Temporary Doc
Calc 3

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Contents

Chapter 1

Vector Valued Functions $f : \mathbb{R} \longrightarrow \mathbb{R}^n$

1.1 Absolute Extrema

For a function f defined on a region R in $\mathbb{R}^2$, a point (a, b) in R is an *absolute maximum* of f if $f(a, b) \geqslant f(x, y)$ for any (x, y) in R. Similarly, (a, b) is an *absolute minimum* if $f(a, b) \leqslant f(x, y)$ for any (x, y) in R.

Theorem 1.1.1 Extreme Value Theorem

If R is closed and bounded, and if f is continuous on R, then the absolute extrema of f on R can be found by examining:

1. Critical points of f (for local extrema within R),
2. Boundary values of f on R.

Example 1.1.1 (Example Extreme Value Theorem)

Consider the function $V(\ell, w) = \ell w(96 - \ell - w)$, where $R = \{(\ell, w) \mid \ell \geqslant 0, w \geqslant 0, \ell + w \leqslant 96\}$.

1. Since R is closed, we can apply the Extreme Value Theorem:
   (a) Find local maxima by setting $\nabla V(\ell, w) = 0$.
   (b) Evaluate V on the boundary of R.

To find critical points:

$$V(\ell, w) = \ell w(96 - \ell - w)$$

The partial derivatives are:

$$\frac{\partial V}{\partial \ell} = w(96 - 2\ell - w), \quad \frac{\partial V}{\partial w} = \ell(96 - \ell - 2w)$$

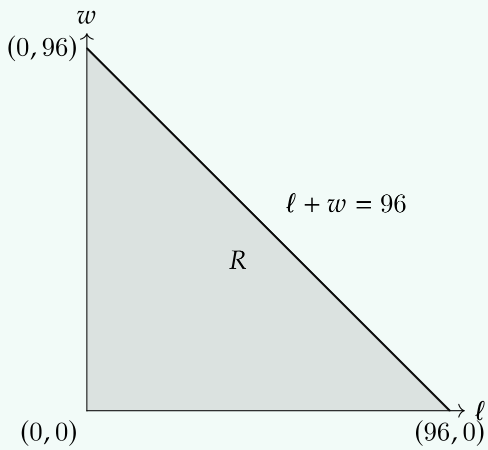
Setting these to zero, we get:

$$\begin{cases} w(96 - 2\ell - w) = 0 \\ \ell(96 - \ell - 2w) = 0 \end{cases}$$

This yields critical points at $(\ell, w) = (32, 32)$.

On the boundary of R:

1. $\ell = 0$: Then $V = 0$ for any w.
2. $w = 0$: Then $V = 0$ for any ℓ.
3. $\ell + w = 96$: Substitute $w = 96 - \ell$ into $V(\ell, w) = \ell(96 - \ell)(96 - \ell - (96 - \ell)) = \ell(96 - \ell)^2$.



Solving this, we find that $V(32, 32) = 32 \cdot 32 \cdot 32 = 32768$, which is the maximum value on R.

To confirm the local maximum at $(32, 32)$, we compute the Hessian:

$$H(\ell, w) = \begin{bmatrix} -2w & 96 - \ell - w \\ 96 - \ell - w & -2\ell \end{bmatrix}$$

At $(32, 32)$, the Hessian determinant is:

$$\det(H) = (-2 \times 32)^2 - (96 - 32 - 32)^2 = (64 \times 64) - (32 \times 32) = 32768 > 0$$

Thus, $V(\ell, w)$ has a local maximum at $(32, 32)$.

Example 1.1.2 (Another Example)
Consider $f(x, y) = 4 - x^2 - y^2$ on $R = \{(x, y) \mid -1 \leqslant x \leqslant 1, x^2 + y^2 < 1\}$. Here, R is an open region without boundaries.

The maximum of $f(x, y)$ occurs at $(0, 0)$, where $f(0, 0) = 4$. There is no absolute minimum because for points approaching the boundary (e.g., $x \approx 0.99$), $f(x, y)$ approaches $-\infty$.
Thus, in R, there is no absolute minimum value for f, illustrating the importance of the region's closedness and boundedness for the Extreme Value Theorem to apply.

1.2 Lagrange Multiplier Method

1.2.1 General Procedure

To maximize or minimize $f(x, y)$ subject to a constraint $g(x, y) = 0$, follow these steps:

1. **Identify Critical Points:** A point (x', y') is a critical point if:
   - There exists $\lambda \in \mathbb{R}$ such that $\nabla f(x', y') = \lambda \nabla g(x', y')$ (using the *method of Lagrange multipliers*).
   - $g(x', y') = 0$.
2. **Evaluate Cases for Critical Points:**
   (a) **Case 1:** The constraint region R is bounded and has no endpoints.
   - In this case, assuming continuity of f, any local extrema within R are also absolute extrema.
   - Select critical points (x', y') and evaluate f at these points.

   (b) **Case 2:** The constraint region R is bounded and has endpoints.

- Example: $x^2 - 4y^2 \leqslant 0$ with $-1 \leqslant x \leqslant 1$.
- Absolute extrema may occur at local extrema or endpoints.
- Check critical points and also evaluate f at the boundary endpoints.

(c) **Case 3:** The constraint region R is unbounded or does not include all endpoints.

- Example: $x^2 - 4y^2$ unbounded or $x^2 + y^2 - 4 = 0$ with $-2 \leqslant x \leqslant 2$.
- It may be possible that absolute extrema do not exist.
- Find any critical points and evaluate $f(x)$ as $x \longrightarrow \pm\infty$ if applicable.

Example 1.2.1 (Lagrange Multiplier Method)

We aim to find the absolute extrema of the function $f(x, y) = x - 2y$ subject to the constraint $g(x, y) = x^2 + y^2 = 4$.
To find the extrema, we use the method of Lagrange multipliers, where we seek points where $\nabla f = \lambda \nabla g$. The gradients of f and g are:

$$\nabla f(x, y) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \nabla g(x, y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

Setting $\nabla f = \lambda \nabla g$, we have:

$$\begin{cases} 1 = \lambda \cdot 2x \\ -2 = \lambda \cdot 2y \end{cases}$$

This simplifies to:

$$\lambda = \frac{1}{2x} = \frac{-1}{y} \Rightarrow y = -2x$$

Substitute $y = -2x$ into the constraint $g(x, y) = x^2 + y^2 = 4$:

$$x^2 + (-2x)^2 = 4 \Rightarrow x^2 + 4x^2 = 4 \Rightarrow 5x^2 = 4 \Rightarrow x = \pm\frac{2}{\sqrt{5}}$$

Then, $y = -2x$ gives $y = \pm\frac{4}{\sqrt{5}}$. So the points are:

$$\left(\frac{2}{\sqrt{5}}, -\frac{4}{\sqrt{5}}\right) \quad \text{and} \quad \left(-\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right)$$

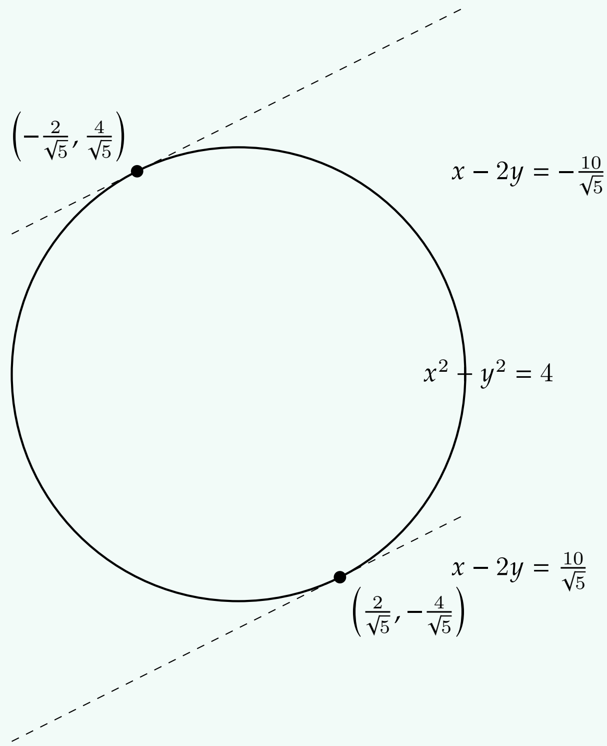
Calculate $f(x, y)$ at the points:

$$f\left(\frac{2}{\sqrt{5}}, -\frac{4}{\sqrt{5}}\right) = \frac{2}{\sqrt{5}} + \frac{8}{\sqrt{5}} = \frac{10}{\sqrt{5}} = 2\sqrt{5}$$

$$f\left(-\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right) = -\frac{2}{\sqrt{5}} - \frac{8}{\sqrt{5}} = -\frac{10}{\sqrt{5}} = -2\sqrt{5}$$

Thus, the **absolute maximum** is $2\sqrt{5}$ and the **absolute minimum** is $-2\sqrt{5}$.

Below is a diagram showing the constraint $x^2 + y^2 = 4$ as a circle and the level curves of $f(x, y) = x - 2y$, specifically showing the two tangent level curves at $z = \pm\frac{10}{\sqrt{5}}$ that represent the maximum and minimum values.



Example 1.2.2 (Example: Finding Extrema of $f(x, y) = e^{x+y}$ with Constraint)
Consider maximizing or minimizing $f(x, y) = e^{x+y}$ subject to the constraint $g(x, y) = x^2 + xy + y^2 - 9 = 0$.

1. Using the method of Lagrange multipliers, we set up the system:

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

   which gives:

$$\begin{cases} e^{x+y} = \lambda(2x + y) \\ e^{x+y} = \lambda(x + 2y) \end{cases}$$

2. Dividing the equations, we get:

$$\frac{2x + y}{x + 2y} = 1 \Rightarrow x = y$$

3. Substitute $x = y$ into the constraint $x^2 + xy + y^2 = 9$:

$$x^2 + x^2 + x^2 = 9 \Rightarrow 3x^2 = 9 \Rightarrow x = \pm\sqrt{3}, \quad y = \pm\sqrt{3}$$

4. The critical points are $(\sqrt{3}, \sqrt{3})$ and $(-\sqrt{3}, -\sqrt{3})$.
5. Evaluating $f(x, y)$ at these points:

$$f(\sqrt{3}, \sqrt{3}) = e^{2\sqrt{3}}, \quad f(-\sqrt{3}, -\sqrt{3}) = e^{-2\sqrt{3}}$$

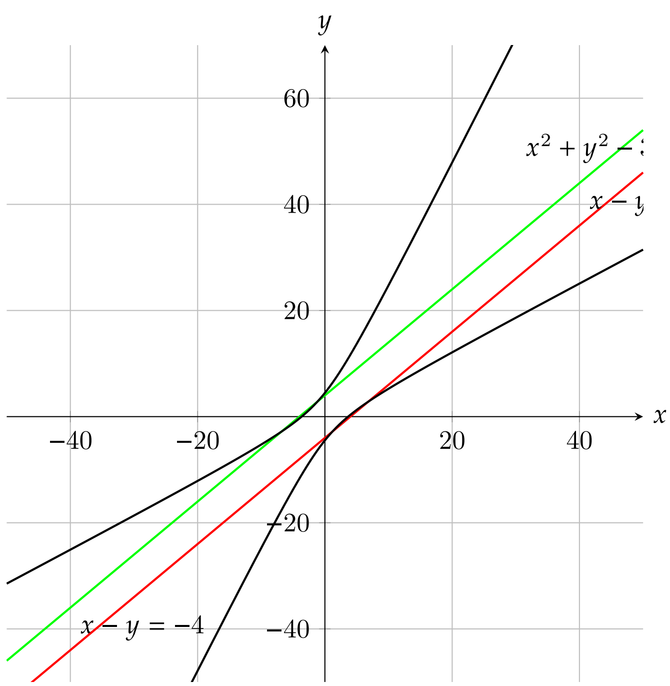
6. Thus, the maximum value is $e^{2\sqrt{3}}$ and the minimum value is $e^{-2\sqrt{3}}$.

Example 1.2.3 (Example: Finding Extrema of $f(x, y) = x - y$ with Constraint)

Consider the function $f(x, y) = x - y$ with the constraint $g(x, y) = x^2 + y^2 - 3xy - 20 = 0$.

1. The gradients are:
$$\nabla f(x, y) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \nabla g(x, y) = \begin{bmatrix} 2x - 3y \\ 2y - 3x \end{bmatrix}$$
2. Set $\nabla f = \lambda \nabla g$:
$$\begin{cases} 1 = \lambda(2x - 3y) \\ -1 = \lambda(2y - 3x) \end{cases}$$
3. Solving this system, we find critical points at $(2, -2)$ and $(-2, 2)$.
4. Evaluating $f(x, y)$ at these points:
$$f(2, 1) = 1, \quad f(-2, -1) = -1$$
5. Therefore, the maximum value is 1 and the minimum value is -1.

By observing the level curves diagram, we can see that the given points do not maximise/minimise the function, as any other line where $x - y < \pm 4$ would further increase/decrease the function.



1.3 Multiple Integration

Note:-

The integral
$$\int_a^b f(x)\,dx = A$$

represents the area under the curve $f(x)$ from a to b, where A can also be approximated as

$$A \approx \sum_{i=1}^{N} f(x_i)\Delta x.$$

Note:-

To find the volume under a surface $z = f(x, y)$ over a region $R = [a, b] \times [c, d]$, we use the double integral

$$\iint_R f(x, y)\, dA.$$

This can be approximated by summing over small subregions within R:

$$\iint_R f(x, y)\, dA = \lim_{\Delta \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\Delta x\, \Delta y.$$

Theorem 1.3.1 Continuity and Limits of Double Integrals

If the limit

$$\lim_{(x,y)\to(a,b)} f(x, y)$$

exists for all points (a, b) in R, then $f(x, y)$ is continuous over R, and the order of integration can be changed under Fubini's theorem.

Note:-

Consider dividing R into n subregions, each with area ΔA_{ij} and height $f(x_i, y_j)$. Then, the volume V is approximated as

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\Delta x\, \Delta y,$$

where n is the number of boxes in the x-direction and m is the number in the y-direction. Taking the limit as $\Delta x, \Delta y \to 0$ gives

$$V = \iint_R f(x, y)\, dA.$$

Definition 1.3.1: Double Integral Definition

The double integral of $f(x, y)$ over $R = [a, b] \times [c, d]$ is defined by

$$\iint_R f(x, y)\, dA = \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_i, y_j)\Delta A_{ij}.$$

Theorem 1.3.2 Fubini's Theorem

The order of integration does not affect the result of the double integral. Thus,

$$\iint_R f(x, y)\, dA = \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx = \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy.$$

Example 1.3.1 (Example of Double Integral Calculation)

Calculate the volume under $f(x, y) = 4 + x + y^2$ over the region $R = [-1, 1] \times [0, 2]$:

$$\iint_R f(x, y)\, dA = \int_{-1}^{1} \int_{0}^{2} (4 + x + y^2)\, dy\, dx.$$

Evaluating the inner integral with respect to y,

$$\int_0^2 (4 + x + y^2)\, dy = 4y + xy + \frac{y^3}{3}\Big|_0^2 = 8 + 2x + \frac{8}{3}.$$

Then, integrating with respect to x,

$$\int_{-1}^{1} \left(8 + 2x + \frac{8}{3}\right) dx = \left(8 + \frac{8}{3}\right) \cdot 2 = \frac{32}{3}.$$

Thus, the volume $V = \frac{32}{3}$.

Example 1.3.2 (Example: Integration Over a Rectangular Domain)
Evaluate

$$\int_0^2 \int_0^1 \frac{xy}{1 + x^2}\, dx\, dy.$$

Using substitution $u = x^2$ with $du = 2x\, dx$, we find

$$\int_0^2 \int_0^1 \frac{xy}{1 + x^2}\, dx\, dy = \int_0^2 y \left(\int_0^1 \frac{x}{1 + x^2}\, dx\right) dy = \int_0^2 y \left[\frac{1}{2} \ln(1 + x^2)\right]_0^1 dy.$$

Simplifying, we get

$$V = \int_0^2 y \cdot \frac{\ln(2)}{2}\, dy = \frac{\ln(2)}{2} \int_0^2 y\, dy = \frac{\ln(2)}{2} \cdot \frac{y^2}{2}\Big|_0^2 = \ln(2).$$

1.3.1 Non-Rectangular Domains

Note:-

Consider $R = \{(x, y) : a \leqslant x \leqslant b, g_1(x) \leqslant y \leqslant g_2(x)\}$, where the boundaries are given by functions $y = g_1(x)$ and $y = g_2(x)$.

Example 1.3.3 (Example: Non-Rectangular Domain Integration)
Suppose $R = \{(x, y) : -1 \leqslant x \leqslant 1, x^2 \leqslant y \leqslant 2 - x^2\}$. We wish to evaluate

$$\int_R (x + y)\, dA.$$

We split R into two regions, R_1 and R_2, with bounds given by

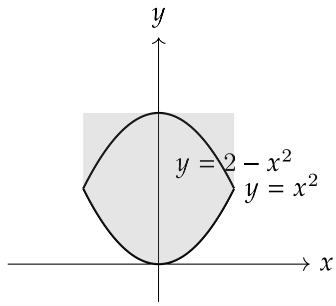
$$R_1 = \{(x, y) : -1 \leqslant x \leqslant 1, x^2 \leqslant y \leqslant 2 - x^2\}.$$

Evaluating each integral, we obtain

$$V = \int_{-1}^{1} \left(\int_{x^2}^{2-x^2} (x + y)\, dy\right) dx.$$

On solving, we get

$$V = \int_{-1}^{1} \left(\int_{x^2}^{2-x^2} (x+y)\, dy \right) dx = \ldots = \frac{8}{35}.$$



Note:-

When changing the order of integration, try dividing the region into smaller regions to make integration simpler.

1.3.2 Volume Between Surfaces

Note:-

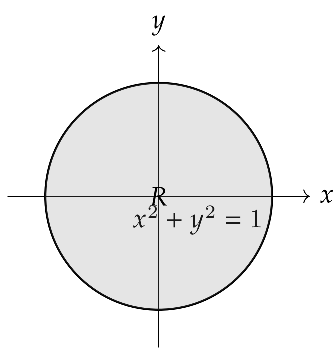
To find the volume of a sphere using double integrals, consider the surface

$$x^2 + y^2 + z^2 = 1.$$

Then $z = \pm\sqrt{1 - x^2 - y^2}$, and we can set up the integral as

$$V = 2 \iint_R \sqrt{1 - x^2 - y^2}\, dA,$$

where $R = \{(x, y) : -1 \leqslant x \leqslant 1, -\sqrt{1 - x^2} \leqslant y \leqslant \sqrt{1 - x^2}\}$.



Example 1.3.4 (Example: Volume Between Surfaces)

Calculate the volume between the surfaces $z = \sqrt{1 - x^2 - y^2}$ and $z = -\sqrt{1 - x^2 - y^2}$:

$$V = \iint_R \left(\sqrt{1 - x^2 - y^2} - (-\sqrt{1 - x^2 - y^2}) \right) dA = 2 \iint_R \sqrt{1 - x^2 - y^2}\, dA.$$

Setting up the limits as before, we integrate over R to find the volume of the sphere.