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0.1 Linear Combinations

Definition 0.1.1: Linear Combination

Given a matrix $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$ where $\mathbf{a}_1$ and $\mathbf{a}_2$ are column vectors, and a vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, the linear combination is given by:

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2$$

In general, for $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$, the linear combination is:

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

Note:-

Intuition: Scaling each column and then adding corresponding rows.

Example 0.1.1 (Example of Linear Combination)

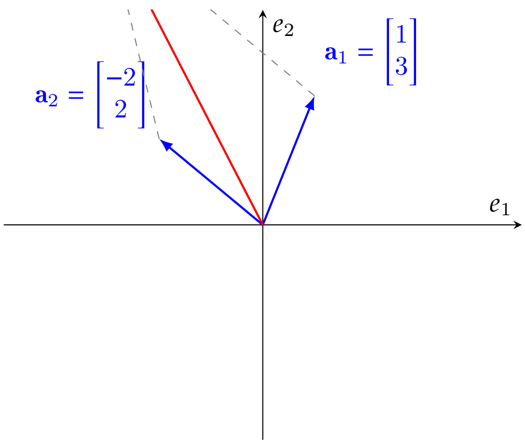
Let $A = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$. Then,

$$A\mathbf{x} = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + k_2 \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1k_1 + (-2)k_2 \\ 3k_1 + 2k_2 \end{bmatrix}$$

For example, if $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, then

$$A\mathbf{x} = 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -4 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$$

0.1.1 Visualizing Linear Combinations in $\mathbb{R}^2$



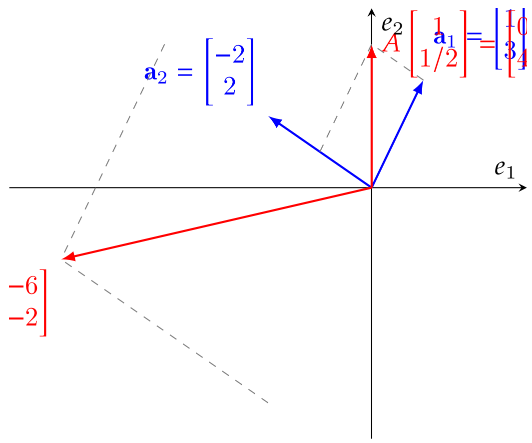
Note:-

If $A = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$, then $A\mathbf{x}$ shows the linear combination of columns of A. For example, if $\mathbf{x} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$, then

$$A\mathbf{x} = 1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

If $\mathbf{x} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, then

$$A\mathbf{x} = -2\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \end{bmatrix} + \begin{bmatrix} -4 \\ 4 \end{bmatrix} = \begin{bmatrix} -6 \\ -2 \end{bmatrix}$$



0.2 Span

Definition 0.2.1: Span

The span of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is the set of all possible linear combinations of these vectors.

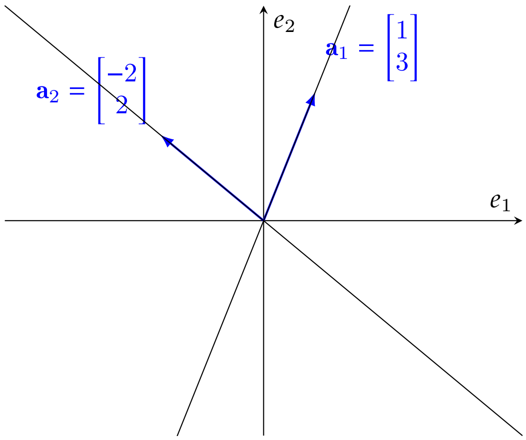
$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p \mid c_1, c_2, \ldots, c_p \in \mathbb{R}\}$$

Note:-

Intuition: All the places vectors can reach through linear combination.

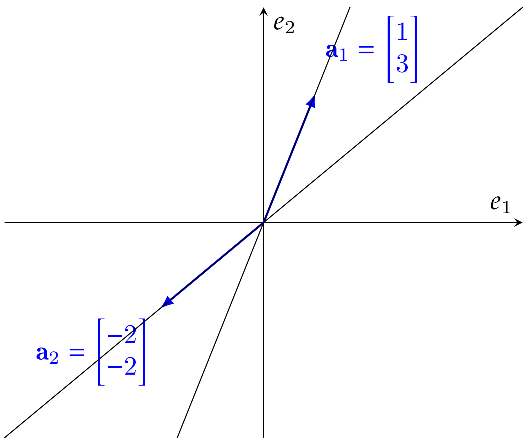
Example 0.2.1 (Span Example in $\mathbb{R}^2$)

Let $A = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$, columns are $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{a}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2\} = \text{Span}\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}\right\}$ is the set of all vectors that can be made by linear combination $A\mathbf{x}$.



Note:-

Intuition: Set of possible vectors $A\mathbf{x}$ with the linear combination $A\mathbf{x}$ are those vectors outside Span are those that are not consistent. Vectors outside Span are those which the columns are overlapping or parallel.



0.3 When is b in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$?

Question 1: When is b in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$?

A vector $\mathbf{b}$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ if there exist weights x_1, x_2 such that $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{b}$. This is equivalent to asking if the vector equation $A\mathbf{x} = \mathbf{b}$ has a solution, where $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Example 0.3.1 (Example: Checking if $\mathbf{b}$ is in Span)

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$. Is $\mathbf{b}$ in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$?

We need to check if there exist x_1, x_2 such that $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{b}$.

$$x_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$$

This leads to the system of equations:

$$\begin{aligned} x_1 - x_2 &= 7 \\ -2x_1 + 3x_2 &= -3 \end{aligned}$$

Solving this system: From the first equation, $x_1 = 7 + x_2$. Substituting into the second equation:

$$-2(7 + x_2) + 3x_2 = -3 - 14 - 2x_2 + 3x_2 = -3x_2 = 11$$

Then $x_1 = 7 + 11 = 18$. So, $18\mathbf{v}_1 + 11\mathbf{v}_2 = \mathbf{b}$. Thus, $\mathbf{b}$ is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

Example 0.3.2 (Example: When **b** is NOT in Span)

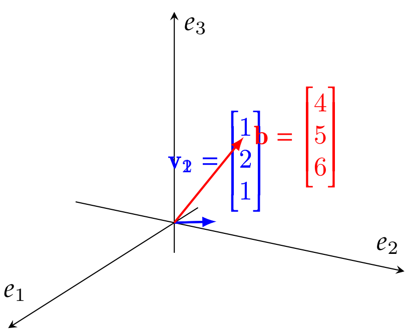
Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$. Is $\mathbf{b}$ in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$?

Here, $\mathbf{v}_1$ and $\mathbf{v}_2$ are the same vector (or linearly dependent if they were different but parallel). If we try to solve $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{b}$, we get:

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$(x_1 + x_2) \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

This would require $x_1 + x_2 = 4$, $2(x_1 + x_2) = 5$, and $x_1 + x_2 = 6$ simultaneously, which is impossible. Therefore, $\mathbf{b}$ is not in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.



Note:-

If $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, then $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is not $\mathbb{R}^3$ because $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a line, and $\mathbf{b}$ is not in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.