Temporary Doc
Calc 3

Giacomo Cappelletto

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Chapter 1

Vector Valued Functions $f : \mathbb{R} \to \mathbb{R}^n$

1.1 Change of Variable for Double and Triple Integrals

Polar Coordinates

$$\iint_D f(x, y)\, dx\, dy \to \iint_S f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Cylindrical Coordinates

$$\iiint_D f(x, y, z)\, dx\, dy\, dz \to \iiint_S f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz$$

Spherical Coordinates

$$\iiint_D f(x, y, z)\, dx\, dy\, dz \to \iiint_S f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

Theorem 1.1.1 Intuition Behind Change of Variables

We use a **mapping** T to transform coordinates in one space S to another R. This is particularly useful when integrating over regions that are easier to describe in new coordinates (e.g., circular or spherical regions).
For example:

$$S = [0, 2\pi] \times [0, 2], \quad T(r, \theta) = (r\cos\theta, r\sin\theta)$$

Here, the mapping T converts a point in S into a point in R.

Area Differential Transformation

Consider a small differential area element in the original space:

$$dA = |\det(J)|\, du\, dv$$

where J is the **Jacobian matrix**, and $|\det(J)|$ accounts for how the transformation scales area.

Definition 1.1.1: Jacobian Matrix

The Jacobian matrix represents the linear transformation of the mapping T at a given point:

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

For a transformation $T(u, v) = (g(u, v), h(u, v))$, the determinant of J is:

$$\det(J) = \begin{vmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{vmatrix} = \frac{\partial g}{\partial u} \cdot \frac{\partial h}{\partial v} - \frac{\partial g}{\partial v} \cdot \frac{\partial h}{\partial u}$$

Geometric Interpretation

- **Local Stretching/Scaling:** $|\det(J)|$ gives the local scaling factor of the area due to the transformation.
- **Orientation:** The sign of $\det(J)$ indicates whether the orientation is preserved or flipped.

Example 1.1.1 (Polar Coordinates)

For the transformation $T(r, \theta) = (r\cos\theta, r\sin\theta)$, the Jacobian matrix is:

$$J = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

The determinant is:

$$\det(J) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

Thus, the area differential in polar coordinates becomes:

$$dx\,dy = r\,dr\,d\theta$$

Definition 1.1.2: General Formula for Transforming Integrals

If $T : S \to R$ is a transformation with Jacobian determinant $|\det(J)|$, then the integral transforms as:

$$\iint_R f(x, y)\,dx\,dy = \iint_S f(T(u, v))\,|\det(J)|\,du\,dv$$

Definition 1.1.3: Intuition for Higher Dimensions

In three dimensions, the Jacobian matrix extends to account for the transformation of volume elements:

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$$

The volume scaling factor is given by $|\det(J)|$, and the integral transforms as:

$$\iiint_R f(x, y, z)\,dx\,dy\,dz = \iiint_S f(T(u, v, w))\,|\det(J)|\,du\,dv\,dw$$

1.2 Non-overlapping from Mapping T

Theorem 1.2.1 Non-overlapping Condition

For any two points Q and P:

$$T(Q) \neq T(P) \quad \text{(This would result in overlapping areas in the domain } R\text{)}$$

However, boundaries (e.g., $y = 2x$) can overlap as long as the bounded region is distinct.

Example 1.2.1 (Integral Transformation Example)

Evaluate:

$$\iint_R 2x(y - 2x)\, dA$$

where R is the parallelogram with vertices $(0, 0), (0, 1), (2, 4), (2, 3)$.
Steps:

1. **Choose a Transformation:** Select a mapping T to simplify the integral.
2. **Define the Mapping:**
$$x = u, \quad y = 2x + v = 2u + v$$
Substituting:
$$(x, y) \longrightarrow (u, v)$$
3. **Boundary Equations:**
$$0 \leqslant x \leqslant 2 \quad \Rightarrow \quad 0 \leqslant u \leqslant 2$$
$$0 \leqslant y - 2x < 1 \quad \Rightarrow \quad 0 \leqslant v < 1$$
4. **Region:**
$$S = [0, 2] \times [0, 1]$$
5. **Transform the Integrand:**
$$f(T(x, y)) = 2u(v)$$
6. **Jacobian Calculation:**
$$J = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad \det(J) = 1 \cdot 1 - 2 \cdot 0 = 1$$
7. **Transformed Integral:**
$$\iint_R 2x(y - 2x)\, dA = \int_0^2 \int_0^1 2uv\, du\, dv$$

1.3 Integral Transformation for a Parallelogram Region

Example 1.3.1 (Example of Transformation)

Evaluate:

$$\iint_R 2x(y - 2x)\, dA$$

where R is the parallelogram defined by the vertices $(0, 0), (0, 1), (2, 4), (2, 3)$.

Steps:

1. **Choose a Transformation:** Select a transformation T that simplifies the integral.

2. **Define x, y in terms of u, v:**
$$x = u, \quad y = 2x + v = 2u + v$$
Substituting:
$$(x, y) \to (u, v)$$
Here, u corresponds to x, and $v = y - 2x$.

3. **Boundary Equations:**
$$0 \leqslant x \leqslant 2 \quad \Rightarrow \quad 0 \leqslant u \leqslant 2$$
$$0 \leqslant y - 2x < 1 \quad \Rightarrow \quad 0 \leqslant v < 1$$

4. **Region in u, v:**
$$S = [0, 2] \times [0, 1]$$
This maps the parallelogram R into a rectangle S in the u, v-plane.

5. **Transform the Integrand:** Substituting $x = u$ and $y - 2x = v$:
$$f(T(x, y)) = 2u(v)$$

6. **Jacobian Calculation:** The Jacobian matrix for the transformation T is:
$$J = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$
The determinant of J is:
$$\det(J) = 1 \cdot 1 - 2 \cdot 0 = 1$$

7. **Transformed Integral:** Using the transformation and the Jacobian determinant:
$$\iint_R 2x(y - 2x)\, dA = \int_0^2 \int_0^1 2uv\, du\, dv$$
The transformed integral simplifies the computation significantly.

1.4 Integral Transformation for a Triangular Region

Example 1.4.1 (Example of Transformation)
Evaluate:
$$\iint_R (x - u)\sqrt{x - 2y}\, dA$$
where R is the triangular region bounded by the lines $y = 0$, $x - 2y = 0$, and $x = y + 1$.

Steps:

1. **Region Definition:** The region R is defined by:
$$y = 0, \quad x - 2y = 0, \quad x = y + 1$$
The boundaries in x and y are:
$$0 \leqslant x \leqslant 2, \quad 0 \leqslant y \leqslant \frac{x}{2}, \quad x \leqslant y + 1$$

2. **Define Transformation:** Let:

$$u = x - 2y, \quad v = x - y$$

Substituting:

$$x = v + u, \quad y = v - u$$

3. **Boundaries in New Coordinates:** Using the transformation:

$$\begin{aligned} u = x - 2y \quad &\Rightarrow \quad 0 \leqslant u \leqslant 1 \\ v = x - y \quad &\Rightarrow \quad u \leqslant v \leqslant 1 \end{aligned}$$

The transformed region S is bounded by $u = 0$, $v = 1$, and $v - u = 1$.

4. **Jacobian Calculation:** The Jacobian matrix for the transformation $T(u, v)$ is:

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

The determinant of J is:

$$\det(J) = (1)(1) - (1)(-2) = 1 + 2 = 3$$

5. **Transform the Integral:** Using the transformation and Jacobian determinant:

$$\iint_R (x - u)\sqrt{x - 2y}\, dA = \int_0^1 \int_0^v \sqrt{u} \cdot 3\, du\, dv$$

Simplify:

$$\int_0^1 \int_0^v \sqrt{u}\, du\, dv = \int_0^1 \left[\frac{2}{3}u^{3/2}\right]_0^v dv = \int_0^1 \frac{2}{3}v^{3/2} dv$$

$$= \left[\frac{2}{3} \cdot \frac{2}{5}v^{5/2}\right]_0^1 = \frac{4}{15}.$$

The result is:

$$\iint_R (x - u)\sqrt{x - 2y}\, dA = \frac{4}{15}.$$