Temporary Doc
Calc 3

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Contents

Chapter 1

Vector Valued Functions $f : \mathbb{R} \to \mathbb{R}^n$

1.1 Change of Variable for Double and Triple Integrals

Polar Coordinates

$$\iint_D f(x, y)\, dx\, dy \to \iint_S f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Cylindrical Coordinates

$$\iiint_D f(x, y, z)\, dx\, dy\, dz \to \iiint_S f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz$$

Spherical Coordinates

$$\iiint_D f(x, y, z)\, dx\, dy\, dz \to \iiint_S f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

Theorem 1.1.1 Intuition Behind Change of Variables

We use a **mapping** T to transform coordinates in one space S to another R. This is particularly useful when integrating over regions that are easier to describe in new coordinates (e.g., circular or spherical regions).
For example:

$$S = [0, 2\pi] \times [0, 2], \quad T(r, \theta) = (r\cos\theta, r\sin\theta)$$

Here, the mapping T converts a point in S into a point in R.

Area Differential Transformation

Consider a small differential area element in the original space:

$$dA = |\det(J)|\, du\, dv$$

where J is the **Jacobian matrix**, and $|\det(J)|$ accounts for how the transformation scales area.

Definition 1.1.1: Jacobian Matrix

The Jacobian matrix represents the linear transformation of the mapping T at a given point:

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

For a transformation $T(u, v) = (g(u, v), h(u, v))$, the determinant of J is:

$$\det(J) = \begin{vmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{vmatrix} = \frac{\partial g}{\partial u} \cdot \frac{\partial h}{\partial v} - \frac{\partial g}{\partial v} \cdot \frac{\partial h}{\partial u}$$

Geometric Interpretation

- **Local Stretching/Scaling:** $|\det(J)|$ gives the local scaling factor of the area due to the transformation.
- **Orientation:** The sign of $\det(J)$ indicates whether the orientation is preserved or flipped.

Example 1.1.1 (Polar Coordinates)

For the transformation $T(r, \theta) = (r\cos\theta, r\sin\theta)$, the Jacobian matrix is:

$$J = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

The determinant is:

$$\det(J) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

Thus, the area differential in polar coordinates becomes:

$$dx\, dy = r\, dr\, d\theta$$

Definition 1.1.2: General Formula for Transforming Integrals

If $T : S \to R$ is a transformation with Jacobian determinant $|\det(J)|$, then the integral transforms as:

$$\iint_R f(x, y)\, dx\, dy = \iint_S f(T(u, v))\, |\det(J)|\, du\, dv$$

Definition 1.1.3: Intuition for Higher Dimensions

In three dimensions, the Jacobian matrix extends to account for the transformation of volume elements:

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$$

The volume scaling factor is given by $|\det(J)|$, and the integral transforms as:

$$\iiint_R f(x, y, z)\, dx\, dy\, dz = \iiint_S f(T(u, v, w))\, |\det(J)|\, du\, dv\, dw$$

1.2 Non-overlapping from Mapping T

Theorem 1.2.1 Non-overlapping Condition

For any two points Q and P:

$$T(Q) \neq T(P) \quad \text{(This would result in overlapping areas in the domain } R\text{)}$$

However, boundaries (e.g., $y = 2x$) can overlap as long as the bounded region is distinct.

Example 1.2.1 (Integral Transformation Example)

Evaluate:

$$\iint_R 2x(y - 2x)\, dA$$

where R is the parallelogram with vertices $(0, 0), (0, 1), (2, 4), (2, 3)$.
Steps:

1. **Choose a Transformation:** Select a mapping T to simplify the integral.
2. **Define the Mapping:**
$$x = u, \quad y = 2x + v = 2u + v$$
Substituting:
$$(x, y) \longrightarrow (u, v)$$
3. **Boundary Equations:**
$$0 \leqslant x \leqslant 2 \quad \Rightarrow \quad 0 \leqslant u \leqslant 2$$
$$0 \leqslant y - 2x < 1 \quad \Rightarrow \quad 0 \leqslant v < 1$$
4. **Region:**
$$S = [0, 2] \times [0, 1]$$
5. **Transform the Integrand:**
$$f(T(x, y)) = 2u(v)$$
6. **Jacobian Calculation:**
$$J = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad \det(J) = 1 \cdot 1 - 2 \cdot 0 = 1$$
7. **Transformed Integral:**
$$\iint_R 2x(y - 2x)\, dA = \int_0^2 \int_0^1 2uv\, du\, dv$$

1.3 Integral Transformation for a Parallelogram Region

Example 1.3.1 (Example of Transformation)

Evaluate:

$$\iint_R 2x(y - 2x)\, dA$$

where R is the parallelogram defined by the vertices $(0, 0), (0, 1), (2, 4), (2, 3)$.

Steps:

1. **Choose a Transformation:** Select a transformation T that simplifies the integral.

2. **Define x, y in terms of u, v:**

$$x = u, \quad y = 2x + v = 2u + v$$

   Substituting:

$$(x, y) \to (u, v)$$

   Here, u corresponds to x, and $v = y - 2x$.

3. **Boundary Equations:**

$$0 \leqslant x \leqslant 2 \quad \Rightarrow \quad 0 \leqslant u \leqslant 2$$

$$0 \leqslant y - 2x < 1 \quad \Rightarrow \quad 0 \leqslant v < 1$$

4. **Region in u, v:**

$$S = [0, 2] \times [0, 1]$$

   This maps the parallelogram R into a rectangle S in the u, v-plane.

5. **Transform the Integrand:** Substituting $x = u$ and $y - 2x = v$:

$$f(T(x, y)) = 2u(v)$$

6. **Jacobian Calculation:** The Jacobian matrix for the transformation T is:

$$J = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

   The determinant of J is:

$$\det(J) = 1 \cdot 1 - 2 \cdot 0 = 1$$

7. **Transformed Integral:** Using the transformation and the Jacobian determinant:

$$\iint_R 2x(y - 2x)\, dA = \int_0^2 \int_0^1 2uv\, du\, dv$$

   The transformed integral simplifies the computation significantly.

1.4 Integral Transformation for a Triangular Region

Example 1.4.1 (Example of Transformation)
Evaluate:

$$\iint_R (x - u)\sqrt{x - 2y}\, dA$$

where R is the triangular region bounded by the lines $y = 0$, $x - 2y = 0$, and $x = y + 1$.

Steps:

1. **Region Definition:** The region R is defined by:

$$y = 0, \quad x - 2y = 0, \quad x = y + 1$$

   The boundaries in x and y are:

$$0 \leqslant x \leqslant 2, \quad 0 \leqslant y \leqslant \frac{x}{2}, \quad x \leqslant y + 1$$

2. **Define Transformation:** Let:
$$u = x - 2y, \quad v = x - y$$
Substituting:
$$x = v + u, \quad y = v - u$$

3. **Boundaries in New Coordinates:** Using the transformation:
$$u = x - 2y \quad \Rightarrow \quad 0 \leqslant u \leqslant 1$$
$$v = x - y \quad \Rightarrow \quad u \leqslant v \leqslant 1$$
The transformed region S is bounded by $u = 0$, $v = 1$, and $v - u = 1$.

4. **Jacobian Calculation:** The Jacobian matrix for the transformation $T(u, v)$ is:
$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$
The determinant of J is:
$$\det(J) = (1)(1) - (1)(-2) = 1 + 2 = 3$$

5. **Transform the Integral:** Using the transformation and Jacobian determinant:
$$\iint_R (x - u)\sqrt{x - 2y}\, dA = \int_0^1 \int_0^v \sqrt{u} \cdot 3\, du\, dv$$
Simplify:
$$\int_0^1 \int_0^v \sqrt{u}\, du\, dv = \int_0^1 \left[\frac{2}{3}u^{3/2}\right]_0^v dv = \int_0^1 \frac{2}{3}v^{3/2} dv$$
$$= \left[\frac{2}{3} \cdot \frac{2}{5}v^{5/2}\right]_0^1 = \frac{4}{15}.$$
The result is:
$$\iint_R (x - u)\sqrt{x - 2y}\, dA = \frac{4}{15}.$$

Chapter 2

Vector Fields $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

2.1 Vector Fields

A **vector field** is a function $\vec{F}$ that takes points in $\mathbb{R}^2$ or $\mathbb{R}^3$ and outputs a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$:

$$\vec{F} : \mathbb{R}^n \to \mathbb{R}^n, \quad \vec{F}(x, y, z) = \langle q(x, y, z), w(x, y, z), t(x, y, z)\rangle.$$

2.1.1 Properties

- Continuous if q, w, t are continuous.
- Differentiable if q, w, t are differentiable.
- Domain is the intersection of domains of q, w, t.

2.1.2 Applications

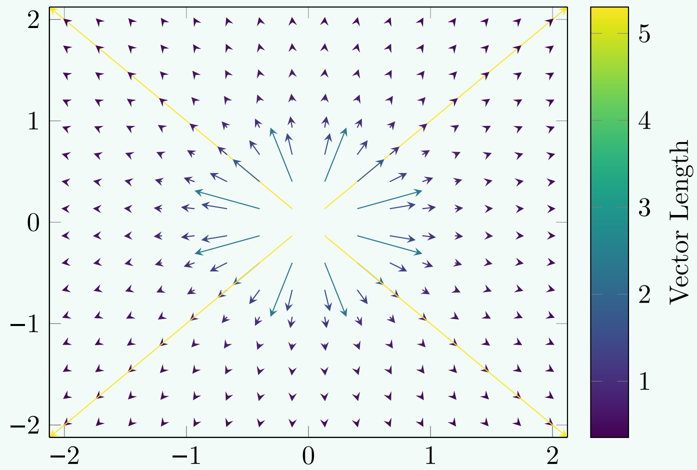
- Water and wind currents.
- Gravitational, electric, and magnetic fields.
- Human circulation, heat propagation.
- Modeling through partial differential equations.

2.1.3 Drawing a Vector Field

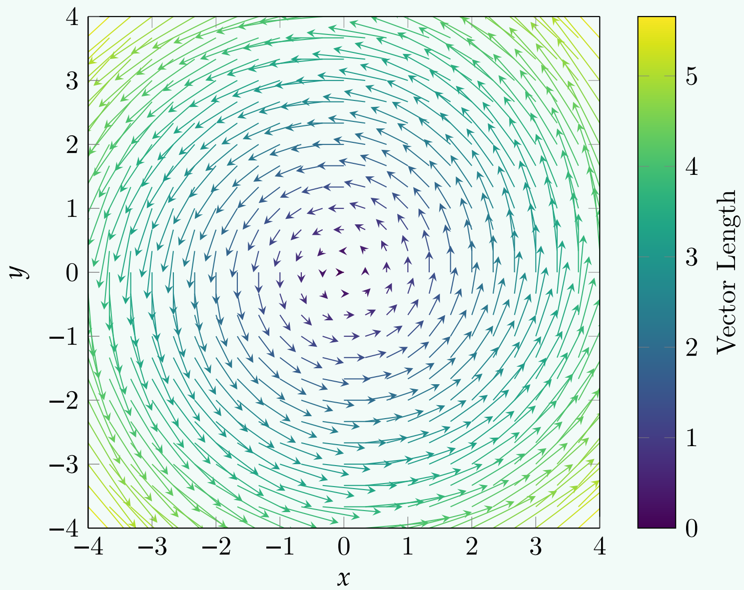
At each point in the domain of $\vec{F}$, draw a vector whose:

- Direction is parallel to $\vec{F}(x, y)$.
- Length is proportional to the magnitude of $\vec{F}(x, y)$.

Example 2.1.1 ($\vec{F}(x, y) = \frac{\langle x,y\rangle}{|\langle x,y\rangle|^3}$)



Example 2.1.2 ($\vec{F}(x,y) = \langle -y, x \rangle$)



2.2 Gradient Vector Fields

A **gradient vector field** $\nabla\varphi(x,y)$ is a vector field:

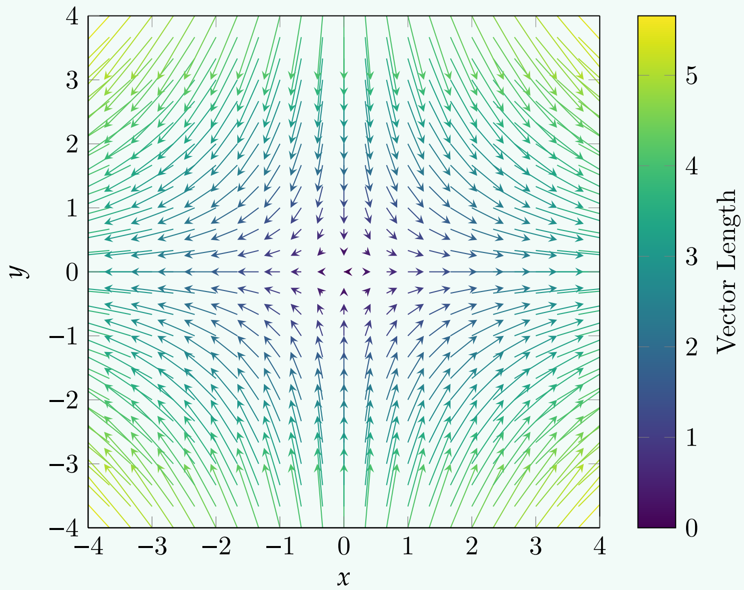
$$\vec{F}(x,y) = \nabla\varphi \quad \text{where } \varphi \text{ is a potential function.}$$

If φ exists, $\vec{F}$ is called a **conservative field**.

Example 2.2.1 ($\varphi = -\frac{x^2+y^2}{2}$)

$$\vec{F}(x,y) = \nabla\varphi = \langle -x, -y \rangle$$

Vector Field $F = [-y, x]$



2.3 Line Integrals

Scalar Case

For a scalar function $f(x, y)$, the line integral over a curve C is defined as:

$$A = \int_C f(x, y)\, ds$$

where $ds = |\mathbf{r}'(t)|dt$, with $\mathbf{r}(t)$ being the parameterization of C.

If $\mathbf{r}(t) = (x(t), y(t))$, then:

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

This allows us to rewrite the line integral as:

$$\int_C f(x, y)\, ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\, dt$$

Example 2.3.1 (Example: Average Temperature on a Plate)

Suppose we have a plate located in $R = \{(x, y) : x^2 + y^2 \leqslant 4\}$, where the temperature at any point (x, y) is $T(x, y) = 100(x^2 + y^2)$. Find the average temperature over the boundary of R.

1. **Parameterize the Boundary:** The boundary C of R is a circle of radius 2, centered at the origin. Parameterize C as:

$$\mathbf{r}(t) = (2\cos t, 2\sin t), \quad 0 \leqslant t \leqslant 2\pi$$

Then:

$$|\mathbf{r}'(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2} = 2$$

2. **Average Temperature:** The average temperature is given by:

$$\text{Average Temperature} = \frac{\int_C T(x, y)\, ds}{\int_C ds}$$

Compute the numerator:

$$\int_C T(x, y)\, ds = \int_0^{2\pi} 100 \cdot 4 \cdot 2\, dt = 800\pi$$

Compute the denominator (arc length of C):

$$\int_C ds = \int_0^{2\pi} 2\, dt = 4\pi$$

Thus:

$$\text{Average Temperature} = \frac{800\pi}{4\pi} = 200$$

Oriented Curves

An oriented curve C includes a direction along the curve. For example, when calculating work done by a force $\mathbf{F}$ along a curve, the direction matters.

Example 2.3.2 (Example: Work)

Work done by a force $\mathbf{F}(x)$ is given by:

$$W = \int_a^b F(x)\, dx$$

For a spring with force $F(x) = -kx$, the work done to stretch the spring from $x = a$ to $x = b$ is:

$$W = \int_a^b -kx\, dx = -\frac{k}{2}\left[b^2 - a^2\right]$$

2.4 Vector Line Integrals

Example 2.4.1 (Example: Earth's Gravitational Field)

The Earth's gravitational field exerts a force $\mathbf{F}(\mathbf{r})$ on an object, pulling it toward the origin. Suppose an object travels along a path C. The work done by $\mathbf{F}$ is:

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

If $\mathbf{F}(\mathbf{r}) = \frac{-Gm}{|\mathbf{r}|^3}\mathbf{r}$, parameterize C as $\mathbf{r}(t)$, and compute:

$$\mathbf{F}(\mathbf{r}) = -\frac{Gm}{|\mathbf{r}|^3}\mathbf{r}, \quad d\mathbf{r} = \mathbf{r}'(t)\,dt$$

Then:

$$W = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$$

If C is along the z-axis from $(0,0,1)$ to $(0,0,4)$, parameterize:

$$\mathbf{r}(t) = (0,0,t), \quad 1 \leqslant t \leqslant 4$$

Then:

$$W = \int_1^4 -\frac{Gm}{t^2} \cdot 1\,dt = -Gm \int_1^4 \frac{1}{t^2}\,dt$$

$$W = -Gm\left[-\frac{1}{t}\right]_1^4 = -Gm\left(-\frac{1}{4} + 1\right) = Gm\left(\frac{3}{4}\right)$$

2.5 Fundamental Theorem of Line Integrals

Theorem 2.5.1 Fundamental Theorem of Line Integrals

If $\mathbf{F}$ is a vector field that can be expressed as $\mathbf{F} = \nabla\phi$, then given an oriented curve C from P to Q:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(Q) - \phi(P)$$

Example 2.5.1 (Example)

Let $\phi(x,y,z) = \frac{-G}{\sqrt{x^2+y^2+z^2}}$, and $\mathbf{F} = \nabla\phi$. Compute:

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

from (a,a,a) to $(1,1,1)$:

$$\phi(x,y,z) = \frac{-G}{\sqrt{x^2+y^2+z^2}}, \quad \Delta\phi = \phi(1,1,1) - \phi(a,a,a)$$

$$\phi(a,a,a) = \frac{-G}{\sqrt{3a^2}} = \frac{-G}{a\sqrt{3}}, \quad \phi(1,1,1) = \frac{-G}{\sqrt{3}}$$

Thus:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(1,1,1) - \phi(a,a,a) = \frac{-G}{\sqrt{3}} - \left(\frac{-G}{a\sqrt{3}}\right)$$

$$= \frac{G}{\sqrt{3}}\left(1 - \frac{1}{a}\right)$$

2.6 Vector Line Integrals and Gradient Fields

2.6.1 Vector Line Integral for a Curve C

Given a vector field $\mathbf{F}(x, y) = (P(x, y), Q(x, y))$ and an oriented curve C, the vector line integral is defined as:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy$$

where:

$$d\mathbf{r} = (dx, dy), \quad \mathbf{F} \cdot d\mathbf{r} = P\,dx + Q\,dy.$$

If C is parameterized as $\mathbf{r}(t) = (x(t), y(t))$, where $t \in [a, b]$, then:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_a^b \left[P(x(t), y(t))\frac{dx}{dt} + Q(x(t), y(t))\frac{dy}{dt} \right] dt$$

Example 2.6.1 (Example: Compute a Vector Line Integral)

Let $\mathbf{F}(x, y) = (x, x^2 + y)$ and let C be the curve defined by $\mathbf{r}(t) = (t, t^2)$, where $t \in [1, 3]$.
Steps:

1. **Parameterize C:** From the parameterization, we have:

$$x(t) = t, \quad y(t) = t^2, \quad \mathbf{r}'(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (1, 2t)$$

2. **Evaluate $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$:** Substitute $x(t)$ and $y(t)$ into $\mathbf{F}(x, y)$:

$$\mathbf{F}(\mathbf{r}(t)) = (t, t^2 + t^2) = (t, 2t^2)$$

Then:

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = t \cdot 1 + (2t^2) \cdot (2t) = t + 4t^3$$

3. **Compute the Integral:**

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_1^3 (t + 4t^3)\,dt$$

Split the integral:

$$\int_1^3 (t + 4t^3)\,dt = \int_1^3 t\,dt + \int_1^3 4t^3\,dt$$

Compute each term:

$$\int_1^3 t\,dt = \left[\frac{t^2}{2}\right]_1^3 = \frac{9}{2} - \frac{1}{2} = 4$$

$$\int_1^3 4t^3\,dt = 4\left[\frac{t^4}{4}\right]_1^3 = [81 - 1] = 80$$

Thus:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 4 + 80 = 84$$

2.6.2 Gradient Fields and Fundamental Theorem of Line Integrals

If $\mathbf{F}$ is a gradient field, meaning $\mathbf{F} = \nabla\phi$, then the line integral over a curve C depends only on the endpoints of C. Specifically:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(B) - \phi(A)$$

where A and B are the start and end points of C.

Intuition:

- The integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ represents the difference in potential between the endpoints of C.
- The path taken does not matter; only the values of ϕ at A and B are relevant.

Example 2.6.2 (Example: Gradient Field Integral)
Let $\mathbf{F} = \nabla\phi$, where $\phi(x, y) = x^2 + xy$. Compute:

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where C is any curve starting at $A = (1, 2)$ and ending at $B = (3, 4)$.
Solution:

1. Compute $\phi(A)$ and $\phi(B)$:
$$\phi(A) = (1)^2 + (1)(2) = 1 + 2 = 3$$
$$\phi(B) = (3)^2 + (3)(4) = 9 + 12 = 21$$

2. Apply the Fundamental Theorem:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(B) - \phi(A) = 21 - 3 = 18$$

2.7 Gradient Vector Fields and Closed Curves

2.7.1 Definition of a Closed Curve

A curve C is said to be **closed** if its start and end points are the same, i.e., $A = B$.

For a closed curve C, the following property holds:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0 \quad \text{if } \mathbf{F} \text{ is a gradient vector field.}$$

If $\mathbf{F} = \nabla\phi$, then:
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(A) - \phi(A) = 0.$$

This implies that **the line integral of a gradient field over a closed curve is always zero**.

2.7.2 How to Identify a Gradient Vector Field

To determine whether a vector field $\mathbf{F} = (P(x, y), Q(x, y))$ is a gradient field:

- $\mathbf{F}$ must be defined on a region R that is **connected** and **simply connected** (no holes in R).
- $P(x, y)$ and $Q(x, y)$ must satisfy the condition:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Theorem: If $\mathbf{F}$ satisfies the above conditions, then $\mathbf{F}$ is a gradient vector field, and there exists a scalar function $\phi(x, y)$ such that $\mathbf{F} = \nabla\phi$.

2.7.3 Finding the Potential Function $\phi(x, y)$

To find $\phi(x, y)$, follow these steps:

1. Check if $\mathbf{F}$ satisfies $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

2. Assume $\phi(x, y)$ exists and satisfies:

$$\frac{\partial \phi}{\partial x} = P(x, y), \quad \frac{\partial \phi}{\partial y} = Q(x, y).$$

3. Integrate $P(x, y)$ with respect to x to find a partial expression for $\phi(x, y)$:

$$\phi(x, y) = \int P(x, y)\, dx + h(y),$$

   where $h(y)$ is an arbitrary function of y.

4. Differentiate the result with respect to y, and equate it to $Q(x, y)$ to find $h'(y)$:

$$\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\left[\int P(x, y)\, dx + h(y)\right] = Q(x, y).$$

   Solve for $h(y)$.

5. Substitute $h(y)$ back into $\phi(x, y)$ to obtain the full potential function.

Example 2.7.1 (Example: Finding the Potential Function)
Given $\mathbf{F} = (2xy, x^2 + 2y)$, determine if $\mathbf{F}$ is a gradient field and, if so, find $\phi(x, y)$.
Step 1: Check the Condition.

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(2xy) = 2x, \quad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x^2 + 2y) = 2x.$$

Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, $\mathbf{F}$ is a gradient field.
Step 2: Integrate $P(x, y)$ with Respect to x.

$$\phi(x, y) = \int P(x, y)\, dx = \int 2xy\, dx = x^2 y + h(y),$$

where $h(y)$ is an arbitrary function of y.
Step 3: Differentiate with Respect to y.

$$\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\left(x^2 y + h(y)\right) = x^2 + h'(y).$$

Step 4: Equate to $Q(x, y)$.

$$x^2 + h'(y) = x^2 + 2y \quad \Rightarrow \quad h'(y) = 2y.$$

Step 5: Solve for $h(y)$.

$$h(y) = \int 2y\, dy = y^2 + C.$$

Step 6: Write the Final Potential Function.

$$\phi(x, y) = x^2 y + y^2 + C.$$

2.8 Non-Gradient Vector Fields and Green's Theorem

2.8.1 Non-Gradient Vector Fields

For a vector field $\mathbf{F} = (P(x, y), Q(x, y))$, if:

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x},$$

then $\mathbf{F}$ is **not a gradient vector field**.

To measure the **rotation** of $\mathbf{F}$, we compute the **curl** of $\mathbf{F}$, defined as:

$$\text{curl}(\mathbf{F}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

The curl describes the tendency of $\mathbf{F}$ to "rotate" around a point.

Example 2.8.1 (Example: Interpreting the Curl)

Suppose $\mathbf{F}(x, y)$ represents the velocity of a fluid. If:

$$\frac{\partial Q}{\partial x} > \frac{\partial P}{\partial y},$$

then $\text{curl}(\mathbf{F}) > 0$, indicating counterclockwise rotation of the fluid flow. Conversely, if:

$$\frac{\partial Q}{\partial x} < \frac{\partial P}{\partial y},$$

then $\text{curl}(\mathbf{F}) < 0$, indicating clockwise rotation of the fluid flow.

Consider a case where $Q_x = 1$ and $P_y = 0$. This means $\text{curl}(\mathbf{F}) = 1$, indicating counterclockwise rotation.

2.8.2 Green's Theorem

Green's Theorem provides a relationship between a line integral over a closed curve C and a double integral over the region R enclosed by C.

Theorem: Let C be a positively oriented, simple, closed curve enclosing a region R. If $\mathbf{F} = (P, Q)$ is a continuously differentiable vector field, then:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Intuition: Green's Theorem states that the circulation of $\mathbf{F}$ along the boundary C equals the sum of the curl of $\mathbf{F}$ over the area R. It connects the local rotation of $\mathbf{F}$ (through curl) to its global behavior along the curve C.

Example 2.8.2 (Example: Application of Green's Theorem)

Let $\mathbf{F} = (-3y, 3x)$, and let R be the triangle with vertices $(0,0), (1,0), (0,2)$, oriented counterclockwise. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ using Green's Theorem.

Solution:

1. Compute the Curl of $\mathbf{F}$:

$$\text{curl}(\mathbf{F}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3 - (-3) = 6.$$

2. Set up the Double Integral over R: The region R is the triangle with vertices $(0,0), (1,0), (0,2)$. Its boundaries can be described by:

$$0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant 2 - 2x.$$

Using Green's Theorem:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \text{curl}(\mathbf{F})\,dA = \iint_R 6\,dA.$$

3. Compute the Double Integral:

$$\iint_R 6\,dA = \int_0^1 \int_0^{2-2x} 6\,dy\,dx.$$

First, integrate with respect to y:

$$\int_0^{2-2x} 6\,dy = 6y\Big|_0^{2-2x} = 6(2-2x).$$

Now integrate with respect to x:

$$\int_0^1 6(2-2x)\,dx = \int_0^1 (12-12x)\,dx = \left[12x - 6x^2\right]_0^1.$$

Evaluate:

$$\left[12(1) - 6(1)^2\right] - \left[12(0) - 6(0)^2\right] = 12 - 6 = 6.$$

4. Final Answer:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 6.$$

2.9 Line Integrals and Area

Area Using Line Integrals

For a vector field $\mathbf{F} = (-y, x)$, the curl is given by:

$$\operatorname{curl}(\mathbf{F}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1.$$

If C is a closed curve oriented counterclockwise, enclosing a region R, the area of R can be computed using the following line integrals:

$$\text{Area}(R) = \iint_R 1\,dA = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

The parametrized curve C is expressed as $\mathbf{r}(t) = (x(t), y(t))$, with:

$$\mathbf{F} \cdot d\mathbf{r} = P\,dx + Q\,dy.$$

Alternative Formulas for Area

Using line integrals:

$$\text{Area}(R) = \int_C x\,dy = -\int_C y\,dx = \frac{1}{2}\int_C (x\,dy - y\,dx).$$

These expressions are equivalent and depend on how C is parameterized.

Example 2.9.1 (Example: Area of an Ellipse)
Consider an ellipse parameterized by:

$$\mathbf{r}(t) = (a\cos t, b\sin t), \quad t \in [0, 2\pi].$$

Here, a and b are the semi-major and semi-minor axes, respectively.
Steps:

1. Compute $x\,dy$:

$$x = a\cos t, \quad dy = \frac{d}{dt}(b\sin t)\,dt = b\cos t\,dt.$$

Thus:

$$x\,dy = a\cos t \cdot b\cos t\,dt = ab\cos^2 t\,dt.$$

2. Compute $y\,dx$:

$$y = b\sin t, \quad dx = \frac{d}{dt}(a\cos t)\,dt = -a\sin t\,dt.$$

Thus:

$$y\,dx = b\sin t \cdot (-a\sin t)\,dt = -ab\sin^2 t\,dt.$$

3. Compute $x\,dy - y\,dx$:

$$x\,dy - y\,dx = ab\cos^2 t\,dt - (-ab\sin^2 t\,dt) = ab(\cos^2 t + \sin^2 t)\,dt = ab\,dt.$$

4. Compute the Line Integral:

$$\text{Area}(R) = \frac{1}{2}\int_0^{2\pi} ab\,dt = \frac{1}{2}ab\int_0^{2\pi} 1\,dt = \frac{1}{2}ab \cdot 2\pi = \pi ab.$$

Conclusion: The area of the ellipse is:

$$\text{Area}(R) = \pi ab,$$

where a and b are the semi-major and semi-minor axes.

Summary

The area of a region R enclosed by a curve C can be computed using line integrals in multiple equivalent ways:

- Directly using $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = (-y, x)$.
- Using $\int_C x\,dy$ or $-\int_C y\,dx$.
- Using the symmetric formula:

$$\text{Area}(R) = \frac{1}{2}\int_C (x\,dy - y\,dx).$$

2.10 Flux of a Vector Field

Definition of Flux

Flux represents the flow of a vector field $\mathbf{F} = (f, g)$ across an oriented curve C. It measures how much of $\mathbf{F}$ passes through C in the normal direction.

The flux of $\mathbf{F}$ across C is defined as:

$$\text{Flux} = \int_C \mathbf{F}\cdot\mathbf{n}\,ds,$$

where:

- $\mathbf{n}$ is the unit normal vector to the curve C.
- ds is the arc length differential.

For a parameterized curve $\mathbf{r}(t) = (x(t), y(t))$, where $t \in [a, b]$, the unit normal vector is given by:

$$\mathbf{n} = \frac{1}{|\mathbf{r}'(t)|}(-y'(t), x'(t)).$$

Thus:

$$\mathbf{F}\cdot\mathbf{n} = \frac{1}{|\mathbf{r}'(t)|}\left[f(x(t), y(t))(-y'(t)) + g(x(t), y(t))x'(t)\right].$$

Flux in Simplified Form

If the curve C is parameterized, the flux integral can be simplified as:

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_a^b \left[f(x(t), y(t))y'(t) - g(x(t), y(t))x'(t) \right] dt.$$

Or equivalently:

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C f\, dy - g\, dx.$$

Green's Theorem Applied to Flux

By applying Green's Theorem, the flux of $\mathbf{F} = (f, g)$ across C can also be related to a double integral over the region R enclosed by C:

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \operatorname{div}(\mathbf{F})\, dA,$$

where:

$$\operatorname{div}(\mathbf{F}) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}.$$

This shows that the flux across the boundary C is equal to the total divergence of $\mathbf{F}$ inside R.

Example 2.10.1 (Example: Flux Around a Circle)
Let $\mathbf{F}(x, y) = (x, y)$, and let C be the circle $x^2 + y^2 = 1$ parameterized by:

$$\mathbf{r}(t) = (\cos t, \sin t), \quad t \in [0, 2\pi].$$

Step 1: Parameterize the Flux Integral. The unit normal vector to C is given by:

$$\mathbf{n} = (-\sin t, \cos t),$$

and $\mathbf{F}(\mathbf{r}(t)) = (\cos t, \sin t)$. Then:

$$\mathbf{F} \cdot \mathbf{n} = \cos t(-\sin t) + \sin t(\cos t) = 0.$$

Step 2: Apply Green's Theorem. Using Green's Theorem:

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_R \operatorname{div}(\mathbf{F})\, dA.$$

The divergence of $\mathbf{F} = (x, y)$ is:

$$\operatorname{div}(\mathbf{F}) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 1 + 1 = 2.$$

The area of the circle is:

$$\operatorname{Area}(R) = \pi \cdot 1^2 = \pi.$$

Thus:

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_R 2\, dA = 2 \cdot \pi = 2\pi.$$

Final Answer: The flux across the circle is:

$$\text{Flux} = 2\pi.$$