ENG EK103: Problem Set 10 (Spring 2025) Due: Before 11:59pm, April 17, 2025

Use MATLAB only to check your answers. **No Explanation = No Credit**. For each problem, all solution steps should be written down and explained clearly.

Bold lowercase letter (such as $\boldsymbol{v}$) represents a vector. Uppercase letter (such as A) represents a matrix.

Q1 (Markov Chains)

In this problem, we are interested in exploring a *Markov Chain* for a population $\mathcal{P}$ whose members at any given time k (where k=0,1,2…) belong to one of three states: S_1, S_2 or S_3. Let the state vector for the given Markov Chain at any particular time k be denoted by $\boldsymbol{x}_k$.

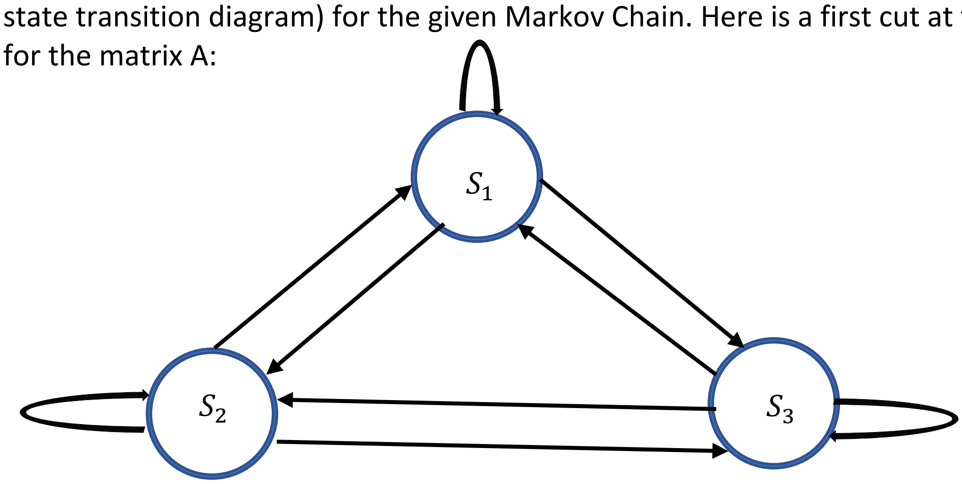
$$\boldsymbol{x}_k = \begin{bmatrix} a_k \\ b_k \\ c_k \end{bmatrix}$$

Here, a_k denotes the proportion (a real number between 0 and 1) of the population $\mathcal{P}$ that belongs to state S_1 at time k, b_k denotes the proportion of the population $\mathcal{P}$ that belongs to state S_2 at time k, and c_k denotes the **proportion** of the population $\mathcal{P}$ that belongs to state S_3 at time k. Note that since each state vector represents the distribution of the population among the three states, each state vector is a *probability vector* (the sum of its elements is 1).
For our Markov Chain, it is also given that:

$$\boldsymbol{x}_{k+1} = \begin{bmatrix} 0.6a_k + 0.2b_k + \alpha c_k \\ 0.2a_k + \beta b_k + 0.2c_k \\ \gamma a_k + 0.2b_k + 0.6c_k \end{bmatrix}$$

(a) Determine the 3×3 *state transition matrix* A for the given Markov Chain, so that $\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k$. You may use the constants α, β, and γ as some of the elements within the matrix A.

(b) For the matrix A you just determined, we can draw a *Markov Diagram* (also known as a state transition diagram) for the given Markov Chain. Here is a first cut at the diagram for the matrix A:



Here, the circles represent the three different states, each arrow represents a transition from a state (at the arrow's tail) at time k to a state (at the arrow's tip) at time $k + 1$. The only thing missing is the labels on the arrows indicating the corresponding transition probabilities. Please complete the diagram by placing the appropriate label near each arrow; You may use the constants $\alpha, \beta, \text{and } \gamma$ as some of the labels in the diagram.

(c) Given that the state transition matrix for any Markov Chain is a *stochastic matrix* (whose columns are *probability vectors*), determine the numerical values of the constants $\alpha, \beta, \text{and } \gamma$.

(d) Show that $|A - \lambda I)| = (\lambda - 1)(-\lambda^2 + 0.8\lambda - 0.16)$

(e) Determine all three eigenvalues of A. Note that $\lambda = 1$ is the "dominant" eigenvalue in the sense that it is the one with the largest magnitude among all the eigenvalues of A.

(f) Explain why the matrix A is diagonalizable and thus can be written as $A = PDP,^{-1}$ where the diagonal matrix D holds the eigenvalues of A and the columns of the matrix P are independent eigenvectors of the matrix A.

(g) What are the eigenvalues of A^k? (HINT: From lecture, if $A = PDP,^{-1}$ then we can conclude that $A^k = PD^kP.^{-1}$)

(h) What happens to the eigenvalues of A^k as $k \to \infty$? (HINT: all the eigenvalues of A^k go to zero except the dominant eigenvalue. In fact, this property forms a foundation of what is known in linear algebra as the *Power Method* for determining the dominant eigenvalue and the corresponding eigenspace).

(i) As $k \to \infty$, determine the fraction of the population $\mathcal{P}$ that ends up in state S_1.(HINT: You can perform RREF on the matrix $A - I$ and solve for the eigenspace corresponding to the dominant eigenvalue ($\lambda = 1$) of A. Then pick the *only* member of that eigenspace that is also a probability vector. This vector represents the final state vector (as $k \to \infty$) of the Markov Chain, regardless of the initial state vector $\boldsymbol{x}_0$).

Q2

In this problem, we are going to explore the *projections* of the vector $\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ onto two different vector spaces of $\mathbb{R}^3$:

- Vector subspace S_1 with a basis set specified as $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$.
- Vector subspace S_2 with a basis set specified as $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(a) Determine the *Projection Matrix* P_1 that maps any vector in $\mathbb{R}^3$ onto the vector subspace S_1.

(b) Apply the matrix P_1 to the vector $\boldsymbol{v}$ given at the beginning of this problem to obtain a vector $\boldsymbol{w}_1 = P_1\boldsymbol{v}$.

(c) What is the *shortest distance* between the (tip of the) vector $\boldsymbol{v}$ and (the tip of) any vector in the subspace S_1

(d) Determine the *Projection Matrix* P_2 that maps any vector in $\mathbb{R}^3$ onto the vector subspace S_2.

(e) Apply the matrix P_2 to the vector $\boldsymbol{v}$ given at the beginning of this problem to obtain a vector $\boldsymbol{w}_2 = P_2\boldsymbol{v}$.

(f) What is the *shortest distance* between the (tip of the) vector $\boldsymbol{v}$ and (the tip of) any vector in the subspace S_2?

Q3

In this problem, we explore the *projection* of the vectors $\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 4 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ onto the vector subspace S that has the basis set specified as $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

(a) Determine the *Projection Matrix* P that maps any vector in $\mathbb{R}^4$ onto the vector subspace S.

(b) Show that the vector $\boldsymbol{v}$ belongs to the subspace S.

(c) Apply the matrix P to the vector $\boldsymbol{v}$ given at the beginning of this problem. Explain how your result makes sense in light of what you showed in the previous part of this problem.

(d) Show that the vector $\boldsymbol{w}$ is orthogonal to every vector in the subspace S

(e) Apply the matrix P to the vector $\boldsymbol{w}$ given at the beginning of this problem. Explain how your result makes sense in light of what you showed in the previous part of this problem.

See Problem 4 on the next page !!

** You will need the 2 *.mat data files for Problem 4(b):

File #1: `AA_test.mat` = Contains 6 matrices of size (3x3)
= You should use this to test your code out

File #2: `AA_homework.mat` = Contains 100 matrices of size (3x3)
= Use this for your homework submission

4 Q4: Matlab application

NOTE: PROBLEM 4 REQUIRES TWO SEPARATE CODE SUBMISSIONS, ONE FOR (a) AND ONE FOR (b).

a) Using `eig()` and `rank()` (do not use `inv()`), write a MATLAB function that takes any square matrix and:

- computes its eigenvalues and eigenvectors
- checks if it is diagonalizable
- returns the tuple (`P` , `D` , `isDiagonalizable`), where `isDiagonalizable` is `true` if $\mathbf{P}^{-1}$ exists, and `false` otherwise.

Publish this Matlab script (see below for script scaffold).

```
====(BEGIN PUBLISHABLE FUNCTION SCRIPT)====

[a,b,c] = yourFunctionNameHere(A);

display(a)
display(b)
display(c)

% We will count off points if your function name isn't renamed to be appropriate AND if
this boilerplate comment isn't replaced by a descriptive comment.
function [..]  = yourFunctionNameHere(A)

end

====(END PUBLISHABLE FUNCTION SCRIPT)====
```

b) Using the `*.mat` data files we provided, write a Matlab function which

- calls your prior function from part (a) above
- loops over all matrices
- sums together the eigenvalues of **only the diagonalizable** matrices.

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For instance, if the eigenvalues of the first matrix were [1,2], the second were [3,4,5], *but it were not diagonalizable*, and the third were [-1,5,10,22] then the answer would be 39 (i.e. 1+2+-1+5+10+22).

NOTE: You will need to use `cell` notation to extract the individual matrices. This is straightforward: to access the first matrix in the cell array, you'll write `A = AA{1}`, for the second one `A = AA{2}`, and so on...

In a brand new script, copy the code from (a) and adapt it. **Publish this script**, using the code scaffold below.

```
====(BEGIN PUBLISHABLE FUNCTION SCRIPT)====

runningSum = 0;

for ..
⋮
end

display(runningSum)

% We will count off points if your function name isn't renamed to be appropriate AND if
this boilerplate comment isn't replaced by a descriptive comment.
function [..]  = yourFunctionNameHere(A)
⋮
end

====(END PUBLISHABLE FUNCTION SCRIPT)====
```