Exam 1 Formulas Sheet - EK 103 Spring 2025

REMINDER!

This sheet (formulas, definitions, and equations) is not a guide for what will be on the exam. You will need to practice, on your own: the procedures for performing calculations, how to approach or solve problems, or how to use the facts on this sheet to interpret your calculations.

1 Definitions

- A vector can be written as a bold lower-case letter, or a lower-case letter with an arrow above it, both are the same. We write the number of elements in the vector using the symbol $\mathbb{R}$, for example, if there are n elements in the vector:

$$\mathbf{x} = \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

  You can also be given two points P and Q and write the vector between them as $\overrightarrow{PQ}$ like this:

$$P = (x_1, y_1), \qquad Q = (x_2, y_2), \qquad \overrightarrow{PQ} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$$

- A system of linear equations occurs when you have a matrix (A) multiplied by a vector of unknowns ($\mathbf{x}$), which must equal another vector ($\mathbf{b}$):

$$A\mathbf{x} = \mathbf{b} \quad \iff \quad \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad \iff \quad \begin{array}{c} a_{11}x_1 + a_{12}x_2 + \dots a_{1m}x_1 = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots a_{2m}x_2 = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots a_{nm}x_m = b_n \end{array}$$

- A homogeneous linear system is when the vector $\mathbf{b} = 0$, like $A\mathbf{x} = 0$
- A nonhomogeneous linear system is when the vector $\mathbf{b}$ is not zero, like $A\mathbf{x} = \mathbf{b}$
- An augmented matrix is when we put the $\mathbf{b}$ vector as an additional column, $[A\ \mathbf{b}]$. You can write a vertical line between the A part and the $\mathbf{b}$ part, or not, both are valid.

$$[A \mid \mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1m} & b_1 \\ a_{21} & a_{22} & \dots & a_{2m} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} & b_n \end{array}\right]$$

- The leading entry of a row in a matrix is its left-most nonzero number.
- The Row Echelon Form (REF) of a matrix is often written as $[U\ \ \mathbf{c}]$ and the Reduced Row Echelon Form (RREF) is often written as $[R\ \mathbf{d}]$, and are defined as:

DEFINITION A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

- A pivot or pivot position is the leading entry in a row when a matrix is in REF, such as the black boxes in this graphic. A pivot column is a column with a pivot in REF.

EXAMPLE 1 The following matrices are in echelon form. The leading entries ($\blacksquare$) may have any nonzero value; the starred entries ($*$) may have any value (including zero).

$$\left[\begin{array}{ccc|c} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{ccccccccc|c} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{array}\right] \longleftarrow \text{REF}$$

Basic Free

Basic Free

The following matrices are in reduced echelon form because the leading entries are 1's, and there are 0's below *and above* each leading 1.

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix} \longleftarrow \text{RREF}$$

- A basic variable is one of the x_i when the corresponding column contains a pivot in REF.

  In the example on the left, x_1 and x_2 are basic variables.

- A free variable is one of the x_i when the corresponding column does not contain a pivot in REF.

  In the example on the left, x_3 is a free variable.

- A linear combination is a sum of scaled vectors, and is another way to interpret matrix-vector multiplication. Here, the $\mathbf{a}_i$ vectors are the columns of the A matrix:

DEFINITION

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the **product of A and $\mathbf{x}$**, denoted by $A\mathbf{x}$, is **the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights**; that is,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

- The trivial solution to a homogeneous linear system, $A\mathbf{x} = \mathbf{0}$, is the vector $\mathbf{x} = 0$. It is always a solution to $A\mathbf{x} = \mathbf{0}$.
- A nontrivial solution to $A\mathbf{x} = \mathbf{0}$ is some other $\mathbf{x}$ that is not 0.

  Fact:

> The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

- The homogeneous solution to an $A\mathbf{x} = \mathbf{b}$ problem is the solution to the $A\mathbf{x} = \mathbf{0}$ problem with the same A matrix.
- The particular solution is a constant vector (*not* multiplied by an unknown) that is a solution to $A\mathbf{x} = \mathbf{b}$.
- The complete solution is the sum of the homogeneous solution and particular solution.

  Fact:

> Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

- A set of vectors is linearly independent if, when put as an $A\mathbf{x} = \mathbf{0}$ problem, the only solution is $\mathbf{x} = 0$. (the trivial solution!)

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = 0$$

If $\mathbf{x} = 0$ is the only solution, then the vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ are linearly independent.

- A set of vectors is linearly dependent if it is not linearly independent. In other words, dependent if nontrivial solutions to $A\mathbf{x} = \mathbf{0}$.
- The span of a set of vectors is all linear combinations of them:

DEFINITION

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned** (or **generated**) **by** $\mathbf{v}_1, \ldots, \mathbf{v}_p$. That is, Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

with $c_1, \ldots, c_p$ scalars.

Asking whether a vector $\mathbf{b}$ is in Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ amounts to asking whether the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{b}$$

has a solution, or, equivalently, asking whether the linear system with augmented matrix $[\,\mathbf{v}_1 \quad \cdots \quad \mathbf{v}_p \quad \mathbf{b}\,]$ has a solution.

2 Formulas

- Transpose: If you have a vector $\mathbf{x}$ that has n components, its transpose is a row:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{x}^\top = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}$$

- Dot Product: If you have two vectors with the same number of components, their dot product is:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \qquad \mathbf{v} \cdot \mathbf{u} = \mathbf{v}^\top \mathbf{u} = v_1u_1 + v_2u_2 + \ldots + v_nu_n$$

  Observe that the dot product is commutative: $\mathbf{u}^\top\mathbf{v}$ is the same calculation as $\mathbf{v}^\top\mathbf{u}$.

- Norm: The norm of a vector is its length, and is calculated by the Pythagorean theorem:

$$||\mathbf{v}|| = \sqrt{\mathbf{v}^\top\mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$$

- Dot Product Using Lengths and Angles: Another definition of the dot product uses the lengths of the two vectors, and and angle between them:

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||\ ||\mathbf{v}||\ \cos(\theta)$$

- Orthogonality: Two vectors are orthogonal if the angle between them is 90° (or some multiple of 90°, like −90° or 270°). Orthogonality can be determined by a dot product calculation:

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are **orthogonal** (to each other) if $\mathbf{u}\cdot\mathbf{v} = 0$.

3 Theorems and Facts

- Can I multiply these two matrices? Two matrices can only be multiplied together when the number of columns of the first matrix equals the number of rows of the second matrix. The "inside dimensions" must match.

$$A \in \mathbb{R}^{m\times n}, \qquad B \in \mathbb{R}^{p\times q}, \qquad AB \in \mathbb{R}^{m\times q} \qquad \text{only possible if } n = p$$

- Existence and Uniqueness: "How many solutions exist to this $A\mathbf{x} = \mathbf{b}$ problem?" There are only three possibilities: either 0, or 1, or ∞ solutions.

  If there is only one solution, we say it is unique. As long as at least one solution exists, we say it is consistent.

THEOREM 2 **Existence and Uniqueness Theorem**

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—that is, if and only if an echelon form of the augmented matrix has *no* row of the form

$$[\,0 \quad \cdots \quad 0 \quad b\,] \qquad \text{with } b \text{ nonzero}$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

This definition is equivalent to other facts related to linear combinations, span, and linear independence:

THEOREM 4

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
c. The columns of A span $\mathbb{R}^m$.
d. A has a pivot position in every row.

Be careful when attempting to 'flip' this theorem. For example: if solutions exist to $A\mathbf{x} = \mathbf{b}$ for a particular $\mathbf{b}$ but not others, there must be at least one row without a pivot. HOWEVER! This tells you nothing about *how many columns* have pivots, and does not tell you if solutions are *unique vs. infinite* for that $\mathbf{b}$.

- Linear Independence and Row Reduction: The linear independence of a set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ can be determined by putting them as columns of a matrix $A = [\mathbf{a}_1\ \mathbf{a}_2\ \ldots \mathbf{a}_n]$ and calculating the REF of that matrix:

The columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution. (3)

This implies two additional facts:

THEOREM 8

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

$$\begin{array}{cc} & p \\ n & \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \end{array}$$

...since at least one column cannot have a pivot in it.

THEOREM 9

If a set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

...since a column of zeros cannot have a pivot in it.

4 Trigonometric Identities

- Pythagorean Theorem with sines and cosines:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

- Negative angles:

$$\sin(\theta) = -\sin(\theta), \qquad \cos(-\theta) = \cos(\theta)$$

- Sums and differences of angles:

$$\begin{aligned} \sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \qquad & \sin(\alpha-\beta) &= \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \\ \cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \qquad & \cos(\alpha-\beta) &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \end{aligned}$$

- Double angle:

$$\begin{aligned} \sin(2\theta) &= 2\sin(\theta)\cos(\theta) = (\sin^2(\theta) + \cos^2(\theta)) - 1 \\ \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) = 1 - 2\sin^2(\theta) \end{aligned}$$

5 Table of Transformations

...is on the next four pages.

SOLUTION Write

$$T(\mathbf{e}_1) = 3\mathbf{e}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = 3\mathbf{e}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

■

EXAMPLE 3 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle φ, with counterclockwise rotation for a positive angle. We could show geometrically that such a transformation is linear. (See Figure 6 in Section 1.8.) Find the standard matrix A of this transformation.

SOLUTION $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ rotates into $\begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ rotates into $\begin{bmatrix} -\sin\varphi \\ \cos\varphi \end{bmatrix}$. See Figure 1. By Theorem 10,

$$A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

Example 5 in Section 1.8 is a special case of this transformation, with $\varphi = \pi/2$. ■

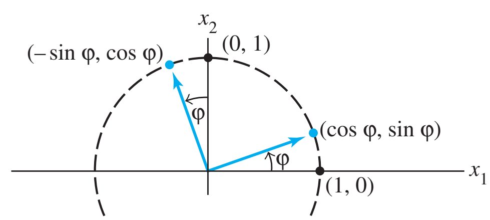


FIGURE 1 A rotation transformation.

Geometric Linear Transformations of $\mathbb{R}^2$

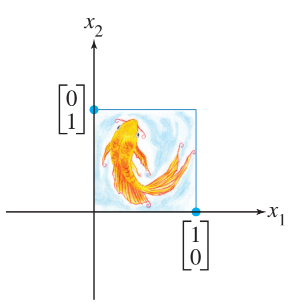


FIGURE 2
The unit square.

Examples 2 and 3 illustrate linear transformations that are described geometrically. Tables 1–4 illustrate other common geometric linear transformations of the plane. Because the transformations are linear, they are determined completely by what they do to the columns of I_2. Instead of showing only the images of $\mathbf{e}_1$ and $\mathbf{e}_2$, the tables show what a transformation does to the unit square (Figure 2).

Other transformations can be constructed from those listed in Tables 1–4 by applying one transformation after another. For instance, a horizontal shear could be followed by a reflection in the x_2-axis. Section 2.1 will show that such a *composition* of linear transformations is linear. (Also, see Exercise 36.)

Existence and Uniqueness Questions

The concept of a linear transformation provides a new way to understand the existence and uniqueness questions asked earlier. The two definitions following Tables 1–4 give the appropriate terminology for transformations.

TABLE 1 Reflections

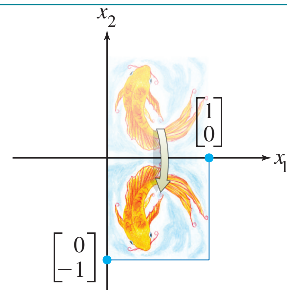
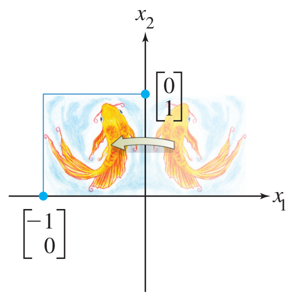
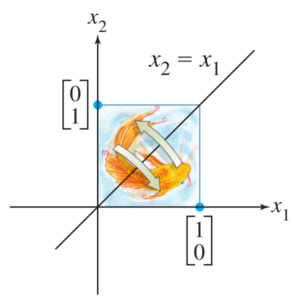
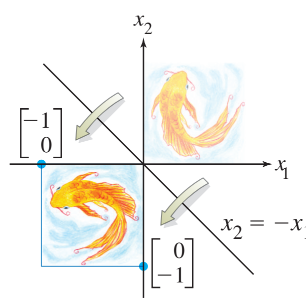
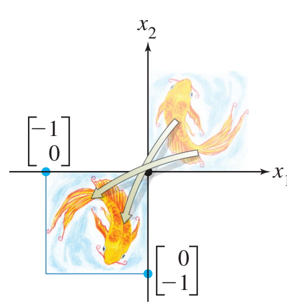
Transformation	Image of the Unit Square	Standard Matrix
Reflection through the x_1-axis		$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$
Reflection through the x_2-axis		$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$
Reflection through the line $x_2 = x_1$		$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$
Reflection through the line $x_2 = -x_1$		$\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$
Reflection through the origin		$\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$

TABLE 2 **Contractions and Expansions**

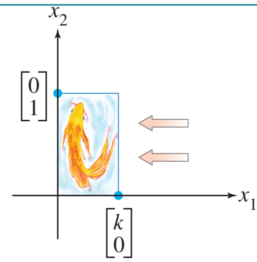
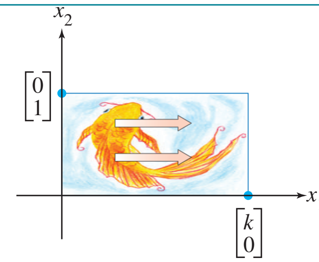
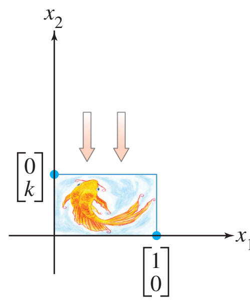
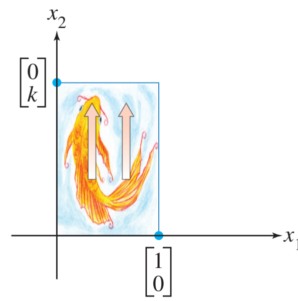
Transformation	Image of the Unit Square		Standard Matrix
Horizontal contraction and expansion	 $0 < k < 1$	 $k > 1$	$\begin{bmatrix}k & 0\\0 & 1\end{bmatrix}$
Vertical contraction and expansion	 $0 < k < 1$	 $k > 1$	$\begin{bmatrix}1 & 0\\0 & k\end{bmatrix}$

TABLE 3 **Shears**

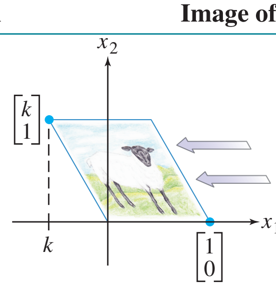
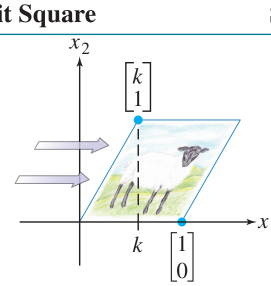
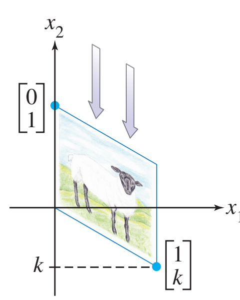
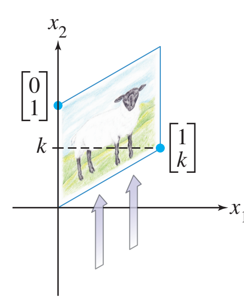
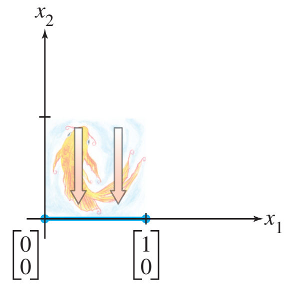
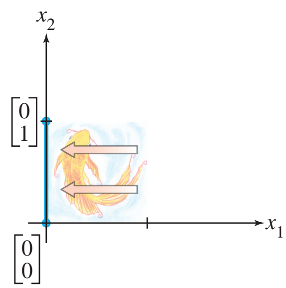
Transformation	Image of the Unit Square		Standard Matrix
Horizontal shear	 $k < 0$	 $k > 0$	$\begin{bmatrix}1 & k\\0 & 1\end{bmatrix}$
Vertical shear	 $k < 0$	 $k > 0$	$\begin{bmatrix}1 & 0\\k & 1\end{bmatrix}$

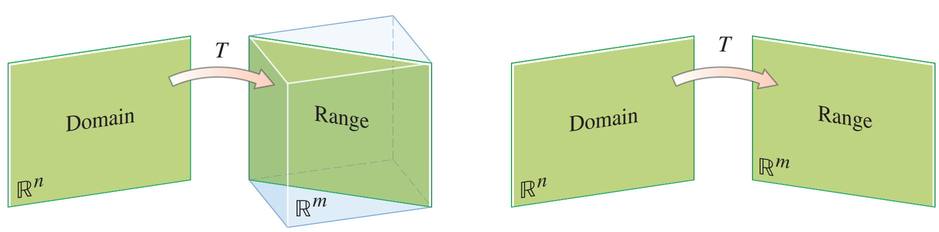
TABLE 4 Projections

Transformation	Image of the Unit Square	Standard Matrix
Projection onto the x_1-axis		$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Projection onto the x_2-axis		$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

DEFINITION

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at least one* $\mathbf{x}$ in $\mathbb{R}^n$.

Equivalently, T is onto $\mathbb{R}^m$ when the range of T is all of the codomain $\mathbb{R}^m$. That is, T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if, for each $\mathbf{b}$ in the codomain $\mathbb{R}^m$, there exists at least one solution of $T(\mathbf{x}) = \mathbf{b}$. "Does T map $\mathbb{R}^n$ onto $\mathbb{R}^m$?" is an existence question. The mapping T is *not* onto when there is some $\mathbf{b}$ in $\mathbb{R}^m$ for which the equation $T(\mathbf{x}) = \mathbf{b}$ has no solution. See Figure 3.



T is *not* onto $\mathbb{R}^m$ T is onto $\mathbb{R}^m$

FIGURE 3 Is the range of T all of $\mathbb{R}^m$?

DEFINITION

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at most one* $\mathbf{x}$ in $\mathbb{R}^n$.