Temporary Doc
Calc 3

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Contents

Chapter 1

Vector Valued Functions $f : \mathbb{R} \longrightarrow \mathbb{R}^n$

1.1 Arc Length in Polar Coordinates

The position vector in polar coordinates is given by:

$$\vec{r}(t) = \langle f(\theta)\cos\theta, f(\theta)\sin\theta \rangle$$

The magnitude of the derivative of the position vector is:

$$|\vec{r}'(t)| = \sqrt{\left(f'(\theta)\cos\theta - f(\theta)\sin\theta\right)^2 + \left(f'(\theta)\sin\theta + f(\theta)\cos\theta\right)^2}$$

Simplifying, this becomes:

$$|\vec{r}'(t)| = \sqrt{\left(f'(\theta)\right)^2 + \left(f(\theta)\right)^2}$$

The arc length S_r is then calculated as:

$$S_r = \int_{\theta_1}^{\theta_2} \sqrt{\left(f'(\theta)\right)^2 + \left(f(\theta)\right)^2}\, d\theta$$

Example 1.1.1 (Circle of Radius a)

Example: Let $r(\theta) = a$, where:

$$r'(\theta) = 0, \quad r(\theta) = a$$

The arc length is:

$$S = \int_0^{2\pi} \sqrt{a^2}\, d\theta = a \int_0^{2\pi} 1\, d\theta = 2\pi a$$

1.2 Multivariate Case

The double integral over a region R:

$$\iint_R f(x, y)\, dA$$

If converted to polar coordinates:

$$\iint_R f(x, y)\, dA = \iint_R f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Example 1.2.1 (Volume under $z = 9 - x^2 - y^2$:)

$$R = \{(x, y) : x^2 + y^2 \leqslant 9\}$$

In polar coordinates:

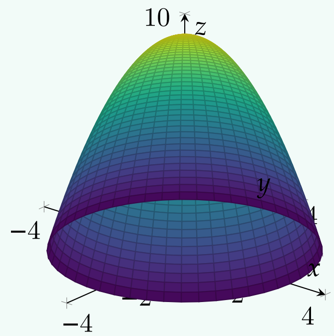
$$R = \{(r, \theta) : 0 \leqslant \theta \leqslant 2\pi, 0 \leqslant r \leqslant 3\}$$

The function in polar coordinates:

$$f(r, \theta) = 9 - r^2$$

The volume is then:

$$\int_R f(r, \theta)\, dA = \int_0^{2\pi} \int_0^3 r(9 - r^2)\, dr\, d\theta$$



Simplified Polar Volume Calculation:

$$\int_R f(r, \theta)\, dA = \int_0^{2\pi} \int_0^3 r(9 - r^2)\, dr\, d\theta$$

1.3 General Procedure for Polar Integration

Given a region $R = \{(r, \theta) : \theta_1 \leqslant \theta \leqslant \theta_2,\ q(\theta) \leqslant r \leqslant h(\theta)\}$, the volume is calculated as:

$$V = \iint_R f(r, \theta)\, dA = \int_{\theta_1}^{\theta_2} \int_{q(\theta)}^{h(\theta)} r f(r, \theta)\, dr\, d\theta$$

where the area element is $\Delta A = r\, \Delta r\, \Delta\theta$.

Example 1.3.1 (Example: Region Outside $r = \frac{1}{2}$ and Inside $r = 1 + \cos\theta$)

The bounds are:

$$\frac{1}{2} \leqslant r \leqslant 1 + \cos\theta$$

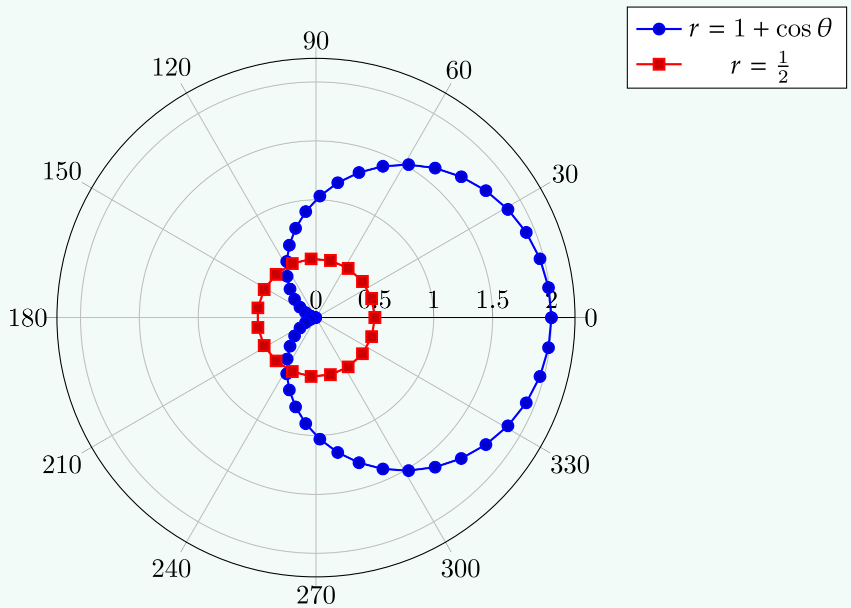
$$-\frac{2\pi}{3} \leqslant \theta \leqslant \frac{2\pi}{3}$$

Cosine conditions are determined by:

$$\cos\theta = -\frac{1}{2}, \quad \theta = -\frac{2\pi}{3}, \frac{2\pi}{3}$$

The integral for the volume is:

$$\iint_R q(r, \theta)\, dA = \int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} \int_{\frac{1}{2}}^{1+\cos\theta} r\, q(r, \theta)\, dr\, d\theta$$


$r = 1 + \cos\theta$
$r = \frac{1}{2}$
90
60
30
0
330
300
270
240
210
180
150
120
0
0.5
1
1.5
2