0 Preface

EK103 Spring 2025, PS9
due TUESDAY, Apr 8th, 11:59PM

As always, show all work.

1 Q1: Diagonalization

Diagonalize into $\mathbf{P}$, $\mathbf{D}$, and $\mathbf{P}^{-1}$, if possible. If not, explain why.

All work must be done by hand, but as always we encourage you to check with Matlab.

a)

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

b)

$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix}$$

c)

$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 0 & 5 \end{bmatrix}$$

d)

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

e)

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

f)

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

g)

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

2 Q2: Powers of A

Using $\mathbf{PDP}^{-1}$ diagonalization, calculate the following.

All work must be done by hand, but as always we encourage you to check with Matlab.

(a)

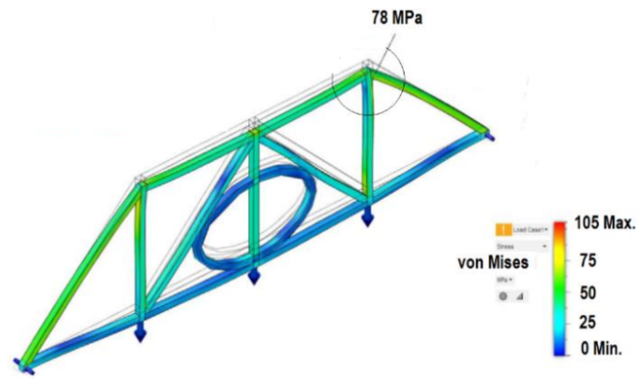
$$\mathbf{A}^5, \text{ where } \mathbf{A} = \begin{bmatrix} 2 & 2 \\ 0 & 4 \end{bmatrix}$$

(b)

$$\mathbf{A}^2, \text{ where } \mathbf{A} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

3 Q3: Another application of Diagonalization

We all know that things break when we stress them too hard. But something you haven't learned yet at BU is how to predict how hard something is being stressed, and whether it will break. It turns out it's, pardon the pun, not hard at all, and you already have the tools at hand!



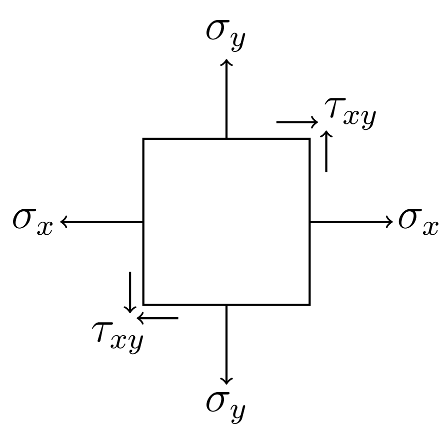
Bridge bending analysis

Background: the Cauchy Stress Matrix

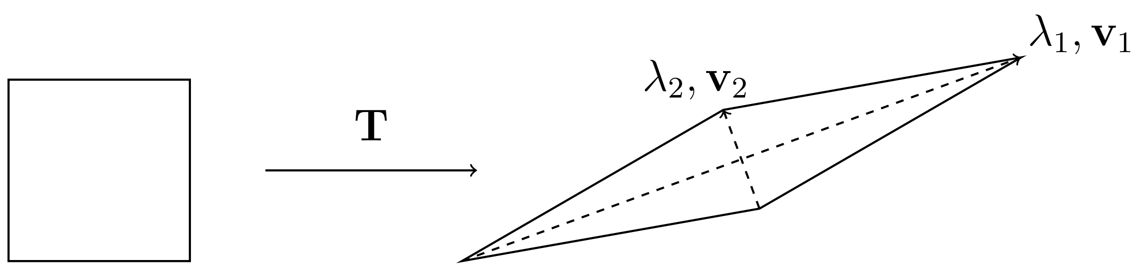
Consider a square element under 2D stress with the following components:

- σ_x is the squishing/stretching stress in the x-direction.
- σ_y is the squishing/stretching stress in the y-direction.
- τ_{xy} is the "shear" stress on the square.

These stresses are shown on the faces of the square element:



You don't have to understand this right now. Just look at it and think how squishing and stretching and shearing this square is going to result in a deformed shape which, it turns out, is going to be some kind of parallelogram.



(You should be getting some major eigenvector stretch/shrink vibes right now!!!)

The way we express the stress on a unit square is below:

$$\mathbf{T} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \tag{1}$$

where you would simply substitute in for the $, \sigma_y, \tau_x y$ stresses you're given.

We call this the "stress" matrix.

For reasons which will come up in a few weeks, **a symmetric matrix is always diagonalizable**. So since the stress matrix is symmetric, we know we can always find a $\mathbf{T} = \mathbf{PDP}^{-1}$ decomposition.

The *eigenvalues* comprising the diagonal of $\mathbf{D}$ are the maximum stretching/shrinking stresses on the material, when considering all the deformation effects. We call these the *principal stresses*.

The *eigenvectors* comprising the columns of $\mathbf{P}$ correspond to the directions of these principal stresses, which are the orientations in which peak stress occurs—crucial information for determining failure points!

Background: the *von Mises* stresses

Lastly, what we really care about is not the stress in the material but **will it break**? To answer that, we calculate the *von Mises stress*, $\sigma_{\text{von Mises}}$.

If the material is stronger than the von Mises stress then it's fine, and if it's weaker then it breaks.

The *von Mises* stress is computed from the eigenvalues matrix $\mathbf{D}$ (now written as $\mathbf{T}_{\text{von Mises}}$):

$$\mathbf{T}_{\text{von Mises}} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$

and is given by the formula:

$$\sigma_{\text{von Mises}} = \frac{\sqrt{2}}{2}\sqrt{(\sigma_1 - \sigma_2)^2 + \sigma_1^2 + \sigma_2^2}$$

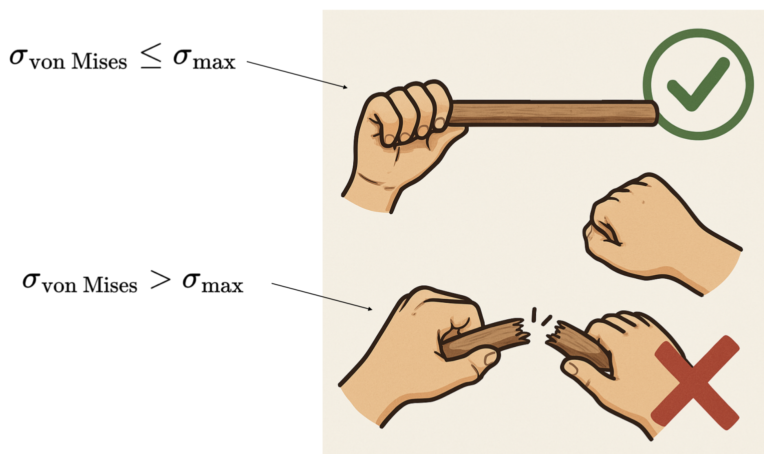


Figure: $\sigma_{\text{von Mises}}$ vs σ_{max}

Questions

a) Starting from equation (1), calculate the magnitude of the *principal stresses* for the following stresses:

- σ_x = 15MPa
- σ_y = 15MPa
- τ_{xy} = -5MPa

Give your answer as $\sigma_1 = ...$ **and** $\sigma_2 = ...$

b) Calculate the direction of the *principal stresses* for the matrix $\mathbf{T}$ (in (1)).

Give your answer as $\mathbf{v}_1 = ...$ **and** $\mathbf{v}_2 = ...$

c) Lastly, assuming that for this material $\sigma_{max} = 17$MPa, will it break?

Why/why not?

Congratulations, now you know how to calculate the stress on a teensy element. If you were to use a computer to repeat this millions of times across a mesh, you would be able to run a simulation determine where a part is most likely to fail. Cool!