CASMA 225
Calc 3

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Chapter 1

Vector Valued Functions $f : \mathbb{R} \longrightarrow \mathbb{R}^n$

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1.1.1 Basics

Definition 1.1.1: Notation

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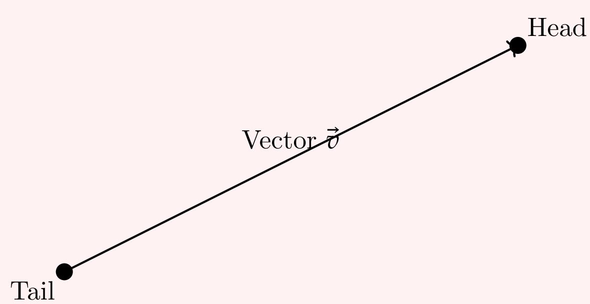
Typed Vectors: $\mathbf{v}$

Definition 1.1.2: Velocity

Magnitude of the velocity: $|\vec{v}|$
Direction of the velocity: $dir(\vec{v})$

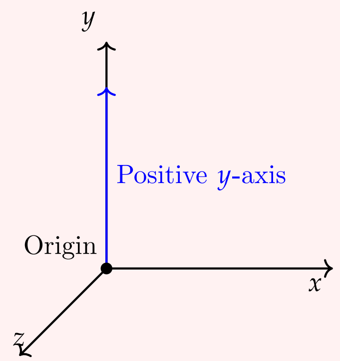
Definition 1.1.3: Heads and Tails



Note:-
Scalar is like a 1 directional vector, either positive or negative, and its magnitude is the absolute value of the scalar

1.1.2 Notation

Definition 1.1.4: Positive y axis



Definition 1.1.5: Standard Basis Vecotrs

In an n-dimensional space $\mathbb{R}^n$, the standard basis vectors are a set of n vectors where each vector has a 1 in one component and 0 in all other components. These vectors are denoted as $\mathbf{e}_i$ for $i = 1, 2, \dots, n$. The i-th standard basis vector in $\mathbb{R}^n$ is written as:

$$\mathbf{e}_i = \begin{pmatrix} & 0 \\ & 0 \\ & \vdots \\ 1 & \text{(in the } i\text{-th position)} \\ & \vdots \\ & 0 \end{pmatrix}$$

For example, in $\mathbb{R}^3$ (three-dimensional space), the standard basis vectors are:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

These vectors span the entire vector space $\mathbb{R}^n$, meaning any vector $\mathbf{v} \in \mathbb{R}^n$ can be written as a linear combination of the standard basis vectors:

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n,$$

where $v_1, v_2, \dots, v_n$ are the components of the vector $\mathbf{v}$.

1.2 Operations

1.2.1 Dot Product

Definition 1.2.1: Dot (Scalar)Product Definitons

The **scalar product** (or **dot product**) of two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$ is defined as:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

In $\mathbb{R}^3$, for vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, the dot product is:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

The dot product can also be expressed in terms of the magnitudes of $\mathbf{a}$ and $\mathbf{b}$ and the angle θ between them:

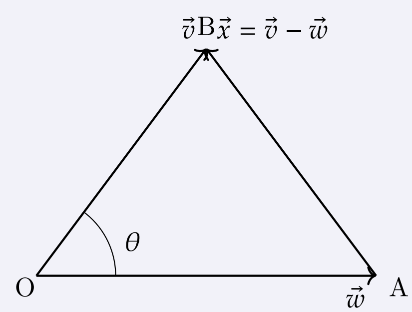
$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

The dot product is a scalar quantity and is zero when the vectors are orthogonal (perpendicular).
Useful to find the angle between the two vectors being dot produced together,

Theorem 1.2.1 Dot Product Proof

We are given the vectors $\vec{v}$ and $\vec{w}$, and we want to express the dot product in terms of their magnitudes and the angle between them.
Start with the relationship:

$$\vec{x} = \vec{v} - \vec{w}$$



The above diagram illustrates the vectors $\vec{v}$, $\vec{w}$, and their difference $\vec{x} = \vec{v} - \vec{w}$, forming a triangle. The angle θ is between $\vec{v}$ and $\vec{w}$.
The magnitude squared of $\vec{x}$ is:

$$|\vec{x}|^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}||\vec{w}| \cos\theta$$

This is the expansion of the law of cosines.
Now, from the equation:

$$|\vec{x}|^2 = \sqrt[2]{\left((v_x - w_x)^2 + (v_y - w_y)^2\right)}^2$$

We conclude:

$$|\vec{x}|^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2(\vec{v} \cdot \vec{w})$$

Thus, we can express the dot product $\vec{v} \cdot \vec{w}$ as:

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta$$

1.2.2 Applications

Note:-

The dot product of two vectors $\vec{v} \cdot \vec{w}$ can take different values, leading to various interpretations of the relationship between the vectors. Below is a table describing some key cases:

Dot Product Value	Interpretation	Relationship Between Vectors
$\vec{v} \cdot \vec{w} = 0$	$\cos\theta = 0$	Vectors are **perpendicular** (orthogonal), $\theta = 90°$
$\vec{v} \cdot \vec{w} > 0$	$0 < \theta < 90°$	Vectors form an **acute angle**, pointing in the same general direction
$\vec{v} \cdot \vec{w} < 0$	$90° < \theta < 180°$	Vectors form an **obtuse angle**, pointing in opposite general directions
$\vec{v} \cdot \vec{w} = \lvert\vec{v}\rvert\lvert\vec{w}\rvert$	$\cos\theta = 1$	Vectors are **parallel** and point in the **same direction**, $\theta = 0°$
$\vec{v} \cdot \vec{w} = -\lvert\vec{v}\rvert\lvert\vec{w}\rvert$	$\cos\theta = -1$	Vectors are **parallel** but point in **opposite directions**, $\theta = 180°$

Definition 1.2.2: Vector Product (Cross Product)

The **vector product** (or **cross product**) of two vectors **a** and **b** in $\mathbb{R}^3$ is a vector **c** that is perpendicular to both **a** and **b**, and its magnitude is given by:

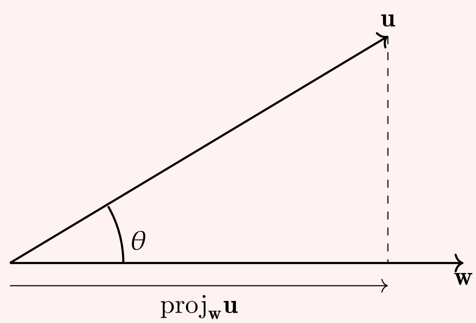
$$|\mathbf{c}| = |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$$

where θ is the angle between **a** and **b**. The cross product is calculated as:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

The result of a cross product is a vector perpendicular to the plane formed by **a** and **b**, with a direction given by the right-hand rule.

Definition 1.2.3: Vector Projections



$$\text{scal}_{\mathbf{w}}\mathbf{u} = |\mathbf{u}| \cdot \cos\theta = \frac{\mathbf{w} \cdot \mathbf{u}}{|\mathbf{w}|}$$

$$\text{proj}_{\mathbf{w}}\mathbf{u} = |\mathbf{u}| \cos\theta \left(\frac{\mathbf{w}}{|\mathbf{w}|}\right)$$

$$\text{proj}_{\mathbf{w}}\mathbf{u} = \left(\frac{\mathbf{w} \cdot \mathbf{u}}{\mathbf{w} \cdot \mathbf{w}}\right) \mathbf{w}$$

1.3 Matrix Determinants

Definition 1.3.1: Matrix Representation

A matrix is a collection of numbers arranged in a grid format, where each element is positioned based on its row and column. A general $m \times n$ matrix is written as:

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

For example, a 2×2 matrix is given by:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

A 3×3 matrix is:

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Matrices can be considered as a collection of vectors where each row or column can represent a vector.

Note:-

Vector Representation
A matrix can also be viewed as a collection of vectors. For instance, a 3×3 matrix can be interpreted as:

$$M = \begin{pmatrix} \vec{v_1} = \langle a, b, c \rangle \\ \vec{v_2} = \langle d, e, f \rangle \\ \vec{v_3} = \langle g, h, i \rangle \end{pmatrix}$$

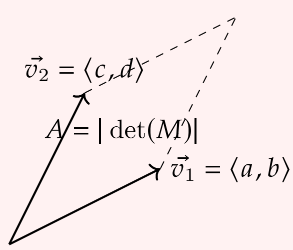
where each row (or column) is treated as a vector in space.

Definition 1.3.2: Determinant of a 2×2 Matrix

The determinant of a 2×2 matrix is given by:

$$\det(M) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

The determinant represents the signed area of the parallelogram formed by the vectors corresponding to the rows (or columns) of the matrix.



Note:-

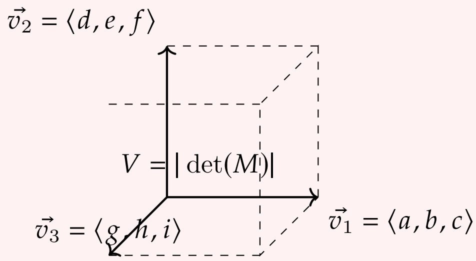
Geometric Interpretation
For a 2×2 matrix, the determinant represents the area A of the parallelogram formed by the two vectors $\vec{v_1} = \langle a, b \rangle$ and $\vec{v_2} = \langle c, d \rangle$. The magnitude of the determinant gives the area of this parallelogram, and the sign of the determinant indicates the orientation (whether the vectors are ordered clockwise or counterclockwise).

Definition 1.3.3: Determinant of a 3×3 Matrix

The determinant of a 3×3 matrix is calculated as:

$$\det(M) = \det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

The determinant represents the signed volume of the parallelepiped formed by the three vectors corresponding to the rows (or columns) of the matrix.



Note:-

Geometric Interpretation for 3×3
In the 3×3 case, the determinant represents the volume V of the parallelepiped formed by three vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$, and the sign indicates whether the orientation is right-handed or left-handed. The magnitude gives the volume.

1.4 Matrix multiplication with 2D Vectors

Definition 1.4.1: Vector Matrix Multiplication

$$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

$$\hat{\mathbf{j}}M = \langle a_{11}V_1 + a_{12}V_2, a_{21}V_1 + a_{22}V_2 \rangle$$

Given:

$$\hat{i} = \langle 1, 0 \rangle \quad \hat{j} = \langle 0, 1 \rangle$$

We can compute:

$$iM = \langle a_{11}, a_{12} \rangle = a_1$$

$$jM = \langle a_{21}, a_{22} \rangle = a_2$$

Where:

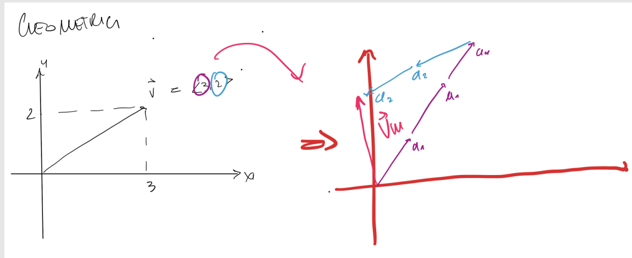
$$\mathbf{V} = V_1\hat{i} + V_2\hat{j}$$

$$\hat{\mathbf{V}}M = \left(V_1\hat{i} + V_2\hat{j}\right)M$$

$$= V_1\hat{i}M + V_2\hat{j}M$$

$$= V_1\mathbf{a}_1 + V_2\mathbf{a}_2$$

Note:-



1.4.1 Effect on Area

Definition 1.4.2: *2D*

The original point $(1, 1)$ is transformed by the matrix M. This transformation impacts the area and orientation as follows:

$$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

The area after transformation is given by the determinant of the matrix:

$$\text{Area} = \det(M)$$

Where the determinant is calculated as:

$$\det(M) = a_{11}a_{22} - a_{12}a_{21}$$

The determinant also determines the orientation:

$$\det(M) = \begin{cases} A & \text{if } a_1 \text{ to } a_2 \text{ is counterclockwise} \\ -A & \text{otherwise} \end{cases}$$

In the example, the original vectors a_1 and a_2 form an area, and the determinant will tell us if the vectors are oriented in a clockwise or counterclockwise fashion.
If the determinant is negative, the orientation is clockwise, as illustrated:

$$\det\left(\begin{pmatrix} a_1 & a_2 \end{pmatrix}\right) < 0$$

Thus, in this case, the transformation results in a clockwise orientation.

1.5 Matrix multiplication with 3D Vectors

Definition 1.5.1: *3D*

The matrix M for a 3D transformation is given as:

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \text{where} \quad \vec{V} = \langle V_1, V_2, V_3 \rangle$$

The transformation of vector $\vec{V}$ under matrix M is:

$$\hat{\vec{V}} M = \langle (a_{11}V_1 + a_{12}V_2 + a_{13}V_3), (a_{21}V_1 + a_{22}V_2 + a_{23}V_3), (a_{31}V_1 + a_{32}V_2 + a_{33}V_3) \rangle$$

This can be written in terms of the basis vectors as:

$$\left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k} \right) M = V_1\vec{a}_1 + V_2\vec{a}_2 + V_3\vec{a}_3$$

Definition 1.5.2: orientation and Volume

- If the determinant of matrix M is negative, the system is **left-handed**, i.e.,

$$\det(M) = -V$$

- The determinant of the matrix M gives the **volume** of the parallelepiped spanned by the vectors a_1, a_2, a_3:

$$\det(M) = \text{Volume}(V)$$

The volume V is given by:

$$V = \begin{cases} +V & \text{if } \vec{a}_1, \vec{a}_2, \vec{a}_3 \text{ are right-handed (RHS)} \\ -V & \text{otherwise (left-handed)} \end{cases}$$

1.6 Cross Product and Volumes

Definition 1.6.1: Cross Product and Volumes

The volume of a parallelepiped defined by three vectors $\vec{u}, \vec{v}, \vec{w}$ is given by:

$$\mathrm{V} = \vec{u} \cdot (\vec{v} \times \vec{w})$$

1.6.1 Link to Matrix Determinants

Definition 1.6.2: Cross Product and Matrix Determinants

Since:

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \det \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix}$$

$$\det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = u_1 \det \begin{pmatrix} v_2 & v_3 \\ w_2 & w_3 \end{pmatrix} - u_2 \det \begin{pmatrix} v_1 & v_3 \\ w_1 & w_3 \end{pmatrix} + u_3 \det \begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix}$$

$$= \vec{u} \cdot \left(\hat{i} \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - \hat{j} \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + \hat{k} \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right)$$

$$= \vec{u} \cdot \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$$

$$= \vec{u} \cdot (\vec{v} \times \vec{w})$$

Therefore:

$$\vec{v} \times \vec{w} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$$

1.7 Cross Product Polynomial Multiplication

Definition 1.7.1: Properties

$$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}$$

$$\hat{j} \times \hat{i} = -\hat{k}, \quad \hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

Example 1.7.1 (Example: Cross Product)

Let $\vec{v} = 2\hat{i} - \hat{j} - 3\hat{k}$ and $\vec{w} = \hat{i} + \hat{j} + \hat{k}$. The cross product $\vec{v} \times \vec{w}$ is computed as:

$$\vec{v} \times \vec{w} = \left(2\hat{i} - \hat{j} - 3\hat{k}\right) \times \left(\hat{i} + \hat{j} + \hat{k}\right)$$

Expanding the cross product term by term:

$$= 2\hat{i} \times \hat{i} + 2\hat{i} \times \hat{j} + 2\hat{i} \times \hat{k} - \hat{j} \times \hat{i} - \hat{j} \times \hat{j} - \hat{j} \times \hat{k} - 3\hat{k} \times \hat{i} - 3\hat{k} \times \hat{j} - 3\hat{k} \times \hat{k}$$

Using the cross product identities:

$$= 0 + 2\hat{k} + 2(-\hat{j}) - (-\hat{k}) + 0 - \hat{i} - 3\hat{j} + 3\hat{i} + 0$$

Combining like terms:

$$= (3\hat{i} - \hat{i}) + (-2\hat{j} - 3\hat{j}) + (2\hat{k} + \hat{k})$$

$$= 2\hat{i} - 5\hat{j} + 3\hat{k}$$

Thus, the final result is:

$$\vec{v} \times \vec{w} = 2\hat{i} - 5\hat{j} + 3\hat{k}$$

1.8 Torque

Definition 1.8.1: Torque and Angular Momentum

Continue with Torque

1.9 Parametric Equations

Definition 1.9.1: Parametric Equations

A parametric equation expresses a set of quantities as explicit functions of an independent parameter. In a two-dimensional case, a parametric equation for a curve can be represented as:

$$\langle x, y\rangle t = \langle f(t), g(t)\rangle$$

and in three dimensions as:

$$\langle x, y, z\rangle t = \langle f(t), g(t), h(t)\rangle$$

Note:-

For example, consider the curve in the plane given by the equation $y = f(x) = x^2+1$. This describes a parabola in Cartesian coordinates.

Theorem 1.9.1 Parametric unit circle in Cartesian coordinates

A unit circle in parametric form can be represented as:

$$x^2 + y^2 = 1$$

which corresponds to the parametric equations:

$$\langle x, y\rangle t = \langle \cos(t), \sin(t)\rangle$$

1.9.1 Examples

Note:-

The parametric equation:

$$\langle x, y\rangle t = \langle 4\cos(t), 3\sin(t)\rangle$$

At specific values of t, we can compute the points:

$$t = 0 \implies \langle 4, 0\rangle$$

$$t = \frac{\pi}{2} \implies \langle 0, 3\rangle$$

$$t = \pi \implies \langle -4, 0\rangle$$

$$t = \frac{3\pi}{2} \implies \langle 0, -3\rangle$$

Definition 1.9.2: Parametric for a Helix

$$\langle x(t), y(t), z(t)\rangle = \langle 4\cos(t), 3\sin(t), 0.1t\rangle$$

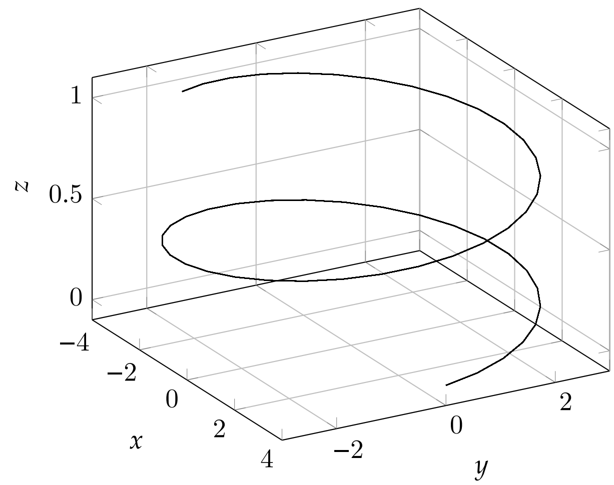


Figure 1.1: 3D plot of a parametric helix.

Theorem 1.9.2 Parametric Equation of a Line

The parametric equation of a line can be expressed as:

$$\langle x, y, z\rangle t = \mathbf{OP} + \mathbf{V}t$$

Where:

- $\mathbf{OP} = \langle x_0, y_0, z_0\rangle$ is the position vector to the initial point P,
- $\mathbf{V} = \langle v_x, v_y, v_z\rangle$ is the direction vector of the line.

Question 1: 3D Parametric Equation of a Line

- $\mathbf{OP} = \langle 1, 2, 3\rangle$ is the position vector to the initial point P,
- $\mathbf{V} = \langle 1, 1, 1\rangle$ is the direction vector of the line.

Thus, the parametric equation of the line becomes:

$$\langle x(t), y(t), z(t)\rangle = \langle 1, 2, 3\rangle + t\langle 1, 1, 1\rangle$$

$$x(t) = 1 + t, \quad y(t) = 2 + t, \quad z(t) = 3 + t$$

Or simply:

$$\langle x, y, z\rangle t = \langle 1 + t, 2 + t, 3 + t\rangle$$

Solution:

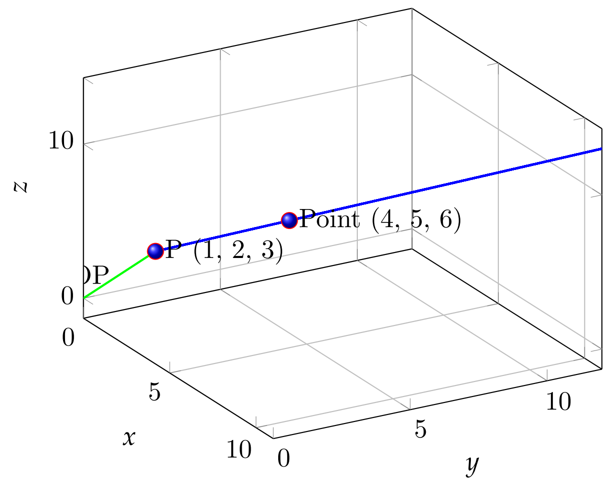


Figure 1.2: 3D plot of a parametric line with vector OP and points.

1.10 Distance from a Point to a Line

Definition 1.10.1: Parametric Equation of the Line

The line is represented by:

$$\mathbf{l} = \mathbf{OP} + t\mathbf{V}$$

Where:

- $\mathbf{OP}$ is the position vector of a point on the line,
- $\mathbf{V}$ is the direction vector of the line.

Theorem 1.10.1 Distance from a Point to a Line

The distance d from a point Q to the line l is given by:

$$d = \frac{|\mathbf{V} \times \mathbf{PQ}|}{|\mathbf{V}|}$$

Where:

- $\mathbf{PQ}$ is the vector from point P on the line to the point Q,
- $\mathbf{V} \times \mathbf{PQ}$ is the cross product of the direction vector $\mathbf{V}$ and the vector $\mathbf{PQ}$.

1.10.1 Example

Question 2: Find the distance from the point $Q = (3, 4, 0)$ to the line l given by the parametric equation

$$\mathbf{l} = \langle t, 1, 2t \rangle = \langle 0, 1, 0 \rangle + t\langle 1, 0, 2 \rangle$$

with point $P = (0, 1, 0)$ and direction vector $\mathbf{V} = \langle 1, 0, 2 \rangle$.

Solution: The vector $\mathbf{PQ}$ from $P = (0, 1, 0)$ to $Q = (3, 4, 0)$ is:

$$\mathbf{PQ} = \langle 3, 4, 0 \rangle - \langle 0, 1, 0 \rangle = \langle 3, 3, 0 \rangle$$

Now, we compute the cross product $\mathbf{V} \times \mathbf{PQ}$:

$$\mathbf{V} \times \mathbf{PQ} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ 3 & 3 & 0 \end{vmatrix} = \hat{i}(0-6) - \hat{j}(0-6) + \hat{k}(3-0) = \langle -6, -6, 3 \rangle$$

Next, calculate the magnitude of the cross product:

$$|\mathbf{V} \times \mathbf{PQ}| = \sqrt{(-6)^2 + (-6)^2 + 3^2} = \sqrt{36 + 36 + 9} = \sqrt{81} = 9$$

The magnitude of the direction vector $\mathbf{V}$ is:

$$|\mathbf{V}| = \sqrt{1^2 + 0^2 + 2^2} = \sqrt{1 + 4} = \sqrt{5}$$

Finally, the distance d is:

$$d = \frac{9}{\sqrt{5}} = \frac{9\sqrt{5}}{5}$$

1.11 Intersection of Two Parametric Lines

Definition 1.11.1: Intersection of Parametric Lines

To find the intersection point of two parametric lines, we need to equate their parametric equations and solve for the parameters.

Question 3: Find the intersection of the lines l_1 and l_2 given by the parametric equations

$$l_1 = \langle x, y \rangle(t) = \langle 0, 1 \rangle + t\langle 1, 0 \rangle$$
$$l_2 = \langle 1, 1 \rangle + s\langle -2, 1 \rangle$$

Solution: Equating the two parametric equations:

$$\langle 0, 1 \rangle + t\langle 1, 0 \rangle = \langle 1, 1 \rangle + s\langle -2, 1 \rangle$$

This gives the system of equations:

$$0 + t = 1 - 2s$$
$$1 + 0 = 1 + s$$

From the second equation, we find:

$$s = 0$$

Substitute $s = 0$ into the first equation:

$$t = 1$$

Thus, the lines intersect when $t = 1$ and $s = 0$.
The intersection point is:

$$\langle 0, 1 \rangle + 1 \cdot \langle 1, 0 \rangle = \langle 1, 1 \rangle$$

Therefore, the lines intersect at $(1, 1)$.

1.12 Planes

Definition 1.12.1: Plane Equation

Given a point $P_0 = (x_0, y_0, z_0)$ on the plane and a normal vector $\vec{n} = \langle a, b, c \rangle$, the equation of the plane can be expressed as:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Note:-

Vector Form
Alternatively, the plane equation can also be derived using the dot product form:

$$\overrightarrow{PQ} \cdot \vec{n} = 0$$

Where $P = (x_0, y_0, z_0)$ and $Q = (x, y, z)$. This leads to the scalar equation of the plane.

Theorem 1.12.1 Equation of a Plane

If a plane passes through the point $P_0 = (x_0, y_0, z_0)$ and has a normal vector $\vec{n} = \langle a, b, c \rangle$, the equation of the plane is:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Question 4: Example

Given the point $P_0 = (1, -2, 3)$ and the normal vector $\vec{n} = \langle -2, -4, -6 \rangle$, find the equation of the plane.

Solution: Using the plane equation $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$, substitute $a = -2$, $b = -4$, $c = -6$, and $P_0 = (1, -2, 3)$:

$$-2(x - 1) - 4(y + 2) - 6(z - 3) = 0$$

Expanding this equation:

$$-2x + 2 - 4y - 8 - 6z + 18 = 0$$

Simplifying:

$$-2x - 4y - 6z + 12 = 0$$

Or:

$$2x + 4y + 6z = 12$$

1.13 Vector Valued Functions

Definition 1.13.1: Parametric Curves

A parametric curve is defined as:

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle = \langle f(t), h(t), g(t) \rangle$$

where t is a real number as input, and the output is a vector.

Question 5: Example

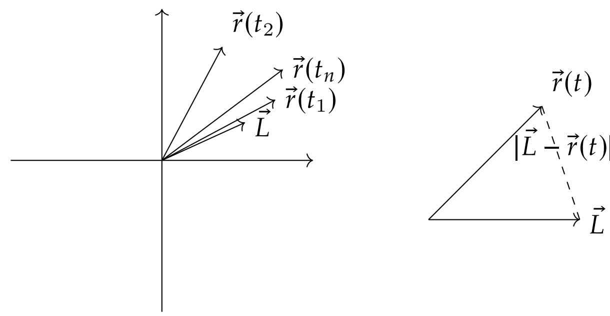
Let $\vec{r}(t) = \langle t^2, t + 1, \sqrt{t - 2} \rangle$. Determine the domain of $\vec{r}(t)$.

Solution: The domain of $\vec{r}(t)$ is $t \geqslant 2$.

Definition 1.13.2: Limits of Vector Functions

The limit of a vector function $\vec{r}(t)$ as $t \longrightarrow a$ can be visualized geometrically as the vector approaching a point $\vec{L}$. If the magnitude of the difference between $\vec{L}$ and $\vec{r}(t)$ approaches zero, we can define:

$$\lim_{t \to a} \vec{r}(t) = \vec{L} \quad \text{if} \quad \lim_{t \to a} |\vec{L} - \vec{r}(t)| = 0$$



Note:-

Component-wise Limits
The limit of a vector function can be evaluated by taking the limit of each of its components:

$$\lim_{t\to a}\vec{r}(t) = \langle \lim_{t\to a} x(t), \lim_{t\to a} y(t), \lim_{t\to a} z(t)\rangle$$

Question 6: Example

Consider the vector function $\vec{r}(t) = \langle \frac{t^2+2t+1}{t+1}, t, t-2\rangle$, as $t \to -1$.

Solution: The limits of each component as $t \to -1$ are:

$$\lim_{t\to -1}\vec{r}(t) = \langle 0, -1, -3\rangle$$

1.14 Calculus and Vector Valued Functions

1.14.1 Derivative

Theorem 1.14.1 Derivative of vector functions

Consider the vector-valued function in 3D:

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

where $f(t), g(t)$, and $h(t)$ are differentiable functions. The derivative of $\vec{r}(t)$ is defined as:

$$\vec{r}'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

Expanding $\vec{r}(t+h) - \vec{r}(t)$:

$$\vec{r}(t+h) - \vec{r}(t) = \langle f(t+h), g(t+h), h(t+h) \rangle - \langle f(t), g(t), h(t) \rangle$$

This gives us:

$$\vec{r}(t+h) - \vec{r}(t) = \langle f(t+h) - f(t), g(t+h) - g(t), h(t+h) - h(t) \rangle$$

Thus, the derivative becomes:

$$\vec{r}'(t) = \lim_{h \to 0} \left\langle \frac{f(t+h) - f(t)}{h}, \frac{g(t+h) - g(t)}{h}, \frac{h(t+h) - h(t)}{h} \right\rangle$$

Since $f(t), g(t)$, and $h(t)$ are differentiable, we apply the definition of derivatives for each component:

$$\vec{r}'(t) = \left\langle \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}, \lim_{h \to 0} \frac{g(t+h) - g(t)}{h}, \lim_{h \to 0} \frac{h(t+h) - h(t)}{h} \right\rangle$$

$$\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

Therefore, the derivative of the vector-valued function $\vec{r}(t)$ in 3D is:

$$\boxed{\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle}$$

Definition 1.14.1: Vector Representation

Let $\vec{r}(t)$ be a vector-valued function that describes the position of a particle over time t:

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

where $x(t), y(t), z(t)$ are differentiable functions. The following terms describe important properties of the function:

1. **Position** at time t: $\vec{r}(t)$
2. **Velocity** (tangent vector) $\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$
3. **Speed** $|\vec{r}'(t)|$, the magnitude of the velocity.
4. **Acceleration** $\vec{r}''(t) = \langle x''(t), y''(t), z''(t) \rangle$

Definition 1.14.2: Unit Tangent Vector

The unit tangent vector is given by:

$$\hat{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

This is the unit vector in the direction of the velocity vector $\vec{r}'(t)$, which represents the direction of motion.

Question 7: Example

Consider the following vector-valued function:

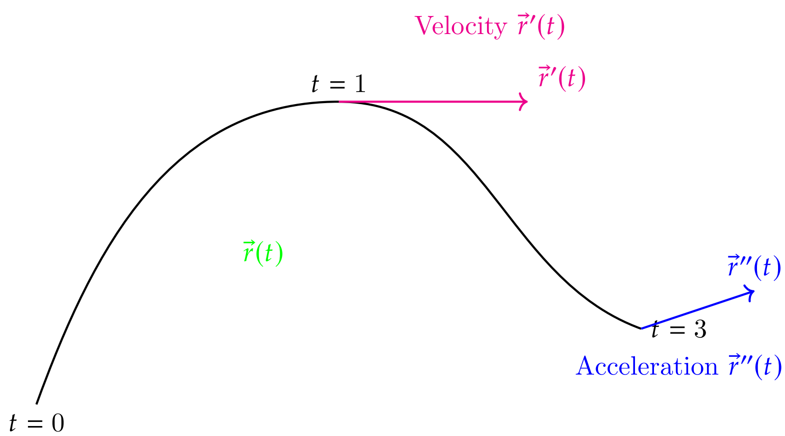
$$\vec{r}(t) = \langle 1 + t^2, 2 + t^2, 3 + t^2 \rangle$$

Taking the derivative:

$$\vec{r}'(t) = \langle 2t, 2t, 2t \rangle$$

This describes a line with a constant acceleration:

$$\vec{r}''(t) = \langle 2, 2, 2 \rangle$$



Definition 1.14.3: Derivatives of Parametric Curves in terms of t

Given a parametric curve defined by:

$$x = f(t), \quad y = g(t)$$

where t is the parameter, we want to find the derivative $\frac{dy}{dx}$, which represents the slope of the tangent to the curve at any point t.

Step 1: Finding the Derivatives of x and y with respect to t To find $\frac{dy}{dx}$, we need to find both $\frac{dx}{dt}$ and $\frac{dy}{dt}$. These are given by:

$$\frac{dx}{dt} = f'(t), \quad \frac{dy}{dt} = g'(t)$$

Step 2: Using the Chain Rule The derivative $\frac{dy}{dx}$ can be found by using the chain rule as follows:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Substituting the expressions for $\frac{dx}{dt}$ and $\frac{dy}{dt}$, we obtain:

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$$

Step 3: Simplifying the Result The expression $\frac{dy}{dx}$ provides the slope of the curve at any point t in terms of the parameter.

Example 1.14.1 (Consider the parametric equations:)

$$x = 6\sin t, \quad y = 6\cos t$$

To find $\frac{dy}{dx}$, we calculate the derivatives with respect to t:

$$\frac{dx}{dt} = 6\cos t, \quad \frac{dy}{dt} = -6\sin t$$

Then, we apply the formula:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-6\sin t}{6\cos t}$$

Simplifying the expression:

$$\frac{dy}{dx} = -\tan t$$

Therefore, the slope of the tangent line to the parametric curve at any point t is:

$$\frac{dy}{dx} = -\tan t$$

Definition 1.14.4: Chain Rule Derivation

$$\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{dx}{dt} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{du}{dt}$$

Question 8: Example

Slope of

$$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$$

At $t = \frac{\pi}{3}$:

Solution:

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, \quad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\vec{r}'(t) = \langle -\sin(t), \cos(t) \rangle$$

At $t = \frac{\pi}{3}$:

$$\vec{r}'\left(\frac{\pi}{3}\right) = \langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$$

Slope Calculation

$$\text{slope} = \frac{\frac{-1}{2}}{\frac{-\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Vector Notation

$$\vec{r}'(t) = \langle -\sin(t), \cos(t) \rangle \quad \text{so} \quad \vec{r}'\left(\frac{\pi}{3}\right) = \langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$$

1.14.2 Chain Rule

Definition 1.14.5: Chain Rule for VV

Let $g : \mathbb{R} \to \mathbb{R}$ and $\vec{r}(t)$ be a vector function. The derivative of the composition of functions is:

$$\frac{d}{dt}[g(\vec{r}(t))] = g'(\vec{r}(t)) \cdot \vec{r}'(t)$$

where $g'(\vec{r}(t))$ is the derivative of g with respect to the vector function $\vec{r}(t)$ and $\vec{r}'(t)$ is the derivative of the vector function $\vec{r}(t)$ with respect to t. This rule is useful for differentiating scalar functions composed with vector-valued functions.

Question 9: Example

$$\vec{r}(t) = \langle \cos^2 t, -3t + \cos^3 t \rangle = \dot{\vec{r}}(g(t))$$

$$y = \cos t, \quad \vec{r} = \langle t^2, -3t, 3t^3 \rangle$$

$$g' = -\sin t, \quad \dot{\vec{r}} = \langle 2t, 3t^2 \rangle$$

Thus, the derivative is:

$$\vec{s}(t) = \frac{d}{dt}\left(g(y(t))\vec{r}(g(t))\right) = g'(t) \cdot \dot{\vec{r}}(g(t))$$

$$= -\sin t \cdot \langle 2\cos t, 3\cos^2 t \rangle$$

1.14.3 Product Rule

Definition 1.14.6: Product rule for VV

Let $g(t) : \mathbb{R} \to \mathbb{R}$, $\vec{r}(t) : \mathbb{R} \to \mathbb{R}^n$, and $\vec{s}(t) : \mathbb{R} \to \mathbb{R}^n$ be vector functions. Then, the product rule is:

$$\frac{d}{dt}\left(g(t)\vec{r}(t)\right) = g'(t)\vec{r}(t) + g(t)\dot{\vec{r}}(t)$$

For vector dot products:

$$\frac{d}{dt}\left(\vec{r}(t) \cdot \vec{s}(t)\right) = \vec{r}'(t) \cdot \vec{s}(t) + \vec{r}(t) \cdot \vec{s}'(t)$$

For vector cross products:

$$\frac{d}{dt}\left(\vec{r}(t) \times \vec{s}(t)\right) = \vec{r}'(t) \times \vec{s}(t) + \vec{r}(t) \times \vec{s}'(t)$$

1.14.4 Integrals

Definition 1.14.7: Integral of Vector-Valued Functions

Let $\vec{r}(t) : [a, b] \to \mathbb{R}^n$ be a continuous vector-valued function. The definite integral of $\vec{r}(t)$ over the interval $[a, b]$ is defined as:

$$\int_a^b \vec{r}(t)\,dt = \langle \int_a^b r_1(t)\,dt, \int_a^b r_2(t)\,dt, \ldots, \int_a^b r_n(t)\,dt \rangle$$

where $\vec{r}(t) = \langle r_1(t), r_2(t), \ldots, r_n(t) \rangle$. This means that the integral of a vector-valued function is the vector whose components are the integrals of the respective component functions.
For example, if $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then:

$$\int_a^b \vec{r}(t)\,dt = \langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \rangle$$

This can be extended to cases where $\vec{r}(t)$ represents the position of a particle over time, and the integral represents the net displacement of the particle over the interval $[a, b]$.

Question 10: Find the integral of the vector-valued function

Calculate the integral of the vector-valued function $\vec{r}(t) = \langle t^2, \sin t, e^t \rangle$ over the interval $t \in [0, 1]$.

Solution:

The vector-valued function is given as:

$$\vec{r}(t) = \langle t^2, \sin t, e^t \rangle$$

We can integrate each component of the vector separately over the interval $[0, 1]$:

$$\int_0^1 \vec{r}(t)\,dt = \langle \int_0^1 t^2\,dt, \int_0^1 \sin t\,dt, \int_0^1 e^t\,dt \rangle$$

Now, solving each integral:
1. For $\int_0^1 t^2\,dt$:

$$\int_0^1 t^2\,dt = \left[\frac{t^3}{3}\right]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

2. For $\int_0^1 \sin t\,dt$:

$$\int_0^1 \sin t \, dt = [-\cos t]_0^1 = -\cos(1) + \cos(0) = 1 - \cos(1)$$

3. For $\int_0^1 e^t \, dt$:

$$\int_0^1 e^t \, dt = \left[e^t\right]_0^1 = e^1 - e^0 = e - 1$$

Thus, the result of the integral is:

$$\int_0^1 \vec{r}(t) \, dt = \langle \frac{1}{3}, 1 - \cos(1), e - 1 \rangle$$

Question 11: Find the position vector of a ball in 3D space

A ball is thrown from the origin with an initial velocity vector $\vec{v}(0) = \langle 1, 3, 5 \rangle$ m/s, and experiences a constant acceleration vector $\vec{a}(t) = \langle 0, 0, -10 \rangle$ m/s^2. Find the position vector $\vec{r}(t)$ of the ball at any time t.

Solution:

The velocity vector $\vec{v}(t)$ is found by integrating the acceleration vector $\vec{a}(t)$:

$$\vec{v}(t) = \int_0^t \vec{a}(s) \, ds + \vec{v}(0)$$

Given that $\vec{a}(t) = \langle 0, 0, -10 \rangle$, we integrate each component:

$$\begin{aligned} \vec{v}(t) &= \int_0^t \langle 0, 0, -10 \rangle \, ds + \langle 1, 3, 5 \rangle \\ &= \langle 0, 0, -10t \rangle + \langle 1, 3, 5 \rangle \\ \vec{v}(t) &= \langle 1, 3, 5 - 10t \rangle \end{aligned}$$

Now, to find the position vector $\vec{r}(t)$, we integrate the velocity vector $\vec{v}(t)$:

$$\begin{aligned} \vec{r}(t) &= \int_0^t \vec{v}(s) \, ds \\ &= \int_0^t \langle 1, 3, 5 - 10s \rangle \, ds \end{aligned}$$

Integrating each component:

1. For the first component:

$$\int_0^t 1 \, ds = t$$

2. For the second component:

$$\int_0^t 3 \, ds = 3t$$

3. For the third component:

$$\int_0^t (5 - 10s) \, ds = \left[5s - 5s^2\right]_0^t = 5t - 5t^2$$

Thus, the position vector is:

$$\vec{r}(t) = \langle t, 3t, 5t - 5t^2 \rangle$$

1.15 Arc Length

The arc length of a smooth curve given by a vector-valued function can be calculated using an integral. Let the position vector of a curve be represented as:

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle,$$

where t is the parameter over some interval $[a, b]$.

Definition 1.15.1: Arc Length Formula

The arc length L of the curve $\vec{r}(t)$ over the interval $[a, b]$ is given by:

$$L = \int_a^b \left|\vec{r}'(t)\right| \, dt,$$

where $\vec{r}'(t)$ is the derivative of the position vector, representing the velocity vector of the curve. To find $\left|\vec{r}'(t)\right|$, we use the following:

$$\left|\vec{r}'(t)\right| = \sqrt{\left(f'(t)\right)^2 + \left(g'(t)\right)^2 + (h'(t))^2}.$$

Therefore, the arc length formula can be explicitly written as:

$$L = \int_a^b \sqrt{\left(f'(t)\right)^2 + \left(g'(t)\right)^2 + (h'(t))^2} \, dt.$$

Question 12: Example: Eagle Spiraling

Consider an example where an eagle is spiraling in space, and its position at time t (in minutes) is given by:

$$\vec{r}(t) = \langle 250\cos(t), 250\sin(t), 100t \rangle \quad \text{(in feet)}.$$

Solution:

To find the velocity vector, we differentiate each component of $\vec{r}(t)$:

$$\vec{r}'(t) = \langle -250\sin(t), 250\cos(t), 100 \rangle \quad \text{(in feet per minute)}.$$

The magnitude of the velocity vector $\left|\vec{r}'(t)\right|$ is:

$$\left|\vec{r}'(t)\right| = \sqrt{(-250\sin(t))^2 + (250\cos(t))^2 + (100)^2}.$$

Simplifying, we obtain:

$$\left|\vec{r}'(t)\right| = \sqrt{250^2\sin^2(t) + 250^2\cos^2(t) + 100^2}.$$

Using the Pythagorean identity $\sin^2(t) + \cos^2(t) = 1$, we find:

$$\left|\vec{r}'(t)\right| = \sqrt{250^2 + 100^2} = \sqrt{62500 + 10000} = \sqrt{72500} \approx 269.3 \quad \text{(feet per minute)}.$$

Therefore, the speed of the eagle is approximately 269.3 feet per minute.

Question 13: Calculating Distance Over Time

Find the total distance over 10 minutes

Solution:

To find the total distance traveled by the eagle over a period of 10 minutes, we compute the arc length:

$$\text{Distance} = \int_0^{10} \left|\vec{r}'(t)\right| \, dt = \int_0^{10} 269.3 \, dt.$$

Evaluating the integral:

$$\text{Distance} = 269.3 \cdot (10 - 0) = 2693 \text{ feet}.$$

Thus, over 10 minutes, the eagle travels approximately 2693 feet.

Definition 1.15.2: Arc Length Function

The arc length function, denoted by $S(t)$, is defined as:

$$S(t) = \text{distance traveled along the curve from } t = a.$$

It is given by the integral:

$$S(t) = \int_a^t \left|\vec{r}'(s)\right| ds.$$

This function essentially converts the vector quantity $\vec{r}(t)$ into a 1-dimensional scalar that measures the total distance traveled without considering direction.
The arc length function $S(t)$ effectively flattens the vector $\vec{r}(t)$ into a scalar by integrating the magnitude of the velocity vector $\vec{r}'(s)$ over the interval $[a, t]$.

Example 1.15.1 (Application: Reparametrization by Arc Length)

One common use of the arc length function is to reparametrize the curve so that it is traveled at unit speed.
Given the inverse function of $S(t)$, denoted as $S^{-1}(t)$, we wish to compare:

$$\vec{r}\left(S^{-1}(t)\right) \quad \text{with} \quad \vec{r}(t).$$

If $S(t)$ is given by:

$$S(t) = \int_a^t \left|\vec{r}'(p)\right| dp,$$

then $S'(t) = \left|\vec{r}'(t)\right|$.
To differentiate $\vec{r}(S^{-1}(t))$, we use the chain rule:

$$\frac{d}{dt}\vec{r}(S^{-1}(t)) = \frac{1}{S'(S^{-1}(t))}\vec{r}'(S^{-1}(t)).$$

Since $S'(S^{-1}(t)) = \left|\vec{r}'(S^{-1}(t))\right|$, we have:

$$\frac{d}{dt}\vec{r}(S^{-1}(t)) = \frac{\vec{r}'(S^{-1}(t))}{\left|\vec{r}'(S^{-1}(t))\right|}.$$

This shows that the derivative of $\vec{r}(S^{-1}(t))$ is a unit vector, implying that $\vec{r}(S^{-1}(t))$ describes the same curve as $\vec{r}(t)$, but at unit speed.
By reparametrizing the curve with respect to arc length, we achieve a uniform traversal of the curve at constant unit speed. The transformation $\vec{r}(S^{-1}(t))$ provides a way to describe the geometry of the curve without the influence of varying speed.

Question 14: Worked Example

Consider the position of an eagle as it spirals upward over time. The position vector $\vec{r}(t)$ is given by:

$$\vec{r}(t) = \langle 250\cos(t), 250\sin(t), 100t\rangle \quad \text{(in feet)},$$

where t is in minutes.

Step 1: Find the Speed of the Eagle

The velocity vector is obtained by differentiating the position vector:

$$\vec{r}'(t) = \langle -250\sin(t), 250\cos(t), 100\rangle.$$

The speed is the magnitude of the velocity vector:

$$\left|\vec{r}'(t)\right| = \sqrt{(-250\sin(t))^2 + (250\cos(t))^2 + 100^2} = \sqrt{62500 + 10000} = 269.3 \quad \text{(feet per minute)}.$$

Step 2: Arc Length Function $S(t)$

To find the arc length function $S(t)$, which represents the distance traveled along the curve from $t = 0$ to $t = t$:

$$S(t) = \int_0^t \left|\vec{r}'(p)\right| dp.$$

Since the speed is constant:

$$S(t) = \int_0^t 269.3\, dp = 269.3t.$$

Thus, at time t, the eagle has traveled $269.3t$ feet.

Step 3: Reparametrize by Arc Length

The inverse of the arc length function $S(t)$, denoted as $S^{-1}(t)$, is given by:

$$S^{-1}(t) = \frac{t}{269.3}.$$

Now, to reparametrize the curve by arc length, we substitute $S^{-1}(t)$ into the original position vector $\vec{r}(t)$:

$$\vec{r}(S^{-1}(t)) = \left\langle 250\cos\left(\frac{t}{269.3}\right), 250\sin\left(\frac{t}{269.3}\right), 100\frac{t}{269.3}\right\rangle.$$

Interpretation

This reparametrization gives the position of the eagle as a function of the distance traveled (in feet), rather than as a function of time. It provides a way to describe the motion of the eagle along the curve such that the eagle is moving at a constant speed of 1 foot per unit of time.

1.16 Curvature of a Vector-Valued Function

Curvature is a measure of the "sharpness" or "bendiness" of a curve. For a smooth curve given by a vector-valued function $\vec{r}(t)$, the curvature at a point is related to how quickly the direction of the curve is changing.

1.16.1 Osculating Circle

The concept of curvature is closely tied to the idea of an **osculating circle**. At any point on the curve, the osculating circle is the circle that best approximates the curve's curvature at that point. This circle: - Has the same tangent as the curve $\vec{r}(t)$ at the point $\vec{r}(t_0)$. - Its radius is a measure of the "tightness" of the curve's bend at that point.

The radius of the osculating circle is denoted as:

$$R(t),$$

where $R(t)$ is the radius of curvature of the curve at time t.

1.16.2 Curvature $\kappa(t)$

The curvature $\kappa(t)$ of the curve $\vec{r}(t)$ at the point $\vec{r}(t_0)$ is defined as the reciprocal of the radius of the osculating circle:

$$\kappa(t) = \frac{1}{R(t)}.$$

A smaller radius $R(t)$ indicates a tighter bend and thus a larger curvature $\kappa(t)$, whereas a larger radius corresponds to a gentler curve and smaller curvature.

The osculating circle provides a geometric interpretation of curvature, serving as the best circular approximation to the curve at any given point. The curvature $\kappa(t)$ quantifies how sharply the curve bends at that point, with the relationship:

$$\kappa(t) = \frac{1}{R(t)}.$$

Definition 1.16.1: Osculating Circle of a Straight Line

A straight line is a curve with no curvature. Thus, the curvature $\kappa(t)$ of a straight line is zero at all points. To understand why, consider the definition of the osculating circle.
For any curve $\vec{r}(t)$, the osculating circle at a point $\vec{r}(t_0)$ is defined as the circle that "best fits" the curve at that point. Mathematically, this means:

$$\kappa(t) = \lim_{\Delta t \to 0} \frac{\text{change in angle}}{\text{arc length}},$$

where $\kappa(t)$ is the curvature of the curve. However, for a straight line, the change in angle is always zero, so:

$$\kappa(t) = 0.$$

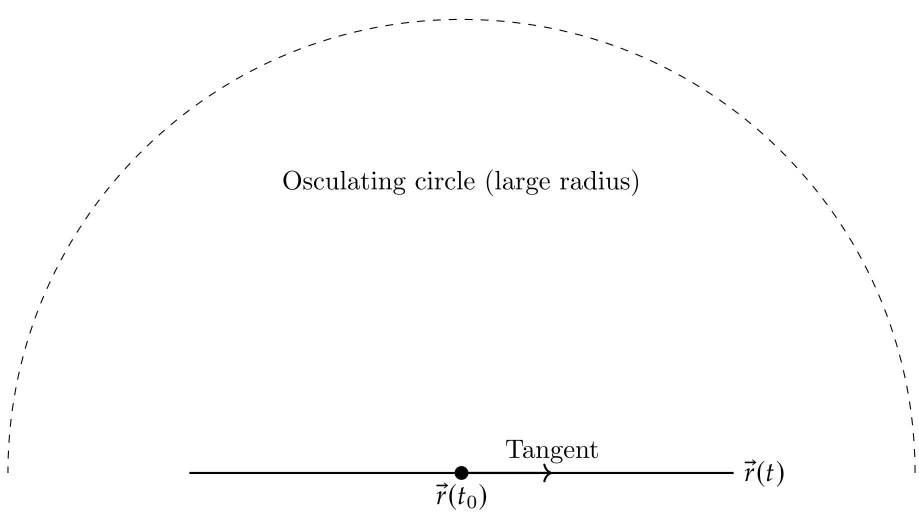
Since curvature is the reciprocal of the radius of the osculating circle $R(t)$, a zero curvature implies an infinite radius:

$$R(t) = \frac{1}{\kappa(t)} \to \infty.$$

Therefore, the osculating circle of a straight line is effectively a circle with infinite radius, meaning it is "flattened" into a straight line.

Diagram

Below is a diagram to illustrate the concept of the osculating circle of a straight line:



The dashed arc represents the osculating circle with an infinitely large radius, approximating the straight line at $\vec{r}(t_0)$.

To approach the concept of osculating circles for vector-valued functions, we break down the analysis into four key points:

1. Tangent Vector at t_0

The tangent vector to the curve at time t_0 is given by:

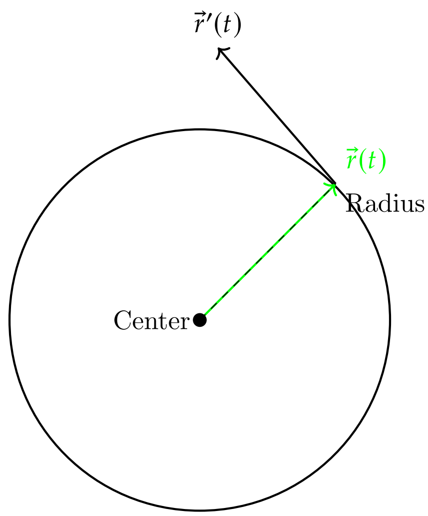
$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}.$$

This vector is the unit tangent vector at any point on the curve.

2. If $\vec{r}(t)$ Describes a Circle

Suppose that $\vec{r}(t)$ describes a circular path. In this case:

- The vector $\vec{r}(t)$ is always parallel to the radius of the circle.
- The velocity vector $\vec{r}'(t)$ is also perpendicular to $\vec{r}(t)$.



3. Acceleration Vector $\vec{a}(t)$

For a circular path, the acceleration vector $\vec{a}(t)$ points toward the center of the circle. Hence,

$$\vec{a}(t) \text{ is opposite to } \vec{r}(t) \text{ and directed towards the center.}$$

This implies:

$$\vec{a}(t) \perp \vec{r}'(t), \quad \vec{a}(t) \perp \vec{T}(t).$$

4. Constant Speed and Perpendicularity

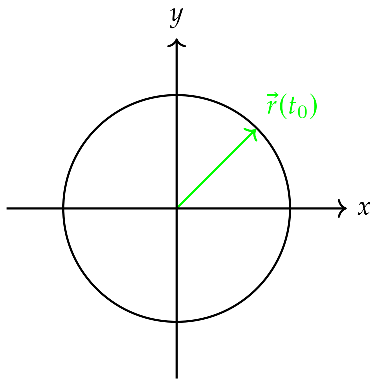
Suppose that $\vec{r}(t)$ describes a curve with constant speed:

$$|\vec{r}'(t)| = C, \quad C \in \mathbb{R}.$$

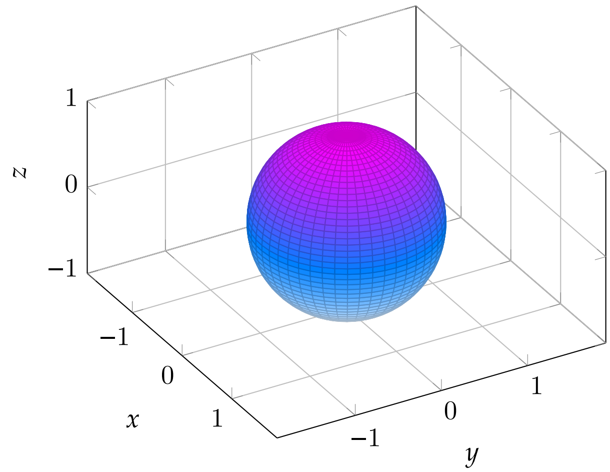
For both 2D and 3D, the curve will either be a circle or a sphere with a fixed radius. The velocity and acceleration vectors, $\vec{r}'(t)$ and $\vec{a}(t)$, maintain a perpendicular relationship:

$$\vec{r}(t) \cdot \vec{v}(t) = 0.$$

2D Circle



3D Sphere



1.16.3 Deriving the Perpendicularity

Let $\vec{v}(t) = \vec{r}'(t)$ be the velocity vector. Then:

$$\vec{r}'(t) \cdot \vec{v}(t) = \vec{r}'(t) \cdot \vec{r}'(t) = \frac{1}{2}\frac{d}{dt}\left(\vec{r} \cdot \vec{r}\right).$$

Differentiating:

$$= \frac{1}{2}\frac{d}{dt}\left(|\vec{r}'(t)|^2\right) = \frac{1}{2}\frac{d}{dt}C^2 = 0,$$

since C is constant.

Therefore, if $\left|\vec{r}'(t)\right| = C$, then:

$$\vec{v}(t) \perp \vec{r}(t).$$

1.17 Calculating Curvature

1.17.1 Case 1: $\vec{r}(t)$ travels at unit speed around a circle

We consider the position vector $\vec{r}(t)$ traveling at unit speed around a circle of radius R.

$$\vec{r}(t) = \left\langle R\cos\left(\frac{t}{R}\right), R\sin\left(\frac{t}{R}\right), 0\right\rangle$$

The velocity vector $\vec{v}(t)$ is then the derivative of the position vector:

$$\vec{v}(t) = \vec{r}'(t) = \left\langle -\sin\left(\frac{t}{R}\right), \cos\left(\frac{t}{R}\right), 0\right\rangle$$

Note that the magnitude of the velocity vector is 1 (unit speed):

$$|\vec{v}(t)| = 1$$

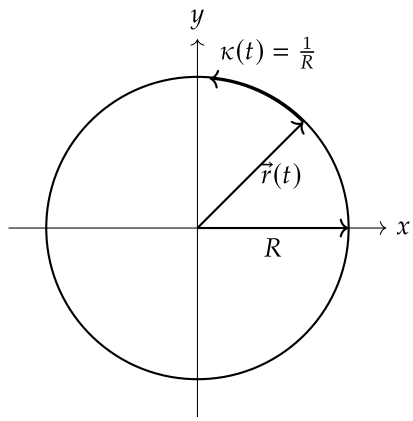
The acceleration vector $\vec{a}(t)$, which is the second derivative of the position vector, is given by:

$$\vec{a}(t) = \vec{r}''(t) = \left\langle -\frac{1}{R}\cos\left(\frac{t}{R}\right), -\frac{1}{R}\sin\left(\frac{t}{R}\right), 0\right\rangle$$

Definition 1.17.1: The curvature $\kappa(t)$ of the circle

The magnitude of the acceleration vector (centripetal acceleration) is:

$$\kappa(t) = |\vec{r}''(t)| = \frac{1}{R}$$



Note:-

- $\vec{r}(t)$ represents the position of a point moving around a circle of radius R.
- The velocity vector $\vec{v}(t)$ is orthogonal to $\vec{r}(t)$ and has a constant magnitude, indicating uniform circular motion.
- The acceleration vector $\vec{a}(t)$ points towards the center of the circle (centripetal acceleration) and its magnitude is inversely proportional to the radius R, showing that a smaller radius corresponds to a greater acceleration.
- The curvature $\kappa(t)$ is defined as the reciprocal of the radius of the circle, $\frac{1}{R}$.

1.17.2 Case 2: General Curve with Unit Speed

Consider a curve given by the position vector $\vec{r}(t)$, which travels at unit speed along a general curve. This means that the magnitude of the velocity vector is constant:

$$|\vec{v}(t)| = 1.$$

Since the speed is constant, the acceleration vector $\vec{a}(t)$ is orthogonal to the velocity vector:

$$\vec{v}(t) \cdot \vec{a}(t) = 0.$$

This implies that all changes in $\vec{a}(t)$ are perpendicular to the direction of motion (velocity), which aligns with the properties of motion at unit speed.

Claim 1.17.1 Acceleration and the Osculating Circle

The acceleration vector $\vec{a}(t)$ is directed towards the center of curvature of the curve at each point, and its magnitude is related to the curvature. In fact, the acceleration "matches" the behavior of the osculating circle at that point. The osculating circle is the circle that best approximates the curve locally, and its radius of curvature R is inversely proportional to the curvature.

Theorem 1.17.1 Curvature and Acceleration

The curvature $\kappa(t)$ of the curve at time t is defined as the magnitude of the acceleration vector:

$$\kappa(t) = |\vec{a}(t)|.$$

This provides a measure of how sharply the curve is bending at any given point. Since the speed is unitary, the curvature directly corresponds to the acceleration's magnitude.

Example 1.17.1 (Matching the Curve with Its Osculating Circle)

The osculating circle at a point on the curve is chosen so that its first and second derivatives match the curve's first and second derivatives, respectively:

$$\text{Choose circle } \vec{C}(t) \text{ such that } \vec{C}'(t) \text{ and } \vec{C}''(t) \text{ match } \vec{r}'(t) \text{ and } \vec{r}''(t).$$

This ensures that the circle is the best local approximation to the curve at that point, providing insight into the curve's behavior through the curvature.

1.17.3 Case 3: General Curve with Variable Speed

In this case, we analyze a general curve with a variable speed. Unlike the previous case, the magnitude of the velocity vector is not constant:

$$|\vec{v}(t)| \neq 1.$$

This implies that the curve's speed is not uniform, and therefore, the analysis of the curvature will require a different approach.

Let $s(t)$ be the arc length parameter, which is a function of t. The position vector can be reparameterized in terms of arc length:

$$\vec{r}(s^{-1}(t)),$$

where $s^{-1}(t)$ is the inverse function of $s(t)$.

Theorem 1.17.2 Calculating Curvature $\kappa(t)$

To find the curvature, we introduce a parameter t_0, which satisfies $t_0 = s^{-1}(t)$. Thus, we have:

$$s^{-1}(t_0) = t.$$

The curvature $\kappa(t)$ is then given by:

$$\kappa(t) = \left| \frac{d}{dt_0} \vec{T}(s^{-1}(t_0)) \right|,$$

where $\vec{T}(t)$ is the unit tangent vector.

This can be rewritten as:

$$\kappa(t) = \left| \frac{\frac{d}{dt}\vec{r}(s^{-1}(t))}{|\vec{r}'(s^{-1}(t))|} \right| = \left| \frac{\vec{T}'(s^{-1}(t_0))}{|\vec{r}'(s^{-1}(t_0))|} \right|.$$

The final expression for the curvature is:

$$\boxed{\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}}$$

where $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$ is the unit tangent vector to the curve.

Alternatively, the curvature can be found with

$$\boxed{\kappa(t) = \frac{|\vec{a} \times \vec{v}|}{|\vec{v}|^3}, \quad \kappa(t) = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right|, \quad \kappa(t) = |\frac{d\vec{T}}{ds}|}$$

if $y = f(x)$ can be parametrized as

$$\vec{r}(t) = \langle t, f(t), 0 \rangle,$$

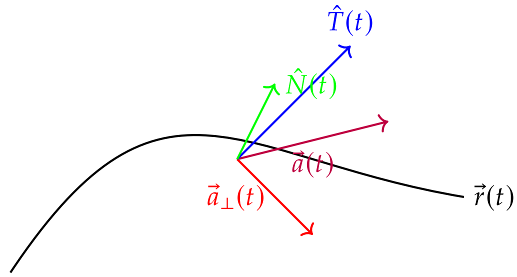
where

$$x'(t) = 1, \quad x''(t) = 0.$$

This parametrization gives the curvature κ of the curve as

$$\boxed{\kappa(t) = \frac{|y''(x)|}{\left(1 + (y'(x))^2\right)^{3/2}}.}$$

1.18 Components of Acceleration



$\boxed{|\vec{a}_\perp(t)| = \frac{|\vec{v}(t)\times\vec{a}(t)|}{|\vec{v}(t)|} = |\vec{v}(t)|^2 \cdot \kappa(t)}$ $\boxed{|\vec{a}_\parallel(t)| = \frac{d}{dt}|\vec{v}(t)| = \frac{d^2}{dt^2}\int_0^t |\vec{v}(p)|dp = \frac{d^2}{dt^2}s(t)}$

$\boxed{\vec{a}_\perp(t) = \text{centripetal acceleration}}$

$\boxed{\vec{a}_\parallel(t) = \text{speed change}}$

$\boxed{\vec{a}(t) = |\vec{a}_\perp(t)| \cdot \frac{\vec{T}'(t)}{|\vec{T}'(t)|} + |\vec{a}_\parallel(t)| \cdot \frac{\vec{r}'(t)}{|\vec{r}'(t)|}}$

Missing a third component of motion

1.19 Binormal Vector

Definition 1.19.1: Binormal Vector

The **binormal vector** $\vec{B}$ is defined as:

$$\vec{B} = \vec{T} \times \vec{N} = \frac{\vec{v} \times \vec{a}}{|\vec{v} \times \vec{a}|}$$

where:

- $\vec{T}$ is the unit tangent vector to the curve, given by:

$$\vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

- $\vec{N}$ is the unit normal vector to the curve, and it is defined as:

$$\vec{N} = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

The binormal vector $\vec{B}$ represents the direction to exit the current plane of motion described by the vectors $\vec{T}$ and $\vec{N}$.

1.19.1 Change in the Plane of Motion

The rate of change of the binormal vector is given by:

$$\frac{d\vec{B}}{dt} = \text{change in the plane of motion}$$

Applying the derivative product rule to $\vec{B} = \vec{T} \times \vec{N}$:

$$\frac{d}{dt}(\vec{T} \times \vec{N}) = \frac{d\vec{T}}{dt} \times \vec{N} + \vec{T} \times \frac{d\vec{N}}{dt}$$

Since:

$$\vec{N} = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

it follows that:

$$\vec{T} \cdot \vec{N} = 0$$

and thus:

$$\frac{d\vec{B}}{dt} \text{ is orthogonal to both } \vec{T} \text{ and } \vec{N}$$

1.19.2 Torsion and its Relation to the Binormal Vector

Since $\vec{N}$ is a unit vector, the vector $\vec{B}'$ is parallel to $\vec{N}$:

$$\frac{d\vec{B}}{dt} = -\tau\vec{N}$$

where τ is the **torsion** of the plane of movement.

Note:-

If $\tau > 0$, the motion is on the plane of movement; if $\tau < 0$, the motion is in the opposite direction.

To find τ, take the dot product of both sides with $\vec{N}$:

$$\frac{d\vec{B}}{dt} \cdot \vec{N} = -\tau(\vec{N} \cdot \vec{N}) = -\tau$$

Thus, at unit speed:

Definition 1.19.2: Torsion τ at unit speed

$$\tau = -\frac{\frac{d\vec{B}}{dt} \cdot \vec{N}}{|\vec{N}|} = -\frac{d\vec{B}}{dt} \cdot \vec{N}$$

1.19.3 General Case and Torsion Formula

For the general case, replace $\vec{r}(t)$ with $\vec{r}(s^{-1}(t))$, where $s(t)$ is the arc-length parameterization.

The derivative of $\vec{B}$ in the general case is given by:

$$\frac{d\vec{B}(s^{-1})}{d(t_0)} = \frac{\vec{B}'(s^{-1}(t_0))}{|\vec{r}'(s^{-1}(t_0))|}$$

or simply:

$$\frac{d\vec{B}(t)}{dt} = \frac{\vec{B}'(t) \cdot \vec{N}(t)}{|\vec{r}'(t)|}$$

The general formula for torsion τ becomes:

Definition 1.19.3: General form for torsion τ

$$\tau = \frac{\vec{B}'(t) \cdot \vec{N}(t)}{|\vec{r}'(t)|}$$

.

1.19.4 Expressing the Binormal Vector with Velocity and Acceleration

The binormal vector $\vec{B}$ can be expressed as:

$$\vec{B} = \frac{\vec{v} \times \vec{a}}{|\vec{v} \times \vec{a}|}$$

where $\vec{v}$ is the velocity vector and $\vec{a}$ is the acceleration vector.

The torsion τ is given by:

$$\tau = \frac{(\vec{v} \times \vec{a}) \cdot \vec{a}'}{|\vec{v} \times \vec{a}|^2}$$

Note:-

The centripetal acceleration determines the curvature of the path.

Example 1.19.1 (Finding the Torsion of a Helix Path)

Consider the helix parameterized by:

$$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$$

The first derivative (velocity vector) is:

$$\vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$$

with magnitude:

$$|\vec{r}'(t)| = \sqrt{2}$$

The unit tangent vector $\vec{T}(t)$ is:

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \left\langle -\frac{\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

The derivative of the tangent vector is:

$$\vec{T}'(t) = \left\langle -\frac{\cos t}{\sqrt{2}}, -\frac{\sin t}{\sqrt{2}}, 0 \right\rangle$$

and its magnitude is:

$$|\vec{T}'(t)| = \frac{1}{\sqrt{2}}$$

The curvature $\kappa(t)$ is given by:

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{\frac{1}{\sqrt{2}}}{\sqrt{2}} = \frac{1}{2}$$

Hence, the radius of curvature $R(t)$ is:

$$R(t) = \frac{1}{\kappa(t)} = 2$$

The normal vector $\vec{N}(t)$ is:

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \langle -\cos t, -\sin t, 0 \rangle$$

which points towards the center of the curvature.
The normal acceleration a_N is:

$$a_N = |\vec{r}'(t)|^2 \cdot \kappa = 1 \quad \text{(constant, points towards the center)}$$

The tangential acceleration is:

$$a_T = s''(t), \quad s(t) = t\sqrt{2}, \quad s'(t) = \sqrt{2}$$

The binormal vector $\vec{B}$ is given by:

$$\vec{B} = \vec{T} \times \vec{N} = \frac{1}{\sqrt{2}}\langle \sin t, -\cos t, 0\rangle$$

The derivative of the binormal vector is:

$$\vec{B}' = \frac{1}{\sqrt{2}}\langle \cos t, \sin t, 0\rangle = -\frac{1}{\sqrt{2}}\vec{N} \implies \tau = \frac{\vec{B}' \cdot \vec{N}}{|\vec{r}'(t)|} = \frac{1}{2}$$

Therefore, the torsion τ is:

$$\tau = \frac{\kappa}{\sqrt{2}} = \frac{1}{2}$$

Chapter 2

Functions of Multiple Random Variables $f : \mathbb{R}^n \to \mathbb{R}$

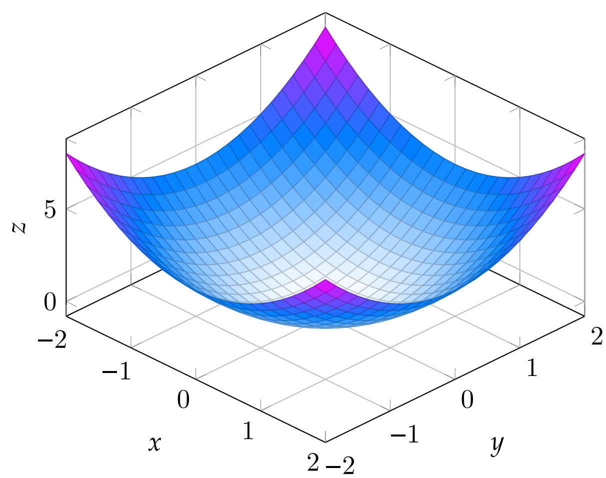
2.1 Plotting Multiple Random Variable Functions

Consider a function $f(x, y) = z$. We aim to plot functions in 3D.

Example

$$f(x, y) = x^2 + y^2$$

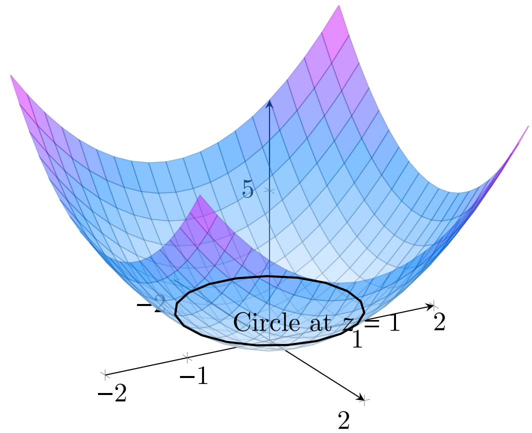
- When $y = 0$, $f(x, 0) = x^2$.
- When $x = 0$, $f(0, y) = y^2$.



For this surface:

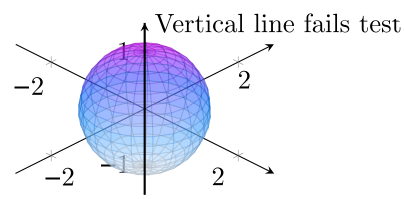
- If $z = 0$, $(x, y) = (0, 0)$.
- If $z = 1$, $x^2 + y^2 = 1$, which represents a **circle**.
- If $z > 0$, $x^2 + y^2 = z$, representing a circle with radius $\sqrt{z}$ in the xy-plane.

The vertical line test still applies: a vertical line intersects the surface at at most 1 point.



Sphere by Self Equation

A sphere cannot be represented as a function in the form $f(x, y) = z$. This is because it fails the vertical line test; a vertical line may intersect the sphere at multiple points.



2.2 Domain of Multiple Random Variable Functions

Definition 2.2.1: Domain

The domain of $f(x, y)$ is the set of pairs $(x, y) \in \mathbb{R}^2$ such that $f(x, y)$ is defined.

Example 2.2.1 (Example of finding domain)

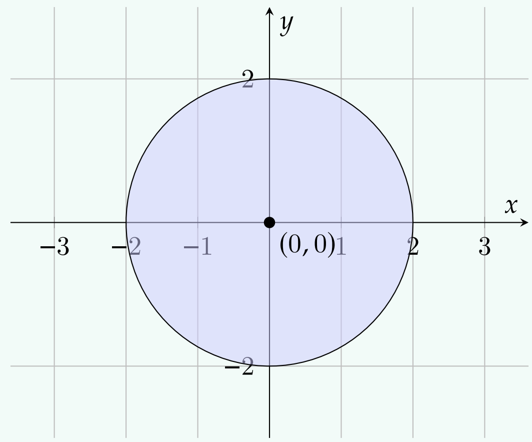
Let

$$f(x, y) = \sqrt{4 - x^2 - y^2}$$

To ensure that $f(x, y)$ is defined, the expression inside the square root must be non-negative:

$$4 - x^2 - y^2 \geqslant 0 \implies x^2 + y^2 \leqslant 4$$

This inequality describes a disk of radius 2 centered at the origin $(0, 0)$.



Therefore, the domain of $f(x, y)$ is:

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant 4\}$$

2.3 Range of Multiple Random Variable Functions

Definition 2.3.1: Range

The range of a function $f(x, y)$ is the set of all z such that $z = f(x, y)$ for some input (x, y).

Example 2.3.1 (Example of finding the range)

Consider the function

$$f(x, y) = \sqrt{4 - x^2 - y^2}$$

Evaluating f at the origin:

$$f(0, 0) = \sqrt{4 - 0^2 - 0^2} = 2$$

The maximum value of f is 2. To find the minimum, consider the boundary of the domain: If $x^2 + y^2 = 4$, then

$$f(x, y) = \sqrt{4 - (x^2 + y^2)} = \sqrt{4 - 4} = 0$$

Therefore, the range of $f(x, y)$ is:

$$\{z \in [0, 2]\}$$

Example 2.3.2 (Another example)

Let

$$f(x, y) = x^2 + y^2$$

Since $f(x, y) \geqslant 0$ for all $(x, y) \in \mathbb{R}^2$, the range of $f(x, y)$ is:

$$\{z \mid z \geqslant 0\}$$

2.4 Level Curves of Functions of Two Variables

Definition 2.4.1: Level Curves

Level curves (or contour lines) of a function $f(x, y)$ are curves in the xy-plane along which the function takes a constant value. For a given value c, a level curve is the set of points (x, y) such that:

$$f(x, y) = c$$

Example 2.4.1 (Example of level curves)

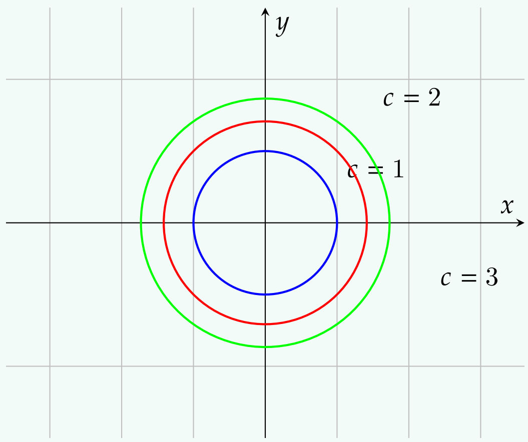
Consider the function:

$$f(x, y) = x^2 + y^2$$

To find the level curves for this function, set $f(x, y)$ equal to a constant c:

$$x^2 + y^2 = c$$

These curves are circles centered at the origin with radius $\sqrt{c}$.



In this case, each level curve is a circle centered at the origin. The radius of each circle is $\sqrt{c}$, indicating that as the value of c increases, the radius of the level curve also increases.

2.5 Limits of Two-Variable Functions

Definition 2.5.1: Limit of a Two-Variable Function

Let $f(x, y)$ be a function of two variables. The limit of $f(x, y)$ as (x, y) approaches (a, b) is denoted as:

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

This means that as the point (x, y) gets arbitrarily close to (a, b), the function values $f(x, y)$ approach the value L. For the limit to exist, the function must approach the same value L regardless of the path taken to reach (a, b).

Definition 2.5.2: Formal Definition

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that:

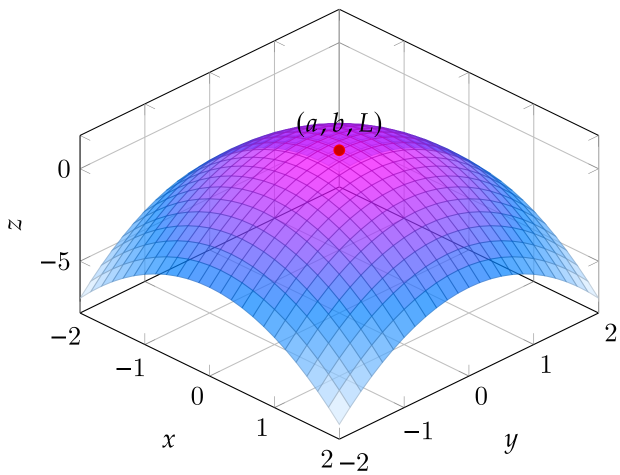
$$|f(x,y) - L| < \varepsilon$$

whenever (x,y) are in the domain of f and:

$$|(x,y) - (a,b)| < \delta$$

Visualizing Limits

The concept of limits for two-variable functions can be visualized in 3D, where $f(x,y)$ is represented as a surface over the xy-plane. The behavior of the function as (x,y) approaches a point (a,b) corresponds to how the surface approaches a specific height, L.



In this diagram, the surface represents a function $f(x,y)$, and the point (a,b,L) is the limit we are trying to approach.

Example 2.5.1 (Example of a Limit)

Consider the function:

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$$

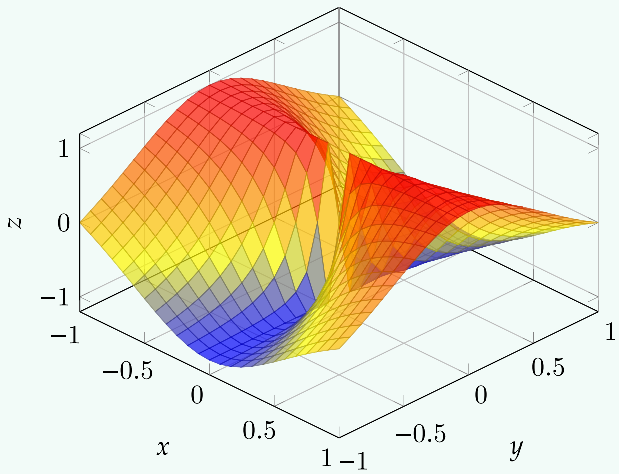
We want to find

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

Evaluating the limit along different paths:

- Along the path $y = 0$, we have $f(x,0) = \frac{x^2}{x^2} = 1$.
- Along the path $x = 0$, we have $f(0,y) = \frac{-y^2}{y^2} = -1$.

Since the limit depends on the path taken to approach $(0,0)$, the limit does not exist.



This diagram shows how the function behaves differently along different paths toward the origin, resulting in no single limit.

Example 2.5.2 (Example of a Limit that Exists)
Consider the function:

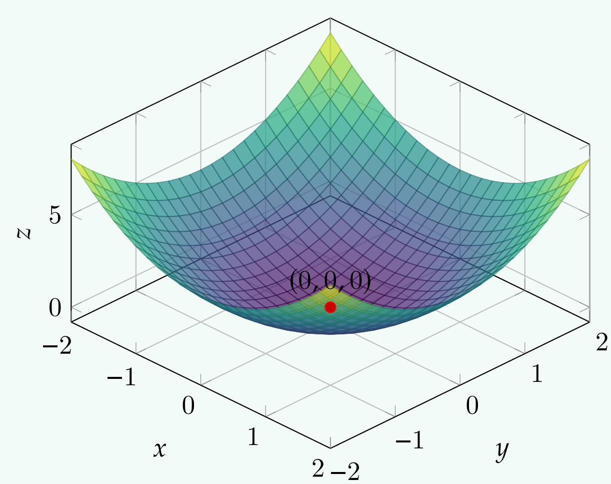
$$f(x, y) = x^2 + y^2$$

We want to find

$$\lim_{(x,y)\to(0,0)} f(x, y)$$

Since $f(x, y) = x^2 + y^2$, regardless of the path taken to approach $(0, 0)$, the value of $f(x, y)$ approaches 0. Therefore,

$$\lim_{(x,y)\to(0,0)} f(x, y) = 0.$$



2.5.1 Limits by substituting with function $f : \mathbb{R} \to \mathbb{R}$

Example 2.5.3 (Changing $g(x)$)
For a function $f(x, y)$, the limit as (x, y) approaches a point (a, b) must be the same regardless of the path taken to approach (a, b).

Consider the function:

$$f(x,y) = \frac{3xy^2}{x^2 + y^4}$$

We wish to find:

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

This limit initially appears to be of the form $\frac{0}{0}$, indicating an indeterminate form.
To proceed, we evaluate the limit along different paths. Let's test two paths to see if the limit is consistent.

Path 1: Linear Path $y = mx$

Substitute $y = mx$ into $f(x,y)$:

$$f(x,y) = \frac{3x(mx)^2}{x^2 + (mx)^4} = \frac{3m^2x^3}{x^2 + m^4x^4}$$

Then, consider the limit as $x \to 0$:

$$\lim_{x\to 0} \frac{3m^2x^3}{x^2 + m^4x^4}$$

Factor out x^2 in the denominator:

$$= \lim_{x\to 0} \frac{3m^2x}{1 + m^4x^2} = 0$$

Path 2: Parabolic Path $y = m\sqrt{x}$

Substitute $y = m\sqrt{x}$ into $f(x,y)$:

$$f(x,y) = \frac{3x(m\sqrt{x})^2}{x^2 + (m\sqrt{x})^4} = \frac{3m^2x^2}{x^2 + m^4x^2}$$

Then, consider the limit as $x \to 0$:

$$\lim_{x\to 0} \frac{3m^2x^2}{(m^4+1)x^2} = \lim_{x\to 0} \frac{3m^2}{m^4+1} = \frac{3m^2}{m^4+1} \neq 0$$

Since the limit depends on the path taken (0 for the linear path and $\frac{3m^2}{m^4+1}$ for the parabolic path), the limit does not exist:

$$\lim_{(x,y)\to(0,0)} f(x,y) \text{ DNE (Does Not Exist)}$$

2.5.2 Limits by substituting with Vector Valued Functions $f : \mathbb{R} \to \mathbb{R}^n$

Example 2.5.4 (Substituting $< g_1(t), g_2(t) >$)

Since a 1 to 1 function $g(t)$ must pass the vertical line test, it is a limiting way to approach the point at which we want to compute the limit of a two variable function. Using a vector valued function will remove this limitation and enable us to approach from any direction.
For a function $f(x,y)$, the limit as (x,y) approaches a point (a,b) must be the same regardless of the path taken to approach (a,b).
Consider the function:

$$f(x,y) = \frac{3xy^2}{x^2 + y^4}$$

We wish to find:

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

This limit initially appears to be of the form $\frac{0}{0}$, indicating an indeterminate form.
To proceed, we evaluate the limit along different paths. Let's test two paths to see if the limit is consistent.

Path 1: Parametric Linear Path $\vec{r}(t) = \langle t, mt \rangle$

Substitute $\vec{r}(t) = \langle t, mt \rangle$ into $f(x, y)$:

$$f(x, y) = \frac{3(t)(mt)^2}{t^2 + (mt)^4} = \frac{3m^2t^3}{t^2 + m^4t^4}$$

Then, consider the limit as $t \to 0$:

$$\lim_{t \to 0} \frac{3m^2t^3}{t^2 + m^4t^4}$$

Factor out t^2 in the denominator:

$$= \lim_{t \to 0} \frac{3m^2t}{1 + m^4t^2} = 0$$

Path 2: Parametric Parabolic Path $\vec{r}(t) = \langle t^2, m\sqrt{t} \rangle$

Substitute $\vec{r}(t) = \langle t^2, m\sqrt{t} \rangle$ into $f(x, y)$:

$$f(x, y) = \frac{3(t^2)(m\sqrt{t})^2}{(t^2)^2 + (m\sqrt{t})^4} = \frac{3m^2t^3}{t^4 + m^4t^2}$$

Then, consider the limit as $t \to 0$:

$$\lim_{t \to 0} \frac{3m^2t^3}{t^4 + m^4t^2}$$

Factor out t^2 in the denominator:

$$= \lim_{t \to 0} \frac{3m^2t}{t^2 + m^4} = \frac{3m^2}{m^4} = \frac{3}{m^2} \neq 0$$

Since the limit depends on the path taken (0 for the linear path and $\frac{3}{m^2}$ for the parabolic path), the limit does not exist:

$$\lim_{(x,y) \to (0,0)} f(x, y) \text{ DNE (Does Not Exist)}$$

Example 2.5.5 (Other Example)

$$f(x, y) = \frac{\sqrt[2]{x+y} + ln(\sqrt[2]{x+y})}{x + y}$$

$$f(x, y) = \begin{cases} \frac{\sqrt[2]{x+y} + ln(\sqrt[2]{x+y})}{x+y} & \text{if } x, y \neq 0 \\ L & \text{if } x = y = 0 \end{cases}$$

If

$$g(t) =< g_1(t), g_2(t) >$$

where $g_1(t) = t, g_2(t) = \sqrt{t}$
and

$$h(s) = \frac{\sqrt[2]{s} + ln(\sqrt[2]{s})}{s}$$

then

$$f(x,y) = h(g_1(x), g_2(y))$$

2.6 Tips for Finding Limits of Two-Variable Functions (2VF)

2.6.1 Polynomials are Continuous

Polynomials in two variables are always continuous everywhere in their domain. For example:

$$f(x,y) = 3x^2 + 2y^2 + 3x^4 + 1$$

is continuous for all $(x,y) \in \mathbb{R}^2$.

2.6.2 Compositions of Continuous Functions are Continuous

If $g(t)$ is a function of one variable and continuous at $t = 0$, and $f(x,y)$ is a continuous function of two variables, then $g(f(x,y))$ is also continuous at any point where $f(x,y)$ is defined. For example:

$$g(t) = \begin{cases} \frac{\sqrt{1+t}-1}{t} & \text{if } t \neq 0 \\ e & \text{if } t = 0 \end{cases}$$

is continuous everywhere.

Now consider the function:

$$f(x,y) = xy^2$$

Since $f(x,y)$ is continuous, the composition $g(f(x,y))$ is also continuous. Specifically:

$$g(f(x,y)) = \begin{cases} \frac{\sqrt{1+xy^2}-1}{xy^2} & \text{if } xy^2 \neq 0 \\ e & \text{if } xy^2 = 0 \end{cases}$$

This composition is continuous.

2.6.3 Rational Functions

A rational function $f(x,y) = \frac{P(x,y)}{Q(x,y)}$ is continuous at all points where the denominator $Q(x,y) \neq 0$. For example:

$$f(x,y) = \frac{3xy^2}{x^2 + y^4}$$

is continuous wherever $x^2 + y^4 \neq 0$.

Note:-

Tip on Factoring and Simplifying
To find the limit of a rational function, try factoring and simplifying the expression. This can help identify points of discontinuity or removable singularities.

2.6.4 Special Limits: Squeeze Theorem

The Squeeze Theorem can be used for two-variable functions. If:

$$h(x,y) \leqslant f(x,y) \leqslant g(x,y)$$

and both $h(x,y)$ and $g(x,y)$ approach the same limit L as $(x,y) \to (a,b)$, then:

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

2.6.5 Use of Polar Coordinates

For limits involving two-variable functions, converting to polar coordinates can often simplify the problem. Let:

$$x = r\cos\theta, \quad y = r\sin\theta$$

Then, $f(x, y)$ becomes a function of r and θ, which can be easier to analyze as $r \to 0$. This is particularly useful when the limit depends on the distance from the origin.

2.6.6 Testing Multiple Paths

When checking if a limit exists, test multiple paths (e.g., $y = mx$, $y = m\sqrt{x}$, $x = 0$, $y = 0$). If the limit differs for any path, then the limit does not exist. However, if all paths give the same result, it is likely that the limit exists (though more thorough testing is required for rigor).

2.6.7 Continuity of Common Functions

Most standard functions (e.g., trigonometric, exponential, logarithmic) are continuous within their domains. If they are composed with continuous functions of x and y, the result is continuous as well.

2.6.8 Rational functions

Consider a rational function of two variables in the form:

$$\frac{h(x, y)}{g(x, y)}$$

where $h(x, y)$ and $g(x, y)$ are polynomials. The behavior of the limit of this function as $(x, y) \to (a, b)$ can be classified into three cases:

2.6.9 Case 1: $g(a, b) \neq 0$

If the denominator $g(x, y)$ is non-zero at the point (a, b), i.e., $g(a, b) \neq 0$, then the function is continuous at (a, b). In this case, we have:

$$\lim_{(x,y)\to(a,b)} \frac{h(x, y)}{g(x, y)} = \frac{h(a, b)}{g(a, b)}.$$

Thus, the limit exists and equals the value of the function at the point (a, b).

2.6.10 Case 2: $g(a, b) = 0$ and $h(a, b) \neq 0$

If the denominator $g(x, y)$ equals zero at (a, b) but the numerator $h(x, y)$ does not, i.e., $h(a, b) \neq 0$, then the limit does not exist. Intuitively, this is because the denominator tends to zero while the numerator tends to a non-zero constant, leading to a blow-up of the function.

$$\lim_{(x,y)\to(a,b)} \frac{h(x, y)}{g(x, y)} \quad \text{does not exist.}$$

2.6.11 Case 3: $h(a, b) = g(a, b) = 0$

If both the numerator and the denominator are zero at (a, b), i.e., $h(a, b) = g(a, b) = 0$, we have an indeterminate form $\frac{0}{0}$. In this case, we try to simplify the expression by factoring out the common terms in $h(x, y)$ and $g(x, y)$.

Example 2.6.1 (Example)

Consider the function:

$$\frac{x^2 - y^2}{x - y}.$$

At the point $(1,1)$, both the numerator and the denominator are zero:

$$h(1,1) = g(1,1) = 0.$$

We factor the numerator:

$$x^2 - y^2 = (x-y)(x+y).$$

Now, cancel the common factor $x - y$ in the numerator and denominator:

$$\frac{x^2 - y^2}{x - y} = x + y.$$

Thus, the limit simplifies to:

$$\lim_{(x,y)\to(1,1)} \frac{x^2 - y^2}{x - y} = \lim_{(x,y)\to(1,1)} (x+y) = 1 + 1 = 2.$$

Hence, by removing the common factor, the limit can be computed as 2.

2.6.12 Case 4: $h(a,b) = g(a,b) = 0$ still

Try several Vector Valued functions, trying to make the denominator small or big compared to the numerator

Example 2.6.2 (Large denominator)

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + y^2}$$

One possible path: Let $y = 0$,

$$\frac{x^2(0)}{x^2 + 0^2} = \frac{0}{x^2} = 0$$

Now try a different path: Let $x = 0$,

$$\frac{0^2 y}{0^2 + y^2} = \frac{0}{y^2} = 0$$

Parametric form:

$$\lim_{t\to 0} f(r_1(t), r_2(t)) = \lim_{t\to 0} f(0,t) = \lim_{t\to 0} \frac{0^2 t}{0^2 + t^2} = \lim_{t\to 0} 0 = 0$$

Thus, the limit is $0 = 0$.

Example 2.6.3 (Small denominator)

$$x = -y^2 + y^n, \quad \text{where } n > 2$$

(for y small)

$$\lim_{t\to 0} f(t^2 + t^n, t) = \frac{(t^4 - 2t^{n+2} + t^{2n})t}{t^2 + t^n + t^2} = \lim_{t\to 0} \frac{t^5 - 2t^{n+3} + t^{2n+1}}{t^n} \quad (n = 5)$$

Simplifying:

$$n = 5: \quad \lim_{t\to 0} \left(1 - 2t^3 + t^6\right) = 1$$

2.6.13 Case 5: Still getting the same limit

Example 2.6.4 (Trying)

Let $\lim_{(x,y)\to(0,0)} = \frac{xy^2}{x^2+y^2}$.
If we divide by y^2 we get

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x}{(\frac{x^2}{y^2})+1} = 0$$

2.7 Properties of Limits of Two-Variable Functions

Let $f(x,y)$ and $g(x,y)$ be two functions of two variables such that:

$$\lim_{(x,y)\to(a,b)} f(x,y) = L_1, \qquad \lim_{(x,y)\to(a,b)} g(x,y) = L_2$$

Then, the following properties hold:

2.7.1 Scaling by a constant:

$$\lim_{(x,y)\to(a,b)} [kf(x,y)] = kL_1, \quad \text{where } k \text{ is any real constant}$$

2.7.2 Addition of functions:

$$\lim_{(x,y)\to(a,b)} [f(x,y)+g(x,y)] = L_1 + L_2$$

2.7.3 Multiplication of functions:

$$\lim_{(x,y)\to(a,b)} [f(x,y)g(x,y)] = L_1L_2$$

2.7.4 Division of functions:

$$\lim_{(x,y)\to(a,b)} \left[\frac{f(x,y)}{g(x,y)}\right] = \frac{L_1}{L_2}, \quad \text{provided } L_2 \neq 0$$

2.7.5 Other

$$\lim_{(x,y)\to(a,b)} \big(f(x,y)\big)^n = L^n \quad (n = 0,1,2,3,\dots)$$

$$\lim_{(x,y)\to(a,b)} \big(f(x,y)\big)^{1/n} = L^{1/n} \quad \text{if } n \text{ is even, } L \geqslant 0$$

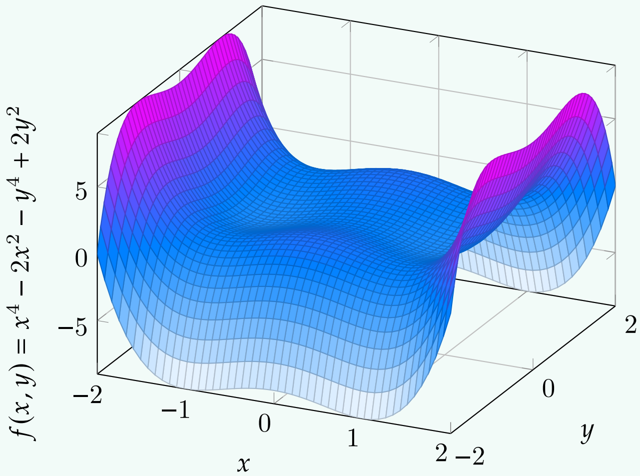
2.8 Partial Derivatives

Definition 2.8.1: Partial Derivative of random two variable function

$$\frac{\partial}{\partial x} f(x,y) = lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$\frac{\partial}{\partial y} f(x,y) = lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

Example 2.8.1 (Example of application of partial derivatives)



The partial derivative of a function $f(x, y)$ with respect to x at a point $P = (x_0, y_0)$ is defined as:

$$f_x(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

Similarly, the partial derivative with respect to y at $P = (x_0, y_0)$ is:

$$f_y(x_0, y_0) = \lim_{k\to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k}$$

For the function $f(x, y) = x^4 - 2x^2 - y^4 + 2y^2$, the partial derivatives are:

$$f_x(x, y) = 4x^3 - 4x, \quad f_y(x, y) = -4y^3 + 4y$$

2.8.1 Differentiability

Theorem 2.8.1 Conditions for Differentiability

Consider a function $f(x, y)$, and we aim to analyze the differentiability at a point (a, b).
To say that $f(x, y)$ is differentiable at the point (a, b), the following conditions must hold:

1. The partial derivatives $f_x(a, b)$ and $f_y(a, b)$ must exist.
2. We must have the following expression for the increment of f at (a, b):

$$f(a + \Delta x, b + \Delta y) - f(a, b) = \Delta x f_x(a, b) + \Delta y f_y(a, b) + \epsilon_1(\Delta x)\Delta x + \epsilon_2(\Delta y)\Delta y$$

   where $\epsilon_1(\Delta x) \to 0$ as $\Delta x \to 0$ and $\epsilon_2(\Delta y) \to 0$ as $\Delta y \to 0$.

This form shows that the change in the function can be approximated by the linear part involving the partial derivatives, with the higher-order terms represented by ϵ_1 and ϵ_2, which tend to zero.

Graphical Interpretation

Consider the following illustration of the change in the function f:

$$\text{Elevation of } f(a, b) \to f(a + \Delta x, b + \Delta y)$$

The increment of f is given by:

$$f(a + \Delta x, b + \Delta y) - f(a, b)$$

Using the definition of differentiability, we can decompose this increment as follows:

$$\Delta f = \Delta x f_x(a,b) + \Delta y f_y(a,b)$$

Graphically, this can be interpreted as moving along the tangent plane at the point (a,b) in the direction of Δx and Δy. The elevation changes linearly based on the partial derivatives.

For the function to be differentiable at (a,b), we require that:

$$\lim_{\Delta x \to 0} \epsilon_1(\Delta x) = 0, \quad \lim_{\Delta y \to 0} \epsilon_2(\Delta y) = 0$$

These conditions ensure that the error terms involving ϵ_1 and ϵ_2 vanish as the changes Δx and Δy become small, ensuring a linear approximation near the point (a,b).

2.8.2 Open and Closed Sets

Consider the set R defined by:

$$R : y < x, \quad x^2 + y^2 < 1$$

The set R is open if we can draw it using **dashed lines**, indicating that the boundary is not included in the set. If we have a set of solutions to inequalities with strict inequalities (e.g., $<$ or $>$), the set is typically **open**.

If R is defined by inequalities involving $\leqslant$ or $\geqslant$, then the set is **closed**.

Example of Open and Closed Sets

Let R_1 be defined as:

$$R_1 : x^2 + y^2 < 1 \quad \text{(open)}$$

Let S be defined as:

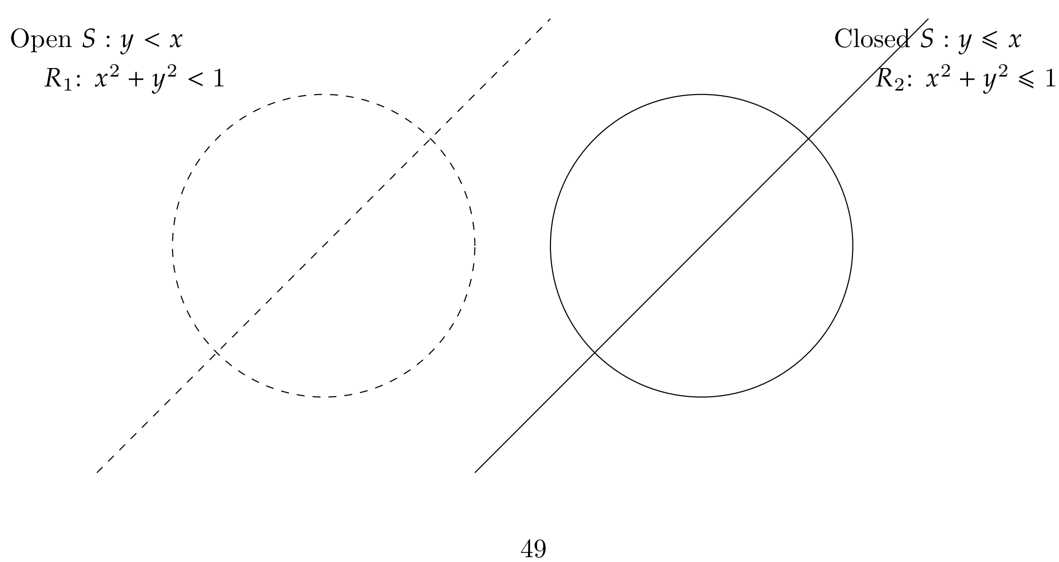
$$S : y < x \quad \text{(open)}$$

We can also consider the closed sets:

$$R_2 : x^2 + y^2 \leqslant 1 \quad \text{(closed)}$$

$$S : y \leqslant x \quad \text{(closed)}$$

Below are illustrations of open and closed sets. Dotted lines indicate that the boundary is **not included** (open), while solid lines indicate the boundary **is included** (closed).



2.8.3 Conditions for Differentiability

A function $f(x, y)$ is differentiable at (a, b) if:

1. The partial derivatives $f_x(a, b)$ and $f_y(a, b)$ exist.
2. The following limit holds:

$$f(a + \Delta x, b + \Delta y) - f(a, b) = \Delta x f_x(a, b) + \Delta y f_y(a, b) + \Delta x \epsilon_1(\Delta x) + \Delta y \epsilon_2(\Delta y)$$

   where $\epsilon_1(\Delta x) \to 0$ as $\Delta x \to 0$ and $\epsilon_2(\Delta y) \to 0$ as $\Delta y \to 0$.

Theorem 2.8.2 Sufficient Conditions for Differentiability

A sufficient condition for differentiability is the following theorem:
If there exists an open set D containing (a, b) such that $f_x(x, y)$ and $f_y(x, y)$ exist at all points in D, and if $f_x(a, b)$ and $f_y(a, b)$ are continuous at (a, b), then $f(x, y)$ is differentiable at (a, b).

Example 2.8.2 (Differentiability of a Given Function)

Consider the function:

$$f(x, y) = e^{xy} + x^3 y$$

We want to check whether this function is differentiable at any point (a, b). First, we compute the partial derivatives:

$$f_x(x, y) = ye^{xy} + 3x^2 y$$
$$f_y(x, y) = xe^{xy} + x^3$$

For any point (a, b), both $f_x(a, b)$ and $f_y(a, b)$ exist in $\mathbb{R}^2$ and are continuous at (a, b) because the terms e^{xy}, x^3, and y are all continuous on $\mathbb{R}^2$.
Thus, according to the theorem for differentiability, we conclude that:

$$f(x, y) \text{ is differentiable at } (a, b)$$

Example 2.8.3 (Discontinuity of a Function)

Consider the function:

$$f(x, y) = \begin{cases} \frac{3xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

We are interested in determining whether this function is continuous and differentiable at the point $(0, 0)$.

Continuity Check

We first evaluate $f(0, 0)$:

$$f(0, 0) = 0$$

Next, we compute the limit of $f(x, y)$ as $(x, y) \to (0, 0)$. To do this, we examine the limit along two different paths:

$$r_1(t) = \langle t, 0 \rangle$$

Along this path, we have:

$$\lim_{t \to 0} f(t, 0) = \lim_{t \to 0} \frac{3t \cdot 0}{t^2 + 0^2} = 0$$

Now consider the path:

$$r_2(t) = \langle t, t \rangle$$

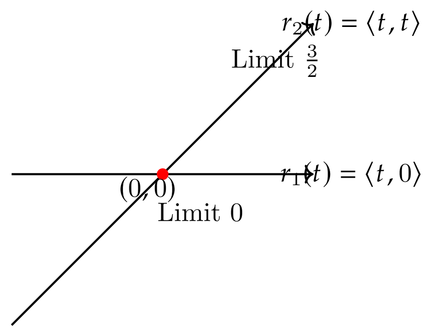
Along this path, we compute:

$$\lim_{t \to 0} f(t,t) = \lim_{t \to 0} \frac{3t \cdot t}{t^2 + t^2} = \frac{3t^2}{2t^2} = \frac{3}{2}$$

Since the limits along different paths do not agree (one is 0, and the other is $\frac{3}{2}$), we conclude that:

$$f(x,y) \text{ is not continuous at } (0,0)$$

Thus, even though the partial derivatives f_x and f_y exist at $(0,0)$, the function is not continuous, and therefore, $f(x,y)$ is not differentiable at $(0,0)$.



Theorem 2.8.3 Differentiability Implies Continuity

If $f(x,y)$ is differentiable at (a,b), then $f(x,y)$ is also continuous at (a,b). This is an important result that will help us analyze the next example.

2.9 Directional Derivative

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector, i.e., $|\mathbf{u}| = 1$. The directional derivative of $f(x,y)$ at the point (a,b) in the direction of $\mathbf{u}$ is defined as:

$$D_{\mathbf{u}} f(a,b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a,b)}{h}$$

This represents the slope of the hill that you are traveling on in the direction of $\mathbf{u}$.
If $f(x,y)$ is differentiable at (a,b), we know that the function f satisfies the following approximation:

$$f(a + \Delta x, b + \Delta y) - f(a,b) = \Delta x f_x(a,b) + \Delta y f_y(a,b) + \Delta x \epsilon_1(\Delta x) + \Delta y \epsilon_2(\Delta y)$$

where $\epsilon_1(\Delta x) \to 0$ as $\Delta x \to 0$ and $\epsilon_2(\Delta y) \to 0$ as $\Delta y \to 0$.

2.9.1 Derivation of the Directional Derivative

Substituting $\Delta x = hu_1$ and $\Delta y = hu_2$, the directional derivative becomes:

$$\lim_{h \to 0} \frac{hu_1 f_x(a,b) + hu_2 f_y(a,b) + hu_1 \epsilon_1(h) + hu_2 \epsilon_2(h)}{h}$$

Simplifying the expression:

$$= u_1 f_x(a,b) + u_2 f_y(a,b) + u_1 \epsilon_1(h) + u_2 \epsilon_2(h)$$

As $h \to 0$, the terms $u_1 \epsilon_1(h)$ and $u_2 \epsilon_2(h)$ tend to 0, so we are left with:

$$D_{\mathbf{u}} f(a,b) = u_1 f_x(a,b) + u_2 f_y(a,b)$$

This can be expressed compactly as the dot product:

$$D_{\mathbf{u}} f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle \cdot \mathbf{u}$$

Definition 2.9.1: Definition of the Gradient

The gradient of $f(x,y)$ at the point (a,b) is defined as the vector:

$$\nabla f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle$$

The directional derivative is then the dot product of the gradient and the direction vector $\mathbf{u}$:

$$D_{\mathbf{u}} f(a,b) = \nabla f(a,b) \cdot \mathbf{u}$$

Thus, the gradient gives the direction of the steepest ascent of the function, and the directional derivative gives the rate of change of the function in the direction of $\mathbf{u}$.

2.10 Application of the Gradient

The gradient $\nabla f(a,b)$ plays an important role in determining the behavior of the function in various directions.

2.10.1 Directional Derivative and the Gradient

If f is differentiable at (a,b), then for any unit vector $\mathbf{u}$, the directional derivative of f in the direction of $\mathbf{u}$ is given by:

$$D_{\mathbf{u}} f(a,b) = \nabla f(a,b) \cdot \mathbf{u}$$

This formula can be used to determine how the function changes as we move in the direction of $\mathbf{u}$.

2.10.2 Dot Product and Angle

We know from vector calculus that:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$$

Thus, for the directional derivative:

$$D_{\mathbf{u}} f(a,b) = |\nabla f(a,b)||\mathbf{u}| \cos\theta$$

where θ is the angle between the gradient vector $\nabla f(a,b)$ and the direction vector $\mathbf{u}$.

- If $\theta > \frac{\pi}{2}$, then $\mathbf{u}$ points downhill (negative directional derivative).
- If $0 < \theta < \frac{\pi}{2}$, then $\mathbf{u}$ points uphill (positive directional derivative).

2.10.3 Maximum and Minimum Values of the Directional Derivative

The directional derivative $D_{\mathbf{u}} f(a,b)$ is maximized when $\theta = 0°$, which means that $\mathbf{u}$ points in the direction of the steepest ascent. This happens when $\mathbf{u}$ is in the same direction as $\nabla f(a,b)$:

$$\nabla f(a,b) \text{ points towards the direction of steepest ascent.}$$

Conversely, the directional derivative is minimized when $\theta = 180°$, which means $\mathbf{u}$ points in the direction of steepest descent. In this case:

$$-\nabla f(a,b) \text{ points towards the direction of steepest descent.}$$

This forms the foundation of gradient descent algorithms, where the negative gradient is used to find minimum values.

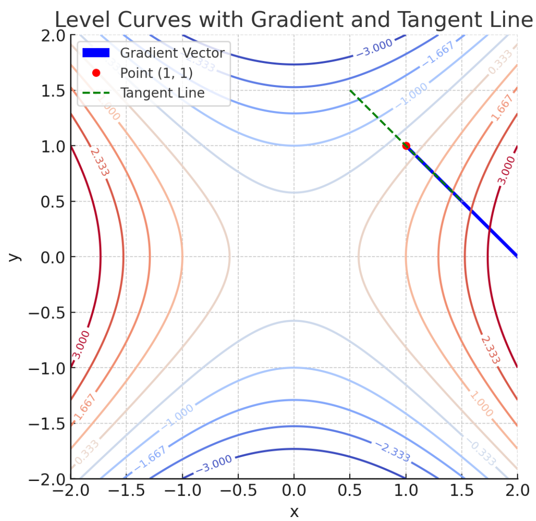
2.10.4 Tangent to a Level Curve

Consider a level curve of the form $f(x, y) = c$. The vector $\nabla f(a, b)$ is perpendicular to the level curve at the point (a, b), which implies that any vector $\mathbf{u}$ tangent to the level curve satisfies:

$$D_{\mathbf{u}} f(a, b) = 0$$

This means that if $\mathbf{u}$ is tangent to the level curve, the directional derivative is zero, implying that there is no change in the value of the function along that curve.

Below is a diagram representing the gradient and the tangents to the level curves:



2.11 Tangent Planes

2.11.1 Tangent Line for Single Variable Functions

For a function of a single variable, the equation of the tangent line at a point $f(a)$ is given by:

$$y = f'(a)(x - a) + f(a)$$

This represents the line that touches the curve $y = f(x)$ at the point $(a, f(a))$, and the slope of this line is determined by the derivative $f'(a)$.

2.11.2 Tangent Plane for Two Variable Functions

For functions of two variables, i.e. $f(x, y)$, the tangent plane at a point $P(a, b, c)$ on the surface can be described. Let the function be $f(a, b) = c$, and we define a point $P(a, b, c)$ that lies on the surface at the tangent point.

The tangent plane equation is derived using partial derivatives and parameter vectors.

Parameter Vector Representation

Let the point on the surface be $P = (a, b, c)$ and consider small increments in x and y, denoted Δx and Δy, respectively. The change in the function value Δz due to these increments can be approximated using the partial derivatives. The tangent plane is spanned by vectors in the x and y directions, leading to:

$$\vec{v} = \langle v_1, v_2 \rangle$$

where:

$$v_1 = \frac{\partial f}{\partial x}(a, b) \quad \text{and} \quad v_2 = \frac{\partial f}{\partial y}(a, b)$$

The equation of the plane then involves computing how much z increments based on x and y, giving us the tangent plane equation:

$$z = c + \nabla f(a, b) \cdot \langle x - a, y - b \rangle$$

where $\nabla f(a,b)$ represents the gradient vector of the function $f(x,y)$ at the point (a,b).

Definition 2.11.1: The Tangent Plane Equation

The explicit equation of the tangent plane can be written as:

$$\frac{\partial f}{\partial x}(a,b)(x-a)+\frac{\partial f}{\partial y}(a,b)(y-b)+z-c=0$$

This expresses how the surface locally behaves near the point (a,b,c), and the coefficients of $x-a$ and $y-b$ are given by the partial derivatives of f at (a,b).

Example 2.11.1 (1)

We are tasked with finding the tangent plane of the surface $z=f(x,y)=\sin(x)\cdot e^y$ at the point $\left(\frac{\pi}{2},0,1\right)$. The general form of the equation of a tangent plane to a surface $z=f(x,y)$ is given by:

$$\frac{\partial f}{\partial x}(a,b)(x-a)+\frac{\partial f}{\partial y}(a,b)(y-b)+z-c=0$$

We will follow the steps below to compute the tangent plane:

1. **Find the point of tangency (a,b,c):**

   The point of tangency is given as $\left(\frac{\pi}{2},0,1\right)$, so:

   $$a=\frac{\pi}{2},\quad b=0,\quad c=1$$

2. **Compute the partial derivative with respect to x:**

   The partial derivative of $f(x,y)=\sin(x)\cdot e^y$ with respect to x is:

   $$\frac{\partial f}{\partial x}(x,y)=\cos(x)\cdot e^y$$

   At the point $\left(\frac{\pi}{2},0\right)$, this becomes:

   $$\frac{\partial f}{\partial x}\left(\frac{\pi}{2},0\right)=\cos\left(\frac{\pi}{2}\right)\cdot e^0=0\cdot 1=0$$

3. **Compute the partial derivative with respect to y:**

   The partial derivative of $f(x,y)=\sin(x)\cdot e^y$ with respect to y is:

   $$\frac{\partial f}{\partial y}(x,y)=\sin(x)\cdot e^y$$

   At the point $\left(\frac{\pi}{2},0\right)$, this becomes:

   $$\frac{\partial f}{\partial y}\left(\frac{\pi}{2},0\right)=\sin\left(\frac{\pi}{2}\right)\cdot e^0=1\cdot 1=1$$

4. **Substitute into the tangent plane equation:**

   Using the general form of the tangent plane equation and substituting $a=\frac{\pi}{2}$, $b=0$, $c=1$, as well as the values of the partial derivatives $\frac{\partial f}{\partial x}(a,b)=0$ and $\frac{\partial f}{\partial y}(a,b)=1$, we have:

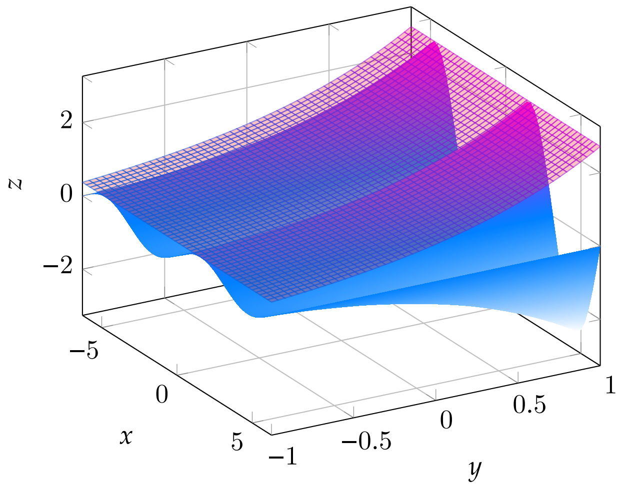
   $$0\cdot(x-\frac{\pi}{2})+1\cdot(y-0)+z-1=0$$

Simplifying the equation:

$$y + z - 1 = 0$$

5. **Final equation of the tangent plane:**
   Thus, the equation of the tangent plane at $\left(\frac{\pi}{2}, 0, 1\right)$ is:

$$y + z - 1 = 0$$



2.11.3 Implicit Case

When dealing with implicit functions such as $f(x, y) = D$, the process involves implicit differentiation to find the slope of the tangent line. For example, given an equation of the form:

$$x^2 + y^2 = 4$$

we differentiate implicitly to obtain the tangent line equation. The process is as follows:

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

For the given equation $x^2 + y^2 = 4$, we compute:

$$\frac{dy}{dx} = -\frac{x}{y}$$

leading to the equation of the tangent line:

$$y - b = -\frac{F_x}{F_y}(x - a)$$

Definition 2.11.2: Tangent Line for Implicit Functions

Finally, for the tangent line in the implicit case, we use the gradient $\nabla f(a, b)$ to express the equation of the tangent line:

$$\nabla f(a, b) \cdot \langle x - a, y - b \rangle = 0$$

This represents a line tangent to the curve at the point (a, b) on the implicit function.

2.11.4 Level Surfaces and Tangent Planes

A **level surface** is represented by the equation $w = f(x, y, z)$, where w is a constant, indicating a 3D surface embedded in 4D space. For example:

$$f(x, y, z) = D, \quad D \in \mathbb{R}$$

This represents a 2D surface in 3D space. An example is the implicit equation:

$$x^2 + y^2 + z^2 = 4$$

which describes a sphere of radius 2 in 3D space.

Definition 2.11.3: Level Curves Tangent Plane

The tangent plane to the surface is given by:

$$\nabla F(a, b, c) \cdot \langle x - a, y - b, z - c \rangle = 0$$

where $\nabla F(a, b, c)$ is the gradient of the function F at the point (a, b, c).

2.11.5 Summary of Cases

Curve/Surface	Equation
$y = f(x)$	$y = f'(a)(x - a) + f(a)$
$z = f(x, y)$	$z = c + \nabla f(a, b) \cdot \langle x - a, y - b \rangle$
$F(x, y, z) = D$	$\nabla F(a, b, c) \cdot \langle x - a, y - b, z - c \rangle = 0$

Example 2.11.2 (1)

Consider the surface given by:

$$z = f(x, y) = x^2y^2 - 2x$$

At the point $(1, 3, 7)$, we compute the gradient:

$$\nabla f(x, y) = \langle y^2 2x - 2, x^2 2y \rangle$$

At the point $(1, 3)$:

$$\nabla f(1, 3) = \langle 9 - 2, 6 \rangle = \langle 7, 6 \rangle$$

The equation of the tangent plane is:

$$7(x - 1) + 6(y - 3) + z - 7 = 0$$

Simplifying this gives:

$$7(x - 1) + 6(y - 3) + z = 7$$

Example 2.11.3 (2)

Now, consider the implicit surface given by:

$$F(x, y, z) = x^3 - 3y + z^2 = 8$$

At the point $(1, 2, 3)$, the gradient is:

$$\nabla F(x, y, z) = \langle 3x^2, -3, 2z \rangle$$

At the point $(1, 2, 3)$, this evaluates to:

$$\nabla F(1, 2, 3) = \langle 3, -3, 6 \rangle$$

The tangent plane equation is:

$$3(x - 1) - 3(y - 2) + 6(z - 3) = 0$$

Simplifying:

$$3(x-1) - 3(y-2) + 6(z-3) = 0$$

$$3(x-1) - 3(y-2) + 6(z-3) = 0$$

This describes the tangent plane to the surface at the given point.

2.12 Applications of Tangent Planes

2.12.1 Linear Approximation

In many cases, we know the value of the function $f(a,b)$ at a particular point and want to estimate the value of $f(x,y)$ near that point. We use the linear approximation formula for this estimation:

$$f(x,y) \approx f(a,b) + \nabla f(a,b) \cdot \langle x-a, y-b \rangle$$

where $\nabla f(a,b)$ is the gradient of the function f at the point (a,b).

Example 2.12.1 (Linear Approximation of $\sin(xy)$)
Let $f(x,y) = \sin(xy)$. We want to estimate $f(0.8, 0.25)$ using linear approximation.

1. First, find $f(a,b) = f(\pi/2, 0.2)$. Since $\sin(\frac{\pi}{2} \cdot 0.2) \approx 0.985$, we get:

$$f(a,b) = \sin\left(\frac{\pi}{2} \times 0.2\right) \approx 0.985$$

2. The gradient $\nabla f(x,y)$ is given by:

$$\nabla f(x,y) = \langle y\cos(xy), x\cos(xy) \rangle$$

At $(\frac{\pi}{2}, 0.2)$, we calculate the partial derivatives:

$$\nabla f\left(\frac{\pi}{2}, 0.2\right) = \langle 0.2\cos(1), \frac{\pi}{2}\cos(1) \rangle$$

Using $\cos(1) \approx 0.5403$, we have:

$$\nabla f\left(\frac{\pi}{2}, 0.2\right) = \langle 0.1081, 0.8474 \rangle$$

3. Now, estimate $f(0.8, 0.25)$ using the formula for linear approximation:

$$f(0.8, 0.25) \approx f\left(\frac{\pi}{2}, 0.2\right) + \nabla f\left(\frac{\pi}{2}, 0.2\right) \cdot \langle 0.8 - \frac{\pi}{2}, 0.25 - 0.2 \rangle$$

Simplifying:

$$f(0.8, 0.25) \approx 0.985 + \langle 0.1081, 0.8474 \rangle \cdot \langle -0.014, 0.05 \rangle$$

$$f(0.8, 0.25) \approx 0.985 + (-0.0015 + 0.0424)$$

$$f(0.8, 0.25) \approx 0.985 + 0.0409 = 1.0259$$

2.12.2 Finding Approximate Solutions to Equations

To find approximate solutions to equations, we can apply linearization techniques similar to the above. For instance, if we know a point (a,b,c) and want to estimate the solution for $f(x,y,z)$, we use the following approximation:

$$f(x,y,z) \approx f(a,b,c) + \nabla f(a,b,c) \cdot \langle x-a, y-b, z-c \rangle$$

where $\nabla f(a,b,c)$ is the gradient of the function at the point (a,b,c).

Example 2.12.2 (Finding the Root of $f(z) = ze^z + 1.02z - e^z - 0.2$)

To find approximate solutions to equations, we can apply linearization techniques similar to the above. For instance, if we know a point (a, b, c) and want to estimate the solution for $f(x, y, z)$, we use the following approximation:

$$f(x, y, z) \approx f(a, b, c) + \nabla f(a, b, c) \cdot \langle x - a, y - b, z - c\rangle$$

where $\nabla f(a, b, c)$ is the gradient of the function at the point (a, b, c).

2.12.3 Example: Linearizing $f(x, y, z) = z\ln(z) + xz - e^{yz}$

Consider the function $f(x, y, z) = z\ln(z) + xz - e^{yz}$. We want to find an approximate solution around the point $(x, y, z) = (1, 0, 1)$.
First, we calculate the value of the function at this point:

$$f(1, 0, 1) = 1\ln(1) + 1 \cdot 1 - e^{0 \cdot 1} = 0 + 1 - 1 = 0.$$

Next, we compute the gradient $\nabla f(x, y, z)$ at the point $(1, 0, 1)$. The gradient is given by:

$$\nabla f(x, y, z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

The partial derivatives are calculated as follows:

$$\frac{\partial f}{\partial x} = z, \quad \frac{\partial f}{\partial y} = -ze^{yz}, \quad \frac{\partial f}{\partial z} = \ln(z) + 1 + x - ye^{yz}.$$

Evaluating these at $(x, y, z) = (1, 0, 1)$:

$$\frac{\partial f}{\partial x}(1, 0, 1) = 1, \quad \frac{\partial f}{\partial y}(1, 0, 1) = -1, \quad \frac{\partial f}{\partial z}(1, 0, 1) = \ln(1) + 1 + 1 = 2.$$

Thus, the gradient at $(1, 0, 1)$ is:

$$\nabla f(1, 0, 1) = (1, -1, 2).$$

Using the linear approximation formula:

$$f(x, y, z) \approx f(1, 0, 1) + \nabla f(1, 0, 1) \cdot \langle x - 1, y, z - 1\rangle,$$

we substitute the values:

$$f(x, y, z) \approx 0 + (1)(x - 1) + (-1)(y) + (2)(z - 1).$$

Simplifying:

$$f(x, y, z) \approx x - 1 - y + 2(z - 1).$$

$$f(x, y, z) \approx x + 2z - y - 3.$$

Thus, the linearized approximation of $f(x, y, z)$ around $(1, 0, 1)$ is:

$$f(x, y, z) \approx x + 2z - y - 3.$$

2.13 Chain Rule in Functions of several random variables $f : \mathbb{R}^n \to \mathbb{R}$

Claim 2.13.1 Differentiation of functions

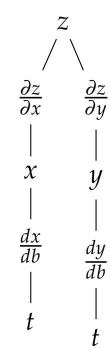
Let z be a differentiable function of x and y on an interval domain, where x and y are differentiable functions of t on interval I. Then:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

We sum the rates of change of z through x and y.

Definition 2.13.1: Single Variable

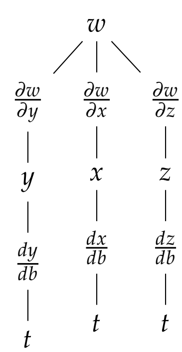
For single-variable functions, we use $\frac{d}{dt}$; for multiple variables, we use $\frac{\partial}{\partial t}$.



Claim 2.13.2 Three Variables

If z is a function of x, y, and t, we have:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} + \frac{\partial z}{\partial t}$$



Example 2.13.1 (Example 1)

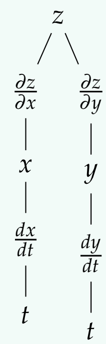
Given $z = x\sin(y),\ x = t^2,\ y = t^3$:

$$\frac{\partial z}{\partial x} = \sin(y), \quad \frac{\partial z}{\partial y} = x\cos(y)$$

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 3t^2$$

Therefore,

$$\frac{dz}{dt} = \sin(t^3)(2t) + t^2\cos(t^3)(3t^2) = 2t\sin(t^3) + 3t^4\cos(t^3)$$



Example 2.13.2 (Example 2)

Given $V = xyz,\ x = e^t,\ y = 2t+3,\ z = \sin(t)$:

$$\frac{\partial V}{\partial x} = yz, \quad \frac{\partial V}{\partial y} = xz, \quad \frac{\partial V}{\partial z} = xy$$

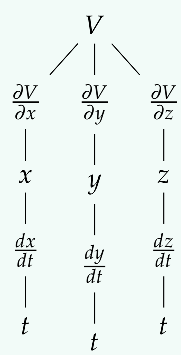
$$\frac{dx}{dt} = e^t, \quad \frac{dy}{dt} = 2, \quad \frac{dz}{dt} = \cos(t)$$

Therefore,

$$\frac{dV}{dt} = yz \cdot e^t + xz \cdot 2 + xy \cdot \cos(t)$$

Substituting:

$$\frac{dV}{dt} = (2t+3)\sin(t) \cdot e^t + (e^t)\sin(t) \cdot 2 + (e^t)(2t+3) \cdot \cos(t)$$

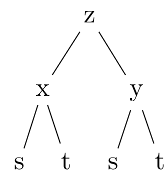


Claim 2.13.3 Two Independent Variables

If $z = f(x,y), x = g(s,t), y = h(s,t)$, then:

$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$



Claim 2.13.4 Implicit Differentiation

Let f be a differentiable function on its domain, and suppose $f(x, y) = 0$ defines y as a differentiable function of x. Provided $F_y \neq 0$, we have:

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Example 2.13.3 (Example)

Given:

$$\frac{dy}{dx} \text{ if } ye^{xy} - 2 = 0$$

We have:

$$F_x = y^2 e^{xy}, \quad F_y = xye^{xy} + e^{xy}$$

Therefore:

$$\frac{dy}{dx} = -\frac{y^2 e^{xy}}{xye^{xy} + e^{xy}} = \frac{-y^2}{x + y}$$