

CASMA 225

Calc 3

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Contents

Chapter 1	Vector Valued Functions $f : \mathbb{R} \rightarrow \mathbb{R}^n$	Page 4
1.1	Review	4
	Basics — 4 • Notation — 5	
1.2	Operations	6
	Dot Product — 6 • Applications — 7	
1.3	Matrix Determinants	8
1.4	Matrix multiplication with 2D Vectors	9
	Effect on Area — 10	
1.5	Matrix multiplication with 3D Vectors	11
1.6	Cross Product and Volumes	11
	Link to Matrix Determinants — 12	
1.7	Cross Product Polynomial Multiplication	12
1.8	Torque	13
1.9	Parametric Equations	13
	Examples — 13	
1.10	Distance from a Point to a Line	15
	Example — 15	
1.11	Intersection of Two Parametric Lines	16
1.12	Planes	16
1.13	Vector Valued Functions	17
1.14	Calculus and Vector Valued Functions	19
	Derivative — 19 • Chain Rule — 22 • Product Rule — 23 • Integrals — 23	
1.15	Arc Length	25
1.16	Curvature of a Vector-Valued Function	27
	Osculating Circle — 27 • Curvature $\kappa(t)$ — 28 • Deriving the Perpendicularity — 30	
1.17	Calculating Curvature	30
	Case 1: $\vec{r}(t)$ travels at unit speed around a circle — 30 • Case 2: General Curve with Unit Speed — 31 • Case 3: General Curve with Variable Speed — 32	
1.18	Components of Acceleration	33
1.19	Binormal Vector	33
	Change in the Plane of Motion — 33 • Torsion and its Relation to the Binormal Vector — 34 • General Case and Torsion Formula — 34 • Expressing the Binormal Vector with Velocity and Acceleration — 35	
Chapter 2	Functions of Multiple Random Variables $f : \mathbb{R}^n \rightarrow \mathbb{R}$	Page 37
2.1	Plotting Multiple Random Variable Functions	37
2.2	Domain of Multiple Random Variable Functions	38
2.3	Range of Multiple Random Variable Functions	39
2.4	Level Curves of Functions of Two Variables	39

2.5	Limits of Two-Variable Functions	40
	Limits by substituting with function $f : \mathbb{R} \rightarrow \mathbb{R}$ — 42 • Limits by substituting with Vector Valued Functions $f : \mathbb{R} \rightarrow \mathbb{R}^n$ — 43	
2.6	Tips for Finding Limits of Two-Variable Functions (2VF)	44
	Polynomials are Continuous — 44 • Compositions of Continuous Functions are Continuous — 44 • Rational Functions — 45 • Special Limits: Squeeze Theorem — 45 • Use of Polar Coordinates — 45 • Testing Multiple Paths — 45 • Continuity of Common Functions — 45 • Rational functions — 45 • Case 1: $g(a, b) \neq 0$ — 46 • Case 2: $g(a, b) = 0$ and $h(a, b) \neq 0$ — 46 • Case 3: $h(a, b) = g(a, b) = 0$ — 46 • Case 4: $h(a, b) = g(a, b) = 0$ still — 46 • Case 5: Still getting the same limit — 47	
2.7	Properties of Limits of Two-Variable Functions	47
	Scaling by a constant: — 47 • Addition of functions: — 47 • Multiplication of functions: — 47 • Division of functions: — 48 • Other — 48	
2.8	Partial Derivatives	48
	Differentiability — 49 • Open and Closed Sets — 49 • Conditions for Differentiability — 50	
2.9	Directional Derivative	52
	Derivation of the Directional Derivative — 52	
2.10	Application of the Gradient	52
	Directional Derivative and the Gradient — 52 • Dot Product and Angle — 53 • Maximum and Minimum Values of the Directional Derivative — 53 • Tangent to a Level Curve — 53	
2.11	Tangent Planes	54
	Tangent Line for Single Variable Functions — 54 • Tangent Plane for Two Variable Functions — 54 • Implicit Case — 56 • Level Surfaces and Tangent Planes — 56 • Summary of Cases — 56	
2.12	Applications of Tangent Planes	57
	Linear Approximation — 57 • Finding Approximate Solutions to Equations — 58 • Example: Linearizing $f(x, y, z) = z \ln(z) + xz - e^{yz}$ — 59 • Differentials — 59	
2.13	Chain Rule in Functions of several random variables $f : \mathbb{R}^n \rightarrow \mathbb{R}$	60
2.14	Maximum and Minimum of 2 Variable Functions $f : \mathbb{R} \rightarrow \mathbb{R}^2$	63
	Critical Points — 63 • Definitions — 63 • Visual Representation of Maxima, Minima and Saddle Points — 63 • Second Derivative Test — 64 • Connection Between the Second Derivative Test and Eigenvalues of the Hessian — 65 • Intuition from Linear Algebra — 65	
2.15	Absolute Extrema	66
2.16	Lagrange Multiplier Method	67
	General Procedure — 67	
2.17	Multiple Integration	70
	Non-Rectangular Domains — 72 • Volume Between Surfaces — 73	
2.18	Applications of Double Integrals	74
	Area of a Surface — 74 • Average Value of a Function — 75	
2.19	Derivation of Triple Integrals through Riemann Sums	76
	Partitioning the Region — 76 • Forming the Riemann Sum — 76	
2.20	Triple Integrals Intuitions and Applications	77
	Integrals Represent Products — 77	
2.21	Polar Coordinate System	81
2.22	Basic Equations	83
	Ellipse — 83 • Line — 83 • Cardioid — 83	
2.23	Slope of a Curve	83
2.24	Deriving the Integral for Area Under a Curve in Polar Coordinates	85
2.25	Arc Length in Polar Coordinates	87
2.26	Multivariate Case	87
2.27	General Procedure for Polar Integration	88
2.28	Cylindrical Coordinate Axes	89
2.29	Spherical Coordinate Axes	90
2.30	Shapes Represented in Spherical Coordinates	90
	Overview of Spherical Coordinates — 90 • Shapes in Spherical Coordinates — 91	
2.31	Change of Variable for Double and Triple Integrals	92

2.32	Non-overlapping from Mapping T	94
2.33	Integral Transformation for a Parallelogram Region	95
2.34	Integral Transformation for a Triangular Region	96

Chapter 3	Vector Fields $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$	Page 98
3.1	Vector Fields	98
	Properties — 98 • Applications — 98 • Drawing a Vector Field — 98	
3.2	Gradient Vector Fields	99
3.3	Line Integrals	100
3.4	Vector Line Integrals	101
3.5	Fundamental Theorem of Line Integrals	102
3.6	Vector Line Integrals and Gradient Fields	103
	Vector Line Integral for a Curve C — 103 • Gradient Fields and Fundamental Theorem of Line Integrals — 103	
3.7	Gradient Vector Fields and Closed Curves	104
	Definition of a Closed Curve — 104 • How to Identify a Gradient Vector Field — 104 • Finding the Potential Function $\phi(x, y)$ — 104	
3.8	Non-Gradient Vector Fields and Green's Theorem	105
	Non-Gradient Vector Fields — 105 • Green's Theorem — 106	
3.9	Line Integrals and Area	107
3.10	Flux of a Vector Field	108

Chapter 1

Vector Valued Functions $f : \mathbb{R} \rightarrow \mathbb{R}^n$

1.1 Review

1.1.1 Basics

Definition 1.1.1: Notation

Drawn Vectors: \vec{v}

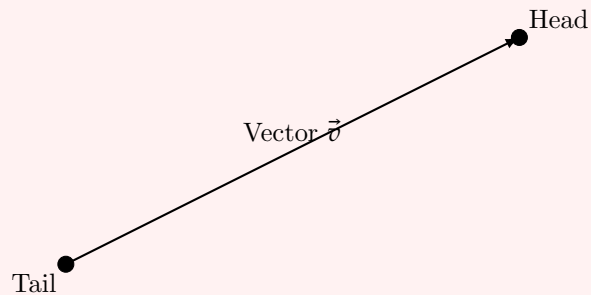
Typed Vectors: \mathbf{v}

Definition 1.1.2: Velocity

Magnitude of the velocity: $|\vec{v}|$

Direction of the velocity: $dir(\vec{v})$

Definition 1.1.3: Heads and Tails

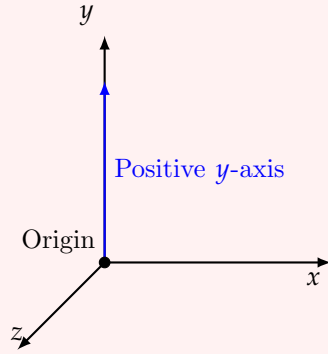


Note:-

Scalar is like a 1 directional vector, either positive or negative, and its magnitude is the absolute value of the scalar

1.1.2 Notation

Definition 1.1.4: Positive y axis



Definition 1.1.5: Standard Basis Vectors

In an n -dimensional space \mathbb{R}^n , the standard basis vectors are a set of n vectors where each vector has a 1 in one component and 0 in all other components. These vectors are denoted as \mathbf{e}_i for $i = 1, 2, \dots, n$. The i -th standard basis vector in \mathbb{R}^n is written as:

$$\mathbf{e}_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \text{ (in the } i\text{-th position)} \\ \vdots \\ 0 \end{pmatrix}$$

For example, in \mathbb{R}^3 (three-dimensional space), the standard basis vectors are:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

These vectors span the entire vector space \mathbb{R}^n , meaning any vector $\mathbf{v} \in \mathbb{R}^n$ can be written as a linear combination of the standard basis vectors:

$$\mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + \dots + v_n \mathbf{e}_n,$$

where v_1, v_2, \dots, v_n are the components of the vector \mathbf{v} .

1.2 Operations

1.2.1 Dot Product

Definition 1.2.1: Dot (Scalar) Product Definition

The **scalar product** (or **dot product**) of two vectors \mathbf{a} and \mathbf{b} in \mathbb{R}^n is defined as:

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

In \mathbb{R}^3 , for vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, the dot product is:

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

The dot product can also be expressed in terms of the magnitudes of \mathbf{a} and \mathbf{b} and the angle θ between them:

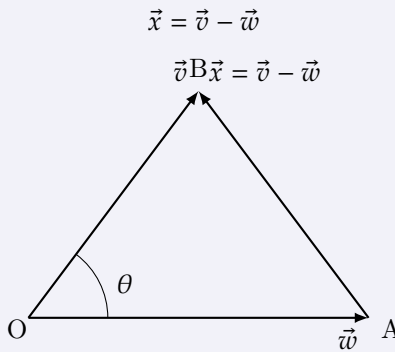
$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

The dot product is a scalar quantity and is zero when the vectors are orthogonal (perpendicular).
Useful to find the angle between the two vectors being dot product together,

Theorem 1.2.1 Dot Product Proof

We are given the vectors \vec{v} and \vec{w} , and we want to express the dot product in terms of their magnitudes and the angle between them.

Start with the relationship:



The above diagram illustrates the vectors \vec{v} , \vec{w} , and their difference $\vec{x} = \vec{v} - \vec{w}$, forming a triangle. The angle θ is between \vec{v} and \vec{w} .

The magnitude squared of \vec{x} is:

$$|\vec{x}|^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}||\vec{w}| \cos \theta$$

This is the expansion of the law of cosines.

Now, from the equation:

$$|\vec{x}|^2 = \sqrt{((v_x - w_x)^2 + (v_y - w_y)^2)}$$

We conclude:

$$|\vec{x}|^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2(\vec{v} \cdot \vec{w})$$

Thus, we can express the dot product $\vec{v} \cdot \vec{w}$ as:

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos \theta$$

1.2.2 Applications

Note:-

The dot product of two vectors $\vec{v} \cdot \vec{w}$ can take different values, leading to various interpretations of the relationship between the vectors. Below is a table describing some key cases:

Dot Product Value	Interpretation	Relationship Between Vectors
$\vec{v} \cdot \vec{w} = 0$	$\cos \theta = 0$	Vectors are perpendicular (orthogonal), $\theta = 90^\circ$
$\vec{v} \cdot \vec{w} > 0$	$0 < \theta < 90^\circ$	Vectors form an acute angle , pointing in the same general direction
$\vec{v} \cdot \vec{w} < 0$	$90^\circ < \theta < 180^\circ$	Vectors form an obtuse angle , pointing in opposite general directions
$\vec{v} \cdot \vec{w} = \vec{v} \vec{w} $	$\cos \theta = 1$	Vectors are parallel and point in the same direction , $\theta = 0^\circ$
$\vec{v} \cdot \vec{w} = - \vec{v} \vec{w} $	$\cos \theta = -1$	Vectors are parallel but point in opposite directions , $\theta = 180^\circ$

Definition 1.2.2: Vector Product (Cross Product)

The **vector product** (or **cross product**) of two vectors \mathbf{a} and \mathbf{b} in \mathbb{R}^3 is a vector \mathbf{c} that is perpendicular to both \mathbf{a} and \mathbf{b} , and its magnitude is given by:

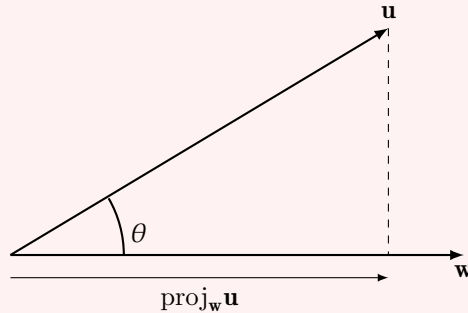
$$|\mathbf{c}| = |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

where θ is the angle between \mathbf{a} and \mathbf{b} . The cross product is calculated as:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

The result of a cross product is a vector perpendicular to the plane formed by \mathbf{a} and \mathbf{b} , with a direction given by the right-hand rule.

Definition 1.2.3: Vector Projections



$$\text{scal}_{\mathbf{w}} \mathbf{u} = |\mathbf{u}| \cdot \cos \theta = \frac{\mathbf{w} \cdot \mathbf{u}}{|\mathbf{w}|}$$

$$\text{proj}_{\mathbf{w}} \mathbf{u} = |\mathbf{u}| \cos \theta \left(\frac{\mathbf{w}}{|\mathbf{w}|} \right)$$

$$\text{proj}_{\mathbf{w}} \mathbf{u} = \left(\frac{\mathbf{w} \cdot \mathbf{u}}{\mathbf{w} \cdot \mathbf{w}} \right) \mathbf{w}$$

1.3 Matrix Determinants

Definition 1.3.1: Matrix Representation

A matrix is a collection of numbers arranged in a grid format, where each element is positioned based on its row and column. A general $m \times n$ matrix is written as:

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

For example, a 2×2 matrix is given by:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

A 3×3 matrix is:

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Matrices can be considered as a collection of vectors where each row or column can represent a vector.

Note:-

Vector Representation

A matrix can also be viewed as a collection of vectors. For instance, a 3×3 matrix can be interpreted as:

$$M = \begin{pmatrix} \vec{v}_1 = \langle a, b, c \rangle \\ \vec{v}_2 = \langle d, e, f \rangle \\ \vec{v}_3 = \langle g, h, i \rangle \end{pmatrix}$$

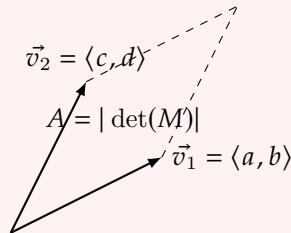
where each row (or column) is treated as a vector in space.

Definition 1.3.2: Determinant of a 2×2 Matrix

The determinant of a 2×2 matrix is given by:

$$\det(M) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

The determinant represents the signed area of the parallelogram formed by the vectors corresponding to the rows (or columns) of the matrix.



Note:-

Geometric Interpretation

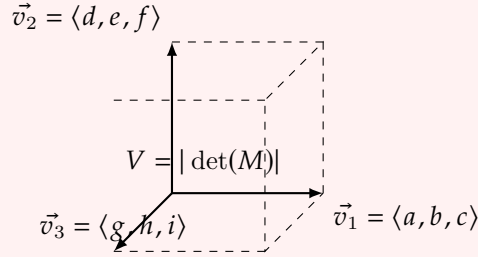
For a 2×2 matrix, the determinant represents the area A of the parallelogram formed by the two vectors $\vec{v}_1 = \langle a, b \rangle$ and $\vec{v}_2 = \langle c, d \rangle$. The magnitude of the determinant gives the area of this parallelogram, and the sign of the determinant indicates the orientation (whether the vectors are ordered clockwise or counterclockwise).

Definition 1.3.3: Determinant of a 3×3 Matrix

The determinant of a 3×3 matrix is calculated as:

$$\det(M) = \det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \det \begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

The determinant represents the signed volume of the parallelepiped formed by the three vectors corresponding to the rows (or columns) of the matrix.



Note:-

Geometric Interpretation for 3×3

In the 3×3 case, the determinant represents the volume V of the parallelepiped formed by three vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$, and the sign indicates whether the orientation is right-handed or left-handed. The magnitude gives the volume.

1.4 Matrix multiplication with 2D Vectors

Definition 1.4.1: Vector Matrix Multiplication

$$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

$$\hat{\mathbf{j}}M = \langle a_{11}V_1 + a_{12}V_2, a_{21}V_1 + a_{22}V_2 \rangle$$

Given:

$$\hat{i} = \langle 1, 0 \rangle \quad \hat{j} = \langle 0, 1 \rangle$$

We can compute:

$$iM = \langle a_{11}, a_{12} \rangle = a_1$$

$$jM = \langle a_{21}, a_{22} \rangle = a_2$$

Where:

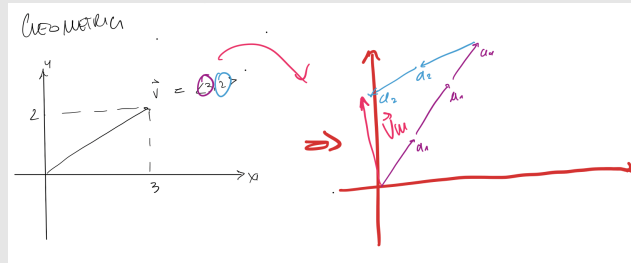
$$\mathbf{V} = V_1\hat{i} + V_2\hat{j}$$

$$\hat{\mathbf{V}}M = (V_1\hat{i} + V_2\hat{j})M$$

$$= V_1\hat{i}M + V_2\hat{j}M$$

$$= V_1\mathbf{a}_1 + V_2\mathbf{a}_2$$

Note:-



1.4.1 Effect on Area

Definition 1.4.2: 2D

The original point $(1, 1)$ is transformed by the matrix M . This transformation impacts the area and orientation as follows:

$$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

The area after transformation is given by the determinant of the matrix:

$$\text{Area} = \det(M)$$

Where the determinant is calculated as:

$$\det(M) = a_{11}a_{22} - a_{12}a_{21}$$

The determinant also determines the orientation:

$$\det(M) = \begin{cases} A & \text{if } a_1 \text{ to } a_2 \text{ is counterclockwise} \\ -A & \text{otherwise} \end{cases}$$

In the example, the original vectors a_1 and a_2 form an area, and the determinant will tell us if the vectors are oriented in a clockwise or counterclockwise fashion.

If the determinant is negative, the orientation is clockwise, as illustrated:

$$\det((a_1 \ a_2)) < 0$$

Thus, in this case, the transformation results in a clockwise orientation.

1.5 Matrix multiplication with 3D Vectors

Definition 1.5.1: 3D

The matrix M for a 3D transformation is given as:

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \text{where} \quad \vec{V} = \langle V_1, V_2, V_3 \rangle$$

The transformation of vector \vec{V} under matrix M is:

$$\hat{V}M = \langle (a_{11}V_1 + a_{12}V_2 + a_{13}V_3), (a_{21}V_1 + a_{22}V_2 + a_{23}V_3), (a_{31}V_1 + a_{32}V_2 + a_{33}V_3) \rangle$$

This can be written in terms of the basis vectors as:

$$(V_1\hat{i} + V_2\hat{j} + V_3\hat{k})M = V_1\vec{a}_1 + V_2\vec{a}_2 + V_3\vec{a}_3$$

Definition 1.5.2: orientation and Volume

- If the determinant of matrix M is negative, the system is **left-handed**, i.e.,

$$\det(M) = -V$$

- The determinant of the matrix M gives the **volume** of the parallelepiped spanned by the vectors a_1, a_2, a_3 :

$$\det(M) = \text{Volume}(V)$$

The volume V is given by:

$$V = \begin{cases} +V & \text{if } \vec{a}_1, \vec{a}_2, \vec{a}_3 \text{ are right-handed (RHS)} \\ -V & \text{otherwise (left-handed)} \end{cases}$$

1.6 Cross Product and Volumes

Definition 1.6.1: Cross Product and Volumes

The volume of a parallelepiped defined by three vectors $\vec{u}, \vec{v}, \vec{w}$ is given by:

$$V = \vec{u} \cdot (\vec{v} \times \vec{w})$$

1.6.1 Link to Matrix Determinants

Definition 1.6.2: Cross Product and Matrix Determinants

Since:

$$\begin{aligned}
 \vec{u} \cdot (\vec{v} \times \vec{w}) &= \det \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix} \\
 \det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} &= u_1 \det \begin{pmatrix} v_2 & v_3 \\ w_2 & w_3 \end{pmatrix} - u_2 \det \begin{pmatrix} v_1 & v_3 \\ w_1 & w_3 \end{pmatrix} + u_3 \det \begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix} \\
 &= \vec{u} \cdot \left(\hat{i} \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - \hat{j} \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + \hat{k} \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right) \\
 &= \vec{u} \cdot \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} \\
 &= \vec{u} \cdot (\vec{v} \times \vec{w})
 \end{aligned}$$

Therefore:

$$\vec{v} \times \vec{w} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$$

1.7 Cross Product Polynomial Multiplication

Definition 1.7.1: Properties

$$\begin{aligned}
 \hat{i} \times \hat{j} &= \hat{k}, & \hat{j} \times \hat{k} &= \hat{i}, & \hat{k} \times \hat{i} &= \hat{j} \\
 \hat{j} \times \hat{i} &= -\hat{k}, & \hat{i} \times \hat{i} &= \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0
 \end{aligned}$$

Example 1.7.1 (Example: Cross Product)

Let $\vec{v} = 2\hat{i} - \hat{j} - 3\hat{k}$ and $\vec{w} = \hat{i} + \hat{j} + \hat{k}$. The cross product $\vec{v} \times \vec{w}$ is computed as:

$$\vec{v} \times \vec{w} = (2\hat{i} - \hat{j} - 3\hat{k}) \times (\hat{i} + \hat{j} + \hat{k})$$

Expanding the cross product term by term:

$$= 2\hat{i} \times \hat{i} + 2\hat{i} \times \hat{j} + 2\hat{i} \times \hat{k} - \hat{j} \times \hat{i} - \hat{j} \times \hat{j} - \hat{j} \times \hat{k} - 3\hat{k} \times \hat{i} - 3\hat{k} \times \hat{j} - 3\hat{k} \times \hat{k}$$

Using the cross product identities:

$$= 0 + 2\hat{k} + 2(-\hat{j}) - (-\hat{k}) + 0 - \hat{i} - 3\hat{j} + 3\hat{i} + 0$$

Combining like terms:

$$\begin{aligned}
 &= (3\hat{i} - \hat{i}) + (-2\hat{j} - 3\hat{j}) + (2\hat{k} + \hat{k}) \\
 &= 2\hat{i} - 5\hat{j} + 3\hat{k}
 \end{aligned}$$

Thus, the final result is:

$$\vec{v} \times \vec{w} = 2\hat{i} - 5\hat{j} + 3\hat{k}$$

1.8 Torque

Definition 1.8.1: Torque and Angular Momentum

Continue with Torque

1.9 Parametric Equations

Definition 1.9.1: Parametric Equations

A parametric equation expresses a set of quantities as explicit functions of an independent parameter. In a two-dimensional case, a parametric equation for a curve can be represented as:

$$\langle x, y \rangle t = \langle f(t), g(t) \rangle$$

and in three dimensions as:

$$\langle x, y, z \rangle t = \langle f(t), g(t), h(t) \rangle$$

Note:-

For example, consider the curve in the plane given by the equation $y = f(x) = x^2 + 1$. This describes a parabola in Cartesian coordinates.

Theorem 1.9.1 Parametric unit circle in Cartesian coordinates

A unit circle in parametric form can be represented as:

$$x^2 + y^2 = 1$$

which corresponds to the parametric equations:

$$\langle x, y \rangle t = \langle \cos(t), \sin(t) \rangle$$

1.9.1 Examples

Note:-

The parametric equation:

$$\langle x, y \rangle t = \langle 4 \cos(t), 3 \sin(t) \rangle$$

At specific values of t , we can compute the points:

$$t = 0 \implies \langle 4, 0 \rangle$$

$$t = \frac{\pi}{2} \implies \langle 0, 3 \rangle$$

$$t = \pi \implies \langle -4, 0 \rangle$$

$$t = \frac{3\pi}{2} \implies \langle 0, -3 \rangle$$

Definition 1.9.2: Parametric for a Helix

$$\langle x(t), y(t), z(t) \rangle = \langle 4 \cos(t), 3 \sin(t), 0.1t \rangle$$

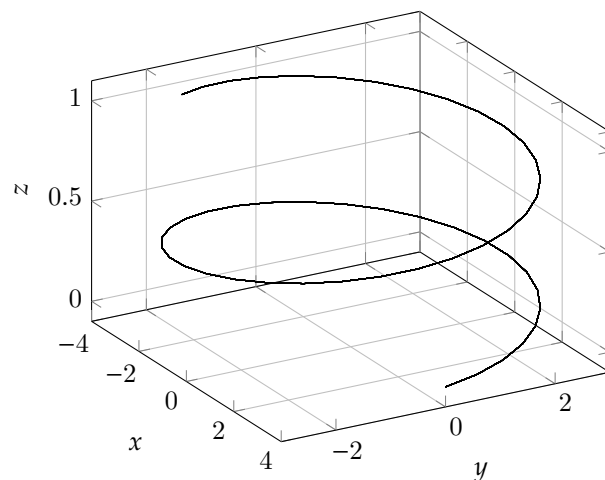


Figure 1.1: 3D plot of a parametric helix.

Theorem 1.9.2 Parametric Equation of a Line

The parametric equation of a line can be expressed as:

$$\langle x, y, z \rangle t = \mathbf{OP} + \mathbf{V}t$$

Where:

- $\mathbf{OP} = \langle x_0, y_0, z_0 \rangle$ is the position vector to the initial point P ,
- $\mathbf{V} = \langle v_x, v_y, v_z \rangle$ is the direction vector of the line.

Question 1: 3D Parametric Equation of a Line

- $\mathbf{OP} = \langle 1, 2, 3 \rangle$ is the position vector to the initial point P ,
- $\mathbf{V} = \langle 1, 1, 1 \rangle$ is the direction vector of the line.

Thus, the parametric equation of the line becomes:

$$\langle x(t), y(t), z(t) \rangle = \langle 1, 2, 3 \rangle + t\langle 1, 1, 1 \rangle$$

$$x(t) = 1 + t, \quad y(t) = 2 + t, \quad z(t) = 3 + t$$

Or simply:

$$\langle x, y, z \rangle t = \langle 1 + t, 2 + t, 3 + t \rangle$$

Solution:

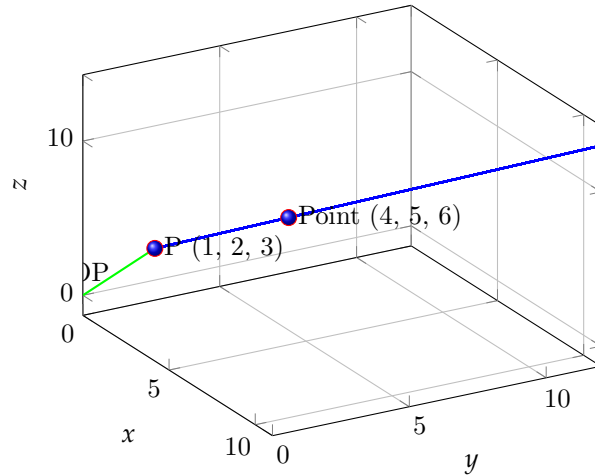


Figure 1.2: 3D plot of a parametric line with vector \mathbf{OP} and points.

1.10 Distance from a Point to a Line

Definition 1.10.1: Parametric Equation of the Line

The line is represented by:

$$\mathbf{l} = \mathbf{OP} + t\mathbf{V}$$

Where:

- \mathbf{OP} is the position vector of a point on the line,
- \mathbf{V} is the direction vector of the line.

Theorem 1.10.1 Distance from a Point to a Line

The distance d from a point Q to the line l is given by:

$$d = \frac{|\mathbf{V} \times \mathbf{PQ}|}{|\mathbf{V}|}$$

Where:

- \mathbf{PQ} is the vector from point P on the line to the point Q ,
- $\mathbf{V} \times \mathbf{PQ}$ is the cross product of the direction vector \mathbf{V} and the vector \mathbf{PQ} .

1.10.1 Example

Question 2: Find the distance from the point $Q = (3, 4, 0)$ to the line l given by the parametric equation

$$\mathbf{l} = \langle t, 1, 2t \rangle = \langle 0, 1, 0 \rangle + t\langle 1, 0, 2 \rangle$$

with point $P = (0, 1, 0)$ and direction vector $\mathbf{V} = \langle 1, 0, 2 \rangle$.

Solution: The vector \mathbf{PQ} from $P = (0, 1, 0)$ to $Q = (3, 4, 0)$ is:

$$\mathbf{PQ} = \langle 3, 4, 0 \rangle - \langle 0, 1, 0 \rangle = \langle 3, 3, 0 \rangle$$

Now, we compute the cross product $\mathbf{V} \times \mathbf{PQ}$:

$$\mathbf{V} \times \mathbf{PQ} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ 3 & 3 & 0 \end{vmatrix} = \hat{i}(0 - 6) - \hat{j}(0 - 6) + \hat{k}(3 - 0) = \langle -6, -6, 3 \rangle$$

Next, calculate the magnitude of the cross product:

$$|\mathbf{V} \times \mathbf{PQ}| = \sqrt{(-6)^2 + (-6)^2 + 3^2} = \sqrt{36 + 36 + 9} = \sqrt{81} = 9$$

The magnitude of the direction vector \mathbf{V} is:

$$|\mathbf{V}| = \sqrt{1^2 + 0^2 + 2^2} = \sqrt{1 + 4} = \sqrt{5}$$

Finally, the distance d is:

$$d = \frac{9}{\sqrt{5}} = \frac{9\sqrt{5}}{5}$$

1.11 Intersection of Two Parametric Lines

Definition 1.11.1: Intersection of Parametric Lines

To find the intersection point of two parametric lines, we need to equate their parametric equations and solve for the parameters.

Question 3: Find the intersection of the lines l_1 and l_2 given by the parametric equations

$$\begin{aligned} l_1 &= \langle x, y \rangle(t) = \langle 0, 1 \rangle + t\langle 1, 0 \rangle \\ l_2 &= \langle 1, 1 \rangle + s\langle -2, 1 \rangle \end{aligned}$$

Solution: Equating the two parametric equations:

$$\langle 0, 1 \rangle + t\langle 1, 0 \rangle = \langle 1, 1 \rangle + s\langle -2, 1 \rangle$$

This gives the system of equations:

$$0 + t = 1 - 2s$$

$$1 + 0 = 1 + s$$

From the second equation, we find:

$$s = 0$$

Substitute $s = 0$ into the first equation:

$$t = 1$$

Thus, the lines intersect when $t = 1$ and $s = 0$.

The intersection point is:

$$\langle 0, 1 \rangle + 1 \cdot \langle 1, 0 \rangle = \langle 1, 1 \rangle$$

Therefore, the lines intersect at $(1, 1)$.

1.12 Planes

Definition 1.12.1: Plane Equation

Given a point $P_0 = (x_0, y_0, z_0)$ on the plane and a normal vector $\vec{n} = \langle a, b, c \rangle$, the equation of the plane can be expressed as:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Note:-**Vector Form**

Alternatively, the plane equation can also be derived using the dot product form:

$$\overrightarrow{PQ} \cdot \vec{n} = 0$$

Where $P = (x_0, y_0, z_0)$ and $Q = (x, y, z)$. This leads to the scalar equation of the plane.

Theorem 1.12.1 Equation of a Plane

If a plane passes through the point $P_0 = (x_0, y_0, z_0)$ and has a normal vector $\vec{n} = \langle a, b, c \rangle$, the equation of the plane is:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Question 4: Example

Given the point $P_0 = (1, -2, 3)$ and the normal vector $\vec{n} = \langle -2, -4, -6 \rangle$, find the equation of the plane.

Solution: Using the plane equation $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$, substitute $a = -2$, $b = -4$, $c = -6$, and $P_0 = (1, -2, 3)$:

$$-2(x - 1) - 4(y + 2) - 6(z - 3) = 0$$

Expanding this equation:

$$-2x + 2 - 4y - 8 - 6z + 18 = 0$$

Simplifying:

$$-2x - 4y - 6z + 12 = 0$$

Or:

$$2x + 4y + 6z = 12$$

1.13 Vector Valued Functions

Definition 1.13.1: Parametric Curves

A parametric curve is defined as:

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle = \langle f(t), h(t), g(t) \rangle$$

where t is a real number as input, and the output is a vector.

Question 5: Example

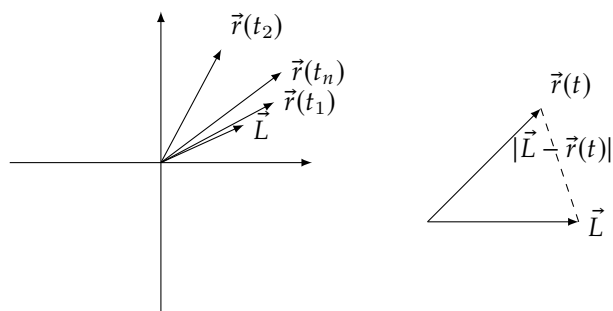
Let $\vec{r}(t) = \langle t^2, t + 1, \sqrt{t - 2} \rangle$. Determine the domain of $\vec{r}(t)$.

Solution: The domain of $\vec{r}(t)$ is $t \geq 2$.

Definition 1.13.2: Limits of Vector Functions

The limit of a vector function $\vec{r}(t)$ as $t \rightarrow a$ can be visualized geometrically as the vector approaching a point \vec{L} . If the magnitude of the difference between \vec{L} and $\vec{r}(t)$ approaches zero, we can define:

$$\lim_{t \rightarrow a} \vec{r}(t) = \vec{L} \quad \text{if} \quad \lim_{t \rightarrow a} |\vec{L} - \vec{r}(t)| = 0$$



Note:-

Component-wise Limits

The limit of a vector function can be evaluated by taking the limit of each of its components:

$$\lim_{t \rightarrow a} \vec{r}(t) = \langle \lim_{t \rightarrow a} x(t), \lim_{t \rightarrow a} y(t), \lim_{t \rightarrow a} z(t) \rangle$$

Question 6: Example

Consider the vector function $\vec{r}(t) = \langle \frac{t^2+2t+1}{t+1}, t, t-2 \rangle$, as $t \rightarrow -1$.

Solution: The limits of each component as $t \rightarrow -1$ are:

$$\lim_{t \rightarrow -1} \vec{r}(t) = \langle 0, -1, -3 \rangle$$

1.14 Calculus and Vector Valued Functions

1.14.1 Derivative

Theorem 1.14.1 Derivative of vector functions

Consider the vector-valued function in 3D:

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

where $f(t)$, $g(t)$, and $h(t)$ are differentiable functions. The derivative of $\vec{r}(t)$ is defined as:

$$\vec{r}'(t) = \lim_{h \rightarrow 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

Expanding $\vec{r}(t+h) - \vec{r}(t)$:

$$\vec{r}(t+h) - \vec{r}(t) = \langle f(t+h), g(t+h), h(t+h) \rangle - \langle f(t), g(t), h(t) \rangle$$

This gives us:

$$\vec{r}(t+h) - \vec{r}(t) = \langle f(t+h) - f(t), g(t+h) - g(t), h(t+h) - h(t) \rangle$$

Thus, the derivative becomes:

$$\vec{r}'(t) = \lim_{h \rightarrow 0} \left\langle \frac{f(t+h) - f(t)}{h}, \frac{g(t+h) - g(t)}{h}, \frac{h(t+h) - h(t)}{h} \right\rangle$$

Since $f(t)$, $g(t)$, and $h(t)$ are differentiable, we apply the definition of derivatives for each component:

$$\vec{r}'(t) = \left\langle \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h}, \lim_{h \rightarrow 0} \frac{g(t+h) - g(t)}{h}, \lim_{h \rightarrow 0} \frac{h(t+h) - h(t)}{h} \right\rangle$$
$$\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

Therefore, the derivative of the vector-valued function $\vec{r}(t)$ in 3D is:

$$\boxed{\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle}$$

Definition 1.14.1: Vector Representation

Let $\vec{r}(t)$ be a vector-valued function that describes the position of a particle over time t :

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

where $x(t)$, $y(t)$, $z(t)$ are differentiable functions. The following terms describe important properties of the function:

1. **Position** at time t : $\vec{r}(t)$
2. **Velocity** (tangent vector) $\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$
3. **Speed** $|\vec{r}'(t)|$, the magnitude of the velocity.
4. **Acceleration** $\vec{r}''(t) = \langle x''(t), y''(t), z''(t) \rangle$

Definition 1.14.2: Unit Tangent Vector

The unit tangent vector is given by:

$$\hat{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

This is the unit vector in the direction of the velocity vector $\vec{r}'(t)$, which represents the direction of motion.

Question 7: Example

Consider the following vector-valued function:

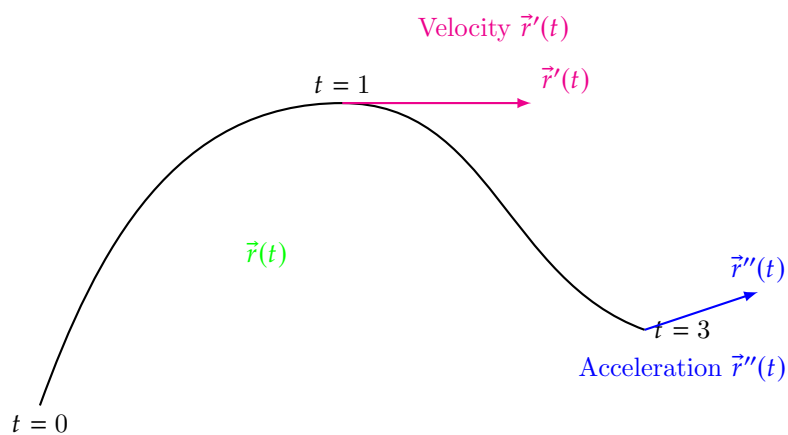
$$\vec{r}(t) = \langle 1 + t^2, 2 + t^2, 3 + t^2 \rangle$$

Taking the derivative:

$$\vec{r}'(t) = \langle 2t, 2t, 2t \rangle$$

This describes a line with a constant acceleration:

$$\vec{r}''(t) = \langle 2, 2, 2 \rangle$$



Definition 1.14.3: Derivatives of Parametric Curves in terms of t

Given a parametric curve defined by:

$$x = f(t), \quad y = g(t)$$

where t is the parameter, we want to find the derivative $\frac{dy}{dx}$, which represents the slope of the tangent to the curve at any point t .

Step 1: Finding the Derivatives of x and y with respect to t To find $\frac{dy}{dx}$, we need to find both $\frac{dx}{dt}$ and $\frac{dy}{dt}$. These are given by:

$$\frac{dx}{dt} = f'(t), \quad \frac{dy}{dt} = g'(t)$$

Step 2: Using the Chain Rule The derivative $\frac{dy}{dx}$ can be found by using the chain rule as follows:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Substituting the expressions for $\frac{dx}{dt}$ and $\frac{dy}{dt}$, we obtain:

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$$

Step 3: Simplifying the Result The expression $\frac{dy}{dx}$ provides the slope of the curve at any point t in terms of the parameter.

Example 1.14.1 (Consider the parametric equations:)

$$x = 6 \sin t, \quad y = 6 \cos t$$

To find $\frac{dy}{dx}$, we calculate the derivatives with respect to t :

$$\frac{dx}{dt} = 6 \cos t, \quad \frac{dy}{dt} = -6 \sin t$$

Then, we apply the formula:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-6 \sin t}{6 \cos t}$$

Simplifying the expression:

$$\frac{dy}{dx} = -\tan t$$

Therefore, the slope of the tangent line to the parametric curve at any point t is:

$$\frac{dy}{dx} = -\tan t$$

Definition 1.14.4: Chain Rule Derivation

$$\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{du}{dt}$$

Question 8: Example

Slope of

$$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$$

At $t = \frac{\pi}{3}$:

Solution:

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, \quad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\vec{r}'(t) = \langle -\sin(t), \cos(t) \rangle$$

At $t = \frac{\pi}{3}$:

$$\vec{r}'\left(\frac{\pi}{3}\right) = \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$$

Slope Calculation

$$\text{slope} = \frac{\frac{-1}{2}}{\frac{-\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Vector Notation

$$\vec{r}'(t) = \langle -\sin(t), \cos(t) \rangle \quad \text{so} \quad \vec{r}'\left(\frac{\pi}{3}\right) = \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$$

1.14.2 Chain Rule

Definition 1.14.5: Chain Rule for VV

Let $g : \mathbb{R} \rightarrow \mathbb{R}$ and $\vec{r}(t)$ be a vector function. The derivative of the composition of functions is:

$$\frac{d}{dt}[g(\vec{r}(t))] = g'(\vec{r}(t)) \cdot \vec{r}'(t)$$

where $g'(\vec{r}(t))$ is the derivative of g with respect to the vector function $\vec{r}(t)$ and $\vec{r}'(t)$ is the derivative of the vector function $\vec{r}(t)$ with respect to t . This rule is useful for differentiating scalar functions composed with vector-valued functions.

Question 9: Example

$$\vec{r}(t) = \langle \cos^2 t, -3t + \cos^3 t \rangle = \vec{r}(g(t))$$

$$y = \cos t, \quad \vec{r} = \langle t^2, -3t, 3t^3 \rangle$$

$$g' = -\sin t, \quad \dot{\vec{r}} = \langle 2t, 3t^2 \rangle$$

Thus, the derivative is:

$$\vec{s}(t) = \frac{d}{dt}(g(y(t))\vec{r}(g(t))) = g'(t) \cdot \dot{\vec{r}}(g(t))$$

$$= -\sin t \cdot \langle 2 \cos t, 3 \cos^2 t \rangle$$

1.14.3 Product Rule

Definition 1.14.6: Product rule for VV

Let $g(t) : \mathbb{R} \rightarrow \mathbb{R}$, $\vec{r}(t) : \mathbb{R} \rightarrow \mathbb{R}^n$, and $\vec{s}(t) : \mathbb{R} \rightarrow \mathbb{R}^n$ be vector functions. Then, the product rule is:

$$\frac{d}{dt} (g(t)\vec{r}(t)) = g'(t)\vec{r}(t) + g(t)\dot{\vec{r}}(t)$$

For vector dot products:

$$\frac{d}{dt} (\vec{r}(t) \cdot \vec{s}(t)) = \vec{r}'(t) \cdot \vec{s}(t) + \vec{r}(t) \cdot \vec{s}'(t)$$

For vector cross products:

$$\frac{d}{dt} (\vec{r}(t) \times \vec{s}(t)) = \vec{r}'(t) \times \vec{s}(t) + \vec{r}(t) \times \vec{s}'(t)$$

1.14.4 Integrals

Definition 1.14.7: Integral of Vector-Valued Functions

Let $\vec{r}(t) : [a, b] \rightarrow \mathbb{R}^n$ be a continuous vector-valued function. The definite integral of $\vec{r}(t)$ over the interval $[a, b]$ is defined as:

$$\int_a^b \vec{r}(t) dt = \left\langle \int_a^b r_1(t) dt, \int_a^b r_2(t) dt, \dots, \int_a^b r_n(t) dt \right\rangle$$

where $\vec{r}(t) = \langle r_1(t), r_2(t), \dots, r_n(t) \rangle$. This means that the integral of a vector-valued function is the vector whose components are the integrals of the respective component functions.

For example, if $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then:

$$\int_a^b \vec{r}(t) dt = \left\langle \int_a^b f(t) dt, \int_a^b g(t) dt, \int_a^b h(t) dt \right\rangle$$

This can be extended to cases where $\vec{r}(t)$ represents the position of a particle over time, and the integral represents the net displacement of the particle over the interval $[a, b]$.

Question 10: Find the integral of the vector-valued function

Calculate the integral of the vector-valued function $\vec{r}(t) = \langle t^2, \sin t, e^t \rangle$ over the interval $t \in [0, 1]$.

Solution:

The vector-valued function is given as:

$$\vec{r}(t) = \langle t^2, \sin t, e^t \rangle$$

We can integrate each component of the vector separately over the interval $[0, 1]$:

$$\int_0^1 \vec{r}(t) dt = \left\langle \int_0^1 t^2 dt, \int_0^1 \sin t dt, \int_0^1 e^t dt \right\rangle$$

Now, solving each integral:

1. For $\int_0^1 t^2 dt$:

$$\int_0^1 t^2 dt = \left[\frac{t^3}{3} \right]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

2. For $\int_0^1 \sin t dt$:

$$\int_0^1 \sin t \, dt = [-\cos t]_0^1 = -\cos(1) + \cos(0) = 1 - \cos(1)$$

3. For $\int_0^1 e^t \, dt$:

$$\int_0^1 e^t \, dt = [e^t]_0^1 = e^1 - e^0 = e - 1$$

Thus, the result of the integral is:

$$\int_0^1 \vec{r}(t) \, dt = \left\langle \frac{1}{3}, 1 - \cos(1), e - 1 \right\rangle$$

Question 11: Find the position vector of a ball in 3D space

A ball is thrown from the origin with an initial velocity vector $\vec{v}(0) = \langle 1, 3, 5 \rangle$ m/s, and experiences a constant acceleration vector $\vec{a}(t) = \langle 0, 0, -10 \rangle$ m/s². Find the position vector $\vec{r}(t)$ of the ball at any time t .

Solution:

The velocity vector $\vec{v}(t)$ is found by integrating the acceleration vector $\vec{a}(t)$:

$$\vec{v}(t) = \int_0^t \vec{a}(s) \, ds + \vec{v}(0)$$

Given that $\vec{a}(t) = \langle 0, 0, -10 \rangle$, we integrate each component:

$$\begin{aligned} \vec{v}(t) &= \int_0^t \langle 0, 0, -10 \rangle \, ds + \langle 1, 3, 5 \rangle \\ &= \langle 0, 0, -10t \rangle + \langle 1, 3, 5 \rangle \\ \vec{v}(t) &= \langle 1, 3, 5 - 10t \rangle \end{aligned}$$

Now, to find the position vector $\vec{r}(t)$, we integrate the velocity vector $\vec{v}(t)$:

$$\begin{aligned} \vec{r}(t) &= \int_0^t \vec{v}(s) \, ds \\ &= \int_0^t \langle 1, 3, 5 - 10s \rangle \, ds \end{aligned}$$

Integrating each component:

1. For the first component:

$$\int_0^t 1 \, ds = t$$

2. For the second component:

$$\int_0^t 3 \, ds = 3t$$

3. For the third component:

$$\int_0^t (5 - 10s) \, ds = [5s - 5s^2]_0^t = 5t - 5t^2$$

Thus, the position vector is:

$$\vec{r}(t) = \langle t, 3t, 5t - 5t^2 \rangle$$

1.15 Arc Length

The arc length of a smooth curve given by a vector-valued function can be calculated using an integral. Let the position vector of a curve be represented as:

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle,$$

where t is the parameter over some interval $[a, b]$.

Definition 1.15.1: Arc Length Formula

The arc length L of the curve $\vec{r}(t)$ over the interval $[a, b]$ is given by:

$$L = \int_a^b |\vec{r}'(t)| dt,$$

where $\vec{r}'(t)$ is the derivative of the position vector, representing the velocity vector of the curve. To find $|\vec{r}'(t)|$, we use the following:

$$|\vec{r}'(t)| = \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}.$$

Therefore, the arc length formula can be explicitly written as:

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2} dt.$$

Question 12: Example: Eagle Spiraling

Consider an example where an eagle is spiraling in space, and its position at time t (in minutes) is given by:

$$\vec{r}(t) = \langle 250 \cos(t), 250 \sin(t), 100t \rangle \quad (\text{in feet}).$$

Solution:

To find the velocity vector, we differentiate each component of $\vec{r}(t)$:

$$\vec{r}'(t) = \langle -250 \sin(t), 250 \cos(t), 100 \rangle \quad (\text{in feet per minute}).$$

The magnitude of the velocity vector $|\vec{r}'(t)|$ is:

$$|\vec{r}'(t)| = \sqrt{(-250 \sin(t))^2 + (250 \cos(t))^2 + (100)^2}.$$

Simplifying, we obtain:

$$|\vec{r}'(t)| = \sqrt{250^2 \sin^2(t) + 250^2 \cos^2(t) + 100^2}.$$

Using the Pythagorean identity $\sin^2(t) + \cos^2(t) = 1$, we find:

$$|\vec{r}'(t)| = \sqrt{250^2 + 100^2} = \sqrt{62500 + 10000} = \sqrt{72500} \approx 269.3 \quad (\text{feet per minute}).$$

Therefore, the speed of the eagle is approximately 269.3 feet per minute.

Question 13: Calculating Distance Over Time

Find the total distance over 10 minutes

Solution:

To find the total distance traveled by the eagle over a period of 10 minutes, we compute the arc length:

$$\text{Distance} = \int_0^{10} |\vec{r}'(t)| dt = \int_0^{10} 269.3 dt.$$

Evaluating the integral:

$$\text{Distance} = 269.3 \cdot (10 - 0) = 2693 \text{ feet.}$$

Thus, over 10 minutes, the eagle travels approximately 2693 feet.

Definition 1.15.2: Arc Length Function

The arc length function, denoted by $S(t)$, is defined as:

$$S(t) = \text{distance traveled along the curve from } t = a.$$

It is given by the integral:

$$S(t) = \int_a^t |\vec{r}'(s)| \, ds.$$

This function essentially converts the vector quantity $\vec{r}(t)$ into a 1-dimensional scalar that measures the total distance traveled without considering direction.

The arc length function $S(t)$ effectively flattens the vector $\vec{r}(t)$ into a scalar by integrating the magnitude of the velocity vector $\vec{r}'(s)$ over the interval $[a, t]$.

Example 1.15.1 (Application: Reparametrization by Arc Length)

One common use of the arc length function is to reparametrize the curve so that it is traveled at unit speed.

Given the inverse function of $S(t)$, denoted as $S^{-1}(t)$, we wish to compare:

$$\vec{r}(S^{-1}(t)) \quad \text{with} \quad \vec{r}(t).$$

If $S(t)$ is given by:

$$S(t) = \int_a^t |\vec{r}'(p)| \, dp,$$

then $S'(t) = |\vec{r}'(t)|$.

To differentiate $\vec{r}(S^{-1}(t))$, we use the chain rule:

$$\frac{d}{dt} \vec{r}(S^{-1}(t)) = \frac{1}{S'(S^{-1}(t))} \vec{r}'(S^{-1}(t)).$$

Since $S'(S^{-1}(t)) = |\vec{r}'(S^{-1}(t))|$, we have:

$$\frac{d}{dt} \vec{r}(S^{-1}(t)) = \frac{\vec{r}'(S^{-1}(t))}{|\vec{r}'(S^{-1}(t))|}.$$

This shows that the derivative of $\vec{r}(S^{-1}(t))$ is a unit vector, implying that $\vec{r}(S^{-1}(t))$ describes the same curve as $\vec{r}(t)$, but at unit speed.

By reparametrizing the curve with respect to arc length, we achieve a uniform traversal of the curve at constant unit speed. The transformation $\vec{r}(S^{-1}(t))$ provides a way to describe the geometry of the curve without the influence of varying speed.

Question 14: Worked Example

Consider the position of an eagle as it spirals upward over time. The position vector $\vec{r}(t)$ is given by:

$$\vec{r}(t) = \langle 250 \cos(t), 250 \sin(t), 100t \rangle \quad (\text{in feet}),$$

where t is in minutes.

Step 1: Find the Speed of the Eagle

The velocity vector is obtained by differentiating the position vector:

$$\vec{r}'(t) = \langle -250 \sin(t), 250 \cos(t), 100 \rangle.$$

The speed is the magnitude of the velocity vector:

$$|\vec{r}'(t)| = \sqrt{(-250 \sin(t))^2 + (250 \cos(t))^2 + 100^2} = \sqrt{62500 + 10000} = 269.3 \quad (\text{feet per minute}).$$

Step 2: Arc Length Function $S(t)$

To find the arc length function $S(t)$, which represents the distance traveled along the curve from $t = 0$ to $t = t$:

$$S(t) = \int_0^t |\vec{r}'(p)| \, dp.$$

Since the speed is constant:

$$S(t) = \int_0^t 269.3 \, dp = 269.3t.$$

Thus, at time t , the eagle has traveled $269.3t$ feet.

Step 3: Reparametrize by Arc Length

The inverse of the arc length function $S(t)$, denoted as $S^{-1}(t)$, is given by:

$$S^{-1}(t) = \frac{t}{269.3}.$$

Now, to reparametrize the curve by arc length, we substitute $S^{-1}(t)$ into the original position vector $\vec{r}(t)$:

$$\vec{r}(S^{-1}(t)) = \left\langle 250 \cos\left(\frac{t}{269.3}\right), 250 \sin\left(\frac{t}{269.3}\right), 100 \frac{t}{269.3} \right\rangle.$$

Interpretation

This reparametrization gives the position of the eagle as a function of the distance traveled (in feet), rather than as a function of time. It provides a way to describe the motion of the eagle along the curve such that the eagle is moving at a constant speed of 1 foot per unit of time.

1.16 Curvature of a Vector-Valued Function

Curvature is a measure of the "sharpness" or "bendiness" of a curve. For a smooth curve given by a vector-valued function $\vec{r}(t)$, the curvature at a point is related to how quickly the direction of the curve is changing.

1.16.1 Osculating Circle

The concept of curvature is closely tied to the idea of an **osculating circle**. At any point on the curve, the osculating circle is the circle that best approximates the curve's curvature at that point. This circle: - Has the same tangent as the curve $\vec{r}(t)$ at the point $\vec{r}(t_0)$. - Its radius is a measure of the "tightness" of the curve's bend at that point.

The radius of the osculating circle is denoted as:

$$R(t),$$

where $R(t)$ is the radius of curvature of the curve at time t .

1.16.2 Curvature $\kappa(t)$

The curvature $\kappa(t)$ of the curve $\vec{r}(t)$ at the point $\vec{r}(t_0)$ is defined as the reciprocal of the radius of the osculating circle:

$$\kappa(t) = \frac{1}{R(t)}.$$

A smaller radius $R(t)$ indicates a tighter bend and thus a larger curvature $\kappa(t)$, whereas a larger radius corresponds to a gentler curve and smaller curvature.

The osculating circle provides a geometric interpretation of curvature, serving as the best circular approximation to the curve at any given point. The curvature $\kappa(t)$ quantifies how sharply the curve bends at that point, with the relationship:

$$\kappa(t) = \frac{1}{R(t)}.$$

Definition 1.16.1: Osculating Circle of a Straight Line

A straight line is a curve with no curvature. Thus, the curvature $\kappa(t)$ of a straight line is zero at all points. To understand why, consider the definition of the osculating circle.

For any curve $\vec{r}(t)$, the osculating circle at a point $\vec{r}(t_0)$ is defined as the circle that "best fits" the curve at that point. Mathematically, this means:

$$\kappa(t) = \lim_{\Delta t \rightarrow 0} \frac{\text{change in angle}}{\text{arc length}},$$

where $\kappa(t)$ is the curvature of the curve. However, for a straight line, the change in angle is always zero, so:

$$\kappa(t) = 0.$$

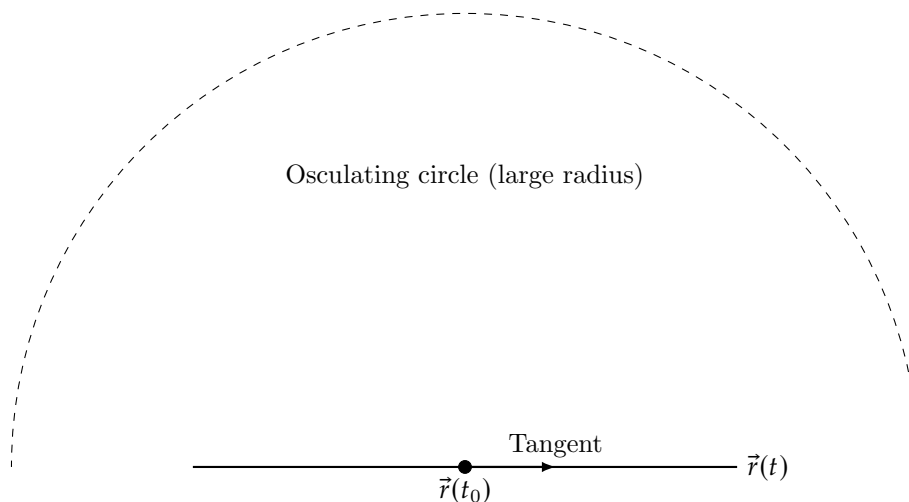
Since curvature is the reciprocal of the radius of the osculating circle $R(t)$, a zero curvature implies an infinite radius:

$$R(t) = \frac{1}{\kappa(t)} \rightarrow \infty.$$

Therefore, the osculating circle of a straight line is effectively a circle with infinite radius, meaning it is "flattened" into a straight line.

Diagram

Below is a diagram to illustrate the concept of the osculating circle of a straight line:



The dashed arc represents the osculating circle with an infinitely large radius, approximating the straight line at $\vec{r}(t_0)$.

To approach the concept of osculating circles for vector-valued functions, we break down the analysis into four key points:

1. Tangent Vector at t_0

The tangent vector to the curve at time t_0 is given by:

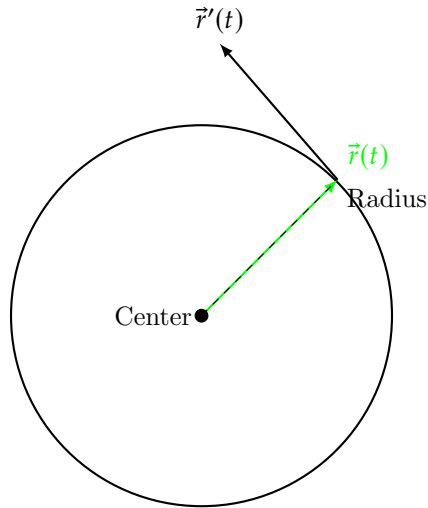
$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}.$$

This vector is the unit tangent vector at any point on the curve.

2. If $\vec{r}(t)$ Describes a Circle

Suppose that $\vec{r}(t)$ describes a circular path. In this case:

- The vector $\vec{r}(t)$ is always parallel to the radius of the circle.
- The velocity vector $\vec{r}'(t)$ is also perpendicular to $\vec{r}(t)$.



3. Acceleration Vector $\vec{a}(t)$

For a circular path, the acceleration vector $\vec{a}(t)$ points toward the center of the circle. Hence,

$\vec{a}(t)$ is opposite to $\vec{r}(t)$ and directed towards the center.

This implies:

$$\vec{a}(t) \perp \vec{r}'(t), \quad \vec{a}(t) \perp \vec{T}(t).$$

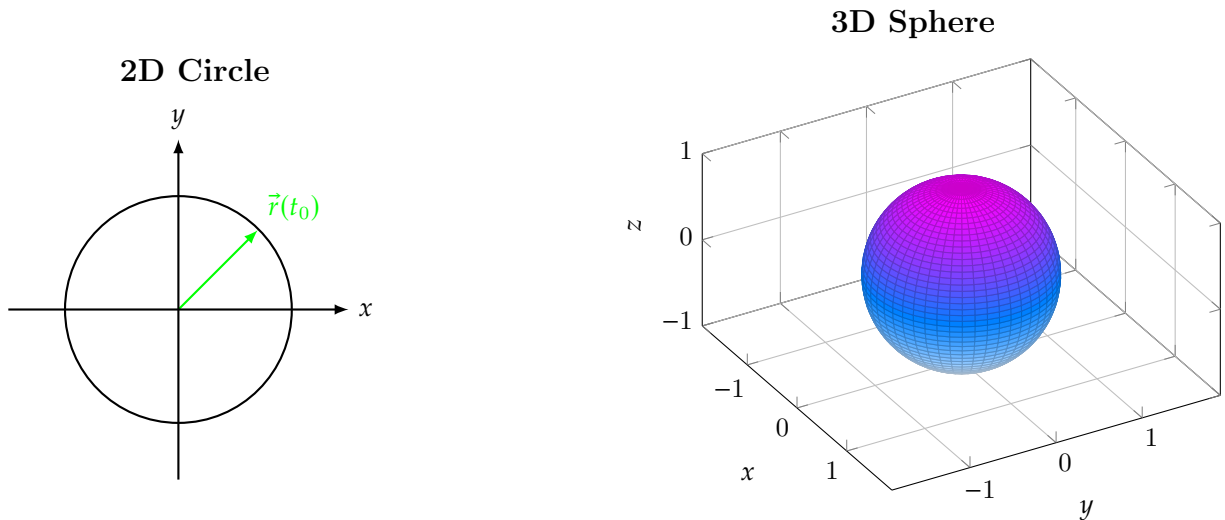
4. Constant Speed and Perpendicularity

Suppose that $\vec{r}(t)$ describes a curve with constant speed:

$$|\vec{r}'(t)| = C, \quad C \in \mathbb{R}.$$

For both 2D and 3D, the curve will either be a circle or a sphere with a fixed radius. The velocity and acceleration vectors, $\vec{r}'(t)$ and $\vec{a}(t)$, maintain a perpendicular relationship:

$$\vec{r}(t) \cdot \vec{v}(t) = 0.$$



1.16.3 Deriving the Perpendicularity

Let $\vec{v}(t) = \vec{r}'(t)$ be the velocity vector. Then:

$$\vec{r}'(t) \cdot \vec{v}(t) = \vec{r}'(t) \cdot \vec{r}'(t) = \frac{1}{2} \frac{d}{dt} (\vec{r} \cdot \vec{r}).$$

Differentiating:

$$= \frac{1}{2} \frac{d}{dt} (|\vec{r}'(t)|^2) = \frac{1}{2} \frac{d}{dt} C^2 = 0,$$

since C is constant.

Therefore, if $|\vec{r}'(t)| = C$, then:

$$\vec{v}(t) \perp \vec{r}(t).$$

1.17 Calculating Curvature

1.17.1 Case 1: $\vec{r}(t)$ travels at unit speed around a circle

We consider the position vector $\vec{r}(t)$ traveling at unit speed around a circle of radius R .

$$\vec{r}(t) = \left\langle R \cos\left(\frac{t}{R}\right), R \sin\left(\frac{t}{R}\right), 0 \right\rangle$$

The velocity vector $\vec{v}(t)$ is then the derivative of the position vector:

$$\vec{v}(t) = \vec{r}'(t) = \left\langle -\sin\left(\frac{t}{R}\right), \cos\left(\frac{t}{R}\right), 0 \right\rangle$$

Note that the magnitude of the velocity vector is 1 (unit speed):

$$|\vec{v}(t)| = 1$$

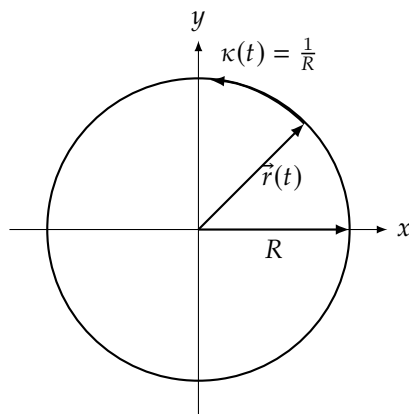
The acceleration vector $\vec{a}(t)$, which is the second derivative of the position vector, is given by:

$$\vec{a}(t) = \vec{r}''(t) = \left\langle -\frac{1}{R} \cos\left(\frac{t}{R}\right), -\frac{1}{R} \sin\left(\frac{t}{R}\right), 0 \right\rangle$$

Definition 1.17.1: The curvature $\kappa(t)$ of the circle

The magnitude of the acceleration vector (centripetal acceleration) is:

$$\kappa(t) = |\vec{r}''(t)| = \frac{1}{R}$$



Note:-

- $\vec{r}(t)$ represents the position of a point moving around a circle of radius R .
- The velocity vector $\vec{v}(t)$ is orthogonal to $\vec{r}(t)$ and has a constant magnitude, indicating uniform circular motion.
- The acceleration vector $\vec{a}(t)$ points towards the center of the circle (centripetal acceleration) and its magnitude is inversely proportional to the radius R , showing that a smaller radius corresponds to a greater acceleration.
- The curvature $\kappa(t)$ is defined as the reciprocal of the radius of the circle, $\frac{1}{R}$.

1.17.2 Case 2: General Curve with Unit Speed

Consider a curve given by the position vector $\vec{r}(t)$, which travels at unit speed along a general curve. This means that the magnitude of the velocity vector is constant:

$$|\vec{v}(t)| = 1.$$

Since the speed is constant, the acceleration vector $\vec{a}(t)$ is orthogonal to the velocity vector:

$$\vec{v}(t) \cdot \vec{a}(t) = 0.$$

This implies that all changes in $\vec{a}(t)$ are perpendicular to the direction of motion (velocity), which aligns with the properties of motion at unit speed.

Claim 1.17.1 Acceleration and the Osculating Circle

The acceleration vector $\vec{a}(t)$ is directed towards the center of curvature of the curve at each point, and its magnitude is related to the curvature. In fact, the acceleration "matches" the behavior of the osculating circle at that point. The osculating circle is the circle that best approximates the curve locally, and its radius of curvature R is inversely proportional to the curvature.

Theorem 1.17.1 Curvature and Acceleration

The curvature $\kappa(t)$ of the curve at time t is defined as the magnitude of the acceleration vector:

$$\kappa(t) = |\vec{a}(t)|.$$

This provides a measure of how sharply the curve is bending at any given point. Since the speed is unitary, the curvature directly corresponds to the acceleration's magnitude.

Example 1.17.1 (Matching the Curve with Its Osculating Circle)

The osculating circle at a point on the curve is chosen so that its first and second derivatives match the

curve's first and second derivatives, respectively:

Choose circle $\vec{C}(t)$ such that $\vec{C}'(t)$ and $\vec{C}''(t)$ match $\vec{r}'(t)$ and $\vec{r}''(t)$.

This ensures that the circle is the best local approximation to the curve at that point, providing insight into the curve's behavior through the curvature.

1.17.3 Case 3: General Curve with Variable Speed

In this case, we analyze a general curve with a variable speed. Unlike the previous case, the magnitude of the velocity vector is not constant:

$$|\vec{v}(t)| \neq 1.$$

This implies that the curve's speed is not uniform, and therefore, the analysis of the curvature will require a different approach.

Let $s(t)$ be the arc length parameter, which is a function of t . The position vector can be reparameterized in terms of arc length:

$$\vec{r}(s^{-1}(t)),$$

where $s^{-1}(t)$ is the inverse function of $s(t)$.

Theorem 1.17.2 Calculating Curvature $\kappa(t)$

To find the curvature, we introduce a parameter t_0 , which satisfies $t_0 = s^{-1}(t)$. Thus, we have:

$$s^{-1}(t_0) = t.$$

The curvature $\kappa(t)$ is then given by:

$$\kappa(t) = \left| \frac{d}{dt_0} \vec{T}(s^{-1}(t_0)) \right|,$$

where $\vec{T}(t)$ is the unit tangent vector.

This can be rewritten as:

$$\kappa(t) = \left| \frac{\frac{d}{dt} \vec{r}(s^{-1}(t))}{|\vec{r}'(s^{-1}(t))|} \right| = \left| \frac{\vec{T}'(s^{-1}(t_0))}{|\vec{r}'(s^{-1}(t_0))|} \right|.$$

The final expression for the curvature is:

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

where $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$ is the unit tangent vector to the curve.

Alternatively, the curvature can be found with

$$\kappa(t) = \frac{|\vec{a} \times \vec{v}|}{|\vec{v}|^3}, \quad \kappa(t) = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right|, \quad \kappa(t) = \left| \frac{d\vec{T}}{ds} \right|$$

if $y = f(x)$ can be parametrized as

$$\vec{r}(t) = \langle t, f(t), 0 \rangle,$$

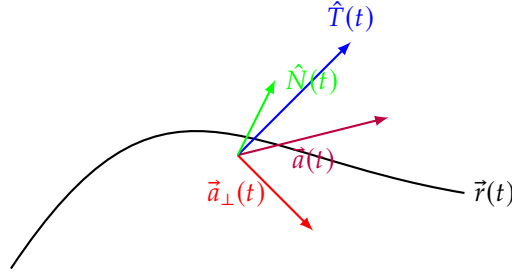
where

$$x'(t) = 1, \quad x''(t) = 0.$$

This parametrization gives the curvature κ of the curve as

$$\kappa(t) = \frac{|y''(x)|}{(1 + (y'(x))^2)^{3/2}}.$$

1.18 Components of Acceleration



$$|\vec{a}_\perp(t)| = \frac{|\vec{v}(t) \times \vec{a}(t)|}{|\vec{v}(t)|} = |\vec{v}(t)|^2 \cdot \kappa(t) \quad |\vec{a}_\parallel(t)| = \frac{d}{dt} |\vec{v}(t)| = \frac{d^2}{dt^2} \int_0^t |\vec{v}(p)| dp = \frac{d^2}{dt^2} s(t)$$

$$\vec{a}_\perp(t) = \text{centripetal acceleration}$$

$$\vec{a}_\parallel(t) = \text{speed change}$$

$$\vec{a}(t) = |\vec{a}_\perp(t)| \cdot \frac{\vec{T}'(t)}{|\vec{T}'(t)|} + |\vec{a}_\parallel(t)| \cdot \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

Missing a third component of motion

1.19 Binormal Vector

Definition 1.19.1: Binormal Vector

The **binormal vector** \vec{B} is defined as:

$$\vec{B} = \vec{T} \times \vec{N} = \frac{\vec{v} \times \vec{a}}{|\vec{v} \times \vec{a}|}$$

where:

- \vec{T} is the unit tangent vector to the curve, given by:

$$\vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

- \vec{N} is the unit normal vector to the curve, and it is defined as:

$$\vec{N} = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

The binormal vector \vec{B} represents the direction to exit the current plane of motion described by the vectors \vec{T} and \vec{N} .

1.19.1 Change in the Plane of Motion

The rate of change of the binormal vector is given by:

$$\frac{d\vec{B}}{dt} = \text{change in the plane of motion}$$

Applying the derivative product rule to $\vec{B} = \vec{T} \times \vec{N}$:

$$\frac{d}{dt}(\vec{T} \times \vec{N}) = \frac{d\vec{T}}{dt} \times \vec{N} + \vec{T} \times \frac{d\vec{N}}{dt}$$

Since:

$$\vec{N} = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

it follows that:

$$\vec{T} \cdot \vec{N} = 0$$

and thus:

$$\frac{d\vec{B}}{dt} \text{ is orthogonal to both } \vec{T} \text{ and } \vec{N}$$

1.19.2 Torsion and its Relation to the Binormal Vector

Since \vec{N} is a unit vector, the vector \vec{B}' is parallel to \vec{N} :

$$\frac{d\vec{B}}{dt} = -\tau \vec{N}$$

where τ is the **torsion** of the plane of movement.

Note:-

If $\tau > 0$, the motion is on the plane of movement; if $\tau < 0$, the motion is in the opposite direction.

To find τ , take the dot product of both sides with \vec{N} :

$$\frac{d\vec{B}}{dt} \cdot \vec{N} = -\tau(\vec{N} \cdot \vec{N}) = -\tau$$

Thus, at unit speed:

Definition 1.19.2: Torsion τ at unit speed

$$\tau = -\frac{\frac{d\vec{B}}{dt} \cdot \vec{N}}{|\vec{N}|} = -\frac{d\vec{B}}{dt} \cdot \vec{N}$$

1.19.3 General Case and Torsion Formula

For the general case, replace $\vec{r}(t)$ with $\vec{r}(s^{-1}(t))$, where $s(t)$ is the arc-length parameterization.

The derivative of \vec{B} in the general case is given by:

$$\frac{d\vec{B}(s^{-1})}{d(t_0)} = \frac{\vec{B}'(s^{-1}(t_0))}{|\vec{r}'(s^{-1}(t_0))|}$$

or simply:

$$\frac{d\vec{B}(t)}{dt} = \frac{\vec{B}'(t) \cdot \vec{N}(t)}{|\vec{r}'(t)|}$$

The general formula for torsion τ becomes:

Definition 1.19.3: General form for torsion τ

$$\tau = \frac{\vec{B}'(t) \cdot \vec{N}(t)}{|\vec{r}'(t)|}$$

1.19.4 Expressing the Binormal Vector with Velocity and Acceleration

The binormal vector \vec{B} can be expressed as:

$$\vec{B} = \frac{\vec{v} \times \vec{a}}{|\vec{v} \times \vec{a}|}$$

where \vec{v} is the velocity vector and \vec{a} is the acceleration vector.

The torsion τ is given by:

$$\tau = \frac{(\vec{v} \times \vec{a}) \cdot \vec{a}'}{|\vec{v} \times \vec{a}|^2}$$

Note:-

The centripetal acceleration determines the curvature of the path.

Example 1.19.1 (Finding the Torsion of a Helix Path)

Consider the helix parameterized by:

$$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$$

The first derivative (velocity vector) is:

$$\vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$$

with magnitude:

$$|\vec{r}'(t)| = \sqrt{2}$$

The unit tangent vector $\vec{T}(t)$ is:

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \left\langle -\frac{\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

The derivative of the tangent vector is:

$$\vec{T}'(t) = \left\langle -\frac{\cos t}{\sqrt{2}}, -\frac{\sin t}{\sqrt{2}}, 0 \right\rangle$$

and its magnitude is:

$$|\vec{T}'(t)| = \frac{1}{\sqrt{2}}$$

The curvature $\kappa(t)$ is given by:

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{\frac{1}{\sqrt{2}}}{\sqrt{2}} = \frac{1}{2}$$

Hence, the radius of curvature $R(t)$ is:

$$R(t) = \frac{1}{\kappa(t)} = 2$$

The normal vector $\vec{N}(t)$ is:

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \langle -\cos t, -\sin t, 0 \rangle$$

which points towards the center of the curvature.

The normal acceleration a_N is:

$$a_N = |\vec{r}'(t)|^2 \cdot \kappa = 1 \quad (\text{constant, points towards the center})$$

The tangential acceleration is:

$$a_T = s''(t), \quad s(t) = t\sqrt{2}, \quad s'(t) = \sqrt{2}$$

The binormal vector \vec{B} is given by:

$$\vec{B} = \vec{T} \times \vec{N} = \frac{1}{\sqrt{2}} \langle \sin t, -\cos t, 0 \rangle$$

The derivative of the binormal vector is:

$$\vec{B}' = \frac{1}{\sqrt{2}} \langle \cos t, \sin t, 0 \rangle = -\frac{1}{\sqrt{2}} \vec{N} \implies \tau = \frac{\vec{B}' \cdot \vec{N}}{|\vec{r}'(t)|} = \frac{1}{2}$$

Therefore, the torsion τ is:

$$\tau = \frac{\kappa}{\sqrt{2}} = \frac{1}{2}$$

Chapter 2

Functions of Multiple Random Variables $f : \mathbb{R}^n \rightarrow \mathbb{R}$

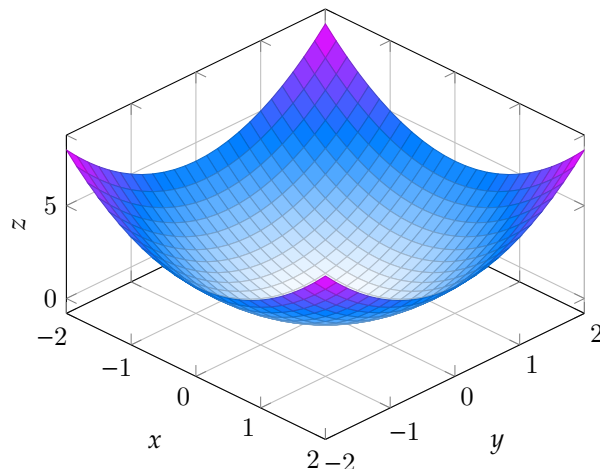
2.1 Plotting Multiple Random Variable Functions

Consider a function $f(x, y) = z$. We aim to plot functions in 3D.

Example

$$f(x, y) = x^2 + y^2$$

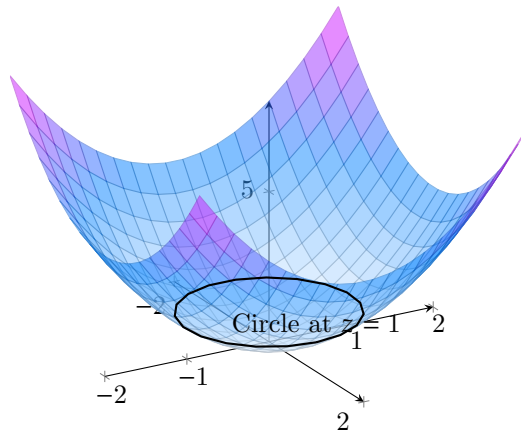
- When $y = 0$, $f(x, 0) = x^2$.
- When $x = 0$, $f(0, y) = y^2$.



For this surface:

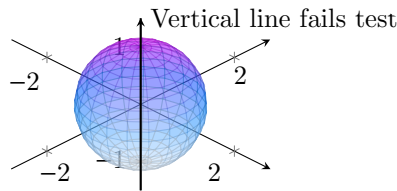
- If $z = 0$, $(x, y) = (0, 0)$.
- If $z = 1$, $x^2 + y^2 = 1$, which represents a **circle**.
- If $z > 0$, $x^2 + y^2 = z$, representing a circle with radius \sqrt{z} in the xy -plane.

The vertical line test still applies: a vertical line intersects the surface at at most 1 point.



Sphere by Self Equation

A sphere cannot be represented as a function in the form $f(x, y) = z$. This is because it fails the vertical line test; a vertical line may intersect the sphere at multiple points.



2.2 Domain of Multiple Random Variable Functions

Definition 2.2.1: Domain

The domain of $f(x, y)$ is the set of pairs $(x, y) \in \mathbb{R}^2$ such that $f(x, y)$ is defined.

Example 2.2.1 (Example of finding domain)

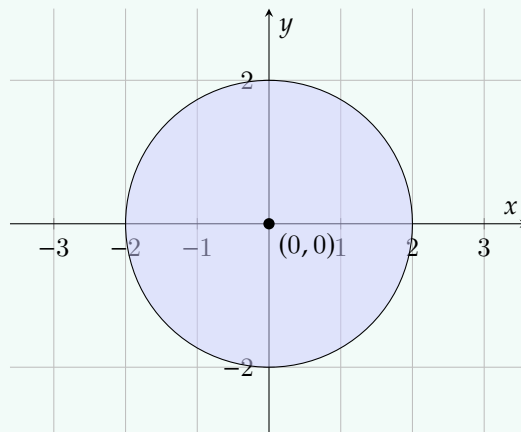
Let

$$f(x, y) = \sqrt{4 - x^2 - y^2}$$

To ensure that $f(x, y)$ is defined, the expression inside the square root must be non-negative:

$$4 - x^2 - y^2 \geq 0 \implies x^2 + y^2 \leq 4$$

This inequality describes a disk of radius 2 centered at the origin $(0, 0)$.



Therefore, the domain of $f(x, y)$ is:

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$$

2.3 Range of Multiple Random Variable Functions

Definition 2.3.1: Range

The range of a function $f(x, y)$ is the set of all z such that $z = f(x, y)$ for some input (x, y) .

Example 2.3.1 (Example of finding the range)

Consider the function

$$f(x, y) = \sqrt{4 - x^2 - y^2}$$

Evaluating f at the origin:

$$f(0, 0) = \sqrt{4 - 0^2 - 0^2} = 2$$

The maximum value of f is 2. To find the minimum, consider the boundary of the domain: If $x^2 + y^2 = 4$, then

$$f(x, y) = \sqrt{4 - (x^2 + y^2)} = \sqrt{4 - 4} = 0$$

Therefore, the range of $f(x, y)$ is:

$$\{z \in [0, 2]\}$$

Example 2.3.2 (Another example)

Let

$$f(x, y) = x^2 + y^2$$

Since $f(x, y) \geq 0$ for all $(x, y) \in \mathbb{R}^2$, the range of $f(x, y)$ is:

$$\{z \mid z \geq 0\}$$

2.4 Level Curves of Functions of Two Variables

Definition 2.4.1: Level Curves

Level curves (or contour lines) of a function $f(x, y)$ are curves in the xy -plane along which the function takes a constant value. For a given value c , a level curve is the set of points (x, y) such that:

$$f(x, y) = c$$

Example 2.4.1 (Example of level curves)

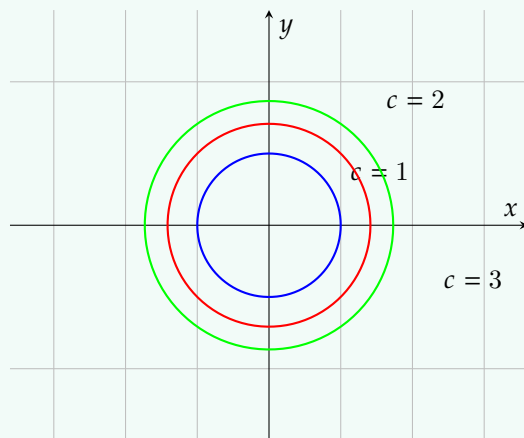
Consider the function:

$$f(x, y) = x^2 + y^2$$

To find the level curves for this function, set $f(x, y)$ equal to a constant c :

$$x^2 + y^2 = c$$

These curves are circles centered at the origin with radius \sqrt{c} .



In this case, each level curve is a circle centered at the origin. The radius of each circle is \sqrt{c} , indicating that as the value of c increases, the radius of the level curve also increases.

2.5 Limits of Two-Variable Functions

Definition 2.5.1: Limit of a Two-Variable Function

Let $f(x, y)$ be a function of two variables. The limit of $f(x, y)$ as (x, y) approaches (a, b) is denoted as:

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$$

This means that as the point (x, y) gets arbitrarily close to (a, b) , the function values $f(x, y)$ approach the value L . For the limit to exist, the function must approach the same value L regardless of the path taken to reach (a, b) .

Definition 2.5.2: Formal Definition

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$$

if for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that:

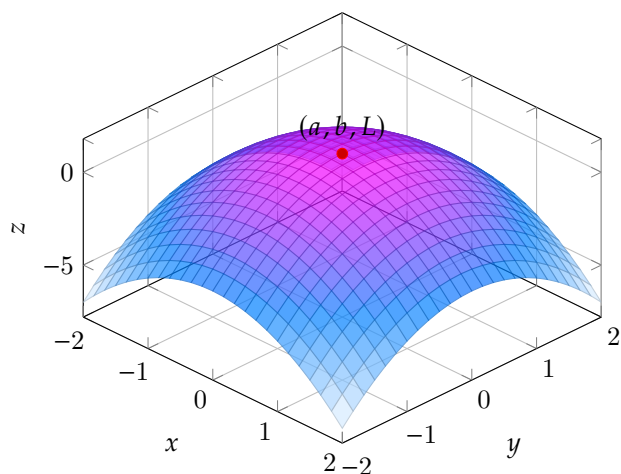
$$|f(x, y) - L| < \varepsilon$$

whenever (x, y) are in the domain of f and:

$$|(x, y) - (a, b)| < \delta$$

Visualizing Limits

The concept of limits for two-variable functions can be visualized in 3D, where $f(x, y)$ is represented as a surface over the xy -plane. The behavior of the function as (x, y) approaches a point (a, b) corresponds to how the surface approaches a specific height, L .



In this diagram, the surface represents a function $f(x, y)$, and the point (a, b, L) is the limit we are trying to approach.

Example 2.5.1 (Example of a Limit)

Consider the function:

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

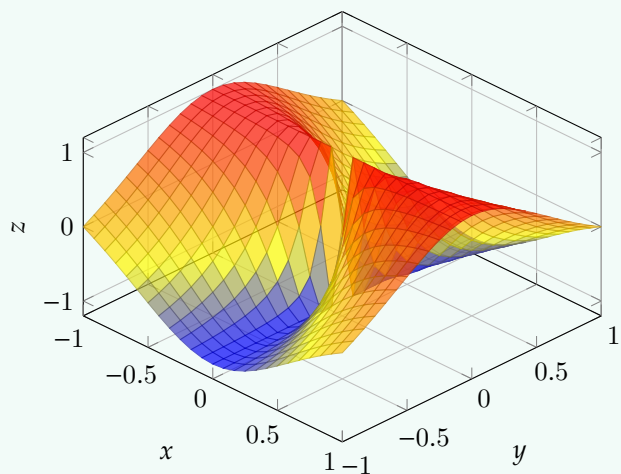
We want to find

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y)$$

Evaluating the limit along different paths:

- Along the path $y = 0$, we have $f(x, 0) = \frac{x^2}{x^2} = 1$.
- Along the path $x = 0$, we have $f(0, y) = \frac{-y^2}{y^2} = -1$.

Since the limit depends on the path taken to approach $(0, 0)$, the limit does not exist.



This diagram shows how the function behaves differently along different paths toward the origin, resulting in no single limit.

Example 2.5.2 (Example of a Limit that Exists)

Consider the function:

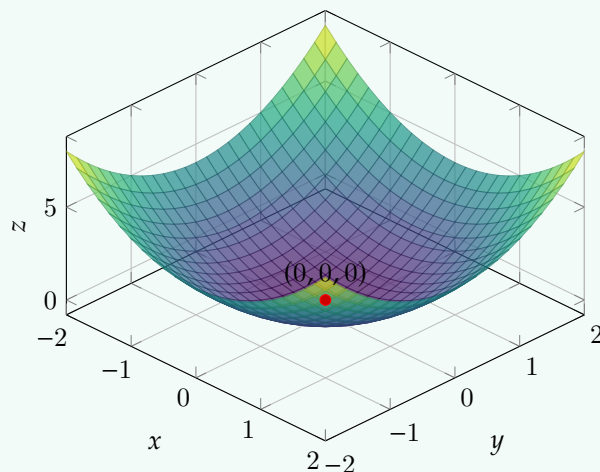
$$f(x, y) = x^2 + y^2$$

We want to find

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y)$$

Since $f(x, y) = x^2 + y^2$, regardless of the path taken to approach $(0, 0)$, the value of $f(x, y)$ approaches 0. Therefore,

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0.$$



2.5.1 Limits by substituting with function $f : \mathbb{R} \rightarrow \mathbb{R}$

Example 2.5.3 (Changing $g(x)$)

For a function $f(x, y)$, the limit as (x, y) approaches a point (a, b) must be the same regardless of the path taken to approach (a, b) .

Consider the function:

$$f(x, y) = \frac{3xy^2}{x^2 + y^4}$$

We wish to find:

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y)$$

This limit initially appears to be of the form $\frac{0}{0}$, indicating an indeterminate form.

To proceed, we evaluate the limit along different paths. Let's test two paths to see if the limit is consistent.

Path 1: Linear Path $y = mx$

Substitute $y = mx$ into $f(x, y)$:

$$f(x, y) = \frac{3x(mx)^2}{x^2 + (mx)^4} = \frac{3m^2x^3}{x^2 + m^4x^4}$$

Then, consider the limit as $x \rightarrow 0$:

$$\lim_{x \rightarrow 0} \frac{3m^2x^3}{x^2 + m^4x^4}$$

Factor out x^2 in the denominator:

$$= \lim_{x \rightarrow 0} \frac{3m^2x}{1 + m^4x^2} = 0$$

Path 2: Parabolic Path $y = m\sqrt{x}$

Substitute $y = m\sqrt{x}$ into $f(x, y)$:

$$f(x, y) = \frac{3x(m\sqrt{x})^2}{x^2 + (m\sqrt{x})^4} = \frac{3m^2x^2}{x^2 + m^4x^2}$$

Then, consider the limit as $x \rightarrow 0$:

$$\lim_{x \rightarrow 0} \frac{3m^2x^2}{(m^4 + 1)x^2} = \lim_{x \rightarrow 0} \frac{3m^2}{m^4 + 1} = \frac{3m^2}{m^4 + 1} \neq 0$$

Since the limit depends on the path taken (0 for the linear path and $\frac{3m^2}{m^4+1}$ for the parabolic path), the limit does not exist:

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) \text{ DNE (Does Not Exist)}$$

2.5.2 Limits by substituting with Vector Valued Functions $f : \mathbb{R} \rightarrow \mathbb{R}^n$

Example 2.5.4 (Substituting $\langle g_1(t), g_2(t) \rangle$)

Since a 1 to 1 function $g(t)$ must pass the vertical line test, it is a limiting way to approach the point at which we want to compute the limit of a two variable function. Using a vector valued function will remove this limitation and enable us to approach from any direction.

For a function $f(x, y)$, the limit as (x, y) approaches a point (a, b) must be the same regardless of the path taken to approach (a, b) .

Consider the function:

$$f(x, y) = \frac{3xy^2}{x^2 + y^4}$$

We wish to find:

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y)$$

This limit initially appears to be of the form $\frac{0}{0}$, indicating an indeterminate form.

To proceed, we evaluate the limit along different paths. Let's test two paths to see if the limit is consistent.

Path 1: Parametric Linear Path $\vec{r}(t) = \langle t, mt \rangle$

Substitute $\vec{r}(t) = \langle t, mt \rangle$ into $f(x, y)$:

$$f(x, y) = \frac{3(t)(mt)^2}{t^2 + (mt)^4} = \frac{3m^2t^3}{t^2 + m^4t^4}$$

Then, consider the limit as $t \rightarrow 0$:

$$\lim_{t \rightarrow 0} \frac{3m^2t^3}{t^2 + m^4t^4}$$

Factor out t^2 in the denominator:

$$= \lim_{t \rightarrow 0} \frac{3m^2t}{1 + m^4t^2} = 0$$

Path 2: Parametric Parabolic Path $\vec{r}(t) = \langle t^2, m\sqrt{t} \rangle$

Substitute $\vec{r}(t) = \langle t^2, m\sqrt{t} \rangle$ into $f(x, y)$:

$$f(x, y) = \frac{3(t^2)(m\sqrt{t})^2}{(t^2)^2 + (m\sqrt{t})^4} = \frac{3m^2t^3}{t^4 + m^4t^2}$$

Then, consider the limit as $t \rightarrow 0$:

$$\lim_{t \rightarrow 0} \frac{3m^2 t^3}{t^4 + m^4 t^2}$$

Factor out t^2 in the denominator:

$$= \lim_{t \rightarrow 0} \frac{3m^2 t}{t^2 + m^4} = \frac{3m^2}{m^4} = \frac{3}{m^2} \neq 0$$

Since the limit depends on the path taken (0 for the linear path and $\frac{3}{m^2}$ for the parabolic path), the limit does not exist:

$$\lim_{(x,y) \rightarrow (0,0)} f(x,y) \text{ DNE (Does Not Exist)}$$

Example 2.5.5 (Other Example)

$$f(x, y) = \frac{\sqrt[3]{x+y} + \ln(\sqrt[3]{x+y})}{x+y}$$

$$f(x, y) = \begin{cases} \frac{\sqrt[3]{x+y} + \ln(\sqrt[3]{x+y})}{x+y} & \text{if } x, y \neq 0 \\ L & \text{if } x = y = 0 \end{cases}$$

If

$$g(t) = \langle g_1(t), g_2(t) \rangle$$

where $g_1(t) = t$, $g_2(t) = \sqrt{t}$
and

$$h(s) = \frac{\sqrt[3]{s} + \ln(\sqrt[3]{s})}{s}$$

then

$$f(x, y) = h(g_1(x), g_2(y))$$

2.6 Tips for Finding Limits of Two-Variable Functions (2VF)

2.6.1 Polynomials are Continuous

Polynomials in two variables are always continuous everywhere in their domain. For example:

$$f(x, y) = 3x^2 + 2y^2 + 3x^4 + 1$$

is continuous for all $(x, y) \in \mathbb{R}^2$.

2.6.2 Compositions of Continuous Functions are Continuous

If $g(t)$ is a function of one variable and continuous at $t = 0$, and $f(x, y)$ is a continuous function of two variables, then $g(f(x, y))$ is also continuous at any point where $f(x, y)$ is defined. For example:

$$g(t) = \begin{cases} \frac{\sqrt{1+t}-1}{t} & \text{if } t \neq 0 \\ e & \text{if } t = 0 \end{cases}$$

is continuous everywhere.

Now consider the function:

$$f(x, y) = xy^2$$

Since $f(x, y)$ is continuous, the composition $g(f(x, y))$ is also continuous. Specifically:

$$g(f(x, y)) = \begin{cases} \frac{\sqrt{1+xy^2}-1}{xy^2} & \text{if } xy^2 \neq 0 \\ e & \text{if } xy^2 = 0 \end{cases}$$

This composition is continuous.

2.6.3 Rational Functions

A rational function $f(x, y) = \frac{P(x, y)}{Q(x, y)}$ is continuous at all points where the denominator $Q(x, y) \neq 0$. For example:

$$f(x, y) = \frac{3xy^2}{x^2 + y^4}$$

is continuous wherever $x^2 + y^4 \neq 0$.

Note:-

Tip on Factoring and Simplifying

To find the limit of a rational function, try factoring and simplifying the expression. This can help identify points of discontinuity or removable singularities.

2.6.4 Special Limits: Squeeze Theorem

The Squeeze Theorem can be used for two-variable functions. If:

$$h(x, y) \leq f(x, y) \leq g(x, y)$$

and both $h(x, y)$ and $g(x, y)$ approach the same limit L as $(x, y) \rightarrow (a, b)$, then:

$$\lim_{(x, y) \rightarrow (a, b)} f(x, y) = L$$

2.6.5 Use of Polar Coordinates

For limits involving two-variable functions, converting to polar coordinates can often simplify the problem. Let:

$$x = r \cos \theta, \quad y = r \sin \theta$$

Then, $f(x, y)$ becomes a function of r and θ , which can be easier to analyze as $r \rightarrow 0$. This is particularly useful when the limit depends on the distance from the origin.

2.6.6 Testing Multiple Paths

When checking if a limit exists, test multiple paths (e.g., $y = mx$, $y = m\sqrt{x}$, $x = 0$, $y = 0$). If the limit differs for any path, then the limit does not exist. However, if all paths give the same result, it is likely that the limit exists (though more thorough testing is required for rigor).

2.6.7 Continuity of Common Functions

Most standard functions (e.g., trigonometric, exponential, logarithmic) are continuous within their domains. If they are composed with continuous functions of x and y , the result is continuous as well.

2.6.8 Rational functions

Consider a rational function of two variables in the form:

$$\frac{h(x, y)}{g(x, y)}$$

where $h(x, y)$ and $g(x, y)$ are polynomials. The behavior of the limit of this function as $(x, y) \rightarrow (a, b)$ can be classified into three cases:

2.6.9 Case 1: $g(a, b) \neq 0$

If the denominator $g(x, y)$ is non-zero at the point (a, b) , i.e., $g(a, b) \neq 0$, then the function is continuous at (a, b) . In this case, we have:

$$\lim_{(x,y) \rightarrow (a,b)} \frac{h(x, y)}{g(x, y)} = \frac{h(a, b)}{g(a, b)}.$$

Thus, the limit exists and equals the value of the function at the point (a, b) .

2.6.10 Case 2: $g(a, b) = 0$ and $h(a, b) \neq 0$

If the denominator $g(x, y)$ equals zero at (a, b) but the numerator $h(x, y)$ does not, i.e., $h(a, b) \neq 0$, then the limit does not exist. Intuitively, this is because the denominator tends to zero while the numerator tends to a non-zero constant, leading to a blow-up of the function.

$$\lim_{(x,y) \rightarrow (a,b)} \frac{h(x, y)}{g(x, y)} \text{ does not exist.}$$

2.6.11 Case 3: $h(a, b) = g(a, b) = 0$

If both the numerator and the denominator are zero at (a, b) , i.e., $h(a, b) = g(a, b) = 0$, we have an indeterminate form $\frac{0}{0}$. In this case, we try to simplify the expression by factoring out the common terms in $h(x, y)$ and $g(x, y)$.

Example 2.6.1 (Example)

Consider the function:

$$\frac{x^2 - y^2}{x - y}.$$

At the point $(1, 1)$, both the numerator and the denominator are zero:

$$h(1, 1) = g(1, 1) = 0.$$

We factor the numerator:

$$x^2 - y^2 = (x - y)(x + y).$$

Now, cancel the common factor $x - y$ in the numerator and denominator:

$$\frac{x^2 - y^2}{x - y} = x + y.$$

Thus, the limit simplifies to:

$$\lim_{(x,y) \rightarrow (1,1)} \frac{x^2 - y^2}{x - y} = \lim_{(x,y) \rightarrow (1,1)} (x + y) = 1 + 1 = 2.$$

Hence, by removing the common factor, the limit can be computed as 2.

2.6.12 Case 4: $h(a, b) = g(a, b) = 0$ still

Try several Vector Valued functions, trying to make the denominator small or big compared to the numerator

Example 2.6.2 (Large denominator)

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^2 + y^2}$$

One possible path: Let $y = 0$,

$$\frac{x^2(0)}{x^2 + 0^2} = \frac{0}{x^2} = 0$$

Now try a different path: Let $x = 0$,

$$\frac{0^2 y}{0^2 + y^2} = \frac{0}{y^2} = 0$$

Parametric form:

$$\lim_{t \rightarrow 0} f(r_1(t), r_2(t)) = \lim_{t \rightarrow 0} f(0, t) = \lim_{t \rightarrow 0} \frac{0^2 t}{0^2 + t^2} = \lim_{t \rightarrow 0} 0 = 0$$

Thus, the limit is $0 = 0$.

Example 2.6.3 (Small denominator)

$$x = -y^2 + y^n, \quad \text{where } n > 2$$

(for y small)

$$\lim_{t \rightarrow 0} f(t^2 + t^n, t) = \frac{(t^4 - 2t^{n+2} + t^{2n})t}{t^2 + t^n + t^2} = \lim_{t \rightarrow 0} \frac{t^5 - 2t^{n+3} + t^{2n+1}}{t^n} \quad (n = 5)$$

Simplifying:

$$n = 5 : \quad \lim_{t \rightarrow 0} (1 - 2t^3 + t^6) = 1$$

2.6.13 Case 5: Still getting the same limit

Example 2.6.4 (Trying)

Let $\lim_{(x,y) \rightarrow (0,0)} = \frac{xy^2}{x^2 + y^2}$.

If we divide by y^2 we get

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{x}{(\frac{x^2}{y^2}) + 1} = 0$$

2.7 Properties of Limits of Two-Variable Functions

Let $f(x, y)$ and $g(x, y)$ be two functions of two variables such that:

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L_1, \quad \lim_{(x,y) \rightarrow (a,b)} g(x, y) = L_2$$

Then, the following properties hold:

2.7.1 Scaling by a constant:

$$\lim_{(x,y) \rightarrow (a,b)} [kf(x, y)] = kL_1, \quad \text{where } k \text{ is any real constant}$$

2.7.2 Addition of functions:

$$\lim_{(x,y) \rightarrow (a,b)} [f(x, y) + g(x, y)] = L_1 + L_2$$

2.7.3 Multiplication of functions:

$$\lim_{(x,y) \rightarrow (a,b)} [f(x, y)g(x, y)] = L_1L_2$$

2.7.4 Division of functions:

$$\lim_{(x,y) \rightarrow (a,b)} \left[\frac{f(x,y)}{g(x,y)} \right] = \frac{L_1}{L_2}, \quad \text{provided } L_2 \neq 0$$

2.7.5 Other

$$\lim_{(x,y) \rightarrow (a,b)} (f(x,y))^n = L^n \quad (n = 0, 1, 2, 3, \dots)$$

$$\lim_{(x,y) \rightarrow (a,b)} (f(x,y))^{1/n} = L^{1/n} \quad \text{if } n \text{ is even, } L \geq 0$$

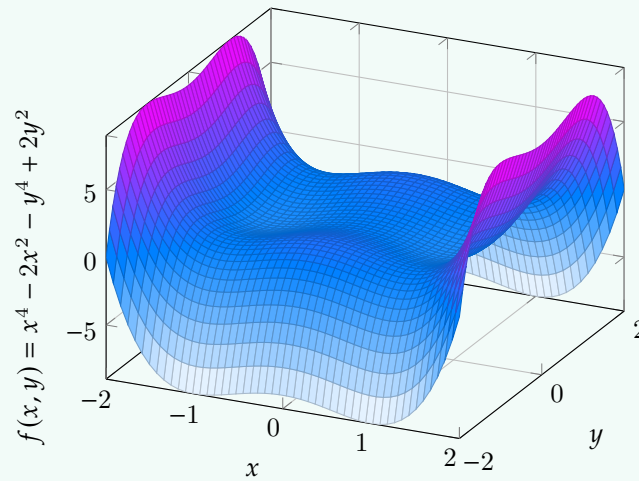
2.8 Partial Derivatives

Definition 2.8.1: Partial Derivative of random two variable function

$$\frac{\partial}{\partial x} f(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\frac{\partial}{\partial y} f(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Example 2.8.1 (Example of application of partial derivatives)



The partial derivative of a function $f(x, y)$ with respect to x at a point $P = (x_0, y_0)$ is defined as:

$$f_x(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

Similarly, the partial derivative with respect to y at $P = (x_0, y_0)$ is:

$$f_y(x_0, y_0) = \lim_{k \rightarrow 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k}$$

For the function $f(x, y) = x^4 - 2x^2 - y^4 + 2y^2$, the partial derivatives are:

$$f_x(x, y) = 4x^3 - 4x, \quad f_y(x, y) = -4y^3 + 4y$$

2.8.1 Differentiability

Theorem 2.8.1 Conditions for Differentiability

Consider a function $f(x, y)$, and we aim to analyze the differentiability at a point (a, b) . To say that $f(x, y)$ is differentiable at the point (a, b) , the following conditions must hold:

1. The partial derivatives $f_x(a, b)$ and $f_y(a, b)$ must exist.
2. We must have the following expression for the increment of f at (a, b) :

$$f(a + \Delta x, b + \Delta y) - f(a, b) = \Delta x f_x(a, b) + \Delta y f_y(a, b) + \epsilon_1(\Delta x)\Delta x + \epsilon_2(\Delta y)\Delta y$$

where $\epsilon_1(\Delta x) \rightarrow 0$ as $\Delta x \rightarrow 0$ and $\epsilon_2(\Delta y) \rightarrow 0$ as $\Delta y \rightarrow 0$.

This form shows that the change in the function can be approximated by the linear part involving the partial derivatives, with the higher-order terms represented by ϵ_1 and ϵ_2 , which tend to zero.

Graphical Interpretation

Consider the following illustration of the change in the function f :

$$\text{Elevation of } f(a, b) \rightarrow f(a + \Delta x, b + \Delta y)$$

The increment of f is given by:

$$f(a + \Delta x, b + \Delta y) - f(a, b)$$

Using the definition of differentiability, we can decompose this increment as follows:

$$\Delta f = \Delta x f_x(a, b) + \Delta y f_y(a, b)$$

Graphically, this can be interpreted as moving along the tangent plane at the point (a, b) in the direction of Δx and Δy . The elevation changes linearly based on the partial derivatives.

For the function to be differentiable at (a, b) , we require that:

$$\lim_{\Delta x \rightarrow 0} \epsilon_1(\Delta x) = 0, \quad \lim_{\Delta y \rightarrow 0} \epsilon_2(\Delta y) = 0$$

These conditions ensure that the error terms involving ϵ_1 and ϵ_2 vanish as the changes Δx and Δy become small, ensuring a linear approximation near the point (a, b) .

2.8.2 Open and Closed Sets

Consider the set R defined by:

$$R : y < x, \quad x^2 + y^2 < 1$$

The set R is open if we can draw it using **dashed lines**, indicating that the boundary is not included in the set. If we have a set of solutions to inequalities with strict inequalities (e.g., $<$ or $>$), the set is typically **open**.

If R is defined by inequalities involving \leq or \geq , then the set is **closed**.

Example of Open and Closed Sets

Let R_1 be defined as:

$$R_1 : x^2 + y^2 < 1 \quad (\text{open})$$

Let S be defined as:

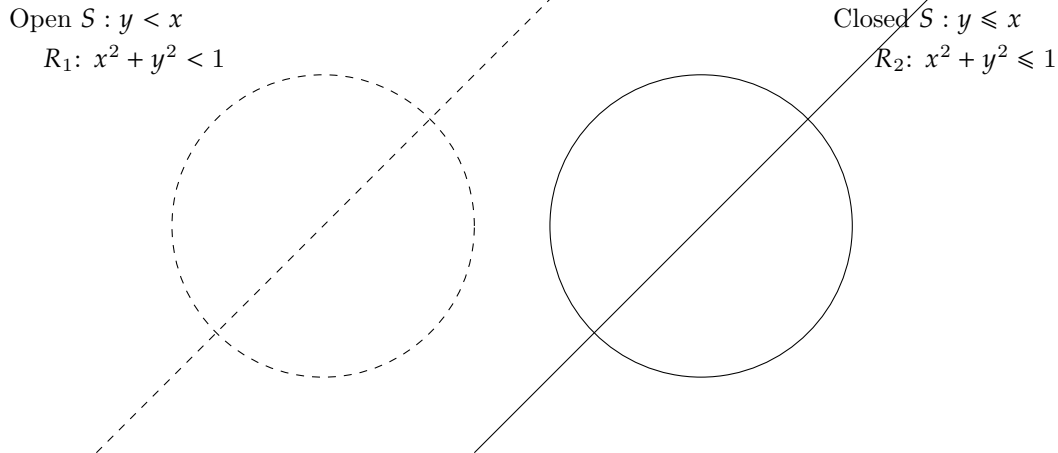
$$S : y < x \quad (\text{open})$$

We can also consider the closed sets:

$$R_2 : x^2 + y^2 \leq 1 \quad (\text{closed})$$

$$S : y \leq x \quad (\text{closed})$$

Below are illustrations of open and closed sets. Dotted lines indicate that the boundary is **not included** (open), while solid lines indicate the boundary **is included** (closed).



2.8.3 Conditions for Differentiability

A function $f(x, y)$ is differentiable at (a, b) if:

1. The partial derivatives $f_x(a, b)$ and $f_y(a, b)$ exist.
2. The following limit holds:

$$f(a + \Delta x, b + \Delta y) - f(a, b) = \Delta x f_x(a, b) + \Delta y f_y(a, b) + \Delta x \epsilon_1(\Delta x) + \Delta y \epsilon_2(\Delta y)$$

where $\epsilon_1(\Delta x) \rightarrow 0$ as $\Delta x \rightarrow 0$ and $\epsilon_2(\Delta y) \rightarrow 0$ as $\Delta y \rightarrow 0$.

Theorem 2.8.2 Sufficient Conditions for Differentiability

A sufficient condition for differentiability is the following theorem:

If there exists an open set D containing (a, b) such that $f_x(x, y)$ and $f_y(x, y)$ exist at all points in D , and if $f_x(a, b)$ and $f_y(a, b)$ are continuous at (a, b) , then $f(x, y)$ is differentiable at (a, b) .

Example 2.8.2 (Differentiability of a Given Function)

Consider the function:

$$f(x, y) = e^{xy} + x^3 y$$

We want to check whether this function is differentiable at any point (a, b) . First, we compute the partial derivatives:

$$f_x(x, y) = ye^{xy} + 3x^2 y$$

$$f_y(x, y) = xe^{xy} + x^3$$

For any point (a, b) , both $f_x(a, b)$ and $f_y(a, b)$ exist in \mathbb{R}^2 and are continuous at (a, b) because the terms e^{xy} , x^3 , and y are all continuous on \mathbb{R}^2 .

Thus, according to the theorem for differentiability, we conclude that:

$f(x, y)$ is differentiable at (a, b)

Example 2.8.3 (Discontinuity of a Function)

Consider the function:

$$f(x, y) = \begin{cases} \frac{3xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

We are interested in determining whether this function is continuous and differentiable at the point $(0, 0)$.

Continuity Check

We first evaluate $f(0, 0)$:

$$f(0, 0) = 0$$

Next, we compute the limit of $f(x, y)$ as $(x, y) \rightarrow (0, 0)$. To do this, we examine the limit along two different paths:

$$r_1(t) = \langle t, 0 \rangle$$

Along this path, we have:

$$\lim_{t \rightarrow 0} f(t, 0) = \lim_{t \rightarrow 0} \frac{3t \cdot 0}{t^2 + 0^2} = 0$$

Now consider the path:

$$r_2(t) = \langle t, t \rangle$$

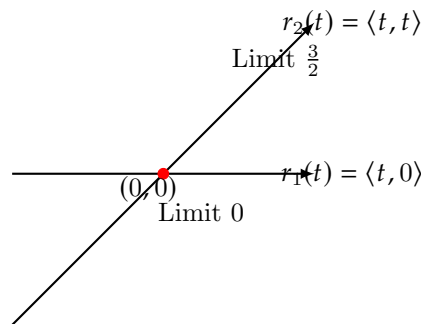
Along this path, we compute:

$$\lim_{t \rightarrow 0} f(t, t) = \lim_{t \rightarrow 0} \frac{3t \cdot t}{t^2 + t^2} = \frac{3t^2}{2t^2} = \frac{3}{2}$$

Since the limits along different paths do not agree (one is 0, and the other is $\frac{3}{2}$), we conclude that:

$$f(x, y) \text{ is not continuous at } (0, 0)$$

Thus, even though the partial derivatives f_x and f_y exist at $(0, 0)$, the function is not continuous, and therefore, $f(x, y)$ is not differentiable at $(0, 0)$.



Theorem 2.8.3 Differentiability Implies Continuity

If $f(x, y)$ is differentiable at (a, b) , then $f(x, y)$ is also continuous at (a, b) . This is an important result that will help us analyze the next example.

2.9 Directional Derivative

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector, i.e., $|\mathbf{u}| = 1$. The directional derivative of $f(x, y)$ at the point (a, b) in the direction of \mathbf{u} is defined as:

$$D_{\mathbf{u}}f(a, b) = \lim_{h \rightarrow 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}$$

This represents the slope of the hill that you are traveling on in the direction of \mathbf{u} .

If $f(x, y)$ is differentiable at (a, b) , we know that the function f satisfies the following approximation:

$$f(a + \Delta x, b + \Delta y) - f(a, b) = \Delta x f_x(a, b) + \Delta y f_y(a, b) + \Delta x \epsilon_1(\Delta x) + \Delta y \epsilon_2(\Delta y)$$

where $\epsilon_1(\Delta x) \rightarrow 0$ as $\Delta x \rightarrow 0$ and $\epsilon_2(\Delta y) \rightarrow 0$ as $\Delta y \rightarrow 0$.

2.9.1 Derivation of the Directional Derivative

Substituting $\Delta x = hu_1$ and $\Delta y = hu_2$, the directional derivative becomes:

$$\lim_{h \rightarrow 0} \frac{hu_1 f_x(a, b) + hu_2 f_y(a, b) + hu_1 \epsilon_1(h) + hu_2 \epsilon_2(h)}{h}$$

Simplifying the expression:

$$= u_1 f_x(a, b) + u_2 f_y(a, b) + u_1 \epsilon_1(h) + u_2 \epsilon_2(h)$$

As $h \rightarrow 0$, the terms $u_1 \epsilon_1(h)$ and $u_2 \epsilon_2(h)$ tend to 0, so we are left with:

$$D_{\mathbf{u}}f(a, b) = u_1 f_x(a, b) + u_2 f_y(a, b)$$

This can be expressed compactly as the dot product:

$$D_{\mathbf{u}}f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle \cdot \mathbf{u}$$

Definition 2.9.1: Definition of the Gradient

The gradient of $f(x, y)$ at the point (a, b) is defined as the vector:

$$\nabla f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle$$

The directional derivative is then the dot product of the gradient and the direction vector \mathbf{u} :

$$D_{\mathbf{u}}f(a, b) = \nabla f(a, b) \cdot \mathbf{u}$$

Thus, the gradient gives the direction of the steepest ascent of the function, and the directional derivative gives the rate of change of the function in the direction of \mathbf{u} .

2.10 Application of the Gradient

The gradient $\nabla f(a, b)$ plays an important role in determining the behavior of the function in various directions.

2.10.1 Directional Derivative and the Gradient

If f is differentiable at (a, b) , then for any unit vector \mathbf{u} , the directional derivative of f in the direction of \mathbf{u} is given by:

$$D_{\mathbf{u}}f(a, b) = \nabla f(a, b) \cdot \mathbf{u}$$

This formula can be used to determine how the function changes as we move in the direction of \mathbf{u} .

2.10.2 Dot Product and Angle

We know from vector calculus that:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta$$

Thus, for the directional derivative:

$$D_{\mathbf{u}}f(a, b) = |\nabla f(a, b)| |\mathbf{u}| \cos \theta$$

where θ is the angle between the gradient vector $\nabla f(a, b)$ and the direction vector \mathbf{u} .

- If $\theta > \frac{\pi}{2}$, then \mathbf{u} points downhill (negative directional derivative).
- If $0 < \theta < \frac{\pi}{2}$, then \mathbf{u} points uphill (positive directional derivative).

2.10.3 Maximum and Minimum Values of the Directional Derivative

The directional derivative $D_{\mathbf{u}}f(a, b)$ is maximized when $\theta = 0^\circ$, which means that \mathbf{u} points in the direction of the steepest ascent. This happens when \mathbf{u} is in the same direction as $\nabla f(a, b)$:

$\nabla f(a, b)$ points towards the direction of steepest ascent.

Conversely, the directional derivative is minimized when $\theta = 180^\circ$, which means \mathbf{u} points in the direction of steepest descent. In this case:

$-\nabla f(a, b)$ points towards the direction of steepest descent.

This forms the foundation of gradient descent algorithms, where the negative gradient is used to find minimum values.

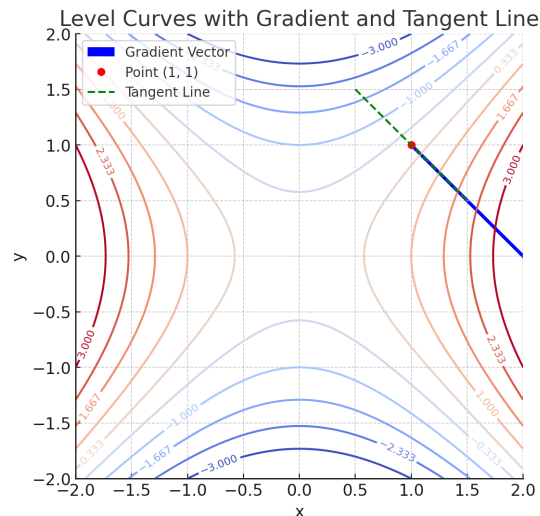
2.10.4 Tangent to a Level Curve

Consider a level curve of the form $f(x, y) = c$. The vector $\nabla f(a, b)$ is perpendicular to the level curve at the point (a, b) , which implies that any vector \mathbf{u} tangent to the level curve satisfies:

$$D_{\mathbf{u}}f(a, b) = 0$$

This means that if \mathbf{u} is tangent to the level curve, the directional derivative is zero, implying that there is no change in the value of the function along that curve.

Below is a diagram representing the gradient and the tangents to the level curves:



2.11 Tangent Planes

2.11.1 Tangent Line for Single Variable Functions

For a function of a single variable, the equation of the tangent line at a point $f(a)$ is given by:

$$y = f'(a)(x - a) + f(a)$$

This represents the line that touches the curve $y = f(x)$ at the point $(a, f(a))$, and the slope of this line is determined by the derivative $f'(a)$.

2.11.2 Tangent Plane for Two Variable Functions

For functions of two variables, i.e. $f(x, y)$, the tangent plane at a point $P(a, b, c)$ on the surface can be described. Let the function be $f(a, b) = c$, and we define a point $P(a, b, c)$ that lies on the surface at the tangent point.

The tangent plane equation is derived using partial derivatives and parameter vectors.

Parameter Vector Representation

Let the point on the surface be $P = (a, b, c)$ and consider small increments in x and y , denoted Δx and Δy , respectively. The change in the function value Δz due to these increments can be approximated using the partial derivatives. The tangent plane is spanned by vectors in the x and y directions, leading to:

$$\vec{v} = \langle v_1, v_2 \rangle$$

where:

$$v_1 = \frac{\partial f}{\partial x}(a, b) \quad \text{and} \quad v_2 = \frac{\partial f}{\partial y}(a, b)$$

The equation of the plane then involves computing how much z increments based on x and y , giving us the tangent plane equation:

$$z = c + \nabla f(a, b) \cdot \langle x - a, y - b \rangle$$

where $\nabla f(a, b)$ represents the gradient vector of the function $f(x, y)$ at the point (a, b) .

Definition 2.11.1: The Tangent Plane Equation

The explicit equation of the tangent plane can be written as:

$$\frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b) + z - c = 0$$

This expresses how the surface locally behaves near the point (a, b, c) , and the coefficients of $x - a$ and $y - b$ are given by the partial derivatives of f at (a, b) .

Example 2.11.1 (1)

We are tasked with finding the tangent plane of the surface $z = f(x, y) = \sin(x) \cdot e^y$ at the point $(\frac{\pi}{2}, 0, 1)$. The general form of the equation of a tangent plane to a surface $z = f(x, y)$ is given by:

$$\frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b) + z - c = 0$$

We will follow the steps below to compute the tangent plane:

1. Find the point of tangency (a, b, c) :

The point of tangency is given as $(\frac{\pi}{2}, 0, 1)$, so:

$$a = \frac{\pi}{2}, \quad b = 0, \quad c = 1$$

2. Compute the partial derivative with respect to x :

The partial derivative of $f(x, y) = \sin(x) \cdot e^y$ with respect to x is:

$$\frac{\partial f}{\partial x}(x, y) = \cos(x) \cdot e^y$$

At the point $(\frac{\pi}{2}, 0)$, this becomes:

$$\frac{\partial f}{\partial x}\left(\frac{\pi}{2}, 0\right) = \cos\left(\frac{\pi}{2}\right) \cdot e^0 = 0 \cdot 1 = 0$$

3. Compute the partial derivative with respect to y :

The partial derivative of $f(x, y) = \sin(x) \cdot e^y$ with respect to y is:

$$\frac{\partial f}{\partial y}(x, y) = \sin(x) \cdot e^y$$

At the point $(\frac{\pi}{2}, 0)$, this becomes:

$$\frac{\partial f}{\partial y}\left(\frac{\pi}{2}, 0\right) = \sin\left(\frac{\pi}{2}\right) \cdot e^0 = 1 \cdot 1 = 1$$

4. Substitute into the tangent plane equation:

Using the general form of the tangent plane equation and substituting $a = \frac{\pi}{2}$, $b = 0$, $c = 1$, as well as the values of the partial derivatives $\frac{\partial f}{\partial x}(a, b) = 0$ and $\frac{\partial f}{\partial y}(a, b) = 1$, we have:

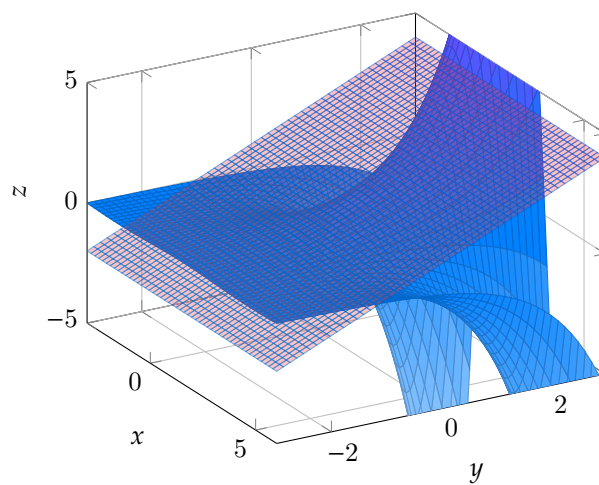
$$0 \cdot \left(x - \frac{\pi}{2}\right) + 1 \cdot (y - 0) + z - 1 = 0$$

Simplifying the equation:

$$y + z - 1 = 0$$

5. Final equation of the tangent plane: Thus, the equation of the tangent plane at $(\frac{\pi}{2}, 0, 1)$ is:

$$y + z - 1 = 0$$



2.11.3 Implicit Case

When dealing with implicit functions such as $f(x, y) = D$, the process involves implicit differentiation to find the slope of the tangent line. For example, given an equation of the form:

$$x^2 + y^2 = 4$$

we differentiate implicitly to obtain the tangent line equation. The process is as follows:

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

For the given equation $x^2 + y^2 = 4$, we compute:

$$\frac{dy}{dx} = -\frac{x}{y}$$

leading to the equation of the tangent line:

$$y - b = -\frac{F_x}{F_y}(x - a)$$

Definition 2.11.2: Tangent Line for Implicit Functions

Finally, for the tangent line in the implicit case, we use the gradient $\nabla f(a, b)$ to express the equation of the tangent line:

$$\nabla f(a, b) \cdot \langle x - a, y - b \rangle = 0$$

This represents a line tangent to the curve at the point (a, b) on the implicit function.

2.11.4 Level Surfaces and Tangent Planes

A **level surface** is represented by the equation $w = f(x, y, z)$, where w is a constant, indicating a 3D surface embedded in 4D space. For example:

$$f(x, y, z) = D, \quad D \in \mathbb{R}$$

This represents a 2D surface in 3D space. An example is the implicit equation:

$$x^2 + y^2 + z^2 = 4$$

which describes a sphere of radius 2 in 3D space.

Definition 2.11.3: Level Curves Tangent Plane

The tangent plane to the surface is given by:

$$\nabla F(a, b, c) \cdot \langle x - a, y - b, z - c \rangle = 0$$

where $\nabla F(a, b, c)$ is the gradient of the function F at the point (a, b, c) .

2.11.5 Summary of Cases

Curve/Surface	Equation
$y = f(x)$	$y = f'(a)(x - a) + f(a)$
$z = f(x, y)$	$z = c + \nabla f(a, b) \cdot \langle x - a, y - b \rangle$
$F(x, y, z) = D$	$\nabla F(a, b, c) \cdot \langle x - a, y - b, z - c \rangle = 0$

Example 2.11.2 (1)

Consider the surface given by:

$$z = f(x, y) = x^2y^2 - 2x$$

At the point $(1, 3, 7)$, we compute the gradient:

$$\nabla f(x, y) = \langle y^2 2x - 2, x^2 2y \rangle$$

At the point $(1, 3)$:

$$\nabla f(1, 3) = \langle 9 - 2, 6 \rangle = \langle 7, 6 \rangle$$

The equation of the tangent plane is:

$$7(x - 1) + 6(y - 3) + z - 7 = 0$$

Simplifying this gives:

$$7(x - 1) + 6(y - 3) + z = 7$$

Example 2.11.3 (2)

Now, consider the implicit surface given by:

$$F(x, y, z) = x^3 - 3y + z^2 = 8$$

At the point $(1, 2, 3)$, the gradient is:

$$\nabla F(x, y, z) = \langle 3x^2, -3, 2z \rangle$$

At the point $(1, 2, 3)$, this evaluates to:

$$\nabla F(1, 2, 3) = \langle 3, -3, 6 \rangle$$

The tangent plane equation is:

$$3(x - 1) - 3(y - 2) + 6(z - 3) = 0$$

Simplifying:

$$3(x - 1) - 3(y - 2) + 6(z - 3) = 0$$

$$3(x - 1) - 3(y - 2) + 6(z - 3) = 0$$

This describes the tangent plane to the surface at the given point.

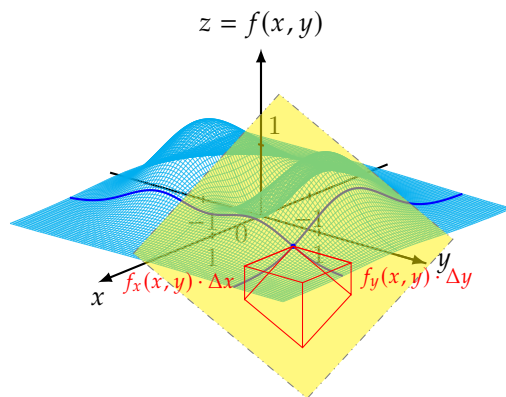
2.12 Applications of Tangent Planes

2.12.1 Linear Approximation

In many cases, we know the value of the function $f(a, b)$ at a particular point and want to estimate the value of $f(x, y)$ near that point. We use the linear approximation formula for this estimation:

$$f(x, y) \approx f(a, b) + \nabla f(a, b) \cdot \langle x - a, y - b \rangle$$

where $\nabla f(a, b)$ is the gradient of the function f at the point (a, b) .



Example 2.12.1 (Linear Approximation of $\sin(xy)$)

Let $f(x, y) = \sin(xy)$. We want to estimate $f(0.8, 0.25)$ using linear approximation.

1. First, find $f(a, b) = f(\pi/2, 0.2)$. Since $\sin(\frac{\pi}{2} \cdot 0.2) \approx 0.985$, we get:

$$f(a, b) = \sin\left(\frac{\pi}{2} \times 0.2\right) \approx 0.985$$

2. The gradient $\nabla f(x, y)$ is given by:

$$\nabla f(x, y) = \langle y \cos(xy), x \cos(xy) \rangle$$

At $(\frac{\pi}{2}, 0.2)$, we calculate the partial derivatives:

$$\nabla f\left(\frac{\pi}{2}, 0.2\right) = \langle 0.2 \cos(1), \frac{\pi}{2} \cos(1) \rangle$$

Using $\cos(1) \approx 0.5403$, we have:

$$\nabla f\left(\frac{\pi}{2}, 0.2\right) = \langle 0.1081, 0.8474 \rangle$$

3. Now, estimate $f(0.8, 0.25)$ using the formula for linear approximation:

$$f(0.8, 0.25) \approx f\left(\frac{\pi}{2}, 0.2\right) + \nabla f\left(\frac{\pi}{2}, 0.2\right) \cdot \langle 0.8 - \frac{\pi}{2}, 0.25 - 0.2 \rangle$$

Simplifying:

$$f(0.8, 0.25) \approx 0.985 + \langle 0.1081, 0.8474 \rangle \cdot \langle -0.014, 0.05 \rangle$$

$$f(0.8, 0.25) \approx 0.985 + (-0.0015 + 0.0424)$$

$$f(0.8, 0.25) \approx 0.985 + 0.0409 = 1.0259$$

2.12.2 Finding Approximate Solutions to Equations

To find approximate solutions to equations, we can apply linearization techniques similar to the above. For instance, if we know a point (a, b, c) and want to estimate the solution for $f(x, y, z)$, we use the following approximation:

$$f(x, y, z) \approx f(a, b, c) + \nabla f(a, b, c) \cdot \langle x - a, y - b, z - c \rangle$$

where $\nabla f(a, b, c)$ is the gradient of the function at the point (a, b, c) .

Example 2.12.2 (Finding the Root of $f(z) = ze^z + 1.02z - e^z - 0.2$)

To find approximate solutions to equations, we can apply linearization techniques similar to the above. For instance, if we know a point (a, b, c) and want to estimate the solution for $f(x, y, z)$, we use the following approximation:

$$f(x, y, z) \approx f(a, b, c) + \nabla f(a, b, c) \cdot \langle x - a, y - b, z - c \rangle$$

where $\nabla f(a, b, c)$ is the gradient of the function at the point (a, b, c) .

2.12.3 Example: Linearizing $f(x, y, z) = z \ln(z) + xz - e^{yz}$

Consider the function $f(x, y, z) = z \ln(z) + xz - e^{yz}$. We want to find an approximate solution around the point $(x, y, z) = (1, 0, 1)$.

First, we calculate the value of the function at this point:

$$f(1, 0, 1) = 1 \ln(1) + 1 \cdot 1 - e^{0 \cdot 1} = 0 + 1 - 1 = 0.$$

Next, we compute the gradient $\nabla f(x, y, z)$ at the point $(1, 0, 1)$. The gradient is given by:

$$\nabla f(x, y, z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right).$$

The partial derivatives are calculated as follows:

$$\frac{\partial f}{\partial x} = z, \quad \frac{\partial f}{\partial y} = -ze^{yz}, \quad \frac{\partial f}{\partial z} = \ln(z) + 1 + x - ye^{yz}.$$

Evaluating these at $(x, y, z) = (1, 0, 1)$:

$$\frac{\partial f}{\partial x}(1, 0, 1) = 1, \quad \frac{\partial f}{\partial y}(1, 0, 1) = -1, \quad \frac{\partial f}{\partial z}(1, 0, 1) = \ln(1) + 1 + 1 = 2.$$

Thus, the gradient at $(1, 0, 1)$ is:

$$\nabla f(1, 0, 1) = (1, -1, 2).$$

Using the linear approximation formula:

$$f(x, y, z) \approx f(1, 0, 1) + \nabla f(1, 0, 1) \cdot \langle x - 1, y, z - 1 \rangle,$$

we substitute the values:

$$f(x, y, z) \approx 0 + (1)(x - 1) + (-1)(y) + (2)(z - 1).$$

Simplifying:

$$f(x, y, z) \approx x - 1 - y + 2(z - 1).$$

$$f(x, y, z) \approx x + 2z - y - 3.$$

Thus, the linearized approximation of $f(x, y, z)$ around $(1, 0, 1)$ is:

$$f(x, y, z) \approx x + 2z - y - 3.$$

2.12.4 Differentials

Theorem 2.12.1 Differentials approximation using gradient

Suppose $z = f(x, y)$, then

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

can be approximated by

$$\Delta z \approx \nabla f(x, y) \cdot \langle \Delta x, \Delta y \rangle$$

which expands to

$$\Delta z = f_x(x, y)\Delta x + f_y(x, y)\Delta y$$

where $f_x(x, y)$ and $f_y(x, y)$ are the partial derivatives of f with respect to x and y , respectively.

Example 2.12.3 (Example)

Consider the volume V of a cone, given by

$$V = \frac{\pi}{2}r^2h$$

where r is the radius and h is the height of the cone. If r decreases by 1.5% and h increases by 2.2%, what is the percentage change in V ?

First, find the change in volume ΔV using the differential approximation:

$$\Delta V = V_r\Delta r + V_h\Delta h$$

where V_r and V_h are the partial derivatives of V with respect to r and h . We have:

$$V_r = \pi rh, \quad V_h = \frac{\pi}{2}r^2$$

Thus,

$$\frac{\Delta V}{V} = \frac{\pi rh\Delta r}{\frac{\pi}{2}r^2h} + \frac{\frac{\pi}{2}r^2\Delta h}{\frac{\pi}{2}r^2h} = \frac{2\Delta r}{r} + \frac{\Delta h}{h}$$

Substituting the percentage changes:

$$\frac{\Delta V}{V} = 2\left(\frac{-1.5}{100}\right) + \left(\frac{2.2}{100}\right) = -0.03 + 0.022 = -0.008$$

Thus, the volume decreases by 0.8%.

2.13 Chain Rule in Functions of several random variables $f : \mathbb{R}^n \rightarrow \mathbb{R}$

Claim 2.13.1 Differentiation of functions

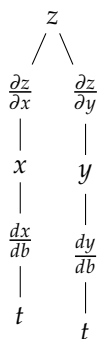
Let z be a differentiable function of x and y on an interval domain, where x and y are differentiable functions of t on interval I . Then:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

We sum the rates of change of z through x and y .

Definition 2.13.1: Single Variable

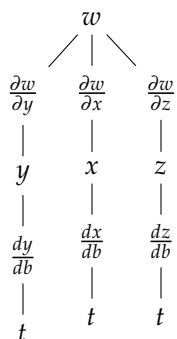
For single-variable functions, we use $\frac{d}{dt}$; for multiple variables, we use $\frac{\partial}{\partial t}$.



Claim 2.13.2 Three Variables

If z is a function of x , y , and t , we have:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} + \frac{\partial z}{\partial t}$$



Example 2.13.1 (Example 1)

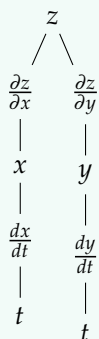
Given $z = x \sin(y)$, $x = t^2$, $y = t^3$:

$$\frac{\partial z}{\partial x} = \sin(y), \quad \frac{\partial z}{\partial y} = x \cos(y)$$

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 3t^2$$

Therefore,

$$\frac{dz}{dt} = \sin(t^3)(2t) + t^2 \cos(t^3)(3t^2) = 2t \sin(t^3) + 3t^4 \cos(t^3)$$



Example 2.13.2 (Example 2)

Given $V = xyz$, $x = e^t$, $y = 2t + 3$, $z = \sin(t)$:

$$\frac{\partial V}{\partial x} = yz, \quad \frac{\partial V}{\partial y} = xz, \quad \frac{\partial V}{\partial z} = xy$$

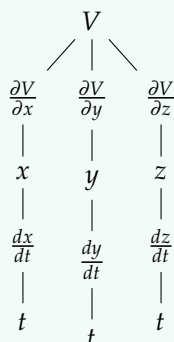
$$\frac{dx}{dt} = e^t, \quad \frac{dy}{dt} = 2, \quad \frac{dz}{dt} = \cos(t)$$

Therefore,

$$\frac{dV}{dt} = yz \cdot e^t + xz \cdot 2 + xy \cdot \cos(t)$$

Substituting:

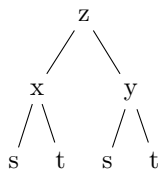
$$\frac{dV}{dt} = (2t + 3)\sin(t) \cdot e^t + (e^t)\sin(t) \cdot 2 + (e^t)(2t + 3) \cdot \cos(t)$$

**Claim 2.13.3** Two Independent Variables

If $z = f(x, y)$, $x = g(s, t)$, $y = h(s, t)$, then:

$$\frac{dz}{ds} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

**Claim 2.13.4** Implicit Differentiation

Let f be a differentiable function on its domain, and suppose $f(x, y) = 0$ defines y as a differentiable function of x . Provided $F_y \neq 0$, we have:

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Example 2.13.3 (Example)

Given:

$$\frac{dy}{dx} \text{ if } ye^{xy} - 2 = 0$$

We have:

$$F_x = y^2 e^{xy}, \quad F_y = xye^{xy} + e^{xy}$$

Therefore:

$$\frac{dy}{dx} = -\frac{y^2 e^{xy}}{xye^{xy} + e^{xy}} = \frac{-y^2}{x + y}$$

2.14 Maximum and Minimum of 2 Variable Functions $f : \mathbb{R} \rightarrow \mathbb{R}^2$

2.14.1 Critical Points

A *critical point* of a function $f(x, y)$ is a point $(a, b) \in \mathbb{R}^2$ such that:

1. $\nabla f(a, b) = \vec{0}$, i.e., the gradient of f at (a, b) is the zero vector $\langle 0, 0 \rangle$.
2. The partial derivatives $f_x(a, b)$ and $f_y(a, b)$ both exist.

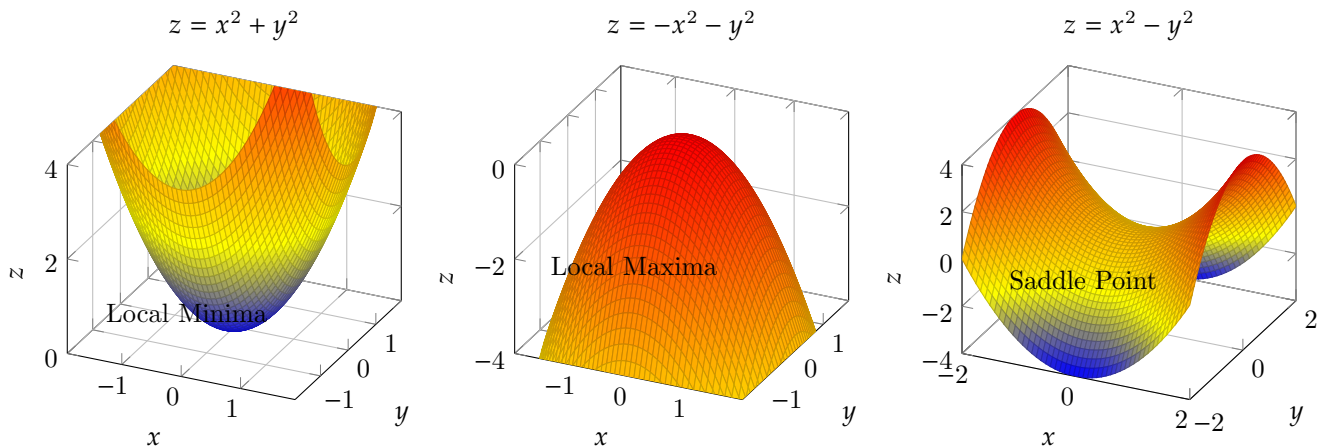
At a critical point, there are three possible types:

1. *Local minimum*: The function value increases as we move away from (a, b) in all directions.
2. *Local maximum*: The function value decreases as we move away from (a, b) in all directions.
3. *Saddle point*: The function behaves such that in some directions, it increases, while in others, it decreases.

2.14.2 Definitions

1. $f(a, b)$ is a *local maximum* if there exists an open disc D , such that for all $(x, y) \in D$, $f(a, b) \geq f(x, y)$.
2. $f(a, b)$ is a *local minimum* if the same condition holds, but with $f(a, b) \leq f(x, y)$.
3. A *saddle point* exists if, for any neighborhood around (a, b) , there exist points (x_1, y_1) and (x_2, y_2) such that $f(a, b) > f(x_1, y_1)$ and $f(a, b) < f(x_2, y_2)$.

2.14.3 Visual Representation of Maxima, Minima and Saddle Points



2.14.4 Second Derivative Test

The second derivative test is used to classify critical points by examining the Hessian matrix. The Hessian matrix, $H(x, y)$, contains the second-order partial derivatives of $f(x, y)$:

$$H(x, y) = \begin{bmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{bmatrix}$$

We evaluate the determinant of the Hessian, $\det(H) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2$, and use the following rules:

1. If $\det(H) > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a *local minimum*.
2. If $\det(H) > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a *local maximum*.
3. If $\det(H) < 0$, then (a, b) is a *saddle point*.
4. If $\det(H) = 0$, the test is *inconclusive*.

Example 2.14.1 (Example)

Consider the function:

$$f(x, y) = x^3 - 3x + y^3 - 3y$$

To find the critical points, we calculate the gradient:

$$\nabla f(x, y) = \begin{bmatrix} 3x^2 - 3 \\ 3y^2 - 3 \end{bmatrix}$$

Setting $\nabla f(x, y) = 0$, we solve:

$$3x^2 - 3 = 0 \quad \Rightarrow \quad x = \pm 1$$

$$3y^2 - 3 = 0 \quad \Rightarrow \quad y = \pm 1$$

Thus, the critical points are $(1, 1), (1, -1), (-1, 1), (-1, -1)$.

Next, we calculate the Hessian matrix:

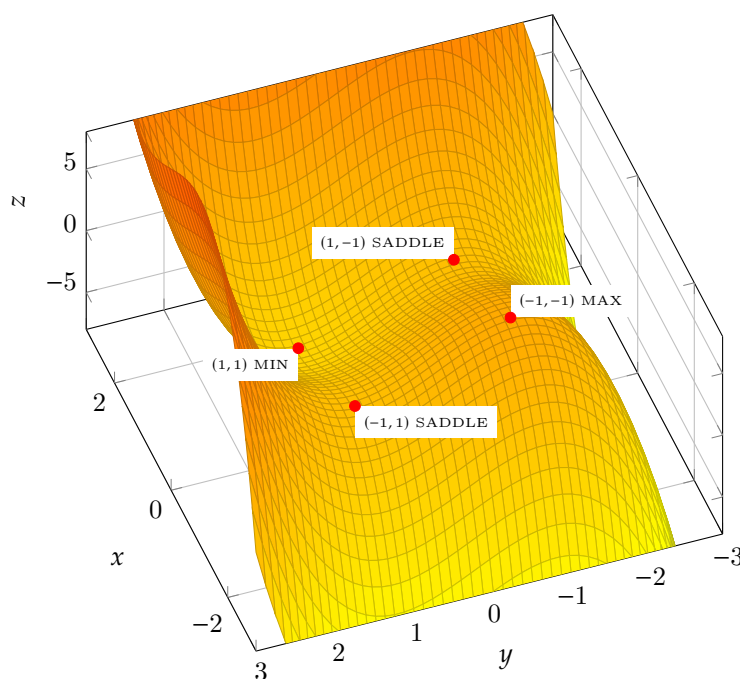
$$H(x, y) = \begin{bmatrix} 6x & 0 \\ 0 & -6y \end{bmatrix}$$

The determinant of the Hessian is:

$$\det(H) = -36xy$$

For the critical points:

- $(1, 1)$: $\det(H) = 36$, so this is a saddle point.
- $(1, -1)$: $\det(H) = -36$, and since $f_{xx}(1, -1) = 6$, this is a local minimum.
- $(-1, 1)$: $\det(H) = -36$, and since $f_{xx}(-1, 1) = -6$, this is a local maximum.
- $(-1, -1)$: $\det(H) = 36$, so this is a saddle point.



2.14.5 Connection Between the Second Derivative Test and Eigenvalues of the Hessian

The Second Derivative Test is closely related to the eigenvalues of the Hessian matrix. To understand this connection, we need to explore the geometric interpretation of eigenvalues and how they inform us about the curvature of a function at a critical point.

Given a Hessian matrix $H(x, y)$, we can compute its eigenvalues, λ_1 and λ_2 . These eigenvalues describe the principal curvatures of the function in the directions of the corresponding eigenvectors. Here's the relationship between the eigenvalues and the nature of the critical point:

1. If both eigenvalues $\lambda_1 > 0$ and $\lambda_2 > 0$, the function $f(x, y)$ is *locally convex* at the critical point, meaning the graph of the function curves upwards in all directions. This corresponds to a **local minimum**.
2. If both eigenvalues $\lambda_1 < 0$ and $\lambda_2 < 0$, the function is *locally concave*, meaning the graph curves downwards in all directions. This corresponds to a **local maximum**.
3. If one eigenvalue is positive and the other is negative ($\lambda_1 \lambda_2 < 0$), the function exhibits a *saddle point*. The function curves upwards in the direction corresponding to the positive eigenvalue and downwards in the direction corresponding to the negative eigenvalue.
4. If any eigenvalue is zero ($\lambda_1 = 0$ or $\lambda_2 = 0$), the Second Derivative Test is *inconclusive*. In this case, higher-order terms or a more detailed analysis is required to classify the critical point.

2.14.6 Intuition from Linear Algebra

Eigenvalues of the Hessian matrix tell us about the principal directions in which the function is changing at a critical point. Consider the Hessian as a local quadratic approximation of the function at a point, where the eigenvalues reflect how steep the function's graph is along particular directions. Specifically:

- Positive eigenvalues correspond to directions where the function increases, resembling an upward-parabola shape.
- Negative eigenvalues correspond to directions where the function decreases, resembling a downward-parabola.
- Mixed signs for the eigenvalues indicate a saddle-like behavior, where some directions exhibit maxima and others minima.

Thus, checking the signs of the eigenvalues is an alternative way to interpret the Second Derivative Test in terms of curvature.

Example 2.14.2 (Example)

Let's revisit the example $f(x, y) = x^3 - 3x + y^3 - 3y$ and calculate the Hessian matrix at one of the critical points. For the critical point $(1, 1)$, we have:

$$H(1, 1) = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

The eigenvalues of this Hessian are $\lambda_1 = 6$ and $\lambda_2 = 6$, both positive, which confirms that $(1, 1)$ is a local minimum as expected from the Second Derivative Test.

For the critical point $(1, -1)$, the Hessian is:

$$H(1, -1) = \begin{bmatrix} 6 & 0 \\ 0 & -6 \end{bmatrix}$$

The eigenvalues are $\lambda_1 = 6$ and $\lambda_2 = -6$, confirming that $(1, -1)$ is a saddle point, as one direction curves upward and the other downward.

2.15 Absolute Extrema

For a function f defined on a region R in \mathbb{R}^2 , a point (a, b) in R is an *absolute maximum* of f if $f(a, b) \geq f(x, y)$ for any (x, y) in R . Similarly, (a, b) is an *absolute minimum* if $f(a, b) \leq f(x, y)$ for any (x, y) in R .

Theorem 2.15.1 Extreme Value Theorem

If R is closed and bounded, and if f is continuous on R , then the absolute extrema of f on R can be found by examining:

1. Critical points of f (for local extrema within R),
2. Boundary values of f on R .

Example 2.15.1 (Example Extreme Value Theorem)

Consider the function $V(\ell, w) = \ell w(96 - \ell - w)$, where $R = \{(\ell, w) \mid \ell \geq 0, w \geq 0, \ell + w \leq 96\}$.

1. Since R is closed, we can apply the Extreme Value Theorem:

- (a) Find local maxima by setting $\nabla V(\ell, w) = 0$.
- (b) Evaluate V on the boundary of R .

To find critical points:

$$V(\ell, w) = \ell w(96 - \ell - w)$$

The partial derivatives are:

$$\frac{\partial V}{\partial \ell} = w(96 - 2\ell - w), \quad \frac{\partial V}{\partial w} = \ell(96 - \ell - 2w)$$

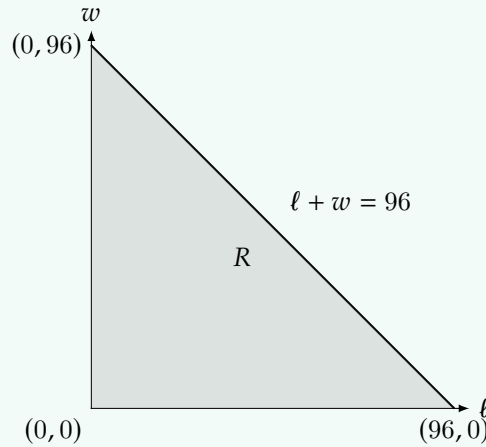
Setting these to zero, we get:

$$\begin{cases} w(96 - 2\ell - w) = 0 \\ \ell(96 - \ell - 2w) = 0 \end{cases}$$

This yields critical points at $(\ell, w) = (32, 32)$.

On the boundary of R :

1. $\ell = 0$: Then $V = 0$ for any w .
2. $w = 0$: Then $V = 0$ for any ℓ .
3. $\ell + w = 96$: Substitute $w = 96 - \ell$ into $V(\ell, w) = \ell(96 - \ell)(96 - \ell - (96 - \ell)) = \ell(96 - \ell)^2$.



Solving this, we find that $V(32, 32) = 32 \cdot 32 \cdot 32 = 32768$, which is the maximum value on R .

To confirm the local maximum at $(32, 32)$, we compute the Hessian:

$$H(\ell, w) = \begin{bmatrix} -2w & 96 - \ell - w \\ 96 - \ell - w & -2\ell \end{bmatrix}$$

At $(32, 32)$, the Hessian determinant is:

$$\det(H) = (-2 \times 32)^2 - (96 - 32 - 32)^2 = (64 \times 64) - (32 \times 32) = 32768 > 0$$

Thus, $V(\ell, w)$ has a local maximum at $(32, 32)$.

Example 2.15.2 (Another Example)

Consider $f(x, y) = 4 - x^2 - y^2$ on $R = \{(x, y) \mid -1 \leq x \leq 1, x^2 + y^2 < 1\}$. Here, R is an open region without boundaries.

The maximum of $f(x, y)$ occurs at $(0, 0)$, where $f(0, 0) = 4$. There is no absolute minimum because for points approaching the boundary (e.g., $x \approx 0.99$), $f(x, y)$ approaches $-\infty$.

Thus, in R , there is no absolute minimum value for f , illustrating the importance of the region's closedness and boundedness for the Extreme Value Theorem to apply.

2.16 Lagrange Multiplier Method

2.16.1 General Procedure

To maximize or minimize $f(x, y)$ subject to a constraint $g(x, y) = 0$, follow these steps:

1. **Identify Critical Points:** A point (x', y') is a critical point if:
 - There exists $\lambda \in \mathbb{R}$ such that $\nabla f(x', y') = \lambda \nabla g(x', y')$ (using the *method of Lagrange multipliers*).
 - $g(x', y') = 0$.
2. **Evaluate Cases for Critical Points:**

- (a) **Case 1:** The constraint region R is bounded and has no endpoints.
- In this case, assuming continuity of f , any local extrema within R are also absolute extrema.
 - Select critical points (x', y') and evaluate f at these points.
- (b) **Case 2:** The constraint region R is bounded and has endpoints.
- Example: $x^2 - 4y^2 \leq 0$ with $-1 \leq x \leq 1$.
 - Absolute extrema may occur at local extrema or endpoints.
 - Check critical points and also evaluate f at the boundary endpoints.
- (c) **Case 3:** The constraint region R is unbounded or does not include all endpoints.
- Example: $x^2 - 4y^2$ unbounded or $x^2 + y^2 - 4 = 0$ with $-2 \leq x \leq 2$.
 - It may be possible that absolute extrema do not exist.
 - Find any critical points and evaluate $f(x)$ as $x \rightarrow \pm\infty$ if applicable.

Example 2.16.1 (Lagrange Multiplier Method)

We aim to find the absolute extrema of the function $f(x, y) = x - 2y$ subject to the constraint $g(x, y) = x^2 + y^2 = 4$.

To find the extrema, we use the method of Lagrange multipliers, where we seek points where $\nabla f = \lambda \nabla g$. The gradients of f and g are:

$$\nabla f(x, y) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \nabla g(x, y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

Setting $\nabla f = \lambda \nabla g$, we have:

$$\begin{cases} 1 = \lambda \cdot 2x \\ -2 = \lambda \cdot 2y \end{cases}$$

This simplifies to:

$$\lambda = \frac{1}{2x} = \frac{-1}{y} \Rightarrow y = -2x$$

Substitute $y = -2x$ into the constraint $g(x, y) = x^2 + y^2 = 4$:

$$x^2 + (-2x)^2 = 4 \Rightarrow x^2 + 4x^2 = 4 \Rightarrow 5x^2 = 4 \Rightarrow x = \pm \frac{2}{\sqrt{5}}$$

Then, $y = -2x$ gives $y = \pm \frac{4}{\sqrt{5}}$. So the points are:

$$\left(\frac{2}{\sqrt{5}}, -\frac{4}{\sqrt{5}} \right) \quad \text{and} \quad \left(-\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}} \right)$$

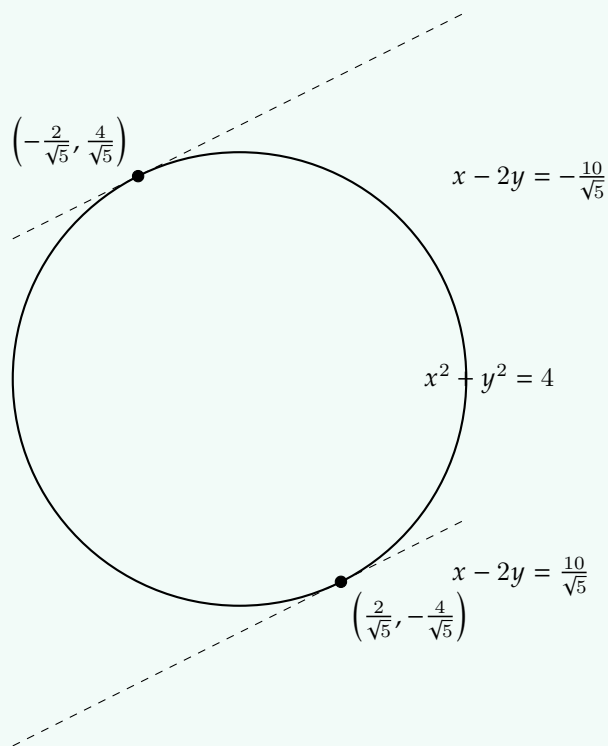
Calculate $f(x, y)$ at the points:

$$f\left(\frac{2}{\sqrt{5}}, -\frac{4}{\sqrt{5}}\right) = \frac{2}{\sqrt{5}} + \frac{8}{\sqrt{5}} = \frac{10}{\sqrt{5}} = 2\sqrt{5}$$

$$f\left(-\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right) = -\frac{2}{\sqrt{5}} - \frac{8}{\sqrt{5}} = -\frac{10}{\sqrt{5}} = -2\sqrt{5}$$

Thus, the **absolute maximum** is $2\sqrt{5}$ and the **absolute minimum** is $-2\sqrt{5}$.

Below is a diagram showing the constraint $x^2 + y^2 = 4$ as a circle and the level curves of $f(x, y) = x - 2y$, specifically showing the two tangent level curves at $z = \pm \frac{10}{\sqrt{5}}$ that represent the maximum and minimum values.



Example 2.16.2 (Example: Finding Extrema of $f(x, y) = e^{x+y}$ with Constraint)

Consider maximizing or minimizing $f(x, y) = e^{x+y}$ subject to the constraint $g(x, y) = x^2 + xy + y^2 - 9 = 0$.

1. Using the method of Lagrange multipliers, we set up the system:

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

which gives:

$$\begin{cases} e^{x+y} = \lambda(2x + y) \\ e^{x+y} = \lambda(x + 2y) \end{cases}$$

2. Dividing the equations, we get:

$$\frac{2x + y}{x + 2y} = 1 \Rightarrow x = y$$

3. Substitute $x = y$ into the constraint $x^2 + xy + y^2 = 9$:

$$x^2 + x^2 + x^2 = 9 \Rightarrow 3x^2 = 9 \Rightarrow x = \pm\sqrt{3}, \quad y = \pm\sqrt{3}$$

4. The critical points are $(\sqrt{3}, \sqrt{3})$ and $(-\sqrt{3}, -\sqrt{3})$.

5. Evaluating $f(x, y)$ at these points:

$$f(\sqrt{3}, \sqrt{3}) = e^{2\sqrt{3}}, \quad f(-\sqrt{3}, -\sqrt{3}) = e^{-2\sqrt{3}}$$

6. Thus, the maximum value is $e^{2\sqrt{3}}$ and the minimum value is $e^{-2\sqrt{3}}$.

Example 2.16.3 (Example: Finding Extrema of $f(x, y) = x - y$ with Constraint)

Consider the function $f(x, y) = x - y$ with the constraint $g(x, y) = x^2 + y^2 - 3xy - 20 = 0$.

1. The gradients are:

$$\nabla f(x, y) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \nabla g(x, y) = \begin{bmatrix} 2x - 3y \\ 2y - 3x \end{bmatrix}$$

2. Set $\nabla f = \lambda \nabla g$:

$$\begin{cases} 1 = \lambda(2x - 3y) \\ -1 = \lambda(2y - 3x) \end{cases}$$

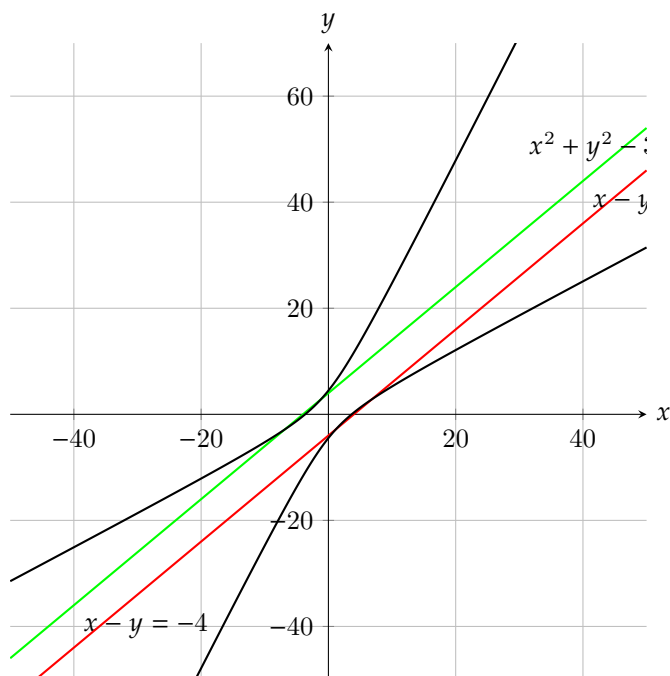
3. Solving this system, we find critical points at $(2, -2)$ and $(-2, 2)$.

4. Evaluating $f(x, y)$ at these points:

$$f(2, -2) = 1, \quad f(-2, 2) = -1$$

5. Therefore, the maximum value is 1 and the minimum value is -1 .

By observing the level curves diagram, we can see that the given points do not maximise/minimise the function, as any other line where $x - y < \pm 4$ would further increase/decrease the function.



2.17 Multiple Integration

Note:-

The integral

$$\int_a^b f(x) dx = A$$

represents the area under the curve $f(x)$ from a to b , where A can also be approximated as

$$A \approx \sum_{i=1}^N f(x_i) \Delta x.$$

Note:-

To find the volume under a surface $z = f(x, y)$ over a region $R = [a, b] \times [c, d]$, we use the double integral

$$\iint_R f(x, y) dA.$$

This can be approximated by summing over small subregions within R :

$$\iint_R f(x, y) dA = \lim_{\Delta \rightarrow 0} \sum_{i=1}^m \sum_{j=1}^n f(x_i, y_j) \Delta x \Delta y.$$

Theorem 2.17.1 Continuity and Limits of Double Integrals

If the limit

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y)$$

exists for all points (a, b) in R , then $f(x, y)$ is continuous over R , and the order of integration can be changed under Fubini's theorem.

Note:-

Consider dividing R into n subregions, each with area ΔA_{ij} and height $f(x_i, y_j)$. Then, the volume V is approximated as

$$V \approx \sum_{i=1}^m \sum_{j=1}^n f(x_i, y_j) \Delta x \Delta y,$$

where n is the number of boxes in the x -direction and m is the number in the y -direction. Taking the limit as $\Delta x, \Delta y \rightarrow 0$ gives

$$V = \iint_R f(x, y) dA.$$

Definition 2.17.1: Double Integral Definition

The double integral of $f(x, y)$ over $R = [a, b] \times [c, d]$ is defined by

$$\iint_R f(x, y) dA = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^n f(x_i, y_j) \Delta A_{ij}.$$

Theorem 2.17.2 Fubini's Theorem

The order of integration does not affect the result of the double integral. Thus,

$$\iint_R f(x, y) dA = \int_a^b \left(\int_c^d f(x, y) dy \right) dx = \int_c^d \left(\int_a^b f(x, y) dx \right) dy.$$

Example 2.17.1 (Example of Double Integral Calculation)

Calculate the volume under $f(x, y) = 4 + x + y^2$ over the region $R = [-1, 1] \times [0, 2]$:

$$\iint_R f(x, y) dA = \int_{-1}^1 \int_0^2 (4 + x + y^2) dy dx.$$

Evaluating the inner integral with respect to y ,

$$\int_0^2 (4 + x + y^2) dy = 4y + xy + \frac{y^3}{3} \Big|_0^2 = 8 + 2x + \frac{8}{3}.$$

Then, integrating with respect to x ,

$$\int_{-1}^1 \left(8 + 2x + \frac{8}{3}\right) dx = \left(8 + \frac{8}{3}\right) \cdot 2 = \frac{32}{3}.$$

Thus, the volume $V = \frac{32}{3}$.

Example 2.17.2 (Example: Integration Over a Rectangular Domain)

Evaluate

$$\int_0^2 \int_0^1 \frac{xy}{1+x^2} dx dy.$$

Using substitution $u = x^2$ with $du = 2x dx$, we find

$$\int_0^2 \int_0^1 \frac{xy}{1+x^2} dx dy = \int_0^2 y \left(\int_0^1 \frac{x}{1+x^2} dx \right) dy = \int_0^2 y \left[\frac{1}{2} \ln(1+x^2) \right]_0^1 dy.$$

Simplifying, we get

$$V = \int_0^2 y \cdot \frac{\ln(2)}{2} dy = \frac{\ln(2)}{2} \int_0^2 y dy = \frac{\ln(2)}{2} \cdot \frac{y^2}{2} \Big|_0^2 = \ln(2).$$

2.17.1 Non-Rectangular Domains

Note:-

Consider $R = \{(x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$, where the boundaries are given by functions $y = g_1(x)$ and $y = g_2(x)$.

Example 2.17.3 (Example: Non-Rectangular Domain Integration)

Suppose $R = \{(x, y) : -1 \leq x \leq 1, x^2 \leq y \leq 2 - x^2\}$. We wish to evaluate

$$\int_R (x + y) dA.$$

We split R into two regions, R_1 and R_2 , with bounds given by

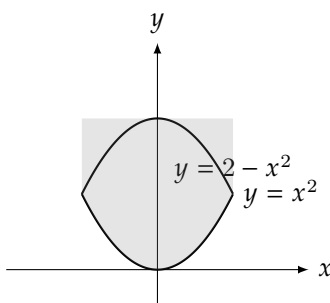
$$R_1 = \{(x, y) : -1 \leq x \leq 1, x^2 \leq y \leq 2 - x^2\}.$$

Evaluating each integral, we obtain

$$V = \int_{-1}^1 \left(\int_{x^2}^{2-x^2} (x + y) dy \right) dx.$$

On solving, we get

$$V = \int_{-1}^1 \left(\int_{x^2}^{2-x^2} (x+y) dy \right) dx = \dots = \frac{8}{35}.$$



Note:-

When changing the order of integration, try dividing the region into smaller regions to make integration simpler.

2.17.2 Volume Between Surfaces

Note:-

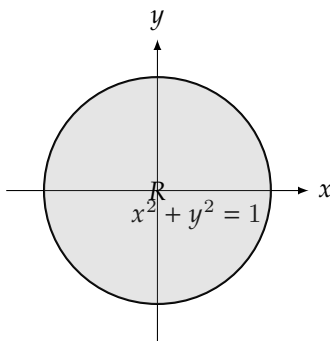
To find the volume of a sphere using double integrals, consider the surface

$$x^2 + y^2 + z^2 = 1.$$

Then $z = \pm\sqrt{1 - x^2 - y^2}$, and we can set up the integral as

$$V = 2 \iint_R \sqrt{1 - x^2 - y^2} dA,$$

where $R = \{(x, y) : -1 \leq x \leq 1, -\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}\}$.



Example 2.17.4 (Example: Volume Between Surfaces)

Calculate the volume between the surfaces $z = \sqrt{1 - x^2 - y^2}$ and $z = -\sqrt{1 - x^2 - y^2}$:

$$V = \iint_R \left(\sqrt{1 - x^2 - y^2} - (-\sqrt{1 - x^2 - y^2}) \right) dA = 2 \iint_R \sqrt{1 - x^2 - y^2} dA.$$

Setting up the limits as before, we integrate over R to find the volume of the sphere.

2.18 Applications of Double Integrals

2.18.1 Area of a Surface

Definition 2.18.1: Area of a Surface with Double integrals

When calculating the area of a surface where the height of the surface in 3D space is constantly 1, the area of the surface is numerically equal to the volume under the surface. Thus, by setting the height $f(x, y) = 1$ over a given region R , the area of R can be computed using a double integral, which effectively measures the "volume" under the constant surface.

$$\text{Area of } R = \iint_R f(x, y) dA = \iint_R 1 dA$$

Example 2.18.1 (Example)

Calculate the area of the region $R = \{(x, y) : x^2 + y^2 \leq 1\}$, which is the unit disk in the xy -plane.

$$\iint_R dA = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy dx$$

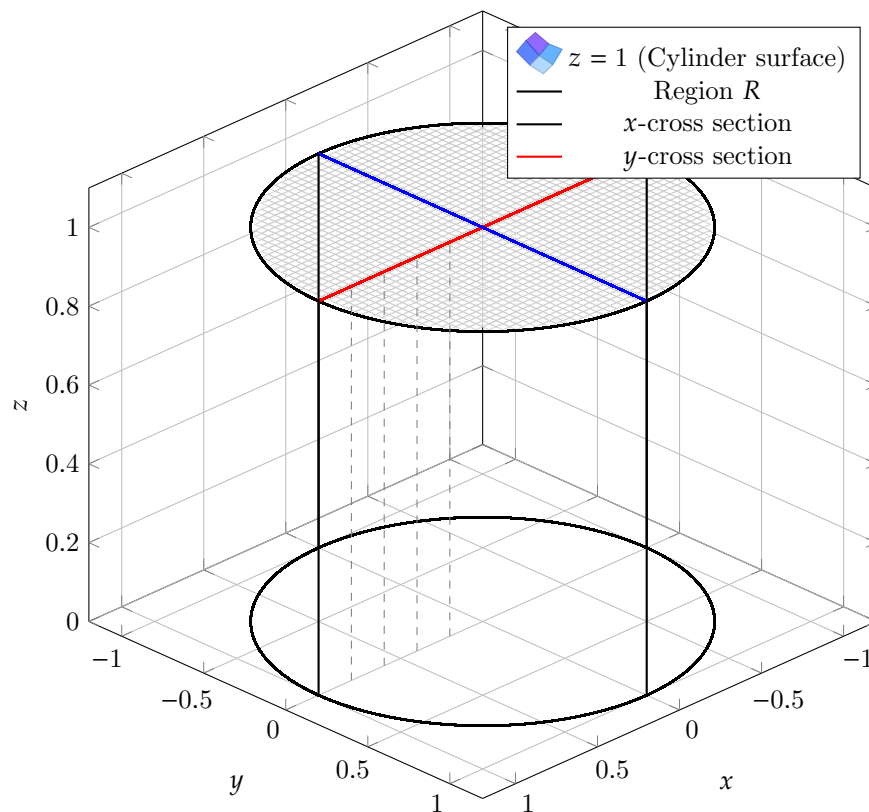
Calculating,

$$= 2 \int_{-1}^1 \sqrt{1-x^2} dx$$

Using substitution, let $x = \cos \theta$, we get:

$$= 2 \int_0^\pi \sin^2 \theta d\theta = \pi$$

Thus, the area of R is π .



2.18.2 Average Value of a Function

Definition 2.18.2: Average Value of a Function

The average value of a function $f(x, y)$ over a region R is defined as the total value of $f(x, y)$ over R divided by the area of R . This can be expressed using double integrals as follows:

$$\bar{f} = \frac{1}{\text{Area of } R} \iint_R f(x, y) dA = \frac{1}{\iint_R dA} \iint_R f(x, y) dA$$

Example 2.18.2 (Example)

Find the average value of $f(x, y) = \sqrt{1 - x^2 - y^2}$ over the region $R = \{(x, y) : x^2 + y^2 \leq 1\}$. First, calculate the area of R :

$$\text{Area of } R = \iint_R dA = \pi.$$

Now, calculate the integral of $f(x, y)$ over R :

$$\iint_R \sqrt{1 - x^2 - y^2} dA.$$

Using polar coordinates, where $x = r \cos \theta$ and $y = r \sin \theta$, we have $dA = r dr d\theta$:

$$= \int_0^{2\pi} \int_0^1 \sqrt{1 - r^2} \cdot r dr d\theta.$$

Breaking down the integral,

$$= 2\pi \int_0^1 r \sqrt{1 - r^2} dr.$$

Using the substitution $u = 1 - r^2$ (with $du = -2r dr$),

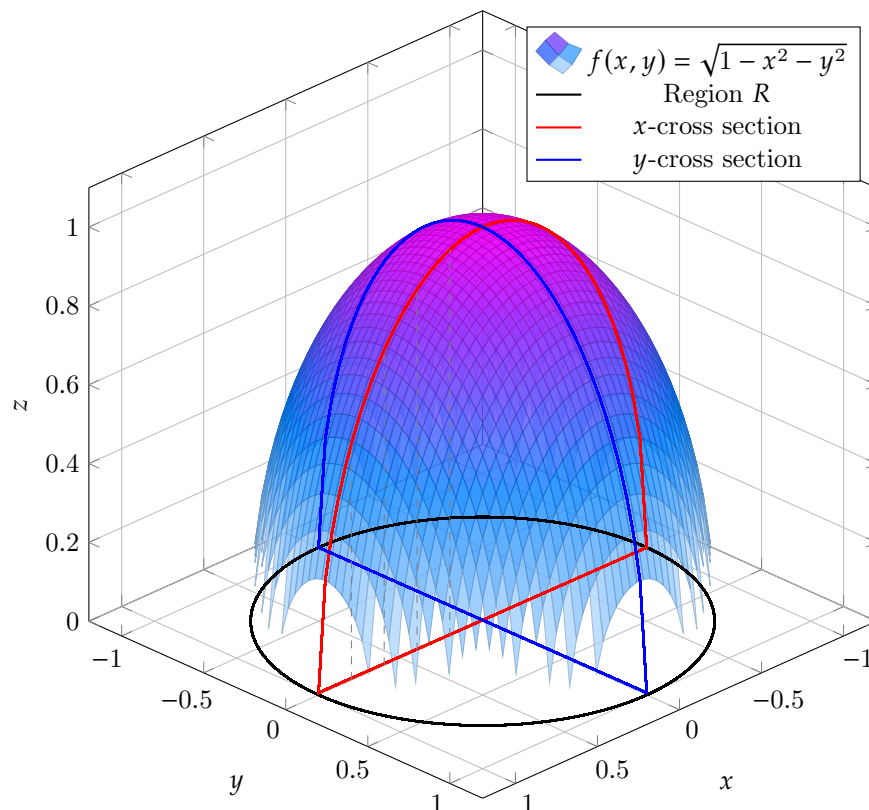
$$= 2\pi \int_1^0 \sqrt{u} \cdot \left(-\frac{du}{2}\right) = \pi \int_0^1 \sqrt{u} du.$$

Evaluating the integral,

$$= \pi \int_0^1 u^{1/2} du = \pi \left[\frac{u^{3/2}}{3/2} \right]_0^1 = \pi \cdot \frac{2}{3} = \frac{2\pi}{3}.$$

Therefore, the average value of $f(x, y)$ over R is

$$\bar{f} = \frac{\frac{2\pi}{3}}{\pi} = \frac{2}{3}.$$



2.19 Derivation of Triple Integrals through Riemann Sums

To derive the concept of a triple integral, we start by considering a bounded, three-dimensional region $D \subset \mathbb{R}^3$ over which we wish to integrate a continuous function $f(x, y, z)$. The idea is to approximate the "volume" under the surface defined by $f(x, y, z)$ over D by partitioning D into smaller subregions, summing up values of f at chosen points within these subregions, and then taking the limit as the partitions become finer.

2.19.1 Partitioning the Region

1. Let D be divided into $n \times m \times l$ subregions, each of volume $\Delta V_{ijk} = \Delta x_i \Delta y_j \Delta z_k$, where:

$$\Delta x_i = x_{i+1} - x_i, \quad \Delta y_j = y_{j+1} - y_j, \quad \Delta z_k = z_{k+1} - z_k$$

2. In each subregion, choose a point (x_i^*, y_j^*, z_k^*) where f will be evaluated. The function value at each of these points, $f(x_i^*, y_j^*, z_k^*)$, approximates the "height" over the corresponding volume element.

2.19.2 Forming the Riemann Sum

With these chosen points and partition, the Riemann sum for f over the region D is given by:

$$\sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^l f(x_i^*, y_j^*, z_k^*) \Delta x_i \Delta y_j \Delta z_k.$$

As the partitions become finer, i.e., $\Delta x_i, \Delta y_j, \Delta z_k \rightarrow 0$ for all i, j, k , the sum approaches the exact "volume" under f over D , which we denote as the triple integral:

$$\iiint_D f(x, y, z) dV = \lim_{n, m, l \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^l f(x_i^*, y_j^*, z_k^*) \Delta x_i \Delta y_j \Delta z_k.$$

Thus, the triple integral represents the limit of the Riemann sum as the volume elements ΔV_{ijk} become infinitesimally small.

2.20 Triple Integrals Intuitions and Applications

$$\begin{aligned}\int f(x) dx &\rightarrow \text{Area} & \int \int f(x, y) dA &\rightarrow \text{Volume under } f(x, y) \\ \iiint f(x, y, z) dV &\rightarrow \text{Volume under solid } w = f(x, y, z)\end{aligned}$$

2.20.1 Integrals Represent Products

Examples:

$$\begin{aligned}\text{Length} = F = P \cdot A &\Rightarrow \iint_R P(x, y) dA \\ \text{Mass} = \rho \cdot V &\Rightarrow m = \iiint_D \rho(x, y, z) dV\end{aligned}$$

Example 2.20.1 (Mass of a Rock)

Suppose a rock with uniform density ρ and volume V . Then

$$m = \rho V$$

If ρ is non-uniform, how do you find the mass?

Suppose $\rho(x, y, z)$ is given by $p(x, y, z)$ over the support of D , then

$$m = \iiint_D p(x, y, z) dV$$

where $D \subseteq \mathbb{R}^3$ is the region occupied by the rock.

Example 2.20.2 (Volume)

Note $A \subseteq \mathbb{R}^2$, then

Volume of D in \mathbb{R}^3

is

$$V = \iiint_D dV$$

Example 2.20.3 (Example 3: Probability Densities)

Let x, y, z be variables representing random quantities. You can find $f(x, y, z)$ called a Probability Density Function (P.D.F.) where

$$P((x, y, z) \in D) = \iiint_D f(x, y, z) dV$$

Definition 2.20.1: Bounds of Integration for triple integrals

Suppose $D = \{(x, y, z) : a \leq x \leq b, g(x) \leq y \leq h(x), G(x, y) \leq z \leq H(x, y)\}$, where the bounds of y depend on x and the bounds of z depend on x and y .

Then

$$V = \iiint_D f(x, y, z) dV = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x, y)}^{H(x, y)} f(x, y, z) dz dy dx$$

Example 2.20.4 (PDF in 3 Continuous Random Variables)

Three customers wait x, y, z minutes.

$$f(x, y, z) = \begin{cases} 10^3 e^{-(x+y+z)} \cdot \frac{1}{10} & \text{if } x, y, z \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(1) **Probability that $(x, y, z) \in D$:**

$$P((x, y, z) \in D) = \iiint_D 10^3 e^{-(x+y+z)} \cdot \frac{1}{10} dV$$

(2) **Probability that customers wait at most 5 minutes:**

$$D = [0, 5] \times [0, 5] \times [0, 5]$$

$$P = \iiint_D 10^3 e^{-(x+y+z)} \cdot \frac{1}{10} dx dy dz = 0.061 = 6.1\%$$

(3) **Probability that $0 \leq x \leq y \leq z \leq 20$:** Define

$$P = \{(x, y, z) : 0 \leq x \leq y \leq z \leq 20\}$$

1. Find the domain of x in D :

$$0 \leq x \leq 20$$

2. Fix some $x = x_0$, define $R = \{(y, z) : 0 \leq x_0 \leq y \leq z \leq 20\}$. 3. Marginalize y in R :

$$x \leq y \leq 20$$

4. Marginalize z in R :

$$y \leq z \leq 20$$

Thus,

$$D = [0, 20] \times [x, 20] \times [y, 20]$$

$$P = \int_0^{20} \int_x^{20} \int_y^{20} 10^3 e^{-(x+y+z)} \cdot \frac{1}{10} dz dy dx$$

Theorem 2.20.1 Fubini's Theorem Extended to Triple Integrals

Fubini's Theorem allows us to evaluate triple integrals as iterated integrals by integrating one variable at a time. If $f(x, y, z)$ is continuous on a rectangular region $D = [a, b] \times [c, d] \times [e, f]$, then

$$\iiint_D f(x, y, z) dV = \int_a^b \int_c^d \int_e^f f(x, y, z) dz dy dx.$$

This can also be rearranged in different orders of integration:

$$\iiint_D f(x, y, z) dV = \int_c^d \int_a^b \int_e^f f(x, y, z) dx dz dy,$$

and similarly for other orders, depending on the bounds and convenience for computation.

Example 2.20.5 (Example of Setting up the Integral with Bounds)

Find the integral $\iiint_D xyz dV$ where $D = \{(x, y, z) | x + y + z \leq 1, x, y, z \geq 0\}$.

To determine the bounds for integration, we analyze the region D defined by $x + y + z \leq 1$ with $x, y, z \geq 0$.

1. First, we set up the bounds for x :

$$0 \leq x \leq 1$$

since x is bounded below by 0 and, based on the constraint $x + y + z \leq 1$, cannot exceed 1.

2. Given a fixed $x = x_0$, we find the bounds for y :

$$0 \leq y \leq 1 - x_0$$

since for any fixed x , y is bounded below by 0 and must satisfy $y + z \leq 1 - x_0$.

3. For a fixed $x = x_0$ and $y = y_0$, we find the bounds for z :

$$0 \leq z \leq 1 - x_0 - y_0$$

since z is bounded below by 0 and must satisfy the remaining part of the inequality, $z \leq 1 - x_0 - y_0$.

With these bounds, we can now write the integral as:

$$\iiint_D xyz \, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} xyz \, dz \, dy \, dx$$

Example 2.20.6 (Volume of a Solid Bounded by Planes)

Find the volume V of the solid bounded by $x = 0$, $y = 2$, $z = 0$, and $2 - x - 4y - z = 0$.

Define the region $D = \{(x, y, z) \mid x \geq 0, y \leq 2, z \geq 0, z \leq 2 - x - 4y\}$.

The volume V is given by

$$V = \iiint_D 1 \, dV$$

Step-by-Step Bounds Determination:

1. Start by determining the bounds for z :
 - The minimum value of z is 0 since $z \geq 0$.
 - The maximum value of z is determined by the plane $z = 2 - x - 4y$, so $0 \leq z \leq 2 - x - 4y$.
2. Next, determine the bounds for y given a fixed z :
 - The plane $y = 2$ limits y from above, so $y \leq 2$.
 - Rearranging the inequality $2 - x - 4y \geq 0$ gives $y \leq \frac{2-x-z}{4}$.
 - Therefore, for a fixed z , y ranges from 0 to $\min\left(2, \frac{2-x-z}{4}\right)$.
3. Finally, determine the bounds for x :
 - Since $x \geq 0$, the minimum value of x is 0.
 - From the inequality $z = 2 - x - 4y \geq 0$, rearranging gives $x \leq 2 - 4y - z$, so $x \leq 1$.

Based on these bounds, the volume integral can be set up as:

$$V = \int_0^1 \int_0^{2-x} \int_0^{2-x-4y} 1 \, dz \, dy \, dx$$

Example 2.20.7 (Problem with Multiple Surfaces)

The solid common to the cylinders bounded by $z = \sin x$ and $z = \sin y$ over the region

$$R = \{(x, y) : 0 \leq x \leq \pi, 0 \leq y \leq \pi\}.$$

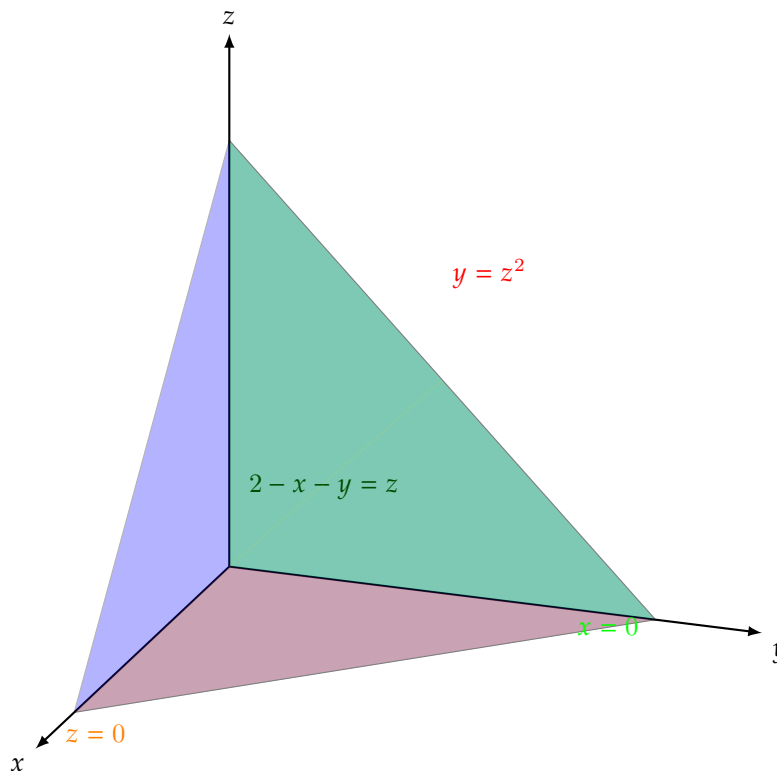


Figure 2.1: Volume bounded by $x = 0$, $z = 0$, $y = z^2$, and $2 - x - y = z$.

We define the domain D as

$$D = \{(x, y, z) : (x, y) \in R, 0 \leq z \leq \min(\sin x, \sin y)\}.$$

To solve this problem, we will:

1. Consider one-fourth of the volume under the intersection of the two cylinders, taking advantage of symmetry.
2. Identify the bounding conditions for z in each quadrant.
3. Set up and compute the triple integral over D to find the volume.

Given the symmetry, we restrict our analysis to the region

$$0 \leq x \leq \frac{\pi}{2}, \quad 0 \leq y \leq \frac{\pi}{2}.$$

In this quadrant, z is bounded by $\min(\sin x, \sin y)$, which means $z \leq \sin y$ in the region R_1 , where $\sin y \leq \sin x$.

For region R_2 (where $\sin x \leq \sin y$), we have $z \leq \sin x$. By observing symmetry, the volume in each quadrant contributes equally, so we can calculate the volume in this restricted region and multiply by 4. The bounds for x and y are:

$$0 \leq y \leq \pi, \quad y \leq x \leq \pi - y.$$

Thus, we set up the volume integral as follows:

$$V = 4 \int_0^{\pi/2} \int_0^{\sin y} \int_y^{\pi-y} dx \, dy \, dz.$$

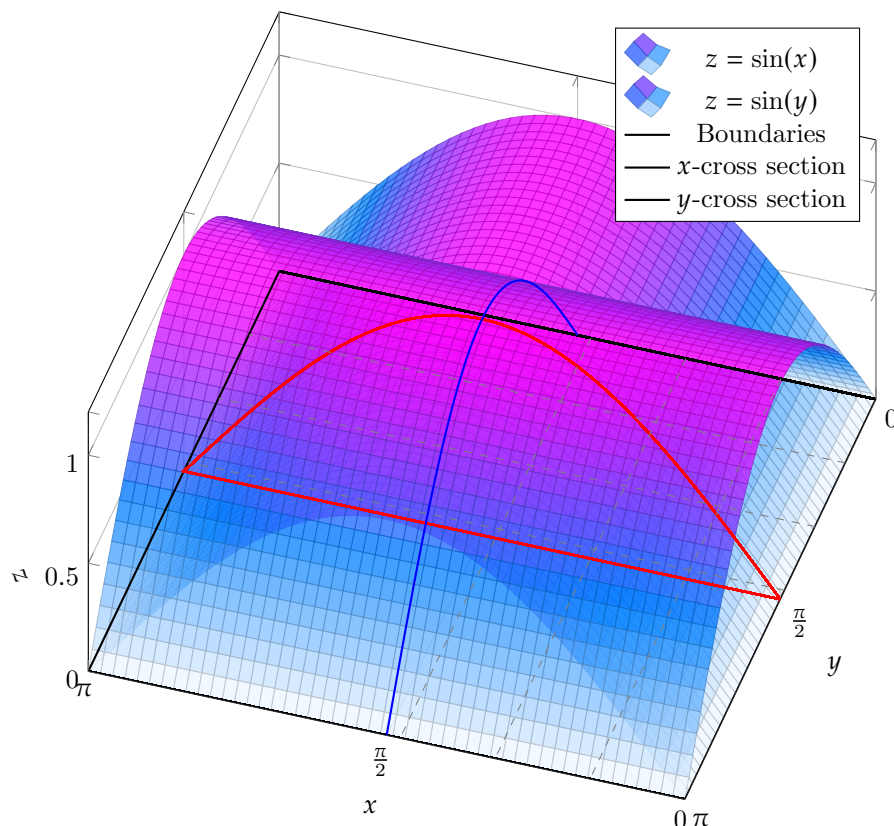
Evaluating the integral:

$$V = 4 \int_0^{\pi/2} \int_0^{\sin y} \left(\int_y^{\pi-y} dx \right) dy dz.$$

Upon computation, this integral yields:

$$V = \pi - 2.$$

The key idea was to leverage the symmetry of the intersection region, focusing on one-fourth of the area and then scaling up by a factor of 4. By analyzing the geometry, we found that z was bounded by $\sin y$ in region R_1 . From there, the triple integral was computed over x , y , and z to yield the final volume of the region.



2.21 Polar Coordinate System

Definition 2.21.1: Polar Coordinates Overview

The polar coordinate system is a two-dimensional coordinate system in which each point on a plane is determined by a distance from a reference point (often called the origin or pole) and an angle from a reference direction (usually the positive x -axis). The two coordinates are:

- r : the radial distance from the origin to the point.
- θ : the angular coordinate, representing the angle in radians (or degrees) between the positive x -axis and the line connecting the origin to the point.

A point P in the plane can therefore be represented in polar coordinates as (r, θ) .

Definition 2.21.2: Relationship Between Cartesian and Polar Coordinates

In Cartesian coordinates, a point P can be represented as (x, y) . We can convert between Cartesian and polar coordinates with the following relations:

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

These equations allow for the translation of a point's position between Cartesian and polar forms, showing the adaptability of the polar system for various types of analyses, especially when working with circular or rotational symmetry.

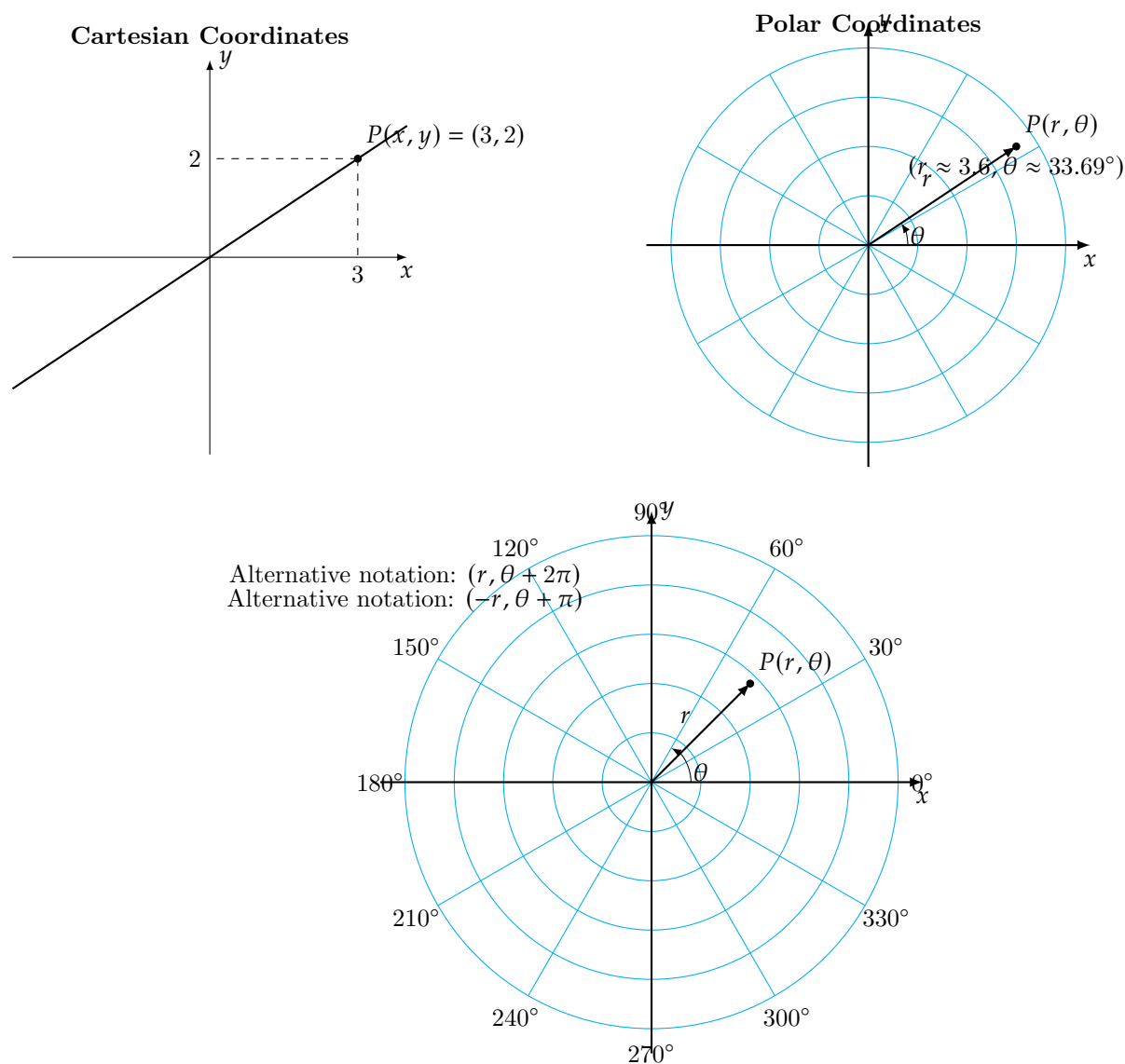


Diagram illustrating the polar coordinate system with concentric circles and angular lines, showing a point P represented as (r, θ) and alternative notations.

Note:-

An interesting feature of polar coordinates is that a single point P can have multiple equivalent representations. For instance, the point (r, θ) can also be expressed as $(-r, \theta + \pi)$ by reversing the radial direction and adjusting the angle. Additionally, due to the periodic nature of angles, adding any integer multiple of 2π to θ results in the same point, i.e., $(r, \theta + 2\pi k)$ for integer k .

Example 2.21.1 (Applications of Polar Coordinates)

Polar coordinates are particularly useful in problems involving symmetry around a central point, such as in physics for modeling circular motion, waves, and fields. They simplify equations and visualizations in cases where Cartesian coordinates might be cumbersome.

2.22 Basic Equations

2.22.1 Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

For parametric representation:

$$x = a \cos \theta, \quad y = b \sin \theta$$

2.22.2 Line

$$r(\theta) = \frac{b}{\sin \theta - \cos \theta}$$

where $y = mx + b$.

2.22.3 Cardioid

The equation for a cardioid:

$$r(\theta) = 1 + \cos \theta$$

Transformations:

$$r = 1 + \cos \theta \Rightarrow r = x + \cos \theta$$

$$r = r \cos \theta \Rightarrow x = \cos \theta$$

2.23 Slope of a Curve

For a function $r = f(\theta)$:

$$\text{slope} = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

Using parametric form $r(\theta) = f(\theta)$, then

$$x(\theta) = f(\theta) \cos \theta, \quad y(\theta) = f(\theta) \sin \theta$$

Therefore,

$$\frac{dy}{d\theta} = f'(\theta) \sin \theta + f(\theta) \cos \theta$$

$$\frac{dx}{d\theta} = f'(\theta) \cos \theta - f(\theta) \sin \theta$$

Definition 2.23.1: Slope of function with Polar Coordinates

$$\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{dy}{dx} = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}$$

Example 2.23.1 (Example: Slope of Cardioid)

For the cardioid $r = 1 + \cos \theta$,

$$f(\theta) = 1 + \cos \theta, \quad f'(\theta) = -\sin \theta$$

Then,

$$\text{slope} = \frac{-\sin \theta (\sin \theta) + (1 + \cos \theta) \cos \theta}{-\sin \theta \cos \theta - (1 + \cos \theta) \sin \theta}$$

Simplifying,

$$\text{slope} = -\frac{\cos^2 \theta}{\sin \theta (2 \cos \theta + 1)}$$

To determine where the slope is undefined, we examine the denominator:

$$\text{slope} = -\frac{\cos^2 \theta}{\sin \theta (2 \cos \theta + 1)}$$

The slope is undefined when the denominator is zero.

At $\theta = \pi$:

$$\sin \pi = 0 \quad \text{and} \quad \cos \pi = -1$$

$$\sin \pi \cdot (2 \cos \pi + 1) = 0 \cdot (2 \cdot (-1) + 1) = 0$$

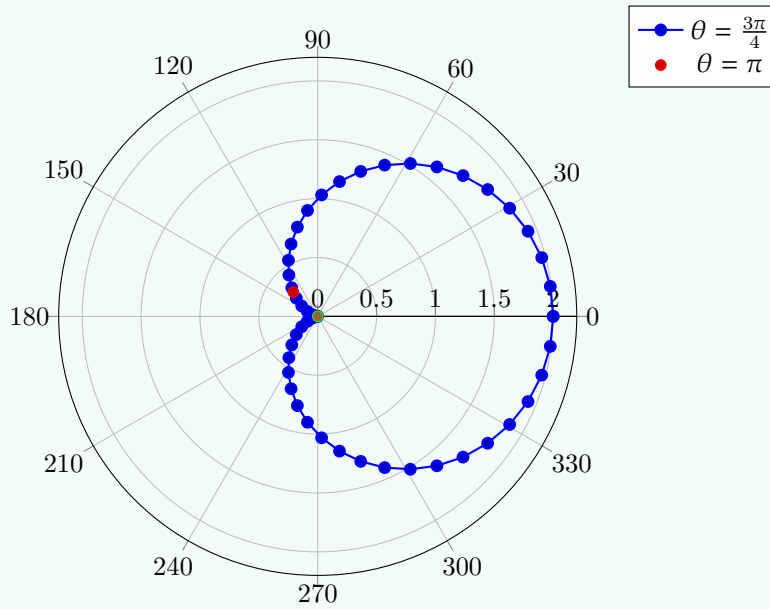
Thus, the slope is undefined at $\theta = \pi$.

At $\theta = \frac{3\pi}{4}$:

$$\sin \frac{3\pi}{4} = \frac{\sqrt{2}}{2} \quad \text{and} \quad \cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$$

$$\sin \frac{3\pi}{4} \cdot \left(2 \cos \frac{3\pi}{4} + 1\right) = \frac{\sqrt{2}}{2} \cdot \left(-\sqrt{2} + 1\right) = 0$$

Thus, the slope is also undefined at $\theta = \frac{3\pi}{4}$.



The slope at $\theta = \frac{3\pi}{4}$ does not exist in the usual Cartesian sense. Here's why:

1. Radial Line: At $\theta = \frac{3\pi}{4}$, the line connecting the point $(1 + \cos(\frac{3\pi}{4}), \frac{3\pi}{4})$ to the origin is a radial line. Radial lines in polar coordinates point directly toward or away from the origin, meaning they lack a well-defined Cartesian slope.
2. Slope in Polar Coordinates: In polar coordinates, the concept of "slope" differs from Cartesian coordinates. We can calculate $\frac{dr}{d\theta}$ at $\theta = \frac{3\pi}{4}$, but this does not correspond to a Cartesian slope.
3. Alternative Interpretation: Converting $r = 1 + \cos(\theta)$ to Cartesian form and calculating derivatives would show a vertical tangent at $\theta = \frac{3\pi}{4}$, indicating an undefined slope.

2.24 Deriving the Integral for Area Under a Curve in Polar Coordinates

To find the area under a curve defined in polar coordinates using Riemann sums, we follow these steps:

In Cartesian coordinates, the area under a curve from $x = a$ to $x = b$ is given by:

$$\int_a^b f(x) dx$$

In polar coordinates, a point is represented by (r, θ) , where r is the radial distance from the origin, and θ is the angle from the positive x -axis. To calculate the area under a curve $r = f(\theta)$ between two angles $\theta = \alpha$ and $\theta = \beta$, we use sectors of circles.

For a small angle $d\theta$, the area of a sector with radius r is approximately:

$$dA \approx \frac{1}{2} r^2 d\theta$$

This comes from the area formula for a sector, $\frac{1}{2} r^2 \theta$, with a small change in angle $d\theta$.

To find the total area under the curve $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$, we can divide the interval $[\alpha, \beta]$ into n small subintervals, each of width $\Delta\theta = \frac{\beta - \alpha}{n}$.

In each subinterval $[\theta_i, \theta_{i+1}]$, we approximate the radius r by $f(\theta_i)$, where θ_i is a sample point in that interval. The area of each small sector is then approximately:

$$\Delta A_i \approx \frac{1}{2} f(\theta_i)^2 \Delta\theta$$

The total area A is the sum of the areas of these small sectors:

$$A \approx \sum_{i=0}^{n-1} \frac{1}{2} f(\theta_i)^2 \Delta\theta$$

As we take the limit where $n \rightarrow \infty$ and $\Delta\theta \rightarrow 0$, this Riemann sum becomes an integral:

$$A = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \frac{1}{2} f(\theta_i)^2 \Delta\theta = \int_{\alpha}^{\beta} \frac{1}{2} f(\theta)^2 d\theta$$

Thus, the area under the curve $r = f(\theta)$ in polar coordinates from $\theta = \alpha$ to $\theta = \beta$ is:

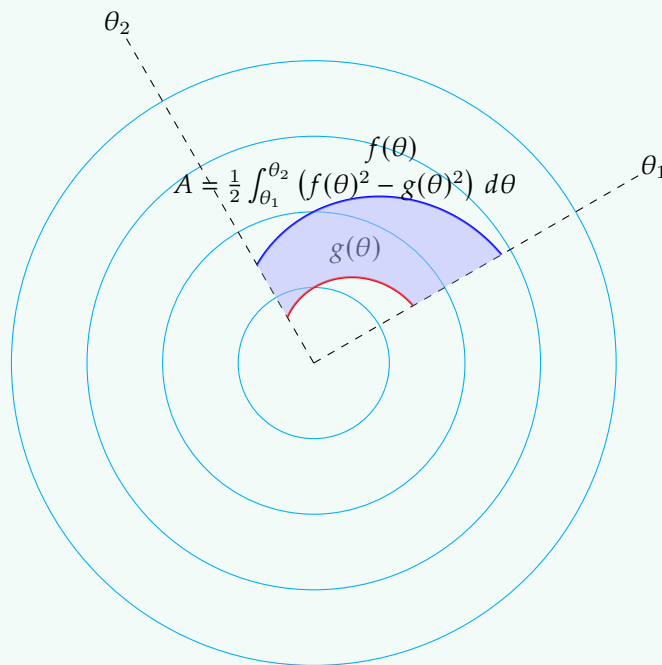
$$A = \int_{\alpha}^{\beta} \frac{1}{2} f(\theta)^2 d\theta$$

This is the formula for the area in polar coordinates, derived using the concept of Riemann sums.

Example 2.24.1 (Example: Circle $r = a$)

$$A = \frac{1}{2} \int_0^{2\pi} a^2 d\theta = \pi a^2$$

Example 2.24.2 (Two curves)



For two curves $r = f(\theta)$ and $r = g(\theta)$:

$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2} f(\theta)^2 d\theta - \int_{\theta_1}^{\theta_2} \frac{1}{2} g(\theta)^2 d\theta$$

which simplifies to

$$A = \frac{1}{2} \int_{\theta_1}^{\theta_2} (f(\theta)^2 - g(\theta)^2) d\theta$$

2.25 Arc Length in Polar Coordinates

The position vector in polar coordinates is given by:

$$\vec{r}(t) = \langle f(\theta) \cos \theta, f(\theta) \sin \theta \rangle$$

The magnitude of the derivative of the position vector is:

$$|\vec{r}'(t)| = \sqrt{(f'(\theta) \cos \theta - f(\theta) \sin \theta)^2 + (f'(\theta) \sin \theta + f(\theta) \cos \theta)^2}$$

Simplifying, this becomes:

$$|\vec{r}'(t)| = \sqrt{(f'(\theta))^2 + (f(\theta))^2}$$

The arc length S_r is then calculated as:

$$S_r = \int_{\theta_1}^{\theta_2} \sqrt{(f'(\theta))^2 + (f(\theta))^2} d\theta$$

Example 2.25.1 (Circle of Radius a)

Example: Let $r(\theta) = a$, where:

$$r'(\theta) = 0, \quad r(\theta) = a$$

The arc length is:

$$S = \int_0^{2\pi} \sqrt{a^2} d\theta = a \int_0^{2\pi} 1 d\theta = 2\pi a$$

2.26 Multivariate Case

The double integral over a region R :

$$\iint_R f(x, y) dA$$

If converted to polar coordinates:

$$\iint_R f(x, y) dA = \iint_R f(r \cos \theta, r \sin \theta) r dr d\theta$$

Example 2.26.1 (Volume under $z = 9 - x^2 - y^2$.)

$$R = \{(x, y) : x^2 + y^2 \leq 9\}$$

In polar coordinates:

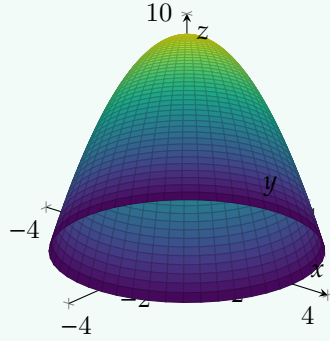
$$R = \{(r, \theta) : 0 \leq \theta \leq 2\pi, 0 \leq r \leq 3\}$$

The function in polar coordinates:

$$f(r, \theta) = 9 - r^2$$

The volume is then:

$$\int_R f(r, \theta) dA = \int_0^{2\pi} \int_0^3 r(9 - r^2) dr d\theta$$



Simplified Polar Volume Calculation:

$$\int_R f(r, \theta) dA = \int_0^{2\pi} \int_0^3 r(9 - r^2) dr d\theta$$

2.27 General Procedure for Polar Integration

Given a region $R = \{(r, \theta) : \theta_1 \leq \theta \leq \theta_2, q(\theta) \leq r \leq h(\theta)\}$, the volume is calculated as:

$$V = \iint_R f(r, \theta) dA = \int_{\theta_1}^{\theta_2} \int_{q(\theta)}^{h(\theta)} r f(r, \theta) dr d\theta$$

where the area element is $\Delta A = r \Delta r \Delta \theta$.

Example 2.27.1 (Example: Region Outside $r = \frac{1}{2}$ and Inside $r = 1 + \cos \theta$)

The bounds are:

$$\frac{1}{2} \leq r \leq 1 + \cos \theta$$

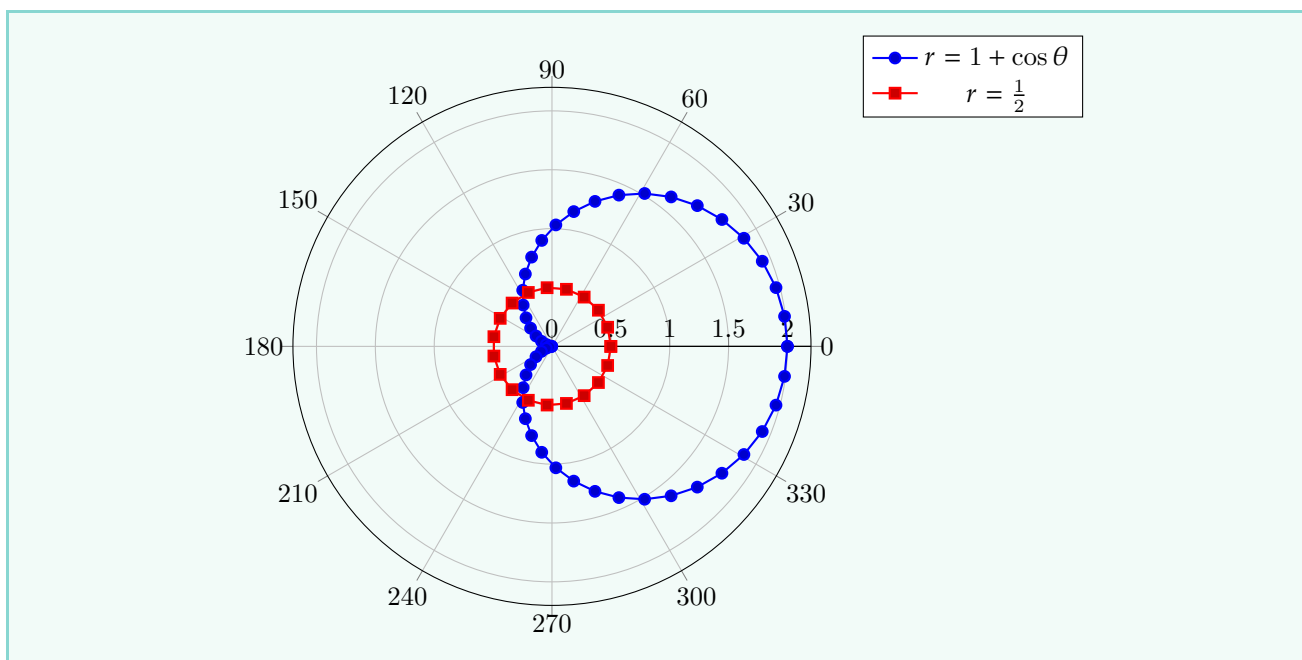
$$-\frac{2\pi}{3} \leq \theta \leq \frac{2\pi}{3}$$

Cosine conditions are determined by:

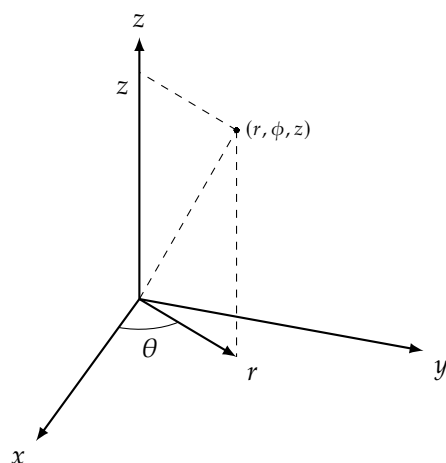
$$\cos \theta = -\frac{1}{2}, \quad \theta = -\frac{2\pi}{3}, \frac{2\pi}{3}$$

The integral for the volume is:

$$\iint_R q(r, \theta) dA = \int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} \int_{\frac{1}{2}}^{1+\cos \theta} r q(r, \theta) dr d\theta$$



2.28 Cylindrical Coordinate Axes



Example 2.28.1 (Cylinders and Cones Example)

Cylinders:

$$r = 1, \quad v = 2$$

Cones:

$$z = \frac{\sqrt{3}}{3}r, \quad z = \sqrt{3}r$$

Ranges:

- r is bounded by the cylinders: $1 \leq r \leq 2$
- z is bounded by the cones: $\frac{\sqrt{3}}{3}r \leq z \leq \sqrt{3}r$
- θ is unbounded: $0 \leq \theta \leq 2\pi$

Volume:

$$V = \int_0^{2\pi} \int_1^2 \int_{\frac{\sqrt{3}}{3}r}^{\sqrt{3}r} r \, dz \, dr \, d\theta$$

(Simplify to evaluate)

2.29 Spherical Coordinate Axes

The relationships between Cartesian coordinates (x, y, z) and spherical coordinates (ρ, ϕ, θ) are as follows:

Definition 2.29.1: Polar - Spherical Relationships

$$x = \rho \cos \phi \cos \theta$$

$$y = \rho \cos \phi \sin \theta$$

$$z = \rho \sin \phi$$

Conversely, the spherical coordinates can be expressed in terms of Cartesian coordinates as:

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

$$\phi = \tan^{-1} \left(\frac{z}{\sqrt{x^2 + y^2}} \right)$$

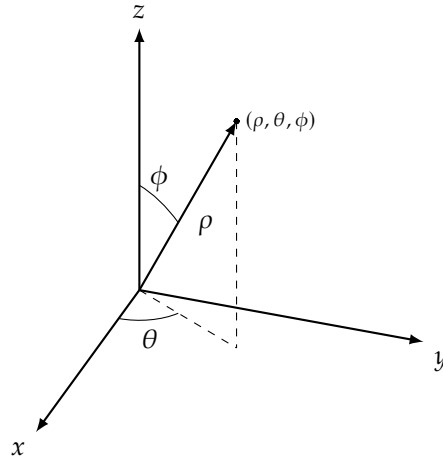
$$\theta = \tan^{-1} \left(\frac{y}{x} \right)$$

The volume element in spherical coordinates is:

$$dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

The integral of a function $f(x, y, z)$ in spherical coordinates becomes:

$$\iiint_P f(x, y, z) \, dV = \iiint_P f(\rho \cos \phi \cos \theta, \rho \cos \phi \sin \theta, \rho \sin \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$



2.30 Shapes Represented in Spherical Coordinates

2.30.1 Overview of Spherical Coordinates

- **Definition:** Spherical coordinates (r, θ, ϕ) are defined as:

- r : Radial distance from the origin.
- θ : Polar angle (angle from the positive z-axis, $0 \leq \theta \leq \pi$).

– ϕ : Azimuthal angle (angle in the xy -plane from the positive x -axis, $0 \leq \phi < 2\pi$).

• **Conversion to Cartesian coordinates:**

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta$$

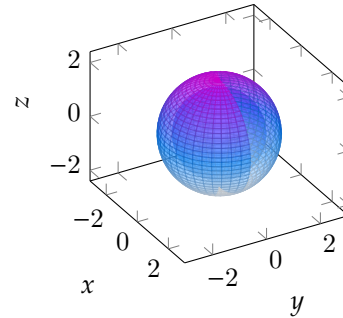
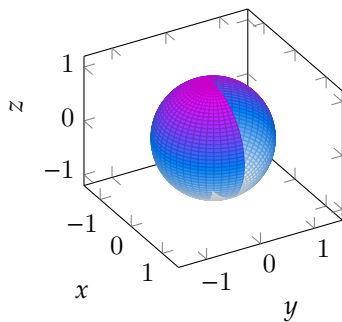
2.30.2 Shapes in Spherical Coordinates

1. Sphere

Equation: $r = R$

Description: A sphere of radius R centered at the origin.

Representation: Independent of θ and ϕ .

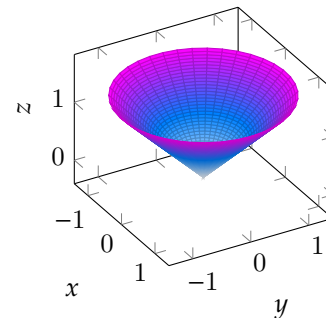


4. Cone

Equation: $\theta = \theta_0$

Description: A cone with its vertex at the origin, opening angle $2\theta_0$, symmetric around the z -axis.

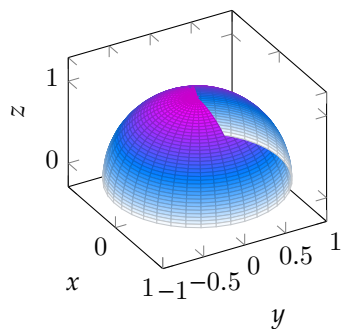
Constraints: $r > 0$, ϕ varies from 0 to 2π .



2. Half-Sphere

Equation: $r = R, 0 \leq \theta \leq \frac{\pi}{2}$

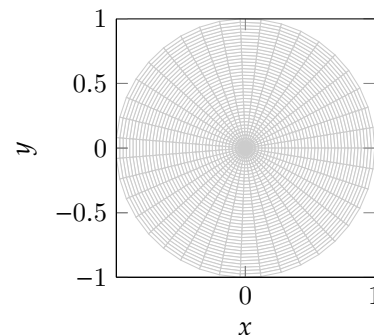
Description: Upper half of a sphere centered at the origin, constrained by the polar angle.



5. Circular Disc (in the xy -Plane)

Equation: $r = R, \theta = \frac{\pi}{2}$

Description: A circular disc of radius R centered on the origin in the xy -plane.



3. Spherical Shell

Equation: $R_1 \leq r \leq R_2$

Description: A hollow spherical region with inner radius R_1 and outer radius R_2 .

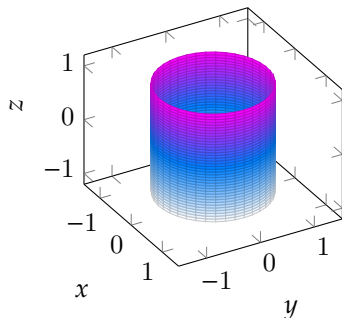
6. Cylinder

Equation: $r \sin \theta = a$

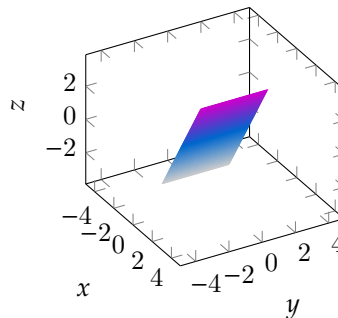
Description: A cylinder of radius a around the z -axis.

Constraints:

- $0 \leq \phi < 2\pi$,
- $r \cos \theta$ is unbounded (representing the z -coordinate).



- $\phi = \text{constant}$: A vertical plane slicing through the z -axis at a fixed azimuthal angle.



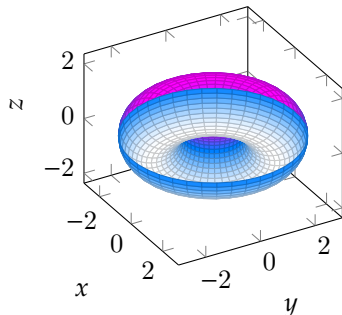
7. Torus (Approximation)

Equation: $r = R + r_0 \sin \theta$

Description: A torus with major radius R and minor radius r_0 , approximately represented in spherical coordinates.

Constraints:

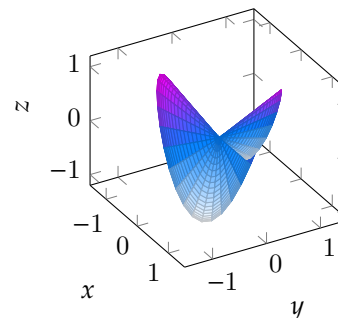
- ϕ varies from 0 to 2π ,
- Specific parametric constraints apply.



9. Lemniscate Shape

Equation: $r = a \sin \theta \sin 2\phi$

Description: A figure-eight or lemniscate shape in spherical coordinates.



8. Plane

Equation: $\theta = \text{constant}$ or $\phi = \text{constant}$

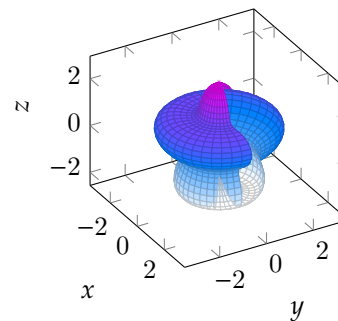
Description:

- $\theta = \text{constant}$: A conical plane cutting through the origin at a fixed polar angle.

10. Wave-like Surfaces

Equation: $r = R + A \cos(k\theta)$

Description: Oscillating surface around a spherical shell, useful in representing waves or perturbations on a sphere.



2.31 Change of Variable for Double and Triple Integrals

Polar Coordinates

$$\iint_D f(x, y) dx dy \rightarrow \iint_S f(r \cos \theta, r \sin \theta) r dr d\theta$$

Cylindrical Coordinates

$$\iiint_D f(x, y, z) dx dy dz \rightarrow \iiint_S f(r \cos \theta, r \sin \theta, z) r dr d\theta dz$$

Spherical Coordinates

$$\iiint_D f(x, y, z) dx dy dz \rightarrow \iiint_S f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi d\rho d\phi d\theta$$

Theorem 2.31.1 Intuition Behind Change of Variables

We use a **mapping** T to transform coordinates in one space S to another R . This is particularly useful when integrating over regions that are easier to describe in new coordinates (e.g., circular or spherical regions).

For example:

$$S = [0, 2\pi] \times [0, 2], \quad T(r, \theta) = (r \cos \theta, r \sin \theta)$$

Here, the mapping T converts a point in S into a point in R .

Area Differential Transformation

Consider a small differential area element in the original space:

$$dA = |\det(J)| du dv$$

where J is the **Jacobian matrix**, and $|\det(J)|$ accounts for how the transformation scales area.

Definition 2.31.1: Jacobian Matrix

The Jacobian matrix represents the linear transformation of the mapping T at a given point:

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

For a transformation $T(u, v) = (g(u, v), h(u, v))$, the determinant of J is:

$$\det(J) = \begin{vmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{vmatrix} = \frac{\partial g}{\partial u} \cdot \frac{\partial h}{\partial v} - \frac{\partial g}{\partial v} \cdot \frac{\partial h}{\partial u}$$

Geometric Interpretation

- **Local Stretching/Scaling:** $|\det(J)|$ gives the local scaling factor of the area due to the transformation.
- **Orientation:** The sign of $\det(J)$ indicates whether the orientation is preserved or flipped.

Example 2.31.1 (Polar Coordinates)

For the transformation $T(r, \theta) = (r \cos \theta, r \sin \theta)$, the Jacobian matrix is:

$$J = \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}$$

The determinant is:

$$\det(J) = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r(\cos^2 \theta + \sin^2 \theta) = r$$

Thus, the area differential in polar coordinates becomes:

$$dx dy = r dr d\theta$$

Definition 2.31.2: General Formula for Transforming Integrals

If $T : S \rightarrow R$ is a transformation with Jacobian determinant $|\det(J)|$, then the integral transforms as:

$$\iint_R f(x, y) dx dy = \iint_S f(T(u, v)) |\det(J)| du dv$$

Definition 2.31.3: Intuition for Higher Dimensions

In three dimensions, the Jacobian matrix extends to account for the transformation of volume elements:

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$$

The volume scaling factor is given by $|\det(J)|$, and the integral transforms as:

$$\iiint_R f(x, y, z) dx dy dz = \iiint_S f(T(u, v, w)) |\det(J)| du dv dw$$

2.32 Non-overlapping from Mapping T

Theorem 2.32.1 Non-overlapping Condition

For any two points Q and P :

$$T(Q) \neq T(P) \quad (\text{This would result in overlapping areas in the domain } R)$$

However, boundaries (e.g., $y = 2x$) can overlap as long as the bounded region is distinct.

Example 2.32.1 (Integral Transformation Example)

Evaluate:

$$\iint_R 2x(y - 2x) dA$$

where R is the parallelogram with vertices $(0, 0), (0, 1), (2, 4), (2, 3)$.

Steps:

1. **Choose a Transformation:** Select a mapping T to simplify the integral.

2. **Define the Mapping:**

$$x = u, \quad y = 2x + v = 2u + v$$

Substituting:

$$(x, y) \rightarrow (u, v)$$

3. **Boundary Equations:**

$$0 \leq x \leq 2 \quad \Rightarrow \quad 0 \leq u \leq 2$$

$$0 \leq y - 2x < 1 \quad \Rightarrow \quad 0 \leq v < 1$$

4. **Region:**

$$S = [0, 2] \times [0, 1]$$

5. **Transform the Integrand:**

$$f(T(x, y)) = 2u(v)$$

6. **Jacobian Calculation:**

$$J = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad \det(J) = 1 \cdot 1 - 2 \cdot 0 = 1$$

7. **Transformed Integral:**

$$\iint_R 2x(y - 2x) \, dA = \int_0^2 \int_0^1 2uv \, du \, dv$$

2.33 Integral Transformation for a Parallelogram Region

Example 2.33.1 (Example of Transformation)

Evaluate:

$$\iint_R 2x(y - 2x) dA$$

where R is the parallelogram defined by the vertices $(0, 0), (0, 1), (2, 4), (2, 3)$.

Steps:

1. **Choose a Transformation:** Select a transformation T that simplifies the integral.

2. **Define x, y in terms of u, v :**

$$x = u, \quad y = 2x + v = 2u + v$$

Substituting:

$$(x, y) \rightarrow (u, v)$$

Here, u corresponds to x , and $v = y - 2x$.

3. **Boundary Equations:**

$$0 \leq x \leq 2 \quad \Rightarrow \quad 0 \leq u \leq 2$$

$$0 \leq y - 2x < 1 \quad \Rightarrow \quad 0 \leq v < 1$$

4. **Region in u, v :**

$$S = [0, 2] \times [0, 1]$$

This maps the parallelogram R into a rectangle S in the u, v -plane.

5. **Transform the Integrand:** Substituting $x = u$ and $y - 2x = v$:

$$f(T(x, y)) = 2u(v)$$

6. **Jacobian Calculation:** The Jacobian matrix for the transformation T is:

$$J = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

The determinant of J is:

$$\det(J) = 1 \cdot 1 - 2 \cdot 0 = 1$$

7. **Transformed Integral:** Using the transformation and the Jacobian determinant:

$$\iint_R 2x(y - 2x) dA = \int_0^2 \int_0^1 2uv \, du \, dv$$

The transformed integral simplifies the computation significantly.

2.34 Integral Transformation for a Triangular Region

Example 2.34.1 (Example of Transformation)

Evaluate:

$$\iint_R (x - u)\sqrt{x - 2y} dA$$

where R is the triangular region bounded by the lines $y = 0$, $x - 2y = 0$, and $x = y + 1$.

Steps:

1. **Region Definition:** The region R is defined by:

$$y = 0, \quad x - 2y = 0, \quad x = y + 1$$

The boundaries in x and y are:

$$0 \leq x \leq 2, \quad 0 \leq y \leq \frac{x}{2}, \quad x \leq y + 1$$

2. **Define Transformation:** Let:

$$u = x - 2y, \quad v = x - y$$

Substituting:

$$x = v + u, \quad y = v - u$$

3. **Boundaries in New Coordinates:** Using the transformation:

$$\begin{aligned} u = x - 2y &\Rightarrow 0 \leq u \leq 1 \\ v = x - y &\Rightarrow u \leq v \leq 1 \end{aligned}$$

The transformed region S is bounded by $u = 0$, $v = 1$, and $v - u = 1$.

4. **Jacobian Calculation:** The Jacobian matrix for the transformation $T(u, v)$ is:

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

The determinant of J is:

$$\det(J) = (1)(1) - (1)(-2) = 1 + 2 = 3$$

5. **Transform the Integral:** Using the transformation and Jacobian determinant:

$$\iint_R (x - u) \sqrt{x - 2y} \, dA = \int_0^1 \int_0^v \sqrt{u} \cdot 3 \, du \, dv$$

Simplify:

$$\begin{aligned} \int_0^1 \int_0^v \sqrt{u} \, du \, dv &= \int_0^1 \left[\frac{2}{3} u^{3/2} \right]_0^v \, dv = \int_0^1 \frac{2}{3} v^{3/2} \, dv \\ &= \left[\frac{2}{3} \cdot \frac{2}{5} v^{5/2} \right]_0^1 = \frac{4}{15}. \end{aligned}$$

The result is:

$$\iint_R (x - u) \sqrt{x - 2y} \, dA = \frac{4}{15}.$$

Chapter 3

Vector Fields $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$

3.1 Vector Fields

A **vector field** is a function \vec{F} that takes points in \mathbb{R}^2 or \mathbb{R}^3 and outputs a vector in \mathbb{R}^2 or \mathbb{R}^3 :

$$\vec{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad \vec{F}(x, y, z) = \langle q(x, y, z), w(x, y, z), t(x, y, z) \rangle.$$

3.1.1 Properties

- Continuous if q, w, t are continuous.
- Differentiable if q, w, t are differentiable.
- Domain is the intersection of domains of q, w, t .

3.1.2 Applications

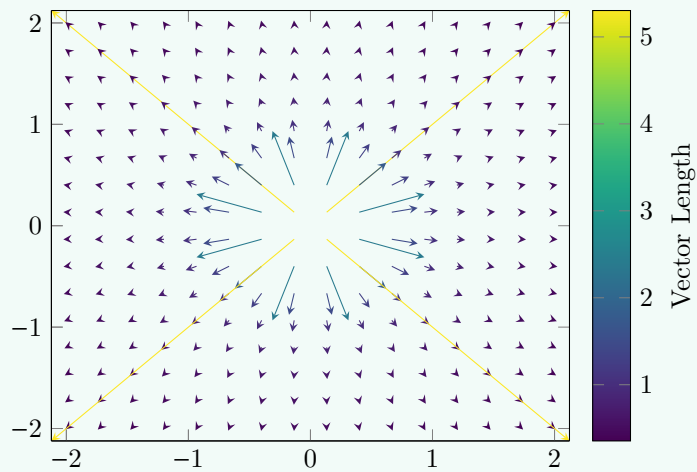
- Water and wind currents.
- Gravitational, electric, and magnetic fields.
- Human circulation, heat propagation.
- Modeling through partial differential equations.

3.1.3 Drawing a Vector Field

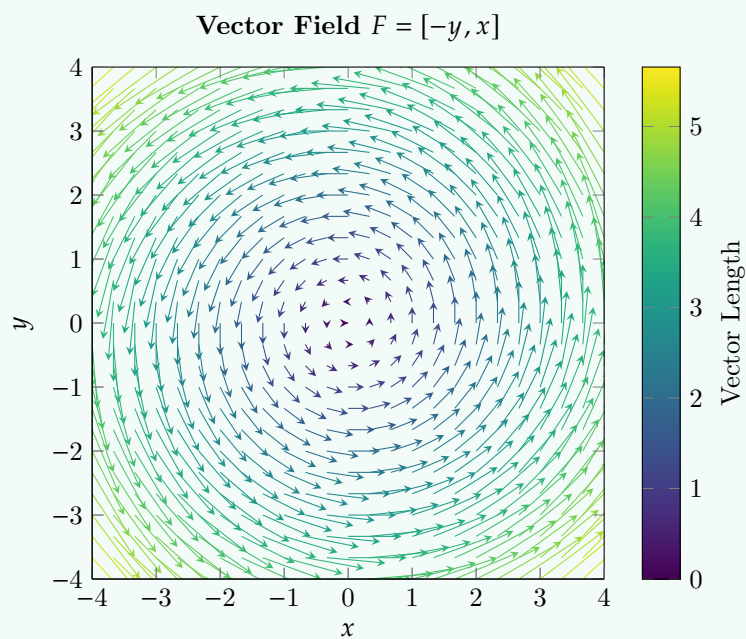
At each point in the domain of \vec{F} , draw a vector whose:

- Direction is parallel to $\vec{F}(x, y)$.
- Length is proportional to the magnitude of $\vec{F}(x, y)$.

Example 3.1.1 ($\vec{F}(x, y) = \frac{\langle x, y \rangle}{|\langle x, y \rangle|^3}$)



Example 3.1.2 ($\vec{F}(x, y) = \langle -y, x \rangle$)



3.2 Gradient Vector Fields

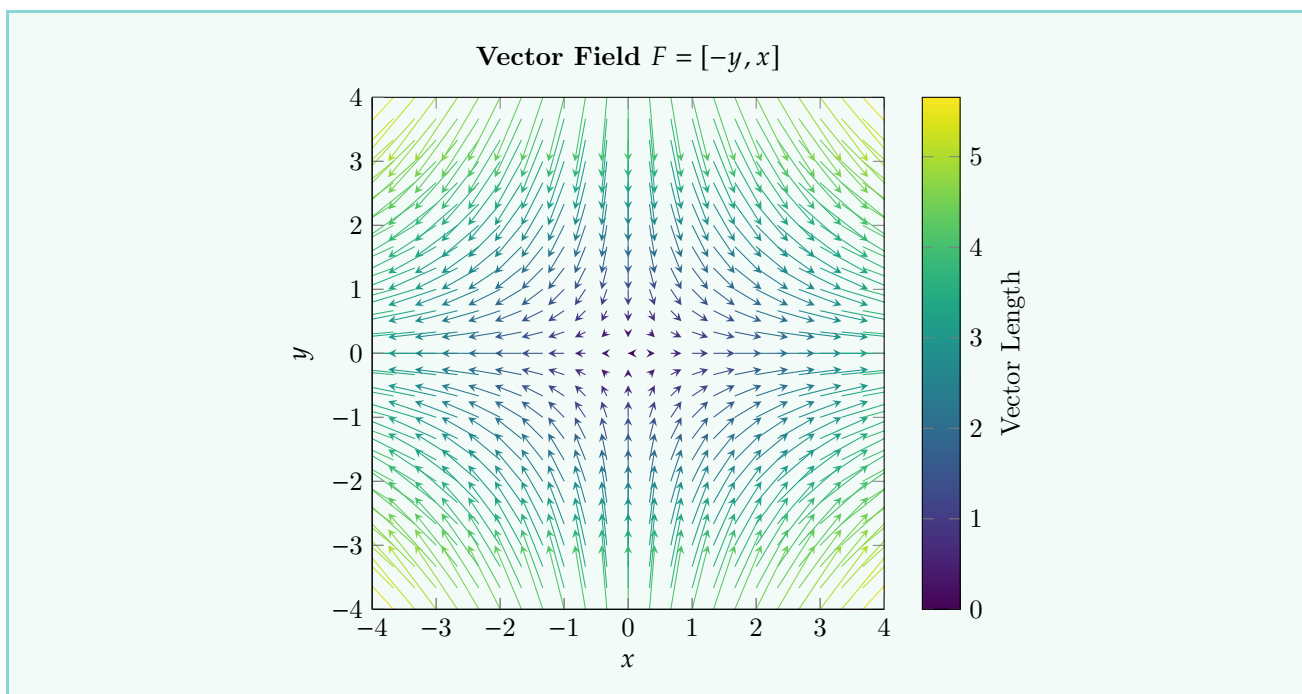
A **gradient vector field** $\nabla\varphi(x, y)$ is a vector field:

$$\vec{F}(x, y) = \nabla\varphi \quad \text{where } \varphi \text{ is a potential function.}$$

If φ exists, \vec{F} is called a **conservative field**.

Example 3.2.1 ($\varphi = -\frac{x^2+y^2}{2}$)

$$\vec{F}(x, y) = \nabla\varphi = \langle -x, -y \rangle$$



3.3 Line Integrals

Scalar Case

For a scalar function $f(x, y)$, the line integral over a curve C is defined as:

$$A = \int_C f(x, y) ds$$

where $ds = |\mathbf{r}'(t)| dt$, with $\mathbf{r}(t)$ being the parameterization of C .

If $\mathbf{r}(t) = (x(t), y(t))$, then:

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

This allows us to rewrite the line integral as:

$$\int_C f(x, y) ds = \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| dt$$

Example 3.3.1 (Example: Average Temperature on a Plate)

Suppose we have a plate located in $R = \{(x, y) : x^2 + y^2 \leq 4\}$, where the temperature at any point (x, y) is $T(x, y) = 100(x^2 + y^2)$. Find the average temperature over the boundary of R .

1. **Parameterize the Boundary:** The boundary C of R is a circle of radius 2, centered at the origin. Parameterize C as:

$$\mathbf{r}(t) = (2 \cos t, 2 \sin t), \quad 0 \leq t \leq 2\pi$$

Then:

$$|\mathbf{r}'(t)| = \sqrt{(-2 \sin t)^2 + (2 \cos t)^2} = 2$$

2. **Average Temperature:** The average temperature is given by:

$$\text{Average Temperature} = \frac{\int_C T(x, y) ds}{\int_C ds}$$

Compute the numerator:

$$\int_C T(x, y) ds = \int_0^{2\pi} 100 \cdot 4 \cdot 2 dt = 800\pi$$

Compute the denominator (arc length of C):

$$\int_C ds = \int_0^{2\pi} 2 dt = 4\pi$$

Thus:

$$\text{Average Temperature} = \frac{800\pi}{4\pi} = 200$$

Oriented Curves

An oriented curve C includes a direction along the curve. For example, when calculating work done by a force \mathbf{F} along a curve, the direction matters.

Example 3.3.2 (Example: Work)

Work done by a force $\mathbf{F}(x)$ is given by:

$$W = \int_a^b F(x) dx$$

For a spring with force $F(x) = -kx$, the work done to stretch the spring from $x = a$ to $x = b$ is:

$$W = \int_a^b -kx dx = -\frac{k}{2} [b^2 - a^2]$$

3.4 Vector Line Integrals

Example 3.4.1 (Example: Earth's Gravitational Field)

The Earth's gravitational field exerts a force $\mathbf{F}(\mathbf{r})$ on an object, pulling it toward the origin. Suppose an object travels along a path C . The work done by \mathbf{F} is:

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

If $\mathbf{F}(\mathbf{r}) = -\frac{Gm}{|\mathbf{r}|^3}\mathbf{r}$, parameterize C as $\mathbf{r}(t)$, and compute:

$$\mathbf{F}(\mathbf{r}) = -\frac{Gm}{|\mathbf{r}|^3}\mathbf{r}, \quad d\mathbf{r} = \mathbf{r}'(t) dt$$

Then:

$$W = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

If C is along the z -axis from $(0, 0, 1)$ to $(0, 0, 4)$, parameterize:

$$\mathbf{r}(t) = (0, 0, t), \quad 1 \leq t \leq 4$$

Then:

$$\begin{aligned} W &= \int_1^4 -\frac{Gm}{t^2} \cdot 1 dt = -Gm \int_1^4 \frac{1}{t^2} dt \\ W &= -Gm \left[-\frac{1}{t} \right]_1^4 = -Gm \left(-\frac{1}{4} + 1 \right) = Gm \left(\frac{3}{4} \right) \end{aligned}$$

3.5 Fundamental Theorem of Line Integrals

Theorem 3.5.1 Fundamental Theorem of Line Integrals

If \mathbf{F} is a vector field that can be expressed as $\mathbf{F} = \nabla\phi$, then given an oriented curve C from P to Q :

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(Q) - \phi(P)$$

Example 3.5.1 (Example)

Let $\phi(x, y, z) = \frac{-G}{\sqrt{x^2 + y^2 + z^2}}$, and $\mathbf{F} = \nabla\phi$. Compute:

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

from (a, a, a) to $(1, 1, 1)$:

$$\phi(x, y, z) = \frac{-G}{\sqrt{x^2 + y^2 + z^2}}, \quad \Delta\phi = \phi(1, 1, 1) - \phi(a, a, a)$$

$$\phi(a, a, a) = \frac{-G}{\sqrt{3a^2}} = \frac{-G}{a\sqrt{3}}, \quad \phi(1, 1, 1) = \frac{-G}{\sqrt{3}}$$

Thus:

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \phi(1, 1, 1) - \phi(a, a, a) = \frac{-G}{\sqrt{3}} - \left(\frac{-G}{a\sqrt{3}} \right) \\ &= \frac{G}{\sqrt{3}} \left(1 - \frac{1}{a} \right) \end{aligned}$$

3.6 Vector Line Integrals and Gradient Fields

3.6.1 Vector Line Integral for a Curve C

Given a vector field $\mathbf{F}(x, y) = (P(x, y), Q(x, y))$ and an oriented curve C , the vector line integral is defined as:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy$$

where:

$$d\mathbf{r} = (dx, dy), \quad \mathbf{F} \cdot d\mathbf{r} = P dx + Q dy.$$

If C is parameterized as $\mathbf{r}(t) = (x(t), y(t))$, where $t \in [a, b]$, then:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_a^b \left[P(x(t), y(t)) \frac{dx}{dt} + Q(x(t), y(t)) \frac{dy}{dt} \right] dt$$

Example 3.6.1 (Example: Compute a Vector Line Integral)

Let $\mathbf{F}(x, y) = (x, x^2 + y)$ and let C be the curve defined by $\mathbf{r}(t) = (t, t^2)$, where $t \in [1, 3]$.

Steps:

1. **Parameterize C:** From the parameterization, we have:

$$x(t) = t, \quad y(t) = t^2, \quad \mathbf{r}'(t) = \left(\frac{dx}{dt}, \frac{dy}{dt} \right) = (1, 2t)$$

2. **Evaluate $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$:** Substitute $x(t)$ and $y(t)$ into $\mathbf{F}(x, y)$:

$$\mathbf{F}(\mathbf{r}(t)) = (t, t^2 + t^2) = (t, 2t^2)$$

Then:

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = t \cdot 1 + (2t^2) \cdot (2t) = t + 4t^3$$

3. **Compute the Integral:**

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_1^3 (t + 4t^3) dt$$

Split the integral:

$$\int_1^3 (t + 4t^3) dt = \int_1^3 t dt + \int_1^3 4t^3 dt$$

Compute each term:

$$\int_1^3 t dt = \left[\frac{t^2}{2} \right]_1^3 = \frac{9}{2} - \frac{1}{2} = 4$$

$$\int_1^3 4t^3 dt = 4 \left[\frac{t^4}{4} \right]_1^3 = [81 - 1] = 80$$

Thus:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 4 + 80 = 84$$

3.6.2 Gradient Fields and Fundamental Theorem of Line Integrals

If \mathbf{F} is a gradient field, meaning $\mathbf{F} = \nabla\phi$, then the line integral over a curve C depends only on the endpoints of C . Specifically:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(B) - \phi(A)$$

where A and B are the start and end points of C .

Intuition:

- The integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ represents the difference in potential between the endpoints of C .
- The path taken does not matter; only the values of ϕ at A and B are relevant.

Example 3.6.2 (Example: Gradient Field Integral)

Let $\mathbf{F} = \nabla\phi$, where $\phi(x, y) = x^2 + xy$. Compute:

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where C is any curve starting at $A = (1, 2)$ and ending at $B = (3, 4)$.

Solution:

1. Compute $\phi(A)$ and $\phi(B)$:

$$\phi(A) = (1)^2 + (1)(2) = 1 + 2 = 3$$

$$\phi(B) = (3)^2 + (3)(4) = 9 + 12 = 21$$

2. Apply the Fundamental Theorem:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(B) - \phi(A) = 21 - 3 = 18$$

3.7 Gradient Vector Fields and Closed Curves

3.7.1 Definition of a Closed Curve

A curve C is said to be **closed** if its start and end points are the same, i.e., $A = B$.

For a closed curve C , the following property holds:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0 \quad \text{if } \mathbf{F} \text{ is a gradient vector field.}$$

If $\mathbf{F} = \nabla\phi$, then:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(A) - \phi(A) = 0.$$

This implies that **the line integral of a gradient field over a closed curve is always zero.**

3.7.2 How to Identify a Gradient Vector Field

To determine whether a vector field $\mathbf{F} = (P(x, y), Q(x, y))$ is a gradient field:

- \mathbf{F} must be defined on a region R that is **connected** and **simply connected** (no holes in R).
- $P(x, y)$ and $Q(x, y)$ must satisfy the condition:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Theorem: If \mathbf{F} satisfies the above conditions, then \mathbf{F} is a gradient vector field, and there exists a scalar function $\phi(x, y)$ such that $\mathbf{F} = \nabla\phi$.

3.7.3 Finding the Potential Function $\phi(x, y)$

To find $\phi(x, y)$, follow these steps:

1. Check if \mathbf{F} satisfies $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

2. Assume $\phi(x, y)$ exists and satisfies:

$$\frac{\partial \phi}{\partial x} = P(x, y), \quad \frac{\partial \phi}{\partial y} = Q(x, y).$$

3. Integrate $P(x, y)$ with respect to x to find a partial expression for $\phi(x, y)$:

$$\phi(x, y) = \int P(x, y) dx + h(y),$$

where $h(y)$ is an arbitrary function of y .

4. Differentiate the result with respect to y , and equate it to $Q(x, y)$ to find $h'(y)$:

$$\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y} \left[\int P(x, y) dx + h(y) \right] = Q(x, y).$$

Solve for $h(y)$.

5. Substitute $h(y)$ back into $\phi(x, y)$ to obtain the full potential function.

Example 3.7.1 (Example: Finding the Potential Function)

Given $\mathbf{F} = (2xy, x^2 + 2y)$, determine if \mathbf{F} is a gradient field and, if so, find $\phi(x, y)$.

Step 1: Check the Condition.

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(2xy) = 2x, \quad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x^2 + 2y) = 2x.$$

Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, \mathbf{F} is a gradient field.

Step 2: Integrate $P(x, y)$ with Respect to x .

$$\phi(x, y) = \int P(x, y) dx = \int 2xy dx = x^2 y + h(y),$$

where $h(y)$ is an arbitrary function of y .

Step 3: Differentiate with Respect to y .

$$\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}(x^2 y + h(y)) = x^2 + h'(y).$$

Step 4: Equate to $Q(x, y)$.

$$x^2 + h'(y) = x^2 + 2y \quad \Rightarrow \quad h'(y) = 2y.$$

Step 5: Solve for $h(y)$.

$$h(y) = \int 2y dy = y^2 + C.$$

Step 6: Write the Final Potential Function.

$$\phi(x, y) = x^2 y + y^2 + C.$$

3.8 Non-Gradient Vector Fields and Green's Theorem

3.8.1 Non-Gradient Vector Fields

For a vector field $\mathbf{F} = (P(x, y), Q(x, y))$, if:

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x},$$

then \mathbf{F} is **not a gradient vector field**.

To measure the **rotation** of \mathbf{F} , we compute the **curl** of \mathbf{F} , defined as:

$$\text{curl}(\mathbf{F}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

The curl describes the tendency of \mathbf{F} to "rotate" around a point.

Example 3.8.1 (Example: Interpreting the Curl)

Suppose $\mathbf{F}(x, y)$ represents the velocity of a fluid. If:

$$\frac{\partial Q}{\partial x} > \frac{\partial P}{\partial y},$$

then $\text{curl}(\mathbf{F}) > 0$, indicating counterclockwise rotation of the fluid flow. Conversely, if:

$$\frac{\partial Q}{\partial x} < \frac{\partial P}{\partial y},$$

then $\text{curl}(\mathbf{F}) < 0$, indicating clockwise rotation of the fluid flow.

Consider a case where $Q_x = 1$ and $P_y = 0$. This means $\text{curl}(\mathbf{F}) = 1$, indicating counterclockwise rotation.

3.8.2 Green's Theorem

Green's Theorem provides a relationship between a line integral over a closed curve C and a double integral over the region R enclosed by C .

Theorem: Let C be a positively oriented, simple, closed curve enclosing a region R . If $\mathbf{F} = (P, Q)$ is a continuously differentiable vector field, then:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Intuition: Green's Theorem states that the circulation of \mathbf{F} along the boundary C equals the sum of the curl of \mathbf{F} over the area R . It connects the local rotation of \mathbf{F} (through curl) to its global behavior along the curve C .

Example 3.8.2 (Example: Application of Green's Theorem)

Let $\mathbf{F} = (-3y, 3x)$, and let R be the triangle with vertices $(0, 0), (1, 0), (0, 2)$, oriented counterclockwise. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ using Green's Theorem.

Solution:

1. Compute the Curl of \mathbf{F} :

$$\text{curl}(\mathbf{F}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3 - (-3) = 6.$$

2. Set up the Double Integral over R : The region R is the triangle with vertices $(0, 0), (1, 0), (0, 2)$. Its boundaries can be described by:

$$0 \leq x \leq 1, \quad 0 \leq y \leq 2 - 2x.$$

Using Green's Theorem:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \text{curl}(\mathbf{F}) dA = \iint_R 6 dA.$$

3. Compute the Double Integral:

$$\iint_R 6 dA = \int_0^1 \int_0^{2-2x} 6 dy dx.$$

First, integrate with respect to y :

$$\int_0^{2-2x} 6 \, dy = 6y \Big|_0^{2-2x} = 6(2-2x).$$

Now integrate with respect to x :

$$\int_0^1 6(2-2x) \, dx = \int_0^1 (12-12x) \, dx = [12x-6x^2]_0^1.$$

Evaluate:

$$[12(1)-6(1)^2] - [12(0)-6(0)^2] = 12-6 = 6.$$

4. Final Answer:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 6.$$

3.9 Line Integrals and Area

Area Using Line Integrals

For a vector field $\mathbf{F} = (-y, x)$, the curl is given by:

$$\text{curl}(\mathbf{F}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1.$$

If C is a closed curve oriented counterclockwise, enclosing a region R , the area of R can be computed using the following line integrals:

$$\text{Area}(R) = \iint_R 1 \, dA = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

The parametrized curve C is expressed as $\mathbf{r}(t) = (x(t), y(t))$, with:

$$\mathbf{F} \cdot d\mathbf{r} = P \, dx + Q \, dy.$$

Alternative Formulas for Area

Using line integrals:

$$\text{Area}(R) = \int_C x \, dy = - \int_C y \, dx = \frac{1}{2} \int_C (x \, dy - y \, dx).$$

These expressions are equivalent and depend on how C is parameterized.

Example 3.9.1 (Example: Area of an Ellipse)

Consider an ellipse parameterized by:

$$\mathbf{r}(t) = (a \cos t, b \sin t), \quad t \in [0, 2\pi].$$

Here, a and b are the semi-major and semi-minor axes, respectively.

Steps:

1. Compute $x \, dy$:

$$x = a \cos t, \quad dy = \frac{d}{dt}(b \sin t) \, dt = b \cos t \, dt.$$

Thus:

$$x \, dy = a \cos t \cdot b \cos t \, dt = ab \cos^2 t \, dt.$$

2. Compute $y \, dx$:

$$y = b \sin t, \quad dx = \frac{d}{dt}(a \cos t) \, dt = -a \sin t \, dt.$$

Thus:

$$y \, dx = b \sin t \cdot (-a \sin t) \, dt = -ab \sin^2 t \, dt.$$

3. Compute $x \, dy - y \, dx$:

$$x \, dy - y \, dx = ab \cos^2 t \, dt - (-ab \sin^2 t \, dt) = ab(\cos^2 t + \sin^2 t) \, dt = ab \, dt.$$

4. Compute the Line Integral:

$$\text{Area}(R) = \frac{1}{2} \int_0^{2\pi} ab \, dt = \frac{1}{2} ab \int_0^{2\pi} 1 \, dt = \frac{1}{2} ab \cdot 2\pi = \pi ab.$$

Conclusion: The area of the ellipse is:

$$\text{Area}(R) = \pi ab,$$

where a and b are the semi-major and semi-minor axes.

Summary

The area of a region R enclosed by a curve C can be computed using line integrals in multiple equivalent ways:

- Directly using $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = (-y, x)$.
- Using $\int_C x \, dy$ or $-\int_C y \, dx$.
- Using the symmetric formula:

$$\text{Area}(R) = \frac{1}{2} \int_C (x \, dy - y \, dx).$$

3.10 Flux of a Vector Field

Definition of Flux

Flux represents the flow of a vector field $\mathbf{F} = (f, g)$ across an oriented curve C . It measures how much of \mathbf{F} passes through C in the normal direction.

The flux of \mathbf{F} across C is defined as:

$$\text{Flux} = \int_C \mathbf{F} \cdot \mathbf{n} \, ds,$$

where:

- \mathbf{n} is the unit normal vector to the curve C .
- ds is the arc length differential.

For a parameterized curve $\mathbf{r}(t) = (x(t), y(t))$, where $t \in [a, b]$, the unit normal vector is given by:

$$\mathbf{n} = \frac{1}{|\mathbf{r}'(t)|} (-y'(t), x'(t)).$$

Thus:

$$\mathbf{F} \cdot \mathbf{n} = \frac{1}{|\mathbf{r}'(t)|} [f(x(t), y(t))(-y'(t)) + g(x(t), y(t))x'(t)].$$

Flux in Simplified Form

If the curve C is parameterized, the flux integral can be simplified as:

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_a^b [f(x(t), y(t))y'(t) - g(x(t), y(t))x'(t)] \, dt.$$

Or equivalently:

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_C f \, dy - g \, dx.$$

Green's Theorem Applied to Flux

By applying Green's Theorem, the flux of $\mathbf{F} = (f, g)$ across C can also be related to a double integral over the region R enclosed by C :

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R \operatorname{div}(\mathbf{F}) \, dA,$$

where:

$$\operatorname{div}(\mathbf{F}) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}.$$

This shows that the flux across the boundary C is equal to the total divergence of \mathbf{F} inside R .

Example 3.10.1 (Example: Flux Around a Circle)

Let $\mathbf{F}(x, y) = (x, y)$, and let C be the circle $x^2 + y^2 = 1$ parameterized by:

$$\mathbf{r}(t) = (\cos t, \sin t), \quad t \in [0, 2\pi].$$

Step 1: Parameterize the Flux Integral. The unit normal vector to C is given by:

$$\mathbf{n} = (-\sin t, \cos t),$$

and $\mathbf{F}(\mathbf{r}(t)) = (\cos t, \sin t)$. Then:

$$\mathbf{F} \cdot \mathbf{n} = \cos t(-\sin t) + \sin t(\cos t) = 0.$$

Step 2: Apply Green's Theorem. Using Green's Theorem:

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_R \operatorname{div}(\mathbf{F}) \, dA.$$

The divergence of $\mathbf{F} = (x, y)$ is:

$$\operatorname{div}(\mathbf{F}) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 1 + 1 = 2.$$

The area of the circle is:

$$\operatorname{Area}(R) = \pi \cdot 1^2 = \pi.$$

Thus:

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_R 2 \, dA = 2 \cdot \pi = 2\pi.$$

Final Answer: The flux across the circle is:

$$\text{Flux} = 2\pi.$$