MA226: Differential Equations

Lecture notes for Differential Equations

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Chapter 1: First-Order Differential Equations

First-order differential equations involve derivatives up to the first derivative only. These form the foundation for understanding more complex differential equations and are ubiquitous in mathematical modeling.

1.1. Modeling and Basic Differential Equations

1.1.1. Three Approaches to Solving Differential Equations

There are three fundamental approaches to tackling differential equations, each with its own strengths:

Three Solution Approaches — Definition 1.1.1.1

1. Analytic ⟶ Formula or equation (exact solutions)
2. Qualitative ⟶ Sketches, describe behavior (understanding without solving)
3. Numerical ⟶ Computing (approximate solutions using algorithms)

Choosing the Right Approach — Note 1.1.1.1

- Use analytic methods when exact solutions are needed and the equation is solvable
- Use qualitative methods to understand long-term behavior and stability
- Use numerical methods when analytic solutions are impossible or impractical

1.1.2. Modeling

Mathematical modeling with differential equations follows a systematic approach to translate real-world phenomena into mathematical language.

1.1.2.1. Types of Models

- Simple models: Easy to analyze; describe the dominant interactions
- Complex models: Capture behavior over a wider domain; less general

1.1.2.2. Model Building Process

Model building typically follows three steps:

1. State assumptions clearly (with units for all quantities)
2. Define variables and parameters WITH UNITS
3. Use assumptions to derive equations relating the variables

Population Modeling — Example 1.1.2.2.1

Target: Population of rabbits $P(t)$ as a function of time t (years).

Key Assumption: The rate of change of population is proportional to the current population size.

Mathematical Model:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP \qquad [1]$$

where k is the growth coefficient (constant parameter).

1.1.3. Fundamental Definitions

Solution and General Solution — Definition 1.1.3.1

A function is a solution of a differential equation on an interval if, when substituted into the equation, it satisfies the equality for every point in that interval.

A general solution contains an arbitrary constant (or constants). Determining the constant(s) from given data yields a particular solution.

Initial Value Problem (IVP) — Definition 1.1.3.2

A differential equation together with an initial condition such as $P(t_0) = P_0$.

Solving the IVP means finding the unique solution that satisfies both the equation and the initial condition on an interval.

1.1.4. Exponential Growth and Decay

1.1.4.1. Solving the Basic Growth Model

Consider the differential equation $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$.

Solution Strategy: Guess that $P(t) = Ce^{\{kt\}}$ for some constant C.

Verification:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(Ce^{\{kt\}}\right) = C \cdot ke^{\{kt\}} = k\left(Ce^{\{kt\}}\right) = kP \quad [2]$$

Therefore, $P(t) = Ce^{\{kt\}}$ is indeed a solution to our differential equation.

General Solution — Note 1.1.4.1.1

Since C is arbitrary, $P(t) = Ce^{\{kt\}}$ represents the general solution to $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$.

The sign of k determines the behavior:

- If $k > 0$: exponential growth
- If $k < 0$: exponential decay

1.1.4.2. Finding Particular Solutions

Complete Solution Process — Example 1.1.4.2.1

Problem: Solve $P' = kP$ with initial conditions $P(0) = 32$ and $P(3) = 47$.

Step 1: Start with general solution $P(t) = Ce^{\{kt\}}$

Step 2: Apply first condition $P(0) = 32$

$$P(0) = Ce^{\{k \cdot 0\}} = Ce^0 = C = 32 \quad [3]$$

So $C = 32$, giving us $P(t) = 32e^{\{kt\}}$.

Step 3: Apply second condition $P(3) = 47$

$$P(3) = 32e^{\{3k\}} = 47 \quad [4]$$

$$e^{\{3k\}} = \frac{47}{32} \quad [5]$$

$$3k = \ln\left(\frac{47}{32}\right) \quad [6]$$

$$k = \frac{1}{3}\ln\left(\frac{47}{32}\right) \quad [7]$$

Final Answer: $P(t) = 32e^{\{\frac{1}{3}\ln(\frac{47}{32}) \cdot t\}}$

Growth vs. Decay Analysis — Note 1.1.4.2.1

- If $k > 0$, then P increases exponentially
- If $k < 0$, then P decreases exponentially
- The time constant $\frac{1}{|}k|$ sets the natural timescale of change

For our example: $k = \frac{1}{3}\ln\left(\frac{47}{32}\right) \approx 0.121 > 0$, so we have exponential growth.

1.1.5. The Logistic Population Model

The simple exponential model $P' = rP$ assumes unlimited resources, leading to unrealistic infinite growth. The logistic model accounts for resource limitations and carrying capacity.

Logistic Equation — Definition 1.1.5.1

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP\left(1 - \frac{P}{K}\right) \qquad [8]$$

where:

- $P(t)$: population size
- $r > 0$: intrinsic growth rate
- $K > 0$: carrying capacity (maximum sustainable population)

Key Insights — Note 1.1.5.1

Per-capita growth rate: $\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}t} = r\left(1 - \frac{P}{K}\right)$

- When $P \approx 0$: growth rate $\approx r$ (nearly exponential)
- When $P = K$: growth rate = 0 (no growth at capacity)
- Growth decreases linearly with population density $\frac{P}{K}$

1.1.5.1. Solution and Behavior

Logistic Solution — Example 1.1.5.1.1

The closed-form solution with initial condition $P(0) = P_0 > 0$ is:

$$P(t) = \frac{K}{1 + \left(\frac{K - P_0}{P_0}\right)e^{-rt}} \qquad [9]$$

This produces the characteristic S-shaped (sigmoidal) curve:

1. Initial phase: Nearly exponential growth when $P \ll K$
2. Transition phase: Growth slows as resources become limited
3. Saturation phase: Population levels off at carrying capacity K

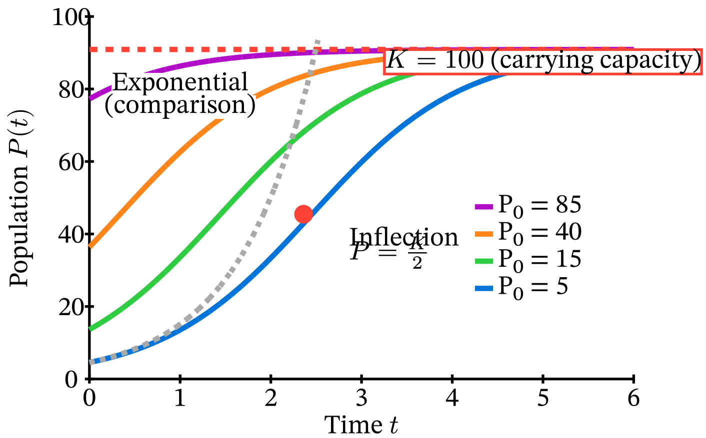

Figure 1: Logistic growth curves showing the characteristic S-shape for different initial populations. The red dashed line shows the carrying capacity K = 100. All logistic curves approach this limit, while the exponential curve (gray, dotted) grows without bound. The inflection point occurs at P = K/2.

1.1.5.2. Harvesting Models

Real populations often face removal through harvesting, hunting, or fishing. We can modify the logistic model by subtracting a harvesting term $H(P)$ from the natural growth rate.

Fish Population with Harvesting Example 1.1.5.2.1

Base Model: Consider a fish population with logistic growth:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = k\left(1 - \frac{P}{N}\right)P \tag{10}$$

where k is the growth rate and N is the carrying capacity.

With Harvesting: We subtract the harvest rate to get:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = k\left(1 - \frac{P}{N}\right)P - H(P) \tag{11}$$

The form of $H(P)$ depends on the harvesting strategy:

(a) Constant Harvesting: 100 fish removed per year

This represents constant-rate removal that doesn't depend on population size.

$$H(P) = 100 \tag{12}$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = k\left(1 - \frac{P}{N}\right)P - 100 \tag{13}$$

Why this form? The ODE assumes continuous removal at a rate of 100 fish per year. If harvesting happened as a discrete once-per-year event, we would need an impulsive model instead.

(b) Proportional Harvesting: One-third of population harvested annually

This is a rate proportional to P, where the harvest rate increases with population size.

$$H(P) = \frac{1}{3}P \tag{14}$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = k\left(1 - \frac{P}{N}\right)P - \frac{P}{3} \tag{15}$$

Why this form? The coefficient $\frac{1}{3}$ has units of year^{-1}, making $H(P)$ have the correct dimensions of fish/year. This models scenarios where harvesting effort scales with population abundance.

(c) Square-Root Harvesting: Harvest proportional to $\sqrt{P}$

This represents a nonlinear harvest rate that's less aggressive than proportional harvesting.

$$H(P) = a\sqrt{P} \quad \text{where } a > 0 \tag{16}$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = k\left(1 - \frac{P}{N}\right)P - a\sqrt{P} \tag{17}$$

Why this form? The parameter a has units of $\frac{\mathrm{fish}^{\frac{1}{2}}}{\mathrm{year}}$ to ensure dimensional consistency. This might model situations where harvesting becomes less efficient at higher population densities, or where there are diminishing returns to fishing effort.

Key Insights on Harvesting Note 1.1.5.2.1

Dimensional Analysis:

- Each $H(P)$ term has units of fish/year, matching $\frac{dP}{dt}$
- Case (a): $H = 100$ has units fish/year directly
- Case (b): $\frac{1}{3}\text{year}^{-1} \times P\ \text{fish} = \text{fish/year}$
- Case (c): $a\frac{\text{fish}^{\frac{1}{2}}}{\text{year}} \times \sqrt{P}\text{fish}^{\frac{1}{2}} = \text{fish/year}$

Continuous vs. Discrete Models:

- Our ODEs assume continuous removal throughout the year
- Real harvesting often occurs in discrete seasons (impulsive events)
- The choice depends on the timescale of interest and harvesting patterns

Equilibrium Effects:

- Constant-rate removal can eliminate equilibria if harvest exceeds maximum growth rate
- Proportional harvesting reduces effective growth rate: $k - \frac{1}{3}$
- Nonlinear harvesting creates complex equilibrium structures

Extinction Thresholds: Excessive harvesting creates minimum viable population sizes below which extinction occurs.

Management Implications: Different harvesting strategies require different sustainability criteria and have distinct economic trade-offs.

1.1.5.3. Predator-Prey Systems with Logistic Growth

The logistic model also appears in multi-species interactions. Predator-prey systems often incorporate logistic growth for the prey species to account for resource limitations.

Predator-Prey System Analysis Example 1.1.5.3.1

Consider the system:

$$\frac{dx}{dt} = \alpha x - \alpha\frac{x^2}{N} - \beta xy \tag{18}$$

$$\frac{dy}{dt} = \gamma y + \delta xy \tag{19}$$

where all parameters $\alpha, \beta, \gamma, \delta, N > 0$.

Algebraic Simplification: The prey equation can be rewritten as:

$$\frac{dx}{dt} = \alpha x - \alpha\frac{x^2}{N} - \beta xy = \alpha x\left(1 - \frac{x}{N}\right) - \beta xy \tag{20}$$

This shows the prey follows logistic growth when alone, modified by predation.

Species Identification:

x is the prey population:

- Natural growth: $+\alpha x$ (exponential when small)
- Self-limitation: $-\alpha\frac{x^2}{N}$ (resource competition/crowding)
- Predation loss: $-\beta xy$ (removed by encounters with predators)

y is the predator population:

- Benefits from encounters: $+\delta xy$ (conversion of prey to predators)
- Alternative food source: $+\gamma y$ (growth independent of prey)

Biological Interpretation Note 1.1.5.3.1

Is prey growth limited by factors other than predators?

Yes. The term $-\alpha\frac{x^2}{N}$ represents logistic self-limitation due to:
- Finite resources (food, territory, nesting sites)
- Carrying capacity N for the environment
- Competition among prey individuals

Even with no predators ($y = 0$), prey follows: $\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha x(1 - \frac{x}{N})$

Do predators have other food sources?

Yes. The term $+\gamma y$ means predators grow even without prey ($x = 0$):

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \gamma y > 0 \qquad [21]$$

This could represent:
- Alternative food sources not modeled explicitly
- Immigration from other regions
- Baseline growth rate from other resources

Equilibria and Stability Attention 1.1.5.3.1

Setting $\frac{\mathrm{d}P}{\mathrm{d}t} = 0$:
- $P^* = 0$: Unstable equilibrium (any $P_0 > 0$ grows away from zero)
- $P^* = K$: Stable equilibrium (all solutions approach carrying capacity)

Maximum growth occurs at $P = \frac{K}{2}$ with rate $\frac{rK}{4}$.

Real-World Applications Note 1.1.5.3.2

- Population ecology: Animal populations in limited habitats
- Epidemiology: Disease spread with finite susceptible population
- Technology adoption: Market saturation models
- Resource management: Sustainable harvesting strategies

1.1.6. Equilibrium Solutions

Equilibrium Solution Definition 1.1.6.1

A constant solution $y(t) \equiv y_*$ such that $y'(t) = 0$ for all t in an interval.

Equilibria correspond to values of y where the right-hand side of $y' = f(t, y)$ is zero for all t.

Finding Equilibrium Solutions Example 1.1.6.1

Consider the differential equation:

$$y' = \frac{(y+2)(y-3)(t-5)}{(y+7)} \quad [22]$$

For an equilibrium solution $y(t) \equiv y_*$, we need the right-hand side to be zero for all t.

Analysis: The right-hand side equals zero when the numerator is zero (and the denominator is non-zero).

The numerator $(y+2)(y-3)(t-5) = 0$ when:

- $y + 2 = 0 \to y = -2$
- $y - 3 = 0 \to y = 3$
- $t - 5 = 0$ (but this depends on t, so doesn't give a constant solution)

Verification: Both $y = -2$ and $y = 3$ make the denominator $y + 7$ non-zero.

Answer: $y \equiv -2$ and $y \equiv 3$ are equilibrium solutions.

Important Note Attention 1.1.6.1

$y \equiv -7$ is NOT a solution because it makes the right-hand side undefined (division by zero).

1.1.7. Key Insights and Intuition

Why Exponential Solutions Work Note 1.1.7.1

The exponential function $e^{\{kt\}}$ has the special property that its derivative is proportional to itself:

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{\{kt\}} = k e^{\{kt\}} \quad [23]$$

This makes it the natural solution to equations of the form $y' = ky$.

Physical Interpretation Note 1.1.7.2

- Population growth: When resources are abundant, growth rate s im current population
- Radioactive decay: Decay rate s im current amount of material
- Bank interest: Continuous compounding gives exponential growth
- Cooling: Newton's law of cooling (with modifications)

1.2. Separable Differential Equations

Separable differential equations are a special class of first-order differential equations that can be solved by separating variables and integrating both sides.

Separable Differential Equation Definition 1.2.1

A first-order differential equation is separable if it can be written in the form:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = g(t)h(y) \quad [24]$$

where $g(t)$ is a function of t only, and $h(y)$ is a function of y only.

Solution Strategy Note 1.2.1

To solve a separable equation $\frac{\mathrm{d}y}{\mathrm{d}t} = g(t)h(y)$:

1. Separate variables: $\frac{\mathrm{d}y}{h(y)} = g(t)\mathrm{d}t$
2. Integrate both sides: $\int \frac{\mathrm{d}y}{h(y)} = \int g(t)\mathrm{d}t$
3. Solve for y (if possible)
4. Apply initial conditions to find particular solutions

1.2.1. Basic Examples

Simple Exponential Growth Example 1.2.1.1

Problem: Solve $y' = 2y$

Step 1: Recognize this is separable with $g(t) = 2$ and $h(y) = y$

Step 2: Separate variables

$$\frac{\mathrm{d}y}{y} = 2\mathrm{d}t \quad [25]$$

Step 3: Integrate both sides

$$\int \frac{\mathrm{d}y}{y} = \int 2\mathrm{d}t \quad [26]$$

$$\ln|y| = 2t + C_1 \quad [27]$$

Step 4: Solve for y

$$|y| = e^{2t+C_1} = e^{C_1}e^{2t} \quad [28]$$

Since $e^{C_1} > 0$, we can write $|y| = Ce^{2t}$ where $C > 0$.

Step 5: Consider both positive and negative solutions

$$y = \pm Ce^{2t} \quad [29]$$

Final Answer: $y = Ce^{2t}$ where C can be any real constant (including negative values and zero).

Non-Separable Counter Example Attention 1.2.1.1

Problem: Is $y' = t + y$ separable?

Analysis: We need to write this as $\frac{\mathrm{d}y}{\mathrm{d}t} = g(t)h(y)$.

We have $\frac{\mathrm{d}y}{\mathrm{d}t} = t + y$. For this to be separable, we need:

$$t + y = g(t) \cdot h(y) \quad [30]$$

But $t + y$ cannot be factored into a product of a function of t only and a function of y only.

Conclusion: This equation is NOT separable and requires different solution methods.

1.2.2. More Complex Examples

Polynomial Growth Factor — Example 1.2.2.1

Problem: Solve $y' = t^4 y$

Step 1: This is separable with $g(t) = t^4$ and $h(y) = y$

Step 2: Separate variables

$$\frac{\mathrm{d}y}{y} = t^4 \mathrm{d}t \quad [31]$$

Step 3: Integrate both sides

$$\int \frac{\mathrm{d}y}{y} = \int t^4 \mathrm{d}t \quad [32]$$

$$\ln|y| = \frac{t^5}{5} + C_1 \quad [33]$$

Step 4: Solve for y

$$|y| = e^{\frac{t^5}{5} + C_1} = e^{C_1} e^{\frac{t^5}{5}} \quad [34]$$

Final Answer: $y = Ce^{\frac{t^5}{5}}$ where C is an arbitrary constant.

Note: The growth becomes extremely rapid for large $|t|$ due to the t^5 term in the exponent.

Linear Decay Model — Example 1.2.2.2

Problem: Solve $y' = 2 - y$

Step 1: Rewrite as $\frac{\mathrm{d}y}{\mathrm{d}t} = 2 - y = -(y - 2)$

This is separable with $g(t) = -1$ and $h(y) = y - 2$.

Step 2: Separate variables

$$\frac{\mathrm{d}y}{y - 2} = -\mathrm{d}t \quad [35]$$

Step 3: Integrate both sides

$$\int \frac{\mathrm{d}y}{y - 2} = \int (-1)\mathrm{d}t \quad [36]$$

$$\ln|y - 2| = -t + C_1 \quad [37]$$

Step 4: Solve for y

$$|y - 2| = e^{-t + C_1} = e^{C_1} e^{-t} \quad [38]$$

$$y - 2 = \pm e^{C_1} e^{-t} = Ce^{-t} \quad [39]$$

Final Answer: $y = Ce^{-t} + 2$

Physical Interpretation: This represents exponential approach to the equilibrium value $y = 2$.

Visualization: The figure below shows several solution curves for different initial conditions, all approaching the equilibrium line $y = 2$.

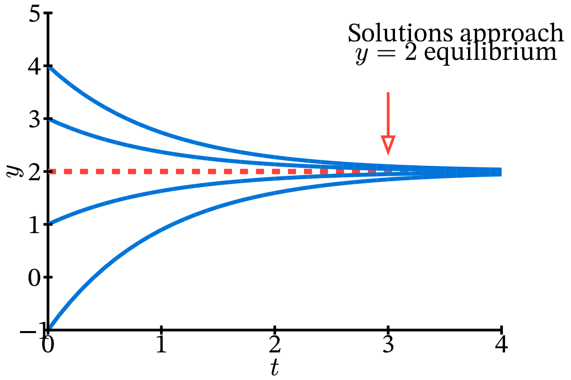

Figure 2: Solution curves for $y' = 2 - y$ with different initial conditions. All solutions exponentially approach the equilibrium $y = 2$ (red dashed line). Initial conditions: $y(0) = 4, 3, 1, -1$ respectively.

1.2.3. Advanced Techniques

Arctangent Integration — Example 1.2.3.1

Problem: Solve $y' = 1 + x^2$ (treating x as the independent variable)

Step 1: This is separable: $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + x^2$

Step 2: Since there's no y dependence, we can integrate directly

$$y = \int (1 + x^2)\mathrm{d}x \tag{40}$$

$$y = x + \frac{x^3}{3} + C \tag{41}$$

Alternative form using arctangent: If we had $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x^2}$, then:

$$y = \int \frac{\mathrm{d}x}{1 + x^2} = \arctan(x) + C \tag{42}$$

Partial Fractions Method — Example 1.2.3.2

Problem: Solve $y' = 12 + 3x^2$ with more complex rational functions

Consider the related problem: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{(2+x)(2-x)} = \frac{1}{4-x^2}$

Step 1: Use partial fractions decomposition

$$\frac{1}{(2 + x)(2 - x)} = \frac{A}{2 + x} + \frac{B}{2 - x} \tag{43}$$

Step 2: Find constants A and B

$$1 = A(2 - x) + B(2 + x) = 2A - Ax + 2B + Bx = (2A + 2B) + (-A + B)x \tag{44}$$

Comparing coefficients:

- Constant term: $2A + 2B = 1$
- Coefficient of x: $-A + B = 0$, so $A = B$

From $A = B$ and $2A + 2B = 1$: $4A = 1$, so $A = B = \frac{1}{4}$

Step 3: Integrate

$$y = \int \left(\frac{1}{4}\frac{1}{2 + x} + \frac{1}{4}\frac{1}{2 - x}\right)\mathrm{d}x \tag{45}$$

$$y = \frac{1}{4}\ln|2 + x| - \frac{1}{4}\ln|2 - x| + C \tag{46}$$

$$y = \frac{1}{4}\ln|\frac{2 + x}{2 - x}| + C \tag{47}$$

Initial Value Problem with Higher Powers **Example 1.2.3.3**

Problem: Solve $y' = t^2y^3$ with $y(0) = 1$

Step 1: Separate variables

$$\frac{\mathrm{d}y}{y^3} = t^2\mathrm{d}t \quad [48]$$

$$y^{-3}\mathrm{d}y = t^2\mathrm{d}t \quad [49]$$

Step 2: Integrate both sides

$$\int y^{-3}\mathrm{d}y = \int t^2\mathrm{d}t \quad [50]$$

$$\frac{y^{-2}}{-2} = \frac{t^3}{3} + C_1 \quad [51]$$

$$-\frac{1}{2y^2} = \frac{t^3}{3} + C_1 \quad [52]$$

Step 3: Solve for y

$$\frac{1}{2y^2} = -\frac{t^3}{3} - C_1 \quad [53]$$

$$\frac{1}{y^2} = -\frac{2t^3}{3} - 2C_1 \quad [54]$$

Let $C = -2C_1$, then:

$$\frac{1}{y^2} = -\frac{2t^3}{3} + C \quad [55]$$

$$y^2 = \frac{1}{-\frac{2t^3}{3} + C} \quad [56]$$

Step 4: Apply initial condition $y(0) = 1$

$$1^2 = \frac{1}{-\frac{2(0)^3}{3} + C} = \frac{1}{C} \quad [57]$$

Therefore $C = 1$, and:

$$y^2 = \frac{1}{1 - \frac{2t^3}{3}} = \frac{3}{3 - 2t^3} \quad [58]$$

Final Answer: $y = \pm\sqrt{\frac{3}{3-2t^3}}$

Since $y(0) = 1 > 0$, we take the positive square root:

$$y = \sqrt{\frac{3}{3 - 2t^3}} \quad [59]$$

Domain: Solution is valid when $3 - 2t^3 > 0$, i.e., when $t^3 < \frac{3}{2}$ or $t < \sqrt[3]{\frac{3}{2}}$.

Key Study Tips **Note 1.2.3.1**

1. Always check separability first - can you factor the right-hand side as $g(t)h(y)$?
2. Be careful with absolute values in logarithmic integration - consider both positive and negative solutions
3. Watch the domain - solutions may have restrictions based on denominators or square roots
4. Initial conditions determine the sign and specific constant value
5. Partial fractions are useful when $h(y)$ is a rational function with distinct linear factors

1.3. Direction Fields (Slope Fields)

When we cannot solve a differential equation analytically, or when we want to understand the qualitative behavior of solutions without solving, direction fields (also called slope fields) provide invaluable geometric insight.

Direction Field — Definition 1.3.1

For a first-order differential equation $y' = f(t, y)$, the direction field is a visual representation where at each point (t, y) in the plane, we draw a short line segment with slope $f(t, y)$.

This field shows the direction in which solutions flow at every point, allowing us to sketch solution curves without solving the equation.

Geometric Intuition — Note 1.3.1

Think of the direction field as a "flow field" - if you were to place a particle at any point (t, y), the direction field tells you which direction the particle would move. Solution curves are the paths particles would follow through this field.

- Positive slopes (red): Solutions increasing
- Negative slopes (blue): Solutions decreasing
- Zero slopes (gray): Horizontal tangent lines, often indicating equilibria

1.3.1. Three Fundamental Types

Direction fields reveal different structural patterns depending on whether the differential equation depends on t, y, or both variables.

1.3.1.1. Type 1: $y' = f(t)$ - Time-Dependent Only

When the differential equation has the form $y' = f(t)$, the slope depends only on the independent variable t.

Quadratic Time Dependence: $y' = t(t + 2)$ — Example 1.3.1.1.1

Consider $y' = t(t + 2) = t^2 + 2t$.

Key Observations:
- At $t = 0$: slope is $0(0 + 2) = 0$
- At $t = -2$: slope is $(-2)(-2 + 2) = 0$
- For $t < -2$ or $t > 0$: slope is positive
- For $-2 < t < 0$: slope is negative

Direction Field Structure: Vertical bands
- All points with the same t-coordinate have identical slopes
- Zero-slope occurs in vertical lines at $t = 0$ and $t = -2$
- Solution curves are independent - they never intersect

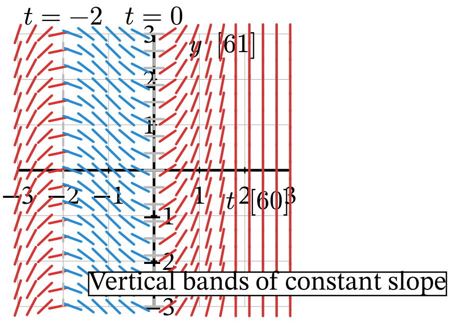

Figure 3: Direction field for $y' = t(t+2)$. Notice the vertical bands structure - all points with the same t-coordinate have identical slopes. Zero slopes occur at $t = 0$ and $t = -2$ (gray dashed lines).

1.3.1.2. Type 2: $y' = f(y)$ - State-Dependent Only

When the equation has the form $y' = f(y)$, the slope depends only on the current value of the dependent variable.

Quadratic Growth with Equilibria: $y' = y^2 - 3$ Example 1.3.1.2.1

Consider $y' = y^2 - 3$.

Equilibrium Analysis: Setting $y' = 0$:

$$y^2 - 3 = 0 \to y = \pm\sqrt{3} \quad [62]$$

Sign Analysis:
- For $y > \sqrt{3}$: $y^2 > 3$, so $y' > 0$ (growth)
- For $-\sqrt{3} < y < \sqrt{3}$: $y^2 < 3$, so $y' < 0$ (decay)
- For $y < -\sqrt{3}$: $y^2 > 3$, so $y' > 0$ (growth away from equilibrium)

Direction Field Structure: Horizontal bands
- All points with the same y-coordinate have identical slopes
- Zero-slope occurs in horizontal lines at $y = \pm\sqrt{3}$
- Solutions cannot cross these equilibrium lines

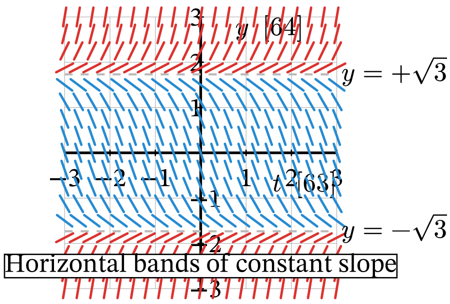

Figure 4: Direction field for $y' = y^2 - 3$. The horizontal band structure shows that slope depends only on y-coordinate. Equilibrium lines at $y = \pm\sqrt{3}$ (gray dashed) separate regions of different behavior.

1.3.1.3. Type 3: $y' = f(t, y)$ - Mixed Dependence

The most general case where slope depends on both variables creates the richest direction field structures.

Linear Mixed Case: $y' = y - t$ Example 1.3.1.3.1

Consider $y' = y - t$.

Equilibrium Curve: Setting $y' = 0$:

$$y - t = 0 \to y = t \qquad [65]$$

This is the diagonal line $y = t$, not just isolated points.

Sign Analysis:
- Above the line $y > t$: $y - t > 0$, so $y' > 0$ (solutions rise)
- On the line $y = t$: $y' = 0$ (horizontal tangents)
- Below the line $y < t$: $y - t < 0$, so $y' < 0$ (solutions fall)

Sample Calculations:

(t, y)	$y - t$	y'	Direction
$(1, 3)$	$3 - 1 = 2$	$+2$	↗ steep rise
$(0, 0)$	$0 - 0 = 0$	0	→ horizontal
$(2, -1)$	$-1 - 2 = -3$	-3	↘ steep fall

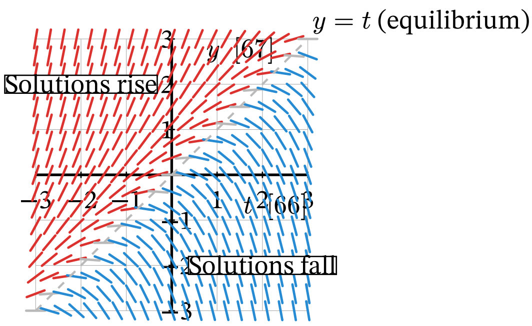

Figure 5: Direction field for $y' = y - t$. The equilibrium set is the diagonal line $y = t$ (gray dashed). Above this line solutions rise (red slopes), below it they fall (blue slopes). This creates a complex flow pattern unlike the simpler band structures.

1.3.2. Sketching Solution Curves

Solution Curve Guidelines Note 1.3.2.1

To sketch approximate solution curves in a direction field:

1. Start at initial condition (t_0, y_0)
2. Follow the flow - move in direction indicated by nearby line segments
3. Stay tangent to the direction field - solution curves should touch each line segment tangentially
4. Respect equilibria - solutions cannot cross equilibrium curves
5. Check consistency - verify your curve makes sense with the equation

Common Mistakes:
- Drawing curves that "cut through" the direction field instead of following it
- Crossing equilibrium lines (impossible for autonomous equations)
- Ignoring the tangent condition at each point

1.3.3. Qualitative Analysis Applications

Direction fields excel at revealing global behavior without solving:

Population Dynamics Analysis — Example 1.3.3.1

For the logistic equation $y' = ry(1 - \frac{y}{K})$:

- Equilibria: $y = 0$ (extinction) and $y = K$ (carrying capacity)
- Direction field reveals: All positive solutions approach carrying capacity
- Stability: $y = 0$ is unstable, $y = K$ is stable
- Solution behavior: S-shaped growth curves visible in the field

This analysis requires no integration - just understanding the direction field structure.

Uniqueness and Existence — Attention 1.3.3.1

Direction fields also reveal where solutions might fail to exist or be unique:

- Intersecting solution curves: Suggests non-uniqueness (requires checking conditions)
- Vertical slopes: May indicate finite-time blowup
- Discontinuities: Points where $f(t, y)$ is undefined create barriers for solutions

1.4. Numerical Methods: Euler's Method

Numerical methods let us approximate solutions when an analytic formula is unavailable or inconvenient. The simplest is Euler's method, which replaces the solution by a polygonal curve whose slope on each subinterval matches the differential equation at the left endpoint.

Euler's Method — Definition 1.4.1

Given an IVP $y' = f(t, y)$ with initial data $y(t_0) = y_0$ and a step size $\Delta t > 0$:

- Define grid points $t_n = t_0 + n\Delta t$.
- Initialize y_0.
- Update recursively by the forward-Euler rule:

$$y_{\{n+1\}} = y_n + \Delta t f(t_n, y_n) \qquad [68]$$

  and

$$t_{\{n+1\}} = t_n + \Delta t \qquad [69]$$

  .

This is the discretized version of the differential relation $\Delta y \approx f(t_n, y_n)\Delta t$ (using the tangent line at (t_n, y_n)).

Intuition and Accuracy — Note 1.4.1

- We follow the tangent at the current point for one step of length Δt.
- Local truncation error is $O(\Delta t^2)$ and the global error after N steps is $O(\Delta t)$.
- Smaller Δt yields higher accuracy but requires more steps (cost).

1.4.1. Worked Example (Forward Euler)

$y' = (3-y)(y+1)$ with $y(0) = 4$ and $\Delta t = 0.5$ Example 1.4.1.1

We compute $f(t, y) = (3-y)(y+1)$ and iterate the Euler update. Values are rounded to 3 decimals for readability.

n	t_n	y_n	$f(t_n, y_n)$	$\Delta y = f\Delta t$	$y_{\{n+1\}}$
0	0.0	4.000	−5.000	−2.500	1.500
1	0.5	1.500	3.750	1.875	3.375
2	1.0	3.375	−1.641	−0.820	2.555
3	1.5	2.555	1.583	0.791	3.346
4	2.0	3.346	−1.503	−0.751	2.595
5	2.5	2.595	1.460	0.730	3.325

Hence the Euler approximation at $t = 3$ (after 6 steps) is

$$y_{E(3)} \approx 3.325. \tag{70}$$

1.4.2. Analytic Solution and Comparison

Separation with Partial Fractions Example 1.4.2.1

For $y' = (3-y)(y+1)$, separate variables:

$$\frac{\mathrm{d}y}{(3-y)(y+1)} = \mathrm{d}t. \tag{71}$$

Decompose:

$$\frac{1}{(3-y)(y+1)} = \frac{1}{4}\frac{1}{y+1} + \frac{1}{4}\frac{1}{3-y}. \tag{72}$$

Integrate:

$$\frac{1}{4}\ln|y+1| - \frac{1}{4}\ln|3-y| = t + C \tag{73}$$

$$\ln\left(\frac{y+1}{3-y}\right) = 4t + C' \tag{74}$$

$$\frac{y+1}{3-y} = Ce^{4t}. \tag{75}$$

Solve for y:

$$y(t) = \frac{3Ce^{4t} - 1}{1 + Ce^{4t}}. \tag{76}$$

Apply $y(0) = 4$ to get $C = -5$, so an equivalent closed form is

$$y(t) = \frac{15e^{4t} + 1}{5e^{4t} - 1} = 3 + \frac{4}{5e^{4t} - 1}. \tag{77}$$

In particular,

$$y(3) = 3 + \frac{4}{5e^{12} - 1} \approx 3.000005. \tag{78}$$

Comparing with Euler above, $y_{E(3)} \approx 3.325$ so the absolute error is about 0.325 with $\Delta t = 0.5$.

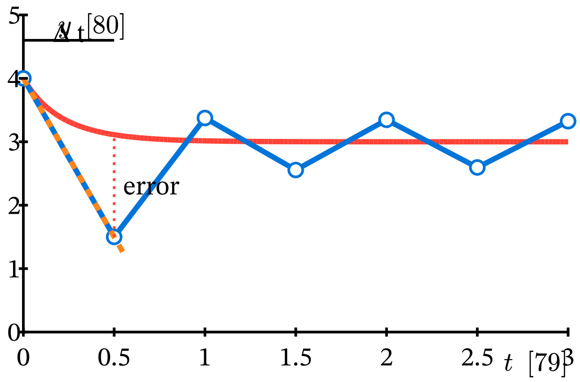

Figure 6: Euler's method for $y' = (3 - y)(y + 1)$ with $\Delta t = 0.5$: exact solution (red), Euler polygon (blue) and the initial tangent step (orange, dashed). The vertical dotted segment at $t = 0.5$ shows the local error.

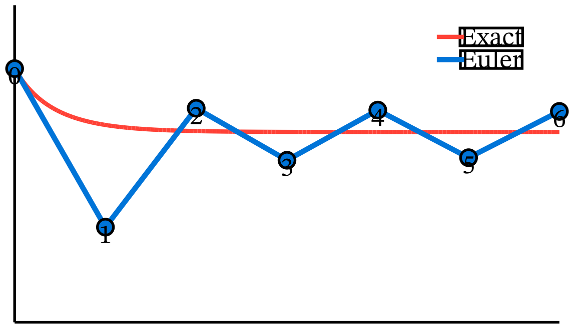

Figure 7: Exact solution (red) vs. Euler polygon (blue) at nodes $t = 0, 0.5, l..., 3$. Filled circles mark the Euler nodes; labels indicate the step index n.

1.5. Existence and Uniqueness for First-Order IVPs

Vocabulary Note 1.5.1

We use ∃ to denote "there exists" and ∃! to denote "there exists exactly one (unique)".

Existence (Peano-type) Theorem 1.5.1

Suppose $f(t, y)$ is continuous on a rectangle

$$R = \{(t, y) : a < t < b, c < y < d\} \quad [81]$$

containing (t_0, y_0). Then $\exists \varepsilon > 0$ and at least one function $y(t)$ defined for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ that solves the IVP

$$\begin{aligned} y' &= f(t, y), \\ y(t_0) &= y_0. \end{aligned} \quad [82]$$

Uniqueness (Picard–Lindelöf) Theorem 1.5.2

If, in addition, $\partial \frac{f}{\partial} y$ is continuous on R (equivalently, f is Lipschitz in y on R), then ∃! a unique solution to the IVP in some interval around t_0.

Rectangles and Domains Note 1.5.2

The interval of guaranteed existence/uniqueness must lie inside a rectangle where the hypotheses hold. If f (or $\partial \frac{f}{\partial} y$) blows up or is undefined, the rectangle—and thus the guaranteed interval—must stop before those singularities.

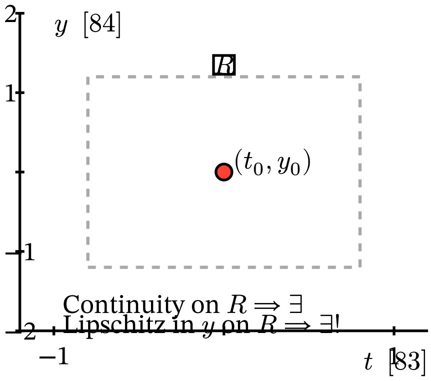

Figure 8: Existence rectangle R around (t_0, y_0). Continuity of f on R gives $\exists$ a solution; a Lipschitz condition in y on R gives $\exists!$ uniqueness near t_0.

1.5.1. Example: $y' = 1 + y^2$

General Solution and Initial Conditions — Example 1.5.1.1

Separate variables:

$$\frac{\mathrm{d}y}{1+y^2} = \mathrm{d}t \tag{85}$$

gives

$$\arctan(y) = t + C \tag{86}$$

and hence

$$y(t) = \tan(t + C) \tag{87}$$

.

- For $y(0) = 0$: $0 = \tan(C) \Rightarrow C = 0 \pmod{\pi}$. The unique solution through $(0, 0)$ near $t = 0$ is
$$y(t) = \tan(t) \tag{88}$$
$(-\frac{\pi}{2},$
, valid on the maximal interval $\frac{\pi}{2})$.
- For $y(\pi) = 0$: $0 = \tan(\pi + C) \Rightarrow C = -\pi$, so
$$y(t) = \tan(t - \pi) \tag{89}$$
$(\pi - \frac{\pi}{2},$
. The natural domain centered at $t_0 = \pi$ is $\pi + \frac{\pi}{2})$.

Here $f(t, y) = 1 + y^2$ and $\partial \frac{f}{\partial} y = 2y$ are continuous for all (t, y), so $\exists!$ a unique solution through any initial condition; the finite domains arise from vertical asymptotes of $\tan$.

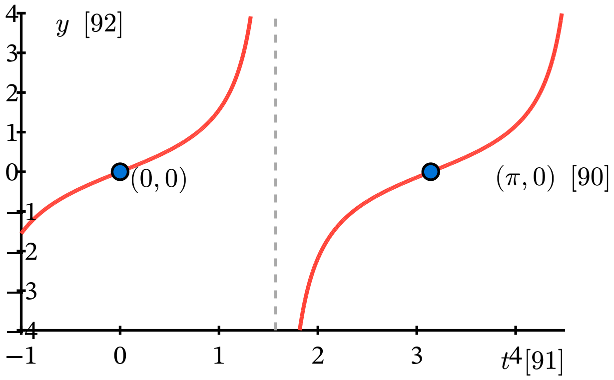

Figure 9: Solutions to $y' = 1 + y^2$ are $y(t) = \tan(t + C)$. Vertical dashed lines mark asymptotes at $t = \pm, \frac{\pi}{2}$ and $t = 3\frac{\pi}{2}$. The points $(0, 0)$ and $(\pi, 0)$ illustrate two initial conditions with distinct valid intervals.

1.5.2. Uniqueness: Consequences and Intuition

No-Crossing Principle Note 1.5.2.1

If the uniqueness hypotheses hold ($\partial \frac{f}{\partial} y$ continuous on a rectangle R), then solution curves through different initial values cannot intersect while they remain in R. Otherwise two different solutions would pass through the same point, contradicting $\exists!$.

Two Solutions That Never Cross Example 1.5.2.1

Consider $y' = y - t$. The general solution is $y(t) = t + 1 + Ce^t$. With $y(0) = -1$ we get $y_1(t) = t + 1 - 2e^t$; with $y(0) = 1$ we get $y_2(t) = t + 1$. Since $y_1(t) = y_2(t)$ would imply $-2e^t = 0$, the solutions never intersect. This illustrates uniqueness and the non-crossing property.

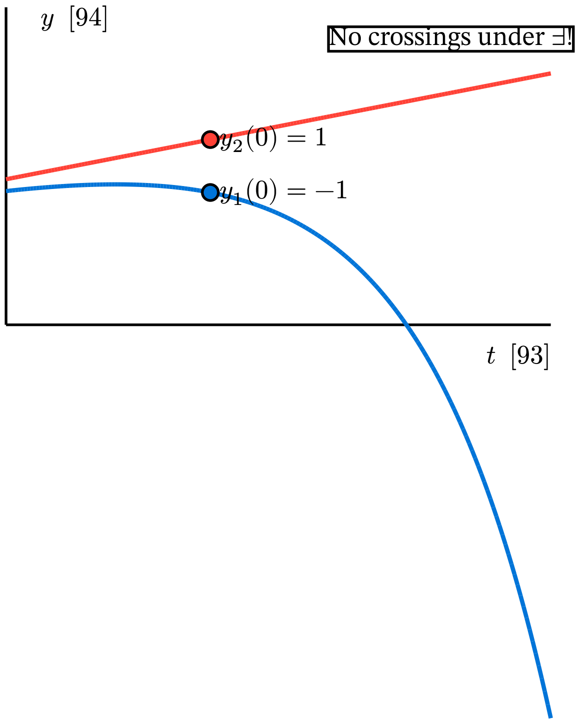

Figure 10: Two distinct solutions of $y' = y - t$ starting at $(0, -1)$ (blue) and $(0, 1)$ (red) never cross, visualizing the non-crossing consequence of uniqueness.

1.5.3. A Classic Non-Uniqueness Example

Failure of Lipschitz at $y = 0$ Example 1.5.3.1

Consider $y' = 3y^{\{\frac{2}{3}\}}$ with $y(0) = 0$.

- Separating: $y^{\{-\frac{2}{3}\}}\mathrm{d}y = 3\mathrm{d}t \Rightarrow 3y^{\{\frac{1}{3}\}} = 3t + C$, hence $y = (t + C')^3$.
- The constant solution $y(t) \equiv 0$ also satisfies the ODE.
- For any $age0$, the piecewise function $y_{a(t)} = 0$ for $tlea$ and $y_{a(t)} = (t - a)^3$ for $tgea$ solves the IVP and matches $y(0) = 0$.

Here $\partial \frac{f}{\partial} y = 2y^{\{-\frac{1}{3}\}}$ is unbounded at $y = 0$ (not Lipschitz), so uniqueness fails.

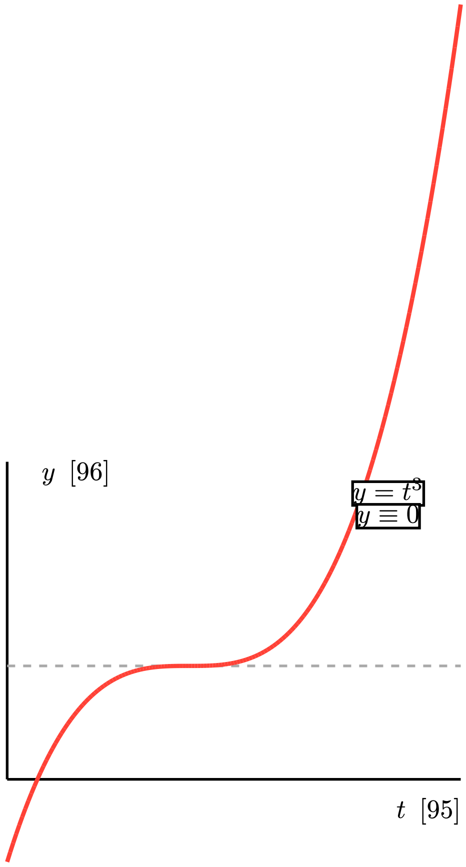

Figure 11: Non-uniqueness for $y' = 3y^{\{\frac{2}{3}\}}$ at $y(0) = 0$: both $y \equiv 0$ (gray dashed) and $y = t^3$ (red) satisfy the IVP, because $\partial \frac{f}{\partial} y$ is unbounded at $y = 0$.

1.5.4. Worked Example with a Singular Line

$y' = \frac{t}{y-2}$ with $y(-1) = 0$ — Example 1.5.4.1

Separate: $(y - 2)\mathrm{d}y = t\mathrm{d}t$ so

$$\frac{1}{2}(y-2)^2 = \frac{1}{2}t^2 + C \quad [97]$$

. Applying $y(-1) = 0$ yields $(y-2)^2 = t^2 + 3$ and hence

$$y(t) = 2 - \sqrt{t^2 + 3}. \quad [98]$$

- The right-hand side $f(t, y) = \frac{t}{y-2}$ and $\partial \frac{f}{\partial} y = -\frac{t}{(y-2)^2}$ are continuous on any rectangle avoiding $y = 2$, so $\exists!$ locally around $(-1, 0)$.
- Our explicit solution remains strictly below $y = 2$ for all t, so it never meets the singular line; the domain is $t \in (-\infty, \infty)$.

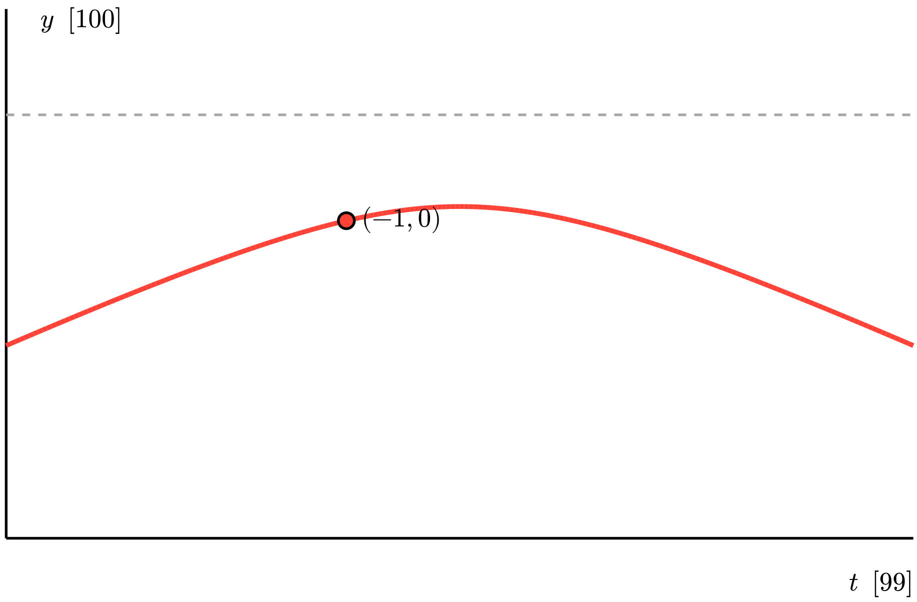

Figure 12: Solution to $y' = \frac{t}{y-2}$ with $y(-1) = 0$ (red). The horizontal dashed line $y = 2$ is a singular barrier and is never crossed.

1.6. Autonomous Equations and Phase Lines

Autonomous equations have the form $y' = f(y)$. Their qualitative behavior can be read from the sign of $f(y)$ using a phase line (a vertical y-axis with arrows up/down where $f(y)$ is positive/negative). Equilibria are zeros of f.

Equilibrium Classification (Phase Line) — Definition 1.6.1

Let y_* be a zero of f.

- sink (stable): arrows point toward y_* from both sides
- source (unstable): arrows point away from y_*
- semi-stable: f touches zero but does not change sign (e.g., repeated root)

Workflow for Phase Lines — Note 1.6.1

1) Factor $f(y)$, find zeros. 2) Determine the sign of f on each interval between zeros. 3) Draw arrows on a vertical y-axis accordingly. 4) Classify each equilibrium. For intuition, also sketch $f(y)$ vs y next to the phase line.

1.6.1. Phase Line 1: $y' = y(1-y)$

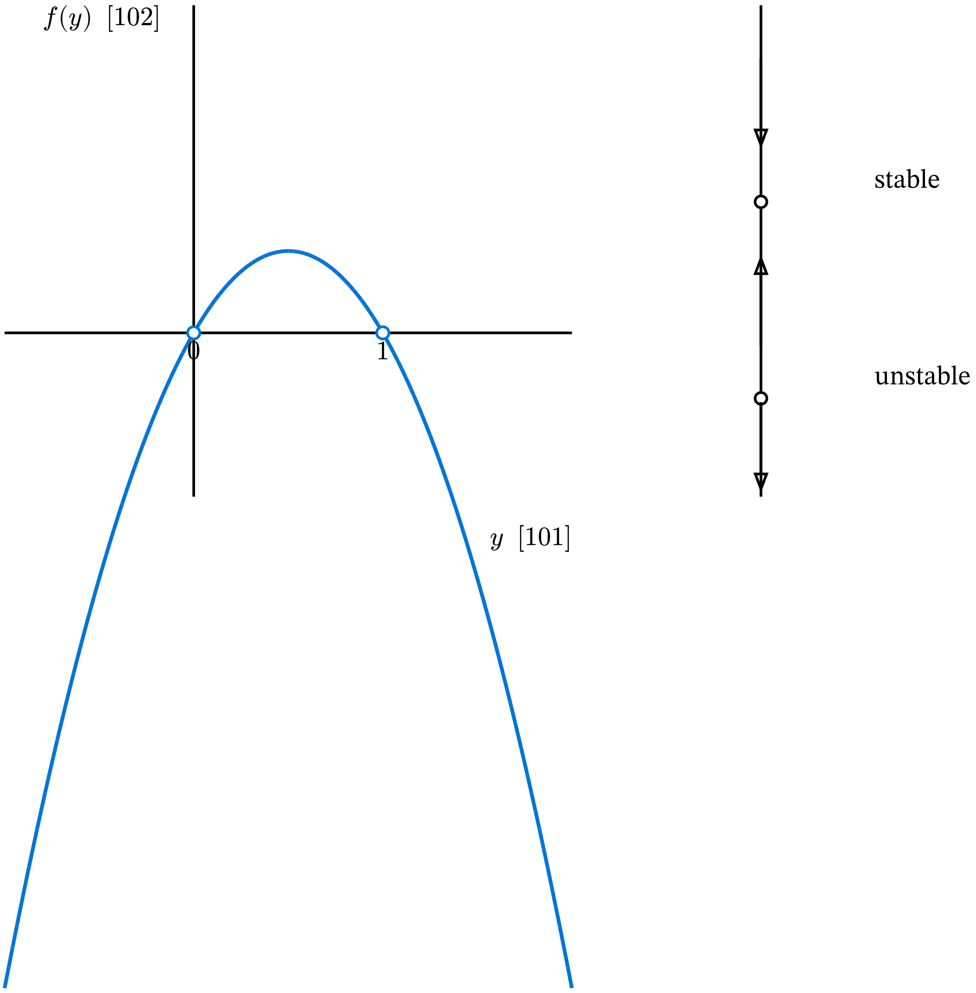

Figure 13: Phase line for $y' = y(1-y)$. Left: $f(y)$ vs y. Right: phase line arrows show $y = 1$ is a sink (stable) and $y = 0$ is a source (unstable).

1.6.2. Phase Line 2: $y' = (y-2)(y+1)$

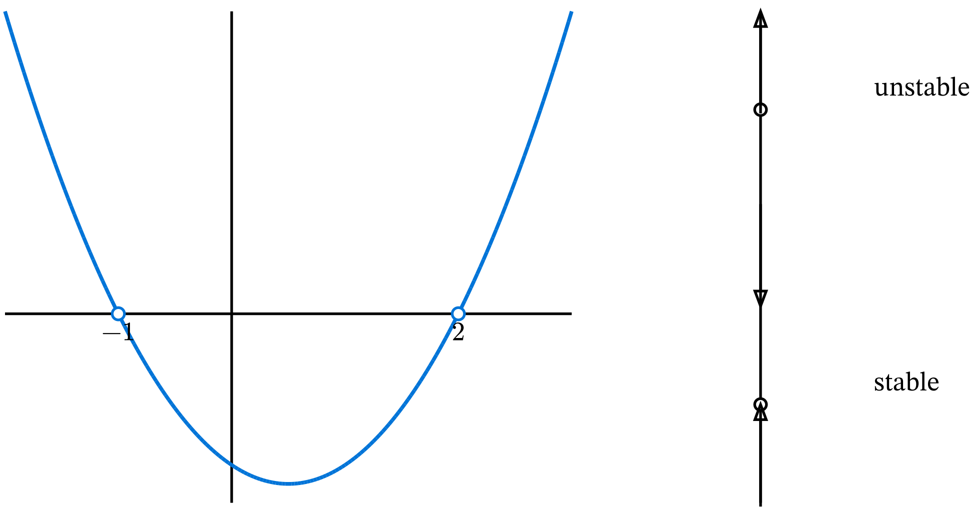

Figure 14: Phase line and $f(y)$ for $y' = (y-2)(y+1)$. The equilibrium $y = -1$ is a sink (stable) while $y = 2$ is a source (unstable).

1.6.3. Phase Line 3: $y' = (y-1)^2(2-y)$

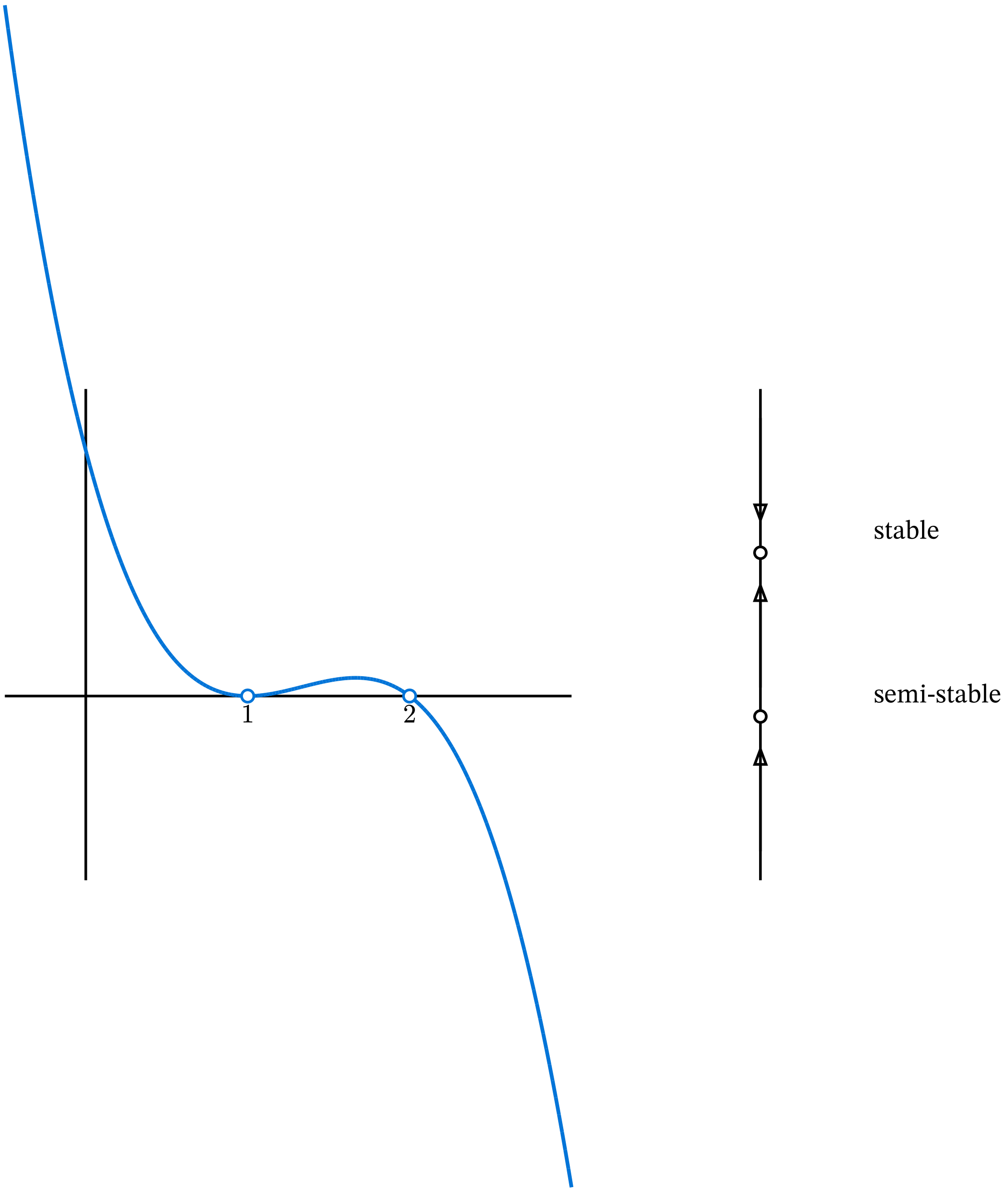

Figure 15: Phase line paired with $f(y)$ for $y' = (y-1)^2(2-y)$. The repeated root at $y = 1$ yields a semi-stable equilibrium (no sign change), while $y = 2$ is a stable sink.