Temporary Doc
Calc 3

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23/10/24

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Chapter 1

Vector Valued Functions $f : \mathbb{R} \longrightarrow \mathbb{R}^n$

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Cylindrical Coordinates

$$\iiint_D f(x, y, z)\, dx\, dy\, dz \to \iiint_S f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz$$

Spherical Coordinates

$$\iiint_D f(x, y, z)\, dx\, dy\, dz \to \iiint_S f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

Theorem 1.1.1 Intuition Behind Change of Variables

We use a **mapping** T to transform coordinates in one space S to another R. This is particularly useful when integrating over regions that are easier to describe in new coordinates (e.g., circular or spherical regions).
For example:

$$S = [0, 2\pi] \times [0, 2], \quad T(r, \theta) = (r\cos\theta, r\sin\theta)$$

Here, the mapping T converts a point in S into a point in R.

Area Differential Transformation

Consider a small differential area element in the original space:

$$dA = |\det(J)|\, du\, dv$$

where J is the **Jacobian matrix**, and $|\det(J)|$ accounts for how the transformation scales area.

Definition 1.1.1: Jacobian Matrix

The Jacobian matrix represents the linear transformation of the mapping T at a given point:

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

For a transformation $T(u, v) = (g(u, v), h(u, v))$, the determinant of J is:

$$\det(J) = \begin{vmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{vmatrix} = \frac{\partial g}{\partial u} \cdot \frac{\partial h}{\partial v} - \frac{\partial g}{\partial v} \cdot \frac{\partial h}{\partial u}$$

Geometric Interpretation

- **Local Stretching/Scaling:** $|\det(J)|$ gives the local scaling factor of the area due to the transformation.
- **Orientation:** The sign of $\det(J)$ indicates whether the orientation is preserved or flipped.

Example 1.1.1 (Polar Coordinates)

For the transformation $T(r, \theta) = (r \cos\theta, r \sin\theta)$, the Jacobian matrix is:

$$J = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

The determinant is:

$$\det(J) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

Thus, the area differential in polar coordinates becomes:

$$dx\,dy = r\,dr\,d\theta$$

Definition 1.1.2: General Formula for Transforming Integrals

If $T : S \to R$ is a transformation with Jacobian determinant $|\det(J)|$, then the integral transforms as:

$$\iint_R f(x, y)\,dx\,dy = \iint_S f(T(u, v))\,|\det(J)|\,du\,dv$$

Definition 1.1.3: Intuition for Higher Dimensions

In three dimensions, the Jacobian matrix extends to account for the transformation of volume elements:

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$$

The volume scaling factor is given by $|\det(J)|$, and the integral transforms as:

$$\iiint_R f(x, y, z)\,dx\,dy\,dz = \iiint_S f(T(u, v, w))\,|\det(J)|\,du\,dv\,dw$$