

Lecture Notes
DP Grand Resume

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Chapter 1

Algebra

1.1 Arithmetic and Geometric Sequences

Definition 1.1.1: Arithmetic Sequence

An **arithmetic sequence** is a sequence of numbers in which the difference between consecutive terms is constant. If a_1 is the first term and d is the common difference, the n -th term a_n of an arithmetic sequence is given by:

$$a_n = a_1 + (n - 1)d$$

The sum S_n of the first n terms of an arithmetic sequence is:

$$S_n = \frac{n}{2} (2a_1 + (n - 1)d)$$

or equivalently:

$$S_n = \frac{n}{2} (a_1 + a_n)$$

Example 1.1.1 (Example of an Arithmetic Sequence)

Consider the arithmetic sequence with the first term $a_1 = 3$ and common difference $d = 5$:

$$3, 8, 13, 18, \dots$$

The 10-th term is:

$$a_{10} = 3 + (10 - 1) \times 5 = 3 + 45 = 48$$

The sum of the first 10 terms is:

$$S_{10} = \frac{10}{2} (3 + 48) = 5 \times 51 = 255$$

Definition 1.1.2: Geometric Sequence

A **geometric sequence** is a sequence of numbers where each term after the first is obtained by multiplying the previous term by a fixed, non-zero number called the **common ratio** r . If a_1 is the first term, the n -th term a_n of a geometric sequence is given by:

$$a_n = a_1 \times r^{n-1}$$

The sum S_n of the first n terms of a geometric sequence is:

$$S_n = a_1 \frac{1 - r^n}{1 - r} \quad \text{for } r \neq 1$$

Example 1.1.2 (Example of a Geometric Sequence)

Consider the geometric sequence with the first term $a_1 = 2$ and common ratio $r = 3$:

$$2, 6, 18, 54, \dots$$

The 6-th term is:

$$a_6 = 2 \times 3^{6-1} = 2 \times 243 = 486$$

The sum of the first 6 terms is:

$$S_6 = 2 \times \frac{1 - 3^6}{1 - 3} = 2 \times \frac{1 - 729}{-2} = 2 \times 364 = 728$$

Definition 1.1.3: Infinite Sum of a Converging Geometric Sequence

If the absolute value of the common ratio r of a geometric sequence is less than 1 ($|r| < 1$), the sequence converges, and the infinite sum S of the sequence can be calculated as:

$$S = \frac{a_1}{1 - r} \quad \text{for } |r| < 1$$

Example 1.1.3 (Example of an Infinite Sum of a Converging Geometric Sequence)

Consider the geometric sequence with the first term $a_1 = 4$ and common ratio $r = \frac{1}{2}$:

$$4, 2, 1, 0.5, \dots$$

The infinite sum of this sequence is:

$$S = \frac{4}{1 - \frac{1}{2}} = \frac{4}{\frac{1}{2}} = 8$$

1.2 Laws of Logarithms

Definition 1.2.1: Laws of Logarithms

- **Product Law:**

$$\log_b(xy) = \log_b x + \log_b y$$

- **Quotient Law:**

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

- **Power Law:**

$$\log_b(x^k) = k \log_b x$$

- **Change of Base Formula:**

$$\log_b x = \frac{\log_a x}{\log_a b}$$

- **Logarithm of 1:**

$$\log_b 1 = 0$$

- **Logarithm of the Base:**

$$\log_b b = 1$$

- **Inverse Property:**

$$\log_b(b^x) = x \quad \text{and} \quad b^{\log_b x} = x$$

1.3 Binomial Theorem

Definition 1.3.1: Binomial Theorem

The Binomial Theorem provides a formula for expanding the power of a binomial expression. For any positive integer n , the Binomial Theorem states:

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

where $\binom{n}{k}$ is the binomial coefficient, defined as:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Definition 1.3.2: Alternative Form

The Binomial Theorem can also be expressed in an alternative form:

$$(x + y)^n = x^n \left(1 + n \left(\frac{y}{x} \right) + \frac{n(n-1)}{2!} \left(\frac{y}{x} \right)^2 + \dots \right)$$

1.4 Partial Fractions

Definition 1.4.1: Partial Fractions

Partial fraction decomposition is a method used to break down a complex rational function into simpler fractions that are easier to integrate or differentiate. This technique is particularly useful in calculus and algebra.

Example 1.4.1 (How to solve Partial Fractions)

Step 1: Factor the Denominator

The first step is to factor the denominator of the rational function completely. The form of the partial fraction decomposition depends on the types of factors in the denominator:

- If the denominator contains distinct linear factors, such as $(x - a)(x - b)$, the partial fractions will have the form:

$$\frac{A}{x - a} + \frac{B}{x - b}$$

- If the denominator contains repeated linear factors, such as $(x - a)^2$, the partial fractions will have the form:

$$\frac{A}{x - a} + \frac{B}{(x - a)^2}$$

- If the denominator contains an irreducible quadratic factor, such as $x^2 + bx + c$, the partial fraction will have the form:

$$\frac{Ax + B}{x^2 + bx + c}$$

Step 2: Set Up the Partial Fraction Decomposition

Write the original fraction as a sum of fractions with unknown coefficients. For example, consider the function:

$$\frac{3x + 5}{(x - 1)(x + 2)}$$

The partial fraction decomposition would be:

$$\frac{3x + 5}{(x - 1)(x + 2)} = \frac{A}{x - 1} + \frac{B}{x + 2}$$

Step 3: Multiply Through by the Common Denominator

Multiply both sides of the equation by the common denominator to eliminate the denominators:

$$3x + 5 = A(x + 2) + B(x - 1)$$

Step 4: Solve for the Coefficients

Expand and collect like terms on the right-hand side:

$$3x + 5 = Ax + 2A + Bx - B$$

Group the terms by powers of x :

$$3x + 5 = (A + B)x + (2A - B)$$

Now, equate the coefficients of like terms (the coefficients of x and the constant term) on both sides:

$$A + B = 3$$

$$2A - B = 5$$

Solve this system of equations to find A and B . For example:

$$A + B = 3 \quad (1)$$

$$2A - B = 5 \quad (2)$$

Add equations (1) and (2) to eliminate B :

$$3A = 8 \Rightarrow A = \frac{8}{3}$$

Substitute $A = \frac{8}{3}$ back into equation (1) to find B :

$$\frac{8}{3} + B = 3 \Rightarrow B = 3 - \frac{8}{3} = \frac{1}{3}$$

Thus, the partial fraction decomposition is:

$$\frac{3x + 5}{(x - 1)(x + 2)} = \frac{\frac{8}{3}}{x - 1} + \frac{\frac{1}{3}}{x + 2}$$

Step 5: Integrate or Use the Decomposition as Needed

Once the partial fractions are found, they can be used for further calculations, such as integration:

$$\int \frac{3x + 5}{(x - 1)(x + 2)} dx = \int \left(\frac{\frac{8}{3}}{x - 1} + \frac{\frac{1}{3}}{x + 2} \right) dx$$

Integrating each term separately gives:

$$\frac{8}{3} \ln|x - 1| + \frac{1}{3} \ln|x + 2| + C$$

1.5 Proof by Induction

Definition 1.5.1: Proof by Induction

Proof by induction is a method of mathematical proof typically used to establish that a given statement is true for all natural numbers. The process involves two main steps:

- **Base Case:** Prove that the statement is true for the initial value (usually $n = 1$ or $n = 0$).
- **Inductive Step:** Assume that the statement holds for some arbitrary positive integer k , and then prove that the statement is true for $k + 1$.

If both steps are successfully completed, the statement is proven true for all natural numbers.

Example 1.5.1 (Challenging Example: Proving a Sum of Powers)

Let's prove by induction that for all $n \geq 1$, the following sum of powers holds:

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n + 1)}{2} \right)^2$$

Base Case:

For $n = 1$, the left-hand side is:

$$1^3 = 1$$

The right-hand side is:

$$\left(\frac{1(1 + 1)}{2} \right)^2 = \left(\frac{1 \times 2}{2} \right)^2 = 1^2 = 1$$

So, the statement is true for $n = 1$.

Inductive Step:

Assume that the statement holds for some $k \geq 1$, i.e., assume:

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = \left(\frac{k(k+1)}{2} \right)^2$$

We need to prove that the statement holds for $k+1$, i.e.:

$$1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2} \right)^2$$

By the inductive hypothesis:

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = \left(\frac{k(k+1)}{2} \right)^2$$

Therefore, adding $(k+1)^3$ to both sides gives:

$$1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \left(\frac{k(k+1)}{2} \right)^2 + (k+1)^3$$

We need to show that:

$$\left(\frac{k(k+1)}{2} \right)^2 + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2} \right)^2$$

Let's simplify the right-hand side:

$$\left(\frac{(k+1)(k+2)}{2} \right)^2 = \left(\frac{k^2 + 3k + 2}{2} \right)^2$$

Expanding both sides:

$$\left(\frac{k(k+1)}{2} \right)^2 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3$$

Factor out $(k+1)^2$ on the left-hand side:

$$\begin{aligned} \frac{k^2(k+1)^2}{4} + (k+1)^3 &= (k+1)^2 \left(\frac{k^2}{4} + (k+1) \right) \\ &= (k+1)^2 \left(\frac{k^2 + 4k + 4}{4} \right) = \frac{(k+1)^2(k+2)^2}{4} \end{aligned}$$

Which simplifies to:

$$\left(\frac{(k+1)(k+2)}{2} \right)^2$$

Thus, the statement holds for $k+1$ if it holds for k .

Conclusion:

Since the base case is true, and the inductive step has been proven, by the principle of mathematical induction, the statement is true for all $n \geq 1$.

1.6 Complex Numbers

Definition 1.6.1: Definition of Complex Numbers

A **complex number** is a number of the form:

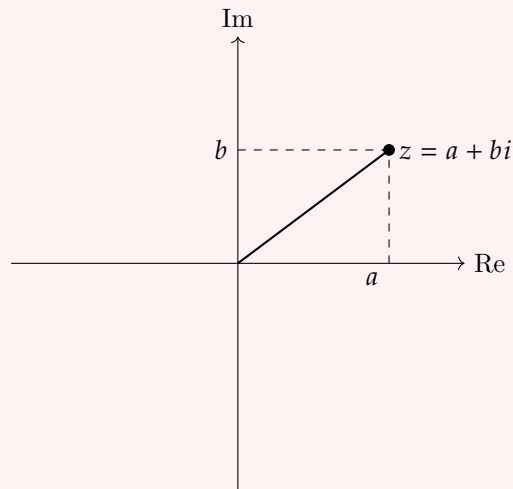
$$z = a + bi$$

where a and b are real numbers, and i is the imaginary unit, defined by the property $i^2 = -1$. In this expression, a is called the **real part** of z , denoted by $\Re(z)$, and b is called the **imaginary part** of z , denoted by $\Im(z)$.

$$\Re(z) = a \quad \Im(z) = b$$

Definition 1.6.2: Depiction on the Complex Plane

A complex number $z = a + bi$ can be represented as a point in the complex plane, where the horizontal axis (Real axis) represents the real part a and the vertical axis (Imaginary axis) represents the imaginary part b .

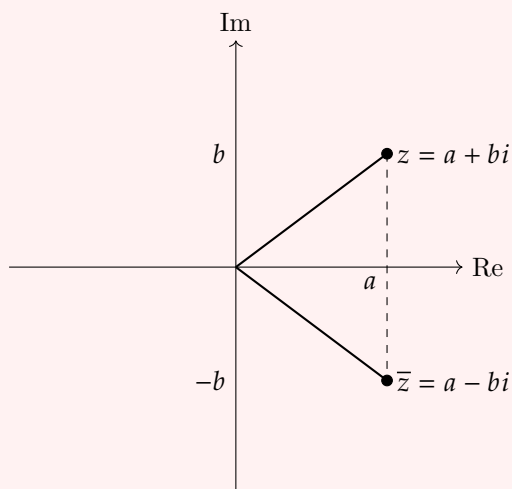


Definition 1.6.3: Conjugate of a Complex Number

The **conjugate** of a complex number $z = a + bi$ is given by:

$$\bar{z} = a - bi$$

Geometrically, the conjugate of z is the reflection of z across the real axis in the complex plane.



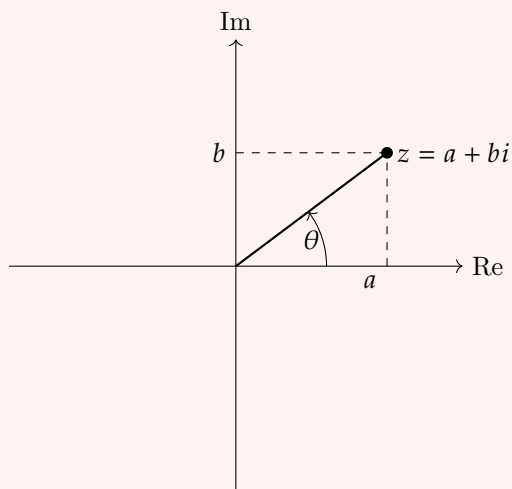
Definition 1.6.4: Modulus and Argument of a Complex Number

The **modulus** (or magnitude) of a complex number $z = a + bi$ is defined as:

$$|z| = \sqrt{a^2 + b^2}$$

The **argument** of z , denoted by $\arg(z)$, is the angle θ that the line connecting the origin to the point z makes with the positive real axis, measured counterclockwise. It can be calculated using:

$$\theta = \arg(z) = \tan^{-1}\left(\frac{b}{a}\right)$$



Example 1.6.1 (Example of Calculating Modulus and Argument)

Let $z = 3 + 4i$. Then:

- The **modulus** of z is:

$$|z| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

- The **argument** of z is:

$$\theta = \arg(z) = \tan^{-1} \left(\frac{4}{3} \right) \approx 53.13^\circ$$

Definition 1.6.5: Polar Form

A complex number can be converted from Cartesian form to polar form, where the number is represented by its modulus r and argument θ :

$$z = r(\cos \theta + i \sin \theta)$$

Here:

- The modulus r is the distance from the origin to the point (a, b) , given by:

$$r = \sqrt{a^2 + b^2}$$

- The argument θ is the angle formed with the positive real axis, given by:

$$\theta = \arg(z) = \tan^{-1} \left(\frac{b}{a} \right)$$

The polar form expresses the complex number in terms of its magnitude and direction in the complex plane.

Definition 1.6.6: Euler's Form

Using Euler's formula, the polar form of a complex number can be rewritten in its Euler form:

$$z = re^{i\theta}$$

where:

$$e^{i\theta} = \cos \theta + i \sin \theta$$

This is a compact and powerful way to represent complex numbers, especially useful in various applications such as signal processing and quantum mechanics.

Example 1.6.2 (Example of Conversion: Cartesian to Polar to Euler)

Consider the complex number $z = 1 + \sqrt{3}i$.

Step 1: Convert to Polar Form

First, calculate the modulus r :

$$r = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1 + 3} = \sqrt{4} = 2$$

Next, calculate the argument θ :

$$\theta = \arg(z) = \tan^{-1} \left(\frac{\sqrt{3}}{1} \right) = \frac{\pi}{3}$$

So, the polar form of z is:

$$z = 2 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$$

Step 2: Convert to Euler Form

Using Euler's formula, the complex number in Euler form is:

$$z = 2e^{i\frac{\pi}{3}}$$

Definition 1.6.7: De Moivre's Theorem

De Moivre's theorem states that for any real number θ and any integer n , the following holds:

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta)$$

This theorem is particularly useful for finding powers and roots of complex numbers.

Definition 1.6.8: Proof of De Moivre's Theorem

We will prove De Moivre's theorem by using mathematical induction on the integer n .

Base Case:

For $n = 1$, the theorem clearly holds since:

$$(\cos \theta + i \sin \theta)^1 = \cos \theta + i \sin \theta$$

which is indeed equal to $\cos(1\theta) + i \sin(1\theta)$.

Inductive Step:

Assume that the theorem holds for some integer $k \geq 1$, i.e., assume:

$$(\cos \theta + i \sin \theta)^k = \cos(k\theta) + i \sin(k\theta)$$

We need to prove that the theorem holds for $k + 1$, i.e., we need to show:

$$(\cos \theta + i \sin \theta)^{k+1} = \cos((k+1)\theta) + i \sin((k+1)\theta)$$

Using the inductive hypothesis, we can write:

$$(\cos \theta + i \sin \theta)^{k+1} = (\cos \theta + i \sin \theta) \cdot (\cos(k\theta) + i \sin(k\theta))$$

Now, apply the distributive property and use the trigonometric identities $\cos A \cos B - \sin A \sin B = \cos(A + B)$ and $\cos A \sin B + \sin A \cos B = \sin(A + B)$ to expand the product:

$$\begin{aligned} (\cos \theta + i \sin \theta) \cdot (\cos(k\theta) + i \sin(k\theta)) &= \cos \theta \cos(k\theta) - \sin \theta \sin(k\theta) + i (\cos \theta \sin(k\theta) + \sin \theta \cos(k\theta)) \\ &= \cos(\theta + k\theta) + i \sin(\theta + k\theta) \\ &= \cos((k+1)\theta) + i \sin((k+1)\theta) \end{aligned}$$

This proves that the theorem holds for $k + 1$ if it holds for k .

Conclusion:

Since the base case holds and the inductive step has been proven, by the principle of mathematical induction, De Moivre's theorem is true for all positive integers n .

Example 1.6.3 (Example Application)

As an example, let's apply De Moivre's theorem to find $(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})^3$.

According to De Moivre's theorem:

$$\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)^3 = \cos\left(3 \times \frac{\pi}{4}\right) + i \sin\left(3 \times \frac{\pi}{4}\right)$$

Simplifying, we get:

$$\cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right)$$

which corresponds to:

$$-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

Hence, $\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)^3 = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$.

Definition 1.6.9: Definition

The n th roots of a complex number z are the solutions to the equation $w^n = z$, where w is a complex number. For a given complex number z , there are n distinct n th roots. Suppose z is expressed in polar form as:

$$z = r(\cos \theta + i \sin \theta)$$

The n th roots of z are given by:

$$w_k = \sqrt[n]{r} \left(\cos \frac{\theta + 2k\pi}{n} + i \sin \frac{\theta + 2k\pi}{n} \right)$$

for $k = 0, 1, 2, \dots, n-1$.

Alternatively, using Euler's form $z = re^{i\theta}$, the n th roots are:

$$w_k = \sqrt[n]{r} e^{i\left(\frac{\theta + 2k\pi}{n}\right)}$$

for $k = 0, 1, 2, \dots, n-1$.

Example 1.6.4 (Example: Finding the Cube Roots of a Complex Number)

Let's find the cube roots of the complex number $z = 8\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$.

First, express z in polar form:

$$z = 8e^{i\frac{\pi}{4}}$$

The cube roots of z are given by:

$$w_k = \sqrt[3]{8} \left(\cos \frac{\frac{\pi}{4} + 2k\pi}{3} + i \sin \frac{\frac{\pi}{4} + 2k\pi}{3} \right)$$

where $k = 0, 1, 2$.

Since $\sqrt[3]{8} = 2$, the roots are:

$$w_0 = 2 \left(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right)$$

$$w_1 = 2 \left(\cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12} \right)$$

$$w_2 = 2 \left(\cos \frac{9\pi}{12} + i \sin \frac{9\pi}{12} \right)$$

Alternatively, in Euler form:

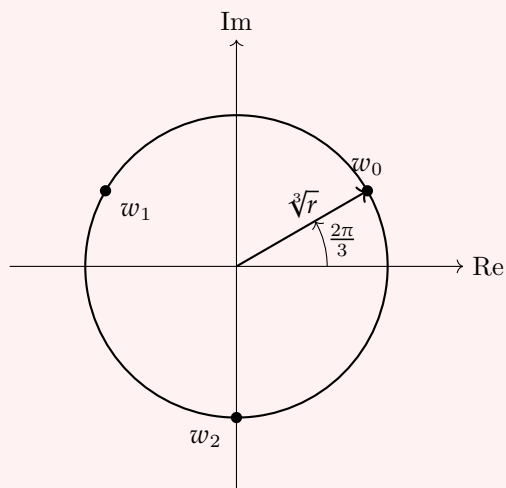
$$w_0 = 2e^{i\frac{\pi}{12}}, \quad w_1 = 2e^{i\frac{5\pi}{12}}, \quad w_2 = 2e^{i\frac{9\pi}{12}}$$

Thus, the three distinct cube roots of $z = 8\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$ are:

$$w_0 = 2e^{i\frac{\pi}{12}}, \quad w_1 = 2e^{i\frac{5\pi}{12}}, \quad w_2 = 2e^{i\frac{9\pi}{12}}$$

Definition 1.6.10: Geometric Interpretation

The n th roots of a complex number z can be interpreted geometrically as points on the complex plane that are evenly spaced around a circle of radius $\sqrt[n]{r}$, where r is the modulus of z . The angle between consecutive roots is $\frac{2\pi}{n}$.

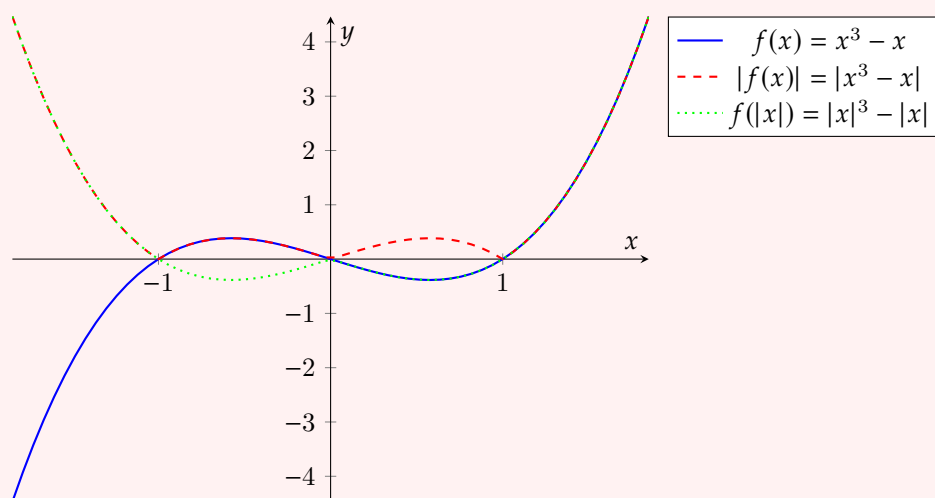


Chapter 2

Functions

2.1 Modulus Functions

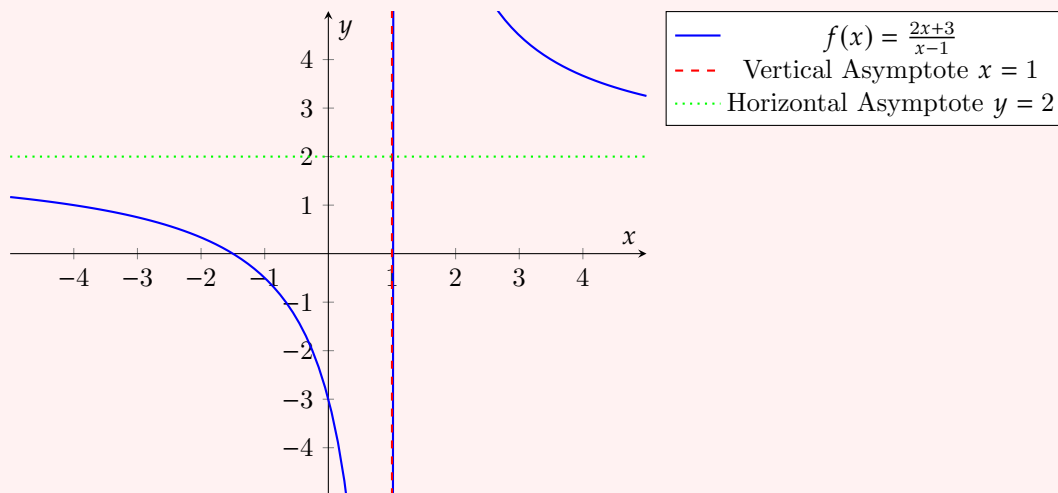
Definition 2.1.1: Modulus function



2.2 Asymptotes

Definition 2.2.1: Asymptotes in Rational Functions

Asymptotes are lines that a function approaches as its input approaches a specific value or infinity. They can be horizontal, vertical, or oblique (with a slope).



For some function:

$$f(x) = \frac{ax + b}{cx - d}$$

Vertical asymptote:

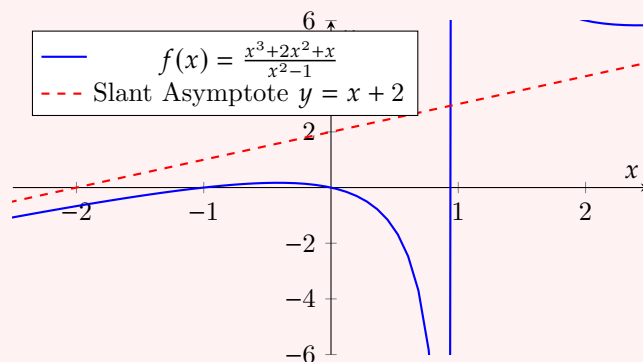
$$x = \frac{-d}{c}$$

Horizontal asymptote:

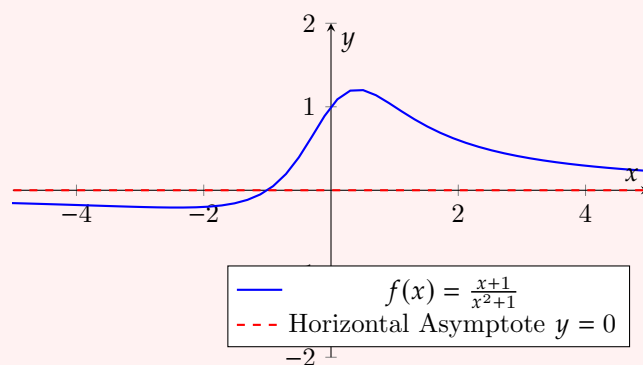
$$y = \frac{a}{c}$$

Definition 2.2.2: Asymptotes in Division of Polynomials

If the degree of the numerator is greater than the degree of the denominator, there is no horizontal asymptote.



To find the oblique asymptote, do the division of the numerator and denominator polynomials
 If the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is $y = 0$



If the degree of the numerator is equal to the degree of the denominator, do as for rational functions

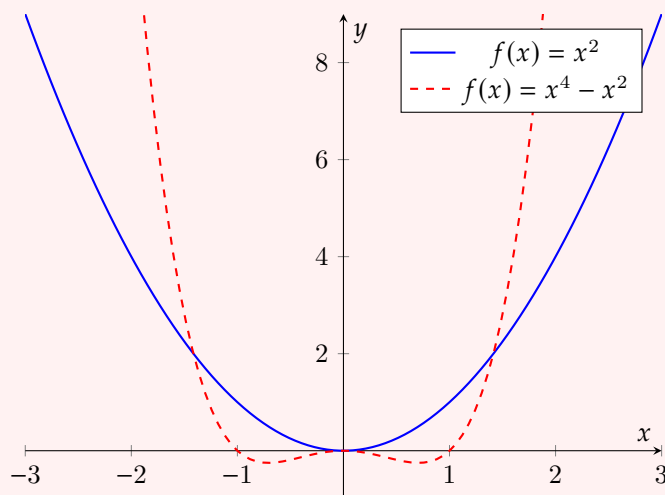
2.3 Odd and Even Functions

Definition 2.3.1: Even Functions

A function $f(x)$ is called an **even function** if it satisfies the condition:

$$f(-x) = f(x) \quad \text{for all } x \text{ in the domain of } f.$$

Geometrically, the graph of an even function is symmetric with respect to the y -axis.

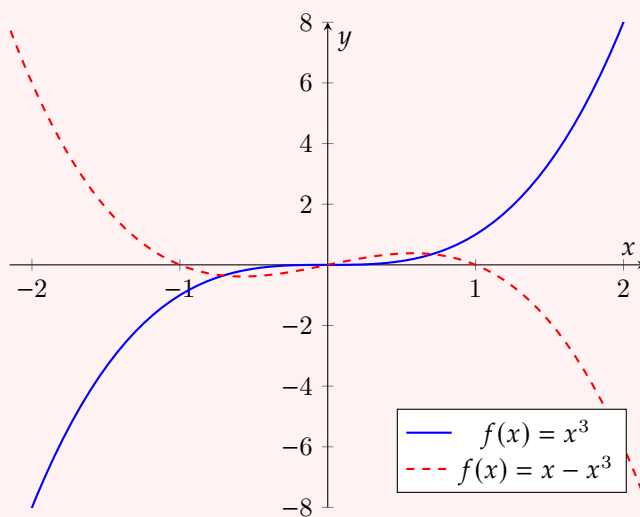


Definition 2.3.2: Odd Functions

A function $f(x)$ is called an **odd function** if it satisfies the condition:

$$f(-x) = -f(x) \quad \text{for all } x \text{ in the domain of } f.$$

Geometrically, the graph of an odd function is symmetric with respect to the origin.



2.4 Reciprocal Functions

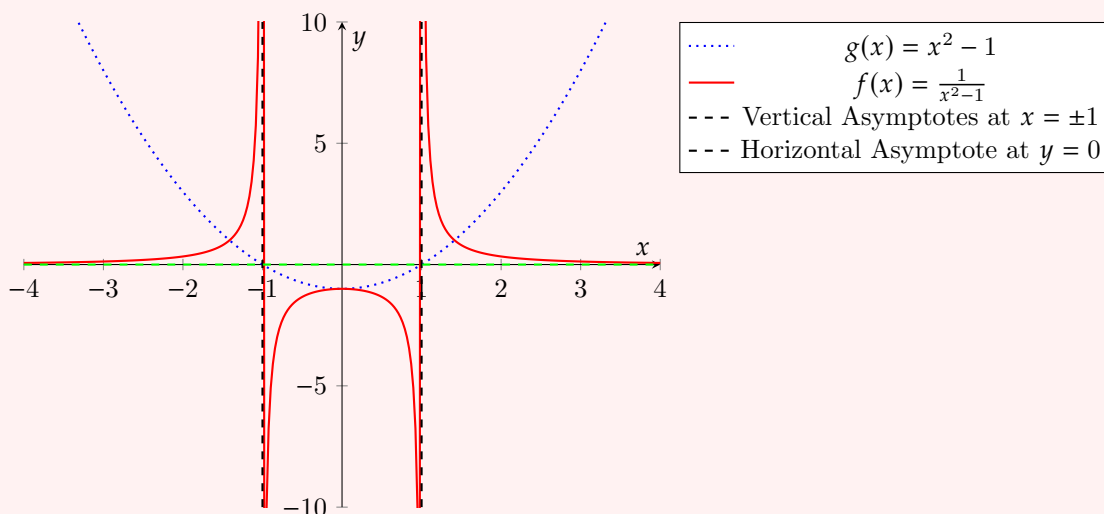
Definition 2.4.1: Reciprocal Function

A **reciprocal function** is a function of the form:

$$f(x) = \frac{1}{g(x)}$$

where $g(x)$ is a non-zero function. The most basic example of a reciprocal function is $f(x) = \frac{1}{x}$. The behavior of a reciprocal function depends heavily on the nature of $g(x)$:

- **Zeros of $g(x)$:** These correspond to vertical asymptotes in $f(x)$ because the function becomes undefined at these points.
- **Maxima and Minima of $g(x)$:** These correspond to minima and maxima of $f(x)$, respectively, but with inverted values.
- **Asymptotes:** If $g(x)$ has a horizontal asymptote, $f(x)$ will have a horizontal asymptote at the reciprocal of this value.



Example 2.4.1 (Example of a Reciprocal Function)

Consider the reciprocal function:

$$f(x) = \frac{1}{x^2 - 1}$$

Let's analyze its key features:

- **Zeros of $g(x)$:** The zeros of $g(x) = x^2 - 1$ are $x = \pm 1$. These correspond to vertical asymptotes in $f(x)$.
- **Minima and Maxima:** The function $g(x)$ has a minimum at $x = 0$, where $g(0) = -1$. The reciprocal function $f(x)$ has a corresponding maximum at $x = 0$, where $f(0) = -1$.
- **Horizontal Asymptote:** As x approaches infinity, $g(x)$ approaches infinity, so $f(x)$ approaches 0, leading to a horizontal asymptote at $y = 0$.

2.5 Factor and Remainder Theorems

Definition 2.5.1: Remainder Theorem

The Remainder Theorem states that if a polynomial $f(x)$ is divided by a linear divisor of the form $x - c$, the remainder of this division is $f(c)$. In other words, to find the remainder when dividing $f(x)$ by $x - c$, simply substitute c into the polynomial.

Example 2.5.1 (Using the Remainder Theorem)

Consider the polynomial:

$$f(x) = 2x^3 - 3x^2 + 4x - 5$$

We want to find the remainder when $f(x)$ is divided by $x - 2$.

According to the Remainder Theorem, the remainder is $f(2)$. Substituting $x = 2$ into $f(x)$ gives:

$$f(2) = 2(2)^3 - 3(2)^2 + 4(2) - 5$$

$$f(2) = 2(8) - 3(4) + 8 - 5$$

$$f(2) = 16 - 12 + 8 - 5 =$$

Thus, the remainder is 7.

Definition 2.5.2: Factor Theorem

The Factor Theorem is a special case of the Remainder Theorem. It states that a polynomial $f(x)$ has a factor $x - c$ if and only if $f(c) = 0$. This means that c is a root (or zero) of the polynomial if and only if $x - c$ is a factor of $f(x)$.

Example 2.5.2 (Using the Factor Theorem)

Let's check if $x - 3$ is a factor of the polynomial:

$$f(x) = x^3 - 7x + 6$$

According to the Factor Theorem, we need to check whether $f(3) = 0$. Substituting $x = 3$ into $f(x)$ gives:

$$f(3) = (3)^3 - 7(3) + 6$$

$$f(3) = 27 - 21 + 6 = 12$$

Since $f(3) \neq 0$, $x - 3$ is not a factor of $f(x)$.

Example 2.5.3 (Finding Factors using the Factor Theorem)

Now, let's find a factor of the polynomial:

$$g(x) = x^3 - 4x^2 + x + 6$$

We test possible values of c by substituting them into $g(x)$ until we find one that makes $g(c) = 0$.

First, let's try $x = 2$:

$$g(2) = (2)^3 - 4(2)^2 + 2 + 6$$

$$g(2) = 8 - 16 + 2 + 6 = 0$$

Since $g(2) = 0$, $x - 2$ is a factor of $g(x)$.

Thus, we can factor $g(x)$ as:

$$g(x) = (x - 2)(x^2 - 2x - 3)$$

Further factoring $x^2 - 2x - 3$ gives:

$$g(x) = (x - 2)(x - 3)(x + 1)$$

So, the complete factorization of $g(x)$ is:

$$g(x) = (x - 2)(x - 3)(x + 1)$$

2.6 Sums and Products of Roots of Polynomials

Definition 2.6.1: Sums and Products of Roots

For any polynomial of degree n with real or complex coefficients, given by:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

where $a_n \neq 0$, the roots r_1, r_2, \dots, r_n of the polynomial satisfy certain relationships, which are derived from Vieta's formulas:

- **Sum of the roots:**

$$r_1 + r_2 + \cdots + r_n = -\frac{a_{n-1}}{a_n}$$

- **Sum of the products of the roots taken two at a time:**

$$r_1 r_2 + r_1 r_3 + \cdots + r_{n-1} r_n = \frac{a_{n-2}}{a_n}$$

- **Product of the roots:**

$$r_1 r_2 \cdots r_n = (-1)^n \frac{a_0}{a_n}$$

Example 2.6.1 (Example for a Quadratic Polynomial)

Consider the quadratic polynomial:

$$P(x) = ax^2 + bx + c = 0$$

with roots r_1 and r_2 . According to Vieta's formulas:

- **Sum of the roots:**

$$r_1 + r_2 = -\frac{b}{a}$$

- **Product of the roots:**

$$r_1 r_2 = \frac{c}{a}$$

Example 2.6.2 (Example for a Cubic Polynomial)

Consider the cubic polynomial:

$$P(x) = ax^3 + bx^2 + cx + d = 0$$

with roots r_1 , r_2 , and r_3 . According to Vieta's formulas:

- **Sum of the roots:**

$$r_1 + r_2 + r_3 = -\frac{b}{a}$$

- Sum of the products of the roots taken two at a time:

$$r_1r_2 + r_1r_3 + r_2r_3 = \frac{c}{a}$$

- Product of the roots:

$$r_1r_2r_3 = -\frac{d}{a}$$

Chapter 3

Vectors and Planes

3.1 Vectors

Definition 3.1.1: Vectors

A **vector** is a mathematical object that has both magnitude and direction. Vectors can be represented in several forms:

- **Component Form:** A vector \mathbf{v} in \mathbb{R}^n can be represented by its components along the coordinate axes. In \mathbb{R}^3 , for example:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$$

where v_1, v_2, v_3 are the components, and $\hat{i}, \hat{j}, \hat{k}$ are the unit vectors along the x, y, and z axes respectively.

- **Magnitude-Direction Form:** A vector \mathbf{v} can also be represented by its magnitude (length) and direction (unit vector):

$$\mathbf{v} = |\mathbf{v}| \hat{v}$$

where $|\mathbf{v}|$ is the magnitude of \mathbf{v} , and \hat{v} is the unit vector in the direction of \mathbf{v} .

- **Position Vector:** The position vector \mathbf{r} represents the position of a point $P(x, y, z)$ in space relative to the origin:

$$\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

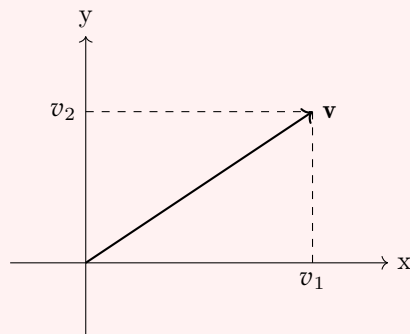
Definition 3.1.2: Vector Notation

Vectors are typically denoted by boldface letters, such as \mathbf{v} , \mathbf{a} , or \mathbf{b} , or by letters with an arrow on top, such as \vec{v} . The magnitude (or length) of a vector \mathbf{v} is denoted by $|\mathbf{v}|$ or $\|\mathbf{v}\|$.

In component form, a vector can be written as:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

or $\mathbf{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ in \mathbb{R}^3 .



3.2 Planes

Definition 3.2.1: Planes

A **plane** is a flat, two-dimensional surface that extends infinitely in all directions. There are several ways to represent a plane in \mathbb{R}^3 :

- General Form: The general equation of a plane is given by:

$$ax + by + cz = d$$

where a, b, c are the coefficients that define the orientation of the plane, and d is the distance from the origin along the normal vector $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

- Point-Normal Form: A plane can also be defined by a point $P_0(x_0, y_0, z_0)$ on the plane and a normal vector $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. The equation of the plane is:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

- Parametric Form: The plane can be represented parametrically as:

$$\mathbf{r}(s, t) = \mathbf{r}_0 + s\mathbf{v}_1 + t\mathbf{v}_2$$

where \mathbf{r}_0 is the position vector of a point on the plane, and $\mathbf{v}_1, \mathbf{v}_2$ are direction vectors lying on the plane.

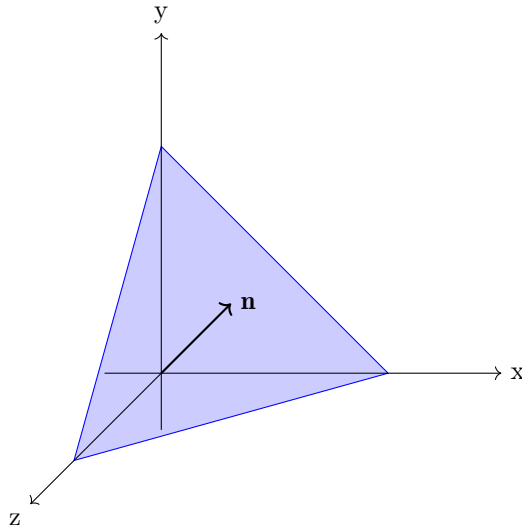
- Intercept Form: When a plane intercepts the x, y, and z axes at points a, b, c respectively, the equation of the plane can be written as:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

- ****Vector-Normal Form****: The equation of a plane can be expressed using the dot product between the position vector $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and the normal vector $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. The equation is:

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$$

where $\mathbf{a} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ is a known point on the plane.



S

3.3 Operations with Vectors

Definition 3.3.1: Scalar Product (Dot Product)

The **scalar product** (or **dot product**) of two vectors \mathbf{a} and \mathbf{b} in \mathbb{R}^n is defined as:

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

In \mathbb{R}^3 , for vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, the dot product is:

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

The dot product can also be expressed in terms of the magnitudes of \mathbf{a} and \mathbf{b} and the angle θ between them:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

The dot product is a scalar quantity and is zero when the vectors are orthogonal (perpendicular).

Definition 3.3.2: Vector Product (Cross Product)

The **vector product** (or **cross product**) of two vectors \mathbf{a} and \mathbf{b} in \mathbb{R}^3 is a vector \mathbf{c} that is perpendicular to both \mathbf{a} and \mathbf{b} , and its magnitude is given by:

$$|\mathbf{c}| = |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

where θ is the angle between \mathbf{a} and \mathbf{b} . The cross product is calculated as:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

The result of a cross product is a vector perpendicular to the plane formed by \mathbf{a} and \mathbf{b} , with a direction given by the right-hand rule.

Definition 3.3.3: Angle Between Two Vectors

The angle θ between two vectors \mathbf{a} and \mathbf{b} can be found using the dot product:

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Solving for θ :

$$\theta = \cos^{-1} \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \right)$$

This formula is valid for any non-zero vectors \mathbf{a} and \mathbf{b} .

Example 3.3.1 (Example: Dot Product and Cross Product)

Consider the vectors $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$.

Dot Product:

$$\mathbf{a} \cdot \mathbf{b} = 2(1) + (-1)(4) + 3(-2) = 2 - 4 - 6 = -8$$

Cross Product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 3 \\ 1 & 4 & -2 \end{vmatrix} = \hat{i}(-1 \times -2 - 3 \times 4) - \hat{j}(2 \times -2 - 3 \times 1) + \hat{k}(2 \times 4 - (-1) \times 1)$$

$$\mathbf{a} \times \mathbf{b} = \hat{i}(2 - 12) - \hat{j}(-4 - 3) + \hat{k}(8 + 1) = -10\hat{i} + 7\hat{j} + 9\hat{k}$$

Angle Between \mathbf{a} and \mathbf{b} :

$$|\mathbf{a}| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{4 + 1 + 9} = \sqrt{14}$$

$$|\mathbf{b}| = \sqrt{1^2 + 4^2 + (-2)^2} = \sqrt{1 + 16 + 4} = \sqrt{21}$$

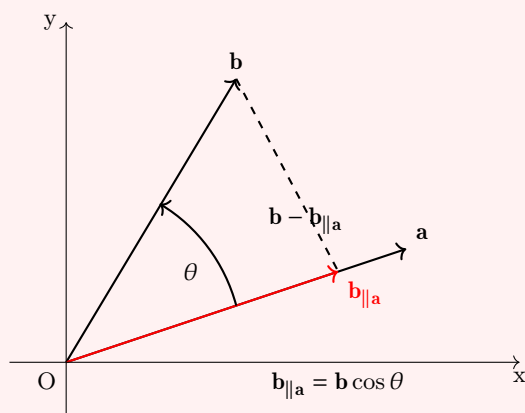
$$\cos \theta = \frac{-8}{\sqrt{14} \times \sqrt{21}}$$

Thus, the angle θ is:

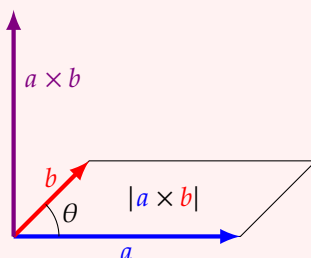
$$\theta = \cos^{-1} \left(\frac{-8}{\sqrt{14} \times \sqrt{21}} \right)$$

Definition 3.3.4: Graphical Representations

Dot Product:



Cross Product:



3.4 Distances in Geometry

Definition 3.4.1: Distance Between Two Lines

The distance between two skew lines (lines that do not intersect and are not parallel) in \mathbb{R}^3 can be found using the following method:

Consider two lines given by their parametric equations:

$$\mathbf{r}_1(t) = \mathbf{r}_0 + t\mathbf{a}, \quad \mathbf{r}_2(s) = \mathbf{r}'_0 + s\mathbf{b}$$

where \mathbf{r}_0 and \mathbf{r}'_0 are points on the respective lines, and \mathbf{a} and \mathbf{b} are direction vectors of the lines.

The distance d between the two lines is given by:

$$d = \frac{|(\mathbf{r}'_0 - \mathbf{r}_0) \cdot (\mathbf{a} \times \mathbf{b})|}{|\mathbf{a} \times \mathbf{b}|}$$

where $\mathbf{a} \times \mathbf{b}$ is the cross product of the direction vectors of the two lines, and $\mathbf{r}'_0 - \mathbf{r}_0$ is the vector connecting points on the two lines.

If the lines are parallel, $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, and the formula is not applicable. In this case, the distance is the distance between any point on one line to the other line.

Definition 3.4.2: Distance Between a Point and a Line

The distance d from a point $\mathbf{P}_0(x_0, y_0, z_0)$ to a line defined by a point $\mathbf{P}_1(x_1, y_1, z_1)$ and a direction vector $\mathbf{d} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is given by:

$$d = \frac{|\mathbf{d} \times (\mathbf{P}_0 - \mathbf{P}_1)|}{|\mathbf{d}|}$$

where $\mathbf{P}_0 - \mathbf{P}_1 = \begin{pmatrix} x_0 - x_1 \\ y_0 - y_1 \\ z_0 - z_1 \end{pmatrix}$ is the vector from the point on the line to the point \mathbf{P}_0 , and $\mathbf{d} \times (\mathbf{P}_0 - \mathbf{P}_1)$ is the cross product of \mathbf{d} and $\mathbf{P}_0 - \mathbf{P}_1$.
Alternatively, the distance can be expressed as:

$$d = \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2 + c^2}}$$

Definition 3.4.3: Distance Between a Point and a Plane

The distance d from a point $\mathbf{P}_0(x_0, y_0, z_0)$ to a plane given by the equation $ax + by + cz + d = 0$ is:

$$d = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

This formula gives the perpendicular distance from the point to the plane.

Example 3.4.1 (Example: Distance Between a Point and a Plane)

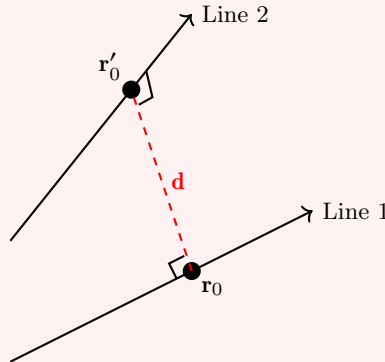
Consider the point $\mathbf{P}_0(1, 2, 3)$ and the plane given by the equation $2x - y + 2z - 4 = 0$. The distance d from the point to the plane is calculated as follows:

$$d = \frac{|2(1) - 1(2) + 2(3) - 4|}{\sqrt{2^2 + (-1)^2 + 2^2}} = \frac{|2 - 2 + 6 - 4|}{\sqrt{4 + 1 + 4}} = \frac{2}{\sqrt{9}} = \frac{2}{3}$$

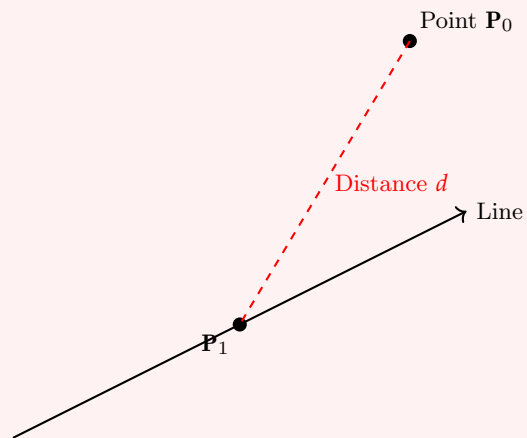
Therefore, the distance from the point $\mathbf{P}_0(1, 2, 3)$ to the plane $2x - y + 2z - 4 = 0$ is $\frac{2}{3}$.

3.5 Graphical Representations (BAD)

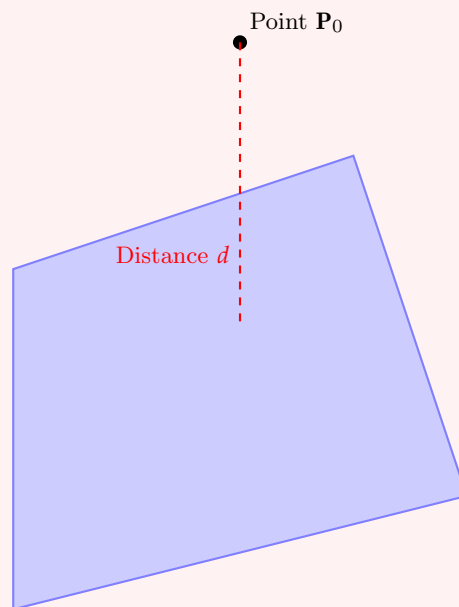
Definition 3.5.1: Distance Between Two Lines (Skew Lines)



Definition 3.5.2: Distance Between a Point and a Line



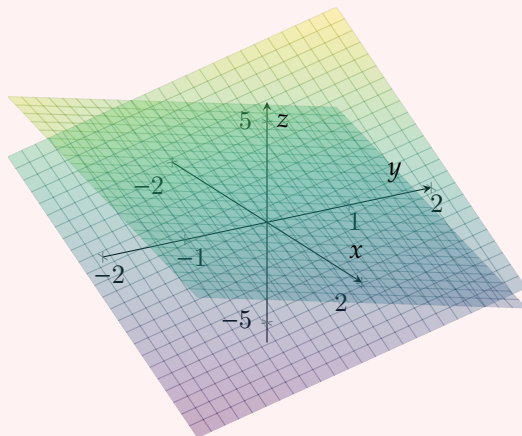
Definition 3.5.3: Distance Between a Point and a Plane



3.6 Intersections between planes

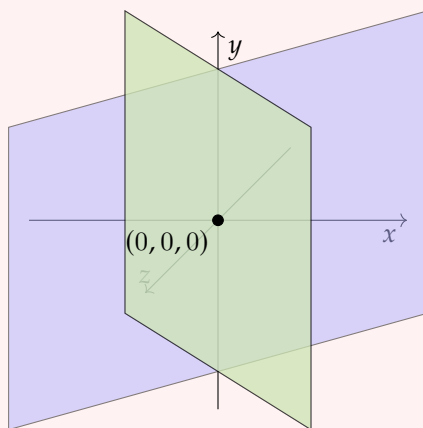
Definition 3.6.1: Intersection Between two planes

The intersection between two planes is always a line which lies in both planes. Its vector equation can be found by computing the vector product of the normal vector of the planes.



Definition 3.6.2: Intersection Between Three Planes

The intersection between three planes is always a point which lies of all three planes. Its vector equation can be found by computing the vector product of the normal vector of the planes, or all pairs of lines which are intersections between the pairs of planes.



Chapter 4

Trigonometry and Circular Functions

4.1 Basic Trigonometry

Definition 4.1.1: Radians and Degrees:

$$\begin{aligned}\text{radians} &= \frac{\pi}{180^\circ} \times \text{degrees} \\ \text{degrees} &= \frac{180^\circ}{\pi} \times \text{radians}\end{aligned}$$

Definition 4.1.2: Area of a sector:

$$\text{Area} = \frac{1}{2}r^2 \cdot \theta$$

Definition 4.1.3: Arc length:

$$\text{Arc length} = r \cdot \theta$$

θ in radians, r = radius.

Definition 4.1.4: Three-figure bearings:

Direction given as an angle of a full circle. North is 0° and the angle is expressed in the clockwise direction from North. So East is 90° , South is 180° and West is 270° .

Definition 4.1.5: Right-angle triangle (triangle with 90° angle):

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} \quad \text{SOH}$$

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \text{CAH}$$

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}} \quad \text{TOA}$$

Definition 4.1.6: Non-right angle triangles:

Sine rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Use this rule when you know: 2 angles and a side (not between the angles) or 2 sides and an angle (not between the sides).

Cosine rule:

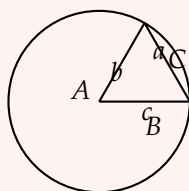
$$c^2 = a^2 + b^2 - 2ab \cos C$$

Use this rule when you know: 3 sides or 2 sides and the angle between them.

Area of a triangle:

$$\text{Area} = \frac{1}{2}ab \sin C$$

Use this rule when you know: 3 sides or 2 sides and the angle between them.



4.2 Circular Functions

Trigonometric Function:

$$y = a \sin(bx + c) + d$$

- Amplitude: a
- Period: $\frac{360^\circ}{b}$ or $\frac{2\pi}{b}$
- Horizontal shift: c
- Vertical shift: d

Trigonometric Identities:

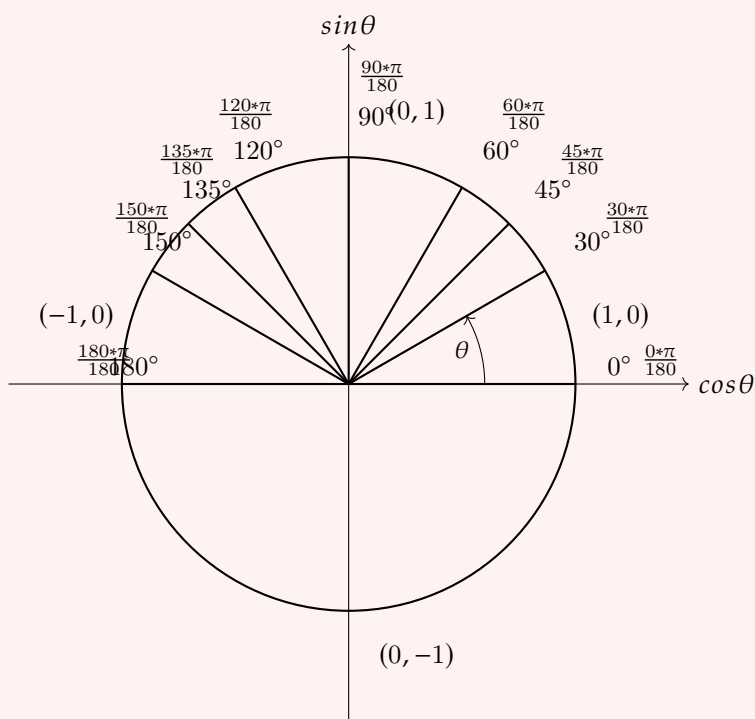
$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$2 \sin \theta \cos \theta = \sin 2\theta$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

Definition 4.2.1: Unit Circle:



Definition 4.2.2: Trigonometric Values:

deg	0°	30°	45°	60°	90°	120°	135°	150°	180°
rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\tan \theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0

4.3 Volumes and Surface Areas of 3D Shapes

Definition 4.3.1: Volumes and Surface Areas of 3D Shapes

Volume of a right pyramid

$$V = \frac{1}{3}Ah$$

Volume of a right cone

$$V = \frac{1}{3}\pi r^2 h$$

Area of the curved surface of a cone

$$A = \pi r l$$

Volume of a sphere

$$V = \frac{4}{3}\pi r^3$$

Surface area of a sphere

$$A = 4\pi r^2$$