PS12

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$$S = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

1

$$\det(S - \lambda I) = 0$$
$$(1-\lambda)(4-\lambda) - 2 \cdot 2 = 0$$
$$\lambda^2 - 5\lambda = 0$$
$$\lambda(\lambda - 5) = 0$$
$$\lambda_1 = 0,\ \lambda_2 = 5$$

$$D = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$$

2

$$\ker(S - 5I) = \text{span}\{v_1\}$$
$$S - 5I = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + \frac{1}{2}R_1} \begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to \frac{-1}{4}R_1} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$
$$x_2 = t \Rightarrow x_1 = \tfrac{1}{2}t \Rightarrow v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\ker(S) = \text{span}\{v_2\}$$
$$S = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$
$$x_2 = t \Rightarrow v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

3

$$u_1 = \tfrac{1}{\sqrt{5}}v_1 = \tfrac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad u_2 = \tfrac{1}{\sqrt{5}}v_2 = \tfrac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$
$$Q = \begin{bmatrix} u_1 & u_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

4

Since Q is orthogonal, $Q^{-1} = Q^T$. Thus

$$Q^{-1} = Q^T = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}.$$

Now compute step by step:

$$QD = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 5\frac{1}{\sqrt{5}} & 0 \\ 5\frac{2}{\sqrt{5}} & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{5} & 0 \\ 2\sqrt{5} & 0 \end{bmatrix}.$$

$$QDQ^{-1} = \begin{bmatrix} \sqrt{5} & 0 \\ 2\sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \sqrt{5}\frac{1}{\sqrt{5}} + 0\left(-\frac{2}{\sqrt{5}}\right) & \sqrt{5}\frac{2}{\sqrt{5}} + 0\frac{1}{\sqrt{5}} \\ 2\sqrt{5}\frac{1}{\sqrt{5}} + 0\left(-\frac{2}{\sqrt{5}}\right) & 2\sqrt{5}\frac{2}{\sqrt{5}} + 0\frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = S.$$