2.18 Applications of Double Integrals

2.18.1 Area of a Surface

Definition 2.18.1: Area of a Surface with Double integrals

When calculating the area of a surface where the height of the surface in 3D space is constantly 1, the area of the surface is numerically equal to the volume under the surface. Thus, by setting the height $f(x, y) = 1$ over a given region R, the area of R can be computed using a double integral, which effectively measures the ”volume” under the constant surface.

$$\text{Area of } R = \iint_R f(x, y)\, dA = \iint_R 1\, dA$$

Example 2.18.1 (Example)

Calculate the area of the region $R = \{(x, y) : x^2 + y^2 \leqslant 1\}$, which is the unit disk in the xy-plane.

$$\iint_R dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\, dx$$

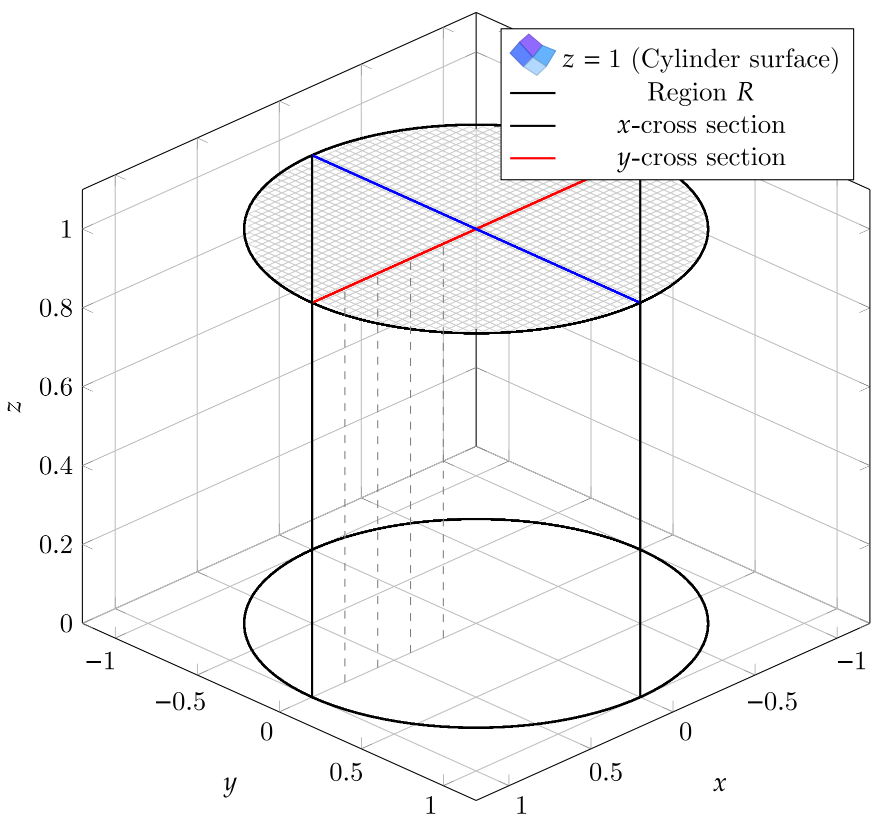
Calculating,

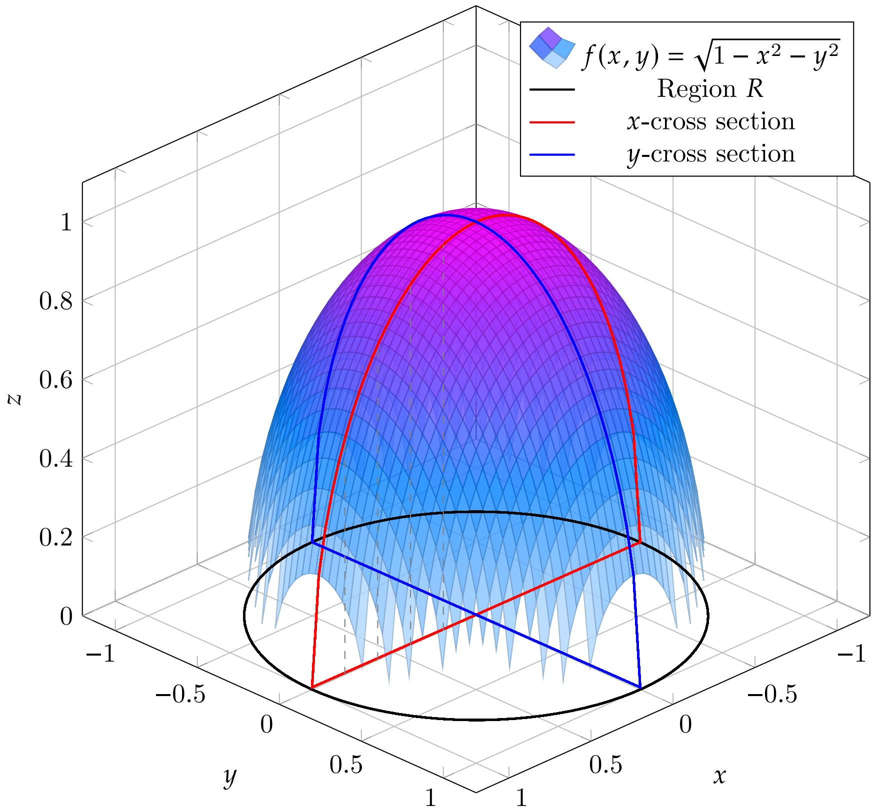
$$= 2 \int_{-1}^{1} \sqrt{1 - x^2}\, dx$$

Using substitution, let $x = \cos\theta$, we get:

$$= 2 \int_0^{\pi} \sin^2\theta\, d\theta = \pi$$

Thus, the area of R is π.





2.19 Derivation of Triple Integrals through Riemann Sums

To derive the concept of a triple integral, we start by considering a bounded, three-dimensional region $D \subset \mathbb{R}^3$ over which we wish to integrate a continuous function $f(x, y, z)$. The idea is to approximate the "volume" under the surface defined by $f(x, y, z)$ over D by partitioning D into smaller subregions, summing up values of f at chosen points within these subregions, and then taking the limit as the partitions become finer.

2.19.1 Partitioning the Region

1. Let D be divided into $n \times m \times l$ subregions, each of volume $\Delta V_{ijk} = \Delta x_i \Delta y_j \Delta z_k$, where:

$$\Delta x_i = x_{i+1} - x_i, \quad \Delta y_j = y_{j+1} - y_j, \quad \Delta z_k = z_{k+1} - z_k$$

2. In each subregion, choose a point (x_i^*, y_j^*, z_k^*) where f will be evaluated. The function value at each of these points, $f(x_i^*, y_j^*, z_k^*)$, approximates the "height" over the corresponding volume element.

2.19.2 Forming the Riemann Sum

With these chosen points and partition, the Riemann sum for f over the region D is given by:

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l} f(x_i^*, y_j^*, z_k^*) \Delta x_i \Delta y_j \Delta z_k.$$

As the partitions become finer, i.e., $\Delta x_i, \Delta y_j, \Delta z_k \to 0$ for all i, j, k, the sum approaches the exact "volume" under f over D, which we denote as the triple integral:

$$\iiint_D f(x, y, z)\, dV = \lim_{n,m,l \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l} f(x_i^*, y_j^*, z_k^*) \Delta x_i \Delta y_j \Delta z_k.$$

Thus, the triple integral represents the limit of the Riemann sum as the volume elements ΔV_{ijk} become infinitesimally small.

$$f(x,y,z) = \begin{cases} 10^3 e^{-(x+y+z)} \cdot \frac{1}{10} & \text{if } x,y,z \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

(1) Probability that $(x,y,z) \in D$:

$$P((x,y,z) \in D) = \iiint_D 10^3 e^{-(x+y+z)} \cdot \frac{1}{10}\, dV$$

(2) Probability that customers wait at most 5 minutes:

$$D = [0,5] \times [0,5] \times [0,5]$$

$$P = \iiint_D 10^3 e^{-(x+y+z)} \cdot \frac{1}{10}\, dx\, dy\, dz = 0.061 = 6.1\%$$

(3) Probability that $0 \leqslant x \leqslant y \leqslant z \leqslant 20$: Define

$$P = \{(x,y,z) : 0 \leqslant x \leqslant y \leqslant z \leqslant 20\}$$

1. Find the domain of x in D:

$$0 \leqslant x \leqslant 20$$

2. Fix some $x = x_0$, define $R = \{(y,z) : 0 \leqslant x_0 \leqslant y \leqslant 20\}$. 3. Marginalize y in R:

$$x \leqslant y \leqslant 20$$

4. Marginalize z in R:

$$y \leqslant z \leqslant 20$$

Thus,

$$D = [0,20] \times [x,20] \times [y,20]$$

$$P = \int_0^{20} \int_x^{20} \int_y^{20} 10^3 e^{-(x+y+z)} \cdot \frac{1}{10}\, dz\, dy\, dx$$

Theorem 2.20.1 Fubini's Theorem Extended to Triple Integrals

Fubini's Theorem allows us to evaluate triple integrals as iterated integrals by integrating one variable at a time. If $f(x,y,z)$ is continuous on a rectangular region $D = [a,b] \times [c,d] \times [e,f]$, then

$$\iiint_D f(x,y,z)\, dV = \int_a^b \int_c^d \int_e^f f(x,y,z)\, dz\, dy\, dx.$$

This can also be rearranged in different orders of integration:

$$\iiint_D f(x,y,z)\, dV = \int_c^d \int_a^b \int_e^f f(x,y,z)\, dx\, dz\, dy,$$

and similarly for other orders, depending on the bounds and convenience for computation.

Example 2.20.5 (Example of Setting up the Integral with Bounds)

Find the integral $\iiint_D xyz\, dV$ where $D = \{(x,y,z) \mid x+y+z \leqslant 1,\ x,y,z \geqslant 0\}$.
To determine the bounds for integration, we analyze the region D defined by $x+y+z \leqslant 1$ with $x,y,z \geqslant 0$.

1. First, we set up the bounds for x:

$$0 \leqslant x \leqslant 1$$

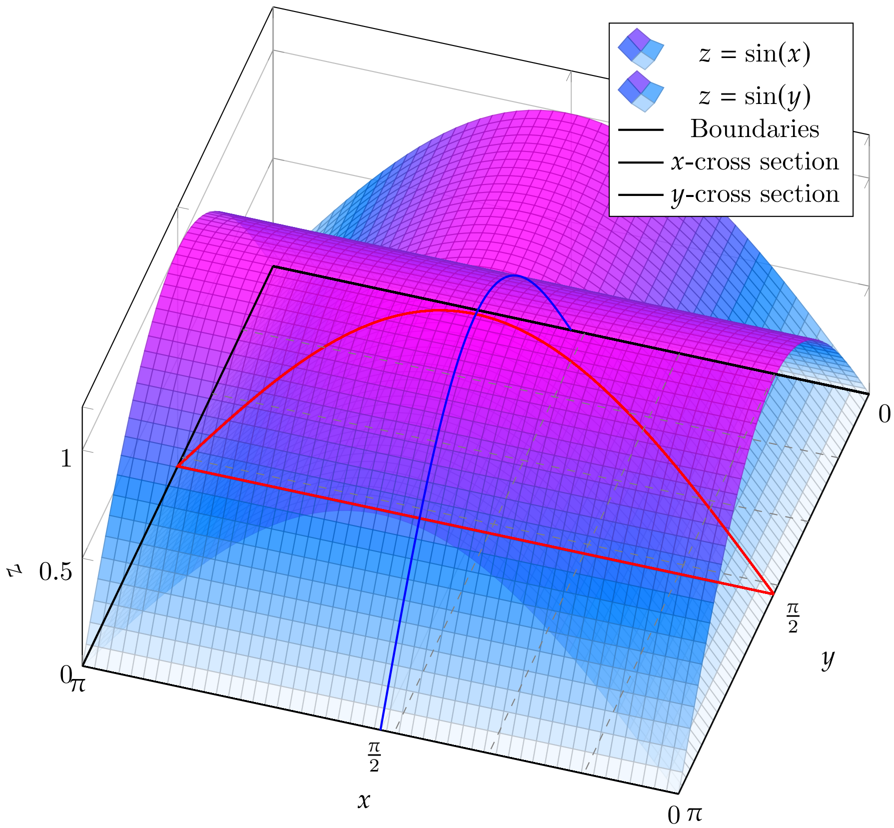
Evaluating the integral:

$$V = 4\int_0^{\pi/2}\int_0^{\sin y}\left(\int_y^{\pi-y} dx\right) dy\, dz.$$

Upon computation, this integral yields:

$$V = \pi - 2.$$

The key idea was to leverage the symmetry of the intersection region, focusing on one-fourth of the area and then scaling up by a factor of 4. By analyzing the geometry, we found that z was bounded by $\sin y$ in region R_1. From there, the triple integral was computed over x, y, and z to yield the final volume of the region.



2.21 Polar Coordinate System

Definition 2.21.1: Polar Coordinates Overview

The polar coordinate system is a two-dimensional coordinate system in which each point on a plane is determined by a distance from a reference point (often called the origin or pole) and an angle from a reference direction (usually the positive x-axis). The two coordinates are:

- r: the radial distance from the origin to the point.
- θ: the angular coordinate, representing the angle in radians (or degrees) between the positive x-axis and the line connecting the origin to the point.

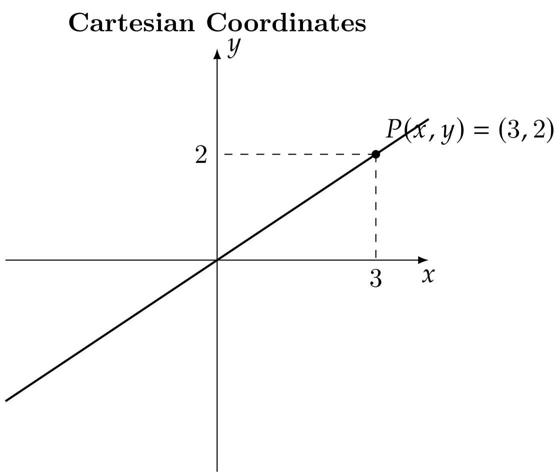
A point P in the plane can therefore be represented in polar coordinates as (r, θ).

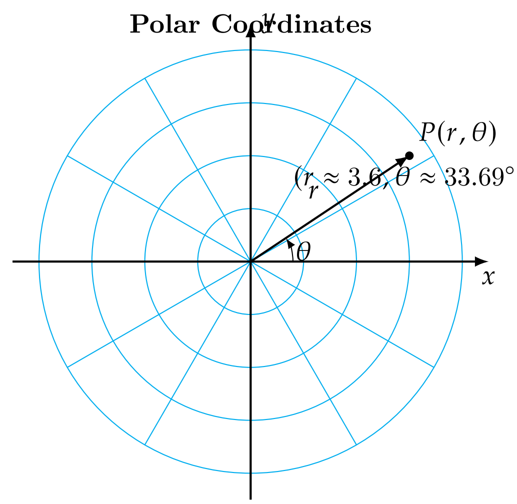
Definition 2.21.2: Relationship Between Cartesian and Polar Coordinates

In Cartesian coordinates, a point P can be represented as (x, y). We can convert between Cartesian and polar coordinates with the following relations:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2}$$
$$\theta = \arctan\left(\frac{y}{x}\right)$$

These equations allow for the translation of a point's position between Cartesian and polar forms, showing the adaptability of the polar system for various types of analyses, especially when working with circular or rotational symmetry.





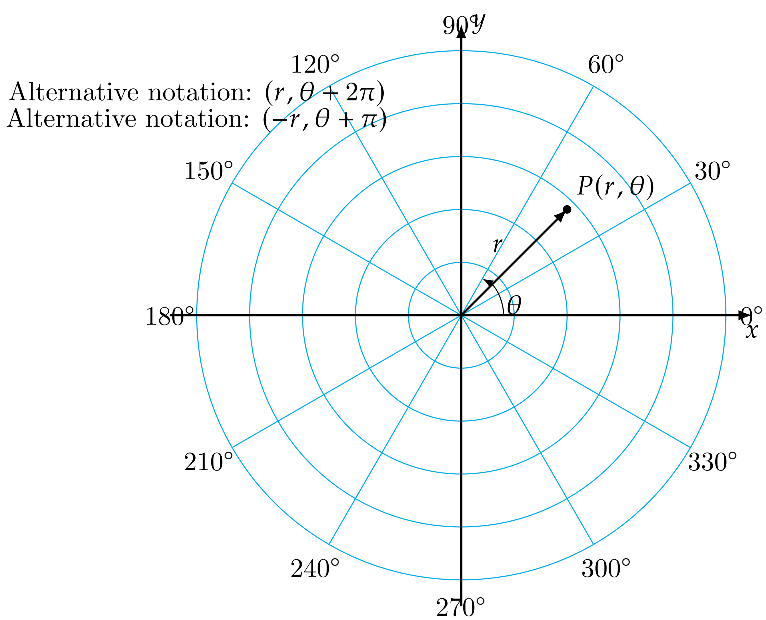


Diagram illustrating the polar coordinate system with concentric circles and angular lines, showing a point P represented as (r, θ) and alternative notations.

(Simplify to evaluate)

2.29 Spherical Coordinate Axes

The relationships between Cartesian coordinates (x, y, z) and spherical coordinates (ρ, ϕ, θ) are as follows:

Definition 2.29.1: Polar - Spherical Relationships

$$x = \rho \cos\phi \cos\theta$$
$$y = \rho \cos\phi \sin\theta$$
$$z = \rho \sin\phi$$

Conversely, the spherical coordinates can be expressed in terms of Cartesian coordinates as:

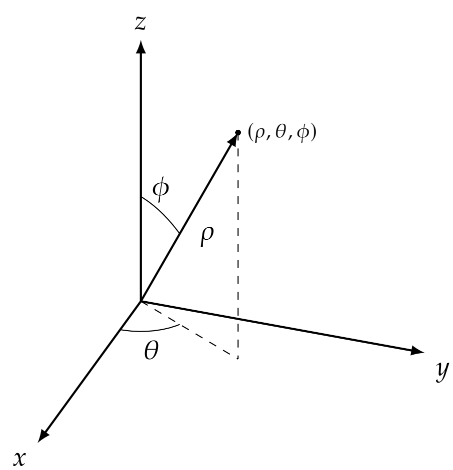
$$\rho = \sqrt{x^2 + y^2 + z^2}$$
$$\phi = \tan^{-1}\left(\frac{z}{\sqrt{x^2 + y^2}}\right)$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

The volume element in spherical coordinates is:

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

The integral of a function $f(x, y, z)$ in spherical coordinates becomes:

$$\iiint_P f(x, y, z)\, dV = \iiint_P f\big(\rho\cos\phi\cos\theta, \rho\cos\phi\sin\theta, \rho\sin\phi\big)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$



2.30 Shapes Represented in Spherical Coordinates

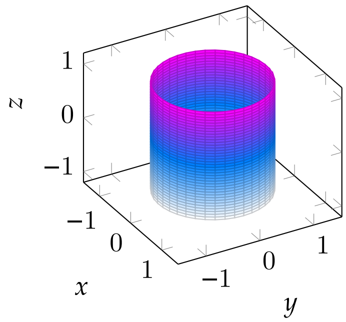
2.30.1 Overview of Spherical Coordinates

- **Definition**: Spherical coordinates (r, θ, ϕ) are defined as:
  - r: Radial distance from the origin.
  - θ: Polar angle (angle from the positive z-axis, $0 \leqslant \theta \leqslant \pi$).

6. Cylinder

Equation: $r \sin\theta = a$
Description: A cylinder of radius a around the z-axis.
Constraints:

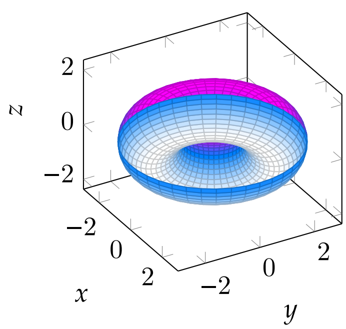
- $0 \leqslant \phi < 2\pi$,
- $r\cos\theta$ is unbounded (representing the z-coordinate).



7. Torus (Approximation)

Equation: $r = R + r_0 \sin\theta$
Description: A torus with major radius R and minor radius r_0, approximately represented in spherical coordinates.
Constraints:

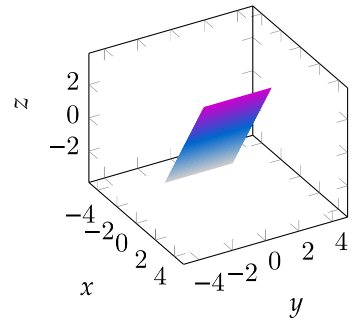
- ϕ varies from 0 to 2π,
- Specific parametric constraints apply.



8. Plane

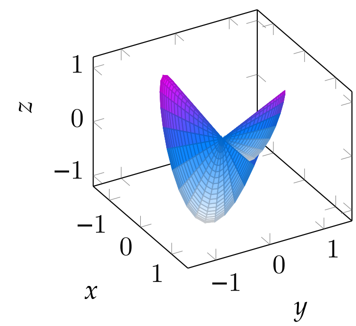
Equation: θ = constant or ϕ = constant
Description:

- θ = constant: A conical plane cutting through the origin at a fixed polar angle.
- ϕ = constant: A vertical plane slicing through the z-axis at a fixed azimuthal angle.



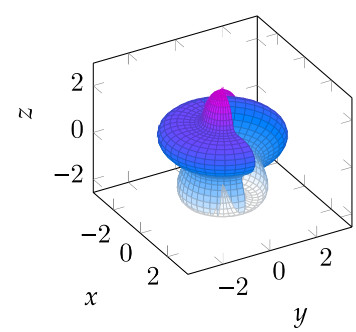
9. Lemniscate Shape

Equation: $r = a\sin\theta\sin 2\phi$
Description: A figure-eight or lemniscate shape in spherical coordinates.



10. Wave-like Surfaces

Equation: $r = R + A\cos(k\theta)$
Description: Oscillating surface around a spherical shell, useful in representing waves or perturbations on a sphere.



2.31 Change of Variable for Double and Triple Integrals

Polar Coordinates

$$\iint_D f(x,y)\,dx\,dy \to \iint_S f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Cylindrical Coordinates

$$\iiint_D f(x, y, z)\, dx\, dy\, dz \to \iiint_S f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz$$

Spherical Coordinates

$$\iiint_D f(x, y, z)\, dx\, dy\, dz \to \iiint_S f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

Theorem 2.31.1 Intuition Behind Change of Variables

We use a **mapping** T to transform coordinates in one space S to another R. This is particularly useful when integrating over regions that are easier to describe in new coordinates (e.g., circular or spherical regions).
For example:

$$S = [0, 2\pi] \times [0, 2], \quad T(r, \theta) = (r\cos\theta, r\sin\theta)$$

Here, the mapping T converts a point in S into a point in R.

Area Differential Transformation

Consider a small differential area element in the original space:

$$dA = |\det(J)|\, du\, dv$$

where J is the **Jacobian matrix**, and $|\det(J)|$ accounts for how the transformation scales area.

Definition 2.31.1: Jacobian Matrix

The Jacobian matrix represents the linear transformation of the mapping T at a given point:

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

For a transformation $T(u, v) = (g(u, v), h(u, v))$, the determinant of J is:

$$\det(J) = \begin{vmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{vmatrix} = \frac{\partial g}{\partial u} \cdot \frac{\partial h}{\partial v} - \frac{\partial g}{\partial v} \cdot \frac{\partial h}{\partial u}$$

Geometric Interpretation

- **Local Stretching/Scaling:** $|\det(J)|$ gives the local scaling factor of the area due to the transformation.
- **Orientation:** The sign of $\det(J)$ indicates whether the orientation is preserved or flipped.

Example 2.31.1 (Polar Coordinates)

For the transformation $T(r, \theta) = (r\cos\theta, r\sin\theta)$, the Jacobian matrix is:

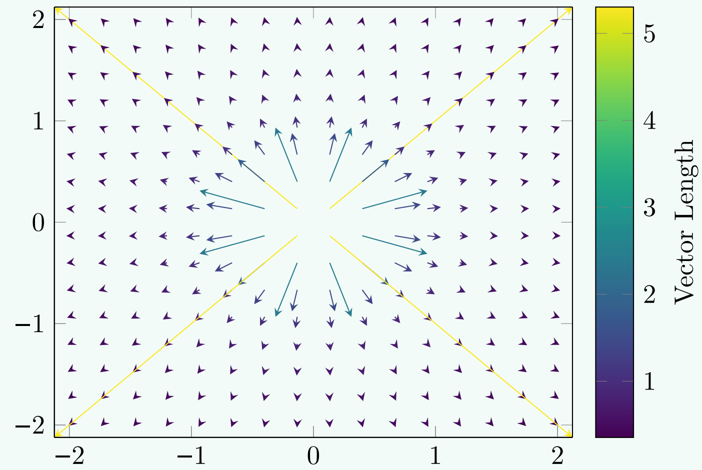
$$J = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

The determinant is:

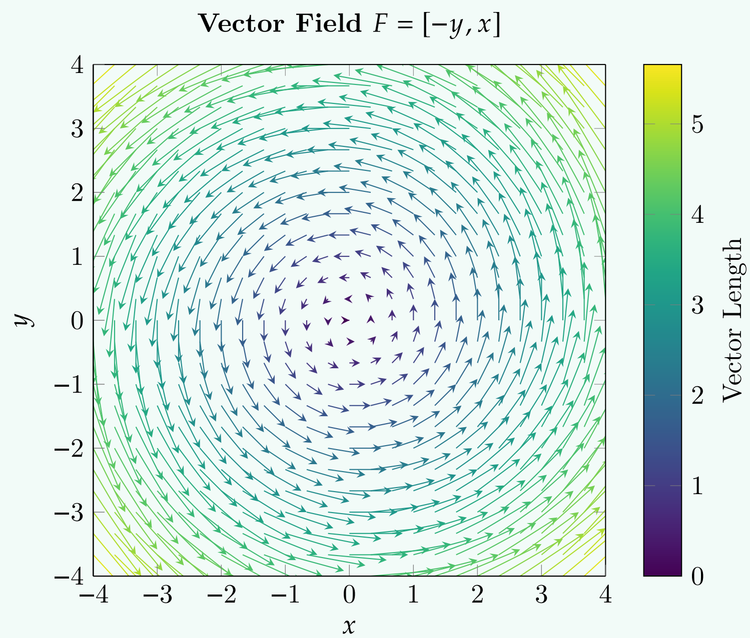
$$\det(J) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

Thus, the area differential in polar coordinates becomes:

$$dx\, dy = r\, dr\, d\theta$$



Example 3.1.2 ($\vec{F}(x,y) = \langle -y, x\rangle$)



3.2 Gradient Vector Fields

A **gradient vector field** $\nabla\varphi(x,y)$ is a vector field:

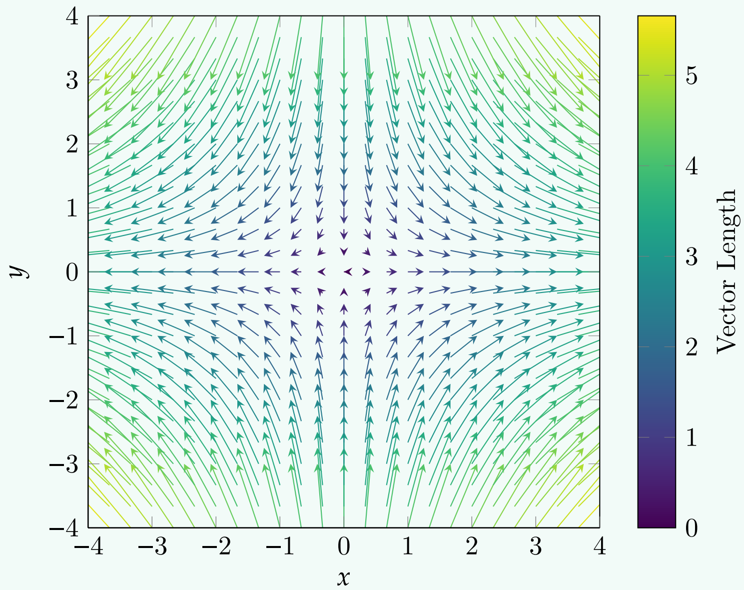
$$\vec{F}(x,y) = \nabla\varphi \quad \text{where } \varphi \text{ is a potential function.}$$

If φ exists, $\vec{F}$ is called a **conservative field**.

Example 3.2.1 ($\varphi = -\frac{x^2+y^2}{2}$)

$$\vec{F}(x,y) = \nabla\varphi = \langle -x, -y\rangle$$

Vector Field $F = [-y, x]$



3.3 Line Integrals

Scalar Case

For a scalar function $f(x, y)$, the line integral over a curve C is defined as:

$$A = \int_C f(x, y)\, ds$$

where $ds = |\mathbf{r}'(t)|dt$, with $\mathbf{r}(t)$ being the parameterization of C.

If $\mathbf{r}(t) = (x(t), y(t))$, then:

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

This allows us to rewrite the line integral as:

$$\int_C f(x, y)\, ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\, dt$$

Example 3.3.1 (Example: Average Temperature on a Plate)

Suppose we have a plate located in $R = \{(x, y) : x^2 + y^2 \leqslant 4\}$, where the temperature at any point (x, y) is $T(x, y) = 100(x^2 + y^2)$. Find the average temperature over the boundary of R.

1. **Parameterize the Boundary:** The boundary C of R is a circle of radius 2, centered at the origin. Parameterize C as:

   $$\mathbf{r}(t) = (2\cos t, 2\sin t), \quad 0 \leqslant t \leqslant 2\pi$$

   Then:

   $$|\mathbf{r}'(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2} = 2$$

2. **Average Temperature:** The average temperature is given by:

   $$\text{Average Temperature} = \frac{\int_C T(x, y)\, ds}{\int_C ds}$$