CASMA 225
Calc 3

Giacomo Cappelletto

4/9/24

Contents

Chapter 1

Vectors

1.1 Review

1.1.1 Basics

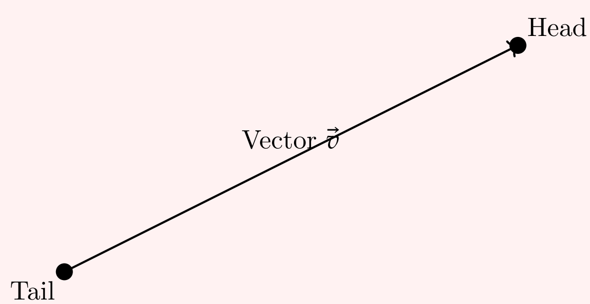
Definition 1.1.1: Notation

Drawn Vectors: $\vec{v}$
Typed Vectors: $\mathbf{v}$

Definition 1.1.2: Velocity

Magnitude of the velocity: $|\vec{v}|$
Direction of the velocity: $dir(\vec{v})$

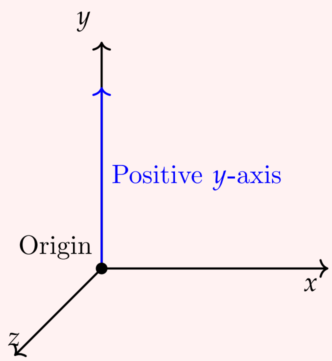
Definition 1.1.3: Heads and Tails



Note:-
Scalar is like a 1 directional vector, either positive or negative, and its magnitude is the absolute value of the scalar

1.1.2 Notation

Definition 1.1.4: Positive y axis



Definition 1.1.5: Standard Basis Vecotrs

In an n-dimensional space $\mathbb{R}^n$, the standard basis vectors are a set of n vectors where each vector has a 1 in one component and 0 in all other components. These vectors are denoted as $\mathbf{e}_i$ for $i = 1, 2, \ldots, n$. The i-th standard basis vector in $\mathbb{R}^n$ is written as:

$$\mathbf{e}_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \quad \text{(in the } i\text{-th position)} \\ \vdots \\ 0 \end{pmatrix}$$

For example, in $\mathbb{R}^3$ (three-dimensional space), the standard basis vectors are:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

These vectors span the entire vector space $\mathbb{R}^n$, meaning any vector $\mathbf{v} \in \mathbb{R}^n$ can be written as a linear combination of the standard basis vectors:

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n,$$

where $v_1, v_2, \ldots, v_n$ are the components of the vector $\mathbf{v}$.

1.2 Operations

1.2.1 Dot Product

Definition 1.2.1: Dot (Scalar)Product Definitons

The **scalar product** (or **dot product**) of two vectors **a** and **b** in $\mathbb{R}^n$ is defined as:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

In $\mathbb{R}^3$, for vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, the dot product is:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

The dot product can also be expressed in terms of the magnitudes of **a** and **b** and the angle θ between them:

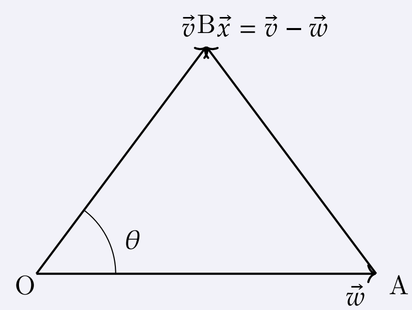
$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

The dot product is a scalar quantity and is zero when the vectors are orthogonal (perpendicular).
Useful to find the angle between the two vectors being dot produced together,

Theorem 1.2.1 Dot Product Proof

We are given the vectors $\vec{v}$ and $\vec{w}$, and we want to express the dot product in terms of their magnitudes and the angle between them.
Start with the relationship:

$$\vec{x} = \vec{v} - \vec{w}$$



The above diagram illustrates the vectors $\vec{v}$, $\vec{w}$, and their difference $\vec{x} = \vec{v} - \vec{w}$, forming a triangle. The angle θ is between $\vec{v}$ and $\vec{w}$.
The magnitude squared of $\vec{x}$ is:

$$|\vec{x}|^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}||\vec{w}| \cos\theta$$

This is the expansion of the law of cosines.
Now, from the equation:

$$|\vec{x}|^2 = \sqrt[2]{\left((v_x - w_x)^2 + (v_y - w_y)^2\right)}^2$$

We conclude:

$$|\vec{x}|^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2(\vec{v} \cdot \vec{w})$$

Thus, we can express the dot product $\vec{v} \cdot \vec{w}$ as:

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta$$

1.2.2 Applications

Note:-

The dot product of two vectors $\vec{v} \cdot \vec{w}$ can take different values, leading to various interpretations of the relationship between the vectors. Below is a table describing some key cases:

Dot Product Value	Interpretation	Relationship Between Vectors
$\vec{v} \cdot \vec{w} = 0$	$\cos\theta = 0$	Vectors are **perpendicular** (orthogonal), $\theta = 90°$
$\vec{v} \cdot \vec{w} > 0$	$0 < \theta < 90°$	Vectors form an **acute angle**, pointing in the same general direction
$\vec{v} \cdot \vec{w} < 0$	$90° < \theta < 180°$	Vectors form an **obtuse angle**, pointing in opposite general directions
$\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\|$	$\cos\theta = 1$	Vectors are **parallel** and point in the **same direction**, $\theta = 0°$
$\vec{v} \cdot \vec{w} = -\|\vec{v}\|\|\vec{w}\|$	$\cos\theta = -1$	Vectors are **parallel** but point in **opposite directions**, $\theta = 180°$

Definition 1.2.2: Vector Product (Cross Product)

The **vector product** (or **cross product**) of two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$ is a vector $\mathbf{c}$ that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, and its magnitude is given by:

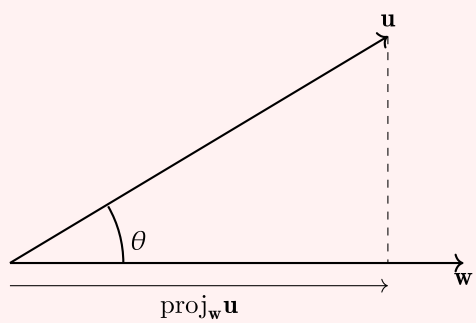
$$|\mathbf{c}| = |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. The cross product is calculated as:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

The result of a cross product is a vector perpendicular to the plane formed by $\mathbf{a}$ and $\mathbf{b}$, with a direction given by the right-hand rule.

Definition 1.2.3: Vector Projections



$$\text{scal}_{\mathbf{w}}\mathbf{u} = |\mathbf{u}| \cdot \cos\theta = \frac{\mathbf{w} \cdot \mathbf{u}}{|\mathbf{w}|}$$

$$\text{proj}_{\mathbf{w}}\mathbf{u} = |\mathbf{u}|\cos\theta\left(\frac{\mathbf{w}}{|\mathbf{w}|}\right)$$

$$\text{proj}_{\mathbf{w}}\mathbf{u} = \left(\frac{\mathbf{w} \cdot \mathbf{u}}{\mathbf{w} \cdot \mathbf{w}}\right)\mathbf{w}$$

1.3 Matrix Determinants

Definition 1.3.1: Matrix Representation

A matrix is a collection of numbers arranged in a grid format, where each element is positioned based on its row and column. A general $m \times n$ matrix is written as:

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

For example, a 2×2 matrix is given by:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

A 3×3 matrix is:

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Matrices can be considered as a collection of vectors where each row or column can represent a vector.

Note:-

Vector Representation
A matrix can also be viewed as a collection of vectors. For instance, a 3×3 matrix can be interpreted as:

$$M = \begin{pmatrix} \vec{v_1} = \langle a, b, c \rangle \\ \vec{v_2} = \langle d, e, f \rangle \\ \vec{v_3} = \langle g, h, i \rangle \end{pmatrix}$$

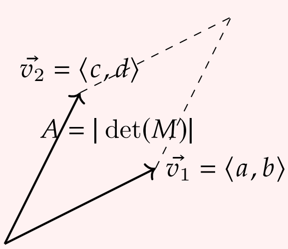
where each row (or column) is treated as a vector in space.

Definition 1.3.2: Determinant of a 2×2 Matrix

The determinant of a 2×2 matrix is given by:

$$\det(M) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

The determinant represents the signed area of the parallelogram formed by the vectors corresponding to the rows (or columns) of the matrix.



Note:-

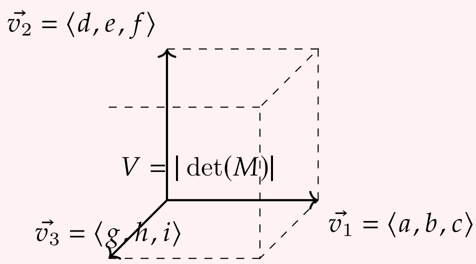
Geometric Interpretation
For a 2×2 matrix, the determinant represents the area A of the parallelogram formed by the two vectors $\vec{v_1} = \langle a, b \rangle$ and $\vec{v_2} = \langle c, d \rangle$. The magnitude of the determinant gives the area of this parallelogram, and the sign of the determinant indicates the orientation (whether the vectors are ordered clockwise or counterclockwise).

Definition 1.3.3: Determinant of a 3×3 Matrix

The determinant of a 3×3 matrix is calculated as:

$$\det(M) = \det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

The determinant represents the signed volume of the parallelepiped formed by the three vectors corresponding to the rows (or columns) of the matrix.



Note:-

Geometric Interpretation for 3×3
In the 3×3 case, the determinant represents the volume V of the parallelepiped formed by three vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$, and the sign indicates whether the orientation is right-handed or left-handed. The magnitude gives the volume.

1.4 Matrix multiplication with 2D Vectors

Definition 1.4.1: Vector Matrix Multiplication

$$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

$$\hat{\mathbf{j}}M = \langle a_{11}V_1 + a_{12}V_2, a_{21}V_1 + a_{22}V_2 \rangle$$

Given:

$$\hat{i} = \langle 1, 0 \rangle \quad \hat{j} = \langle 0, 1 \rangle$$

We can compute:

$$iM = \langle a_{11}, a_{12} \rangle = a_1$$

$$jM = \langle a_{21}, a_{22} \rangle = a_2$$

Where:

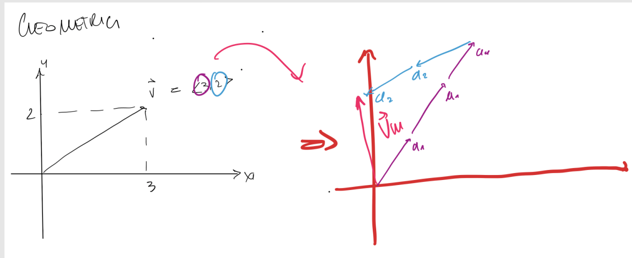
$$\mathbf{V} = V_1\hat{i} + V_2\hat{j}$$

$$\hat{\mathbf{V}}M = \left(V_1\hat{i} + V_2\hat{j}\right)M$$

$$= V_1\hat{i}M + V_2\hat{j}M$$

$$= V_1\mathbf{a}_1 + V_2\mathbf{a}_2$$

Note:-



1.4.1 Effect on Area

Definition 1.4.2: *2D*

The original point $(1, 1)$ is transformed by the matrix M. This transformation impacts the area and orientation as follows:

$$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

The area after transformation is given by the determinant of the matrix:

$$\text{Area} = \det(M)$$

Where the determinant is calculated as:

$$\det(M) = a_{11}a_{22} - a_{12}a_{21}$$

The determinant also determines the orientation:

$$\det(M) = \begin{cases} A & \text{if } a_1 \text{ to } a_2 \text{ is counterclockwise} \\ -A & \text{otherwise} \end{cases}$$

In the example, the original vectors a_1 and a_2 form an area, and the determinant will tell us if the vectors are oriented in a clockwise or counterclockwise fashion.
If the determinant is negative, the orientation is clockwise, as illustrated:

$$\det\left(\begin{pmatrix} a_1 & a_2 \end{pmatrix}\right) < 0$$

Thus, in this case, the transformation results in a clockwise orientation.

1.5 Matrix multiplication with 3D Vectors

Definition 1.5.1: *3D*

The matrix M for a 3D transformation is given as:

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \text{where} \quad \vec{V} = \langle V_1, V_2, V_3 \rangle$$

The transformation of vector $\vec{V}$ under matrix M is:

$$\hat{\vec{V}} M = \langle (a_{11}V_1 + a_{12}V_2 + a_{13}V_3), (a_{21}V_1 + a_{22}V_2 + a_{23}V_3), (a_{31}V_1 + a_{32}V_2 + a_{33}V_3) \rangle$$

This can be written in terms of the basis vectors as:

$$\left(V_1\hat{\imath} + V_2\hat{\jmath} + V_3\hat{k} \right) M = V_1\vec{a}_1 + V_2\vec{a}_2 + V_3\vec{a}_3$$

Definition 1.5.2: orientation and Volume

- If the determinant of matrix M is negative, the system is **left-handed**, i.e.,

$$\det(M) = -V$$

- The determinant of the matrix M gives the **volume** of the parallelepiped spanned by the vectors a_1, a_2, a_3:

$$\det(M) = \text{Volume}(V)$$

The volume V is given by:

$$V = \begin{cases} +V & \text{if } \vec{a}_1, \vec{a}_2, \vec{a}_3 \text{ are right-handed (RHS)} \\ -V & \text{otherwise (left-handed)} \end{cases}$$

1.6 Cross Product and Volumes

Definition 1.6.1: Cross Product and Volumes

The volume of a parallelepiped defined by three vectors $\vec{u}, \vec{v}, \vec{w}$ is given by:

$$\text{V} = \vec{u} \cdot (\vec{v} \times \vec{w})$$

1.6.1 Link to Matrix Determinants

Definition 1.6.2: Cross Product and Matrix Determinants

Since:

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \det \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix}$$

$$\begin{aligned}
\det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} &= u_1 \det \begin{pmatrix} v_2 & v_3 \\ w_2 & w_3 \end{pmatrix} - u_2 \det \begin{pmatrix} v_1 & v_3 \\ w_1 & w_3 \end{pmatrix} + u_3 \det \begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix} \\
&= \vec{u} \cdot \left(\hat{i} \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - \hat{j} \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + \hat{k} \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right) \\
&= \vec{u} \cdot \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} \\
&= \vec{u} \cdot (\vec{v} \times \vec{w})
\end{aligned}$$

Therefore:

$$\vec{v} \times \vec{w} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$$

1.7 Cross Product Polynomial Multiplication

Definition 1.7.1: Properties

$$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}$$

$$\hat{j} \times \hat{i} = -\hat{k}, \quad \hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

Example 1.7.1 (Example: Cross Product)
Let $\vec{v} = 2\hat{i} - \hat{j} - 3\hat{k}$ and $\vec{w} = \hat{i} + \hat{j} + \hat{k}$. The cross product $\vec{v} \times \vec{w}$ is computed as:

$$\vec{v} \times \vec{w} = \left(2\hat{i} - \hat{j} - 3\hat{k}\right) \times \left(\hat{i} + \hat{j} + \hat{k}\right)$$

Expanding the cross product term by term:

$$= 2\hat{i} \times \hat{i} + 2\hat{i} \times \hat{j} + 2\hat{i} \times \hat{k} - \hat{j} \times \hat{i} - \hat{j} \times \hat{j} - \hat{j} \times \hat{k} - 3\hat{k} \times \hat{i} - 3\hat{k} \times \hat{j} - 3\hat{k} \times \hat{k}$$

Using the cross product identities:

$$= 0 + 2\hat{k} + 2(-\hat{j}) - (-\hat{k}) + 0 - \hat{i} - 3\hat{j} + 3\hat{i} + 0$$

Combining like terms:

$$= (3\hat{i} - \hat{i}) + (-2\hat{j} - 3\hat{j}) + (2\hat{k} + \hat{k})$$

$$= 2\hat{i} - 5\hat{j} + 3\hat{k}$$

Thus, the final result is:

$$\vec{v} \times \vec{w} = 2\hat{i} - 5\hat{j} + 3\hat{k}$$

1.8 Torque

Definition 1.8.1: Torque and Angular Momentum

Continue with Torque

Parametric Equations

Definition 1.8.2: Parametric Equations

A parametric equation expresses a set of quantities as explicit functions of an independent parameter. In a two-dimensional case, a parametric equation for a curve can be represented as:

$$\langle x, y\rangle t = \langle f(t), g(t)\rangle$$

and in three dimensions as:

$$\langle x, y, z\rangle t = \langle f(t), g(t), h(t)\rangle$$

Note:-

For example, consider the curve in the plane given by the equation $y = f(x) = x^2+1$. This describes a parabola in Cartesian coordinates.

Theorem 1.8.1 Parametric unit circle in Cartesian coordinates

A unit circle in parametric form can be represented as:

$$x^2 + y^2 = 1$$

which corresponds to the parametric equations:

$$\langle x, y\rangle t = \langle \cos(t), \sin(t)\rangle$$

1.8.1 Examples

Note:-

The parametric equation:

$$\langle x, y\rangle t = \langle 4\cos(t), 3\sin(t)\rangle$$

At specific values of t, we can compute the points:

$$t = 0 \implies \langle 4, 0\rangle$$

$$t = \frac{\pi}{2} \implies \langle 0, 3\rangle$$

$$t = \pi \implies \langle -4, 0\rangle$$

$$t = \frac{3\pi}{2} \implies \langle 0, -3\rangle$$

Definition 1.8.3: Parametric for a Helix

$$\langle x(t), y(t), z(t)\rangle = \langle 4\cos(t), 3\sin(t), 0.1t\rangle$$

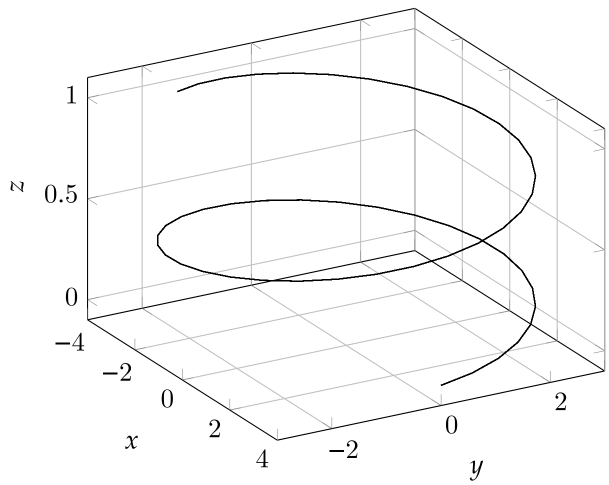


Figure 1.1: 3D plot of a parametric helix.

Theorem 1.8.2 Parametric Equation of a Line

The parametric equation of a line can be expressed as:

$$\langle x, y, z\rangle t = \mathbf{OP} + \mathbf{V}t$$

Where:

- $\mathbf{OP} = \langle x_0, y_0, z_0\rangle$ is the position vector to the initial point P,
- $\mathbf{V} = \langle v_x, v_y, v_z\rangle$ is the direction vector of the line.

Question 1: 3D Parametric Equation of a Line

- $\mathbf{OP} = \langle 1, 2, 3\rangle$ is the position vector to the initial point P,
- $\mathbf{V} = \langle 1, 1, 1\rangle$ is the direction vector of the line.

Thus, the parametric equation of the line becomes:

$$\langle x(t), y(t), z(t)\rangle = \langle 1, 2, 3\rangle + t\langle 1, 1, 1\rangle$$

$$x(t) = 1 + t, \quad y(t) = 2 + t, \quad z(t) = 3 + t$$

Or simply:

$$\langle x, y, z\rangle t = \langle 1 + t, 2 + t, 3 + t\rangle$$

Solution:

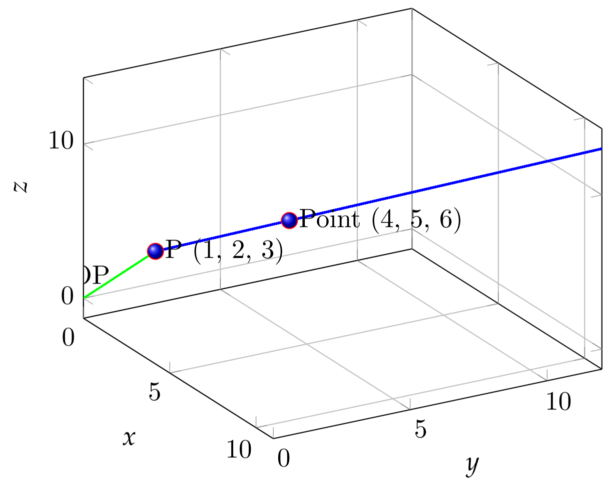


Figure 1.2: 3D plot of a parametric line with vector OP and points.

1.9 Distance from a Point to a Line

Definition 1.9.1: Parametric Equation of the Line

The line is represented by:

$$\mathbf{l} = \mathbf{OP} + t\mathbf{V}$$

Where:

- $\mathbf{OP}$ is the position vector of a point on the line,
- $\mathbf{V}$ is the direction vector of the line.

Theorem 1.9.1 Distance from a Point to a Line

The distance d from a point Q to the line l is given by:

$$d = \frac{|\mathbf{V} \times \mathbf{PQ}|}{|\mathbf{V}|}$$

Where:

- $\mathbf{PQ}$ is the vector from point P on the line to the point Q,
- $\mathbf{V} \times \mathbf{PQ}$ is the cross product of the direction vector $\mathbf{V}$ and the vector $\mathbf{PQ}$.

1.9.1 Example

Question 2: Find the distance from the point $Q = (3, 4, 0)$ to the line l given by the parametric equation

$$\mathbf{l} = \langle t, 1, 2t \rangle = \langle 0, 1, 0 \rangle + t\langle 1, 0, 2 \rangle$$

with point $P = (0, 1, 0)$ and direction vector $\mathbf{V} = \langle 1, 0, 2 \rangle$.

Solution: The vector $\mathbf{PQ}$ from $P = (0, 1, 0)$ to $Q = (3, 4, 0)$ is:

$$\mathbf{PQ} = \langle 3, 4, 0 \rangle - \langle 0, 1, 0 \rangle = \langle 3, 3, 0 \rangle$$

Now, we compute the cross product $\mathbf{V} \times \mathbf{PQ}$:

$$\mathbf{V} \times \mathbf{PQ} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ 3 & 3 & 0 \end{vmatrix} = \hat{i}(0-6) - \hat{j}(0-6) + \hat{k}(3-0) = \langle -6, -6, 3 \rangle$$

Next, calculate the magnitude of the cross product:

$$|\mathbf{V} \times \mathbf{PQ}| = \sqrt{(-6)^2 + (-6)^2 + 3^2} = \sqrt{36 + 36 + 9} = \sqrt{81} = 9$$

The magnitude of the direction vector $\mathbf{V}$ is:

$$|\mathbf{V}| = \sqrt{1^2 + 0^2 + 2^2} = \sqrt{1+4} = \sqrt{5}$$

Finally, the distance d is:

$$d = \frac{9}{\sqrt{5}} = \frac{9\sqrt{5}}{5}$$

1.10 Intersection of Two Parametric Lines

Definition 1.10.1: Intersection of Parametric Lines

To find the intersection point of two parametric lines, we need to equate their parametric equations and solve for the parameters.

Question 3: Find the intersection of the lines l_1 and l_2 given by the parametric equations

$$l_1 = \langle x, y \rangle(t) = \langle 0, 1 \rangle + t\langle 1, 0 \rangle$$
$$l_2 = \langle 1, 1 \rangle + s\langle -2, 1 \rangle$$

Solution: Equating the two parametric equations:

$$\langle 0, 1 \rangle + t\langle 1, 0 \rangle = \langle 1, 1 \rangle + s\langle -2, 1 \rangle$$

This gives the system of equations:

$$0 + t = 1 - 2s$$
$$1 + 0 = 1 + s$$

From the second equation, we find:

$$s = 0$$

Substitute $s = 0$ into the first equation:

$$t = 1$$

Thus, the lines intersect when $t = 1$ and $s = 0$.
The intersection point is:

$$\langle 0, 1 \rangle + 1 \cdot \langle 1, 0 \rangle = \langle 1, 1 \rangle$$

Therefore, the lines intersect at $(1, 1)$.