Q1

A

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 2 & 4 \\ 1 & 1 & 2 & 3 & 4 & 11 \\ 1 & 1 & -3 & -2 & -6 & -9 \end{bmatrix} \xrightarrow{R_1 \Leftrightarrow R_3} \begin{bmatrix} 1 & 1 & -3 & -2 & -6 & -9 \\ 1 & 1 & 2 & 3 & 4 & 11 \\ 0 & 0 & 1 & 1 & 2 & 4 \end{bmatrix} \xrightarrow{R_2 = R_2 - R_1} \begin{bmatrix} 1 & 1 & -3 & -2 & -6 & -9 \\ 0 & 0 & 5 & 5 & 10 & 20 \\ 0 & 0 & 1 & 1 & 2 & 4 \end{bmatrix}$$

$$\xrightarrow{R_3 = R_3 - \frac{1}{5}R_2} \begin{bmatrix} 1 & 1 & -3 & -2 & -6 & -9 \\ 0 & 0 & 5 & 5 & 10 & 20 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = REF(A')$$

B

$$\begin{bmatrix} 1 & 1 & -3 & -2 & -6 & -9 \\ 0 & 0 & 5 & 5 & 10 & 20 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Column	**Type**
1	Pivot
2	Free Variable
3	Pivot
4	Free Variable
5	Free Variable

C

$$\xrightarrow{R_2 = \frac{1}{5}R_2} \begin{bmatrix} 1 & 1 & -3 & -2 & -6 & -9 \\ 0 & 0 & 1 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 = R_1 + 3R_2} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = RREF(A')$$

D

Assigning free variables to free columns:

$$\begin{aligned} x_2 &= t_2 \\ x_4 &= t_4 \\ x_5 &= t_5 \end{aligned}$$

Then from $RREF(A')$

$$\begin{aligned} x_1 &= 3 - t_2 - t_4 \\ x_3 &= 4 - t_3 - 2t_5 \end{aligned}$$

So the complete solution is

$$\vec{x_c} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 4 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t_5 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

E

Since there are 3 linearly independent vectors in $\mathbb{R}^5$ corresponding to 3 free variables in the homogeneous part of the solution, the set of all possible solutions $\vec{x_c}$ spans a 3 dimensional solid or subspace in $\mathbb{R}^5$.

F

Any given 3×5 matrix B where $B\vec{x} = \vec{b}$ is consistent will yield an infinite set of solutions, since the number of pivots cannot exceed the number of rows, which is 3 in this case, which implies that the solution will have a minimum of 2 free variables, and therefore a non-unique solution set.

Q2

A

One solution to $A\vec{x} = 0$ is always $\vec{x} = \vec{0}$, which is the trivial solution. So yes, regardless of the value of p, it will have at least one solution, the trivial one.

B

$$\begin{bmatrix} 3 & 0 & 1 & 5 \\ -5 & 1 & 1 & -2 \\ 8 & -1 & 0 & 7 \\ -1 & 2 & p & 11 \end{bmatrix} \xrightarrow{\substack{R_2 = R_2 + \frac{5}{3}R_1 \\ R_3 = R_3 - \frac{8}{3}R_1 \\ R_4 = R_4 + \frac{1}{3}R_1}} \begin{bmatrix} 3 & 0 & 1 & 5 \\ 0 & 1 & \frac{8}{3} & \frac{19}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 2 & p + \frac{1}{3} & \frac{38}{8} \end{bmatrix} \xrightarrow{R_4 = R_4 - 2R_2} \begin{bmatrix} 3 & 0 & 1 & 5 \\ 0 & 1 & \frac{8}{3} & \frac{19}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p - 5 & 0 \end{bmatrix} \xrightarrow{R_3 \Leftrightarrow R_4} \begin{bmatrix} 3 & 0 & 1 & 5 \\ 0 & 1 & \frac{8}{3} & \frac{19}{3} \\ 0 & 0 & p - 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

C

If $A\vec{x} = b$ is consistent and $\vec{x_c}$ is a line in $\mathbb{R}^3$, then the SLE must have one free variable. Therefore

$$\begin{aligned} p - 5 &= 0 \\ p &= 5 \end{aligned}$$

Then the new $REF(A')$ is

$$\begin{bmatrix} 3 & 0 & 1 & 5 \\ 0 & 1 & \frac{8}{3} & \frac{19}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

D

$$\xrightarrow{R_1 = \frac{1}{3}R_1} \begin{bmatrix} 1 & 0 & \frac{1}{3} & \frac{5}{3} \\ 0 & 1 & \frac{8}{3} & \frac{19}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

E

Since there is a free variable column for x_3 we assign it to the scalar t_3. Then the solutions are

$$\begin{aligned} x_1 &= \frac{5}{3} - \frac{1}{3}t_3 \\ x_2 &= \frac{19}{3} - \frac{8}{3}t_3 \\ x_3 &= t_3 \end{aligned}$$

And therefore the complete solution, a line in $\mathbb{R}^3$, is

$$x_c = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{19}{3} \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} -\frac{1}{3} \\ -\frac{8}{3} \\ 1 \end{bmatrix}$$

F

No, $B\vec{x} = b$ is not guaranteed to have an infinite number of solutions as there can be one pivot per column in the B matrix, and therefore only a particular solution

Q3

A

$$A = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$$

B

$$\xrightarrow{R_1=\frac{1}{a}R_1(a\neq0)} \begin{bmatrix} a & b & c \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix} \xrightarrow{R_2=R_2-\frac{b}{a}R_1, R_3=R_3-\frac{c}{a}R_1} \begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

C

$$\xrightarrow{R_2=\frac{1}{b}R_2(b\neq0)} \begin{bmatrix} a^2 & ab & ac \\ a & b & c \\ ac & bc & c^2 \end{bmatrix} \xrightarrow{R_1=R_1-\frac{a}{b}R_2, R_3=R_3-\frac{c}{b}R_2} \begin{bmatrix} 0 & 0 & 0 \\ a & b & c \\ 0 & 0 & 0 \end{bmatrix}$$

D

$$\xrightarrow{R_3=\frac{1}{c}R_2(c\neq0)} \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ c & b & c \end{bmatrix} \xrightarrow{R_1=R_1-\frac{a}{c}R_3, R_2=R_2-\frac{b}{c}R_3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & b & c \end{bmatrix}$$

E

In any of these cases we will have $Rank(A) = 1$, so the span will be a line through the origin

F

If $\vec{v} = 0$ then the span of A will just be the point $(0, 0, 0)$, as there is no possible way to multiply the zero vector to get it to point to any other point.

Q4

A

The span of A is a subspace of $\mathbb{R}^2$, whereas that of B is in $\mathbb{R}^3$. So the span of the two cannot be the same.

B

Since A spans all of $\mathbb{R}^2$ and A is contained in B, their spans are the same.

C

The first 3 columns of both matrices are not linearly independent, in fact they are the same vectors. Therefore, the span of A is just the line passing through origin with homogeneous part equal to one of those entries. Conversely, B has a fourth entry which may be linearly independent of the first three and therefore result in a different span than that of A.

D

Since b_2 is a linear combination of the entries in A, and the remaining entries are also present in A, then the two must have the same span.