ENG EK 103: Computational Linear Algebra: Final exam

Name: ____________________

BU ID number: ____________________

Please circle your EK 103 section:

- A1 (Belta)
- A2 (Sabelhaus)
- A3 (Kaper)
- A4 (Belta)
- A5 (Nishimura)
- A6 (Sen)
- A7 (Fan)

Here are some important ground rules:

- The exam is open notes, open book, open computer, but *not open internet*
- Show all your work !! Answers with no work to support them will receive zero credit
- Be sure you write your name and BU ID # in the given spaces above
- Every page (except this one) has a space in the header to write your name. Please do so !
- There are a lot of empty pages after each problem for your work. Please start each problem on the page of that problem !

Some useful matlab commands for checking your work are:

rref(A), rref([A b]), inv(A), det(A), , roots( ), [V , D] = eig(A)

[U , SIGMA , V] = svd(A)

Problem 1: (25 points): An $A\vec{x} = \vec{b}$ problem, with the 4 subspaces of matrix A

Consider the following matrix A and vector $\vec{b}$:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \qquad and \qquad \vec{b} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

a) (8 points) Find the complete solutions to $A\vec{x} = \vec{b}$

b) (3 points) What is the rank of A ?

c) (6 points) Find the bases for the following subspaces related to matrix A: The column space $C(A)$, nullspace $N(A)$, and the row space $R(A)$

d) (3 points) What are the dimensions of C(A), N(A), and R(A) ?

f) (5 points) Which subspace is orthogonal to the nullspace $N(A)$? Demonstrate this orthogonality using a non-zero vector from each subspace.

Problem 2: (25 points): Projections and least squares.

In part (a), consider the measurements b at times t as specified in the table below. h is the **second** digit of your BU ID number, Ux**X**-xx-xxxx.

t	b
-2	2
-1	0
0	0
1	2
2	h

(a) (4 points) Suppose the data points in the table above can be fit by a quadratic function $c + dt + et^2 = b$, where c, d and e are scalars. Create the linear system of equations, and rewrite the system in the matrix form $A\boldsymbol{x} = \boldsymbol{b}$.

In parts (b)$-$(d), consider another set $B = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 1 \end{bmatrix}$ and $\boldsymbol{c} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(b) (4 points) Determine if the linear system $B\boldsymbol{x} = \boldsymbol{c}$ has a unique solution, infinitely many solutions, or no solution.

(c) (5 points) Find the least squares solution(s) $\hat{\boldsymbol{x}}$ of the linear system $B\boldsymbol{x} = \boldsymbol{c}$.

(d) (4 points) Find the projected vector $\hat{\boldsymbol{c}} = B\hat{\boldsymbol{x}}$, where $\hat{\boldsymbol{c}}$ is the orthogonal projection of $\hat{\boldsymbol{c}}$ into the column space of B.

In parts (e)$-$(f), consider another set $C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}$ and $\boldsymbol{d} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$. The orthogonal projection of $\boldsymbol{d}$ into the column space of C is $\hat{\boldsymbol{d}} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$.

(e) (4 points) Find the error vector $\boldsymbol{e}$ of the projection.

(f) (4 points) Find the angle between $\boldsymbol{e}$ and the column space of C.

Problem 3: (25 points): Consider the following symmetric matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) (7 points) Find the eigenvalues of A.

(b) (4 points) Determine if A is invertible.

(c) (7 points) Find bases for the eigenspaces of A. Show that the bases are orthogonal to each other.

(d) (7 points) Is A diagonalizable? If so, justify your answer and find a diagonal matrix D and an orthogonal matrix P such that $A = PDP^T$. Note that matrix P should contain orthonormal column vectors. If A is not diagonalizable, justify your answer in 2 or 3 sentences.

Problem 4: (25 points): A problem on SVD.

Given the following A matrix:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & h \end{bmatrix}$$

Select the value for h for this problem based on the ***first*** *digit of your BU ID number: UXx-xx-xxxx. Please circle this choice below.*

First digit is 0, 1, 2, 3:	First digit is 4, 5, 6:	First digit is 7, 8, 9:
$h = 2$	$h = 4$	$h = 5$

Please answer the following, showing or explaining how you solved the problem. IMPORTANT: Read these questions carefully, you may avoid performing unnecessary calculations.

! If you get square roots or fractions, you can leave them as-is, no need to rationalize square roots or denominators. Do not answer using decimals, so we know you're not using MATLAB.

Remember that a singular value decomposition is $A = U\Sigma V^\top$

(a) (3 points) What are the sizes of the U, Σ, and V matrices for this A? Explain.

(b) (5 points) Calculate the singular values σ of this A matrix. Insert them into Σ, arranging σ_1 and σ_2 from largest to smallest. Show all your steps.

(c) (5 points) Find the matrix V. Show all your steps.

(d) (5 points) Find the first column of the U matrix. Show all your steps. **! First column only!**

Next, for parts (e) and (f): we're giving you an eigenvector corresponding to the second singular value, σ_2, for this U matrix:

$$\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(e) (5 points) Find the last column of the U matrix. Show all your steps.

! There are multiple ways to solve this problem. If you use $AA^\top$, remember that the last column corresponds with the $\lambda = 0$ eigenvalue.

(f) (2 points) Please write the full SVD as $A = U\Sigma V^\top$.