MA193
Discrete Mathematics

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Contents

Chapter 1

Fundamental Principles of Counting

1.1 Counting with Repetitions

Note:-

These notes cover the basic counting principles (often called the "Rule of Sum" and the "Rule of Product"), along with a brief discussion of permutations and combinations, including the formula for permutations of multiset objects (i.e., repeated elements).

Definition 1.1.1: Rule of Sum ("OR")

If a certain task can be done in n ways and another independent task can be done in m ways, and these tasks are mutually exclusive, then there are $n + m$ ways to do *one* of the two tasks.

Definition 1.1.2: Rule of Product ("AND")

If a procedure can be broken into two consecutive steps such that the first step can be done in n ways and the second step can be done in m ways (independently of how the first step is done), then there are $n \times m$ ways to do the entire procedure.

Note:-

In many counting problems, we break a larger procedure into a series of smaller steps and then apply either the Rule of Sum or the Rule of Product (or both) as needed.

Definition 1.1.3: Arrangements of n Distinct Objects

If you want to arrange n distinct objects in a row (i.e., an ordered list), there are $n!$ ways to do so. Order matters here, and this number is referred to as the number of permutations of n distinct items.

Definition 1.1.4: Permutations of Multisets

Suppose we have n total objects, but they are not all distinct. Instead, let there be n_1 objects of type 1, n_2 objects of type 2, …, and n_k objects of type k. Clearly

$$n_1 + n_2 + \cdots + n_k = n.$$

Then the number of distinct ways to arrange all n objects is

$$\frac{n!}{n_1!\, n_2! \ldots n_k!}.$$

Example 1.1.1 (Examples of Counting with Repetitions)

1. **ABCD:** All letters are distinct, so the number of ways to arrange A, B, C, D is
$$4! = 24.$$
2. **AABC:** Here we have 4 total letters, with A repeated twice. The number of distinct arrangements is
$$\frac{4!}{2!} = \frac{24}{2} = 12.$$
3. **AABB:** Now we have 2 A's and 2 B's (4 letters total). The number of arrangements is
$$\frac{4!}{2!\,2!} = \frac{24}{2 \times 2} = 6.$$
4. **SUCCESS:** The word "SUCCESS" has 7 letters total: 3 S's, 2 C's, 1 U, and 1 E. The number of distinct permutations is
$$\frac{7!}{3!\,2!\,1!\,1!} = \frac{5040}{(6)(2)(1)(1)} = 420.$$

1.2 Committee-Choosing Problems

Definition 1.2.1: Combinations

A *combination* is a way to choose r objects from n distinct objects where order does *not* matter. The number of ways to do so is given by
$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}.$$
In many committee-selection problems, we use combinations because the particular order in which people are chosen does not matter.

Definition 1.2.2: When to Use Permutations vs. Combinations

- *Permutations* ($P(n,r)$ or $\frac{n!}{(n-r)!}$) are used when we care about the *order* of the chosen items (for instance, arranging people in a line for a photo).
- *Combinations* ($\binom{n}{r}$) are used when we only care about *which* items are chosen, not the order in which they appear (e.g., forming committees).

Definition 1.2.3: Basic Subset Counting

Note that the total number of distinct subsets of a set with n elements is 2^n. This comes from the fact that, for each element, we independently choose to include it or not in a given subset. For an r-element subset, specifically, we use $\binom{n}{r}$.

Example 1.2.1 (Choosing a Simple Committee)

Suppose we have 8 people in a group, and we want to choose a committee of 3 individuals. Since order does not matter, the number of ways to choose the committee is
$$\binom{8}{3} = \frac{8!}{3!(8-3)!} = \frac{8!}{3!5!} = 56.$$

Example 1.2.2 (Committee with Restrictions: Gender Balance)

Imagine we have 5 men and 6 women, and we want to form a committee of 4 people that has at least 2 women. We can break it down by the number of women selected:

$$\binom{6}{2}\binom{5}{2} + \binom{6}{3}\binom{5}{1} + \binom{6}{4}\binom{5}{0}.$$

Here, $\binom{6}{k}$ chooses the women, and $\binom{5}{4-k}$ chooses the men for each scenario. Summing these counts gives the total number of ways to form such a committee.

Definition 1.2.4: Labeled Teams

When teams themselves have distinct "names" or labels (e.g. Team A vs. Team B). A configuration where Person 1 is on Team A and Person 2 is on Team B is then different from Person 1 on Team B and Person 2 on Team A.

Definition 1.2.5: Unlabeled Teams

When the teams are treated as identical (no distinct labels). In that scenario, swapping membership between "Team A" and "Team B" would not create a new configuration.

Example 1.2.3 (Committee with Subgroups Required)

Say we have 10 people, of whom 3 are Teaching Assistants (TAs) and 7 are Professors, and we wish to form a committee of 4 people *with exactly 2 TAs*. We select 2 from the 3 TAs and 2 from the 7 Professors, yielding

$$\binom{3}{2} \times \binom{7}{2}$$

possible committees.

Note:-

Labeled vs. Unlabeled Teams. "3 people, 2 teams with a label?" Here, we wonder if putting Alice on Team A and Bob on Team B differs from Alice on Team B and Bob on Team A. If teams are unlabeled, then swapping them would not create a different outcome. If teams *are* labeled, we count each distinct assignment as different.

Example 1.2.4 (Larger Committees: Multiple Constraints)

Suppose there are 12 people divided into three categories: 4 from group A, 5 from group B, and 3 from group C. We want a committee of 5 that has at least 1 person from each group. One way to count is to enumerate possible splits of 5 among (A, B, C), ensuring each category has at least one representative. For instance:

- 1 from A, 3 from B, 1 from C,
- 1 from A, 2 from B, 2 from C,
- 2 from A, 2 from B, 1 from C,
- and so forth,

and sum the corresponding products of binomial coefficients $\binom{4}{\dots}\binom{5}{\dots}\binom{3}{\dots}$.

Note:-

These examples illustrate the typical approaches to *committee choosing* problems:

- Identify if order matters (usually it does not, hence combinations).
- If there are constraints (e.g. minimum number of members from a certain group), split the problem into valid cases and sum their respective counts.

1.3 Gambling

Example 1.3.1 (Probability of Getting a Flush in Poker)

A *flush* in poker is a hand where all 5 cards are of the same suit. To calculate the probability of being dealt a flush, we proceed as follows:

- There are 4 suits in a deck (hearts, diamonds, clubs, spades).
- For each suit, there are $\binom{13}{5}$ ways to choose 5 cards from the 13 available.
- Thus, the total number of ways to get a flush is $4 \times \binom{13}{5}$.
- The total number of 5-card hands from a 52-card deck is $\binom{52}{5}$.

Therefore, the probability P of being dealt a flush is given by

$$P(\text{flush}) = \frac{\binom{4}{1} \times \binom{13}{5}}{\binom{52}{5}}.$$

So, the probability of being dealt a flush in poker is approximately 0.198%.

Example 1.3.2 (Example: 3 people into 2 teams)

Suppose we have 3 people (say Alice, Bob, and Carol). If the teams are *labeled*, we count every distinct assignment into Team A vs. Team B. If the teams are *unlabeled*, we only care about who ends up together, not which group is called "Team A." Hence the total counts differ, because labeling typically doubles the number of distinct ways (unless one team is empty).

Note:-

N

ext, consider two common 5-card poker hands:

Definition 1.3.1: One Pair

A 5-card hand containing exactly two cards of the same rank and the other three cards all of different ranks (and each different from the pair's rank).

Example 1.3.3 (Counting One Pair)

To choose exactly one pair out of a standard deck, we do:

$$\binom{13}{1}\binom{4}{2}$$

to pick which rank we have a pair of (13 ways) and which 2 suits out of the 4. For a complete 5-card hand with "exactly one pair," we then choose the other 3 cards of distinct ranks, leading to the well-known formula $\binom{13}{1}\binom{4}{2} \times \binom{12}{3}$ (each chosen rank has $\binom{4}{1}$ ways).

Definition 1.3.2: Full House

A 5-card hand consisting of 3 cards of one rank plus 2 cards of another rank.

Example 1.3.4 (Counting a Full House)

We choose the rank for the three-of-a-kind in $\binom{13}{1}$ ways, and then choose 3 suits out of 4 for that rank: $\binom{4}{3}$. Next, we choose the rank for the pair out of the remaining 12 ranks: $\binom{12}{1}$, and choose 2 suits out of 4 for that pair: $\binom{4}{2}$. Hence,

$$\text{Number of Full Houses} = \binom{13}{1}\binom{4}{3} \times \binom{12}{1}\binom{4}{2}.$$

1.4 Binomial identities

Definition 1.4.1: Binomial Symmetry

$$\binom{n}{k} = \binom{n}{n-k}.$$

Note:-

The intuition is that choosing k objects out of n is equivalent to choosing which $n-k$ to *leave out*. Hence $\binom{n}{k}$ equals $\binom{n}{n-k}$.

Example 1.4.1 (Example)

For instance, $\binom{5}{2} = \binom{5}{3}$. Choosing 2 items from 5 is the same as deciding which 3 are excluded.

Definition 1.4.2: Pascal's Rule

$$\binom{n}{v} = \binom{n-1}{v-1} + \binom{n-1}{v}.$$

Note:-

A combinatorial interpretation is to imagine a set of n items where one particular "special" item can either be in your chosen subset or not. If you include the special item, then you need to pick the remaining $v-1$ items from the other $n-1$. If you exclude the special item, then you pick all v items from the other $n-1$. Summing these counts gives $\binom{n}{v}$.

Example 1.4.2 (Example)

$\binom{5}{3} = \binom{4}{2} + \binom{4}{3}$. Either include a special item (then choose 2 more from the other 4) or exclude it (then choose all 3 from the other 4).

Definition 1.4.3: Hockey-Stick Identity

$$\sum_{j=r}^{n} \binom{j}{r} = \binom{n+1}{r+1}.$$

Note:-

This identity is sometimes called the "Christmas Stocking" or "Hockey-Stick" identity because of the pattern it creates in Pascal's triangle. It can be viewed as a cumulative version of Pascal's Rule: once you fix $\binom{n+1}{r+1}$, you can "unroll" it into the sum $\binom{r}{r} + \binom{r+1}{r} + \cdots + \binom{n}{r}$.

Example 1.4.3 (Example)

$\binom{3}{3} + \binom{4}{3} + \binom{5}{3} = \binom{6}{4}$. In general, if you sum along a diagonal in Pascal's triangle, you end up at a binomial coefficient one row down and one column over.

Note:-

A useful application of these identities is to evaluate sums of products like $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + (u-2)(u-1)u$. Note that

$$k(k+1)(k+2) = 3!\binom{k+2}{3},$$

so

$$\sum_{k=1}^{u-2} k(k+1)(k+2) = 3!\sum_{k=1}^{u-2}\binom{k+2}{3} = 3!\binom{u+1}{4}.$$

Example 1.4.4 (Binomial Expansion)

$(x+y)^n = \displaystyle\sum_{k=0}^{n}\binom{n}{k} x^{n-k}\, y^k$. For instance, $(2x+3y)^9$ expands as

$$\sum_{k=0}^{9}\binom{9}{k}(2x)^{9-k}(3y)^k.$$

Note:-

Recall the binomial expansion:

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n.$$

If we substitute $x = -2$, we obtain

$$\sum_{k=0}^{n}\binom{n}{k}(-2)^k.$$

This is simply an application of the binomial theorem with a negative value for x.

Definition 1.4.4: Sum of Squares

A well-known formula for the sum of the first n squares is

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

One way to derive or interpret this is via Riemann sums, though it can also be shown using induction or other combinatorial arguments.

Note:-

We can also consider the derivative approach related to $(1+x)^n$. Note that

$$\frac{d}{dx}(1+x)^n = n(1+x)^{n-1}.$$

Expanding $(1+x)^n$ term-by-term:

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n,$$

so taking a derivative and then comparing coefficients often provides a way to form identities involving sums of binomial coefficients.

Definition 1.4.5: Multinomial Coefficients

For nonnegative integers n and nonnegative integers $k_1, k_2, \ldots, k_m$ such that $k_1 + k_2 + \cdots + k_m = n$, the multinomial coefficient is defined as

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1!\, k_2! \cdots k_m!}.$$

This appears when expanding expressions of the form

$$(x_1 + x_2 + \cdots + x_m)^n.$$

Example 1.4.5 (Expansion of $(a+b+c)^4$)

Consider the expansion

$$(a+b+c)^4.$$

If we want, for example, the coefficient of $b^2\,c^2$ in this expansion, we look at

$$\binom{4}{0,2,2} = \frac{4!}{0!\,2!\,2!} = 6.$$

So the term corresponding to b^2c^2 in $(a+b+c)^4$ is $6\,b^2c^2$.

Example 1.4.6 (Expansion of $(a+2b+3c-2d+5)^{16}$)

As a more general example, for the expansion of

$$(a+2b+3c-2d+5)^{16},$$

one can seek a specific term's coefficient, say the coefficient of $a^2b^3c^1d^5$. One would use the multinomial theorem:

$$\binom{16}{2,3,1,5,\ 16-(2+3+1+5)}(a)^2(2b)^3(3c)^1(-2d)^5(5)^{\,16-(2+3+1+5)}.$$

Every factor (such as $2b$, $3c$, or $-2d$) contributes appropriately to the power and to the overall coefficient.

Note:-

A classic combinatorial problem is *dealing cards in a Bridge game*. A standard deck has 52 cards, dealt evenly to 4 players (each player receives 13 cards). For instance:

$$\binom{52}{13,13,13,13}$$

is the total number of ways to deal the entire deck to 4 players.

Example 1.4.7 (Probability that two specific players get all the spades)

There are 13 spades in the deck. If we ask for the probability that two given players split all 13 spades between them (i.e., they collectively get all spades), we count how the 13 spades can be distributed to those two players, and how the remaining 39 cards are distributed among all four players. Such problems are typically approached using multinomial coefficients or combinations, depending on how strictly the spades must be divided.

Note:-

Another recurring theme is *distributing indistinguishable objects* into distinct boxes. For example, distributing 20 donuts (indistinguishable) among r different people or "flavors." The number of ways to do this (allowing any number of donuts per person) is given by:

$$\binom{20 + r - 1}{r - 1}$$

in the classical "stars and bars" argument. More generally, the number of nonnegative integer solutions to

$$A + B + \cdots + (\text{n variables}) = N$$

is

$$\binom{N + \text{n} - 1}{\text{n} - 1}.$$

Note:-

Placing Distinct vs. Indistinguishable Objects into Containers

- *Distinct objects into distinct containers:* If there are r distinct objects and n distinct containers, each object can go into any of the n containers. Hence there are

$$n^r$$

ways to place them.

- *Distinct objects into identical containers:* This is related to set partitions. We ask: in how many ways can a set of r distinct elements be partitioned into (up to) n unlabeled subsets? Such counting typically involves the Stirling numbers of the second kind, denoted $S(r, n)$.
- *Indistinguishable objects into distinct containers:* This is a "stars and bars" problem. If r identical objects are to be placed in n distinct containers, with no restriction on how many objects go in each container (i.e. allowing zero), the number of ways is

$$\binom{r + n - 1}{n - 1}.$$

- *Indistinguishable objects into identical containers:* This typically counts the number of partitions of an integer r into at most n parts. Equivalently, it's the number of ways to write

$$r = x_1 + x_2 + \cdots + x_n$$

where $x_i \geq 0$ and the order of parts does not matter. These numbers connect to the theory of integer partitions, which can be considerably more involved.

Example 1.4.8 (Two Distinct Objects in Three Identical Containers)

We have two distinct objects (call them A and B) and three identical "boxes" (unlabeled). One way to count the distributions is to look at all possible groupings:

- Both A and B in the same container (there are 3 ways if the containers were distinct, but only 1 way when the containers are identical).

- A and B in different containers (again, if containers were distinct, we'd count more, but with identical containers we see that splitting them up is effectively one arrangement).

Hence there are 2 possible ways:

$$\{A, B\}, \quad \{A\}, \{B\}.$$

Example 1.4.9 (Distribution of Toys and Candy)

Suppose we have 7 identical toys and 8 identical candy bars, and 4 kids. Each kid must receive exactly 1 toy (so that accounts for 4 toys). Then 3 toys remain to be distributed freely among the 4 kids. The number of ways to distribute these 3 identical toys into 4 distinct kids is

$$\binom{3+4-1}{4-1} = \binom{6}{3} = 20.$$

Similarly, all 8 candies are to be distributed among 4 kids. If there is no restriction (i.e. any kid can get any number of candies), the count is

$$\binom{8+4-1}{4-1} = \binom{11}{3}.$$

The overall number of ways to distribute both sets of items is then the product of these two binomial coefficients (because the toy distribution and candy distribution choices are independent).

Definition 1.4.6: Compositions of an Integer

A *composition* of a positive integer n is a way of writing n as an ordered sum of positive integers. For instance, the integer 4 has the following compositions:

$$4, \quad 3+1, \quad 1+3, \quad 2+2, \quad 2+1+1, \quad 1+2+1, \quad 1+1+2, \quad 1+1+1+1.$$

If one writes n as $x_1 + x_2 + \cdots + x_k$ with each $x_i \geq 1$, the order matters for compositions. The total number of compositions of n is 2^{n-1}.

- If we allow zeros as parts, i.e. $x_i \geq 0$, we get related formulas often solved by "stars and bars."
- Partitions, on the other hand, disregard order; they look at the sum of positive parts in a non-increasing arrangement, etc.

Example 1.4.10 (Number of 15-coin-flip sequences with exactly 7 runs)

We want all sequences of 15 coin flips (H or T) that produce exactly 7 runs. Suppose there are 5 heads (H) and 10 tails (T) in total. Label the runs by their lengths $x_1, x_2, \ldots, x_7$, each $x_i \geq 1$.

1. *Case 1: First flip is H.* Then the runs alternate

$$H,\ T,\ H,\ T,\ H,\ T,\ H,$$

so there are 4 head-runs and 3 tail-runs. Hence

$$x_1 + x_3 + x_5 + x_7 = 5 \quad \text{and} \quad x_2 + x_4 + x_6 = 10,$$

with each $x_i \geq 1$. The number of integer solutions is

$$\binom{5-1}{4-1} \times \binom{10-1}{3-1} = \binom{4}{3} \times \binom{9}{2} = 4 \times 36 = 144.$$

2. *Case 2: First flip is T.* Then the runs alternate

$$T,\ H,\ T,\ H,\ T,\ H,\ T,$$

so there are 4 tail-runs and 3 head-runs. Now

$$x_1 + x_3 + x_5 + x_7 = 10 \quad \text{and} \quad x_2 + x_4 + x_6 = 5,$$

again with each $x_i \geq 1$. The number of solutions is

$$\binom{10-1}{4-1} \times \binom{5-1}{3-1} = \binom{9}{3} \times \binom{4}{2} = 84 \times 6 = 504.$$

Adding both cases, the total number of 15-flip sequences with exactly 7 runs is

$$144 + 504 = 648.$$

Chapter 2

Set Theory

$$B \subset A \quad \longrightarrow \quad \text{All elements of } B \text{ are in } A, \text{ but } B \neq A.$$

$$B \subseteq A \quad \longrightarrow \quad \text{All elements of } B \text{ are in } A, \text{ and } B \text{ may be equal to } A.$$

$$A \cup B \quad \longrightarrow \quad \text{All elements of } A \text{ or } B \text{ or both.}$$

$$A \cap B \quad \longrightarrow \quad \text{All elements of } A \text{ and of } B.$$

$$A \setminus B \quad \longrightarrow \quad \text{All elements of } A \text{ not in } B.$$

$$A \Delta B \quad \longrightarrow \quad \text{All elements of } A \text{ or of } B, \text{ but not both.}$$

2.1 Relations - Cartesian Cross Product of 2 Sets

2.1.1 Definitions and Notations

Definition 2.1.1: Cartesian Product

Given two sets A and B, the Cartesian product, denoted by $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$. Formally:

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

Note:-

The cardinality of the Cartesian product of two sets A and B is given by the product of the cardinalities of the individual sets:

$$|A \times B| = |A| \cdot |B|$$

2.1.2 Examples

Example 2.1.1 (Example 1: Cartesian Product)

Let $A = \{2, 3, 4\}$ and $B = \{g, s\}$. Find $A \times B$.

Solution:

$$A \times B = \{(2, g), (2, s), (3, g), (3, s), (4, g), (4, s)\}$$

Example 2.1.2 (Example 2: Cartesian Product with Itself)
Let $B = \{g, s\}$. Find $B^2 = B \times B$.

Solution:

$$B^2 = B \times B = \{(g, g), (g, s), (s, g), (s, s)\}$$

2.1.3 Binary Relations

Definition 2.1.2: Binary Relation

A binary relation from set A to set B is any subset of the Cartesian product $A \times B$.

Note:-

Any subset of $A \times B$ is a binary relation from A to B or on A if the relation is a subset of $A \times A$.

Example 2.1.3 (Example of Relations)
Consider the sets $A = \{2, 3, 4\}$ and $B = \{g, s\}$. Since $|A \times B| = 6$, there are 2^6 possible relations.

Note:-

In the previous example, since $|A \times B| = 6$, there are 2^6 possible relations.

2.2 Further Elaboration

2.2.1 Detailed Explanation of Cartesian Product

The Cartesian Product is a fundamental concept in set theory that allows us to combine elements of two sets in an ordered manner.

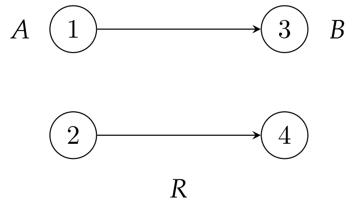
- **Ordered Pairs:** The key feature of the Cartesian product is that it forms ordered pairs. This means the pair (a, b) is different from the pair (b, a) unless $a = b$.
- **Cardinality:** The number of elements in the Cartesian product $A \times B$ is the product of the number of elements in set A and the number of elements in set B. This shows how the Cartesian product scales with the size of the input sets.

2.2.2 Understanding Binary Relations

Binary relations are subsets of the Cartesian product and form the basis for many mathematical structures and concepts.

- **Subsets:** Any subset of $A \times B$ qualifies as a binary relation. This includes the empty set, the full set $A \times B$, and any combination of ordered pairs in between.
- **Representations:** Binary relations can be represented in various ways, including:
  - **Set Notation:** As a set of ordered pairs.
  - **Graphical Representation:** Using directed graphs (digraphs) where elements of the sets are nodes, and ordered pairs are directed edges.

Example 2.2.1 (Graphical Representation of a Relation)
Consider sets $A = \{1, 2\}$ and $B = \{3, 4\}$. Let R be a relation from A to B defined by $R = \{(1, 3), (2, 4)\}$.



Solution: The diagram shows a graphical representation of the relation R. The nodes on the left represent elements of set A, and the nodes on the right represent elements of set B. The directed edges (arrows) represent the ordered pairs in the relation R.

2.2.3 Number of Possible Relations

For finite sets A and B, the number of possible binary relations from A to B is determined by the number of subsets of $A \times B$.

Note:-

If $|A| = m$ and $|B| = n$, then $|A \times B| = m \cdot n$. The number of possible relations is the number of subsets of $A \times B$, which is $2^{|A \times B|} = 2^{m \cdot n}$.

Example 2.2.2 (Counting Relations)

If $A = \{1, 2\}$ and $B = \{x, y, z\}$, how many possible relations are there from A to B?

Solution: $|A| = 2$ and $|B| = 3$, so $|A \times B| = 2 \times 3 = 6$. The number of possible relations is $2^6 = 64$.

2.2.4 Types of Relations

- **Reflexive Relation:** A relation R on a set A is reflexive if $(a, a) \in R$ for all $a \in A$.
- **Symmetric Relation:** A relation R on a set A is symmetric if $(a, b) \in R$ implies $(b, a) \in R$ for all $a, b \in A$.
- **Transitive Relation:** A relation R on a set A is transitive if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in A$.
- **Equivalence Relation** A relation that is reflexive, symmetric and transitive.

Properties That Relations Can Satisfy

Reflexive Relations

Definition 2.2.1: Reflexive Relation

A relation R on a set A is called **reflexive** if for all $x \in A$, xRx (or $(x, x) \in R$). In other words, every element in the set is related to itself.

Example 2.2.3 (Example of Reflexive Relations)

Let $A = \{1, 2, 3\}$.

- $R_1 = \{(1, 1), (2, 2), (3, 3)\}$ is reflexive.
- $R_2 = \{(1, 1), (2, 2), (3, 3), (2, 3)\}$ is also reflexive.

Note:-

The key characteristic of a reflexive relation is that every element of the set *must* be related to itself. The presence of other relations (like (2,3) in R_2) doesn't affect the reflexivity.

Question 1: How many possible reflexive relations are there for a set A if $|A| = n$?

Solution:

- If $|A| = n$, then $|A \times A| = n^2$. This represents all possible ordered pairs in the Cartesian product of A with itself.
- All possible relations on A is 2^{n^2}, since a relation is a subset of $A \times A$, and the number of subsets of a set with n^2 elements is 2^{n^2}.
- If a relation R on A is reflexive, then for all $a_i \in A$, (a_i, a_i) must be in R, where $1 \leq i \leq n$.
- We must consider the rest of the pairs, which are $n^2 - n$.
- Therefore, the number of reflexive relations is $2^{(n^2-n)}$. This is because the n pairs of the form (a_i, a_i) are *required* for reflexivity, leaving $n^2 - n$ other pairs that may or may not be present.

Symmetric Relations

Definition 2.2.2: Symmetric Relation

A relation R on a set A is **symmetric** if for all $x, y \in A$, if $(x, y) \in R$, then $(y, x) \in R$. In simpler terms, if x is related to y, then y must also be related to x.

Example 2.2.4 (Examples of Symmetric Relations)
Let $A = \{1, 2, 3\}$.

- $R_1 = \{(1, 2), (2, 1), (3, 1), (1, 3)\}$ is symmetric, but not reflexive.
- $R_2 = \{(1, 1), (2, 2), (3, 3)\}$ is reflexive and symmetric.

Question 2: How many symmetric relations are there?

Solution:

- For pairs of the form (a_i, a_i), we can either include or exclude them. There are n such pairs, leading to 2^n possibilities.
- For the remaining $n^2 - n$ elements, they must come in pairs: (a_i, a_j) and (a_j, a_i).
- There are $\frac{1}{2}(n^2 - n)$ such pairs.
- For each of these $\frac{1}{2}(n^2 - n)$ pairs, we can either include both (a_i, a_j) and (a_j, a_i) or exclude both.
- Thus, there are $2^{\frac{1}{2}(n^2-n)}$ choices for these pairs.
- The total number of symmetric relations is $2^n \cdot 2^{\frac{1}{2}(n^2-n)} = 2^{n+\frac{1}{2}(n^2-n)} = 2^{\frac{1}{2}(n^2+n)}$.

Antisymmetric Relations

Definition 2.2.3: Antisymmetric Relation

A relation R on a set A is **antisymmetric** if for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$. This means that the only way for both (a, b) and (b, a) to be in the relation is if a and b are the same element.

Example 2.2.5 (Examples of Antisymmetric Relations)
Let $A = \{1, 2, 3\}$.

- $R_1 = \{(1, 2), (2, 1), (1, 1)\}$ is not symmetric and not antisymmetric.
- $R_2 = \{(1, 2), (1, 3)\}$ is antisymmetric.
- $R_3 = \{(1, 1), (2, 2)\}$ is symmetric and antisymmetric.

Question 3: How many antisymmetric relations are there?

Solution:

- For pairs (a_i, a_i), it doesn't matter if we include them or not. There are n such pairs, so 2^n possibilities.
- For pairs (a_i, a_j) where $i \neq j$, if we include one pair, we cannot include the other.
- There are $\frac{1}{2}(n^2 - n)$ such pairs of pairs.
- For each pair of pairs, there are three options:
  1. Place $(a_i, a_j) \in R$.
  2. Place $(a_j, a_i) \in R$.
  3. Place neither.
- Thus, there are $3^{\frac{1}{2}(n^2-n)}$ options for these pairs.
- The total number of antisymmetric relations will be $2^n \cdot 3^{\frac{1}{2}(n^2-n)}$.

Note:-

Note	
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T $\longrightarrow$ F	: F
F $\longrightarrow$ T	: T
F $\longrightarrow$ F	: T

Relations and Divisibility

Defining a Relation

Example 2.2.6 (Example: Divisors of 12)
Let $A = \{1, 2, 3, 4, 6, 12\}$. This set contains all the positive divisors of 12.

Definition 2.2.4: Definition: Relation on A

Define a relation R on A as $(x, y) \in R$ if and only if $x|y$, which means "x exactly divides y".

Note:-
The notation $(x, y) \in R$ is often written as xRy. The relation R in this context represents the divisibility relation.

Properties of the Relation

> **Note:-**
> We will check if the relation R is reflexive, antisymmetric, and transitive.

- **Reflexive:** For all $x \in A$, $x|x$, so $(x, x) \in R$. Thus, R is reflexive.
- **Antisymmetric:** If $x|y$ and $y|x$, then $x = y$. So, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$. Thus, R is antisymmetric.
- **Transitive:** If $x|y$ and $y|z$, then $x|z$. So, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$. Thus, R is transitive.

Example 2.2.7 (Examples of Pairs in R)

Some pairs in R include $(2, 4)$, $(4, 12)$, and consequently, $(2, 12)$ due to transitivity.

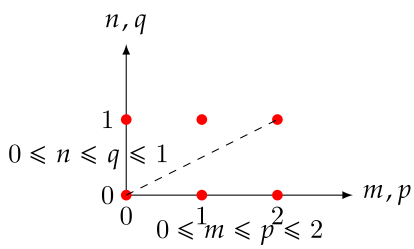
Counting Ordered Pairs

Question 4: Question: Number of Ordered Pairs

How many ordered pairs occur in R?

Solution: We can use prime factorization to determine the number of ordered pairs. Given $12 = 2^2 \cdot 3^1$, if $(c, d) \in R$, then $c = 2^m 3^n$ and $d = 2^p 3^q$ where:

- $0 \leqslant m \leqslant p \leqslant 2$
- $0 \leqslant n \leqslant q \leqslant 1$



For m and p, we can pick from $\{0, 1, 2\}$. The number of ways to choose m and p such that $0 \leqslant m \leqslant p \leqslant 2$ is given by $\binom{3-1+2}{2} = \binom{4}{2} = 6$. For n and q, we pick from $\{0, 1\}$. The number of ways to choose n and q such that $0 \leqslant n \leqslant q \leqslant 1$ is given by $\binom{2-1+2}{2} = \binom{3}{2} = 3$. Total number of ordered pairs $= 6 \times 3 = 18$.

Generalization for 1800

Question 5: Question: Generalization

Consider the number $1800 = 2^3 \cdot 3^2 \cdot 5^2$. How many ordered pairs are there such that one divides the other?

Solution: Let $1800 = 2^3 \cdot 3^2 \cdot 5^2$. If $(c, d) \in R$ such that $c|d$ and both divide 1800, we have $c = 2^r 3^s 5^t$ and $d = 2^u 3^v 5^w$. Then

- $0 \leqslant r \leqslant u \leqslant 3$
- $0 \leqslant s \leqslant v \leqslant 2$
- $0 \leqslant t \leqslant w \leqslant 2$

The number of ways to choose r and u is $\binom{4-1+2}{2} = \binom{5}{2} = 10$. The number of ways to choose s and v is $\binom{3-1+2}{2} = \binom{4}{2} = 6$. The number of ways to choose t and w is $\binom{3-1+2}{2} = \binom{4}{2} = 6$. Thus, the total number of such ordered pairs is $10 \times 6 \times 6 = 360$.