ENG EK 103: Computational Linear Algebra: Problem set 3

problem 1: Some mini-drill exercises on using row reductions to solve $Ax = b$ problems !

Section A: Suppose you were given these 3 systems:

System #1	System 2	System 3
$x_1 + x_2 = 5$	$x_1 + x_2 = 1$	$x_1 - x_2 = -2$
$x_1 - x_2 = 1$	$x_1 - x_2 = 3$	$x_1 + x_2 = 2$
	$x_1 - 3x_2 = 5$	$x_1 + 3x_2 = 0$

Tasks	System	System 2	System 3
(a)	Draw out the 2 lines on graph paper (or on a gridded page in your tablet's app)	Same, but now you have to sketch out 3 lines…	Same, but now you have to sketch out 3 lines…
(b)	Do you expect the system to have: • A unique solution ? • Infinite # of solutions? • No solutions ??	Same	Same
(c)	Write it in matrix form: $Ax = b$	Same	Same
(d)	Rewrite it in augmented matrix form: $[\ A \mid b\ ]$	Same	Same
(e)	Reduce the system down to its upper-triangle form $Ux = c$. Write out all row-reduction steps !! In augmented matrix notation, this means we want to see these row reduction steps: $[\ A \mid b\ ] \xrightarrow{show\ work} [\ U \mid c\ ]$	Reduce the system down to its REF form $(REF)x = c$. Write out all row-reduction steps !! In augmented matrix notation, this means we want to see these row reduction steps: $[\ A \mid b\ ] \xrightarrow{show\ work} [\ REF \mid c\ ]$	Reduce the system down to its REF form $(REF)x = c$. Write out all row-reduction steps !! In augmented matrix notation, this means we want to see these row reduction steps: $[\ A \mid b\ ] \xrightarrow{show\ work} [\ REF \mid c\ ]$
(f)	Using your results from $[\ U \mid c\ ]$, solve for the solution vector x	Using your results from $[\ REF \mid c\ ]$, solve for the solution vector x if you can. If not, explain why not.	Using your results from $[\ REF \mid c\ ]$, solve for the solution vector x if you can. If not, explain why not.

Section B: Suppose you were given these 3 systems:

System #4

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 + x_2 &= -1 \end{aligned}$$

System 5

$$\begin{aligned} x_1 + x_2 &= 3 \\ x_1 - x_2 &= 3 \\ x_1 - 7x_2 &= 3 \\ x_1 + 4x_2 &= 3 \end{aligned}$$

System 6

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 4 \\ -x_1 + x_2 - x_3 + x_4 + x_5 &= 2 \\ 2x_1 + 2x_2 \qquad - x_4 + x_5 &= 4 \end{aligned}$$

Tasks	System 4	System 5	System 6
(a)	Draw out the 2 lines on graph paper (or on a gridded page in your tablet's app)	Same, but now you have to sketch out 4 lines…	** *No need* to sketch this guy out =)
(b)	Do you expect the system to have: • A unique solution ? • Infinite # of solutions? • No solutions ??	Same	Same
(c)	Reduce the system down to its full RREF form $R\boldsymbol{x} = \boldsymbol{d}$. Write out all row-reduction steps !! In augmented matrix notation, this means we want to see these row reduction steps: $[\ A \mid \boldsymbol{b}\ ] \xrightarrow{\text{show work}} [\ R \mid \boldsymbol{d}\ ]$	Same (get it into RREF form)	Same (get it into RREF form)
(d)	Using matlab, check your answer for matrix R using this command (no need to turn this in): `rref( A )`	Same (no need to turn this in)	Same (no need to turn this in)
(e)	You can also check your answer for $[\ R \mid \boldsymbol{d}\ ]$ using this command: `rref( [ A b ] )`	Same (no need to turn this in)	Same (no need to turn this in)
(f)	• Circle all pivots in your R-matrix • Identify all pivot columns and free columns of R	Same	Same
(g)	• Does your results for $[\ R \mid \boldsymbol{d}\ ]$ support what you claimed in Part (b) (where $\boldsymbol{x}$ = unique, infinite, or no solutions) ? Justify your results in 2 – 3 sentences	Same	Same

problem 2: When you have a unique solution for an $A\boldsymbol{x} = \boldsymbol{b}$ problem

Suppose you have this system of equations, where each equation represents a 2D plane residing in 3D space:

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 2 \qquad (Plane\ \#1) \\ x_1 & - & 3\,x_2 & & & = & 1 \qquad (Plane\ \#2) \\ -\,x_1 & + & 2\,x_2 & + & 6x_3 & = & 5 \qquad (Plane\ \#3) \end{array}$$

Before you attempt this problem, let's think about the overall geometry:

- If these 3 planes intersect only at 1 single point in 3D space
- Then, we will have a ***unique*** (single) solution for our $A\boldsymbol{x} = \boldsymbol{b}$ problem, right ?? =)

Your job is to prove that this is the case, and then, you will prove it graphically by plotting them in matlab ! !

(a) Rewrite the above in matrix form: $A\boldsymbol{x} = \boldsymbol{b}$. Then, rewrite the system in augmented form, where it looks like this:

$$[\ A \ |\ \boldsymbol{b}\ ]$$

(b) Using row reductions (in augmented form), please reduce the system $A\boldsymbol{x} = \boldsymbol{b}$ down into an upper-triangle form, where the equation becomes $U\boldsymbol{x} = \boldsymbol{c}$. When expressed in augmented matrix form, it looks like this:

$$[\ A \ |\ \boldsymbol{b}\ ] \xrightarrow{row\ reductions} [\ U \ |\ \boldsymbol{c}\ ]$$

where the upper-triangle matrix U should look something like this:

$$U = \begin{bmatrix} \blacksquare & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare \\ 0 & 0 & \blacksquare \end{bmatrix}, \quad \blacksquare = some\ numbers$$

(c) Circle the pivots in your U-matrix

(d) Using your results for $U\boldsymbol{x} = \boldsymbol{c}$, find the unique solution for $A\boldsymbol{x} = \boldsymbol{b}$.

(e) Using matlab's \ command, check to see if your complete solution for $\boldsymbol{x}$ was correct (you don't have to turn this in).

★★ For EK 103: You should always use matlab to check your answers !!
If you don't, you're literally throwing away points on your homeworks.... =\

It's time for some matlab!! Open up the file: `Problem2_2Dplanes_unique.m`

Briefly scan through the code to see what it entails. You might find some of the plotting commands useful for your other classes, such as how to "auto-construct" legend strings using the *num2str( )* command, plotting polygons in 3D space using the *patch* command, etc.

Then, run the code. It will generate Figure 1 below, where it will plot the 3 intersecting planes defined by our $A\boldsymbol{x} = \boldsymbol{b}$ equation. Now, hover your mouse cursor over the upper-right corner of the figure. Then, click on the *"Rotate 3D" button* to manually rotate the 3D plot…. and you should try to get a feel for the overall geometry of the problem.

Next, you will also notice a lone black dot. *Your job is to move the black dot to the intersection of the 3 planes* (the unique solution vector $\boldsymbol{x}$).

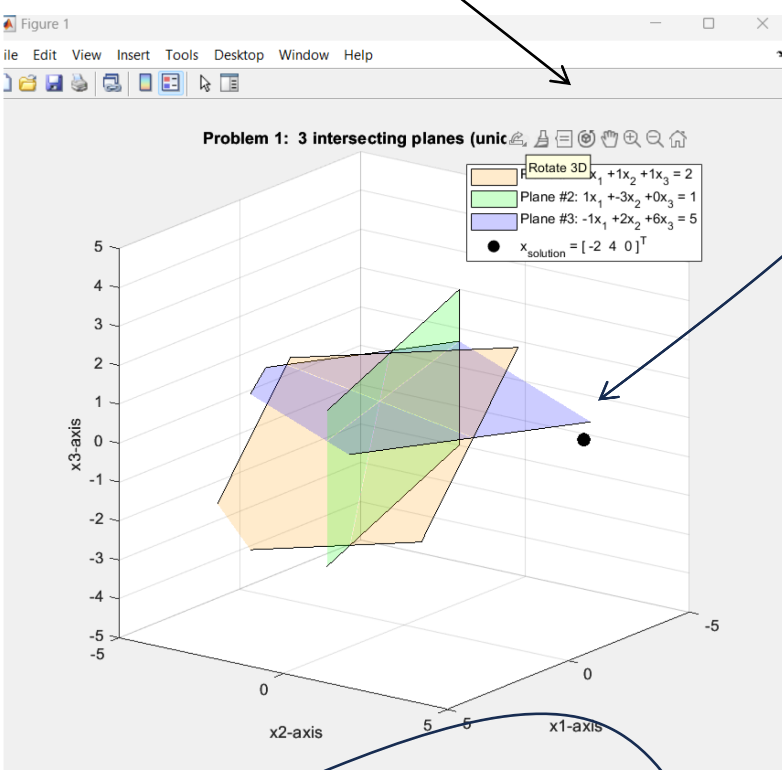


Figure 1: You will edit the matlab code to move the black dot towards the correct solution location for $\boldsymbol{x}$

(f) Edit *Line 159* of the code so that the solution vector x (the black dot) will be moved to the unique intersection of the 3 planes.

(g) Re-run your code, rotate the 3D figure again, and make sure the plot makes sense geometrically (ie. The black dot should be at the intersection of the 3 planes !!). Then, ***publish your code*** into a PDF, and combine it with your handwritten work.

```
148  % \\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\
149  %
150  %   Task #2:  Plot the solution vector x  (the answer) as a
151  %             big black dot
152  %
153  %   ** You will have to change the "my_x" vector to have it
154  %      plotted correctly !!
155  %
156  % \\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\
157
158  % -- Define solution vector
159  x_solution = [ -2  4  0 ]';   %  <---  you need to change this !!
160
161
162  % -- Plot it as a black dot using the "plot3" command:
163
164  x_solution_plothandle = plot3( x_solution(1), x_solution(2), x_solution(3), '.');
165  set(x_solution_plothandle, 'Color', 'black', 'Markersize', 30);
166
```

problem 3: When you have an infinite # of solutions for an $A\boldsymbol{x} = \boldsymbol{b}$ problem (and practice with RREF)

Suppose you have this system of equations, where each equation represents a 2D plane residing in 3D space:

$$\begin{array}{ccccccccc} 2x_1 & - & x_2 & + & x_3 & = & 2 & & (Plane\ \#1\,) \\ x_1 & + & x_2 & - & 2\,x_3 & = & 0 & & (Plane\ \#2\,) \end{array}$$

Before you attempt this problem, let's think about the overall geometry:

- Can these 2 planes intersect only at 1 point in 3D space ? Probably not, right ??…… =\ (unique solution is *__not possible__* for $\boldsymbol{x}$). We can also see this from another perspective: We have 3 unknowns x_1 , x_2, and x_3 in our system, but we only have 2 equations to work with.

- Can these 2 planes not intersect each other at all ? Possibly !! (no solutions for $\boldsymbol{x}$)
- Can these 2 planes intersect at an infinite # of points ? Possibly !! (infinite # of solutions for $\boldsymbol{x}$)

Your job is to prove that we have an *__infinite # of solutions__* for $\boldsymbol{x}$, and then, you will prove it graphically by plotting them in matlab !!

(a) Rewrite the above in matrix form: $A\boldsymbol{x} = \boldsymbol{b}$. Then, rewrite the system in augmented form, where it looks like this:

$$[\ A \mid \boldsymbol{b}\ ]$$

(b) Using row reductions (in augmented form), please reduce the system $A\boldsymbol{x} = \boldsymbol{b}$ down into its reduced row-echelon (RREF) equivalent $R\boldsymbol{x} = \boldsymbol{d}$ When expressed in augmented matrix form, this means:

$$[\ A \mid \boldsymbol{b}\ ] \xrightarrow{\ row\ reductions\ } [\ R \mid \boldsymbol{d}\ ]$$

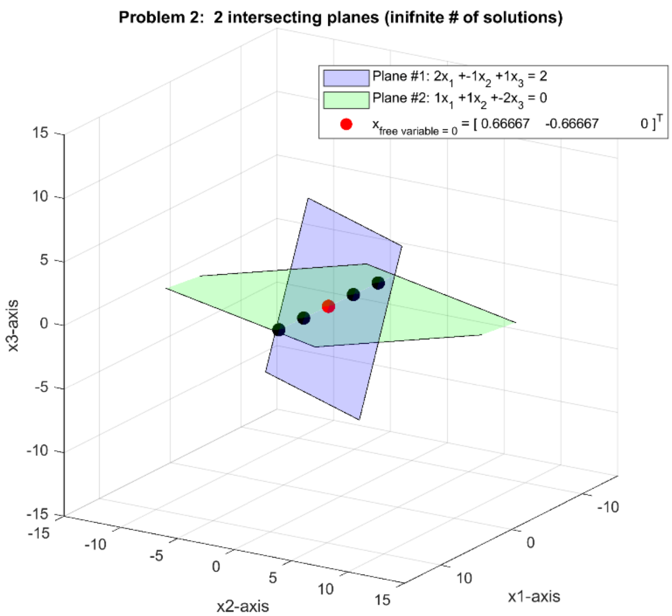
(c) Using matlab's *rref* command, check your handwritten answers (you don't have to turn this in). Type in this command to check for the values of $[\ R \mid \boldsymbol{d}\ ]$:

```
rref( [A b] )
```

Very useful command for all of your future homeworks !!

(d) Circle the pivots in your R-matrix, and indicate the locations of the "free columns," if they exist.

(e) Using your results for $R\boldsymbol{x} = \boldsymbol{d}$, give an 1 - 2 sentence explanation on why you think this system does indeed have an infinite number of solutions.



Now, let's visualize this system !! Open up the file:

```
Problem3_2Dplanes_infinite.m
```

Run the code, and you will see the 2 intersecting planes. You will also see 1 red dot and 4 black dots (see the attached figure to the left).

Manually-rotate the figure around to get a feel for the geometry at hand, and then, try to answer the following questions:

** You don't need to turn in any matlab publish files for Problem 3 !! This file is only for your viewing pleasure =)

(f) In the context of our $A\boldsymbol{x} = \boldsymbol{b}$ problem what you do you think those red and black dots represent ? Explain your answer in 1 or 2 sentences.

(g) In the legend, you see that the red dot currently has the label:

$$\boldsymbol{x}_{solution} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -2/3 \\ 0 \end{bmatrix}$$

Multiply this $\boldsymbol{x}$ by matrix A on paper, and manually show that:

$$A\ \boldsymbol{x}_{solution} = \boldsymbol{b}$$

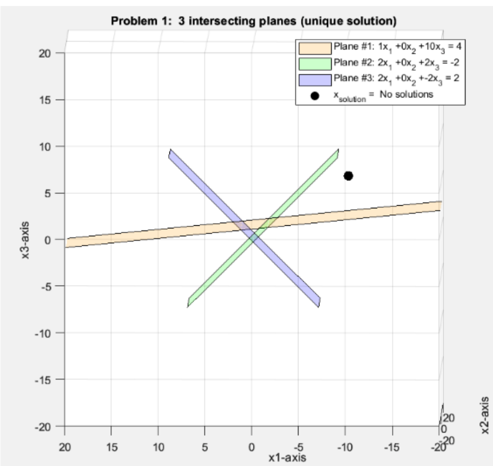
(ie. You have now proven that the red dot is one of many valid solutions for our $A\boldsymbol{x} = \boldsymbol{b}$ problem !! =)

problem 4: When you have no solutions for an $A\boldsymbol{x} = \boldsymbol{b}$ problem (and see how REF / RREF works)

Suppose you have this system of equations, where each equation represents a 2D plane residing in 3D space:

$$\begin{aligned} x_1 \quad &+ 10\ x_3 = 4 \qquad (Plane\ \#1) \\ 2\ x_1 \quad &+ \ 2\ x_3 = -2 \qquad (Plane\ \#2) \\ 2\ x_1 \quad &- \ 2\ x_3 = 2 \qquad (Plane\ \#3) \end{aligned}$$

To begin with, run the code: `Problem4_2Dplanes_no_intersection.m`



You will see that that these 3 planes *will not intersect* each other at a single, common point in 3D space. This means our $A\boldsymbol{x} = \boldsymbol{b}$ problem will have no solutions.

Your job is: Mathematically prove that $A\boldsymbol{x} = \boldsymbol{b}$ has no solutions !! =)

** You don't need to turn in any matlab publish files for Problem 4 !!
This file is only for your viewing pleasure =)

(a) Rewrite the above in matrix form: $A\boldsymbol{x} = \boldsymbol{b}$. Then, rewrite the system in augmented form, where it looks like this:

$$[\ A \mid \boldsymbol{b}\ ]$$

(b) Using row reductions (in augmented form), please reduce the system $A\boldsymbol{x} = \boldsymbol{b}$ down into an REF (not RREF), where the equation becomes $(REF)\ \boldsymbol{x} = \boldsymbol{c}$. When expressed in augmented matrix form, it looks like this:

$$[\ A \mid \boldsymbol{b}\ ] \xrightarrow{row\ reductions} [\ REF \mid \boldsymbol{c}\ ]$$

Hint: Since we expect no solutions for this problem, you will not be able to get matrix A into a pure upper-triangle form U because the last row will be all zeros. Instead, you will have to settle with REF on the left side of the augmented matrix.

$$[\ REF \mid \boldsymbol{c}\ ] = \left[\begin{array}{ccc|c} \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 0 & 0 & \blacksquare \end{array}\right], \quad \blacksquare = some\ numbers$$

(c) In 2 or 3 sentences, explain why your answer for $[\ REF \mid \boldsymbol{c}\ ]$ suggests that there are no solutions for our $A\boldsymbol{x} = \boldsymbol{b}$ problem.

(d) You can also verify this using matlab. Type in the command:

```
rref( [A b] )
```

Write the answer down on paper.

(e) In 2 or 3 sentences, explain why the results from `rref( [A b])` suggests that there are no solutions to our $A\boldsymbol{x} = \boldsymbol{b}$ problem.