P.S. 6

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1

A

$$\begin{bmatrix} -3 & 6 \\ 4 & 8 \\ 5 & -10 \end{bmatrix} \xrightarrow{R_2=R_2+\frac{4}{3}R_1} \begin{bmatrix} -3 & 6 \\ 0 & 16 \\ 1 & -2 \end{bmatrix} \xrightarrow{R_3=R_3+\frac{1}{3}R_1} \begin{bmatrix} -3 & 6 \\ 0 & 16 \\ 0 & -8 \end{bmatrix} \xrightarrow{R_3=R_3+\frac{1}{2}R_2} \begin{bmatrix} -3 & 6 \\ 0 & 16 \\ 0 & 0 \end{bmatrix}$$

$$Col(A) = span\left\{ \begin{bmatrix} -3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 8 \\ -10 \end{bmatrix} \right\}$$

B

$$Basis(Col(A)) = \left\{ \begin{bmatrix} -3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 8 \\ -10 \end{bmatrix} \right\}$$

$$Dim(Col(A)) = 2$$

$Col(A)$ will be a plane in $\mathbb{R}^3$.

C

From

$$Col(A) = \left\{ t_1 \begin{bmatrix} -3 \\ 4 \\ 5 \end{bmatrix} + t_2 \begin{bmatrix} 6 \\ 8 \\ -10 \end{bmatrix} \right\}$$

Choose

$$t_1 = 3, t_2 = 2$$

Then

$$b_1 = \begin{bmatrix} 3 \\ 28 \\ -5 \end{bmatrix}$$

And from MATLAB

```
c_1= [-3;4;5];
c_2=[6;8;-10];

B = c_1*3 + c_2*2;

Fin = [c_1,c_2,B];

disp(rref(Fin))
```

```
   1     0     3
   0     1     2
   0     0     0
```

The system of equations is consistent because $b1$ is a linear combination of the two columns of A, and therefore another vector in the plane that is $Col(A)$.

D

Taking the cross product of the columns of A we get

```
c_1= [-3;4;5];
c_2=[6;8;-10];

disp(cross(c_1, c_2))
```

```
-80
0
-48
```

And scaling down to get

$$b_2 = \begin{bmatrix} -10 \\ 0 \\ -6 \end{bmatrix}$$

Which is perpendicular to the plane spanned by $Col(A)$, and therefore not in the span of $Col(A)$. Consequently, plugging b_2 into the system of equations will yield an inconsistent system.
And proof from MATLAB

```
c_1= [-3;4;5];
c_2=[6;8;-10];

B = cross(c_1 ,c_2);

Fin = [c_1,c_2,B];

disp(rref(Fin))
```

```
   1     0     0
   0     1     0
   0     0     1
```

Which is an inconsistent system.

2

A

$$\begin{bmatrix} 3 & 2 & 5 & -5 \\ 0 & 4 & -2 & 8 \\ 2 & 0 & 4 & -6 \end{bmatrix} \xrightarrow{R_3=R_3-\frac{2}{3}R_1} \begin{bmatrix} 3 & 2 & 5 & -5 \\ 0 & 4 & -2 & 8 \\ 0 & -\frac{4}{3} & \frac{2}{3} & -\frac{8}{3} \end{bmatrix} \xrightarrow{R_3=R_3+\frac{1}{3}R_2} \begin{bmatrix} 3 & 2 & 5 & -5 \\ 0 & 4 & -2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1 = \frac{1}{3}R_1$$
$$R_2 = \frac{1}{4}R_2 \atop \longrightarrow \begin{bmatrix} 1 & \frac{2}{3} & \frac{5}{3} & -\frac{5}{3} \\ 0 & 1 & \frac{-1}{2} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 = R_1 - \frac{2}{3}R_2} \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & \frac{-1}{2} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_3 = t_3\,,\, x_4 = t_4 \Rightarrow \vec{x} = \begin{bmatrix} -2t_3 + 3t_4 \\ -\frac{1}{2}t_3 - 2t_4 \\ t_3 \\ t_4 \end{bmatrix} = t_3 \begin{bmatrix} -2 \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + t_4 \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore Null(B) = span \left\{ \begin{bmatrix} -2 \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

B

$$Basis(Null(B)) = \left\{ \begin{bmatrix} -2 \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$
$$Dim(Null(B)) = 2$$

The dimension of the nullspace of B corresponds to a plane spanned by the two vectors that are the basis of the nullspace.

C

$$\vec{x_h} = t_3 \begin{bmatrix} -2 \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + t_4 \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$\vec{x_h}$ is exactly $Null(B)$

D

$$t_3 = 2\,,\, t_4 = 1$$
$$\vec{x} = \begin{bmatrix} -4 \\ -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 2 \\ 1 \end{bmatrix}$$

3

A

The RREF of B is:

$$\begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -\frac{1}{2} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The non-zero rows of this matrix form a basis for the row space. Therefore, the row space of B is:

$$\text{Row } B = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -\frac{1}{2} \\ 2 \end{bmatrix} \right\}$$

B

The row space of B is a 2-dimensional subspace of $\mathbb{R}^4$. Geometrically, this corresponds to a **plane** in $\mathbb{R}^4$.

C

The number of pivots in matrix B is equal to the number of non-zero rows in its RREF, which is 2. The number of vectors in a basis for the row space (the dimension of the row space) is also 2.

4

```
C = [3  5  -2  -1  1;
  2  0   2   4  2;
 -2  7  -9  -5  4];

R = rref(C);
disp(R);

null_vec = null(sym(C));
disp(null_vec);

disp('C * (nullspace vector) = 0:');
[nRows, nVec] = size(null_vec);
for i = 1:nVec
 xn = null_vec(:, i);
 product = C * xn;
 fprintf('Result for nullspace vector %d:\n', i);
 disp(product);
end
```

```
  1     0     1     0    -1
  0     1    -1     0     1
  0     0     0     1     1

[-1,  1]
[ 1, -1]
[ 1,  0]
[ 0, -1]
[ 0,  1]

C * (nullspace vector) = 0:
Result for nullspace vector 1:
0
0
0

Result for nullspace vector 2:
0
0
0
```