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Adaptive MPC for Uncertain Discrete-Time LTI MIMO Systems with Incremental Input Constraints

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Abstract: In this paper, an adaptive model predictive control (MPC) strategy is proposed for controlling a discrete-time linear MIMO system with parametric uncertainties and subjected to actuator constraints. Compared to previous results in literature which either solve the constrained MPC problem for stable uncertain systems or the unconstrained MPC problem for unstable uncertain systems, this result, presents a solution approach for constrained MPC problems for fully uncertain and unstable systems. An adaptive law, designed to update the estimated parameters of the plant, is combined with a constrained MPC for an estimated system. A sufficient condition is imposed on the adaptation gain to account for feasibility of the MPC optimization problem in the presence of the actuator constraint. Stability analysis of the closed loop system with the proposed adaptive MPC strategy has been shown to guarantee the ultimate boundedness of the parameter estimation errors and boundedness as well as asymptotic convergence of the tracking errors to zero.

Keywords: Model Predictive Control, Adaptive Control, Input Constraints.

1. INTRODUCTION

Model predictive control (MPC) has gained much attention in the recent years due to its simple and effective approach towards various control problems and its robust properties towards external disturbances. The main advantage of MPC is that it exploits within its design methodology the practical problem of exercising control actions in constrained environment . Some excellent overviews of MPC can be found in (Garcia et al., 1989) and (Kouvaritakis and Cannon, 2015). The design methodology of this control strategy is well depicted in (Wang, 2009). MPC has found its applications in various fields of engineering, such as in flight control (Kale and Chipperfield, 2005), control of electrical drives (Linder and Kennel, 2005), hybrid electric vehicles (Borhan et al., 2012), etc.

However, as the name suggests, MPC is a model dependent control strategy. For this reason, systems with parametric uncertainties in the model cannot be directly tackled by the conventional MPC strategy, owing to the difficulty in predicting the future states of the plant. For systems with unknown model, system identification has become an well accepted methodology to obtain a viable model for designing controllers. However, it is difficult to do offline system identification for an inherently unstable plant. In contrast, adaptive control does online adjustments of the controller/system parameters on the basis of measured input-output data, at the same time ensuring stability. Hence, a possible solution for the control of uncertain and constrained systems can be the amalgamation of adaptive control with MPC. Some contributions to this area of research include adaptive MPC based on comparison model (Fukushima et al., 2007), adaptive predictive control of nonlinear systems (Adetola et al., 2009) and adaptive receding horizon predictive control for discrete time linear systems (Kim and Sugie, 2008).

Although there have been significant contributions in the field of adaptive MPC, the control of a completely uncertain MIMO systems, subjected to constraints, has been a challenging problem. Control strategies proposed in (Fukushima et al., 2007) and (Kim and Sugie, 2008) have guaranteed closed loop stability for partially uncertain and constrained linear systems. Some recent works (Tanaskovic et al., 2014), (Zhu and Xia, 2016) have proposed effective adaptive MPC strategies to control fully uncertain systems. However, in (Tanaskovic et al., 2014), FIR approximation method is used to obtain a tractable approximated model corresponding to the uncertain plant for which the stability of the plant is a prerequisite. In (Zhu and Xia, 2016), the system under consideration is unconstrained.

This paper presents a method by which adaptive MPC can be implemented for fully uncertain discrete-time linear MIMO time-invariant plant subjected to incremental input constraint. The main contribution of this paper is the introduction of a sufficient condition on the adaptation gain in order to satisfy the constraint conditions in the MPC optimization problem. The adaptive law guarantees the ultimate boundedness of the parameter estimation errors.

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It is further theoretically proved that the tracking errors of the closed loop system, with the proposed adaptive MPC, are bounded and asymptotically converging to zero.

2. PROBLEM STATEMENT

In this paper, the linear discrete time MIMO system is considered to be of the form

$$x_r(k+1) = A_r x_r(k) + B_r u(k) \tag{1}$$

$$y_r(k) = C_r x_r(k) \tag{2}$$

where $x_r(k) \in \mathbb{R}^n$ is the system state vector, $u(k) \in \mathbb{R}^m$ is the input vector and $y_r(k) \in \mathbb{R}^l$ is the output vector. $A_r \in \mathbb{R}^{n\times n}$ and $B_r \in \mathbb{R}^{n\times m}$ are the uncertain and constant system matrix and input gain matrix, respectively. It is assumed that the pair (A_r, B_r) is controllable. Full state feedback is considered to be available and $C_r \in \mathbb{R}^{l\times n}$ is assumed to be known. The target is to make the system output $y_r(k)$ track a given set point r, where $r = [r_1, r_2, ..., r_l]^T$ and $\|r\| \leq \bar{r}$, in the presence of the following incremental input constraint

$$\|\Delta u(k)\| \leq \Delta U_{max} \tag{3}$$

For the feasibility of the MPC problem, the desired set point must be reachable with the given input constraints, i.e., there must exist some sequence of $\Delta u(k)$, satisfying (3) at all k, such that $u(k)$ reaches some steady control input u_{ss} that can keep the output $y_r(k)$ equal to r as $k \to \infty$. Let the tracking error associated with the system be denoted by $e_r(k)$, given as

$$e_r(k) = y_r(k) - r \tag{4}$$

In case of set point tracking, the knowledge of the steady value u_{ss} is required for solving the optimization problem in MPC. Since the system (1) is uncertain, obtaining exact knowledge of u_{ss} is not possible. Instead, the system given in equations (1) and (2) can be modified and represented in an incremental form (Wang, 2009), given as

$$x(k+1) = Ax(k) + B\Delta u(k) \tag{5}$$

$$y(k) = Cx(k) \tag{6}$$

where, $x(k) \triangleq [\Delta x_r(k)^T, e_r(k)^T]^T$, $\Delta x_r(k) \triangleq x_r(k) - x_r(k-1)$, $\Delta u(k) \triangleq u_r(k) - u_r(k-1)$ and $y(k) = e_r(k)$; and

$$A = \begin{bmatrix} A_r & 0_{n\times l} \\ C_r A_r & I_{l\times l} \end{bmatrix}, B = \begin{bmatrix} B_r \\ C_r B_r \end{bmatrix}, \quad C = [0_{l\times n} \; I_{l\times l}]$$

where $A \in \mathbb{R}^{(n+l)\times(n+l)}$, $B \in \mathbb{R}^{(n+l)\times m}$ and $C \in \mathbb{R}^{l\times(n+l)}$. So $y(k)$ of the augmented system is in fact the tracking error $e_r(k)$ of the original system. Such a formulation aids the concerned MPC problem as for successful set point tracking, $\Delta u(k)$ will always go to zero. It can be seen that A and B are dependent on A_r and B_r. Consequently, in the incremental model (5), we consider A and B as the uncertain constant system parameters. It can be proved that if pair (A_r, B_r) is controllable, then the pair (A, B) will also be controllable. Define a lumped system parameter matrix Θ as $\Theta^T \triangleq [A, B]$. The system (5) can be represented using the lumped parameter matrix as

$$x(k+1) = \Theta^T \Phi(k) \tag{7}$$

where $\Phi(k) = [x(k)^T \quad \Delta u(k)^T]^T \in \mathbb{R}^{n+l+m}$ is the regressor vector.

Objective: To design a control input $\Delta u(k)$ (or $u(k)$), which makes the output $y(k)$ of the uncertain system (5)-(6) track zero (which corresponds to $y_r(k)$ tracking the set point r), in the presence of the incremental input constraint (3).

Definition 1. For any two vector $a = [a_1, a_2, \cdots, a_n]$ and $b = [b_1, b_2, \cdots, b_n]$ with $a, b \in \mathbb{R}^n$ for some finite $n \in \mathbb{I}$, the relation $a \leq b$ is equivalent to $a_i \leq b_i$ where $i = 1, 2, \cdots, n$.

Assumption 1. It is assumed that Θ is bounded as $\|\Theta\| \leq \overline{\Theta}$, where $\overline{\Theta}$ is a known positive scalar.

3. PRELIMINARIES

3.1 Adaptive Update Law Development

An estimated system corresponding to (5) is designed as follows

$$\widehat{x}(k+1) = \widehat{A}(k)x(k) + \widehat{B}(k)\Delta u(k) \tag{8}$$

where $\widehat{A}(k) \in \mathbb{R}^{(n+l)\times(n+l)}$ and $\widehat{B}(k) \in \mathbb{R}^{(n+l)\times m}$ are the time-varying estimated parameters corresponding to A and B, respectively. Define $\widehat{\Theta}(k)^T \triangleq [\widehat{A}(k), \widehat{B}(k)] \in \mathbb{R}^{(n+l)\times(n+l+m)}$. The estimated system state equation (8) can be written as

$$\widehat{x}(k+1) = \widehat{\Theta}(k)^T \Phi(k) \tag{9}$$

Subtracting equation (9) from equation (7) yields

$$\widetilde{x}(k+1) = \widetilde{\Theta}^T(k)\Phi(k) \tag{10}$$

where $\widetilde{x}(k) \triangleq x(k) - \widehat{x}(k)$ and $\widetilde{\Theta}(k) \triangleq \Theta - \widehat{\Theta}(k)$. A gradient descent based update law, given as

$$\widehat{\Theta}(k+1) = \widehat{\Theta}(k) + \lambda\Phi(k)\widetilde{x}(k+1)^T \tag{11}$$

where $\lambda > 0$, can be used to minimize the cost function $J_x(k) = \widetilde{x}(k)^T \widetilde{x}(k)$.

Theorem 2. For the uncertain system defined by (1)-(2), if there exists an input $u_r(k)$ (or $\Delta u(k)$) satisfying the following constraint

$$\Phi(k)^T \Phi(k) \leq \frac{2-\alpha}{\lambda} \tag{12}$$

where $0 < \alpha < 2$ and $\lambda > 0$, then with the adaptive update law (11) the following holds true

- The error $\widetilde{\Theta}(k)$ is ultimately bounded;
- The error $\widetilde{x}(k)$ is asymptotically stable.

Proof. For the detailed proof of the above theorem, refer to (Zhu and Xia, 2016).

3.2 Model Predictive Control for Estimated System

In this section, an MPC is developed for the estimated system (8) .

Formulation of Predictive Equations for Estimated System. Suppose N_p is the length of the prediction horizon and N_c is the length of the control horizon. Let $\widehat{x}(k+i|k)$ denote the predicted state of the estimated system (8) at the $(k+i)^{th}$ instant, using the knowledge of $\widehat{x}(k)$ (here $\widehat{x}(k) = x(k)$). The predictive equations can be represented in a compact form as

$$\widehat{x}(k+m|k) = \widehat{A}(k)^m x(k) + \sum_{j=0}^{\sigma} \widehat{A}(k)^{\sigma-j}\widehat{B}(k)\Delta u(k+j|k) \tag{13}$$

Where $\sigma = min(m-1, N_c-1)$ and $m = 1, 2, ..., N_p$. The predictive equation associated with (13) can be written in a compact form as

$$\widehat{X}(k) = \widehat{F}(k)x(k) + \widehat{\phi}(k)\Delta U(k) \tag{14}$$

where $\widehat{X}(k) \triangleq [\widehat{x}(k+1|k)^T, \widehat{x}(k+2|k)^T, ..., \widehat{x}(k+N_p|k)^T]^T$ and $\Delta U(k) \triangleq [\Delta u(k)^T, \Delta u(k+1)^T, ..., \Delta u(k+N_c-1)^T]^T$ and the predictive matrices are given as

$$\widehat{F}(k) = \begin{bmatrix} \widehat{A} \\ \widehat{A}^2 \\ . \\ \widehat{A}^{N_p} \end{bmatrix}, \quad \widehat{\phi}(k) = \begin{bmatrix} \widehat{B} & 0 & . & 0 \\ \widehat{A}\widehat{B} & \widehat{B} & . & 0 \\ . & . & . & . \\ \widehat{A}^{N_p-1}\widehat{B} & \widehat{A}^{N_p-2}\widehat{B} & . & \widehat{A}^{N_p-N_c}\widehat{B} \end{bmatrix} \tag{15}$$

where $\widehat{\Theta}$ represents $\widehat{\Theta}(k)$, $\widehat{F}(k) \in \mathbb{R}^{lN_p\times(n+l)}$ and $\widehat{\phi}(k) \in \mathbb{R}^{lN_p\times mN_c}$.

Choice of Cost Function and the Optimization Problem The cost function $J(k)$ associated with the system (5) is chosen as

$$J(k) = X(k)^T QX(k) + \Delta U(k)^T R_u \Delta U(k) \tag{16}$$

where $Q = C^TC$, $X(k) \triangleq [x(k+1|k)^T, x(k+2|k)^T, ..., x(k+N_p|k)^T]^T$ and $R_u = diag(r_u) \in \mathbb{R}^{(mN_c)\times(mN_c)}$ is a diagonal weight matrix with $r_u \in \mathbb{R}^{m\times m}$ being positive definite. Corresponding to (16), an estimated cost function $\widehat{J}(k)$, associated with the estimated system (8), is chosen as

$$\widehat{J}(k) = \widehat{X}(k)^T Q\widehat{X}(k) + \Delta U(k)^T R_u \Delta U(k) \tag{17}$$

Replacing $\widehat{X}(k)$ from (14) in (17) and solving, the following expression is obtained

$$\widehat{J}(k) = \Delta U(k)^T \widehat{E}(k)\Delta U(k) + 2\Delta U(k)^T \widehat{H}(k) \tag{18}$$

where $\widehat{E}(k) \triangleq \widehat{\phi}(k)^T\widehat{\phi}(k)+R_u$ and $\widehat{H}(k) \triangleq \widehat{\phi}^T(k)\big(\widehat{F}(k)x(k)\big)$. The objective is to minimize (18) with respect to $\Delta U(k)$ subject to (3) and (12).

4. MAIN RESULTS (FEASIBILITY AND STABILITY)

4.1 Feasibility

The constrained MPC problem can be formulated as the following

$$\Delta U^*(k) = \underset{\Delta U(k)}{\operatorname{argmin}}\, \widehat{J}(k) \tag{19}$$

subject to the constraints given in (3) and (12).

Let $\Phi_m = \max(\Phi(i|0))$, where $\Phi(i|0)$ is obtained by processing the optimization (19) subjected to only a stabilizing terminal constraint as given in (31). According to (Remark 4., (Zhu and Xia, 2016)), for the feasibility of the optimization problem (19), λ must be chosen such that

$$\frac{2-\alpha}{\lambda} >> \frac{2-\alpha}{\lambda_0} \tag{20}$$

where, λ_0 is given as

$$\lambda_0 = \frac{2-\alpha}{\max(\Phi_m^T\Phi_m, \frac{\bar{r}}{\overline{\Theta}})} = \frac{2-\alpha}{\Phi_{max}} \tag{21}$$

where $\Phi_{max} = \max(\Phi_m^T\Phi_m, \frac{\bar{r}}{\overline{\Theta}})$, $\bar{r}$ is the bound on the reference signal $r(k)$ and $\overline{\Theta}$ is the bound on the uncertain parameter matrix Θ. Φ_{max} can be considered as some upper bound of $\|\Phi(k)\|^2$. Therefore, it can be claimed

$$\|\Phi(k)\|^2 \le \Phi_{max} \tag{22}$$

The above inequality is in accordance with the relation given in (12). Therefore,

$$\|x(k)\|^2 < \|x(k)\|^2 + \|\Delta u(k)\|^2 \le \Phi_{max} \tag{23}$$

From (23), a bound on $\|x(k)\|$ can be given as

$$\|x(k)\|^2 < \Phi_{max} \tag{24}$$

Lemma 3. For the optimization problem (19) to be always feasible, subject to (3) and (12), the following condition must hold

$$\lambda < \frac{2-\alpha}{\Delta U_{max}^2 + \Phi_{max}} \tag{25}$$

Proof. The constraint equation (12) can be written as

$$\|\Delta u(k)\|^2 + \|x(k)\|^2 < \frac{2-\alpha}{\lambda}$$

$$\|\Delta u(k)\|^2 < \frac{2-\alpha}{\lambda} - \|x(k)\|^2 \tag{26}$$

Inequality (26) is a state dependent constraint on $\Delta u(k)$. Using the result from (24) and (26), the smallest bound on $\Delta u(k)$ can be found as

$$\|\Delta u(k)\|^2 < \frac{2-\alpha}{\lambda} - \Phi_{max} \tag{27}$$

The objective is to make the above constraint inactive for all instances. This can be achieved by imposing the following condition

$$\|\Delta u(k)\|^2 \le \Delta U_{max}^2 < \frac{2-\alpha}{\lambda} - \Phi_{max}$$

Therefore, the condition on λ is derived as

$$\lambda < \frac{2-\alpha}{\Delta U_{max}^2 + \Phi_{max}} \tag{28}$$

Remark 1. The above result is true when ΔU_{max} is finite. For unconstrained adaptive MPC problem, results from (Zhu and Xia, 2016) must be followed.

Remark 2. The condition (28) is a sufficient condition, which makes the constraint (12) inactive for all instants. Therefore, it is only the actuator constraint (3) that needs to be checked and satisfied when solving the optimization problem.

Following *Definition* 1, the constraint (3) can be reformulated to be represented in vector form as

$$\begin{bmatrix} I_{m\times m} \\ -I_{m\times m} \end{bmatrix} \Delta u(k) \le \frac{\Delta U_{max}}{\sqrt{m}} \begin{bmatrix} \mathbf{1}_m \\ \mathbf{1}_m \end{bmatrix} \tag{29}$$

The last inequality can be written in compact form as

$$M\Delta U(k) \le \gamma \tag{30}$$

where $M = \begin{bmatrix} I_{m\times m} & 0 & ... & 0 \\ -I_{m\times m} & 0 & ... & 0 \end{bmatrix} \in \mathbb{R}^{2m\times(mN_c)}$

$\gamma = \frac{\Delta U_{max}}{\sqrt{m}} \begin{bmatrix} \mathbf{1}_m \\ \mathbf{1}_m \end{bmatrix} \in \mathbb{R}^{2m}$ and $\mathbf{1}_m \triangleq [1, 1, ..., 1]^T \in \mathbb{R}^m$. In order to guarantee stability of the closed loop estimated system, the following terminal constraint is introduced

$$\widehat{x}(k+N_c|k) = \mathbf{0}_l \tag{31}$$

where $\mathbf{0}_l = [0, 0, ..., 0]^T \in \mathbb{R}^l$. In terms of $\Delta U(k)$, the above constraint can be written as

$$M_t\Delta U(k) = \gamma_t \tag{32}$$

where $M_t = C[\widehat{A}^{N_c-1}\widehat{B}, \widehat{A}^{N_c-2}\widehat{B}, ..., \widehat{B}]$ and $\gamma_t = -C\widehat{A}^{N_c}$ $x(k)$. Therefore, the MPC optimization can be formulated as (19) subject to (30) and (32). The proposed control strategy is implemented using receding horizon control

$$\Delta u(k) = [I_{m\times m}, 0, ..., 0]\Delta U^*(k) \tag{33}$$

Remark 3. The optimization problem mentioned above will be feasible at all instants for proper choice of λ corresponding to a some particular initial condition. If the initial conditions are changed, λ must also be changed following the condition in *Lemma* 3.

Definition 2. The MPC optimization problem (19), subjected to (30) and (32), is said to be feasible at some instant k, if there exists some control input sequence $\Delta U(k)$ (dependent on $x(k)$), which satisfies the constraints (30) and (32) and returns a finite value of the cost function.

Assumption 4. If there exists some $x(k)$, for which the MPC optimization problem is feasible for the initial choice $\widehat{\Theta}(k)$, then the optimization problem will be feasible for all $\widehat{\Theta}(k+i)$ and $x(k)$, where $i > k$.

It is assumed that for some initial choices of $\widehat{x}(k)$ and $\widehat{\Theta}(k)$, the optimization problem (19) is feasible at the k^{th} instant and there exists an optimal $\Delta U^*(k)$ satisfying (30) and (32), where

$$\begin{aligned}\Delta U^*(k) =&[\Delta u^*(k|k)^T, \Delta u^*(k+1|k)^T, ..\\ &.., \Delta u^*(k+N_p-1|k)^T]^T\end{aligned} \tag{34}$$

At the $(k+1)^{th}$ instant, the initial value of the state would be $\widehat{x}(k+1) = \widehat{x}^*(k+1|k)$, for which a feasible sequence exists with the system parameters being $\widehat{\Theta}(k)$. Hence, using *Assumption* 4, it can be claimed that, there exists some feasible control input sequence for the estimated system at $(k+1)^{th}$ instant with the initial states and system parameters respectively being $\widehat{x}(k+1)$ and $\widehat{\Theta}(k+1)$. Therefore, a feasible solution for the optimization problem (19) at $(k+1)^{th}$ instant can be given as

$$\begin{aligned}&\Delta U(k+1) = [\Delta u(k+1|k+1)^T, .., \Delta u(k+N_p|k+1)^T]^T\\ &=[\Delta u^*(k+1|k)^T, ..., \Delta u^*(k+N_p-1|k)^T\\ &\ , \Delta u(k+N_p|k+1)^T]^T\end{aligned} \tag{35}$$

where $\Delta u(k+m|k+1) = \Delta u^*(k+m|k)$, $m = 1(1)N_p-1$. If *Assumption* 4 holds, then some $\Delta u(k+N_p|k+1)$, satisfying (30) and the terminal constraint $\widehat{x}(k+N_p+1|k+1) = 0$, will exist. Thus recursive feasibility of the adaptive MPC problem is established.

4.2 Stability of the Closed-Loop System

Theorem 5. If the optimization (19), subject to (30) and (32), is feasible at the initial time, then with the receding horizon control (33), the tracking error of the closed loop system defined by (1) and (2) (or (5) and (6)), with the adaptive update law (11), is bounded and asymptotically converging to zero.

Proof. Without loss of generality, it is assumed that $N_p = N_c$. The cost function described in (17) can be reformulated as

$$\begin{aligned}\widehat{J}_y(k) = &\sum_{m=1}^{N_p} \widehat{x}(k+m|k)^T Q \widehat{x}(k+m|k) +\\ &\sum_{m=0}^{N_p-1} \Delta u(k+m|k)^T r_u \Delta u(k+m|k)\end{aligned} \tag{36}$$

where $Q = C^T C$ and $r_u \in \mathbb{R}^{m\times m}$ is positive definite. It is assumed that the optimization (19) is possible at the k^{th} instant and there exists a $\Delta U^*(k)$, satisfying (30) (by following *Lemma* 3) and (32), so that the cost function (17) reaches its optimal value $\widehat{J}^*_y(k)$. Considering $\widehat{x}^*(k|k) = x(k)$ a Lyapunov function candidate is now chosen as

$$\begin{aligned}V(k) = x(k)^T Q x(k) + \widehat{J}^*_y(k) = &\sum_{m=0}^{N_p} \widehat{x}^*(k+m|k)^T Q\\ \widehat{x}^*(k+m|k) + &\sum_{m=0}^{N_p-1} \Delta u^*(k+m|k)^T r_u \Delta u^*(k+m|k)\end{aligned} \tag{37}$$

$V(k)$ can be upper and lower bounded as (Grune and Pannek, 2011)

$$\left.\begin{aligned}V(k) \geq \alpha_1\Big(\|e_r(k)\|^2 + \|\Delta u^*(k)\|^2\Big)\\ V(k) \leq \alpha_2\Big(\|e_r(k)\|^2 + \|\Delta u^*(k)\|^2\Big)\end{aligned}\right\} \tag{38}$$

where $\alpha_1(\cdot), \alpha_2(\cdot) \in \kappa_\infty$ are invertible functions. Accordingly, the Lyapunov function candidate, at $(k+1)^{th}$ instant, is then formulated as

$$\begin{aligned}&V(k+1) = x(k+1)^T Q x(k+1) + \widehat{J}^*_y(k+1)\\ &= \sum_{m=1}^{N_p+1} \widehat{x}^*_0(k+m|k+1)^T Q \widehat{x}^*_0(k+m|k+1) +\\ &\sum_{m=1}^{N_p} \Delta u^*(k+m|k+1)^T r_u \Delta u^*(k+m|k+1)\end{aligned} \tag{39}$$

A feasible control sequence at $(k+1)^{th}$ instant can be chosen according to (35) with the assumption that some feasible $\Delta u(k+N_p|k+1)$ exists, satisfying

$$\widehat{x}(k+1+N_p|k+1) = 0 \tag{40}$$

Thus $\Delta U(k+1)$ satisfies the constraint given by (32). The constraint given by (30) can be satisfied by following *Lemma* 3. Hence, feasibility of the optimization problem (19) is guaranteed at $(k+1)^{th}$ instant. Hence at the $(k+1)^{th}$ instant, some positive definite function $\overline{V}(k+1)$, is obtained as

$$\begin{aligned}\overline{V}(k+1) = &\sum_{m=1}^{N_p} \widehat{x}(k+m|k+1)^T Q \widehat{x}(k+m|k+1)\\ &+ \sum_{m=1}^{N_p-1} \Delta u^*(k+m|k)^T r_u \Delta u^*(k+m|k)\\ &+ \Delta u(k+N_p|k+1)^T r_u \Delta u(k+N_p|k+1)\end{aligned} \tag{41}$$

It is to be noted that, (41) is obtained by replacing the optimal input sequence $\Delta U^*(k+1)$ by the feasible sequnece $\Delta U(k+1)$ in (39). Therefore, $\overline{V}(k+1)$ is an upper bound on $V(k+1)$ and hence the following hold true

$$V(k+1) - V(k) \leq \overline{V}(k+1) - V(k) \tag{42}$$

Due to the update of the system parameters from $[\widehat{A}(k), \widehat{B}(k)]$ to $[\widehat{A}(k+1), \widehat{B}(k+1)]$ from k^{th} instant to $(k+1)^{th}$ instant, the following holds true

$$\left.\begin{aligned}\widehat{x}(k+1|k+1) &= \widehat{x}^*(k+1|k) + \delta_0(\Delta\Theta(k+1))\\ \widehat{x}(k+2|k+1) &= \widehat{x}^*(k+2|k) + \delta_1(\Delta\Theta(k+1))\\ &\vdots\\ \widehat{x}(k+N_p|k+1) &= \widehat{x}^*(k+N_p|k) + \delta_{N_p-1}(\Delta\Theta(k+1))\end{aligned}\right\} \tag{43}$$

where $\Delta\Theta(k+1) \triangleq \widehat{\Theta}(k+1) - \widehat{\Theta}(k)$ and

$$\delta_i(k+1) = \Delta\Theta_{\widehat{A}}(k+1,i)\widehat{x}^*(k+1|k) + \Delta\Theta_{\widehat{A}\widehat{B}}(k+1,i) \quad (44)$$

where $\Delta\Theta_{\widehat{A}}(k+1,c) \triangleq \widehat{A}(k+1)^c - \widehat{A}(k)^c$ and $\Delta\Theta_{\widehat{A}\widehat{B}}(k+1,c) \triangleq \sum_{j=1}^{c}\left(\widehat{A}(k+1)^{j-1}\widehat{B}(k+1) - \widehat{A}(k)^{j-1}\widehat{B}(k)\right)\Delta u^*(k+j|k)$. $\delta_i(k+1)$ is the explicit expression for $\delta_i(\Delta\Theta(k+1))$. It is seen that $\Delta\Theta(k+1) \to 0$ asymptotically, because of (11) and *Theorem* 2, which results in $\left(\Delta\Theta_{\widehat{A}}(k+1,c), \Delta\Theta_{\widehat{A}\widehat{B}}(k+1,c)\right) \to 0$. Consequently, it is claimed that

$$\lim_{k\to\infty} \delta_i(k+1) = 0 \quad (45)$$

Using the result from (13), (37), (41) and (43)

$$\overline{V}(k+1) - V(k) = -x(k)^T Q x(k) - \Delta u^*(k)^T r_u \Delta u^*(k) + 2\sum_{m=1}^{N_p}\left(\widehat{A}(k)^m x(k) + \sum_{j=0}^{\sigma}\widehat{A}(k)^{\sigma-j}\widehat{B}(k)\Delta u^*(k+j|k)\right)^T Q\delta_{m-1}(k+1) + \sum_{m=0}^{N_{p-1}}\delta_m(k+1)^T Q\delta_m(k+1) + \Delta u(k+N_p|k+1)^T r_u \Delta u(k+N_p|k+1) \quad (46)$$

Define

$$\xi_1\left(k, \widehat{\Theta}(k), \delta_i(k+1)\right) \triangleq 2\sum_{m=1}^{N_p}\left(\widehat{A}(k)^m\right)^T Q\delta_{m-1}(k+1)$$

$$\xi_2\left(k, \Delta U^*(k), \widehat{\Theta}(k), \delta_i(k+1)\right) \triangleq 2\sum_{m=1}^{N_p}\left(\sum_{j=0}^{\sigma}\widehat{A}(k)^{\sigma-j}\widehat{B}(k)\Delta u^*(k+j|k)\right)^T Q\delta_{m-1}(k+1)$$

For ease of representation $\xi_1\left(k, \widehat{\Theta}(k), \delta_i(k+1)\right)$ is represented as $\xi_1(k)$ and $\xi_2\left(k, \Delta U^*(k), \widehat{\Theta}(k), \delta_i(k+1)\right)$ is represented as $\xi_2(k)$. According to *Theorem* 2, $\widehat{\Theta}(k)$ is ultimately bounded and due to the constraints (30) and (32) imposed on the input, $\Delta U^*(k)$ will also be always bounded. Hence, $\xi_1(k)$ and $\xi_2(k)$ will also be bounded at all instants. Using these assumptions, $\overline{V}(k+1, x(k+1) - V(k, x(k))$ can be upper bounded as follows

$$\overline{V}(k+1) - V(k) \le -\lambda_{min}(Q)\|\widehat{x}(k)\|^2 - r_u\|\Delta u^*(k)\|^2 + \|\xi_1(k)\|\|x(k)\| + \|\xi_2(k)\| + \sum_{m=0}^{N_{p-1}}\|Q\|\|\delta_m\|^2 + r_u\|\Delta u(k+N_p|k+1)\|^2 \quad (47)$$

where $\lambda_{min}(\cdot)$ is the minimum eigen value of the argument matrix. Since Q is a positive definite matrix, $\Theta_{min}(Q) > 0$. Suppose $\lambda_{min}(Q) = \lambda_1 + \lambda_2$, where $\lambda_1, \lambda_2 > 0$. Using this result in (47), the following is obtained

$$\overline{V}(k+1) - V(k) \le -\lambda_1\|x(k)\|^2 - \lambda_2\|x(k)\|^2 - r_u\|\Delta u(k)\|^2 + \|\xi_1(k)\|\|x(k)\| + \|\xi_2(k)\| + \sum_{m=0}^{N_{p-1}}\|Q\|\|\delta_m\|^2 + r_u\|\Delta u(k+N_p|k+1)\|^2$$

Completing the squares using the second and the fourth terms on the right hand side of the last inequality, the following is obtained

$$\overline{V}(k+1) - V(k) \le -\lambda_1\|x(k)\|^2 - r_u\|\Delta u(k)\|^2 + \frac{\|\xi_1(k)\|^2}{4\lambda_2} + \|\xi_2(k)\| + \sum_{m=0}^{N_{p-1}}\|Q\|\|\delta_m\|^2 + r_u\|\Delta u(k+N_p|k+1)\|^2 \quad (48)$$

Let

$$\Omega = \frac{\|\xi_1(k)\|^2}{4\lambda_2} + \|\xi_2(k)\| + \sum_{m=0}^{N_{p-1}}\|Q\|\|\delta_m\|^2 + r_u\|\Delta u(k+N_p|k+1)\|^2$$

Therefore, the inequality (48) is reformulated as

$$\overline{V}(k+1) - V(k) \le -\lambda_1\|x(k)\|^2 - r_u\|\Delta u(k)\|^2 + \Omega \quad (49)$$

Since $\xi_1(k), \xi_2(k)$ and δ_m are bounded and $\Delta u(k+N_p|k+1)$ is finite, Ω is also bounded. Moreover, as $\delta_m \to 0$ asymptotically, it implies

$$\lim_{k\to\infty} \xi_1(k), \xi_2(k) = 0$$

Also, the control input $\Delta u(k+N_p|k+1)$ is present due to the presence of model mismatch in between two consecutive instants. Since $\Delta\Theta(k+1) \to 0$ as $k \to \infty$ (from *Theorem* 2), $\Delta u(k+N_p|k+1) \to 0$ as $k \to \infty$. Therefore, it can be claimed that

$$\lim_{k\to\infty} \Omega = 0 \quad (50)$$

Consequently, from (42) and (49) it follows

$$V(k+1) - V(k) \le -\varrho\Big[\|x(k)\|^2 + \|\Delta u^*(k)\|^2\Big] + \Omega \quad (51)$$

where $\varrho = \min(\lambda_1, r_u)$. Since $x(k) \triangleq [\Delta x_r(k)^T, e_r(k)^T]^T$, it implies $\|e_r(k)\| < \|x(k)\|$. Therefore,

$$V(k+1) - V(k) \le -\varrho\Big[\|e_r(k)\|^2 + \|\Delta u^*(k)\|^2\Big] + \Omega \quad (52)$$

Using the result from (38), equation (52) can be reformulated as

$$V(k+1) - V(k) \le -\varrho\alpha_1^{-1}\Big(V(k)\Big) + \Omega \quad (53)$$

Using *Definition* 3.2 of (Jiang and Wang, 2001), it is claimed that $V(k)$ is an ISS-Lyapunov function. Since Ω is bounded and asymptotically converging to zero, it is further claimed that $V(k)$ is bounded and asymptotically converging to zero.Therefore, it can be claimed that the state $x(k)$ and consequently tracking error $e_r(k)$ of the actual system (1)-(2) is bounded and asymptotically converging to zero.

5. SIMULATION RESULTS

The following simulation example is used from (Zhu and Xia, 2016), to illustrate the results as claimed in the theory. The plant under consideration is MIMO with two inputs, given as $[u_{r_1}, u_{r_2}]^T$ and two outputs given as $[y_{r_1}, y_{r_2}]^T$. The uncertain system matrix, input matrix and the known output matrix are given as

$$A_r = \begin{bmatrix} 0.8 & 0.4 & 1.1 \\ 0.6 & 1.5 & -0.1 \\ 0.1 & -1.2 & 1.8 \end{bmatrix}; \ B_r = \begin{bmatrix} 0.7 & 0 \\ 0.2 & 0 \\ -0.6 & 1.4 \end{bmatrix}; C_r = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The initial values of the estimated parameter matrices are considered as

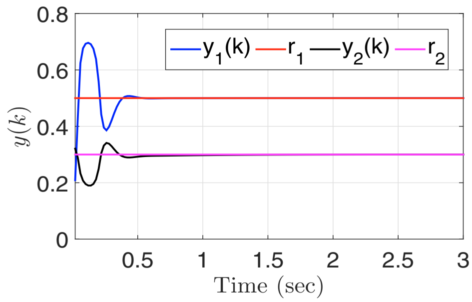

Fig. 1. Actual and desired trajectories.

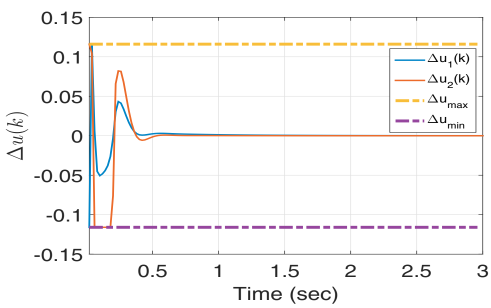

Fig. 2. The incremental control inputs

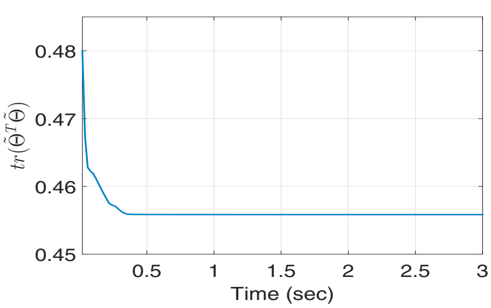

Fig. 3. The norm and variation of parameter estimation errors.

$$\widehat{A}_r(0) = \begin{bmatrix} 1 & 0.5 & 0.75 \\ 0.2 & 0.8 & 0.5 \\ 0 & 0 & 2 \end{bmatrix}; \ \widehat{B}_r(0) = \begin{bmatrix} 0.5 & 0.1 \\ 1 & 0.1 \\ -0.5 & 1.75 \end{bmatrix}$$

The initial conditions of the system states are chosen as $x(0) = [0.15, 0.1, -0.2, 0, 0]^T$ and that of the estimated states are chosen as $\widehat{x}(0) = [0, 0, 0, 0, 0]^T$. The prediction horizon and the control horizon are chosen as $N_p = N_c = 6$. The sampling interval is chosen as $h = 0.02$. The actuator constraint imposed on the system is given as

$$\|\Delta u(k)\| \leq 0.164$$

Therefore, $\Delta u_i(k)$, where $i = 1, 2$, is bounded as

$$-0.12 \leq \Delta u_i(k) \leq 0.12, \ i = 1, 2 \tag{54}$$

The variable α is chosen as $\alpha = 1.8$ and correspondingly λ is chosen such that *Lemma* 3 is satisfied. In this problem $\lambda = 0.1$ and $r_u = 0.1$. The reference time varying signals are given as $r = [0.5 \ 0.3]^T$. The tracking performance of the closed loop system is shown in Fig. 1. It is displayed in Fig. 2 that the rate of change of control inputs are contained within the bounds imposed on them. The parameter estimation error, as seen in Fig. 3, is also bounded and its variation also converges to zero.

6. CONCLUSION

An adaptive MPC strategy is proposed for controlling a fully uncertain discrete-time linear system and subjected to actuator rate constraints. The update rate of the gradient descent adaptive law is upper bounded to satisfy the actuator rate constraint and to make the constraint imposed by the adaptive law inactive for all instants. This work is a contribution over existing results in literature, where an adaptive MPC problem for completely uncertain linear system has not been considered for the constrained case. With the proposed adaptive MPC, it is proved that the parameter estimation errors are ultimately bounded and the tracking errors of the closed loop system are bounded and asymptotically converging to zero. The derived results are substantiated by the provided simulation results.

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