Temporary Doc
Calc 3

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Chapter 1

Vector Valued Functions $f : \mathbb{R} \longrightarrow \mathbb{R}^n$

1.1 Change of Variable for Double and Triple Integrals

Polar Coordinates

$$\iint_D f(x, y)\, dx\, dy \to \iint_S f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Cylindrical Coordinates

$$\iiint_D f(x, y, z)\, dx\, dy\, dz \to \iiint_S f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz$$

Spherical Coordinates

$$\iiint_D f(x, y, z)\, dx\, dy\, dz \to \iiint_S f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

Theorem 1.1.1 Intuition Behind Change of Variables

We use a **mapping** T to transform coordinates in one space S to another R. This is particularly useful when integrating over regions that are easier to describe in new coordinates (e.g., circular or spherical regions).
For example:

$$S = [0, 2\pi] \times [0, 2], \quad T(r, \theta) = (r\cos\theta, r\sin\theta)$$

Here, the mapping T converts a point in S into a point in R.

Area Differential Transformation

Consider a small differential area element in the original space:

$$dA = |\det(J)|\, du\, dv$$

where J is the **Jacobian matrix**, and $|\det(J)|$ accounts for how the transformation scales area.

Definition 1.1.1: Jacobian Matrix

The Jacobian matrix represents the linear transformation of the mapping T at a given point:

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

For a transformation $T(u,v) = (g(u,v), h(u,v))$, the determinant of J is:

$$\det(J) = \begin{vmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{vmatrix} = \frac{\partial g}{\partial u} \cdot \frac{\partial h}{\partial v} - \frac{\partial g}{\partial v} \cdot \frac{\partial h}{\partial u}$$

Geometric Interpretation

- **Local Stretching/Scaling:** $|\det(J)|$ gives the local scaling factor of the area due to the transformation.
- **Orientation:** The sign of $\det(J)$ indicates whether the orientation is preserved or flipped.

Example 1.1.1 (Polar Coordinates)
For the transformation $T(r,\theta) = (r\cos\theta, r\sin\theta)$, the Jacobian matrix is:

$$J = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

The determinant is:

$$\det(J) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

Thus, the area differential in polar coordinates becomes:

$$dx\,dy = r\,dr\,d\theta$$

Definition 1.1.2: General Formula for Transforming Integrals

If $T : S \to R$ is a transformation with Jacobian determinant $|\det(J)|$, then the integral transforms as:

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(T(u,v))\,|\det(J)|\,du\,dv$$

Definition 1.1.3: Intuition for Higher Dimensions

In three dimensions, the Jacobian matrix extends to account for the transformation of volume elements:

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$$

The volume scaling factor is given by $|\det(J)|$, and the integral transforms as:

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_S f(T(u,v,w))\,|\det(J)|\,du\,dv\,dw$$

1.2 Non-overlapping from Mapping T

Theorem 1.2.1 Non-overlapping Condition

For any two points Q and P:

$$T(Q) \neq T(P) \quad \text{(This would result in overlapping areas in the domain } R\text{)}$$

However, boundaries (e.g., $y = 2x$) can overlap as long as the bounded region is distinct.

Example 1.2.1 (Integral Transformation Example)

Evaluate:

$$\iint_R 2x(y-2x)\,dA$$

where R is the parallelogram with vertices $(0,0), (0,1), (2,4), (2,3)$.
Steps:

1. **Choose a Transformation:** Select a mapping T to simplify the integral.
2. **Define the Mapping:**
$$x = u, \quad y = 2x + v = 2u + v$$
Substituting:
$$(x,y) \to (u,v)$$
3. **Boundary Equations:**
$$0 \leqslant x \leqslant 2 \quad \Rightarrow \quad 0 \leqslant u \leqslant 2$$
$$0 \leqslant y - 2x < 1 \quad \Rightarrow \quad 0 \leqslant v < 1$$
4. **Region:**
$$S = [0,2] \times [0,1]$$
5. **Transform the Integrand:**
$$f(T(x,y)) = 2u(v)$$
6. **Jacobian Calculation:**
$$J = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad \det(J) = 1 \cdot 1 - 2 \cdot 0 = 1$$
7. **Transformed Integral:**
$$\iint_R 2x(y-2x)\,dA = \int_0^2 \int_0^1 2uv\,du\,dv$$

1.3 Integral Transformation for a Parallelogram Region

Example 1.3.1 (Example of Transformation)

Evaluate:

$$\iint_R 2x(y-2x)\,dA$$

where R is the parallelogram defined by the vertices $(0,0), (0,1), (2,4), (2,3)$.

Steps:

1. **Choose a Transformation:** Select a transformation T that simplifies the integral.

2. **Define x, y in terms of u, v:**

$$x = u, \quad y = 2x + v = 2u + v$$

   Substituting:

$$(x, y) \to (u, v)$$

   Here, u corresponds to x, and $v = y - 2x$.

3. **Boundary Equations:**

$$0 \leqslant x \leqslant 2 \quad \Rightarrow \quad 0 \leqslant u \leqslant 2$$

$$0 \leqslant y - 2x < 1 \quad \Rightarrow \quad 0 \leqslant v < 1$$

4. **Region in u, v:**

$$S = [0, 2] \times [0, 1]$$

   This maps the parallelogram R into a rectangle S in the u, v-plane.

5. **Transform the Integrand:** Substituting $x = u$ and $y - 2x = v$:

$$f(T(x, y)) = 2u(v)$$

6. **Jacobian Calculation:** The Jacobian matrix for the transformation T is:

$$J = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

   The determinant of J is:

$$\det(J) = 1 \cdot 1 - 2 \cdot 0 = 1$$

7. **Transformed Integral:** Using the transformation and the Jacobian determinant:

$$\iint_R 2x(y - 2x)\, dA = \int_0^2 \int_0^1 2uv\, du\, dv$$

   The transformed integral simplifies the computation significantly.

1.4 Integral Transformation for a Triangular Region

Example 1.4.1 (Example of Transformation)
Evaluate:

$$\iint_R (x - u)\sqrt{x - 2y}\, dA$$

where R is the triangular region bounded by the lines $y = 0$, $x - 2y = 0$, and $x = y + 1$.

Steps:

1. **Region Definition:** The region R is defined by:

$$y = 0, \quad x - 2y = 0, \quad x = y + 1$$

   The boundaries in x and y are:

$$0 \leqslant x \leqslant 2, \quad 0 \leqslant y \leqslant \frac{x}{2}, \quad x \leqslant y + 1$$

2. **Define Transformation:** Let:
$$u = x - 2y, \quad v = x - y$$
Substituting:
$$x = v + u, \quad y = v - u$$

3. **Boundaries in New Coordinates:** Using the transformation:
$$\begin{aligned} u = x - 2y \quad &\Rightarrow \quad 0 \leqslant u \leqslant 1 \\ v = x - y \quad &\Rightarrow \quad u \leqslant v \leqslant 1 \end{aligned}$$
The transformed region S is bounded by $u = 0$, $v = 1$, and $v - u = 1$.

4. **Jacobian Calculation:** The Jacobian matrix for the transformation $T(u, v)$ is:
$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$
The determinant of J is:
$$\det(J) = (1)(1) - (1)(-2) = 1 + 2 = 3$$

5. **Transform the Integral:** Using the transformation and Jacobian determinant:
$$\iint_R (x - u)\sqrt{x - 2y}\, dA = \int_0^1 \int_0^v \sqrt{u} \cdot 3\, du\, dv$$
Simplify:
$$\int_0^1 \int_0^v \sqrt{u}\, du\, dv = \int_0^1 \left[\frac{2}{3}u^{3/2}\right]_0^v dv = \int_0^1 \frac{2}{3}v^{3/2} dv$$
$$= \left[\frac{2}{3} \cdot \frac{2}{5}v^{5/2}\right]_0^1 = \frac{4}{15}.$$
The result is:
$$\iint_R (x - u)\sqrt{x - 2y}\, dA = \frac{4}{15}.$$

Chapter 2

Vector Fields $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

2.1 Vector Fields

A **vector field** is a function $\vec{F}$ that takes points in $\mathbb{R}^2$ or $\mathbb{R}^3$ and outputs a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$:

$$\vec{F} : \mathbb{R}^n \to \mathbb{R}^n, \quad \vec{F}(x, y, z) = \langle q(x, y, z), w(x, y, z), t(x, y, z)\rangle.$$

2.1.1 Properties

- Continuous if q, w, t are continuous.
- Differentiable if q, w, t are differentiable.
- Domain is the intersection of domains of q, w, t.

2.1.2 Applications

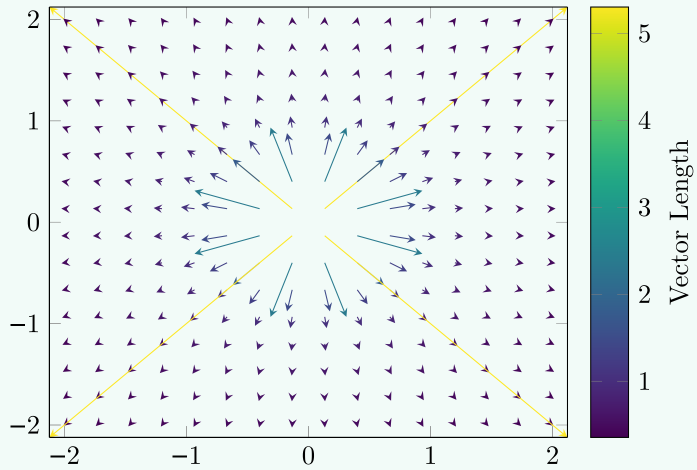
- Water and wind currents.
- Gravitational, electric, and magnetic fields.
- Human circulation, heat propagation.
- Modeling through partial differential equations.

2.1.3 Drawing a Vector Field

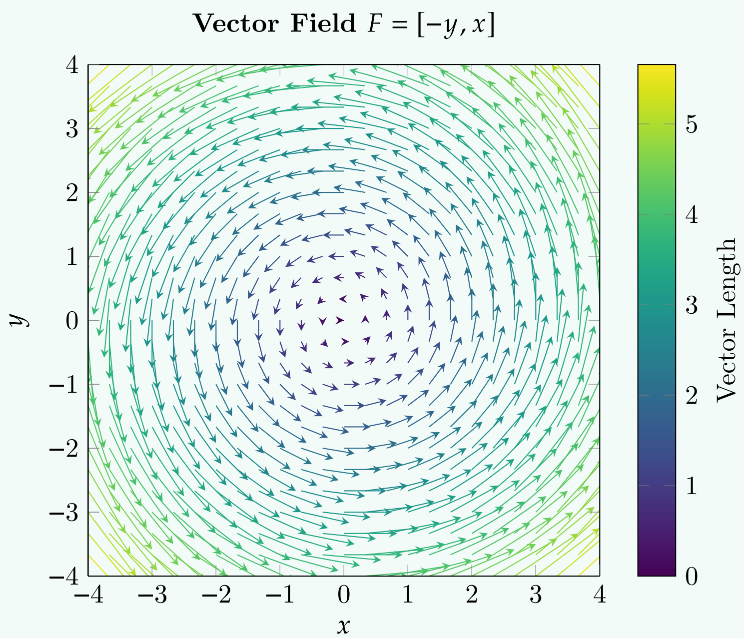
At each point in the domain of $\vec{F}$, draw a vector whose:

- Direction is parallel to $\vec{F}(x, y)$.
- Length is proportional to the magnitude of $\vec{F}(x, y)$.

Example 2.1.1 ($\vec{F}(x, y) = \frac{\langle x,y\rangle}{|\langle x,y\rangle|^3}$)



Example 2.1.2 ($\vec{F}(x,y) = \langle -y, x \rangle$)



2.2 Gradient Vector Fields

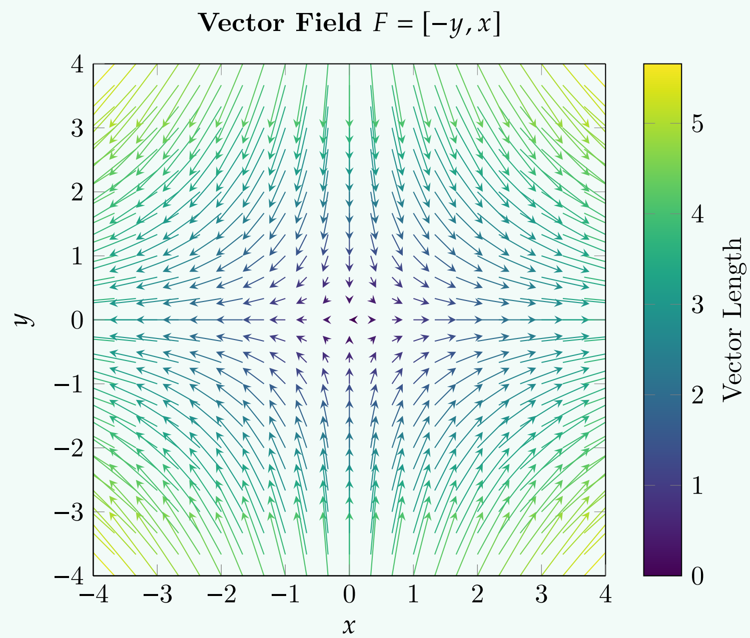
A **gradient vector field** $\nabla\varphi(x,y)$ is a vector field:

$$\vec{F}(x,y) = \nabla\varphi \quad \text{where } \varphi \text{ is a potential function.}$$

If φ exists, $\vec{F}$ is called a **conservative field**.

Example 2.2.1 ($\varphi = -\frac{x^2+y^2}{2}$)

$$\vec{F}(x,y) = \nabla\varphi = \langle -x, -y \rangle$$



2.3 Line Integrals

Scalar Case

For a scalar function $f(x, y)$, the line integral over a curve C is defined as:

$$A = \int_C f(x, y)\, ds$$

where $ds = |\mathbf{r}'(t)|dt$, with $\mathbf{r}(t)$ being the parameterization of C.

If $\mathbf{r}(t) = (x(t), y(t))$, then:

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

This allows us to rewrite the line integral as:

$$\int_C f(x, y)\, ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\, dt$$

Example 2.3.1 (Example: Average Temperature on a Plate)

Suppose we have a plate located in $R = \{(x, y) : x^2 + y^2 \leqslant 4\}$, where the temperature at any point (x, y) is $T(x, y) = 100(x^2 + y^2)$. Find the average temperature over the boundary of R.

1. **Parameterize the Boundary:** The boundary C of R is a circle of radius 2, centered at the origin. Parameterize C as:

   $$\mathbf{r}(t) = (2\cos t, 2\sin t), \quad 0 \leqslant t \leqslant 2\pi$$

   Then:

   $$|\mathbf{r}'(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2} = 2$$

2. **Average Temperature:** The average temperature is given by:

   $$\text{Average Temperature} = \frac{\int_C T(x, y)\, ds}{\int_C ds}$$

Compute the numerator:

$$\int_C T(x, y)\, ds = \int_0^{2\pi} 100 \cdot 4 \cdot 2\, dt = 800\pi$$

Compute the denominator (arc length of C):

$$\int_C ds = \int_0^{2\pi} 2\, dt = 4\pi$$

Thus:

$$\text{Average Temperature} = \frac{800\pi}{4\pi} = 200$$

Oriented Curves

An oriented curve C includes a direction along the curve. For example, when calculating work done by a force $\mathbf{F}$ along a curve, the direction matters.

Example 2.3.2 (Example: Work)

Work done by a force $\mathbf{F}(x)$ is given by:

$$W = \int_a^b F(x)\, dx$$

For a spring with force $F(x) = -kx$, the work done to stretch the spring from $x = a$ to $x = b$ is:

$$W = \int_a^b -kx\, dx = -\frac{k}{2}\left[b^2 - a^2\right]$$

2.4 Vector Line Integrals

Example 2.4.1 (Example: Earth's Gravitational Field)

The Earth's gravitational field exerts a force $\mathbf{F}(\mathbf{r})$ on an object, pulling it toward the origin. Suppose an object travels along a path C. The work done by $\mathbf{F}$ is:

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

If $\mathbf{F}(\mathbf{r}) = \frac{-Gm}{|\mathbf{r}|^3}\mathbf{r}$, parameterize C as $\mathbf{r}(t)$, and compute:

$$\mathbf{F}(\mathbf{r}) = -\frac{Gm}{|\mathbf{r}|^3}\mathbf{r}, \quad d\mathbf{r} = \mathbf{r}'(t)\,dt$$

Then:

$$W = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$$

If C is along the z-axis from $(0, 0, 1)$ to $(0, 0, 4)$, parameterize:

$$\mathbf{r}(t) = (0, 0, t), \quad 1 \leqslant t \leqslant 4$$

Then:

$$W = \int_1^4 -\frac{Gm}{t^2} \cdot 1\,dt = -Gm \int_1^4 \frac{1}{t^2}\,dt$$

$$W = -Gm \left[-\frac{1}{t}\right]_1^4 = -Gm\left(-\frac{1}{4} + 1\right) = Gm\left(\frac{3}{4}\right)$$

2.5 Fundamental Theorem of Line Integrals

Theorem 2.5.1 Fundamental Theorem of Line Integrals

If $\mathbf{F}$ is a vector field that can be expressed as $\mathbf{F} = \nabla\phi$, then given an oriented curve C from P to Q:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(Q) - \phi(P)$$

Example 2.5.1 (Example)

Let $\phi(x, y, z) = \frac{-G}{\sqrt{x^2+y^2+z^2}}$, and $\mathbf{F} = \nabla\phi$. Compute:

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

from (a, a, a) to $(1, 1, 1)$:

$$\phi(x, y, z) = \frac{-G}{\sqrt{x^2 + y^2 + z^2}}, \quad \Delta\phi = \phi(1, 1, 1) - \phi(a, a, a)$$

$$\phi(a, a, a) = \frac{-G}{\sqrt{3a^2}} = \frac{-G}{a\sqrt{3}}, \quad \phi(1, 1, 1) = \frac{-G}{\sqrt{3}}$$

Thus:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(1, 1, 1) - \phi(a, a, a) = \frac{-G}{\sqrt{3}} - \left(\frac{-G}{a\sqrt{3}}\right)$$

$$= \frac{G}{\sqrt{3}}\left(1 - \frac{1}{a}\right)$$