ENG EK 103: Computational Linear Algebra: Final exam

Name: ____

BU ID number: ____

Please circle your EK 103 section:

- A1 (Sabelhaus)
- A3 (Kaper)
- A4 (Sebesta)
- A5 (Nawab)
- A6 (Sen)
- A7 (Fan)

Here are some important ground rules:

- The exam is open notes, open book, open computer, but *not open internet*
- Show all your work !! Answers with no work to support them will receive zero credit
- Be sure you write your name and BU ID # in the given spaces above
- Every page (except this one) has a space in the header to write your name. Please do so !
- There are a lot of empty pages after each problem for your work. Please start each problem on the page of that problem !

Some useful matlab commands for checking your work are:

rref(A), *rref([A b]),* *inv(A),* *det(A),* *, roots( ),* *[V , D] = eig(A)*

[U , SIGMA , V] = svd(A)

Problem 1 (25 points): In this problem, you are to consider the following matrix:

$$A = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \tag{1}$$

and the vector $\mathbf{b} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix}$.

a) (10 points) Performing the row reduction by hand and showing the steps in your work, write down the complete solution $\mathbf{x}_{\text{complete}}$ of $A\mathbf{x} = \mathbf{b}$.

b) (5 points) Find the rank of A...... then, find the dimension of each of the 4 fundamental subspaces:

- The column space of A
- The null space of A
- The column space of A^T (also known as the the row space of A)
- The null space of A^T (also known as the left null space of A)

c) (5 points) Find basis vectors for these 3 subspaces:

- Column space of A
- The null space of A, and
- The row space of A

d) (5 points) Which 2 of the 3 subspaces in part (c) are orthogonal subspaces? Demonstrate the orthogonality between those 2 particular subspaces by picking 1 basis vector from each subspace and taking the dot product between them.

Q2 (25 points)

*(**TIP**: throughout this question, write the formal equations first before replacing symbolic expressions by concrete numbers. When properly done, the arithmetic is easy but a mistake in execution will make it very complicated, so for partial credit it's important we see that you understand the core approach.)*

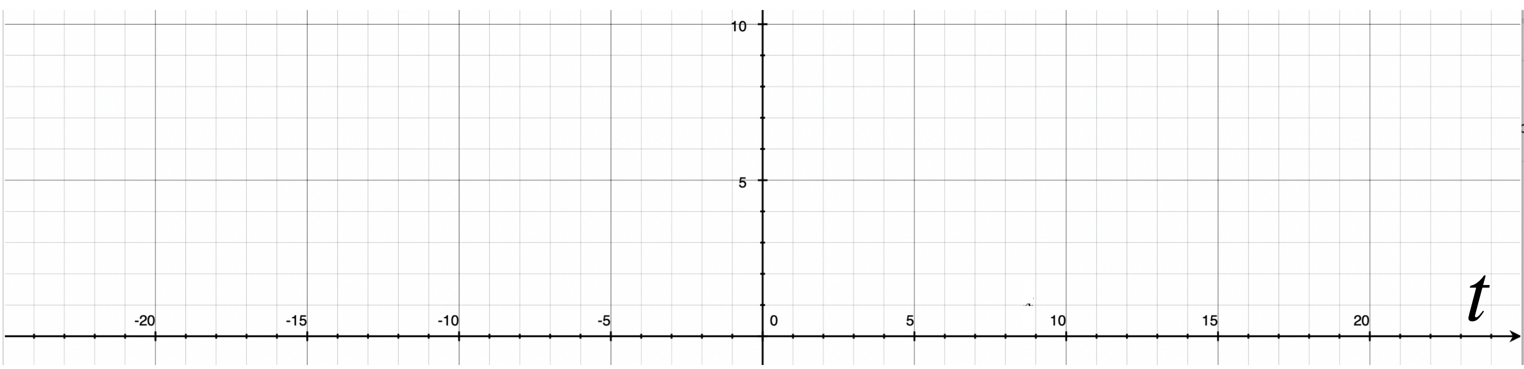
Problem setup:

To the right is a series of height data points relative to time:

t	h
-20	5
-10	3
10	3
20	5

(a) (3 points)

i) Plot the above data points on the graph below.
ii) Sketch a line which will best fit these points.



iii) Eyeballing it, and without using any math, give a best guess as to the line's equation, in the form $h = c_1 t + c_0$. Write the coefficients below:

$c_1 =$ __________ $c_0 =$ __________

(b) (5 points)

Please calculate a $h = c_1 t + c_0$ (aka *line*) least squares fit for the data given above. Show your work, and write the coefficients below:

$c_1 =$ __________ $c_0 =$ __________

HINT: Leave any denominators factored out until right at the very end. Trust us, the calculation will be much easier this way.

TIP: You are strongly encouraged to use the coefficients you find here to refine your "guesstimated" answer to part **(a)(iii)** of this question.

(c) (5 points)

Having written the above curve fitting in $\mathbf{Ax} = \mathbf{b}$ form, now find $\hat{\mathbf{b}}$, the projection of $\mathbf{b}$ onto the column space of $\mathbf{A}$ (see image at right). (NOTE: In some EK103 sections, $\hat{\mathbf{b}}$ is called $\mathbf{p}$.)

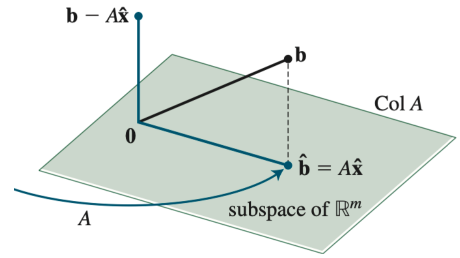


FIGURE 2 The least-squares solution $\hat{\mathbf{x}}$ is in $\mathbb{R}^n$.

(d) (5 points)

i) Find the error vector, defined as $\mathbf{e} = \mathbf{b} - \hat{\mathbf{b}}$

ii) Find the squared error, defined as $||\mathbf{e}||^2$

(e) (5 points)

Set up an equation to find a $h = c_2t^2 + c_1t + c_0$ (aka *quadratic*) least-squares fit for the data given above. DO NOT SOLVE, all we require is your answer in the matrix form $\mathbf{Ax} = \mathbf{b}$

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(f) (2 points)

(This question is independent of the previous parts of this section.)

Using the following basis vectors:

$$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\} = \left\{ \frac{\mathbf{1}}{\sqrt{\mathbf{4}}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \frac{\mathbf{1}}{\sqrt{\mathbf{4}}}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}, \frac{\mathbf{1}}{\sqrt{\mathbf{4}}}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}, \frac{\mathbf{1}}{\sqrt{\mathbf{4}}}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix} \right\}$$

and letting $\mathbf{U} = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3 \quad \mathbf{u}_4]$

i) Find $(\mathbf{U}^\top\mathbf{U})^{-1}$

ii) Find the *projection matrix*, $\mathbf{P} = \mathbf{U}(\mathbf{U}^\top\mathbf{U})^{-1}\mathbf{U}^\top$

Problem 3 (25 points): Singular Value Decomposition.

If you get square roots or fractions, you can leave them as-is, no need to rationalize denominators.

Hint: MATLAB may be less useful for checking your answers than you think. The algorithm inside the `svd()` command can choose a different set of vectors for U and V from us when writing by hand. You *must* show your work.

Remember that a singular value decomposition is $A = U\Sigma V^\top$

Given the following A matrix:

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ \frac{-3\sqrt{2}}{2} & 0 & 0 & \frac{3\sqrt{2}}{2} \end{bmatrix}$$

I have calculated the following for you. These will help you save time, but you may or may not need all of them.

$$AA^\top = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}, \qquad A^\top A = \begin{bmatrix} 4.5 & 0 & 0 & -4.5 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -4.5 & 0 & 0 & 4.5 \end{bmatrix}$$

Four linearly independent, orthogonal, eigenvectors for $A^\top A$, arranged from largest eigenvalue to smallest eigenvalue:

$$\begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ 0 \\ \sqrt{2}/2 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

Questions:

(a) (5 points) What are the eigenvalues of $AA^\top$ and $A^\top A$? Explain.
HINT: **No calculations are required for this part of the question.**

(b) (5 points) Find the matrix Σ. Show all your steps.

(c) (5 points) Find the matrix V. Show all your steps.

(d) (10 points) Find the matrix U. Show all your steps. Write the final SVD as $U\Sigma V^\top$.

Problem 4: Eigenvalues and eigenvectors (25 points)

This problem has two independent parts (PART ONE and PART TWO), i.e., you can do each part without needing any information from the other part.

PART ONE:
Throughout this part you should assume that $A = \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix}$, a <u>symmetric</u> matrix.

(1a) (3 points) Determine whether or not the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of the matrix A. Show your work.

(1b) (5 points) Determine the numerical values of c_1 and c_2 so that the following equation holds:

$$det(A - \lambda I) \ = \lambda^2 + c_1\lambda + c_2$$

Show your work.

(1c) (5 points) Determine a 2×2 *orthonormal* matrix whose columns are eigenvectors of A. Show your work.

PART TWO:
Throughout this part, you should assume that $A = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$, where A is a <u>symmetric</u> matrix and $a, b,$ and c are scalars. Furthermore, we are given two facts about the matrix A:

<u>*FACT 1*</u>: The vector $\boldsymbol{w_1} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is in the *nullspace* of the matrix $B = A - I$, where I is the 2x2 identity matrix.

<u>*FACT 2*</u>: There exist *non-zero* vectors $\boldsymbol{u}$ such that $A\boldsymbol{u} = 2\boldsymbol{u}$.

(2a) (5 points) On the basis of *FACT 1*, one can claim that the vector $\boldsymbol{w_1}$ is in fact an *eigenvector* of matrix A. What is the corresponding *eigenvalue*? Show your reasoning.

(2b) (5 points) Calculate a vector $\boldsymbol{w_2}$ that is *orthogonal* to the vector $\boldsymbol{w_1}$. Show your work.

(2c) (2 points) Give an example of a vector $\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ that satisfies *FACT 2*. In your answer, you should specify a specific numerical value for u_1 and a specific numerical value for u_2. Show your reasoning.