Computational Linear Algebra
EK103

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Contents

Chapter 1

Basics

1.1 Vectors, Norms and Products

Note:-

Let us consider two vectors in $\mathbb{R}^3$:

$$u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

We wish to compute their magnitudes (norms and norm-squared), the angle between them, and the plane that they span. These methods are directly applicable to computational tools such as MATLAB.

Definition 1.1.1: Norm of a Vector

For a vector $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, its norm is

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

In many programming languages (including MATLAB), this is computed via `norm(x)`, while the square of the norm is $\|x\|^2 = x \cdot x = x_1^2 + \cdots + x_n^2$.

Example 1.1.1 (Norms and Norm-Squared of u and v)

$$\|u\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}, \quad \|v\| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}.$$

Thus, both vectors have the same magnitude $\sqrt{3}$. Their squared norms are

$$\|u\|^2 = 3, \quad \|v\|^2 = 3.$$

In MATLAB notation, one could write:

- `norm(u)` or `norm(u,2)` for the norm of u.
- `dot(u,u)` or `norm(u)^2` for $\|u\|^2$.

Definition 1.1.2: Angle Between Two Vectors

The angle θ between two nonzero vectors u and v in $\mathbb{R}^n$ is given by

$$\theta = \arccos\left(\frac{u \cdot v}{\|u\|\|v\|}\right).$$

Example 1.1.2 (Angle Between u and v)
First, compute the dot product:

$$u \cdot v \;=\; (1)(1) + (1)(-1) + (1)(1) \;=\; 1 - 1 + 1 \;=\; 1.$$

Hence,

$$\theta \;=\; \arccos\Big(\frac{u \cdot v}{\|u\|\|v\|}\Big) \;=\; \arccos\Big(\frac{1}{\sqrt{3}\,\sqrt{3}}\Big) \;=\; \arccos\big(\tfrac{1}{3}\big).$$

In MATLAB, one could write:

```
theta = acos(dot(u,v)/(norm(u)*norm(v)));
```

Definition 1.1.3: Plane Spanned by Two Vectors

The plane containing vectors u and v and passing through the origin is given by

$$\{\, \alpha\, u + \beta\, v \;\mid\; \alpha, \beta \in \mathbb{R}\}.$$

An equivalent description is all points $x \in \mathbb{R}^3$ such that $x \cdot (u \times v) = 0$.

Example 1.1.3 (Plane Containing u and v)

- *Span form:*

$$\text{Plane} = \Big\{\, \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \;+\; \beta \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \;\Big|\; \alpha, \beta \in \mathbb{R} \Big\}.$$

- *Normal form:* The cross product

$$u \times v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = (2,\, 0,\, -2).$$

  Hence, the plane also can be described by the set of points $x = (x_1, x_2, x_3)$ for which

$$(2,\, 0,\, -2) \cdot (x_1, x_2, x_3) = 0 \quad\Longrightarrow\quad 2\,x_1 - 2\,x_3 = 0 \quad\Longrightarrow\quad x_1 = x_3.$$

In many computational environments, one simply keeps the span form or uses a symbolic package to compute the cross product and normal equation.

Note:-

In summary, for vectors u and v:

- $\|u\|$ and $\|v\|$ each equal $\sqrt{3}$.
- $\|u\|^2 = \|v\|^2 = 3$.
- The angle between them is $\arccos\big(\frac{1}{3}\big)$.
- The plane is spanned by $\{u, v\}$, or equivalently described by the normal vector $u \times v$.

All these computations can be done in a straightforward manner in a software package such as MATLAB, using `dot`, `norm`, `acos`, and `cross`.