PS10

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1

A

We require each row of A to sum to 1:

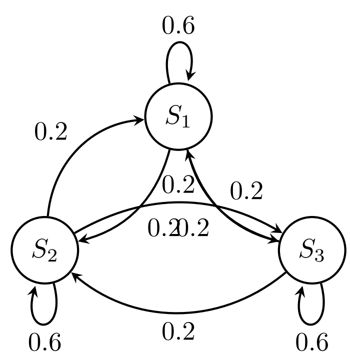
$$\begin{cases} 0.6 + 0.2 + \alpha = 1, \\ 0.2 + \beta + 0.2 = 1, \\ \gamma + 0.2 + 0.6 = 1. \end{cases} \quad \Longrightarrow \quad \alpha = 0.2,\ \beta = 0.6,\ \gamma = 0.2,$$

hence

$$A = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}.$$

B

Write $A = (a_{ij})$ and draw its transition diagram:



C

From (A): $\alpha = 0.2,\ \beta = 0.6,\ \gamma = 0.2.$

D

Compute the characteristic polynomial:

$$\det(A - \lambda I) = \det \begin{bmatrix} 0.6 - \lambda & 0.2 & 0.2 \\ 0.2 & 0.6 - \lambda & 0.2 \\ 0.2 & 0.2 & 0.6 - \lambda \end{bmatrix}$$

$$
\begin{aligned}
&= (0.6-\lambda)\big[(0.6-\lambda)^2 - 0.04\big] - 0.2\big[0.2(0.6-\lambda) - 0.04\big] + 0.2\big[0.04 - 0.2(0.6-\lambda)\big] \\
&= (0.6-\lambda)(\lambda^2 - 1.2\lambda + 0.32) - 0.2(-0.12 + 0.2\lambda) + 0.2(0.04 - 0.2\cdot 0.6 + 0.2\lambda) \\
&= (0.6-\lambda)(\lambda^2 - 1.2\lambda + 0.32) + 0.024 - 0.04\lambda + 0.008 - 0.04\lambda \\
&= -\lambda^3 + 1.8\lambda^2 - 0.96\lambda + 0.16 \\
&= -(\lambda - 1)(\lambda^2 + 0.8\lambda - 0.16).
\end{aligned}
$$

E

Solve the quadratic factor:

$$\lambda^2 + 0.8\lambda - 0.16 = 0 \quad \Longrightarrow \quad -\big(\lambda^2 - 0.8\lambda + 0.16\big) = 0 \quad \Longrightarrow \quad (\lambda - 0.4)^2 = 0,$$

so

$$\lambda_1 = 1, \quad \lambda_2 = \lambda_3 = 0.4.$$

F

Since $A^T = A$, A is symmetric and thus orthogonally diagonalizable.

G

If $A = PDP^{-1}$ then

$$D^k = \begin{bmatrix} \lambda_1^k & 0 & 0 \\ 0 & \lambda_2^k & 0 \\ 0 & 0 & \lambda_3^k \end{bmatrix}$$

H

As $k \to \infty$,

$$1^k \to 1, \qquad 0.4^k \to 0.$$

I

Find the stationary distribution by solving $(A - I)x = 0$:

$$A - I = \begin{bmatrix} -0.4 & 0.2 & 0.2 \\ 0.2 & -0.4 & 0.2 \\ 0.2 & 0.2 & -0.4 \end{bmatrix} \xrightarrow{\text{row-reduce}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad x_1 = x_2 = x_3.$$

Normalized,

$$\pi = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$