Q1

A

$$\begin{bmatrix}0&0&1&1&2&4\\1&1&2&3&4&11\\1&1&-3&-2&-6&-9\end{bmatrix}\xrightarrow{R_1\Leftrightarrow R_3}\begin{bmatrix}1&1&-3&-2&-6&-9\\1&1&2&3&4&11\\0&0&1&1&2&4\end{bmatrix}\xrightarrow{R_2=R_2-R_1}\begin{bmatrix}1&1&-3&-2&-6&-9\\0&0&5&5&10&20\\0&0&1&1&2&4\end{bmatrix}$$

$$\xrightarrow{R_3=R_3-\frac{1}{5}R_2}\begin{bmatrix}1&1&-3&-2&-6&-9\\0&0&5&5&10&20\\0&0&0&0&0&0\end{bmatrix}=REF(A')$$

B

$$\begin{bmatrix}1&1&-3&-2&-6&-9\\0&0&5&5&10&20\\0&0&0&0&0&0\end{bmatrix}$$

Column	Type
1	Pivot
2	Free Variable
3	Pivot
4	Free Variable
5	Free Variable

C

$$\xrightarrow{R_2=\frac{1}{5}R_2}\begin{bmatrix}1&1&-3&-2&-6&-9\\0&0&1&1&2&4\\0&0&0&0&0&0\end{bmatrix}\xrightarrow{R_1=R_1+3R_2}\begin{bmatrix}1&1&0&1&0&3\\0&0&1&1&2&4\\0&0&0&0&0&0\end{bmatrix}=RREF(A')$$

D

Assigning free variables to free columns:

$$x_2 = t_2$$
$$x_4 = t_4$$
$$x_5 = t_5$$

Then from $RREF(A')$

$$x_1 = 3 - t_2 - t_4$$
$$x_3 = 4 - t_3 - 2t_5$$

So the complete solution is

$$\vec{x_c}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}3\\0\\4\\0\\0\end{bmatrix}+t_2\begin{bmatrix}-1\\1\\0\\0\\0\end{bmatrix}+t_4\begin{bmatrix}-1\\0\\-1\\1\\0\end{bmatrix}+t_5\begin{bmatrix}0\\0\\-2\\0\\1\end{bmatrix}$$

E

Since there are 3 linearly independent vectors in $\mathbb{R}^5$ corresponding to 3 free variables in the homogeneous part of the solution, the set of all possible solutions $\vec{x_c}$ spans a 3 dimensional solid or subspace in $\mathbb{R}^5$.

F

Any given 3×5 matrix B where $B\vec{x} = \vec{b}$ is consistent will yield an infinite set of solutions, since the number of pivots cannot exceed the number of rows, which is 3 in this case, which implies that the solution will have a minimum of 2 free variables, and therefore a non-unique solution set.

Q2

A

One solution to $A\vec{x} = 0$ is always $\vec{x} = \vec{0}$, which is the trivial solution. So yes, regardless of the value of p, it will have at least one solution, the trivial one.

B

$$\begin{bmatrix} 3 & 0 & 1 & 5 \\ -5 & 1 & 1 & -2 \\ 8 & -1 & 0 & 7 \\ -1 & 2 & p & 11 \end{bmatrix} \xrightarrow{\substack{R_2 = R_2 + \frac{5}{3}R_1 \\ R_3 = R_3 - \frac{8}{3}R_1 \\ R_4 = R_4 + \frac{1}{3}R_1}} \begin{bmatrix} 3 & 0 & 1 & 5 \\ 0 & 1 & \frac{8}{3} & \frac{19}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 2 & p + \frac{1}{3} & \frac{38}{8} \end{bmatrix} \xrightarrow{R_4 = R_4 - 2R_2} \begin{bmatrix} 3 & 0 & 1 & 5 \\ 0 & 1 & \frac{8}{3} & \frac{19}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p-5 & 0 \end{bmatrix} \xrightarrow{R_3 \Leftrightarrow R_4} \begin{bmatrix} 3 & 0 & 1 & 5 \\ 0 & 1 & \frac{8}{3} & \frac{19}{3} \\ 0 & 0 & p-5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

C

If $A\vec{x} = b$ is consistent and $\vec{x_c}$ is a line in $\mathbb{R}^3$, then the SLE must have one free variable. Therefore

$$\begin{aligned} p - 5 &= 0 \\ p &= 5 \end{aligned}$$

Then the new $REF(A')$ is

$$\begin{bmatrix} 3 & 0 & 1 & 5 \\ 0 & 1 & \frac{8}{3} & \frac{19}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

D

$$\xrightarrow{R_1 = \frac{1}{3}R_1} \begin{bmatrix} 1 & 0 & \frac{1}{3} & \frac{5}{3} \\ 0 & 1 & \frac{8}{3} & \frac{19}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

E

Since there is a free variable column for x_3 we assign it to the scalar t_3. Then the solutions are

$$\begin{aligned} x_1 &= \frac{5}{3} - \frac{1}{3}t_3 \\ x_2 &= \frac{19}{3} - \frac{8}{3}t_3 \\ x_3 &= t_3 \end{aligned}$$

And therefore the complete solution, a line in $\mathbb{R}^3$, is

$$x_c = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{19}{3} \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} -\frac{1}{3} \\ -\frac{8}{3} \\ 1 \end{bmatrix}$$

F

No, $B\vec{x} = b$ is not guaranteed to have an infinite number of solutions as there can be one pivot per column in the B matrix, and therefore only a particular solution