

# CASMA 225

## Calc 3

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# Chapter 1

## Vectors

### 1.1 Review

#### 1.1.1 Basics

##### Definition 1.1.1: Notation

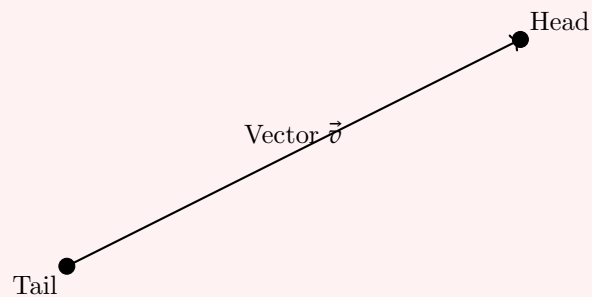
Drawn Vectors:  $\vec{v}$

Typed Vectors:  $\mathbf{v}$

##### Definition 1.1.2: Velocity

Magnitude of the velocity:  $|\vec{v}|$   
Direction of the velocity:  $dir(\vec{v})$

##### Definition 1.1.3: Heads and Tails

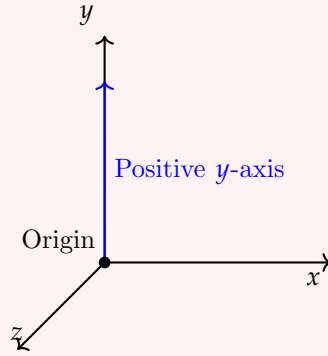


##### Note:-

Scalar is like a 1 directional vector, either positive or negative, and its magnitude is the absolute value of the scalar

### 1.1.2 Notation

#### Definition 1.1.4: Positive y axis



#### Definition 1.1.5: Standard Basis Vectors

In an  $n$ -dimensional space  $\mathbb{R}^n$ , the standard basis vectors are a set of  $n$  vectors where each vector has a 1 in one component and 0 in all other components. These vectors are denoted as  $\mathbf{e}_i$  for  $i = 1, 2, \dots, n$ . The  $i$ -th standard basis vector in  $\mathbb{R}^n$  is written as:

$$\mathbf{e}_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \text{ (in the } i\text{-th position)} \\ \vdots \\ 0 \end{pmatrix}$$

For example, in  $\mathbb{R}^3$  (three-dimensional space), the standard basis vectors are:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

These vectors span the entire vector space  $\mathbb{R}^n$ , meaning any vector  $\mathbf{v} \in \mathbb{R}^n$  can be written as a linear combination of the standard basis vectors:

$$\mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + \dots + v_n \mathbf{e}_n,$$

where  $v_1, v_2, \dots, v_n$  are the components of the vector  $\mathbf{v}$ .

## 1.2 Operations

### 1.2.1 Dot Product

#### Definition 1.2.1: Dot (Scalar) Product Definition

The **scalar product** (or **dot product**) of two vectors  $\mathbf{a}$  and  $\mathbf{b}$  in  $\mathbb{R}^n$  is defined as:

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

In  $\mathbb{R}^3$ , for vectors  $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$  and  $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ , the dot product is:

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

The dot product can also be expressed in terms of the magnitudes of  $\mathbf{a}$  and  $\mathbf{b}$  and the angle  $\theta$  between them:

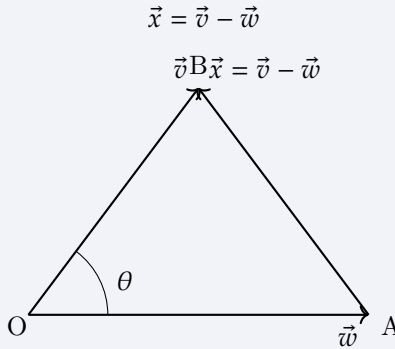
$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

The dot product is a scalar quantity and is zero when the vectors are orthogonal (perpendicular).  
Useful to find the angle between the two vectors being dot producted together,

#### Theorem 1.2.1 Dot Product Proof

We are given the vectors  $\vec{v}$  and  $\vec{w}$ , and we want to express the dot product in terms of their magnitudes and the angle between them.

Start with the relationship:



The above diagram illustrates the vectors  $\vec{v}$ ,  $\vec{w}$ , and their difference  $\vec{x} = \vec{v} - \vec{w}$ , forming a triangle. The angle  $\theta$  is between  $\vec{v}$  and  $\vec{w}$ .

The magnitude squared of  $\vec{x}$  is:

$$|\vec{x}|^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}||\vec{w}| \cos \theta$$

This is the expansion of the law of cosines.

Now, from the equation:

$$|\vec{x}|^2 = \sqrt{((v_x - w_x)^2 + (v_y - w_y)^2)}$$

We conclude:

$$|\vec{x}|^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2(\vec{v} \cdot \vec{w})$$

Thus, we can express the dot product  $\vec{v} \cdot \vec{w}$  as:

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos \theta$$

## 1.2.2 Applications

### Note:-

The dot product of two vectors  $\vec{v} \cdot \vec{w}$  can take different values, leading to various interpretations of the relationship between the vectors. Below is a table describing some key cases:

Dot Product Value	Interpretation	Relationship Between Vectors
$\vec{v} \cdot \vec{w} = 0$	$\cos \theta = 0$	Vectors are <b>perpendicular</b> (orthogonal), $\theta = 90^\circ$
$\vec{v} \cdot \vec{w} > 0$	$0 < \theta < 90^\circ$	Vectors form an <b>acute angle</b> , pointing in the same general direction
$\vec{v} \cdot \vec{w} < 0$	$90^\circ < \theta < 180^\circ$	Vectors form an <b>obtuse angle</b> , pointing in opposite general directions
$\vec{v} \cdot \vec{w} =  \vec{v}  \vec{w} $	$\cos \theta = 1$	Vectors are <b>parallel</b> and point in the <b>same direction</b> , $\theta = 0^\circ$
$\vec{v} \cdot \vec{w} = - \vec{v}  \vec{w} $	$\cos \theta = -1$	Vectors are <b>parallel</b> but point in <b>opposite directions</b> , $\theta = 180^\circ$

### Definition 1.2.2: Vector Product (Cross Product)

The **vector product** (or **cross product**) of two vectors  $\mathbf{a}$  and  $\mathbf{b}$  in  $\mathbb{R}^3$  is a vector  $\mathbf{c}$  that is perpendicular to both  $\mathbf{a}$  and  $\mathbf{b}$ , and its magnitude is given by:

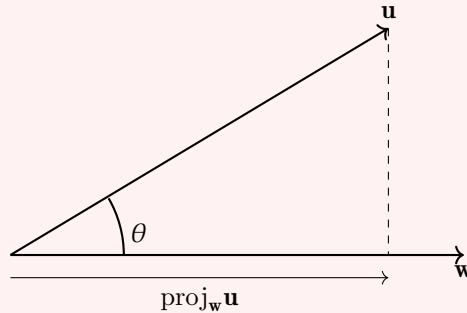
$$|\mathbf{c}| = |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

where  $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b}$ . The cross product is calculated as:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

The result of a cross product is a vector perpendicular to the plane formed by  $\mathbf{a}$  and  $\mathbf{b}$ , with a direction given by the right-hand rule.

### Definition 1.2.3: Vector Projections



$$\text{scal}_{\mathbf{w}} \mathbf{u} = |\mathbf{u}| \cdot \cos \theta = \frac{\mathbf{w} \cdot \mathbf{u}}{|\mathbf{w}|}$$

$$\text{proj}_{\mathbf{w}} \mathbf{u} = |\mathbf{u}| \cos \theta \left( \frac{\mathbf{w}}{|\mathbf{w}|} \right)$$

$$\text{proj}_{\mathbf{w}} \mathbf{u} = \left( \frac{\mathbf{w} \cdot \mathbf{u}}{\mathbf{w} \cdot \mathbf{w}} \right) \mathbf{w}$$

## 1.3 Matrix Determinants

### Definition 1.3.1: Matrix Representation

A matrix is a collection of numbers arranged in a grid format, where each element is positioned based on its row and column. A general  $m \times n$  matrix is written as:

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

For example, a  $2 \times 2$  matrix is given by:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

A  $3 \times 3$  matrix is:

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Matrices can be considered as a collection of vectors where each row or column can represent a vector.

### Note:-

#### Vector Representation

A matrix can also be viewed as a collection of vectors. For instance, a  $3 \times 3$  matrix can be interpreted as:

$$M = \begin{pmatrix} \vec{v}_1 = \langle a, b, c \rangle \\ \vec{v}_2 = \langle d, e, f \rangle \\ \vec{v}_3 = \langle g, h, i \rangle \end{pmatrix}$$

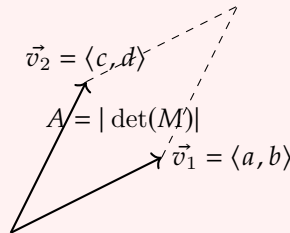
where each row (or column) is treated as a vector in space.

### Definition 1.3.2: Determinant of a $2 \times 2$ Matrix

The determinant of a  $2 \times 2$  matrix is given by:

$$\det(M) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

The determinant represents the signed area of the parallelogram formed by the vectors corresponding to the rows (or columns) of the matrix.



### Note:-

#### Geometric Interpretation

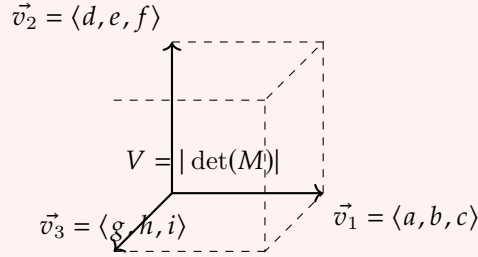
For a  $2 \times 2$  matrix, the determinant represents the area  $A$  of the parallelogram formed by the two vectors  $\vec{v}_1 = \langle a, b \rangle$  and  $\vec{v}_2 = \langle c, d \rangle$ . The magnitude of the determinant gives the area of this parallelogram, and the sign of the determinant indicates the orientation (whether the vectors are ordered clockwise or counterclockwise).

### Definition 1.3.3: Determinant of a $3 \times 3$ Matrix

The determinant of a  $3 \times 3$  matrix is calculated as:

$$\det(M) = \det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \det \begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

The determinant represents the signed volume of the parallelepiped formed by the three vectors corresponding to the rows (or columns) of the matrix.



#### Note:-

#### Geometric Interpretation for $3 \times 3$

In the  $3 \times 3$  case, the determinant represents the volume  $V$  of the parallelepiped formed by three vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3$ , and the sign indicates whether the orientation is right-handed or left-handed. The magnitude gives the volume.

## 1.4 Matrix multiplication with 2D Vectors

### Definition 1.4.1: Vector Matrix Multiplication

$$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

$$\hat{\mathbf{j}}M = \langle a_{11}V_1 + a_{12}V_2, a_{21}V_1 + a_{22}V_2 \rangle$$

Given:

$$\hat{i} = \langle 1, 0 \rangle \quad \hat{j} = \langle 0, 1 \rangle$$

We can compute:

$$iM = \langle a_{11}, a_{12} \rangle = a_1$$

$$jM = \langle a_{21}, a_{22} \rangle = a_2$$

Where:

$$\mathbf{V} = V_1\hat{i} + V_2\hat{j}$$

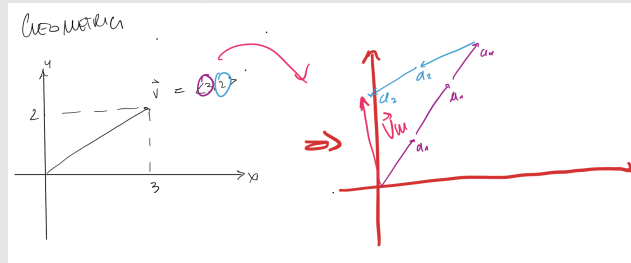
$$\hat{\mathbf{V}}M = (V_1\hat{i} + V_2\hat{j})M$$

$$= V_1\hat{i}M + V_2\hat{j}M$$

$$= V_1\mathbf{a}_1 + V_2\mathbf{a}_2$$



**Note:-**



### 1.4.1 Effect on Area

#### Definition 1.4.2: 2D

The original point  $(1,1)$  is transformed by the matrix  $M$ . This transformation impacts the area and orientation as follows:

$$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

The area after transformation is given by the determinant of the matrix:

$$\text{Area} = \det(M)$$

Where the determinant is calculated as:

$$\det(M) = a_{11}a_{22} - a_{12}a_{21}$$

The determinant also determines the orientation:

$$\det(M) = \begin{cases} A & \text{if } a_1 \text{ to } a_2 \text{ is counterclockwise} \\ -A & \text{otherwise} \end{cases}$$

In the example, the original vectors  $a_1$  and  $a_2$  form an area, and the determinant will tell us if the vectors are oriented in a clockwise or counterclockwise fashion.

If the determinant is negative, the orientation is clockwise, as illustrated:

$$\det \begin{pmatrix} a_1 & a_2 \end{pmatrix} < 0$$

Thus, in this case, the transformation results in a clockwise orientation.

## 1.5 Matrix multiplication with 3D Vectors

### Definition 1.5.1: 3D

The matrix  $M$  for a 3D transformation is given as:

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \text{where} \quad \vec{V} = \langle V_1, V_2, V_3 \rangle$$

The transformation of vector  $\vec{V}$  under matrix  $M$  is:

$$\hat{V}M = \langle (a_{11}V_1 + a_{12}V_2 + a_{13}V_3), (a_{21}V_1 + a_{22}V_2 + a_{23}V_3), (a_{31}V_1 + a_{32}V_2 + a_{33}V_3) \rangle$$

This can be written in terms of the basis vectors as:

$$(V_1\hat{i} + V_2\hat{j} + V_3\hat{k})M = V_1\vec{a}_1 + V_2\vec{a}_2 + V_3\vec{a}_3$$

### Definition 1.5.2: orientation and Volume

- If the determinant of matrix  $M$  is negative, the system is **left-handed**, i.e.,

$$\det(M) = -V$$

- The determinant of the matrix  $M$  gives the **volume** of the parallelepiped spanned by the vectors  $a_1, a_2, a_3$ :

$$\det(M) = \text{Volume}(V)$$

The volume  $V$  is given by:

$$V = \begin{cases} +V & \text{if } \vec{a}_1, \vec{a}_2, \vec{a}_3 \text{ are right-handed (RHS)} \\ -V & \text{otherwise (left-handed)} \end{cases}$$

## 1.6 Cross Product and Volumes

### Definition 1.6.1: Cross Product and Volumes

The volume of a parallelepiped defined by three vectors  $\vec{u}, \vec{v}, \vec{w}$  is given by:

$$V = \vec{u} \cdot (\vec{v} \times \vec{w})$$

### 1.6.1 Link to Matrix Determinants

#### Definition 1.6.2: Cross Product and Matrix Determinants

Since:

$$\begin{aligned}
 \vec{u} \cdot (\vec{v} \times \vec{w}) &= \det \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix} \\
 \det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} &= u_1 \det \begin{pmatrix} v_2 & v_3 \\ w_2 & w_3 \end{pmatrix} - u_2 \det \begin{pmatrix} v_1 & v_3 \\ w_1 & w_3 \end{pmatrix} + u_3 \det \begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix} \\
 &= \vec{u} \cdot \left( \hat{i} \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - \hat{j} \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + \hat{k} \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right) \\
 &= \vec{u} \cdot \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} \\
 &= \vec{u} \cdot (\vec{v} \times \vec{w})
 \end{aligned}$$

Therefore:

$$\vec{v} \times \vec{w} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$$

## 1.7 Cross Product Polynomial Multiplication

#### Definition 1.7.1: Properties

$$\begin{aligned}
 \hat{i} \times \hat{j} &= \hat{k}, & \hat{j} \times \hat{k} &= \hat{i}, & \hat{k} \times \hat{i} &= \hat{j} \\
 \hat{j} \times \hat{i} &= -\hat{k}, & \hat{i} \times \hat{i} &= \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0
 \end{aligned}$$

#### Example 1.7.1 (Example: Cross Product)

Let  $\vec{v} = 2\hat{i} - \hat{j} - 3\hat{k}$  and  $\vec{w} = \hat{i} + \hat{j} + \hat{k}$ . The cross product  $\vec{v} \times \vec{w}$  is computed as:

$$\vec{v} \times \vec{w} = (2\hat{i} - \hat{j} - 3\hat{k}) \times (\hat{i} + \hat{j} + \hat{k})$$

Expanding the cross product term by term:

$$= 2\hat{i} \times \hat{i} + 2\hat{i} \times \hat{j} + 2\hat{i} \times \hat{k} - \hat{j} \times \hat{i} - \hat{j} \times \hat{j} - \hat{j} \times \hat{k} - 3\hat{k} \times \hat{i} - 3\hat{k} \times \hat{j} - 3\hat{k} \times \hat{k}$$

Using the cross product identities:

$$= 0 + 2\hat{k} + 2(-\hat{j}) - (-\hat{k}) + 0 - \hat{i} - 3\hat{j} + 3\hat{i} + 0$$

Combining like terms:

$$\begin{aligned}
 &= (3\hat{i} - \hat{i}) + (-2\hat{j} - 3\hat{j}) + (2\hat{k} + \hat{k}) \\
 &= 2\hat{i} - 5\hat{j} + 3\hat{k}
 \end{aligned}$$

Thus, the final result is:

$$\vec{v} \times \vec{w} = 2\hat{i} - 5\hat{j} + 3\hat{k}$$

## 1.8 Torque

### Definition 1.8.1: Torque and Angular Momentum

Continue with Torque