Computational Anatomy

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Statistical Learning Study Group

Overview

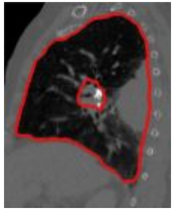
- Computational Anatomy?
- Basic concepts
- Diffeomorphic image registration → key technique!
  - LDDMM
  - Geodesic shooting
  - Stationary velocity fields
- Statistics on nonlinear manifolds

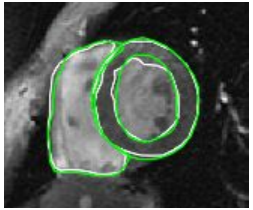
What is Computational Anatomy?

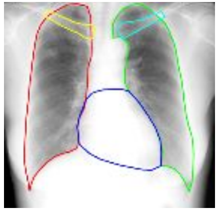
Definition by Michael I. Miller:

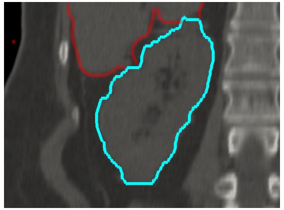
> *Computational Anatomy* is the mathematical study of anatomy […] under groups of diffeomorphisms [...] of anatomical exemplars.

Anatomical shapes









MI Miller: Computational anatomy: shape, growth, and atrophy comparison via diffeomorphisms, NeuroImage 2004

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Two new questions:

1. Mathematical study of anatomy? → Anatomical shapes and their variability
2. Groups of diffeomorphisms?

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Anatomical Shapes and their Variability

Traditional field of study in Medicine & Biology

- Dates back to ancient times
- e.g. based on dissected bodies

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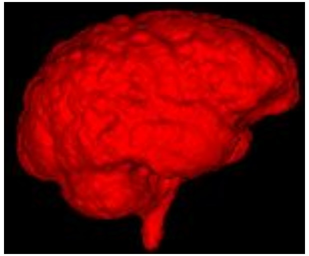
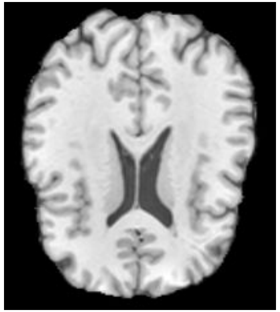
Nowadays: Computer-based shape representations



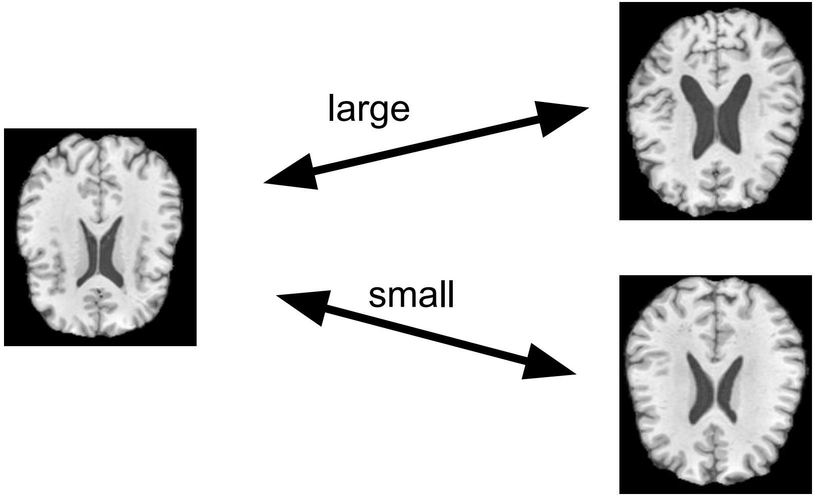
Explicit: Surface mesh



Implicit: Image volume

What do we need to perform statistics?

Pairwise distances:

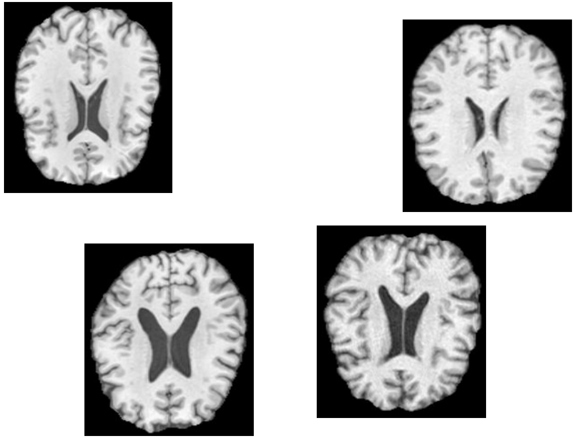


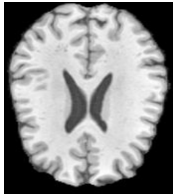
How to quantify shape similarity/dissimilarity?

What do we need to perform statistics?

Pairwise distances

Population studies (mean, variability, classification, ...):





Average shape

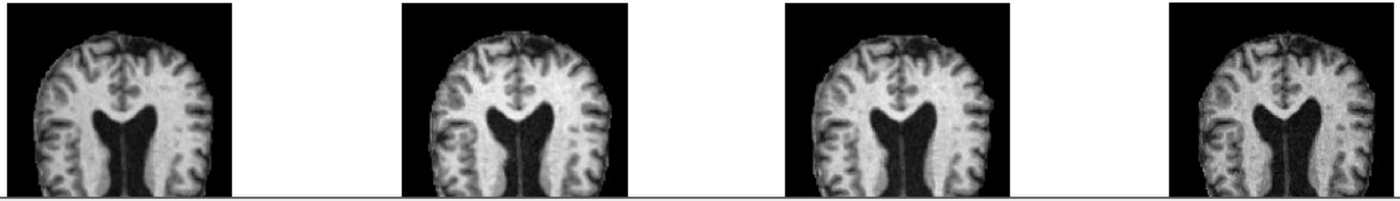
PCA, ICA, ... ?

What do we need to perform statistics?

Pairwise distances

Population studies (mean, variability, classification, ...)

Longitudinal studies (description/quantification of temporal development):

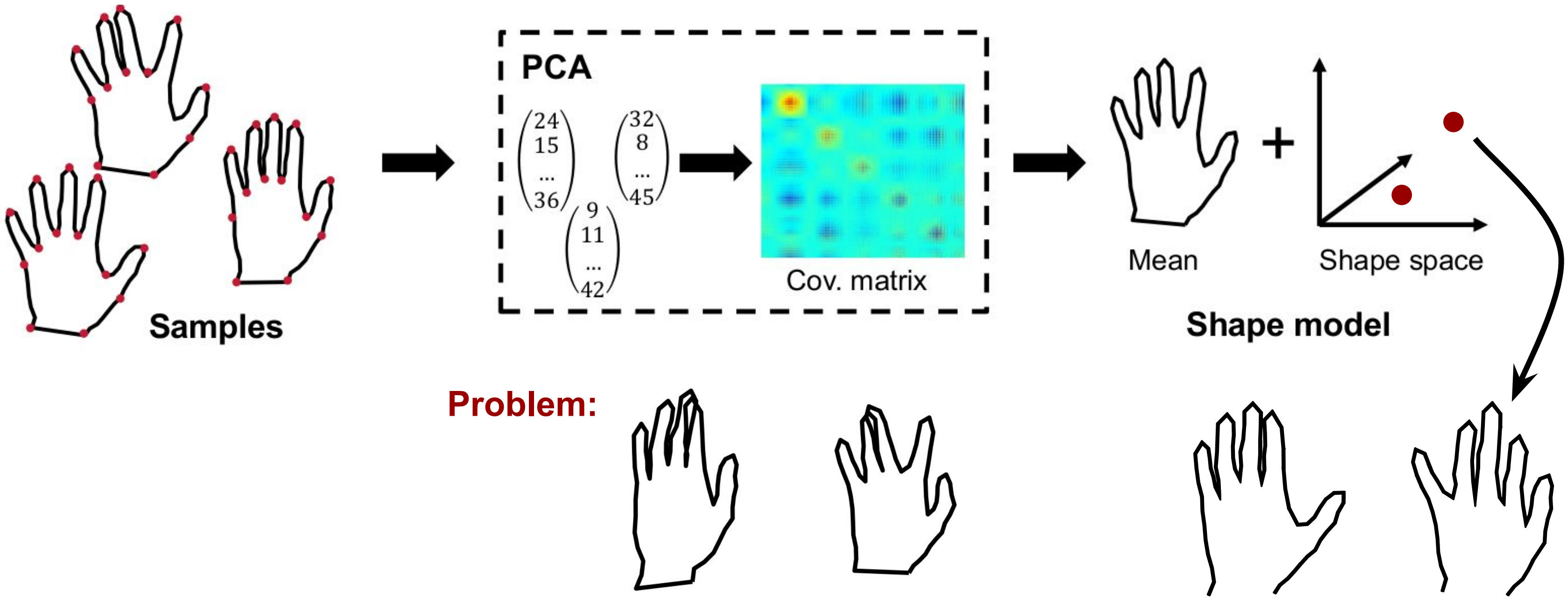


Computational Anatomy provides the mathematical tools to achieve this!

Image source: Hong et al.: Simple Geodesic Regression for Image Time-Series, WBIR 2012

Example: Statistical Shape Models

Widely used PCA-based modeling technique

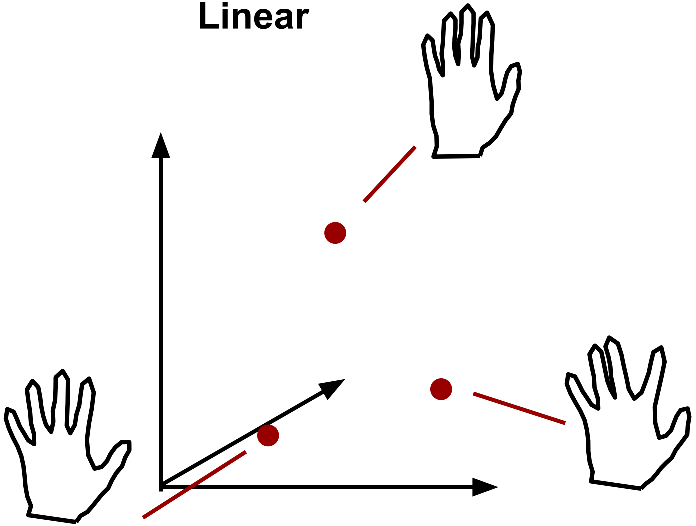


Cootes et al.:Training Models of Shape from Sets of Examples, BMVC, 1992

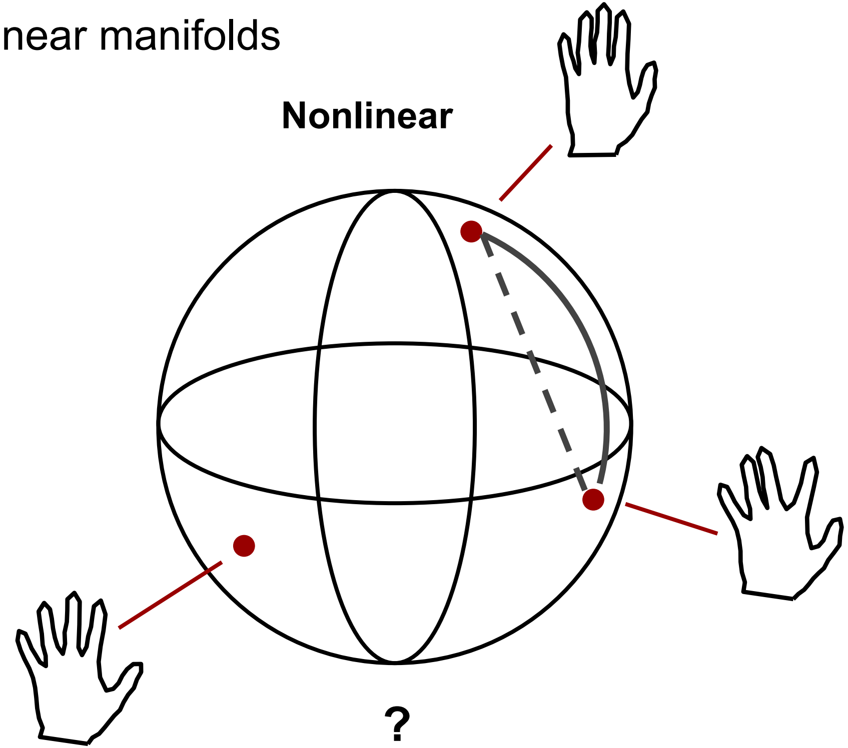
Example: Statistical Shape Models (2)

Problem: Shapes often “live” on or close to nonlinear manifolds

Linear vs. **Nonlinear**



Statistics based on Euclidean distances



?

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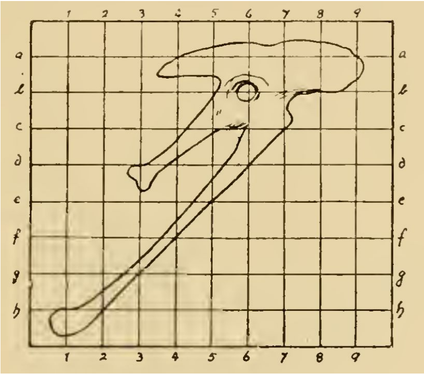
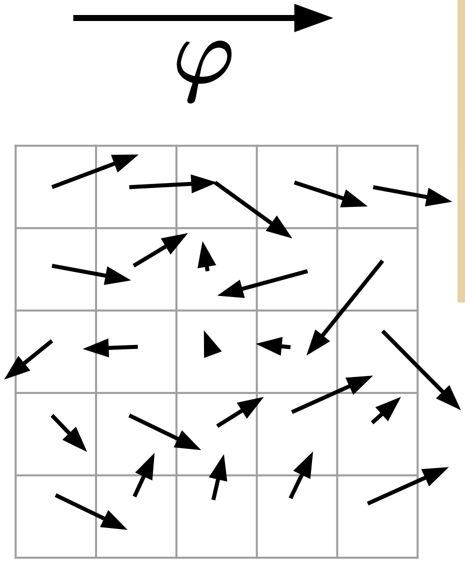
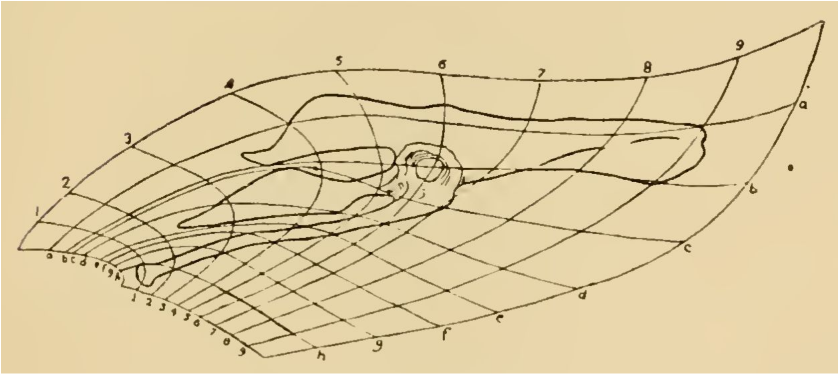
Basic Ideas of Computational Anatomy

1. Shape differences should be described via spatial transformations:







Pelvis of Archaeopteryx

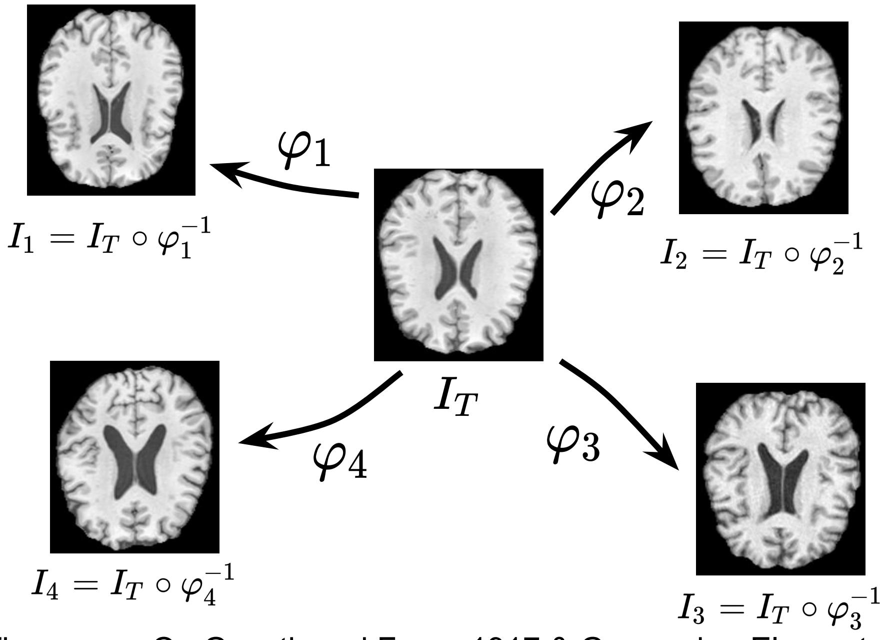
Pelvis of Apatornis

Thompson: On Growth and Form, 1917 & Grenander: Elements of Pattern Theory, 1996

Basic Ideas of Computational Anatomy

1. Shape differences can be described via spatial transformations
2. Deformable template orbit model:

$$\Omega \subset \mathrm{R}^3$$
$$I : \Omega \to \mathrm{R}$$
$$\varphi : \Omega \to \Omega$$



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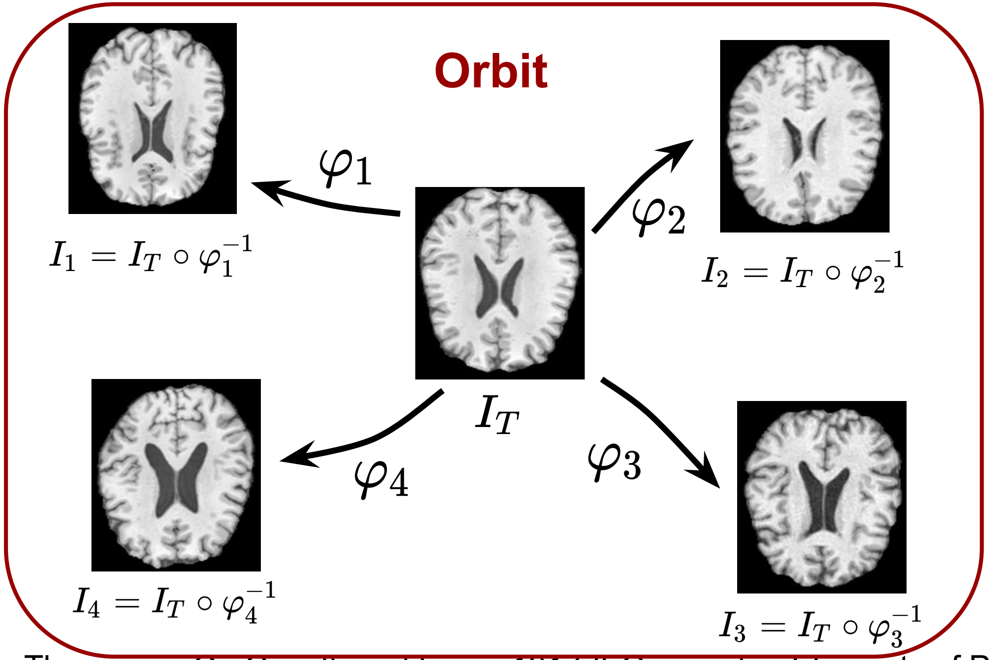


$$\Omega \subset \mathrm{R}^3$$
$$I : \Omega \to \mathrm{R}$$
$$\varphi : \Omega \to \Omega$$

Set of diffeomorphisms

Orbit:

$$O_{I_T} := \{ I_T \circ \varphi^{-1} | \varphi \in \mathrm{Diff}(\Omega) \}$$

$$(\mathrm{Diff}(\Omega), \circ)$$

Group of diffeomorphic transformations

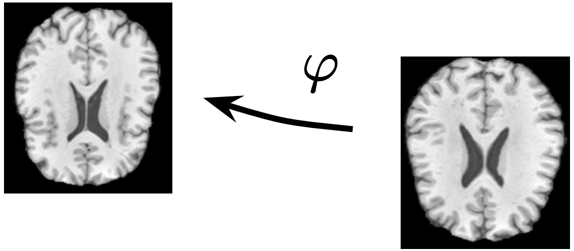
Thompson: On Growth and Form, 1917 & Grenander: Elements of Pattern Theory, 1996

Diffeomorphic Transformations

Properties of $\varphi : \Omega \rightarrow \Omega$:

- Bijection → one-to-one & onto
- differentiable → smooth
- Inverse φ^{-1} exists & is differentiable
- Id is a diffeomorphism

$\Rightarrow \varphi$ is topology preserving!



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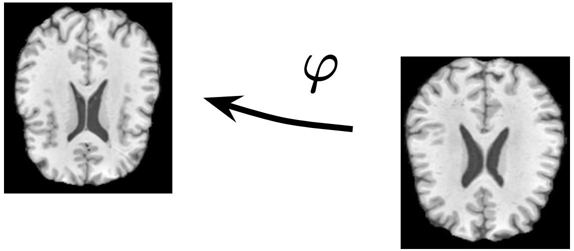
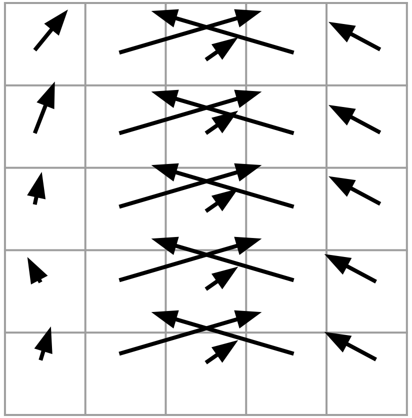


$\Rightarrow \varphi$ is topology preserving!

➢ No local changes of orientation (folding, ...)
➢ No new/disappearing structures
➢ …



Diffeomorphic Transformations (2)

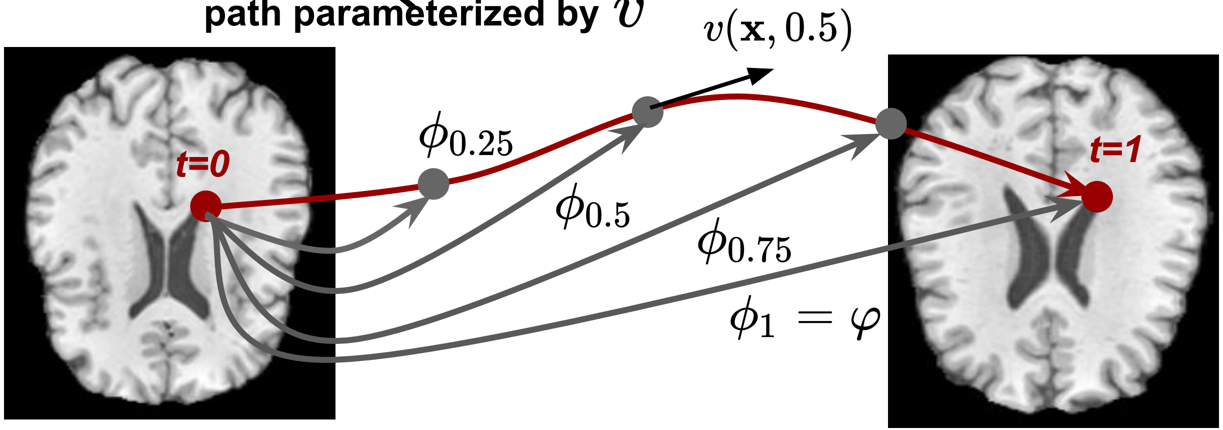
Elements of $\mathrm{Diff}\,(\Omega)$ can be generated via smooth flows ϕ_t

$$\frac{\partial}{\partial t}\phi_t(\mathbf{x}) = v(\phi_t(\mathbf{x}), t) \quad \text{with } \phi_0(\mathbf{x}) = \mathbf{x} \text{ and } \; t \in [0,1]$$

smooth, time-dependent velocity field!

$$\varphi(\mathbf{x}) = \phi_1(\mathbf{x}) = \phi_0(\mathbf{x}) + \int_0^1 v(\phi_t(\mathbf{x}), t)dt$$

path parameterized by v



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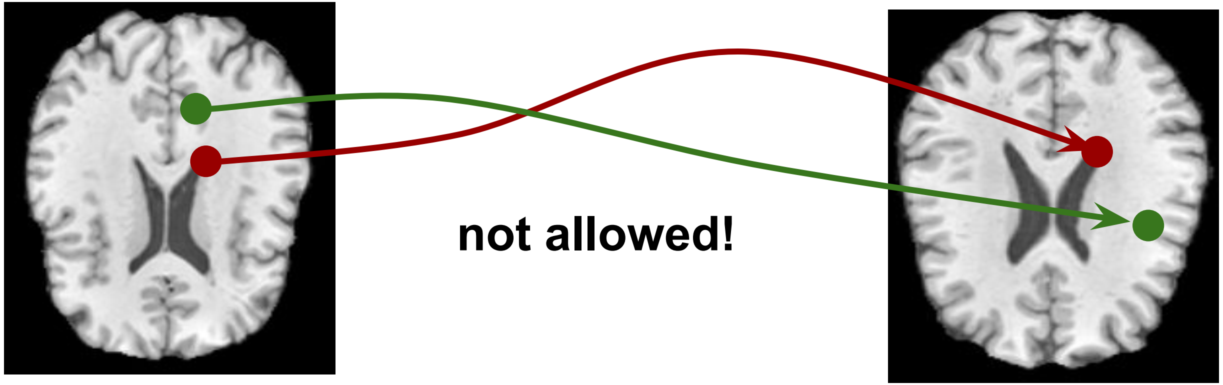
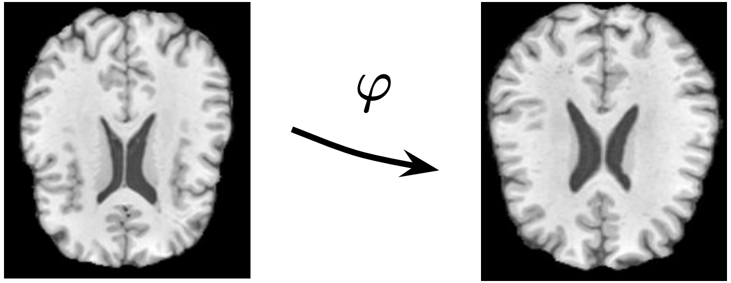


Diffeomorphic Image Registration

Goal: Estimate a $\varphi = \phi_1 \in \mathrm{Diff}(\Omega)$ that *optimally* aligns I_1 and I_2

Optimization problem:



I_1 I_2

$$\arg\min_{v} \; \mathcal{D}(I_1 \circ \varphi^{-1}, I_2) + \alpha \mathcal{S}(v)$$

$$\text{s.t } \frac{\partial}{\partial t}\phi_t = v_t \; \forall t \in [0,1] \text{ and } \phi_0 = Id$$

⇒ Large Deformation Diffeomorphic Metric Mapping (LDDMM)

Beg et al.: Computing Large Deformation Metric Mappings via Geodesic Flows, IJCV 2005

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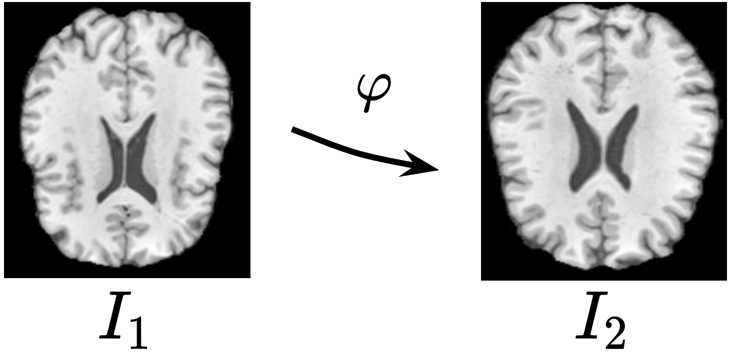


Distance measure (SSD, ...)

Regularizer

$$\int_0^1 \|v_t\|_V^2 \, dt = \int_0^1 \|Lv_t\|_2^2 \, dt$$

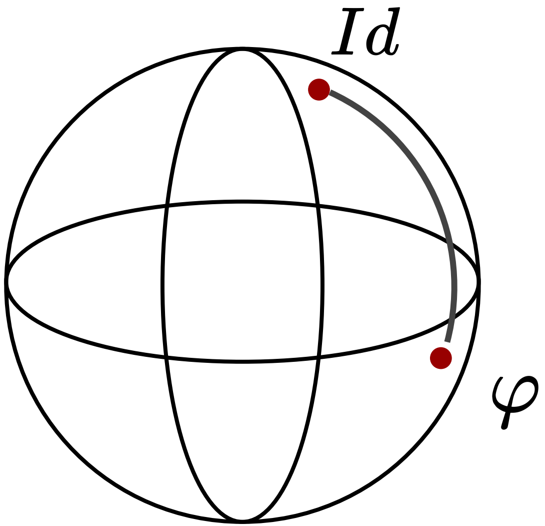
$\Rightarrow$ geodesic shortest length path!

Beg et al.: Computing Large Deformation Metric Mappings via Geodesic Flows, IJCV 2005 shortest path in a manifold

Geodesic Distances

Distance between an optimal φ and Id:

$$d(Id, \varphi)^2_V = \int_0^1 \|Lv_t\|_2^2 \, dt$$



Younes et al.: Evolutions Equations in Computational Anatomy, NeuroImage 2008

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Distance between two diffeomorphisms:

$$d(\varphi_1,\varphi_2)^2_V = d(Id,\varphi_2 \circ \varphi_1^{-1})^2_V$$

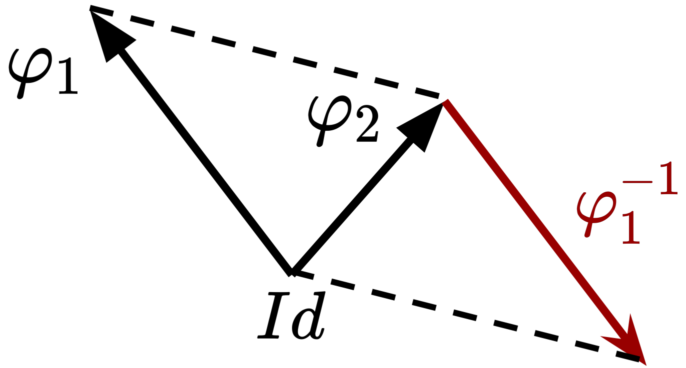

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Image registration!

Distance between two images:

$$d(I_1, I_2)_V^2 = \inf_{\varphi} \{d(Id, \varphi)_V^2 : I_1 \circ \varphi^{-1} = I_2\}$$

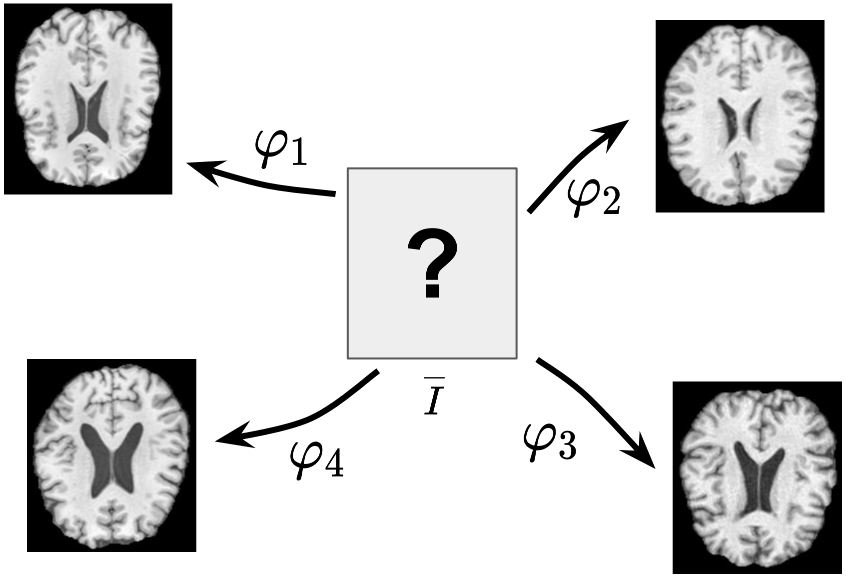
Younes et al.: Evolutions Equations in Computational Anatomy, NeuroImage 2008

Statistics based on Geodesic Distances

Average shape (first order moment)

- no closed form solution!
- Alternative: Fréchet mean

$$\bar{I} = \arg\min_{I} \sum_{i=1}^{N} d(I, I_i)^2$$



Joshi et al.: Unbiased diffeomorphic atlas construction for computational anatomy, NeuroImage 2004

Statistics based on Geodesic Distances

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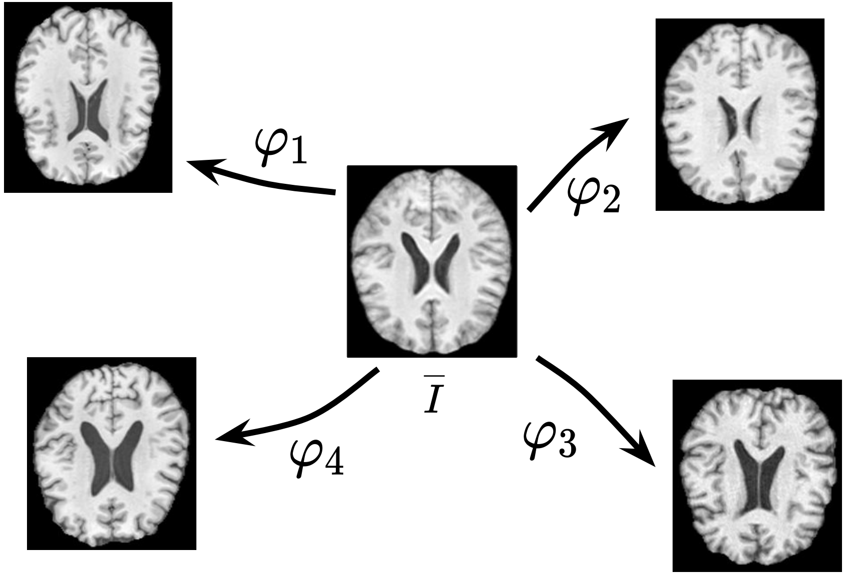
$$\{\bar{v}_i, \bar{I}\} = \arg\min_{v_i, I} \sum_{i=1}^{N} \mathcal{D}(I \circ \varphi_i^{-1}, I_i) + \alpha \mathcal{S}(v_i)$$

Template of an orbit!



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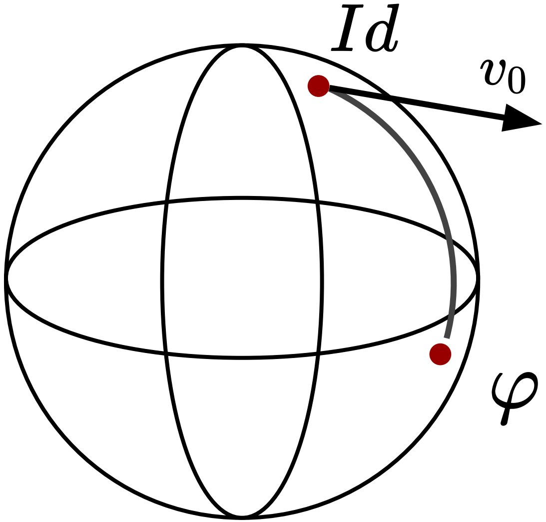
Geodesic Shooting for LDDMM

Problem: Statistics based on original LDDMM formulation difficult to use

Alternative: Flow can be parameterized by initial velocity (momentum)!



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$$\int_0^1 \|v_t\|_V^2 \, dt = \int_0^1 \|Lv_t\|_2^2 \, dt$$

Kinetic energy spent during the deformation

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Kinetic energy spent during the deformation

$$m_t = Lv_t$$

Kinetic momentum

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Geodesic Shooting for LDDMM

Problem: Statistics based on original LDDMM formulation computationally expensive

Alternative: Flow can be parameterized by initial velocity (momentum)!

$$\int_0^1 \|v_t\|_V^2 \, dt = \int_0^1 \|Lv_t\|_2^2 \, dt$$

Kinetic energy spent during the deformation

$$m_t = Lv_t$$

Kinetic momentum

Assumption: No external forces = conservation of momentum

$$v_t = L^{-1} J(m_0 \circ \varphi_t^{-1})$$

Miller et al.: Geodesic Shooting for Computational Anatomy, J Math Vis Imaging 2006

Reformulation of LDDMM

$$\arg\min_{v} \; \mathcal{D}(I_1 \circ \varphi^{-1}, I_2) + \alpha \int_0^1 \|Lv_t\|_2^2 \, dt$$

becomes

$$m_0 = Lv_0$$

$$\arg\min_{v_0} \; \mathcal{D}(I_1 \circ \varphi^{-1}, I_2) + \alpha \|Lv_0\|_2^2$$

Ashburner et al.: Diffeomorphic registration using geodesic shooting and Gauss–Newton optimisation, NeuroImage 2011

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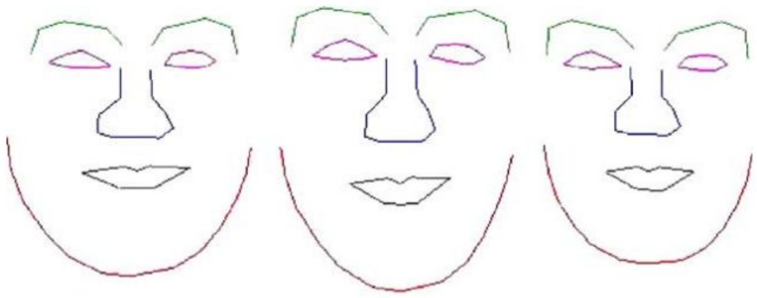
Advantage:

Space of initial momentum fields (velocity fields) <u>is linear</u>! → Euclidean statistics

Ashburner et al.: Diffeomorphic registration using geodesic shooting and Gauss–Newton optimisation, NeuroImage 2011

Example: Diffeomorphic Shape Models

- Mean shape and PCA computed on initial momenta
- Correctly handle the nonlinearity of the shape manifold



1st eigenvector   Mean shape   3rd eigenvector

Possible linear combinations

Standard:

Diffeomorphic:

Vaillant et al.: Statistics on diffeomorphisms via tangent space representations, NeuroImage 2004

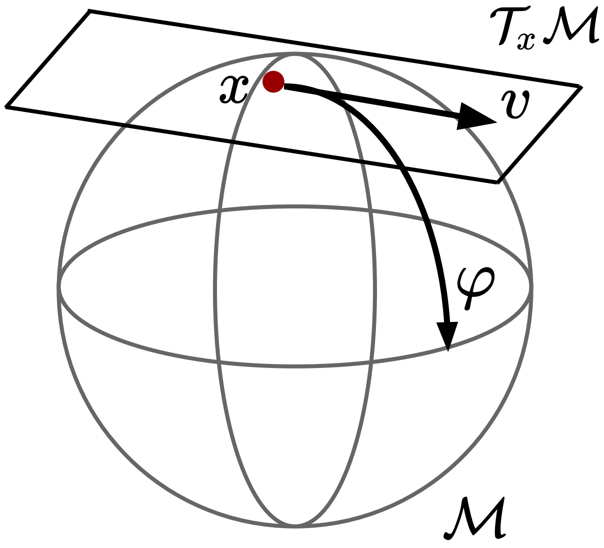
Basic Concepts of Riemannian Geometry

Some properties:

- Smooth manifold $\mathcal{M}$ (e.g. sphere)
- Tangent space $\mathcal{T}_x\mathcal{M}$ defined at $x \in \mathcal{M}$
- Tangent space and manifold are connected via maps:

$$\exp_x : \mathcal{T}_x\mathcal{M} \to \mathcal{M}$$
$$\log_x : \mathcal{M} \to \mathcal{T}_x\mathcal{M}$$

- Tangent spaces are linear!



Log-Euclidean Framework

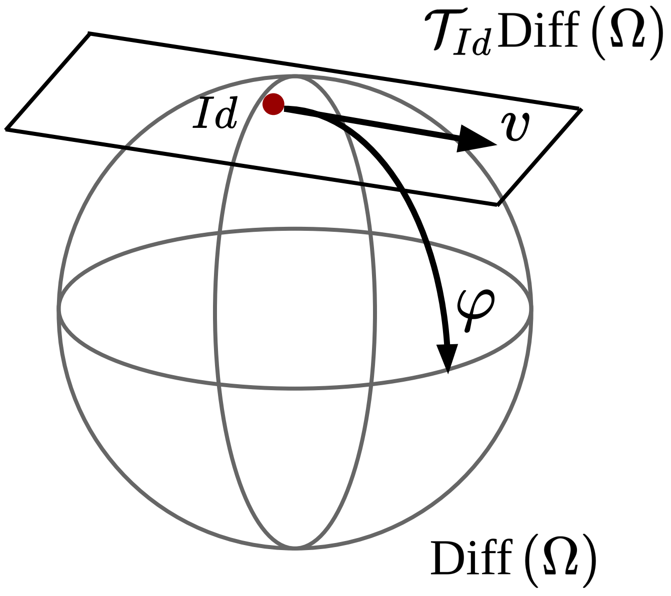
Key ideas:

- Parameterization of diffeomorphisms using stationary velocity fields

$$\frac{\partial}{\partial t}\phi_t(\mathbf{x}) = v(\phi_t(\mathbf{x}), \cancel{t})$$

- $(\mathrm{Diff}(\Omega), \circ)$ interpreted as a Lie group
- Lie algebra is isomorphic to $\mathcal{T}_{Id}\mathrm{Diff}(\Omega)$

$$\varphi = \phi_1 = \exp_{Id}(v)$$
$$v = \log_{Id}(\varphi)$$
$$\varphi^{-1} = \exp_{Id}(-v)$$



Arsigny et al.: A Log-Euclidean Framework for Statistics on Diffeomorphisms, MICCAI 2006

Image Registration based on Stationary Velocities

Energy minimization:

$$\arg\min_{v} \mathcal{D}(I_1 \circ \varphi^{-1}, I_2) + \alpha \mathcal{S}(v)$$

$$\exp_{Id}(-v)$$

Vercauteren et al.: Diffeomorphic demons: efficient non-parametric image registration, NeuroImage 2009

Image Registration based on Stationary Velocities

Energy minimization:

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$$\exp_{Id}(-v)$$

Exponentiation can be solved efficiently using “Scaling and Squaring”!

$$\exp_{Id}(v) = \exp_{Id}(\tfrac{v}{2^N})^{2N}$$

$$\varphi = (\tfrac{v}{2^2} \circ \tfrac{v}{2^2}) \circ (\tfrac{v}{2^2} \circ \tfrac{v}{2^2})$$

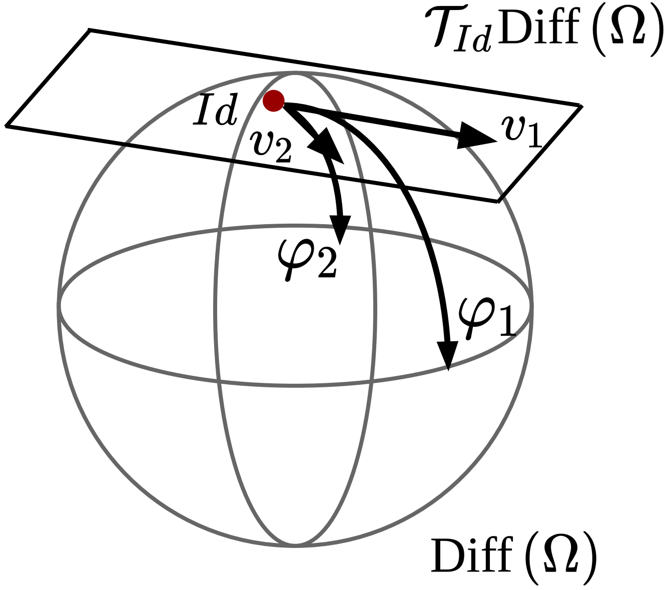
Vercauteren et al.: Diffeomorphic demons: efficient non-parametric image registration, NeuroImage 2009

Statistics based on Stationary Velocities

Distance defined in $\mathcal{T}_{Id}\mathrm{Diff}\,(\Omega)$:

$$\begin{aligned} d(\varphi_1, \varphi_2)^2 &= \|\log_{Id}(\varphi_1) - \log_{Id}(\varphi_2)\|_2^2 \\ &= \|v_1 - v_2\|_2^2 \end{aligned}$$

$\Rightarrow$ Euclidean statistics useable!

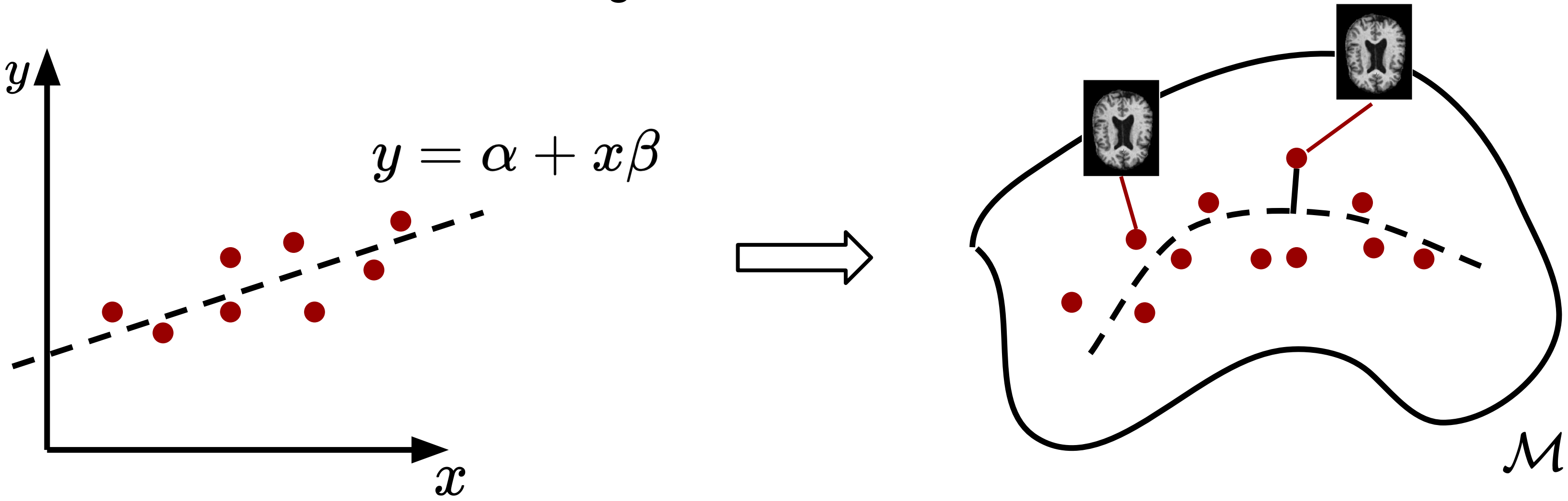


Limitation: Not all diffeomorphisms can be generated/reached when using stationary fields!

Arsigny et al.: A Log-Euclidean Framework for Statistics on Diffeomorphisms, MICCAI 2006

Geodesic Regression

Goal: Generalization of linear regression to manifold-valued data



$$(\hat{\alpha}, \hat{\beta}) = \underset{\alpha,\beta}{\arg\min} \sum_{i=1}^{N} \|y_i - \alpha - \beta x_i\|_2^2$$

Fletcher: Geodesic Regression and the Theory of Least Squares on Riemannian Manifolds, IJCV 2013

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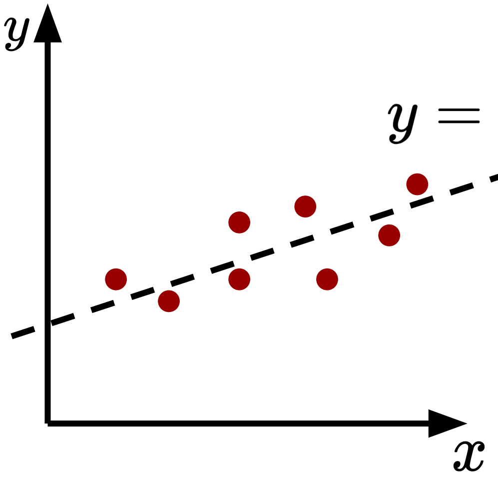

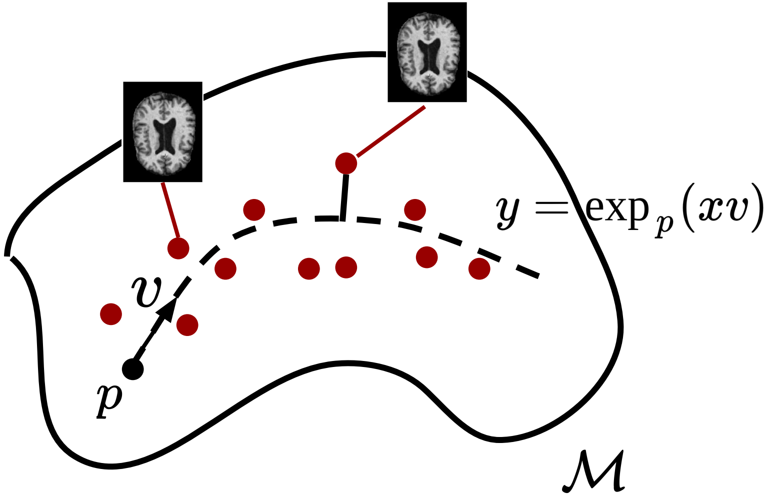

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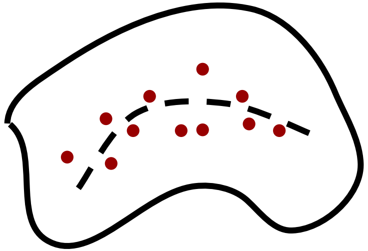
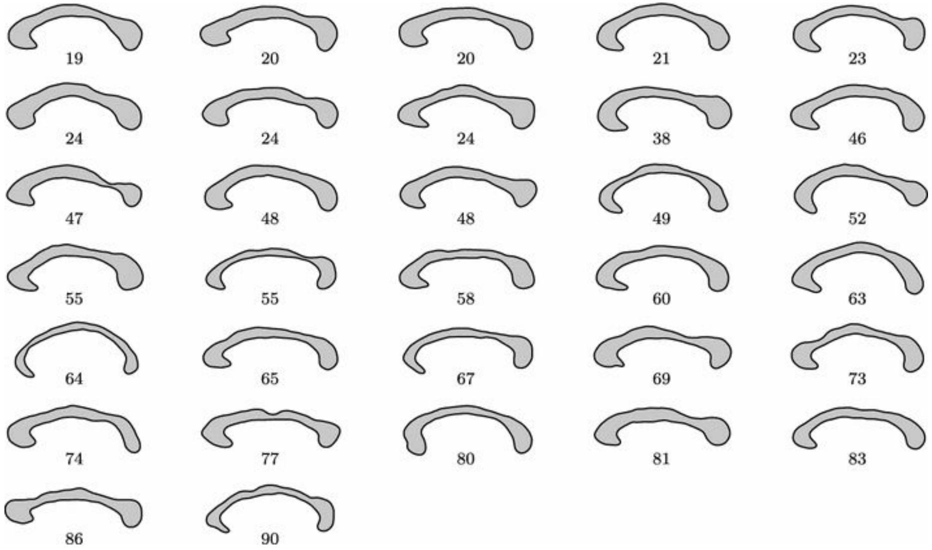
Fletcher: Geodesic Regression and the Theory of Least Squares on Riemannian Manifolds, IJCV 2013

Geodesic Regression: Example

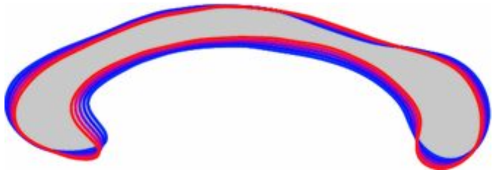


Age-related shape changes of the corpus callosum

Training data



Sampled function



Age 19⟶Age 90

old patient = thin shape

Fletcher: Geodesic Regression and the Theory of Least Squares on Riemannian Manifolds, IJCV 2013

Software

LDDMM-like registration:

- ➢ Advanced Normalization Tools (ANTs)
- ➢ Lagomorph (PyTorch)

Registration based on stationary velocity fields:

- ➢ VariationalRegistration toolkit (part of ITK)
- ➢ DARTEL (part of SPM)

Geodesic regression and more:

- ➢ pyCA (Python 2 only)
- ➢ Deformetrica

Additional Sources and Further Reading

- Presentation by S. Allassonière at Ecole GEOMDATA, September 2018
  https://geomdata.sciencesconf.org/data/AnatComput_Cours1.pdf
- Presentation by X. Pennec at IPAM Workshop on Geometric Processing, April 2019
- MICCAI Educational Challenge 2014 entry by N. Miolane & B. Khanal
  https://www.youtube.com/watch?v=XUz59y6HDEk
- New Book: Riemannian Geometric Statistics in Medical Image Analysis, Elsevier 2019. Edited by X. Pennec, S. Sommer, and T. Fletcher. Many chapters have been made available online (use google!)
- PhD thesis by N. Molane, Geometric statistics for computational anatomy, U Côte d'Azur, 2016
  https://tel.archives-ouvertes.fr/tel-01411886v2
- PhD thesis by T. Polzin, Large Deformation Diffeomorphic Metric Mappings – Theory, Numerics, and Applications, U Lübeck, 2018
  https://www.researchgate.net/publication/328042072_Large_Deformation_Diffeomorphic_Metric_Mappings_-_Theory_Numerics_and_Applications