively.

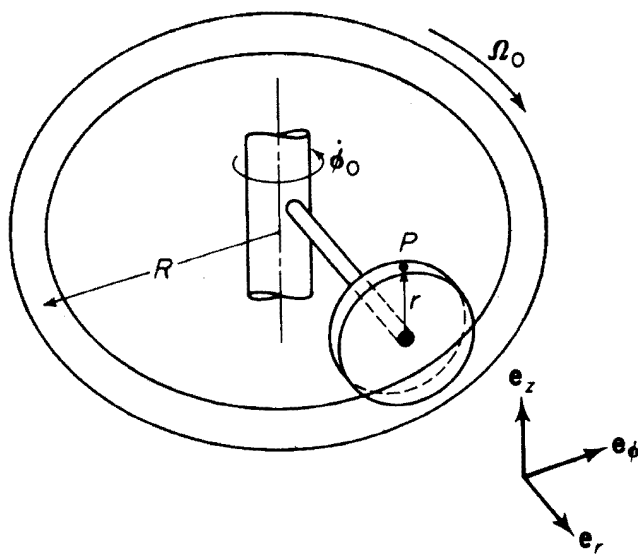


Derive a **general** expression for the vectors:

$${}^B\vec{V}^{Oq}, \quad {}^E\vec{V}^{Oq}, \quad \text{and} \quad {}^C\vec{a}^{Os}$$

Write these using the unit vectors  $\hat{b}_1, \hat{b}_2, \hat{b}_3$ .

5. A vertical disk of radius  $r$  rolls around a horizontal track of radius  $R$ . This track rotates at a constant rate of  $\Omega_0$  about the vertical axis. The center of the disk moves at a constant angular rate of  $\dot{\phi}_0$  about the same vertical axis. Using the set of unit vectors,  $\{\hat{e}_r, \hat{e}_\phi, \hat{e}_z\}$ , obtain the acceleration of point P, located on the edge of the disk when the point is at its highest position. (from Greenwood, *Principles of Dynamics*, 1988)



# The Terrestrial Reference Frame and the Dynamic Earth

As early as the 15th century, Swedes noticed that rocks in their harbors were slowly rising out of the sea [Ekman, 1991]. These local observations were not sufficient to distinguish whether the rocks were rising or the sea level falling. Later, it was realized that Fennoscandia was still rebounding from the last Ice Age. This historical observation is still relevant today. How can you know whether a point on the Earth's surface is slowly moving up, down, or horizontally? One must relate local measurements to a stable and accurate reference frame, one whose scale is much larger than the problem at hand. We remain concerned with sea-level variations, but present-day studies recognize that change must be measured from a global point of view and with respect to a globally well-defined reference frame. Thus, the regional and national geodetic datums developed over the past 200 years are inappropriate for studying global-level problems.

One of the main tasks of modern geodesy—the science of measuring and mapping the Earth's surface—is to define and maintain this global terrestrial reference frame. How well the reference frame can be realized has important implications for our ability to study both regional and global properties of the Earth, including post-glacial rebound, sea-level change, plate tectonics, regional subsidence and loading, plate boundary deformation, and Earth orientation excitation.

## What is a Terrestrial Reference Frame?

Two coordinate frames are commonly used: a rotating system fixed to the Earth's surface (terrestrial; used for most practical applications) and an essentially inertial system fixed to the stars (celestial; where dynamical equations of motion can be solved). The celestial reference frame was historically specified by the equator, ecliptic, and pole of rotation of the Earth and was realized by the two-dimensional coordinates of a large number of stars. The present-day International Celestial Reference Frame (ICRF) is defined by the coordinates of a much smaller set of essentially stationary quasars whose positions are far better known.

The terrestrial reference frame, on the other hand, was realized mostly through national conventions. In 1895, the International Latitude

Service (ILS) formed, marking the development of the first international group of globally distributed stations to define and monitor the evolution of the frame. In 1962, this group evolved into the much broader International Polar Motion Service (IPMS). The establishment of the International Earth Rotation Service (IERS) in 1988 shifted the responsibility for establishing and maintaining both frames to a single international authority. This was the genesis of the International Terrestrial Reference Frame (ITRF).

In a break with the past, IERS based its definition of the ITRF and ICRF on modern observational techniques: Doppler and laser tracking of satellites, very long baseline interferometry (VLBI) astrometry, and later, global positioning system (GPS) tracking. Although far more accurate than classical techniques, none of these techniques is sensitive to all degrees of freedom—translational, rotational, and scale—of a reference frame. We thus need to use the relevant information from each to define the frame in its entirety. Since one of the primary reasons behind establishing a unique global reference frame is to provide a common reference in comparing observations and results from different locations and epochs, “continuity” between the older realizations and the new ones is of utmost importance. Thus, it is not by chance that while the new frames do not rely on astronomical observations, the Greenwich meridian is still used as the “primary meridian” containing the  $x$  and  $z$  axes of the ITRF. Similarly, the  $z$ -axis is defined to coincide with the Conventional International Origin (CIO), which was defined by ILS as the mean pole of rotation over the period 1900–1905.

To relate the celestial and terrestrial reference frames, knowledge of the Earth's variable rotation vector in space is required. This includes motions with respect to both inertial space and the crust. Earth orientation variations, particularly those with respect to the crust, cannot be accurately predicted. Developments in space geodetic techniques over the past 2 decades now allow these processes to be monitored at a level comparable to the accuracy of the terrestrial and celestial frame realizations. IERS coordinates the programs to monitor the Earth orientation parameters that relate the ICRF and ITRF.

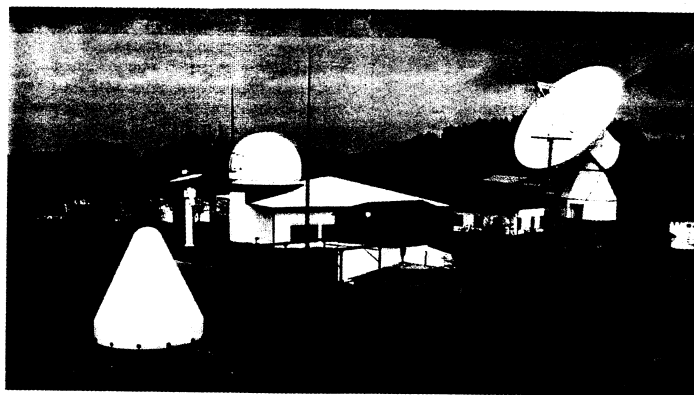


Fig. 1. ITRF sites in Wettzell, Germany, with GPS in the foreground, and SLR and VLBI in the background.

### How Is the ITRF Formed?

The ITRF is realized through the global Cartesian coordinates and linear velocities of a global set of sites equipped with various space geodetic observing systems, and it is maintained by participating agencies (Figure 1). It is assembled by combining sets of results from independent techniques as analyzed by a number of separate groups organized under the IERS and cooperating services. The space geodetic techniques used at present are lunar and satellite laser ranging (LLR, SLR), VLBI, GPS, and Doppler orbit determination and radio positioning integrated on satellites (DORIS). Each technique is organized as a service by the International Association of Geodesy. In addition to these observations, the frame also depends on the surveyed tie vectors that relate co-located systems at a subset of the ITRF sites. Without these ties, each geodetic technique would realize individual terrestrial frames rather than a single unified one. The complementarity of the independent techniques used by the IERS requires an integrated approach to achieve the highest possible accuracy and consistency for the ITRF. All techniques can contribute to the ITRF given appropriate weights and allowing for possible systematic differences. The advantage of using as many different techniques and solutions as possible is that the errors of the combined ITRF can be significantly smaller than for any of the individual contributors, if the error sources are largely independent. This also improves stability from one ITRF realization to the next and improves reliability.

However, it is expected that in many cases the dominant errors in individual solutions will be mainly systematic rather than random. This can make the determination of appropriate solution weights problematic. Since the ITRF should be as accurate as possible, and not merely stable from one realization to the next, care must be taken to ensure that the effects of systematic errors are identified and controlled as much as possible. Beginning with the ITRF94 realization, only those reference frame solutions that provide full variance-covariance information, accompanied by complete *a priori* constraint matrices are used. In this way, the ITRF is formed and maintained in a rigorous fashion. Alignments in orientation between successive updates rely on the full covariance information, which is also provided to the user community with the site coordinates and velocities.

Unlike some of the older terrestrial reference frames, the ITRF allows for the relative motions of sites on the Earth's surface, due to plate tectonics as well as other local effects. These observations of contemporary motions can be compared to plate motion models, which are based on geological data spanning the past 3 million years. Outside of plate boundary deformation zones, the rates generally agree very well. Difficulties do arise in certain cases, particularly when a station's motion is not at a constant velocity. Position offsets can be introduced to account for episodic events such as earthquakes, but the assumption of constant velocities is not always adequate for describing post-seismic motions. Such situations must be treated as special cases.

**Reference Frame (cont. on page 278)**

### Evolution of the ITRF to Present: ITRF2000

Space geodesy at the sub-decimeter level began with SLR and VLBI in the 1970s, and it benefitted particularly from the vigorous support of NASA's Crustal Dynamics Program. The first detection of contemporary intercontinental plate motion was one product of this program. In the mid-1980s, GPS began to be used for regional crustal deformation measurements, expanding quickly to form a global network of continuously operating receivers by the early 1990s. DORIS, which was accepted as an IERS technique in the mid-1990s and first used in the ITRF94 realization, enjoys an especially robust network of globally well-distributed sites. The work of combining contributed sets of coordinates and velocities from the various techniques and analyses is performed by the IERS Terrestrial Reference System Product Center, which is hosted by the Institut Géographique National in Paris. The first ITRF realization was prepared shortly after the founding of IERS in 1988. In the past decade, a new ITRF has been prepared approximately every 2 years.

In March of 2000, ITRF2000 was released. The locations of the ITRF2000 sites are shown in Figure 2; sites where collocated techniques are operating, such as shown in Figure 1, are also highlighted. ITRF2000 is the most extensive

and accurate terrestrial reference frame ever developed and includes positions and velocities for about 800 stations located at about 500 sites.

As stated above, a terrestrial reference frame requires the definition of its scale, its origin, and the orientation of the coordinate axes. The sensitivities of some techniques are better suited for observing certain aspects of the frame. For example, the scale of ITRF2000 was established by a combination of VLBI and SLR results. The orientation of the frame has been aligned with the preceding realization, ITRF97, and its orientation rate is defined, by convention, so that there is no net rotation of the frame with respect to the Earth's lithosphere. To do so, the ITRF2000 orientation rate is aligned to the geological tectonic model NNR-NUVEL-1A [Argus and Gordon, 1991]. Clearly, many of the sites in Figure 2 are in plate boundary zones and thus were not used to specify the no-net-rotation condition. Sites whose velocities showed significant discrepancies with respect to NNR-NUVEL-1A or were observed only briefly were also removed from the rotational constraint.

In addition to the rotational definition, the translational origin must also be precisely defined and maintained. For the ITRF, the origin is chosen to be the center of mass of the Earth, which is determined by the mass distribution of the solid Earth (including the Earth's interior), the oceans, and the atmosphere. This is the point about which

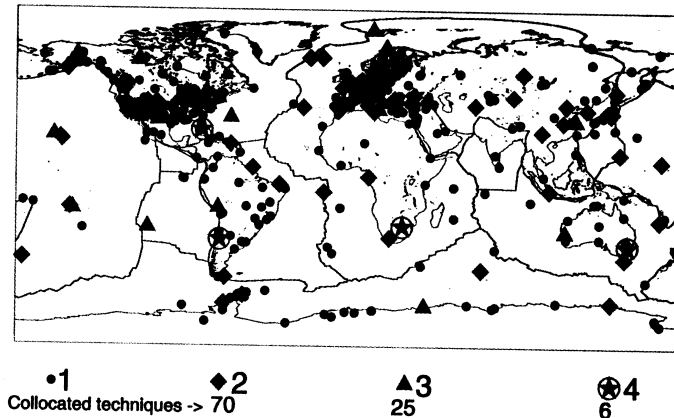


Fig. 2. Location of geodetic sites defined in ITRF2000.

a satellite dynamically orbits, although the tracking sites are all located on the lithosphere only, and so the network origin will not generally coincide with the center of mass [Watkins and Eanes, 1997]. ITRF2000 uses SLR tracking of the Lageos spacecraft to determine the translational origin of the reference frame. Only the linear evolution of the geocenter is currently modeled, but future realizations may include periodic variations as well.

A strength of ITRF2000 is its combination of station positions and velocities that are free from any external constraint, thus reflecting the actual precision of the space geodetic techniques. Although GPS was not used in the ITRF2000 origin or scale definition, it makes a large contribution in terms of the velocity field (about 150 IGS stations, as well as many regional GPS networks). DORIS, with its homogeneous network coverage, provides an excellent tracking system for low Earth orbiting satellites. Based on the internal consistency of the independent solutions included in ITRF2000, the global frame scale and origin stability over 10 years is estimated to be accurate in scale to better than 0.5 parts per billion. This is equivalent to a shift of about 3 mm in station height and better than 4 mm in origin. Maintenance and improvement of a global frame of this accuracy requires continued collection of high-quality geodetic data.

### How is a Reference Frame Used by Scientists?

Space geodesy is used to investigate a wide range of scientific questions and applications, particularly those related to crustal motions and the rotational dynamics of the Earth. In addition to the well-known tectonic deformations being monitored by space geodetic techniques, there are other geodynamic processes that are also measurable with space geodesy, including mantle elasticity and viscosity, structure and properties of the core and core-mantle boundary, and variations in water storage. In many cases, the surface expression of these processes can be less than 1 mm/yr. Current space geodetic technology is capable of measuring these rates, but only with fairly lengthy observation campaigns, ~5–15 years. To achieve the highest accuracy possible, the reference frame must be consistently defined throughout this observation period. The recent space geodetic studies of post-glacial rebound [Argus *et al.*, 1999; Johansson *et al.*, 2001] are just a few examples in which millimeter-per-year level vertical results depend heavily on the accuracy and stability of the terrestrial reference frame.

It is straightforward to see how measuring positions on the Earth's surface—given enough time—will provide the kinematic expression needed to study the dynamics of Earth deformation. What is perhaps less appreciated is how important the terrestrial reference frame is to other scientific experiments. One

Reference Frame (cont. on next page)

## Reference Frame

example is Topex/Poseidon, which uses SLR and DORIS as its primary tracking systems. Assessments of sea-level rise derived from Topex/Poseidon studies directly benefit from accurate ITRF coordinates being used for the SLR and DORIS tracking sites. Likewise, the upcoming JASON-1 radar altimeter mission, the Gravity Recovery and Climate Experiment (GRACE) mission to measure the Earth's time variable gravity, and the ICES mission to measure ice sheet elevation all require orbits that are accurately referenced to the Earth at the centimeter level, which in turn requires the precise reference frame that the ITRF provides. The precise orbit knowledge required by these missions is a critical prerequisite to their ability to significantly contribute to hydrology, oceanography, and glaciology.

## Maintaining Support for the Future

It is easy to overlook the importance of the terrestrial reference frame and take for granted that a well-determined frame will always be available. However, a considerable amount of analysis from a variety of institutions around the world is required to determine and maintain the modern terrestrial reference frame. This effort takes place quietly in the background but is nonetheless critical for much of today's high-accuracy geodesy-based results. Each technique brings unique contributions, and the past few decades have been devoted to improving the quality of the data and the analysis methods. As we approach, and hope to exceed, the part-per-billion level in knowledge of the Earth's shape, the importance of continuing the support of each of these techniques cannot be underestimated. Returning to our initial example, when we ask whether a tide gauge is sinking or the sea level is rising, and we want to know this better than the few millimeter-per-year level, only an accurate, stable global terrestrial reference frame will give the answer.

For more information on space geodesy and reference frames, refer to International Terrestrial Reference Frame, ITRF (<http://lareg.ensg.ign.fr/ITRF/>); International Earth Rotation Service, IERS (<http://www.iers.org/>); International GPS Service, IGS (<http://igs.csb.jpl.nasa.gov/>); International VLBI Service for Geodesy and Astrometry (<http://ivscc.gsfc.nasa.gov/>); International DORIS Service, IDS (<http://ids.cls.fr/welcome.html>); and International Laser Ranging Service, ILRS (<http://ilrs.gsfc.nasa.gov/>).

## Acknowledgments

The ITRF Working Group thanks the many institutions and individuals that contributed to ITRF2000 (<http://lareg.ensg.ign.fr/ITRF/ITRF2000/submissions.html>). We also thank Wolfgang Schlüter for providing Figure 1 and Duncan Agnew for reviewing the article.

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## References

- Argus, D., and R. Gordon, No-net rotation model of current plate velocities incorporating plate motion model NUVEL-1, *Geophys. Res. Lett.*, **18**, 2039–2042, 1991.
- Argus, D. F., W. R. Peltier, and M. M. Watkins, Global isostatic adjustment observed using very long baseline interferometry and satellite laser ranging, *J. Geophys. Res.*, **104**, 29,077–29,083, 1999.
- Ekman, M., A concise history of postglacial land uplift research (from its beginning to 1950), *Terra Nova*, **3**, 358–365, 1991.
- Johansson, J. M., et al., Continuous GPS measurements of postglacial adjustment in Fennoscandia, 1. Geodetic results, *J. Geophys. Res.*, in press, 2001.
- Watkins, M. M., and R. J. Eanes, Observations of Tidally Coherent Variations in the Geocenter, *Geophys. Res. Lett.*, **24**, 2231–2234, 1997.

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Insert Likins pgs 138 to 141 here.

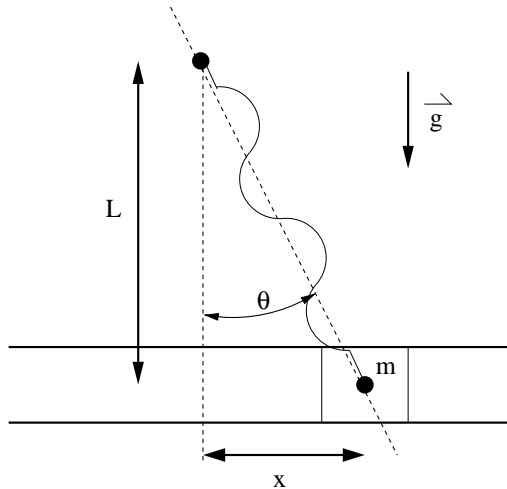


## Chapter 7

# Dynamics

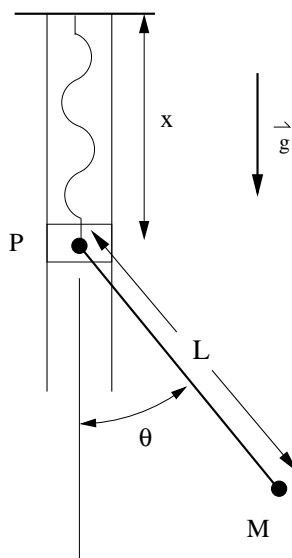
### 7.1 Practice Problems in Dynamics

1. On the following mechanics, the mass can slide along frictionless rails, a distance given by the variable  $x$ . A linear spring with constant  $K$  and unextended length  $L$  is attached to the mass and to a fixed point on the ceiling as shown.

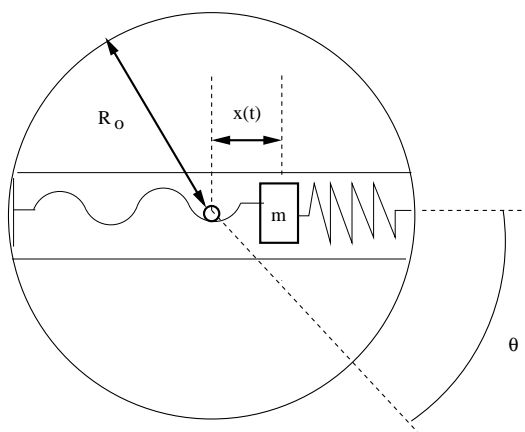


- (a) Derive the **equation(s) of motion** for this system.
- (b) How many **degrees of freedom** does this system have ?
- (c) How many **states** does this system have ?
- (d) Write the E.O.M.'s in **state variable** form.

2. Derive the equations of motion for the following mechanism. The length of the pendulum,  $L$  is constant, and it can rotate through the angle  $\theta$  as shown. The massless slider at point P can move along the frictionless rails, extending the spring a length of  $x$ . The spring constant is  $K$  and the un-stretched length of the spring is  $l$ . Assume a uniform gravity field with acceleration of  $\vec{g}$  in the direction shown in the drawing.



- Derive the **equation(s) of motion** for this system.
  - How many **degrees of freedom** does this system have ?
  - How many **states** does this system have ?
  - Write the E.O.M.'s in **state variable** form.
3. The following mechanism lies in the horizontal plane (such that the acceleration of gravity is normal to the page) and can rotate freely through the angle  $\theta$ . The mass can slide along frictionless rails, and is attached to 2 springs as shown. Each spring has a spring constant of  $K$  and an unextended length of  $L$ .

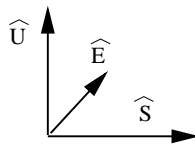
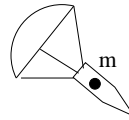


- Derive the **equation(s) of motion** for this system.
- How many **degrees of freedom** does this system have ?
- How many **states** does this system have ?
- Write the E.O.M.'s in **state variable** form.

4. A model rocket, following flight, has deployed its parachute and is now descending to the ground as shown. Assume a uniform acceleration of gravity, and that the surface of the Earth can be considered an inertial observer. The force of aerodynamic drag on the parachute is given by:

$$\mathbf{F}_D = -c \left| I\vec{V}^{PR} \right|^2 \frac{I\vec{V}^{PR}}{\left| I\vec{V}^{PR} \right|}$$

The position vector from the launch pad to the rocket ( $\vec{R}^{PR}$ ) is given using the unit vectors  $\hat{S}$ ,  $\hat{E}$ ,  $\hat{U}$ .



- Derive the **equation(s) of motion** for this system.
- How many **degrees of freedom** does this system have ?
- How many **states** does this system have ?
- Write the E.O.M.'s in **state variable** form.

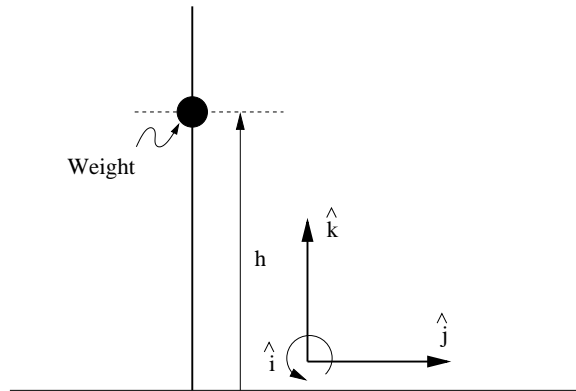
## Chapter 8

# Numerical Integration in MATLAB

Equations of Motion (E.O.M's) have been introduced at this point in the course. Recall that these are **differential equations**, (D.E.'s) which describe the evolution of the **state** of a mechanical system, given the **initial conditions** at some time.

In your differential equations course, you will see more of the theory behind D.E.'s and some methods to analytically obtain a solution for certain types of D.E.'s. In most engineering problems, however, it is not possible or practical to obtain a closed form solution. Therefore, in this course, we will look at how to apply computational methods and obtain a **numerical** solution to an E.O.M. To illustrate this, we choose a simple example problem for which the analytical solution is easily found. This will allow us to compare the results from each method.

First, derive the E.O.M. for a weight attached to a rail such that it is constrained to fall only in the vertical direction, under a uniform gravity field. This problem is illustrated below:



The E.O.M. for this problem can easily be found to be:

$$\ddot{h} = -g$$

In which the position of the weight is given by  $\vec{R}^{OP} = x\hat{k}$  and  $g = -9.8\text{m/sec}^2$  is the acceleration of gravity. This problem has one degree of freedom (motion in the  $\hat{k}$  direction) and is second order.

The first step in numerically integrating E.O.M.'s (and doing a lot more, which you will learn about in later courses) is to write all of the E.O.M.'s in **state variable form**. Since the E.O.M. is a 2nd order D.E. this will produce 2 first order D.E.'s. We do this by defining the states:

$$\begin{aligned}x_1 &= h \\x_2 &= \dot{h}\end{aligned}$$

This gives the state variable equations as:

$$\dot{x}_1 = x_2 \quad (1)$$

$$\dot{x}_2 = -g \quad (2)$$

Note that this has the form of a function, in which the input is the state ( $x_2$  in this case) and the output is the first derivatives of the states ( $\dot{x}_1$  and  $\dot{x}_2$ ).

MATLAB has a built-in function, ode45 which performs numerical integration of systems of equations in state variable form. The syntax of this function is described in the following excerpts from the online help pages:

$[T, Y] = \text{ode45}(\text{ODEFUN}, \text{TSPAN}, Y0)$  with  $\text{TSPAN} = [T0 \text{ TFINAL}]$  integrates the system of differential equations  $\frac{dy}{dt} = f(t, y)$  from time  $T0$  to  $\text{TFINAL}$  with initial conditions  $Y0$ . Function  $\text{ODEFUN}(T, Y)$  must return a column vector corresponding to  $f(t, y)$ . Each row in the solution array  $Y$  corresponds to a time returned in the column vector  $T$ . To obtain solutions at specific times  $T0, T1, \dots, \text{TFINAL}$  (all increasing or all decreasing), use  $\text{TSPAN} = [T0 \ T1 \ \dots \ \text{TFINAL}]$ .

Now, we will apply this to the falling weight problem above. First, a matlab function in the file “eom.m” is created which contains the E.O.M.’s in equation 2 above. This function takes two variables as input: time ( $t$ ) and one  $2 \times 1$  matrix ( $x$ ) that contains the states  $x_1$  and  $x_2$ . It produces one variable as output: a  $2 \times 1$  matrix ( $\text{xdot}$ ) that contains the first derivative of each state.

(Listing of function eom.m)

```
function xdot = eom(t, x)
xdot(1) = x(2);
xdot(2) = -9.8;
xdot = xdot';
return
```

The **name** of the function containing the E.O.M.’s (in this case “eom”) is given as an input to ode45. A short script file “runeom” shows this syntax.

(Listing of runeom.m)

```
x0 = [p0, v0]; % initial conditions
ti = [t0, tf];
[t, xhist] = ode45('eom', ti, x0);
phist = xhist(:,1);
vhist = xhist(:,2);
```

In this script, the initial conditions for the position ( $h(0)$ ) and speed ( $\dot{h}(0)$ ) of the weight are given as  $p0$  and  $v0$  respectively. These are placed into the  $2 \times 1$  array  $x0$ , representing the state at time 0. This is one input to ode45. The other input is the array  $Ti$ , containing the start and end times. The outputs are two arrays:  $xhist$  which contains an  $n \times 2$  array, each row gives the values of the 2 states at time steps between the start and end time.  $t$  is an  $n \times 1$  array which gives the values for the time corresponding to when the states in  $xhist$  occur.

In the following example, this is demonstrated with initial conditions of  $h(0) = 100\text{m}$  and  $\dot{h}(0) = 0\text{m/s}$  (the weight is lifted 100 meters up and released). This is integrated for 4 second.

```

>> t0 = 0

t0 =

    0

>> tf = 4

tf =

    4

>> p0 = 100

p0 =

   100

>> v0 = 0

v0 =

    0

>> runeom

```

After `runeom` completes, the arrays `t` and `xhist` are checked and it is found that 65 time steps were generated. The two states are then plotted as functions of time, in the following graphs:

```

>> size(t)

ans =

    65     1

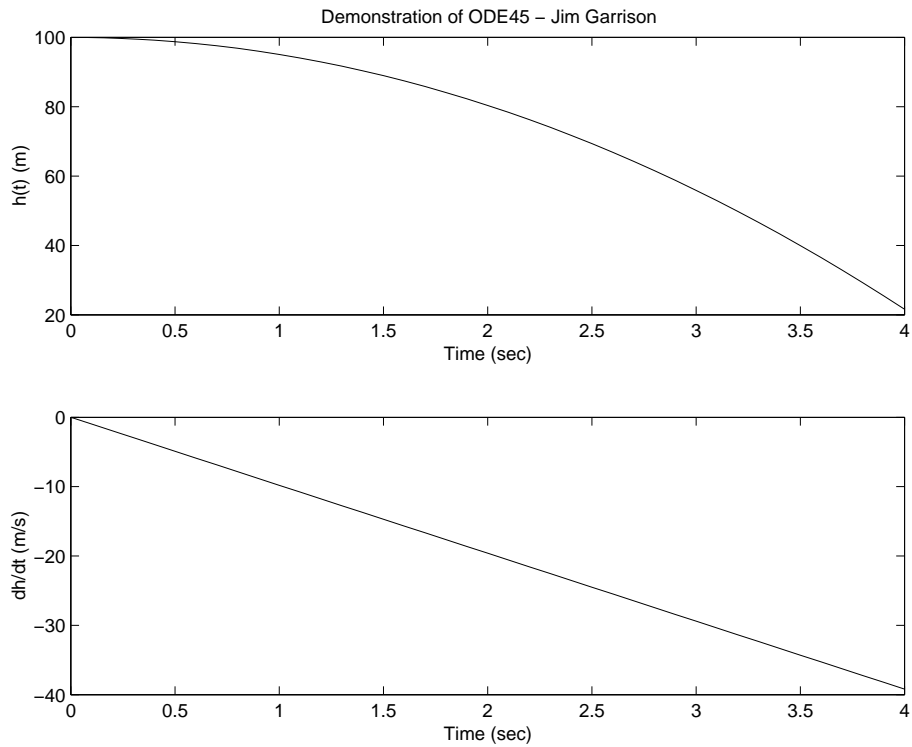
>> size(xhist)

ans =

    65     2

>> subplot(2,1,1)
>> plot(t, xhist(:,1))
>> ylabel('h(t) (m)')
>> xlabel('Time (sec)')
>> title('Demonstration of ODE45 - Jim Garrison')
>> subplot(2,1,2)
>> plot(t, xhist(:,2))
>> xlabel('Time (sec)')
>> ylabel('dh/dt (m/s)')

```



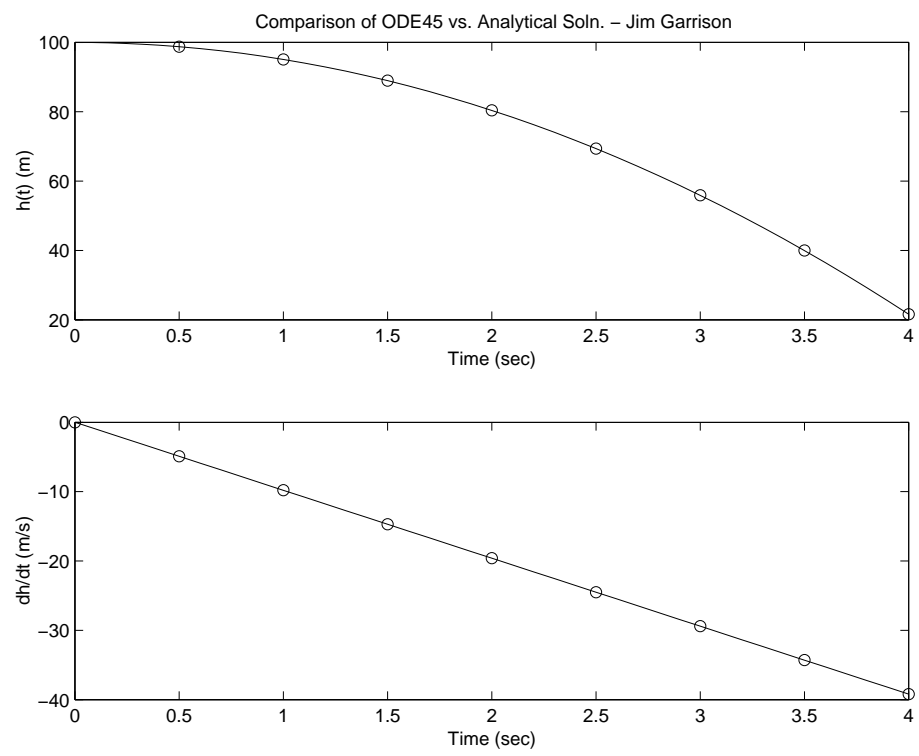
The analytical solution to this problem can be easily found to be:

$$h(t) = h(0) + \dot{h}(0)t - \frac{1}{2}gt^2$$

$$\dot{h}(t) = -gt$$

This function is computed and plotted on top of the solution from ode45 and, as expected, the two give the same results.

```
>> t_calc = [0:0.5:4];
>> h_calc = p0 + v0 * t_calc - 0.5 * (9.8) * t_calc.^2;
>> hdot_calc = -9.8 * t_calc;
>> subplot(2,1,2)
>> plot(t, xhist(:,2), t_calc, hdot_calc, 'o')
>> subplot(2,1,1)
>> plot(t, xhist(:,1), t_calc, h_calc, 'o')
>> xlabel('Time (sec)')
>> ylabel('h(t) (m)')
>> title('Comparison of ODE45 vs. Analytical Soln. - Jim Garrison')
>> subplot(2,1,2)
>> xlabel('Time (sec)')
>> ylabel('dh/dt (m/s)')
```





## Chapter 9

# Final Thoughts

Of the thousands of ways computers have been misused, *simulation* tops the list. The reasons for this widespread abuse are not hard to find:

1. Every simulation simulates *something*, but there's no particular reason it should simulate what the simulator had in mind.
2. Computer outputs are readily mistaken for gospel, especially by people who are working in the dark and seeking any sort of beacon.
3. Simulation languages have succeeded in making it easier to achieve impressive simulations, without making it easier to achieve valid simulations.
4. There are no established curricula based on extensive practical experience. Thus, everyone is an "expert" after writing one simulation, of anything, in any language, with any sort of result.
5. The promise of simulation is so great that it's easy to confuse hope with achievement.

- *Simulation: Principles and Methods*, Graybeal and Pooch, 1980