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Abstract

Gain scheduling (GS) is one of the most popular approaches to nonlinear control design and it is known that GS controllers have a better performance than robust ones. Following the terminology of control engineering, linear parameter-varying (LPV) systems are time-varying plants whose state space matrices are fixed functions of some vector of varying parameters. Our approach is based on considering that the LPV system, scheduling parameters and their derivatives with respect to time lie in a priori given hyper rectangles. To guarantee the performance we use the notion of guaranteed costs. The class of control structure includes centralized, decentralized fixed order output feedbacks like PID controller. Numerical examples illustrate the effectiveness of the proposed approach.

Keywords: Gain scheduled control, controller design, structured controller, decentralized control, MIMO LPV systems.

1. Introduction

Linear parameter-varying systems are time-varying plants whose state space matrices are fixed functions of some vector of varying parameters $\theta(t)$. Linear parameter varying (LPV) systems have the following interpretations:

- they can be viewed as linear time invariant (LTI) plants subject to time-varying known parameters $\theta(t) \in \langle \underline{\theta}\ \overline{\theta} \rangle$,
- they can be models of linear time-varying plants,
- they can be LTI plant models resulting from linearization of the nonlinear plants along trajectories of the parameter $\theta(t) \in \langle \underline{\theta}\ \overline{\theta} \rangle$ which can be measured.

For the first and third class of systems, parameter θ can be exploited for the control strategy to increase the performance of closed - loop systems. Hence, in this paper the following LPV system will be used:

$$\begin{aligned} \dot{x} &= A(\theta(t))x + B(\theta(t))u \\ y &= Cx \end{aligned} \tag{1}$$

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where for the affine case

$$A(\theta(t)) = A_0 + A_1\theta_1(t) + \cdots + A_p\theta_p(t) \quad (2)$$

$$B(\theta(t)) = B_0 + B_1\theta_1(t) + \cdots + B_p\theta_p(t) \quad (3)$$

and $x \in R^n$ is the state, $u \in R^m$ is a control input, $y = R^l$ is the measurement output vector, A_0,B_0,A_i,B_i, $i = 1, 2 \ldots p$, C are constant matrices of appropriate dimension, $\theta(t) \in \langle \underline{\theta}\ \overline{\theta} \rangle \in \Omega$ and $\dot{\theta}(t) \in \langle \underline{\dot{\theta}}\ \overline{\dot{\theta}} \rangle \in \Omega_t$ are vectors of time-varying plant parameters which belong to the known boundaries.

In the case of nonlinear dynamics a widely used idea among control engineers is to linearize the plant around several operating points and to use linear control tools to design a controller for each of these points. The actual controller is implemented using the gain scheduling approach. Success of such an approach depends on establishing the relationship between a nonlinear system and a family of linear ones. There are two main problems:

1. Stability results: stability of the closed-loop nonlinear system and of the closed-loop family of linear systems, when scheduled parameters are changes.
2. Approximation results which provide a direct relationship between the solution of closed-loop nonlinear systems and the solution of associated linear systems [1], [2]

The main motivation for our work lies in [3], [4], [5], [6], [7], [8], where in [3] the LPV controller is designed using the bounded real lemma for continuous and discrete time LPV systems such as to guarantee H_∞ performance.

Paper [4] discusses extensions of H_∞ synthesis techniques to allow for controller dependence on time-varying but measured parameters. In this case a higher performance can be achieved by control laws that incorporate measurements of θ to the control algorithm. Main results can be formulated as follows: Find a control structure such that the LPV controller satisfies closed-loop stability and minimizes of the induced L_2 norm of corresponding closed-loop systems. The author's approach [5] uses a bounding technique based on parameter-dependent Lyapunov function for design of PD controllers. Note that if LPV synthesis problem is solvable, then the induced L_2-norm of the closed-loop system is less than some given constant. The proposed approach represents generalization of the standard sub-optimal H_∞ control problem. In paper [6] the author shows that the performance of LPV systems with LPV controller can be improved by combining this LMI method with MPC techniques and optimizing the H_2 (H_∞) norm. The author [8] tackles the design problem of gain scheduled controllers for LPV systems via parameter-dependent Lyapunov function. The author proposed a new design method as a set LMIs with single line search parameters. The author tackles two problems: H_∞ type problem and H_2. Recently, [9] proposed the design method for the gain scheduled problem using a similar technique to [8]. In the above paper the LPV controller is given in time domain with the same or lower order than the LPV systems using H_∞ optimization approach. The gain scheduling controller design for discrete-time systems is given in [10]. Paper [11] presents the design of gain-scheduled PI controller, when the uncertainty of the system is assumed to be the difference between the nonlinear model and the nominal linear model. PI controller is designed using quadratic Lyapunov H_∞ performance where index γ is H_∞ norm of closed-loop system, considered as closed-loop performance measure. Minimizing γ via LMI the gain scheduled controller is obtained. In [12] the authors design a novel gain scheduling controller for synchronous generator. Improved stability analysis and gain scheduled controller synthesis for parameter-dependent systems are proposed in [7]. Sufficient conditions for robust stability as well as conditions for the existence of a gain-scheduled controller are given in terms of a set of LMIs. The author's approach is based on the notion of quadratic stability and linear fractional representation for parameter dependent systems. The survey of scheduled controller analysis and synthesis can be found in excellent papers [1] and [2].

In this paper our approach is based on:

- A consideration of the LPV systems (1). The scheduling parameters θ_i, $i = 1, 2, \ldots p$ and their derivatives with respect to time are supposed to lie in a priori given hyper rectangles.
- Affine quadratic stability (AQS) introduced by [13].
- To guarantee the performance we use the notion of guaranteed cost to optimize the given cost function.
- The class of control structure includes centralized, decentralized fixed order output feedback like PID controller.

The gain-scheduled controller design procedure is in the form of BMI. A feasible solution for closed-loop system ensures the affine quadratic stability [13] and guaranteed cost when the performance is defined in Q, R, S structure (see eq. (10)).

Quadratic stability (one Lyapunov function with one constant positive definite matrix cover all affine controller design procedure) is more conservative than AQS in general. AQS (Lyapunov function has an affine structure like (2)) incorporates information about the rate of variation $\dot{\theta}(t)$ to reduce conservatism. As we mentioned, in this paper the AQS approach will be used.

Our notations are standard. $D \in R^{m\times n}$ denotes the set of real $m \times n$ matrices. I_m is an $m\times m$ identity matrix. If the size can be determined from the context, we will omit the subscript. $P > 0$ ($P \geq 0$) is a real symmetric, positive definite (semi-definite) matrix.

The paper is organized as follows. Section 2 brings preliminaries and problem formulation. The main result is presented in Section 3. In Section 4, numerical examples illustrate the effectiveness of the proposed approach.

2. Preliminaries and problem formulation

Suppose that the state-space representation of an LPV system with p independent scheduling parameters is governed by (1). The scheduling parameters θ_i and their derivatives with respect to time $\dot{\theta}_i$ are supposed to lie in given hyper rectangles Ω and Ω_t, respectively. For design of the I part of the controller system, equation (1) has to be augmented, see [14] and example 1. Without change of notation the new augmented matrices dimensions are $A(\theta) \in R^{(n+l)\times(n+l)}$, $B(\theta) \in R^{(n+l)\times m}$, $C \in R^{2l\times 2l}$ and $C_d \in R^{l\times l}$ is the output matrix for D part of controller. The output feedback gain-scheduled control law is considered for PID controller in the form

$$u(t) = F(\theta)y + F_d(\theta)\dot{y}_d = F(\theta)Cx + F_dC_d\dot{x} \tag{4}$$

where $y_d = C_dx$ is the output feedback for the D part of the controller,

$$F(\theta) = F_0 + \sum_{i=1}^{p} F_i\theta_i \quad \in R^{m\times 2l} \tag{5}$$

is the static output feedback gain scheduled matrix for the PI controller and

$$F_d(\theta) = F_{d_0} + \sum_{i=1}^{p} F_{d_i}\theta_i \quad \in R^{m\times m} \tag{6}$$

is the static output feedback gain scheduled matrix for the D part of controller.

Remark 1. Since the reference signal does not influence the closed-loop stability, we assume that it is equal to zero.

Remark 2. If the derivative part of the controller includes some filter, the model of this filter can be included in the system model.

The closed-loop system is then

$$[I - B(\theta)F_d(\theta)C_d]\dot{x} = [A(\theta) + B(\theta)F(\theta)C]x \tag{7}$$
$$A_d(\theta)\dot{x} = A_c(\theta)x \tag{8}$$
$$\dot{x} = A_{cd}(\theta)x \tag{9}$$

where

$$\begin{aligned} A_{cd}(\theta) &= A_d(\theta)^{-1}A_c(\theta)x \\ A_d(\theta) &= I - B(\theta)F_d(\theta)C_d \\ A_c(\theta) &= A(\theta) + B(\theta)F(\theta)C \end{aligned}$$

It is well known that the fixed order dynamic output feedback control design problem is a special case of the static output feedback problem. To access the performance quality a quadratic cost function [15] known from LQ theory is often used in the form

$$J = \int_0^\infty (x^T Q x + u^T R u + \dot{x}^T S \dot{x})dt \tag{10}$$

with $Q = Q^T \geq 0$, $R > 0$ and $S = S^T \geq 0$. The guaranteed cost is defined in a standard way.

Definition 1. Consider system (1) with control algorithm (4). If there exists a control law u^* and a positive scalar J^* such that the closed-loop system (7) is stable and the value of closed-loop cost function (10) satisfies $J \leq J^*$, then J^* is said to be a guaranteed cost and u^* is said to be guaranteed cost control law for system (1). □

Definition 2. [13] The linear closed-loop system (7) for $\theta \in \Omega$ and $\dot{\theta} \in \Omega_t$ is affinely quadratically stable if and only if there exist $p + 1$ symmetric matrices $P_0, P_1, \ldots, P_p$ such that

$$P(\theta) = P_0 + \sum_{i=1}^{p} P_i\theta_i > 0 \tag{11}$$

and for the first derivative of Lyapunov function $V(\theta) = x^T P(\theta)x$ along the trajectory of closed-loop system (7) it holds

$$\frac{dV(x,\theta)}{dt} = x^T \left(A_{cd}(\theta)^T P(\theta) + P(\theta)A_{cd}(\theta) + \frac{dP(\theta)}{dt} \right) x < 0 \tag{12}$$

where

$$\frac{dP(\theta)}{dt} = \sum_{i=1}^{p} P_i\dot{\theta}_i \leq \sum_{i=1}^{p} P_i\rho_i$$

□

From LQ theory we introduce the well known results.

Lemma 1. Consider the closed-loop system (7). Closed-loop system (7) is affinely quadratically stable with guaranteed cost if and only if the following inequality holds

$$B_e = \min_u \left\{ \frac{dV(\theta)}{dt} + x^T Q x + u^T R u + \dot{x}^T S \dot{x} \right\} \leq 0 \tag{13}$$

for all $\theta \in \Omega$ and $\dot{\theta} \in \Omega_t$ □

3. Main results

In this section the gain scheduled controller design procedure which guarantees the affine quadratic stability and guaranteed cost for $\theta \in \Omega$ and $\dot{\theta} \in \Omega_t$ is presented. The main results for the case of gain scheduled closed-loop stability analysis reduce to LMI condition and for gain scheduled controller synthesis to BMI one.

The main result of this section, the gain scheduled design procedure, relies in the concept of multi-convexity, that is, convexity along each direction θ_i of the parameter space. The implications of multiconvexity for scalar quadratic functions are given in the next lemma [13].

Lemma 2. Consider a scalar quadratic function of $\theta \in R^p$.

$$f(\theta_1, \ldots, \theta_p) = a_0 + \sum_{i=1}^{p} a_i \theta_i + \sum_{i,j=1}^{p} b_{ij} \theta_i \theta_j + \sum_{i=1}^{p} c_i \theta_i^2 \tag{14}$$

and assume that $f(\theta_1, \ldots, \theta_p)$ is multi-convex, that is

$$\frac{\partial^2 f(\theta)}{\partial \theta_i^2} = 2c_i \geq 0 \tag{15}$$

for $i = 1, 2, \ldots, p$. Then $f(\theta)$ is negative for all $\theta \in \Omega$ if and only if it takes negative values at the corners of θ. □

Using Lemma 2 the following theorem is obtained

Theorem 1. *Closed-loop system (7) is AQS with guaranteed cost if there exist $p+1$ definite matrices $P_0, P_1, P_2, \ldots, P_p$ such that $P(\theta)$ (11) is positive defined for all $\theta \in \Omega$, matrices N_1, N_2, Q, R, S and controller gain scheduled matrices $F(\theta)$ and $F_d(\theta)$, satisfying*

$$M(\theta) < 0; \quad \theta \in \Omega \tag{16a}$$

$$M_{ii} \geq 0; \quad i = 1, 2, \ldots, p \tag{16b}$$

where

$$M(\theta) = M_0 + \sum_{i=1}^{p} M_i \theta_i + \sum_{i=1}^{p} \sum_{j=1}^{p} M_{ij} \theta_i \theta_j$$

$$M_0 = \begin{bmatrix} W_{110} & W_{120} \\ W_{120}^T & W_{220} \end{bmatrix}$$

$$M_i = \begin{bmatrix} W_{11i} & W_{12i} \\ W_{12i}^T & W_{22i} \end{bmatrix}$$

$$M_{ij} = \begin{bmatrix} W_{11ij} & W_{12ij} \\ W_{12ij}^T & W_{22ij} \end{bmatrix}$$

$$
\begin{aligned}
W_{110} &= N_1 A_{d0} + A_{d0}^T N_1^T + C_d^T F_{d0}^T R F_{d0} C_d + S \\
W_{11i} &= N_1 A_{di} + A_{di}^T N_1^T + C_d^T F_{d0}^T R F_{di} C_d \\
&\quad + C_d^T F_{di}^T R F_{d0} C_d \\
W_{11ij} &= N_1 A_{dij} + A_{dij}^T N_1^T + C_d^T F_{di}^T R F_{dj} C_d \\
W_{120} &= P_0 + A_{d0}^T N_2^T - N_1 A_{c0} + C_d^T F_{di}^T R F_0 C \\
W_{12i} &= P_i + A_{di}^T N_2^T - N_1 A_{ci} + C_d^T F_{d0}^T R F_i C \\
&\quad + C_d^T F_{di}^T R F_0 C \\
W_{12ij} &= A_{dij}^T N_2^T - N_1 A_{cij} + C_d^T F_{di}^T R F_j C \\
W_{220} &= \sum_{k=1}^{p} P_k \rho_k - N_2 A_{c0} - A_{c0}^T N_2^T + Q \\
&\quad + C^T F_0^T R F_0 C; \rho_k \in \Omega_t \\
W_{22i} &= -N_2 A_{ci} - A_{ci}^T N_2^T + C^T F_0^T R F_i C \\
&\quad + C^T F_i^T R F_0 C \\
W_{22ij} &= -N_2 A_{cij} - A_{cij}^T N_2^T + C^T F_i^T R F_j C \\
A_{c0} &= A_0 + B_0 F_0 C \\
A_{ci} &= A_i + B_0 F_i C + B_i F_0 C \\
A_{cij} &= B_i F_j C \\
A_{d0} &= I - B_0 F_{d0} C_d \\
A_{di} &= -B_0 F_{di} C_d - B_i F_{d0} C_d \\
A_{dij} &= -B_i F_{dj} C_d
\end{aligned}
$$

□

Proof. Proof is based on *Lemma 1* and *2*. From (8) and (12) we can obtain

$$[2N_1\dot{x} + 2N_2 x]^T [A_d(\theta)\dot{x} - A_c(\theta)x] = 0 \tag{17}$$

and

$$\frac{dV}{dt} = \dot{x}^T P(\theta) x + x^T P(\theta)\dot{x} + x^T \dot{P}(\theta) x \tag{18}$$

Summarizing the above two equations, for the time derivative of Lyapunov function one obtains

$$\frac{dV}{dt} = z^T \begin{bmatrix} N_1 A_d(\theta) + A_d(\theta)^T N_1^T & -N_1 A_c(\theta) + A_d^T N_2^T + P(\theta) \\ * & -N_2 A_c(\theta) - A_c^T(\theta) N_2^T + \sum_{i=1}^{p} P_i \rho_i \end{bmatrix} z \tag{19}$$

where $N_1, N_2 \in R^{n\times n}$ are auxiliary matrices and $z^T = \begin{bmatrix} \dot{x}^T & x^T \end{bmatrix}$. When one substitutes control algorithm (4) to the right hand side of (13) and then the obtained result is combined with (19) and substituted to (13), after some manipulation, using *Lemma 2* we obtain (16), which proofs the *Theorem 1.* □

Let us denote $\theta_m = \sum_{i=1}^{p} \theta_i$, multiplying (16a) with $\sum_{i=1}^{p} \frac{\theta_i}{\theta_m}$ assuming that $\theta_m \neq 0$ and $\theta_m \in \langle \underline{\theta}_m, \overline{\theta}_m \rangle$ we obtain

$$\frac{M_0}{\theta_m^2} \sum_{i=1}^{p}\sum_{j=1}^{p} \theta_i\theta_j + \sum_{i=1}^{p}\sum_{j=1}^{p} \frac{M_i}{\theta_m}\theta_i\theta_j + \sum_{i=1}^{p}\sum_{j=1}^{p} M_{ij}\theta_i\theta_j < 0 \tag{20}$$

After small manipulation

$$\sum_{i=1}^{p}\sum_{j=1}^{p}\left[M_0 + M_i\theta_m + M_{ij}\theta_m^2\right]\theta_i\theta_j < 0 \tag{21}$$

The closed-loop system will be stable or (21) holds if

$$K_{ij} + K_{ji} < 0, \quad i = 1, 2, \cdots, p, \quad j = i, i+1, \cdots, p \tag{22}$$

where

$$K_{ij} = M_0 + M_i\theta_m + M_{ij}\theta_m^2$$

Using stability conditions (22) and *Lemma 2* if the following inequalities are met, the closed-loop system is affine quadratically stable

$$\begin{aligned} 2M_0 + (M_i + M_j)\,\underline{\theta}_m + (M_{ij} + M_{ji})\,\underline{\theta}_m^2 < 0 \\ 2M_0 + (M_i + M_j)\,\overline{\theta}_m + (M_{ij} + M_{ji})\,\overline{\theta}_m^2 < 0 \\ M_{ij} + M_{ji} \geq 0 \end{aligned} \tag{23}$$

for $i = 1, 2, \cdots, p$, $j = i, i+1, \cdots, p$.

Lemma 3. Closed-loop system (7) is AQS with guaranteed cost if there exist $p+1$ definite matrices $P_0, P_1, \cdots, P_p$ such that for all $\theta \in \Omega$, $P(\theta)$ (11) is positive definite, matrices N_1, N_2 and gain scheduled matrices $F(\theta)$ and $F_d(\theta)$ are satisfying (23). □

If the solution of *Theorem 1.* or *Lemma 3.* are feasible:

- For the case of closed-loop system stability, with respect to matrices N_1, N_2 and positive definite matrix $P(\theta)$ the closed-loop system is affine quadratically stable with guaranteed cost and for $\theta \in \Omega$, $\dot{\theta} \in \Omega_t$. For this case gain matrices (4), (5) and (6) are known and (16), (23) reduces to LMI.
- For the gain-scheduled controller design with respect to matrices $F(\theta)$, $F_d(\theta)$, N_1, N_2 and positive definite matrix $P(\theta)$, the closed-loop system is affine quadratically stable with guaranteed cost and for $\theta \in \Omega$, $\dot{\theta} \in \Omega_t$. For this case (16) and (23) are BMI.

4. Examples

The first example is taken from paper [16]. Consider a simple linear time-varying plant with parameter varying coefficients

$$\begin{aligned} \dot{x}(t) &= a(\alpha)x(t) + b(\alpha)u(t) \\ y(t) &= x(t) \end{aligned} \tag{24}$$

where $\alpha(t) \in R$ is an exogenous signal that changes the parameters of the plant as follows

$$a(\alpha) = -6 - \frac{2}{\pi}\arctan\left(\frac{\alpha}{20}\right) \tag{25}$$

$$b(\alpha) = \frac{1}{2} + \frac{5}{\pi}\arctan\left(\frac{\alpha}{20}\right) \tag{26}$$

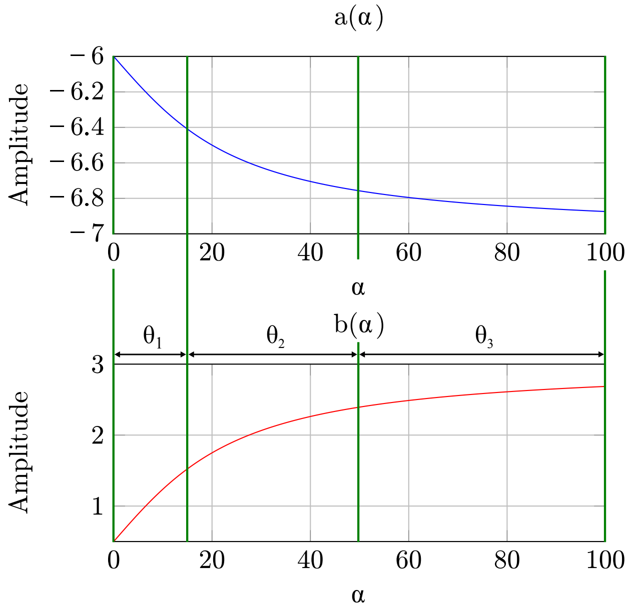

Figure 1: Exogenous signal $\alpha(t)$

Let the problem be the design of a gain scheduled PID controller which will guarantee the closed-loop stability and guaranteed cost for $\alpha \in< 0, 100 >$. We will demonstrate that with the gain-scheduled controller we will obtain practically identical behaviour for closed-loop system. To be able to demonstrate this feature, let us divide the working area to 3 sections with 4 transfer functions in points $\alpha = 0, 15, 50, 100$ so that in each area, where the plant parameter changes, they are nearly linear (Fig. 1.).

In these working points the calculated transfer functions are:

$$\begin{aligned} G_{s1}|_{\alpha=0} &= \frac{0.5}{s+6}, & G_{s2}|_{\alpha=15} &= \frac{1.5242}{s+6.4097} \\ G_{s3}|_{\alpha=50} &= \frac{2.0642}{s+6.6257}, & G_{s4}|_{\alpha=100} &= \frac{2.6858}{s+6.8743} \end{aligned} \tag{27}$$

We transform the above transfer functions to the time domain to obtain the scheduling model in the form (1). The obtained model was extended for the gain-scheduled PID controller design. The extended model is given as follows

$$\begin{aligned} A_0 &= \begin{bmatrix} -6.4370 & 0 \\ 1 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} -0.2050 & 0 \\ 0 & 0 \end{bmatrix} \\ A_2 &= \begin{bmatrix} -0.1080 & 0 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} -0.1240 & 0 \\ 0 & 0 \end{bmatrix} \\ B_0 &= \begin{bmatrix} 1.5930 \\ 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0.5120 \\ 0 \end{bmatrix} \\ B_2 &= \begin{bmatrix} 0.2700 \\ 0 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 0.3110 \\ 0 \end{bmatrix} \\ C &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad D = 0 \end{aligned}$$

Using *Theorem 1* for $\theta_i \in \langle -1, 1\rangle$, $i = 1, 2, 3$ we obtain gain scheduled controller in the form:

$$G_{rGS} = G_{r0} + G_{r1}\theta_1 + G_{r2}\theta_2 + G_{r3}\theta_3 \tag{28}$$

where

$$\begin{aligned}
G_{r0} &= \frac{0.4386s^2 + 2.8850s + 4.4678}{s} \\
G_{r1} &= -\frac{1.72 \times 10^{-6}s^2 + 9.49 \times 10^{-5}s + 5.26 \times 10^{-5}}{s} \\
G_{r2} &= -\frac{0.0283s^2 + 1.5645s + 0.8676}{s} \\
G_{r3} &= -\frac{0.0056s^2 + 0.3085s + 0.1711}{s}
\end{aligned}$$

Note that if plant models in all working points are equal, in this case we obtain $G_{ri} = 0$, $i = 1, 2, \ldots, p$. If some of $G_{ri} \approx 0$ it indicates that some parameters of plant model are close to other ones.

Using (1) and control algorithm

$$u = F(\theta)\left(Cx - w\right) + F_d(\theta)C_d\dot{x} \tag{29}$$

one obtains the structure for simulation of the closed-loop system with gain scheduled PID controller.

Simulation results (Fig. 2.,3.) confirm that *Theorem 1* holds. Fig. 2 demonstrates that with the gain-scheduled controller we have obtained practically identical behaviour for closed-loop system even if α changes as shown in Fig. 3. In figures, $y(t)$ is the output signal, $w(t)$ is the setpoint, $u(t)$ is the controller output, $\alpha(t)$ is exogenous signal on which the system depends and θ is the gain scheduled parameter.

The second example is taken from paper [8]. The model in the form (1) is extended for gain scheduled PID controller design. The extended model is given as follows for $\theta_1 \in \langle -1, 1\rangle$

$$A_0 = \begin{bmatrix} -4 & 3 & 5 & 0 \\ 0 & 7 & -5 & 0 \\ 0.1 & -2 & -3 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 0 & -5 & 0 \\ 2 & 5 & 1.5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 0 \\ 16 \\ 10 \\ 0 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ -5 \\ 3.5 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Using *Lemma 3.* we have obtained the gain-scheduled controller in the form (4) which after small manipulation can be transformed to the form

$$G_{rGS} = G_{r0} + G_{r1}\theta_1 \tag{30}$$

where

$$\begin{aligned}
G_{r0} &= \frac{0,139s^2 + 2,0381s + 0,2401}{s} \\
G_{r1} &= -\frac{0,0027s^2 + 0,014s + 0,004}{s}
\end{aligned}$$

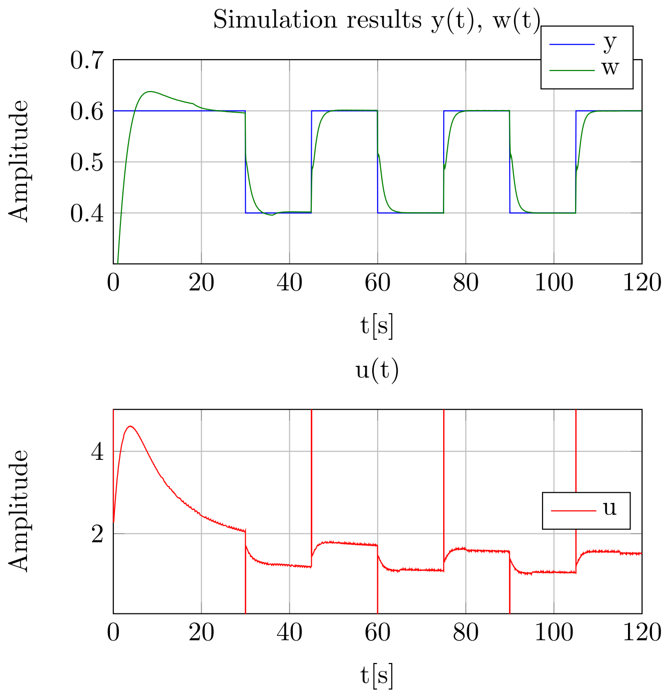

Figure 2: Simulation results

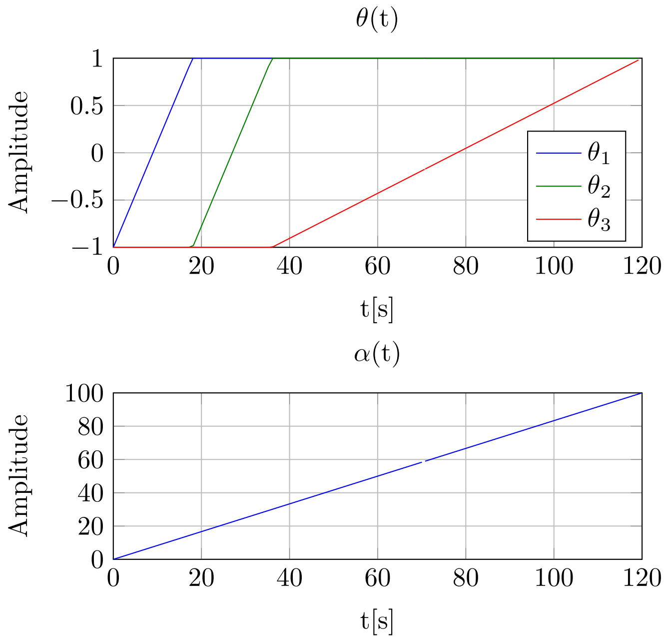

Figure 3: $\theta(t), \alpha(t)$

The simulation results (Fig. 4., 5., 6., 7) confirm, that *Lemma 3.* holds. Figs. 4., 5., 6 demonstrate that with the gain-scheduled controller designed using *Lemma 3* we are able to stabilize and control system with such parameter changes. We can see in Fig. 4. that at $\theta = 1$ the system is slow, and the controller output is positive although when $\theta = 0$ or $\theta = -1$ the system is rapidly fast and the controller output is negative as shown in Fig. 5., 6. Fig. 7. shows a case, when θ is changing linearly in interval $\langle -1, 1\rangle$.

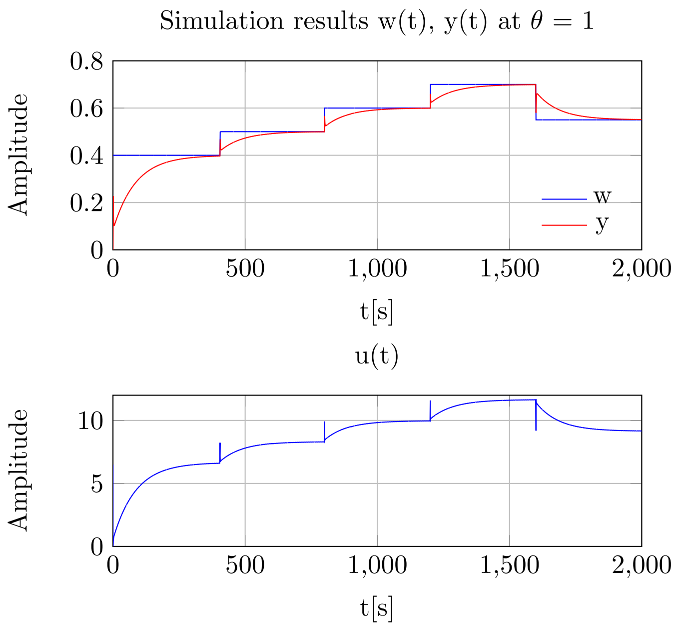

Figure 4: Simulation results for $\theta = 1$

The third example is a realistic model from Humusoft (magnetic levitation, for more detail see www.humusoft.com). The model consists of a coil and a steel ball levitating in a magnetic field. Position of the steel ball is affected by the intensity of the magnetic field and is measured by a linear induction sensor connected to A / D converter. In terms of system theory it is an unstable nonlinear dynamic system with one input (amplifier voltage for coil) and one output (ball position).

We split the ball position (voltage converted by the data acquisition card and scaled to 0÷1 machine unit [MU]) into 3 operating points

1. Position: *0.3 MU* $\longrightarrow$ $\theta_1 = -1, \quad \theta_2 = -1$
2. Position: *0.5 MU* $\longrightarrow$ $\theta_1 = +1, \quad \theta_2 = -1$
3. Position: *0.7 MU* $\longrightarrow$ $\theta_1 = +1, \quad \theta_2 = +1$

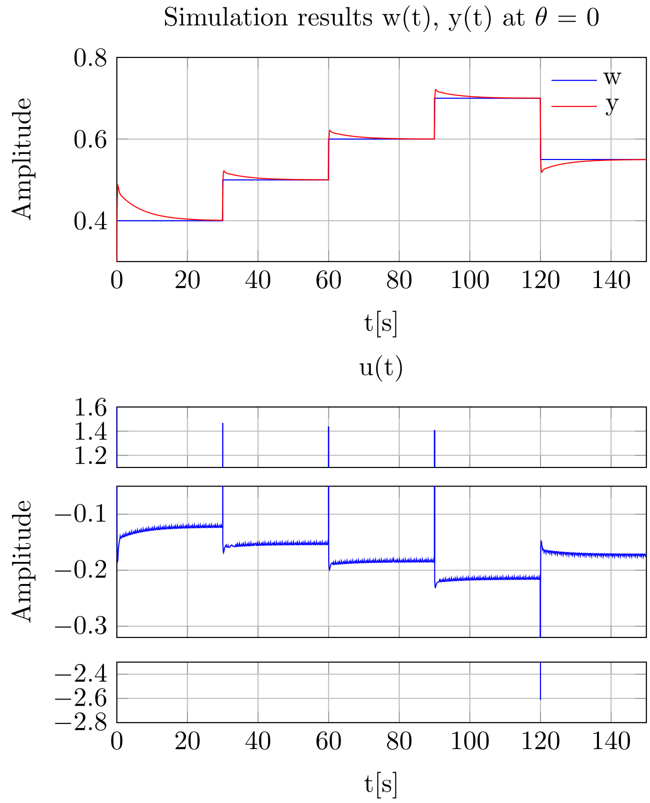

Figure 5: Simulation results for $\theta = 0$

In these working points identified[1] plant transfer functions are

$$
\begin{aligned}
G_{s1} &= \frac{2.0921}{0.000264s^2 + 0.0004s - 1} \\
G_{s2} &= \frac{2.2487}{0.00027s^2 + 0.0032s - 1} \\
G_{s3} &= \frac{2.1205}{0.000155s^2 + 0.0065s - 1}
\end{aligned} \tag{31}
$$

The above transfer functions are transformed to the time domain to obtain the scheduling model in the form (1). The obtained model is extended for the gain-scheduled PID controller design. The extended model is given as follows

[1] Transfer functions were identified in closed-loop system

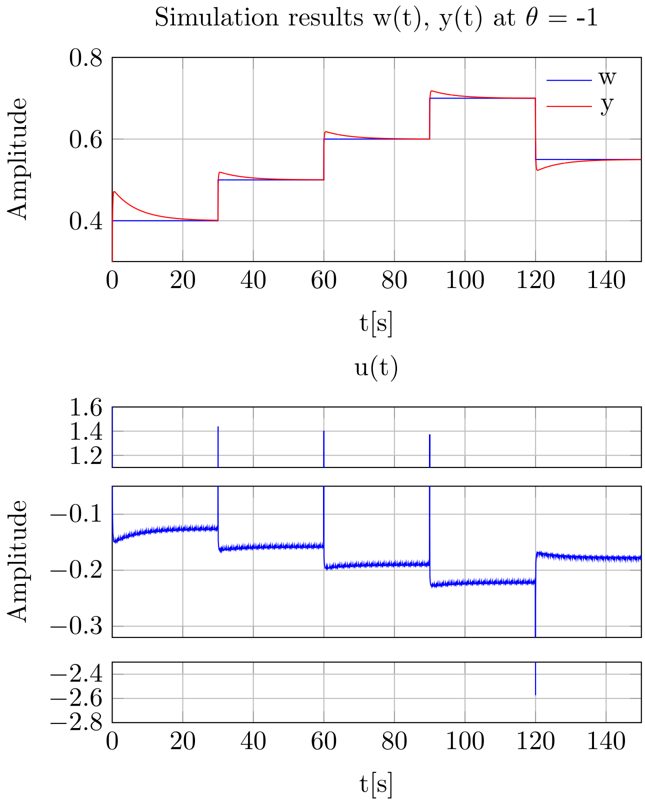

Figure 6: Simulation results for $\theta = -1$

$$A_0 = \begin{bmatrix} 0 & 4.1667 \cdot 10^3 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -5.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 833.3333 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 8.7083 \cdot 10^3 \\ 0 \\ 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -805 \\ 0 \\ 0 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 2.7667 \cdot 10^3 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using *Lemma 3.* with weighting matrices $R = rI$, $r = 1$, $Q = qI$, $q = 1 \times 10^{-1}$, $S = sI$, $s = 1 \times 10^{-3}$ (when increasing q or s with respect to r in the first case dynamic behaviour of the closed-loop system becomes faster and in the second case the overshoot of closed-loop system is smaller, for more detail see [17]) we have obtained gain scheduled controller in the form (4) which after small manipulation can be transformed to the form

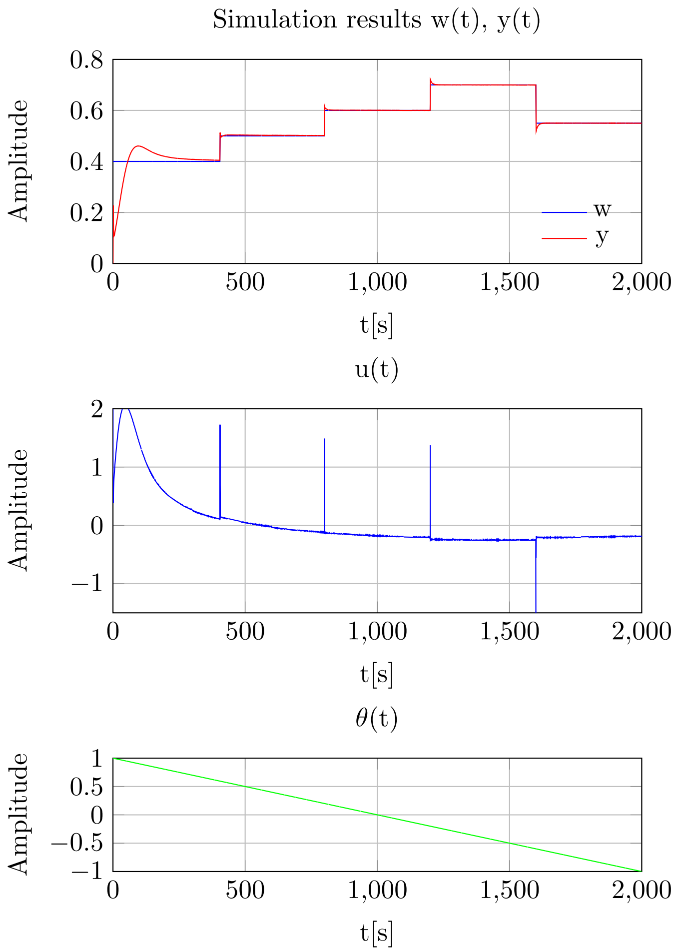

Figure 7: Simulation results for $\theta \in \langle -1, 1\rangle$

$$G_{rGS} = G_{r0} + G_{r1}\theta_1 + G_{r2}\theta_2 \tag{32}$$

where

$$G_{r0} = \frac{0.0926s^2 + 2.2966s + 1.7304}{s}$$
$$G_{r1} = -\frac{0.02s^2 + 0.0103s - 0.0007}{s}$$
$$G_{r2} = \frac{0.0017s^2 - 0.0009s + 0.0016}{s}$$

Simulation results are shown in Fig. 8.

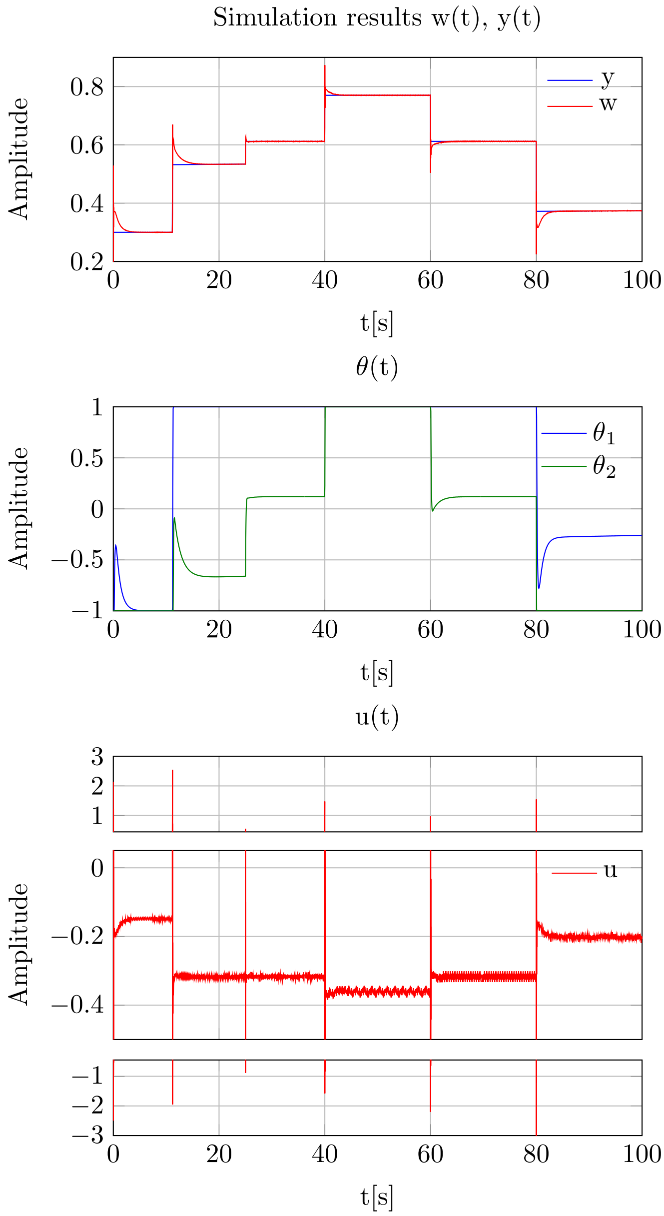

Figure 8: Simulation results for $R = 1, Q = 1 \cdot 10^{-1}, S = 1 \cdot 10^{-3}$

5. Conclusion

The paper addresses the problem of the gain-scheduled controller design which ensures the closed-loop stability and guaranteed cost for all scheduled parameter changes. The proposed procedure is based on the Lyapunov theory of stability, guaranteed cost and BMI. In the gain-scheduled controller design procedure one can include the maximal value of the rate of gain-scheduled parameter changes, which allows to decrease conservativeness and obtain the controller with a given performance. The obtained simulation results show that the gain-scheduled controller may give a better performance of closed-loop system for all changes of scheduled parameter than a classical one including robust controller. Another advantage of this method is the fact that we can affect the quality and cost with weighting matrices R, Q, S. Numerical examples illustrate the effectiveness of the proposed approach.

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