# ENG EK 103: Computational Linear Algebra: Problem set 7

Only use MATLAB when we tell you to do so. **No Explanation = No Credit**. For each problem, all derivation processes should be written down and described in a clear fashion. For accurate grading, be sure to write down your name, BU ID, and homework number (PS7) on all pages you submit.

1. Intuition about determinants.

   (a) Calculate the determinant of each of the following transformation matrices, using the handy formula for determinants of $2 \times 2$ matrices.

   (i) $A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

   (ii) $A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

   (iii) $A_3 = \begin{bmatrix} 1 & 0 \\ -0.5 & 1 \end{bmatrix}$

   (iv) $A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

   (v) $A_5 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

   (vi) $A_6 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ (Note: your answer should not have a $\theta$ in it.)

   (b) Which of these transformations are invertible, and why? Do not perform additional calculations to answer this question. **No Explanation = No Credit.**

   Next, let's use MATLAB to make observations about determinants.

   *Context:* In engineering, we want to write equations about how an object moves around in space over time. The example object is often affectionately called the "engineering potato," a squiggly shape that someone can draw easily on a whiteboard.

   Follow these steps to set up the rest of this problem:

   - Extract `ps7_2025.zip` and open `potato_points.csv` in a spreadsheet program or text editor. You will see it is a list of 122 vectors with two coordinates each. These represent points on an engineering potato that I drew.
   - Open `ps7_problem1d.m` in MATLAB and run it. You will see a plot of the potato, and example code that applies a transformation matrix to the potato's points (vectors).

Then:

(c) Say I put all 122 of my potato vectors into a matrix $B$. Given any of the $A$ matrices above, what is the size of $C = AB$, and why?

(d) For each of the **five** transformations with specific values, (i)-(v), fill in the code in `ps7_problem1d.m` in MATLAB to apply each transformation to the potato.

**Use MATLAB's "Publish" button to create a PDF of your results to submit. Do not upload pictures or screenshots**.

(e) Which of these transformations preserve the volume (or area) of the potato, and why? Justify your answer using technical terminology and/or theorems from the book.

*Note:* We sometimes write determinants using straight vertical lines in place of brackets surrounding a matrix. However, where your book talks about area and determinants, the straight vertical lines around det($A$) are absolute value, $|\det(A)| = \text{abs}(\det(A))$, since det($A$) is a scalar. This is needed to think about reflections.

(f) Often, we want to write equations about parts in an engineering design that are made from metal or hard plastic - parts that are *rigid* and do not bend or change their shape as they move. A transformation that preserves the shape of a set of vectors is called a *rigid transformation*.

*Question:* Is

$$\text{abs}(\det(A)) = 1$$

enough to confirm that a matrix $A$ is a rigid transformation? Why or why not?

(You can answer intuitively, referencing one of the (i)-(vi) above and the name of that transformation from the table in Ch. 1.9.)

2. Calculating determinants.

(a) Use the "cofactor expansion" method to calculate det($A$) for

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 5 & 5 & 2 \\ 2 & 3 & 4 \end{bmatrix}$$

(b) Use the "product of the pivots" method, **without switching rows or scaling rows**, to calculate the determinant of the same $A$.

**Note:** You can use MATLAB's `det()` function to check your answers if you want... but you need to show your work by hand for credit!

(c) Find the determinant of the following matrices using the properties and facts from Chapter 3.2 of the textbook. **Do not** perform cofactor expansion or row reduction, and do not use MATLAB. Explain how you did so. **No Explanation = No Credit**.

(i) $A = \begin{bmatrix} 4 & 5 & 0 & 7 \\ 0 & -3 & 0 & 10 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

(ii) $B = \begin{bmatrix} 4 & 5 & 6 & 7 \\ 0 & 0 & -6 & 10 \\ 0 & 0 & 0 & 13 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(iii) $D = AA$

(iv) $G = AA^\top$

3. Intuition about eigenvectors and eigenvalues.

(a) Here's a matrix:

$$A = \begin{bmatrix} 3 & -1 \\ -3 & 5 \end{bmatrix}$$

I calculated for you its eigenvalues, and an eigenvector for each eigenvalue. They are:

$$\lambda_1 = 6, \quad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \qquad \lambda_2 = 2, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Here are three other vectors:

$$\mathbf{r}_1 = \begin{bmatrix} -4 \\ -4 \end{bmatrix}, \qquad \mathbf{r}_2 = \begin{bmatrix} -0.5 \\ 1.5 \end{bmatrix}, \qquad \mathbf{r}_3 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

First, calculate the result of applying $A$ to these vectors:

$$\mathbf{p}_1 = A\mathbf{r}_1, \qquad \mathbf{p}_2 = A\mathbf{r}_2, \qquad \mathbf{p}_3 = A\mathbf{r}_3$$

Then, sketch three plots, one for each pair of $\mathbf{r}$-to-$\mathbf{p}$. Your plots should contain:

- The eigenvectors of $A$
- The original vector $\mathbf{r}$
- The transformed vector $\mathbf{p}$

Try to keep your plots to scale (graph paper helps here). Label all points.

(b) Use what you observed in part (a) to answer:

What happens when multiplying $A\mathbf{x}$ if the vector $\mathbf{x}$ is a multiple of one of $A$'s eigenvectors? Use technical terminology such as **eigenvalue**.

(c) Say that a vector $\mathbf{v}$ is an eigenvector of a matrix $A$ associated with eigenvalue $\lambda$. Consider a multiple of this vector, $\mathbf{p} = c\mathbf{v}$, with $c \neq 0$. Use the definition of eigenvalues and eigenvectors:

$$A\mathbf{v} = \lambda\mathbf{v}$$

to show that $\mathbf{p}$ is or is not an eigenvector of $A$. If it is, what's its corresponding eigenvalue?

(d) You've shown that both $\mathbf{v}$ and $\mathbf{p}$ are eigenvectors associated with $\lambda$. But are there more? Fill in the blanks below. *Hint:* we haven't specified a value of the constant $c$...

"If a vector $\mathbf{v}$ is an eigenvector of a matrix $A$, then I also know that any

BLANK

...is an eigenvector of $A$, so there are

BLANK

...-many eigenvectors associated with that $\lambda$."

(e) In the same folder where you extracted `ps7_2025.zip`, open `ps7_problem3d.m` in MATLAB.

Use MATLAB's `eig()` function to calculate the eigenvectors and eigenvalues of the matrix $A$ from part (a). Please use the example code fill in the "Your Code Here" section. **Use MATLAB's "Publish" button to create a PDF of your results to submit**.

Your answer may/will look different than what I gave you above. Explain why both are correct. Your observation from part (c) will help. *Hint:* $1/\sqrt{10} = 0.3162$ and $1/\sqrt{2} = 0.7071$.

4. <u>Calculating an eigenspace given an eigenvalue.</u>

Consider the following two matrices.

$$A_1 = \begin{bmatrix} 3 & -1 \\ -3 & 5 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 2 & 14 \\ 0 & -5 \end{bmatrix},$$

For $A_1$, I already gave you its eigenvalues above. They are $\lambda_1 = 6$ and $\lambda_2 = 2$.

(a) Find the eigenvalues for $A_2$ and justify your answer.

(We haven't talked about calculating eigenvalues yet. This problem asks you to "find." What is special about the form of $A_2$?)

(b) Use row reduction to confirm that the $\lambda_1$ and $\lambda_2$ I gave you for $A_1$ are indeed eigenvalues of $A_1$. Write all row operations. Justify your answer using technical terms such as **pivots**, or **linear independence**, etc.

(c) Do the same for $A_2$, using the eigenvalues you found in part (a).

(d) Find the eigenspaces for $A_1$ corresponding to the eigenvalues $\lambda_1$ and $\lambda_2$.

*Important:* Remember that the eigenspace is all possible eigenvectors. Please parameterize your solutions. (You did parameterization on Exam 1's homogeneous equation question.)

*Notes:* If you did the row reduction above, no need to repeat it here. For $A_1$, these should match question 3 up to a scaling factor.

(e) Do the same for $A_2$, using the eigenvalues you found in part (a).