**ENG EK103: Problem Set 4 (Spring 2025) Due: February 20**

Use MATLAB only to check your answers. **No Explanation = No Credit**. For each problem, all solution steps should be written down and explained clearly.

Bold lowercase letter (such as $\boldsymbol{b}$ ) represents a vector. Uppercase letter (such as $A$) represents a matrix.

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**Q1**

In this problem, you are to work out the *complete solution* to a system of linear equations (SLE) specified as $A\boldsymbol{x} = \boldsymbol{b}$ with

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 3 & 4 \\ 1 & 1 & -3 & -2 & -6 \end{bmatrix}$$

and

$$\boldsymbol{b} = \begin{bmatrix} 4 \\ 11 \\ -9 \end{bmatrix}$$

The matrix $A$ is said to be a *wide* matrix because it has more columns than rows; as a consequence, the system of equations $A\boldsymbol{x} = \boldsymbol{b}$ has fewer equations than unknowns**.**

(a) Showing the main steps, perform ***forward row reduction*** to put the ***augmented matrix*** for $A\boldsymbol{x} = \boldsymbol{b}$ into a ***row echelon form*** (REF). Be sure to specify each *elementary row operation* that you perform on the way to obtaining the row echelon form of the augmented matrix.

(b) Circle the ***pivots*** in the REF of the augmented matrix you worked out in the previous part and use it to list the ***basic*** and ***free*** unknowns (variables) of the SLE.

(c) Showing the main steps, perform ***backward row reduction*** on the REF of the augmented matrix for $A\boldsymbol{x} = \boldsymbol{b}$ and work out the ***reduced row echelon form*** (RREF) of the augmented matrix. Be sure to specify each elementary row operation that you perform on the way to obtaining the RREF of the augmented matrix from its REF.

(d) Using the RREF of the augmented matrix that you worked out in the previous part, find a **complete solution** for the given SLE at the top of this problem. Please be sure to write the complete solution as $\boldsymbol{x}_c = \boldsymbol{x}_p + \boldsymbol{x}_h$, where $\boldsymbol{x}_p$ is a **particular solution** (representing a single solution vector) and $\boldsymbol{x}_h$ is the **homogeneous solution** (representing a family of possibly infinite number of vectors) to the given SLE.

(e) State whether the set of all solutions to $A\boldsymbol{x} = \boldsymbol{b}$ at the top of this problem **spans** a point, a line, a plane, or a volume in $\mathbb{R}^5$. Make sure to explain how you arrived at your answer.

(f) Given any *arbitrary* $3 \times 5$ *wide* matrix $B$ and assuming that $B\boldsymbol{x} = \boldsymbol{b}$ is a *consistent* system of equations, is $B\boldsymbol{x} = \boldsymbol{b}$ *guaranteed* to have an infinite number of solutions? Why or why not?

**Q2**

In this problem, you are to work out the *complete solution* to a system of linear equations (SLE) specified as $A\boldsymbol{x} = \boldsymbol{b}$ with

$$A = \begin{bmatrix} 3 & 0 & 1 \\ -5 & 1 & 1 \\ 8 & -1 & 0 \\ -1 & 2 & p \end{bmatrix}$$

and

$$\boldsymbol{b} = \begin{bmatrix} 5 \\ -2 \\ 7 \\ 11 \end{bmatrix}$$

Please note that $p$ is an unspecified element of the matrix $A$. Also, the matrix $A$ is said to be a "tall" matrix because it has more rows than columns.

(a) Regardless of the value of $p$, Is $A\boldsymbol{x} = \boldsymbol{0}$ guaranteed to have at least one solution? Explain.

(b) Determine the *row echelon form* of the augmented matrix $[A|\boldsymbol{b}]$. The matrix you obtain may have some of its elements expressed in terms of the unspecified element $p$ in the matrix $A$.

(c) Determine the value of $p$ in the matrix $A$ for the case when $A\boldsymbol{x} = \boldsymbol{b}$ is given to be a *consistent* SLE with a complete solution that is a straight line in $\mathbb{R}^3$.

(d) Using the value of $p$ you determined in the previous part, determine the RREF of $[A|\boldsymbol{b}]$ by applying backward row reduction to the REF of $[A|\boldsymbol{b}]$.

(e) Using the result from the previous part, write the *complete solution* to $A\boldsymbol{x} = \boldsymbol{b}$ using the value of $p$ you determined in part (c). Please be sure to write the complete solution as $\boldsymbol{x}_c = \boldsymbol{x}_p + \boldsymbol{x}_h$, where $\boldsymbol{x}_p$ is a **particular solution** (representing a single solution vector) and $\boldsymbol{x}_h$ is the **homogeneous solution** (representing a family of possibly infinite number of vectors) to the given SLE.

(f) Given any *arbitrary* $4 \times 3$ *tall* matrix $B$ and assuming that $B\boldsymbol{x} = \boldsymbol{b}$ is a *consistent* system of equations, is $B\boldsymbol{x} = \boldsymbol{b}$ *guaranteed* to have an infinite number of solutions? Why or why not?

**Q3**

Given any $n \times 1$ vector $\boldsymbol{v}$, its dot product with itself is given as the *scalar* $\boldsymbol{v}^T\boldsymbol{v}$. In this problem, we explore the product $\boldsymbol{v}\boldsymbol{v}^T$ for the case of $3 \times 1$ vectors $\boldsymbol{v}$, even though the results can be easily generalized to the case of $n \times 1$ vectors $\boldsymbol{v}$.

(a) Determine the $3 \times 3$ matrix $A = \boldsymbol{v}\boldsymbol{v}^T$ where $\boldsymbol{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

(b) Assuming $a \neq 0$, use *forward row reduction* to turn the matrix $A$ from part (a) into a Row Echelon Form (REF) and show that REF ($A$) has only 1 pivot.

(c) Assuming $b \neq 0$, use *forward row reduction* to turn the matrix $A$ from part (a) into a Row Echelon Form (REF) and show that REF ($A$) has only 1 pivot.

(d) Assuming $c \neq 0$, use *forward row reduction* to turn the matrix $A$ from part (a) into a Row Echelon Form (REF) and show that REF ($A$) has only 1 pivot.

(e) Assuming that the vector $\boldsymbol{v}$ is non-zero, is the **span** of the columns of matrix $A$ a point, a line, a plane, or entire $\mathbb{R}^3$? Explain why. HINT: If the vector $\boldsymbol{v}$ is non-zero, it is guaranteed that either $a \neq 0$ or $b \neq 0$ or $c \neq 0$, cases that were respectively dealt with in parts (b), (c), and (d) of this problem.

(f) The only case in which the span worked out in the previous part does not apply is when $\boldsymbol{v} = \boldsymbol{0}$. Would the **span** of the column vectors of matrix $A$ be a point, a line, a plane or entire $\mathbb{R}^3$ when $\boldsymbol{v} = \boldsymbol{0}$? Explain why.

**Q4**

In each part of this problem, determine whether or not the **span** of the columns of the given matrix $A$ is *guaranteed* to be the same as the span of the columns of the given matrix $B$, *regardless* of the values of the scalars $p$ and $q$. Explain each answer.

(a) $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 3 & 2 \\ p & q \end{bmatrix}$. HINT: Consider the size of the vectors in the span of each matrix.

(b) $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 & p \\ 2 & 5 & q \end{bmatrix}$. HINT: Compare the row echelon forms of the two matrices.

(c) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 & 3 & p \\ 2 & 5 & 6 & q \end{bmatrix}$. HINT: Compare the row echelon forms of the two matrices.

(d) $A = \begin{bmatrix} p & 3 \\ 2 & 2 \\ q & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 3-p & p \\ 2 & 0 & 2 \\ 1 & 1-q & q \end{bmatrix}$. HINT: The middle column of B can be obtained by linearly combining the columns of $A$.