

Homework 3

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February 11, 2025

Problem 1 - Section A

System 1

A

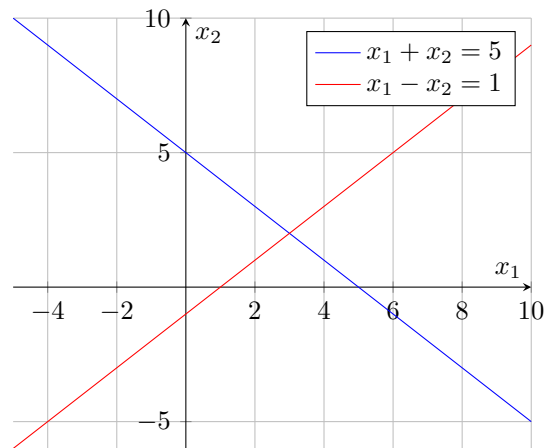


Figure 1: Plot of $x_1 + x_2 = 5$ and $x_1 - x_2 = 1$

B

A unique solution since there is one intersection point between the lines.

C

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

D

$$\begin{bmatrix} 1 & 1 & 5 \\ 1 & -1 & 1 \end{bmatrix}$$

E

$$\begin{bmatrix} 1 & 1 & 5 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow{R_2=R_2-R_1} \begin{bmatrix} 1 & 1 & 5 \\ 0 & -2 & -4 \end{bmatrix}$$

F

$$\begin{bmatrix} 1 & 1 & 5 \\ 0 & -2 & -4 \end{bmatrix} \xrightarrow{R_2 = -\frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 = R_1 - R_2} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

Therefore, given this RREF, it follows that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

System 2

A

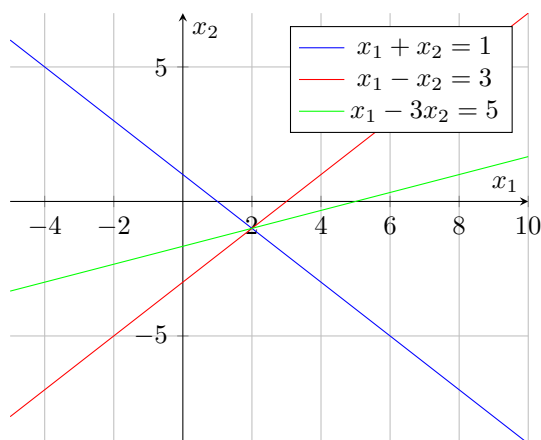


Figure 2: Plot of $x_1 + x_2 = 1$, $x_1 - x_2 = 3$, and $x_1 - 3x_2 = 5$

B

The SLE will have one solution since all lines intersect at the same point.

C

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

D

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ 1 & -3 & 5 \end{bmatrix}$$

E

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ 1 & -3 & 5 \end{bmatrix} \xrightarrow{R_2 = R_2 - R_1, R_3 = R_3 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & -4 & 4 \end{bmatrix} \xrightarrow{R_3 = R_3 - 2R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 = -\frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 = R_1 - R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, given this RREF, it follows that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

System 3

A

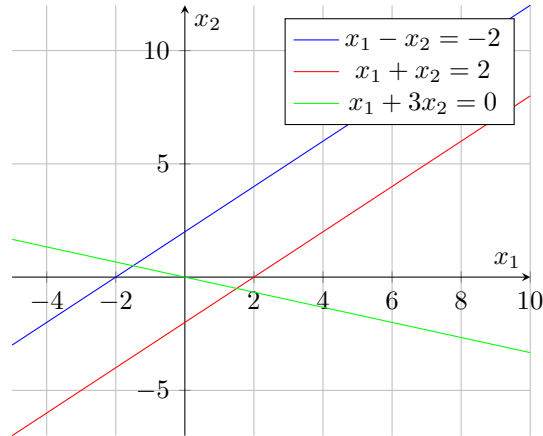


Figure 3: Plot of $x_1 - x_2 = -2$, $x_1 + x_2 = 2$, and $x_1 + 3x_2 = 0$

B

No solution to SLE since the lines do not all have common intersection point.

C

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$$

D

$$\begin{bmatrix} 1 & -1 & -2 \\ 1 & 1 & 2 \\ 1 & 3 & 0 \end{bmatrix}$$

E

$$\begin{bmatrix} 1 & -1 & -2 \\ 1 & 1 & 2 \\ 1 & 3 & 0 \end{bmatrix} \xrightarrow{R_2=R_2-R_1, R_3=R_3-R_1} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{bmatrix} \xrightarrow{R_3=R_3-2R_2} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & -6 \end{bmatrix}$$

F

The last column of the REF effectively shows that $x_1 \cdot 0 = -6$, which is sufficient to determine that the SLE is not consistent.

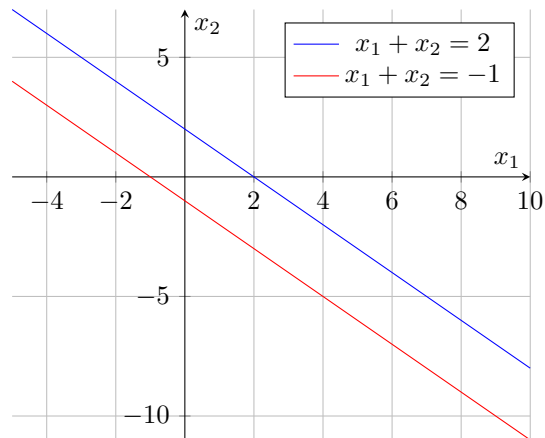


Figure 4: Plot of $x_1 + x_2 = 2$ and $x_1 + x_2 = -1$

Problem 1 - Section B

System 4

B

The SLE will have no solutions since the lines are parallel and therefore will never intersect.

C

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & -2 \end{bmatrix} \xrightarrow{R_2=R_2-R_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} \xrightarrow{R_2=-\frac{1}{4}R_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1=R_1-2R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

D

```
A = [1,1;1,1];
disp(rref(A))
```

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

E

```
A = [1,1,2;1,1,-2];
disp(rref(A))
```

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

F

$$\begin{bmatrix} \textcircled{1} & 1 & 0 \\ 0 & 0 & \textcircled{1} \end{bmatrix}$$

Column 1: Pivot Column 2: Free variable Column 3: Pivot

G

Since there is a pivot in the last column, from which it follows that $x_2 \cdot 0 = 1$, the SLE is in fact not consistent.

System 5

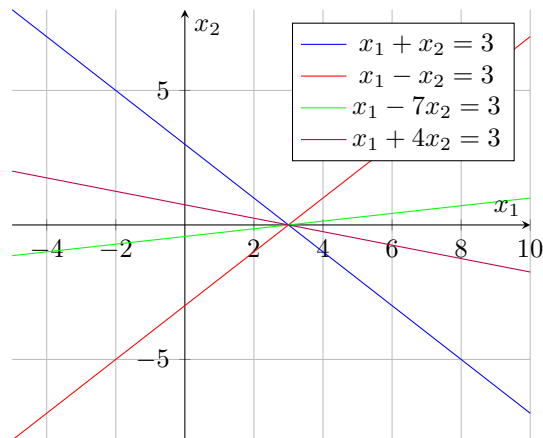


Figure 5: Plot of $x_1 + x_2 = 3$, $x_1 - x_2 = 3$, $x_1 - 7x_2 = 3$, and $x_1 + 4x_2 = 3$

B

Unique solution since there exists a common intersection point of the lines.

C

$$\begin{aligned}
 & \begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 3 \\ 1 & -7 & 3 \\ 1 & 4 & 3 \end{bmatrix} \xrightarrow{R_2=R_2-R_1, R_3=R_3-R_1, R_4=R_4-R_1} \begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & 0 \\ 0 & -8 & 0 \\ 0 & 3 & 0 \end{bmatrix} \xrightarrow{R_4=2R_4} \begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & 0 \\ 0 & -8 & 0 \\ 0 & 6 & 0 \end{bmatrix} \\
 & \xrightarrow{R_4=R_4+3R_3, R_3=R_3-4R_2} \begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2=-\frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1=R_1-R_2} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
 \end{aligned}$$

D

```
A = [1,1;1,-1;1,3;1,4];
disp(rref(A))
```

```
1    0
0    1
0    0
0    0
```

E

```
A = [1,1,3;1,-1,3;1,-7,3;1,4,3];
disp(rref(A))
```

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

F

$$\begin{bmatrix} \textcircled{1} & 0 & 3 \\ 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Column 1: Pivot Column 2: Pivot

G

Yes, the SLE is indeed consistent and unique as predicted. By expanding the augmented matrix back to two equations we are left with $x_1 = 3, x_2 = 0$, which is indeed the same point shown in Fig.5.

System 6

A

-

B

Since there are 3 equations in 5 unknowns, if the system is consistent it must have infinitely many solutions (with two free variables).

C

$$\begin{aligned} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 4 \\ -1 & 1 & -1 & 1 & 1 & 2 \\ 2 & 2 & 0 & -1 & 5 & 4 \end{bmatrix} &\xrightarrow{R_2=R_2+R_1, \quad R_3=R_3-2R_1} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 4 \\ 0 & 2 & 0 & 2 & 2 & 6 \\ 0 & 0 & -2 & -3 & 3 & -4 \end{bmatrix} \\ &\xrightarrow{R_2=\frac{1}{2}R_2, \quad R_3=-\frac{1}{2}R_3} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{3}{2} & 2 \end{bmatrix} \\ &\xrightarrow{R_1=R_1-R_2} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{3}{2} & 2 \end{bmatrix} \\ &\xrightarrow{R_1=R_1-R_3} \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} & \frac{3}{2} & -1 \\ 0 & 1 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{3}{2} & 2 \end{bmatrix} \end{aligned}$$

D

```
A = [1,1,1,1,1;-1,1,-1,1,1;2,2,0,-1,5];  
disp(rref(A))
```

```
1.0000    0          0   -1.5000    1.5000  
0      1.0000          0    1.0000    1.0000  
0          0      1.0000    1.5000   -1.5000
```

E

```
A = [1,1,1,1,1,4;-1,1,-1,1,1,2;2,2,0,-1,5,4];  
disp(rref(A))
```

```
1.0000    0          0   -1.5000    1.5000   -1.0000  
0      1.0000          0    1.0000    1.0000    3.0000  
0          0      1.0000    1.5000   -1.5000    2.0000
```

F

Column 1: Pivot Column 2: Pivot Column 3: Pivot Column 4: Free Variable Column 5: Free Variable

G

Given the RREF, and assuming $x_4 = t_4, x_5 = t_5$, it can be deduced that

$$\begin{aligned}x_1 &= -1 + \frac{3}{2}t_4 - \frac{3}{2}t_5, \\x_2 &= 3 - t_4 - t_5, \\x_3 &= 2 - \frac{3}{2}t_4 + \frac{3}{2}t_5.\end{aligned}$$

Therefore, demonstrating the aforementioned proposition system is consistent and not unique, but has infinitely many solutions.

Problem 2

0.1 A

$$A\vec{v} = \vec{b}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -3 & 0 \\ -1 & 2 & 6 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

$$[A|\vec{v}]$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & -3 & 0 & 1 \\ -1 & 2 & 6 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & -3 & 0 & 1 \\ -1 & 2 & 6 & 5 \end{bmatrix} \xrightarrow{R_2=R_2-R_1, R_3=R_3+R_1} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -4 & -1 & -1 \\ 0 & 3 & 7 & 7 \end{bmatrix} \xrightarrow{R_3=R_3+\frac{3}{4}R_2} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -4 & -1 & -1 \\ 0 & 0 & \frac{25}{4} & \frac{25}{4} \end{bmatrix}$$
$$\begin{bmatrix} \textcircled{1} & 1 & 1 & 2 \\ 0 & \textcircled{-4} & -1 & -1 \\ 0 & 0 & \textcircled{\frac{25}{4}} & \frac{25}{4} \end{bmatrix}$$
$$\begin{aligned} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -4 & -1 & -1 \\ 0 & 0 & \frac{25}{4} & \frac{25}{4} \end{bmatrix} &\xrightarrow{R_2 = -\frac{1}{4}R_2, R_3 = \frac{4}{25}R_3} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 = R_2 - \frac{1}{4}R_3} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \\ &\xrightarrow{R_1 = R_1 - R_3} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 = R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{aligned}$$
$$\begin{matrix} 1 \\ 0 \\ 1 \end{matrix}$$


```

% Plane #1:      x1 + x2 + x3 = 2
% Plane #2:      x1 - x2 + 2x3 = 3
% Plane #3:      -x1 + 2x2 + 4x3 = 3

% -- Define matrix A

A = [ 1  1  1 ;      % -- row vector r1
      1 -3  0 ;      % -- row vector r2
     -1  2  6 ]      % -- row vector r3

%   a1 a2 a3        % -- Column vectors a1, a2, a3

% -- Define target vector b
b = [ 2 1 5 ]'

% -- We can solve for x by using the \ command:

disp('The solution to our Ax = b problem is');
x = A\b

% //////////////////////////////////////
%
% Task 1:   Drawing the 3 planes of interest
%
% We will learn how to do this later when we talk about nullspace
% solutions. For now, just trust the code here =>
%
%
% Sub-tasks:
% -----
%
% -- 1. Extract the row vectors r1, r2, r3 from matrix A
%
% -- 2. Then, we will find the nullspace vectors to the equation
%       r' * xn1 = 0
%       r' * xn2 = 0
%
% -- 3. Then, will draw the plane spanned by the 2 nullspace vectors
%       xn1 and xn2
%
% //////////////////////////////////////

% -- Define my own color pallet

my_colormap = [ 1  0.6  0 ; % --- orange
                0  1  0 ; % -- green
                0  0  1 ]; % -- blue

% -- Create a new figure

```

```

figure;

% -- Forcing the new figure to have a certain size on your screen
%    and at a certain position
set(gcf, 'Position', [100, 100, 700, 600])

% -- Iterate over each row vector r1, r2, r3

for count = 1:1:3

    r1 = A(count,:);    % -- Extract the row vector (r1, r2, r3)

    % -- First, we will auto-calculate the:
    %    a) Particular solution: my_xp
    %    b) Nullspace vector #1: my_xn1
    %    c) Nullspace vector #2: my_xn2

    xp = [ b(count)      0  0 ]' / r1(1);
    xn1 = [ -r1(2)/r1(1)  1  0 ]' ;
    xn2 = [ -r1(3)/r1(1)  0  1 ]' ;

    % -- Then, we will plot the plane spanned by my_xn1
    %                                     my_xn2
    %    with the centroid of the plane at:    xp

    % -- Now, we will use the "patch" command to draw a 2D plane that
    %    represents the span{xn1, xn2}
    %
    %    These are the coordinates for the 4 corners of the plane
    %
    %
    %    Point Q                                     Point M
    %
    %    2*(-xn1+xn2)    -----  2*(xn1 + xn2)
    %
    %
    %    |               |               |
    %
    %    <-----  xp  ----->
    %
    %    |               |               |
    %    Point P                                     Point N
    %
    %    2*(-xn1-xn2)    -----  2*(xn1 - xn2)
    %
    %

    % -- We can change the size + scale of our plane by using this
    %    magnification factor

    my_scale = 2;

```

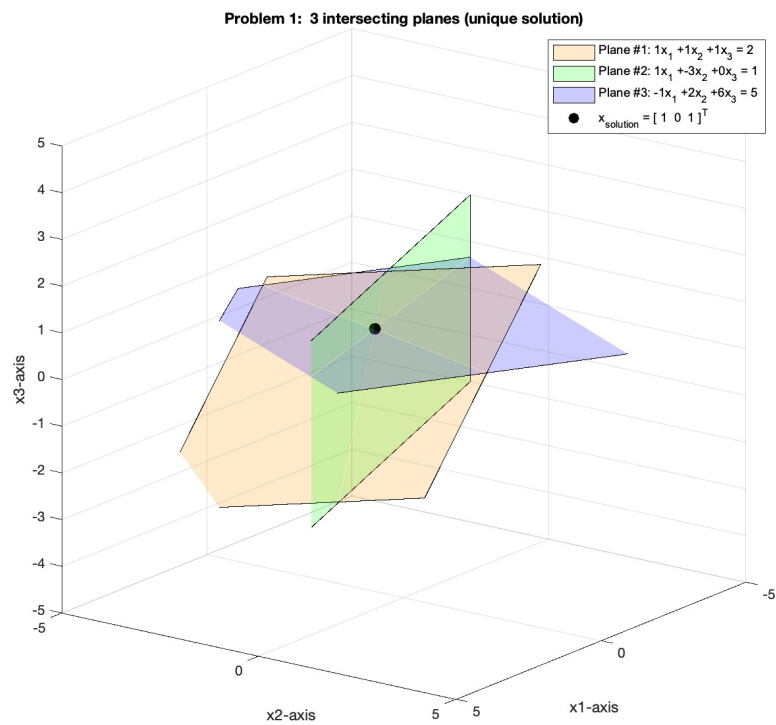

b =

2
1
5

The solution to our $Ax = b$ problem is

x =

1
0
1



Problem 3

A

$$A\vec{x} = \vec{b}$$

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$[A|\vec{b}]$$

$$\begin{bmatrix} 2 & -1 & 1 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

B

$$\begin{aligned} & \begin{bmatrix} 2 & -1 & 1 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \xrightarrow{R_2=R_2-2R_1} \begin{bmatrix} 2 & -1 & 1 & 2 \\ -3 & 3 & -4 & -4 \end{bmatrix} \xrightarrow{R_1=\frac{1}{2}R_1, R_2=-\frac{1}{3}R_2} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & 1 \\ 1 & -1 & \frac{4}{3} & \frac{4}{3} \end{bmatrix} \\ & \xrightarrow{R_2=R_2-R_1} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & 1 \\ 0 & -\frac{1}{2} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \xrightarrow{R_2=-2R_2} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & -\frac{5}{3} & -\frac{2}{3} \end{bmatrix} \xrightarrow{R_1=R_1+\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & -\frac{5}{3} & -\frac{2}{3} \end{bmatrix} \end{aligned}$$

C

```
A = [2,-1,1,2; 1,1,-2,0];
disp(rref(A))
```

```
1.0000  0      -0.3333  0.6667
0      1.0000 -1.6667 -0.6667
```

D

$$\begin{bmatrix} \textcircled{1} & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & \textcircled{1} & -\frac{5}{3} & -\frac{2}{3} \end{bmatrix}$$

Column 1: Pivot Column 2: Pivot Column 3: Free Variable

E

Given the RREF matrix, setting the free variable $x_3 = t_3$, we can then formulate the solution to the SLE As

$$\begin{aligned} x_1 - \frac{1}{3}t_3 &= \frac{2}{3} \\ x_2 - \frac{5}{3}t_3 &= -\frac{2}{3}t_3 = t_3 \end{aligned}$$

Which solved for x_1, x_2 gives

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} \frac{1}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix}$$

Which is the parametric form of a line, which is indeed the intersection between two planes, and therefore has an infinite number of points lying on it.

F

Since the intersection between the planes is a line (they are not parallel or identical) then we can parametrise this line and iterate over it with a scalar parameter (t_3) to get an infinite number of points which satisfies the SLE.

G

$$\begin{aligned} \begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ 0 \end{bmatrix} &= \begin{bmatrix} 2 \cdot \frac{2}{3} + (-1) \cdot \left(-\frac{2}{3}\right) + 1 \cdot 0 \\ 1 \cdot \frac{2}{3} + 1 \cdot \left(-\frac{2}{3}\right) + (-2) \cdot 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{4}{3} + \frac{2}{3} \\ \frac{2}{3} - \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{6}{3} \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \end{aligned}$$

Problem 4

A

$$A\vec{x} = \vec{b}$$

$$\begin{bmatrix} 1 & 0 & 10 \\ 2 & 0 & 2 \\ 2 & 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$$

$$[A|\vec{b}]$$

$$\begin{bmatrix} 1 & 0 & 10 & 4 \\ 2 & 0 & 2 & -2 \\ 2 & 0 & -2 & 2 \end{bmatrix}$$

B