MA226: Differential Equations

Lecture notes for Differential Equations

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Contents

List of Figures

*   *   *

Chapter 1: First-Order Differential Equations

First-order differential equations involve derivatives up to the first derivative only. These form the foundation for understanding more complex differential equations and are ubiquitous in mathematical modeling.

1.1. Modeling and Basic Differential Equations

1.1.1. Three Approaches to Solving Differential Equations

There are three fundamental approaches to tackling differential equations, each with its own strengths:

> **Three Solution Approaches**   Definition 1.1.1.1
>
> 1. Analytic $\rightarrow$ Formula or equation (exact solutions)
> 2. Qualitative $\rightarrow$ Sketches, describe behavior (understanding without solving)
> 3. Numerical $\rightarrow$ Computing (approximate solutions using algorithms)

> **Choosing the Right Approach**   Note 1.1.1.1
>
> - Use analytic methods when exact solutions are needed and the equation is solvable
> - Use qualitative methods to understand long-term behavior and stability
> - Use numerical methods when analytic solutions are impossible or impractical

1.1.2. Modeling

Mathematical modeling with differential equations follows a systematic approach to translate real-world phenomena into mathematical language.

1.1.2.1. Types of Models

- Simple models: Easy to analyze; describe the dominant interactions
- Complex models: Capture behavior over a wider domain; less general

1.1.2.2. Model Building Process

Model building typically follows three steps:
1. State assumptions clearly (with units for all quantities)
2. Define variables and parameters WITH UNITS
3. Use assumptions to derive equations relating the variables

> **Population Modeling**   Example 1.1.2.2.1
>
> Target: Population of rabbits $P(t)$ as a function of time t (years).
>
> Key Assumption: The rate of change of population is proportional to the current population size.
>
> Mathematical Model:
> $$\frac{\mathrm{d}P}{\mathrm{d}t} = kP \qquad [1]$$
> where k is the growth coefficient (constant parameter).

1.1.3. Fundamental Definitions

> **Solution and General Solution**   Definition 1.1.3.1
>
> A function is a solution of a differential equation on an interval if, when substituted into the equation, it satisfies the equality for every point in that interval.
>
> A general solution contains an arbitrary constant (or constants). Determining the constant(s) from given data yields a particular solution.

> **Initial Value Problem (IVP)**   Definition 1.1.3.2
>
> A differential equation together with an initial condition such as $P(t_0) = P_0$.
>
> Solving the IVP means finding the unique solution that satisfies both the equation and the initial condition on an interval.

1.1.4. Exponential Growth and Decay

1.1.4.1. Solving the Basic Growth Model

Consider the differential equation $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$.

Solution Strategy: Guess that $P(t) = Ce^{\{kt\}}$ for some constant C.

Verification:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(Ce^{\{kt\}}\right) = C \cdot ke^{\{kt\}} = k\left(Ce^{\{kt\}}\right) = kP \qquad [2]$$

Therefore, $P(t) = Ce^{\{kt\}}$ is indeed a solution to our differential equation.

General Solution   Note 1.1.4.1.1

Since C is arbitrary, $P(t) = Ce^{\{kt\}}$ represents the general solution to $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$.

The sign of k determines the behavior:

- If $k > 0$: exponential growth
- If $k < 0$: exponential decay

1.1.4.2. Finding Particular Solutions

Complete Solution Process   Example 1.1.4.2.1

Problem: Solve $P' = kP$ with initial conditions $P(0) = 32$ and $P(3) = 47$.

Step 1: Start with general solution $P(t) = Ce^{\{kt\}}$

Step 2: Apply first condition $P(0) = 32$

$$P(0) = Ce^{\{k \cdot 0\}} = Ce^0 = C = 32 \qquad [3]$$

So $C = 32$, giving us $P(t) = 32e^{\{kt\}}$.

Step 3: Apply second condition $P(3) = 47$

$$P(3) = 32e^{\{3k\}} = 47 \qquad [4]$$

$$e^{\{3k\}} = \frac{47}{32} \qquad [5]$$

$$3k = \ln\left(\frac{47}{32}\right) \qquad [6]$$

$$k = \frac{1}{3}\ln\left(\frac{47}{32}\right) \qquad [7]$$

Final Answer: $P(t) = 32e^{\{\frac{1}{3}\ln\left(\frac{47}{32}\right) \cdot t\}}$

Growth vs. Decay Analysis   Note 1.1.4.2.1

- If $k > 0$, then P increases exponentially
- If $k < 0$, then P decreases exponentially
- The time constant $\frac{1}{|k|}$ sets the natural timescale of change

For our example: $k = \frac{1}{3}\ln\left(\frac{47}{32}\right) \approx 0.121 > 0$, so we have exponential growth.

1.1.5. The Logistic Population Model

The simple exponential model $P' = rP$ assumes unlimited resources, leading to unrealistic infinite growth. The logistic model accounts for resource limitations and carrying capacity.

Logistic Equation   Definition 1.1.5.1

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP\left(1 - \frac{P}{K}\right) \qquad [8]$$

where:
- $P(t)$: population size
- $r > 0$: intrinsic growth rate
- $K > 0$: carrying capacity (maximum sustainable population)

Key Insights   Note 1.1.5.1

Per-capita growth rate: $\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}t} = r\left(1 - \frac{P}{K}\right)$

- When $P \approx 0$: growth rate $\approx r$ (nearly exponential)
- When $P = K$: growth rate $= 0$ (no growth at capacity)
- Growth decreases linearly with population density $\frac{P}{K}$

1.1.5.1. Solution and Behavior

Logistic Solution   Example 1.1.5.1.1

The closed-form solution with initial condition $P(0) = P_0 > 0$ is:

$$P(t) = \frac{K}{1 + \left(\frac{K - P_0}{P_0}\right)e^{-rt}} \qquad [9]$$

This produces the characteristic S-shaped (sigmoidal) curve:

1. Initial phase: Nearly exponential growth when $P \ll K$
2. Transition phase: Growth slows as resources become limited
3. Saturation phase: Population levels off at carrying capacity K
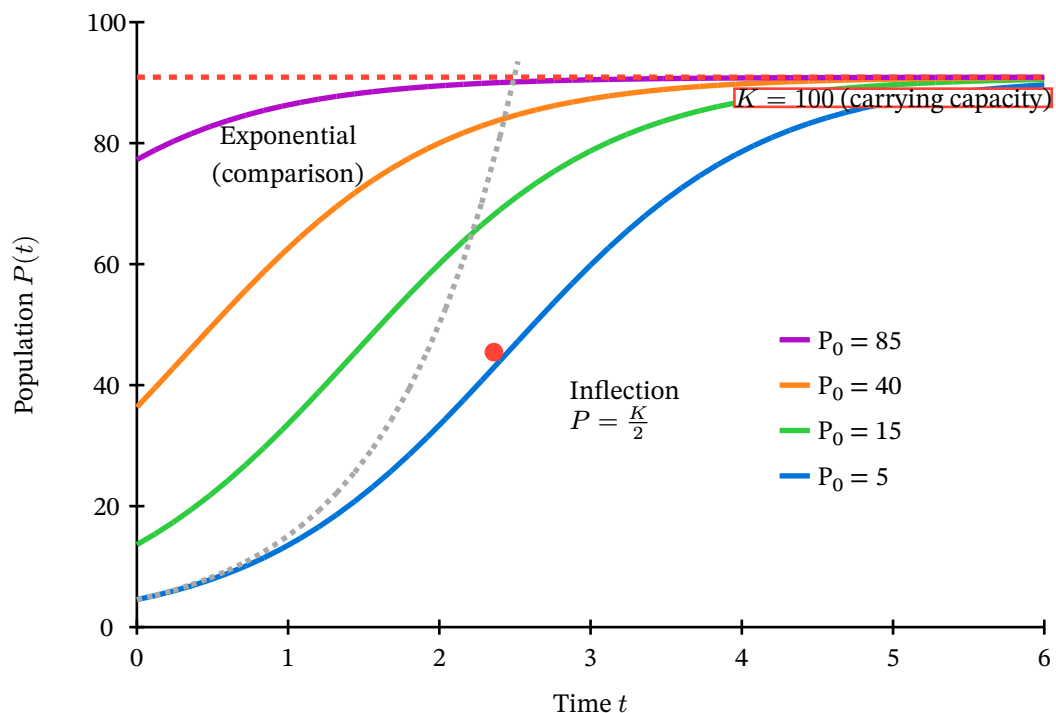

Figure 1: Logistic growth curves showing the characteristic S-shape for different initial populations. The red dashed line shows the carrying capacity K = 100. All logistic curves approach this limit, while the exponential curve (gray, dotted) grows without bound. The inflection point occurs at P = K/2.

1.1.5.2. Harvesting Models

Real populations often face removal through harvesting, hunting, or fishing. We can modify the logistic model by subtracting a harvesting term $H(P)$ from the natural growth rate.

Fish Population with Harvesting   Example 1.1.5.2.1

Base Model: Consider a fish population with logistic growth:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = k\left(1 - \frac{P}{N}\right)P \tag{10}$$

where k is the growth rate and N is the carrying capacity.

With Harvesting: We subtract the harvest rate to get:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = k\left(1 - \frac{P}{N}\right)P - H(P) \tag{11}$$

The form of $H(P)$ depends on the harvesting strategy:

(a) Constant Harvesting: 100 fish removed per year

This represents constant-rate removal that doesn't depend on population size.

$$H(P) = 100 \tag{12}$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = k\left(1 - \frac{P}{N}\right)P - 100 \tag{13}$$

Why this form? The ODE assumes continuous removal at a rate of 100 fish per year. If harvesting happened as a discrete once-per-year event, we would need an impulsive model instead.

(b) Proportional Harvesting: One-third of population harvested annually

This is a rate proportional to P, where the harvest rate increases with population size.

$$H(P) = \frac{1}{3}P \tag{14}$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = k\left(1 - \frac{P}{N}\right)P - \frac{P}{3} \tag{15}$$

Why this form? The coefficient $\frac{1}{3}$ has units of year^{-1}, making $H(P)$ have the correct dimensions of fish/year. This models scenarios where harvesting effort scales with population abundance.

(c) Square-Root Harvesting: Harvest proportional to $\sqrt{P}$

This represents a nonlinear harvest rate that's less aggressive than proportional harvesting.

$$H(P) = a\sqrt{P} \quad \text{where } a > 0 \tag{16}$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = k\left(1 - \frac{P}{N}\right)P - a\sqrt{P} \tag{17}$$

Why this form? The parameter a has units of $\frac{\text{fish}^{\frac{1}{2}}}{\text{year}}$ to ensure dimensional consistency. This might model situations where harvesting becomes less efficient at higher population densities, or where there are diminishing returns to fishing effort.

> **Key Insights on Harvesting** Note 1.1.5.2.1
>
> Dimensional Analysis:
> - Each $H(P)$ term has units of fish/year, matching $\frac{\mathrm{d}P}{\mathrm{d}t}$
> - Case (a): $H = 100$ has units fish/year directly
> - Case (b): $\frac{1}{3}$year$^{-1} \times P$ fish $=$ fish/year
> - Case (c): $a\frac{\text{fish}^{\frac{1}{2}}}{\text{year}} \times \sqrt{P}\text{fish}^{\frac{1}{2}} =$ fish/year
>
> Continuous vs. Discrete Models:
> - Our ODEs assume continuous removal throughout the year
> - Real harvesting often occurs in discrete seasons (impulsive events)
> - The choice depends on the timescale of interest and harvesting patterns
>
> Equilibrium Effects:
> - Constant-rate removal can eliminate equilibria if harvest exceeds maximum growth rate
> - Proportional harvesting reduces effective growth rate: $k - \frac{1}{3}$
> - Nonlinear harvesting creates complex equilibrium structures
>
> Extinction Thresholds: Excessive harvesting creates minimum viable population sizes below which extinction occurs.
>
> Management Implications: Different harvesting strategies require different sustainability criteria and have distinct economic trade-offs.

> **Equilibria and Stability** Attention 1.1.5.2.1
>
> Setting $\frac{\mathrm{d}P}{\mathrm{d}t} = 0$:
>
> - $P^* = 0$: Unstable equilibrium (any $P_0 > 0$ grows away from zero)
> - $P^* = K$: Stable equilibrium (all solutions approach carrying capacity)
>
> Maximum growth occurs at $P = \frac{K}{2}$ with rate $\frac{rK}{4}$.

> **Real-World Applications** Note 1.1.5.2.2
>
> - Population ecology: Animal populations in limited habitats
> - Epidemiology: Disease spread with finite susceptible population
> - Technology adoption: Market saturation models
> - Resource management: Sustainable harvesting strategies

1.1.6. Equilibrium Solutions

> **Equilibrium Solution** Definition 1.1.6.1
>
> A constant solution $y(t) \equiv y_*$ such that $y'(t) = 0$ for all t in an interval.
>
> Equilibria correspond to values of y where the right-hand side of $y' = f(t, y)$ is zero for all t.

Finding Equilibrium Solutions   Example 1.1.6.1

Consider the differential equation:

$$y' = \frac{(y+2)(y-3)(t-5)}{(y+7)} \quad [18]$$

For an equilibrium solution $y(t) \equiv y_*$, we need the right-hand side to be zero for all t.

Analysis: The right-hand side equals zero when the numerator is zero (and the denominator is non-zero).

The numerator $(y+2)(y-3)(t-5) = 0$ when:

- $y + 2 = 0 \rightarrow y = -2$
- $y - 3 = 0 \rightarrow y = 3$
- $t - 5 = 0$ (but this depends on t, so doesn't give a constant solution)

Verification: Both $y = -2$ and $y = 3$ make the denominator $y + 7$ non-zero.

Answer: $y \equiv -2$ and $y \equiv 3$ are equilibrium solutions.

Important Note   Attention 1.1.6.1

$y \equiv -7$ is NOT a solution because it makes the right-hand side undefined (division by zero).

1.1.7. Key Insights and Intuition

Why Exponential Solutions Work   Note 1.1.7.1

The exponential function $e^{\{kt\}}$ has the special property that its derivative is proportional to itself:

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{\{kt\}} = ke^{\{kt\}} \quad [19]$$

This makes it the natural solution to equations of the form $y' = ky$.

Physical Interpretation   Note 1.1.7.2

- Population growth: When resources are abundant, growth rate s im current population
- Radioactive decay: Decay rate s im current amount of material
- Bank interest: Continuous compounding gives exponential growth
- Cooling: Newton's law of cooling (with modifications)

1.2. Separable Differential Equations

Separable differential equations are a special class of first-order differential equations that can be solved by separating variables and integrating both sides.

Separable Differential Equation   Definition 1.2.1

A first-order differential equation is separable if it can be written in the form:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = g(t)h(y) \quad [20]$$

where $g(t)$ is a function of t only, and $h(y)$ is a function of y only.

Solution Strategy   Note 1.2.1

To solve a separable equation $\frac{dy}{dt} = g(t)h(y)$:

1. Separate variables: $\frac{dy}{h(y)} = g(t)dt$
2. Integrate both sides: $\int \frac{dy}{h(y)} = \int g(t)dt$
3. Solve for y (if possible)
4. Apply initial conditions to find particular solutions

1.2.1. Basic Examples

Simple Exponential Growth   Example 1.2.1.1

Problem: Solve $y' = 2y$

Step 1: Recognize this is separable with $g(t) = 2$ and $h(y) = y$

Step 2: Separate variables

$$\frac{dy}{y} = 2dt \tag{21}$$

Step 3: Integrate both sides

$$\int \frac{dy}{y} = \int 2dt \tag{22}$$

$$\ln|y| = 2t + C_1 \tag{23}$$

Step 4: Solve for y

$$|y| = e^{2t+C_1} = e^{C_1}e^{2t} \tag{24}$$

Since $e^{C_1} > 0$, we can write $|y| = Ce^{2t}$ where $C > 0$.

Step 5: Consider both positive and negative solutions

$$y = \pm Ce^{2t} \tag{25}$$

Final Answer: $y = Ce^{2t}$ where C can be any real constant (including negative values and zero).

Non-Separable Counter Example   Attention 1.2.1.1

Problem: Is $y' = t + y$ separable?

Analysis: We need to write this as $\frac{dy}{dt} = g(t)h(y)$.

We have $\frac{dy}{dt} = t + y$. For this to be separable, we need:

$$t + y = g(t) \cdot h(y) \tag{26}$$

But $t + y$ cannot be factored into a product of a function of t only and a function of y only.

Conclusion: This equation is NOT separable and requires different solution methods.

1.2.2. More Complex Examples

Polynomial Growth Factor   Example 1.2.2.1

Problem: Solve $y' = t^4 y$

Step 1: This is separable with $g(t) = t^4$ and $h(y) = y$

Step 2: Separate variables

$$\frac{\mathrm{d}y}{y} = t^4 \mathrm{d}t \tag{27}$$

Step 3: Integrate both sides

$$\int \frac{\mathrm{d}y}{y} = \int t^4 \mathrm{d}t \tag{28}$$

$$\ln|y| = \frac{t^5}{5} + C_1 \tag{29}$$

Step 4: Solve for y

$$|y| = e^{\frac{t^5}{5} + C_1} = e^{C_1} e^{\frac{t^5}{5}} \tag{30}$$

Final Answer: $y = Ce^{\frac{t^5}{5}}$ where C is an arbitrary constant.

Note: The growth becomes extremely rapid for large $|t|$ due to the t^5 term in the exponent.

Linear Decay Model   Example 1.2.2.2

Problem: Solve $y' = 2 - y$

Step 1: Rewrite as $\frac{\mathrm{d}y}{\mathrm{d}t} = 2 - y = -(y - 2)$

This is separable with $g(t) = -1$ and $h(y) = y - 2$.

Step 2: Separate variables

$$\frac{\mathrm{d}y}{y - 2} = -\mathrm{d}t \tag{31}$$

Step 3: Integrate both sides

$$\int \frac{\mathrm{d}y}{y - 2} = \int (-1)\mathrm{d}t \tag{32}$$

$$\ln|y - 2| = -t + C_1 \tag{33}$$

Step 4: Solve for y

$$|y - 2| = e^{-t + C_1} = e^{C_1} e^{-t} \tag{34}$$

$$y - 2 = \pm e^{C_1} e^{-t} = Ce^{-t} \tag{35}$$

Final Answer: $y = Ce^{-t} + 2$

Physical Interpretation: This represents exponential approach to the equilibrium value $y = 2$.

Visualization: The figure below shows several solution curves for different initial conditions, all approaching the equilibrium line $y = 2$.
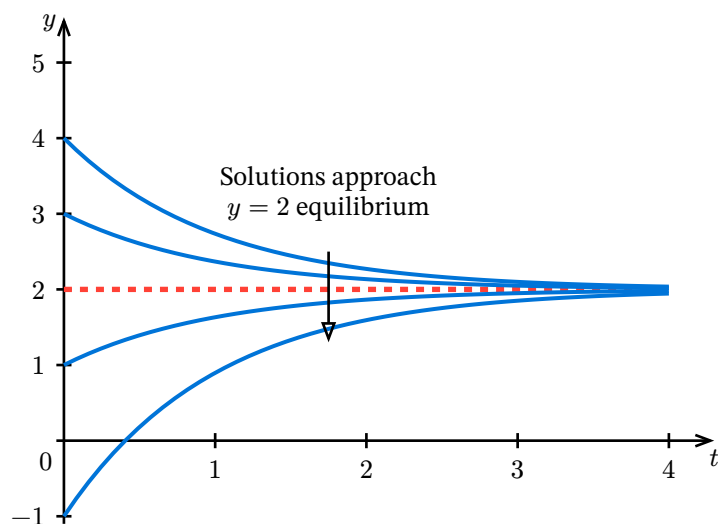
Figure 2: Solution curves for $y' = 2 - y$ with different initial conditions. All solutions exponentially approach the equilibrium $y = 2$ (red dashed line). Initial conditions: $y(0) = 4, 3, 1, -1$ respectively.

1.2.3. Advanced Techniques

Arctangent Integration   Example 1.2.3.1

Problem: Solve $y' = 1 + x^2$ (treating x as the independent variable)

Step 1: This is separable: $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + x^2$

Step 2: Since there's no y dependence, we can integrate directly

$$y = \int (1 + x^2)\,\mathrm{d}x \tag{36}$$

$$y = x + \frac{x^3}{3} + C \tag{37}$$

Alternative form using arctangent: If we had $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x^2}$, then:

$$y = \int \frac{\mathrm{d}x}{1 + x^2} = \arctan(x) + C \tag{38}$$

| Partial Fractions Method | Example 1.2.3.2 |

Problem: Solve $y' = 12 + 3x^2$ with more complex rational functions

Consider the related problem: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{(2+x)(2-x)} = \frac{1}{4-x^2}$

Step 1: Use partial fractions decomposition

$$\frac{1}{(2+x)(2-x)} = \frac{A}{2+x} + \frac{B}{2-x} \tag{39}$$

Step 2: Find constants A and B

$$1 = A(2-x) + B(2+x) = 2A - Ax + 2B + Bx = (2A + 2B) + (-A + B)x \tag{40}$$

Comparing coefficients:

- Constant term: $2A + 2B = 1$
- Coefficient of x: $-A + B = 0$, so $A = B$

From $A = B$ and $2A + 2B = 1$: $4A = 1$, so $A = B = \frac{1}{4}$

Step 3: Integrate

$$y = \int \left(\frac{1}{4} \frac{1}{2+x} + \frac{1}{4} \frac{1}{2-x} \right) \mathrm{d}x \tag{41}$$

$$y = \frac{1}{4} \ln|2 + x| - \frac{1}{4} \ln|2 - x| + C \tag{42}$$

$$y = \frac{1}{4} \ln|\frac{2+x}{2-x}| + C \tag{43}$$

Initial Value Problem with Higher Powers   Example 1.2.3.3

Problem: Solve $y' = t^2 y^3$ with $y(0) = 1$

Step 1: Separate variables

$$\frac{\mathrm{d}y}{y^3} = t^2 \mathrm{d}t \tag{44}$$

$$y^{-3}\mathrm{d}y = t^2 \mathrm{d}t \tag{45}$$

Step 2: Integrate both sides

$$\int y^{-3}\mathrm{d}y = \int t^2 \mathrm{d}t \tag{46}$$

$$\frac{y^{-2}}{-2} = \frac{t^3}{3} + C_1 \tag{47}$$

$$-\frac{1}{2y^2} = \frac{t^3}{3} + C_1 \tag{48}$$

Step 3: Solve for y

$$\frac{1}{2y^2} = -\frac{t^3}{3} - C_1 \tag{49}$$

$$\frac{1}{y^2} = -\frac{2t^3}{3} - 2C_1 \tag{50}$$

Let $C = -2C_1$, then:

$$\frac{1}{y^2} = -\frac{2t^3}{3} + C \tag{51}$$

$$y^2 = \frac{1}{-\frac{2t^3}{3} + C} \tag{52}$$

Step 4: Apply initial condition $y(0) = 1$

$$1^2 = \frac{1}{-\frac{2(0)^3}{3} + C} = \frac{1}{C} \tag{53}$$

Therefore $C = 1$, and:

$$y^2 = \frac{1}{1 - \frac{2t^3}{3}} = \frac{3}{3 - 2t^3} \tag{54}$$

Final Answer: $y = \pm\sqrt{\frac{3}{3-2t^3}}$

Since $y(0) = 1 > 0$, we take the positive square root:

$$y = \sqrt{\frac{3}{3 - 2t^3}} \tag{55}$$

Domain: Solution is valid when $3 - 2t^3 > 0$, i.e., when $t^3 < \frac{3}{2}$ or $t < \sqrt[3]{\frac{3}{2}}$.

Key Study Tips   Note 1.2.3.1

1. Always check separability first - can you factor the right-hand side as $g(t)h(y)$?

2. Be careful with absolute values in logarithmic integration - consider both positive and negative solutions

3. Watch the domain - solutions may have restrictions based on denominators or square roots

4. Initial conditions determine the sign and specific constant value

5. Partial fractions are useful when $h(y)$ is a rational function with distinct linear factors