Homework 10

EK307

Giacomo Cappelletto

Problem 1:

Solution:

$$V_s = 8\langle -40°$$
$$V_s = 8\cos(-40°) + j8\sin(-40°) = 6.128 - j5.142\text{V}$$
$$|V_s| = 8.00\text{V}, \quad \angle V_s = -40.0°$$
$$Z_L = j\omega L = j(2)(3) = j6\Omega$$
$$Z_C = \frac{1}{j\omega C} = \frac{1}{j(2)(0.25)} = \frac{1}{j0.5} = -j2\Omega$$
$$Z_{R_2C} = R_2 + Z_C = 2 - j2\Omega$$
$$Z_{\text{parallel}} = \frac{Z_L \cdot Z_{R_2C}}{Z_L + Z_{R_2C}} = \frac{j6 \cdot (2 - j2)}{j6 + 2 - j2}$$
$$= \frac{j12 + 12}{2 + j4} = \frac{12 + j12}{2 + j4}$$
$$= \frac{(12 + j12)(2 - j4)}{(2 + j4)(2 - j4)} = \frac{24 - j48 + j24 + 48}{4 + 16} = \frac{72 - j24}{20}$$
$$Z_{\text{parallel}} = 3.6 - j1.2\Omega$$
$$|Z_{\text{parallel}}| = 3.795\Omega, \quad \angle Z_{\text{parallel}} = -18.43°$$
$$Z_{\text{total}} = R_1 + Z_{\text{parallel}} = 1 + 3.6 - j1.2 = 4.6 - j1.2\Omega$$
$$|Z_{\text{total}}| = 4.754\Omega, \quad \angle Z_{\text{total}} = -14.62°$$
$$I_s = \frac{V_s}{Z_{\text{total}}} = \frac{6.128 - j5.142}{4.6 - j1.2}$$
$$= 1.520 - j0.721\text{A}$$
$$|I_s| = 1.683\text{A}, \quad \angle I_s = -25.38°$$
$$V_{R_1} = I_s \cdot R_1 = (1.520 - j0.721)(1) = 1.520 - j0.721\text{V}$$
$$|V_{R_1}| = 1.683\text{V}, \quad \angle V_{R_1} = -25.38°$$
$$V_{\text{par}} = V_s - V_{R_1} = (6.128 - j5.142) - (1.520 - j0.721)$$
$$= 4.608 - j4.421\text{V}$$
$$|V_{\text{par}}| = 6.386\text{V}, \quad \angle V_{\text{par}} = -43.81°$$
$$I_L = \frac{V_{\text{par}}}{Z_L} = \frac{4.608 - j4.421}{j6} = -0.737 - j0.768\text{A}$$
$$|I_L| = 1.064\text{A}, \quad \angle I_L = -133.81°$$
$$I_{R_2C} = \frac{V_{\text{par}}}{R_2 + Z_C} = \frac{4.608 - j4.421}{2 - j2}$$
$$= 2.257 + j0.047\text{A}$$
$$|I_{R_2C}| = 2.258\text{A}, \quad \angle I_{R_2C} = 1.19°$$

$$V_{R_2} = I_{R_2C} \cdot R_2 = (2.257 + j0.047)(2) = 4.514 + j0.093\text{V}$$
$$|V_{R_2}| = 4.515\text{V}, \quad \angle V_{R_2} = 1.19°$$
$$V_C = I_{R_2C} \cdot Z_C = (2.257 + j0.047)(-j2) = 0.093 - j4.514\text{V}$$
$$|V_C| = 4.515\text{V}, \quad \angle V_C = -88.81°$$
$$P_{R_1} = \frac{1}{2}\frac{|V_{R_1}|^2}{R_1} = \frac{1}{2}\frac{(1.683)^2}{1} = 1.416\text{W}$$
$$P_{R_2} = \frac{1}{2}\frac{|V_{R_2}|^2}{R_2} = \frac{1}{2}\frac{(4.515)^2}{2} = 5.097\text{W}$$
$$P_L = 0\text{W} \quad (\text{reactive element})$$
$$P_C = 0\text{W} \quad (\text{reactive element})$$

$\boxed{P_{R_1} = 1.416\text{ W}}$ $\boxed{P_{R_2} = 5.097\text{ W}}$ $\boxed{P_L = 0\text{ W}}$ $\boxed{P_C = 0\text{ W}}$

Problem 2:

Solution:

$$Z_{5\parallel L} = \frac{R_5 \cdot Z_L}{R_5 + Z_L} = \frac{5 \cdot j2}{5 + j2}$$
$$= \frac{j10}{5 + j2} = \frac{j10(5 - j2)}{(5 + j2)(5 - j2)} = \frac{j50 + 20}{25 + 4} = \frac{20 + j50}{29}$$
$$Z_{5\parallel L} = 0.6897 + j1.7241\Omega$$
$$Z_{\text{series}} = Z_C + Z_{5\parallel L} = -j3 + 0.6897 + j1.7241$$
$$Z_{\text{series}} = 0.6897 - j1.2759\Omega$$
$$Z_{\text{th}} = R_4 \parallel Z_{\text{series}} = \frac{4 \cdot (0.6897 - j1.2759)}{4 + 0.6897 - j1.2759}$$
$$= \frac{2.7586 - j5.1034}{4.6897 - j1.2759}$$
$$= \frac{(2.7586 - j5.1034)(4.6897 + j1.2759)}{(4.6897 - j1.2759)(4.6897 + j1.2759)}$$
$$= \frac{12.9383 + j3.5205 - j23.9369 + 6.5119}{22.0013 + 1.6279}$$
$$= \frac{19.4502 - j20.4164}{23.6292}$$
$$Z_{\text{th}} = 0.8234 - j0.8642\Omega$$
$$|Z_{\text{th}}| = 1.194\Omega, \quad \angle Z_{\text{th}} = -46.39°$$

Node B: $\frac{V_B - V_s}{R_4} + \frac{V_B - V_C}{Z_C} = 0$

Node C: $\frac{V_C - V_s}{Z_L} + \frac{V_C - V_B}{Z_C} + \frac{V_C}{R_5} = 0$

$$V_B\left(\frac{1}{R_4} + \frac{1}{Z_C}\right) - V_C\left(\frac{1}{Z_C}\right) = \frac{V_s}{R_4}$$
$$-V_B\left(\frac{1}{Z_C}\right) + V_C\left(\frac{1}{Z_L} + \frac{1}{Z_C} + \frac{1}{R_5}\right) = \frac{V_s}{Z_L}$$
$$a_{11} = \frac{1}{4} + \frac{1}{-j3} = 0.25 + j0.3333$$

$$a_{12} = -\frac{1}{-j3} = -j0.3333$$

$$a_{21} = -\frac{1}{-j3} = -j0.3333$$

$$a_{22} = \frac{1}{j2} + \frac{1}{-j3} + \frac{1}{5} = -j0.5 + j0.3333 + 0.2 = 0.2 - j0.1667$$

$$b_1 = \frac{165}{4} = 41.25$$

$$b_2 = \frac{165}{j2} = -j82.5$$

$$\begin{aligned} \det = a_{11}a_{22} - a_{12}a_{21} &= (0.25 + j0.3333)(0.2 - j0.1667) - (-j0.3333)(-j0.3333) \\ &= 0.05 - j0.0417 + j0.0667 + 0.0556 - 0.1111 \\ &= -0.0056 + j0.025 \end{aligned}$$

$$\begin{aligned} V_B = \frac{b_1a_{22} - b_2a_{12}}{\det} &= \frac{41.25(0.2 - j0.1667) - (-j82.5)(-j0.3333)}{-0.0056 + j0.025} \\ &= \frac{8.25 - j6.875 - 27.5}{-0.0056 + j0.025} = \frac{-19.25 - j6.875}{-0.0056 + j0.025} \\ &= \frac{(-19.25 - j6.875)(-0.0056 - j0.025)}{(-0.0056 + j0.025)(-0.0056 - j0.025)} \\ &= \frac{0.1078 + j0.4813 + j0.0385 - 0.1719}{0.0000314 + 0.000625} \\ &= \frac{-0.0641 + j0.5198}{0.0006564} \end{aligned}$$

$$V_{\text{th}} = V_B = 159.22 - j50.10\text{V}$$

$$|V_{\text{th}}| = 166.92\text{V}, \quad \angle V_{\text{th}} = -17.47°$$

$$Z_L^{\text{opt}} = Z_{\text{th}}^* = 0.8234 + j0.8642\Omega$$

$$R_{\text{th}} = 0.8234\Omega, \quad X_{\text{th}} = -0.8642\Omega$$

$$P_{\max} = |V_{\text{th}}\frac{|^2}{4R_{\text{th}}} = \frac{(166.92)^2}{4 \cdot 0.8234}$$

$$P_{\max} = \frac{27860.95}{3.2935} = 8459.54\text{W}$$

$\boxed{Z_L^{\text{opt}} = 0.823 + j\,0.864\;\Omega}$ $\boxed{P_{\max} = 8459.5\text{ W}}$

Problem 3:

3.a)

Solution:

$$H_{L(s)} = \frac{V_{o(s)}}{V_{i(s)}} = \frac{sL}{R+sL}$$

$$H_{L(j\omega)} = \frac{j\omega L}{R+j\omega L}$$

$$\left|H_{L(j\omega)}\right| = \frac{\omega L}{\sqrt{R^2+(\omega L)^2}}$$

Low: $\omega \to 0 \Rightarrow |H_L| \to 0$

High: $\omega \to \infty \Rightarrow |H_L| \to 1$

$$\omega_c = \frac{R}{L} = \frac{200}{0.1} = 2000\text{ rad/s}$$

$f_c = \frac{\omega_c}{2\pi} = \frac{2000}{2\pi} \approx 3.18 \times 10^2 \text{ Hz}$

Filter type: high-pass.

3.b)

Solution:

$$H_{R(s)} = \frac{V_{o(s)}}{V_{i(s)}} = \frac{R}{R+sL}$$
$$H_{R(j\omega)} = \frac{R}{R+j\omega L}$$
$$\left|H_{R(j\omega)}\right| = \frac{R}{\sqrt{R^2+(\omega L)^2}}$$

Low: $\omega \to 0 \Rightarrow |H_R| \to 1$

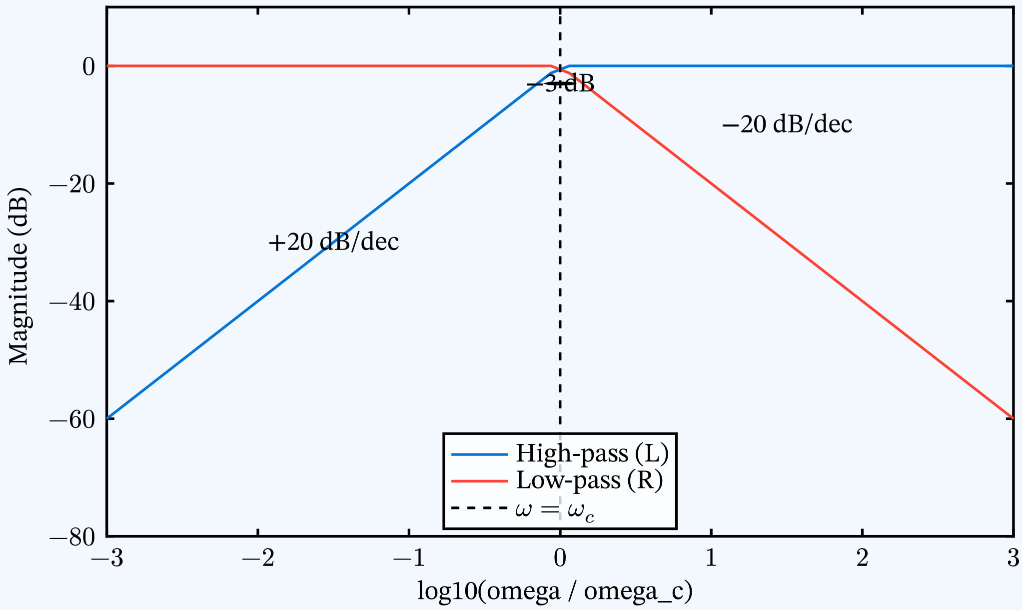
High: $\omega \to \infty \Rightarrow |H_R| \to 0$

At $\omega = \omega_c$: $|H_R| = \frac{1}{\sqrt{2}} \approx -3 \text{ dB}$

Filter type: low-pass.

3.c)

Solution:



3.d)

Solution:

For RC low-pass: $\omega_c = \frac{1}{R_{\{RC\}}C}$

$C = \frac{1}{R_{\{RC\}}\omega_c} = \frac{1}{(10^4)(2000)} = 5 \times 10^{\{-8\}}\text{F} = 50 \text{ nF}$

Circuit: series 10kΩ from v_i to node, capacitor $C = 50$ nF from node to ground, v_o across C.

Problem 4:

4.a)

Solution:

$$Z_L = j\omega L$$
$$Z_C = \frac{1}{j\omega C}$$
$$Z_R = R$$
$$Z_{\{RC\}} = \left(\frac{1}{R} + j\omega C\right)^{\{-1\}}$$

$H(j\omega) = \frac{V_o}{V_i} = \frac{Z_{\{RC\}}}{Z_L + Z_{\{RC\}}} = \frac{\frac{1}{\frac{1}{R}+j\omega C}}{j\omega L + \frac{1}{\frac{1}{R}+j\omega C}}$

$H(j\omega) = \frac{1}{1+j\omega L\left(\frac{1}{R}+j\omega C\right)}$

$j\omega L\left(\frac{1}{R} + j\omega C\right) = j\omega \frac{L}{R} + j^2\omega^2 LC = j\omega \frac{L}{R} - \omega^2 LC$

$$H(j\omega) = \frac{1}{1-\omega^2 LC + j\omega \frac{L}{R}}$$
$$\lim_{\omega\to 0} |H(j\omega)| = \frac{1}{|1|} = 1$$

$\lim_{\omega\to\infty} |H(j\omega)| = \lim_{\omega\to\infty} \frac{1}{\sqrt{\left(1-\omega^2 LC\right)^2 + \left(\omega\frac{L}{R}\right)^2}} = 0$

∴ Low-pass filter.

4.b)

Solution:

$$L = 1, \quad C = 1, \quad R = 0.25$$
$$LC = 1$$
$$\frac{L}{R} = 4$$
$$H(j\omega) = \frac{1}{1-\omega^2 + j4\omega}$$
$$a(\omega) = 1 - \omega^2$$
$$b(\omega) = 4\omega \text{ for } H(\omega) = \frac{1}{a(\omega) + jb(\omega)}$$

Problem 5:

Solution:

$$\omega = 4$$
$$Z_L = j4$$
$$Z_C = \frac{1}{j4 \cdot (0.25)} = -j$$
$$Z_R = 1$$
$$Z_{CR} = 1 - j$$
$$I_{CR} = \frac{V_A}{Z_{CR}}$$
$$I_x = I_L$$

$$I_{\{CR\}} - 0.5I_x - I_x = 0$$
$$I_{\{CR\}} = 1.5I_x$$
$$V_A = V_s - I_x Z_L$$
$$V_A = 1.5I_x Z_{CR}$$
$$I_x(j4 + 1.5(1 - j)) = 24\angle 45°$$
$$I_x = \frac{24\angle 45°}{j4+1.5-1.5j}$$
$$I_x = 8.23\angle(-14°)$$
$$I_{CR} = 1.5I_x = 12.35\angle(-14°)$$
$$V_o = I_{CR}$$
$$v_{o(t)} = 12.35\cos(4t - 14°)$$

End of Homework 10