MA193
Discrete Mathematics

Giacomo Cappelletto

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Contents

Chapter 1

Fundamental Principles of Counting

1.1 Counting with Repetitions

Note:-

These notes cover the basic counting principles (often called the "Rule of Sum" and the "Rule of Product"), along with a brief discussion of permutations and combinations, including the formula for permutations of multiset objects (i.e., repeated elements).

Definition 1.1.1: Rule of Sum ("OR")

If a certain task can be done in n ways and another independent task can be done in m ways, and these tasks are mutually exclusive, then there are $n + m$ ways to do *one* of the two tasks.

Definition 1.1.2: Rule of Product ("AND")

If a procedure can be broken into two consecutive steps such that the first step can be done in n ways and the second step can be done in m ways (independently of how the first step is done), then there are $n \times m$ ways to do the entire procedure.

Note:-

In many counting problems, we break a larger procedure into a series of smaller steps and then apply either the Rule of Sum or the Rule of Product (or both) as needed.

Definition 1.1.3: Arrangements of n Distinct Objects

If you want to arrange n distinct objects in a row (i.e., an ordered list), there are $n!$ ways to do so. Order matters here, and this number is referred to as the number of permutations of n distinct items.

Definition 1.1.4: Permutations of Multisets

Suppose we have n total objects, but they are not all distinct. Instead, let there be n_1 objects of type 1, n_2 objects of type 2, …, and n_k objects of type k. Clearly

$$n_1 + n_2 + \cdots + n_k = n.$$

Then the number of distinct ways to arrange all n objects is

$$\frac{n!}{n_1!\, n_2! \ldots n_k!}.$$

Example 1.1.1 (Examples of Counting with Repetitions)

1. **ABCD:** All letters are distinct, so the number of ways to arrange A, B, C, D is
$$4! = 24.$$
2. **AABC:** Here we have 4 total letters, with A repeated twice. The number of distinct arrangements is
$$\frac{4!}{2!} = \frac{24}{2} = 12.$$
3. **AABB:** Now we have 2 A's and 2 B's (4 letters total). The number of arrangements is
$$\frac{4!}{2!\,2!} = \frac{24}{2 \times 2} = 6.$$
4. **SUCCESS:** The word "SUCCESS" has 7 letters total: 3 S's, 2 C's, 1 U, and 1 E. The number of distinct permutations is
$$\frac{7!}{3!\,2!\,1!\,1!} = \frac{5040}{(6)(2)(1)(1)} = 420.$$

Committee-Choosing Problems

Definition 1.1.5: Combinations

A *combination* is a way to choose r objects from n distinct objects where order does *not* matter. The number of ways to do so is given by
$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}.$$
In many committee-selection problems, we use combinations because the particular order in which people are chosen does not matter.

Definition 1.1.6: When to Use Permutations vs. Combinations

- *Permutations* ($P(n,r)$ or $\frac{n!}{(n-r)!}$) are used when we care about the *order* of the chosen items (for instance, arranging people in a line for a photo).
- *Combinations* ($\binom{n}{r}$) are used when we only care about *which* items are chosen, not the order in which they appear (e.g., forming committees).

Definition 1.1.7: Basic Subset Counting

Note that the total number of distinct subsets of a set with n elements is 2^n. This comes from the fact that, for each element, we independently choose to include it or not in a given subset. For an r-element subset, specifically, we use $\binom{n}{r}$.

Example 1.1.2 (Choosing a Simple Committee)

Suppose we have 8 people in a group, and we want to choose a committee of 3 individuals. Since order does not matter, the number of ways to choose the committee is
$$\binom{8}{3} = \frac{8!}{3!(8-3)!} = \frac{8!}{3!5!} = 56.$$

Example 1.1.3 (Committee with Restrictions: Gender Balance)

Imagine we have 5 men and 6 women, and we want to form a committee of 4 people that has at least 2 women. We can break it down by the number of women selected:

$$\binom{6}{2}\binom{5}{2} + \binom{6}{3}\binom{5}{1} + \binom{6}{4}\binom{5}{0}.$$

Here, $\binom{6}{k}$ chooses the women, and $\binom{5}{4-k}$ chooses the men for each scenario. Summing these counts gives the total number of ways to form such a committee.

Example 1.1.4 (Committee with Subgroups Required)

Say we have 10 people, of whom 3 are Teaching Assistants (TAs) and 7 are Professors, and we wish to form a committee of 4 people *with exactly 2 TAs*. We select 2 from the 3 TAs and 2 from the 7 Professors, yielding

$$\binom{3}{2} \times \binom{7}{2}$$

possible committees.

Example 1.1.5 (Larger Committees: Multiple Constraints)

Suppose there are 12 people divided into three categories: 4 from group A, 5 from group B, and 3 from group C. We want a committee of 5 that has at least 1 person from each group. One way to count is to enumerate possible splits of 5 among (A, B, C), ensuring each category has at least one representative. For instance:

- 1 from A, 3 from B, 1 from C,
- 1 from A, 2 from B, 2 from C,
- 2 from A, 2 from B, 1 from C,
- and so forth,

and sum the corresponding products of binomial coefficients $\binom{4}{\dots}\binom{5}{\dots}\binom{3}{\dots}$.

Note:-

These examples illustrate the typical approaches to *committee choosing* problems:

- Identify if order matters (usually it does not, hence combinations).
- If there are constraints (e.g. minimum number of members from a certain group), split the problem into valid cases and sum their respective counts.

Example 1.1.6 (Probability of Getting a Flush in Poker)

A *flush* in poker is a hand where all 5 cards are of the same suit. To calculate the probability of being dealt a flush, we proceed as follows:

- There are 4 suits in a deck (hearts, diamonds, clubs, spades).
- For each suit, there are $\binom{13}{5}$ ways to choose 5 cards from the 13 available.
- Thus, the total number of ways to get a flush is $4 \times \binom{13}{5}$.
- The total number of 5-card hands from a 52-card deck is $\binom{52}{5}$.

Therefore, the probability P of being dealt a flush is given by

$$P(\text{flush}) = \frac{\binom{4}{1} \times \binom{13}{5}}{\binom{52}{5}}.$$

So, the probability of being dealt a flush in poker is approximately 0.198%.