Temporary Doc
Calc 3

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Contents

Chapter 1

Vector Valued Functions $f : \mathbb{R} \longrightarrow \mathbb{R}^n$

Example 1.0.1 (Problem with Multiple Surfaces)

The solid common to the cylinders bounded by $z = \sin x$ and $z = \sin y$ over the region

$$R = \{(x, y) : 0 \leqslant x \leqslant \pi,\ 0 \leqslant y \leqslant \pi\}.$$

We define the domain D as

$$D = \{(x, y, z) : (x, y) \in R,\ 0 \leqslant z \leqslant \min(\sin x, \sin y)\}.$$

To solve this problem, we will:

1. Consider one-fourth of the volume under the intersection of the two cylinders, taking advantage of symmetry.
2. Identify the bounding conditions for z in each quadrant.
3. Set up and compute the triple integral over D to find the volume.

Given the symmetry, we restrict our analysis to the region

$$0 \leqslant x \leqslant \frac{\pi}{2}, \quad 0 \leqslant y \leqslant \frac{\pi}{2}.$$

In this quadrant, z is bounded by $\min(\sin x, \sin y)$, which means $z \leqslant \sin y$ in the region R_1, where $\sin y \leqslant \sin x$.
For region R_2 (where $\sin x \leqslant \sin y$), we have $z \leqslant \sin x$. By observing symmetry, the volume in each quadrant contributes equally, so we can calculate the volume in this restricted region and multiply by 4.
The bounds for x and y are:

$$0 \leqslant y \leqslant \pi, \quad y \leqslant x \leqslant \pi - y.$$

Thus, we set up the volume integral as follows:

$$V = 4 \int_0^{\pi/2} \int_0^{\sin y} \int_y^{\pi - y} dx\, dy\, dz.$$

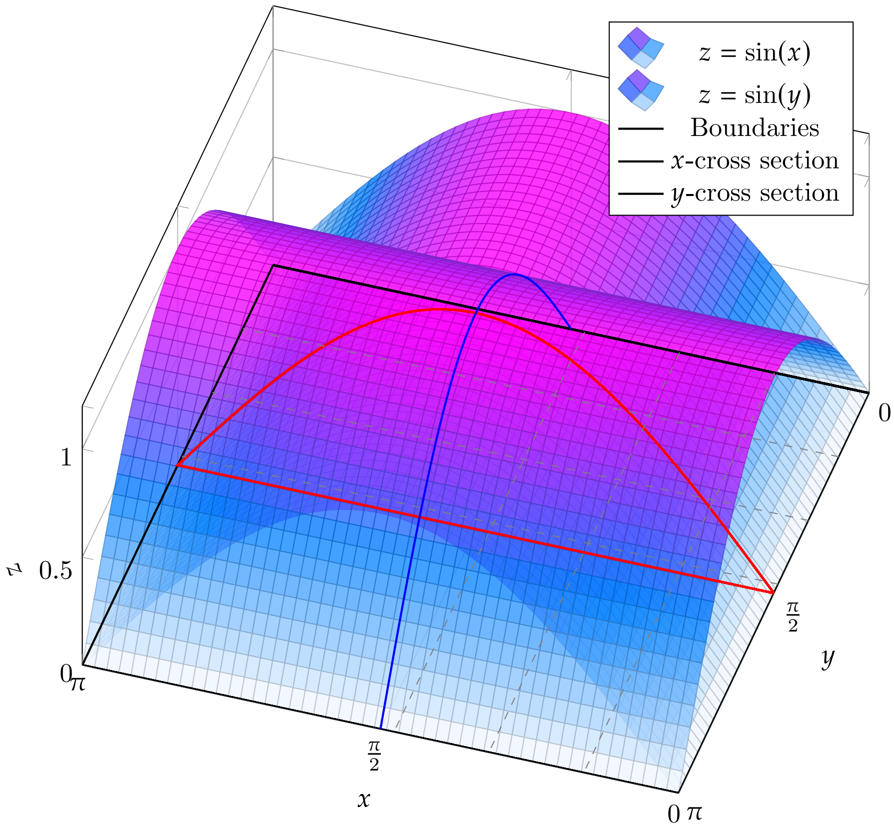
Evaluating the integral:

$$V = 4 \int_0^{\pi/2} \int_0^{\sin y} \left(\int_y^{\pi - y} dx \right) dy\, dz.$$

Upon computation, this integral yields:

$$V = \pi - 2.$$

The key idea was to leverage the symmetry of the intersection region, focusing on one-fourth of the area and then scaling up by a factor of 4. By analyzing the geometry, we found that z was bounded by $\sin y$ in region R_1. From there, the triple integral was computed over x, y, and z to yield the final volume of the region.



1.1 Polar Coordinate System

Definition 1.1.1: Polar Coordinates Overview

The polar coordinate system is a two-dimensional coordinate system in which each point on a plane is determined by a distance from a reference point (often called the origin or pole) and an angle from a reference direction (usually the positive x-axis). The two coordinates are:

- r: the radial distance from the origin to the point.
- θ: the angular coordinate, representing the angle in radians (or degrees) between the positive x-axis and the line connecting the origin to the point.

A point P in the plane can therefore be represented in polar coordinates as (r, θ).

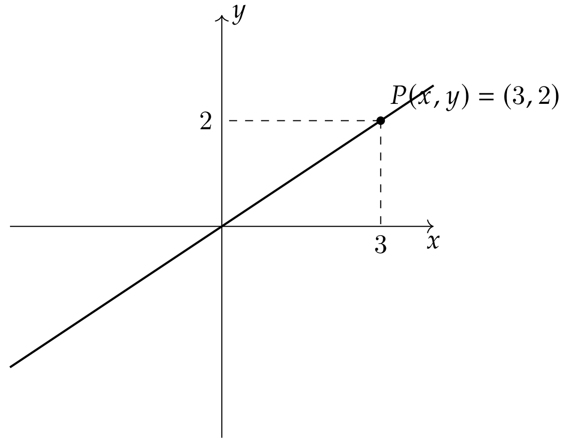
Definition 1.1.2: Relationship Between Cartesian and Polar Coordinates

In Cartesian coordinates, a point P can be represented as (x, y). We can convert between Cartesian and polar coordinates with the following relations:

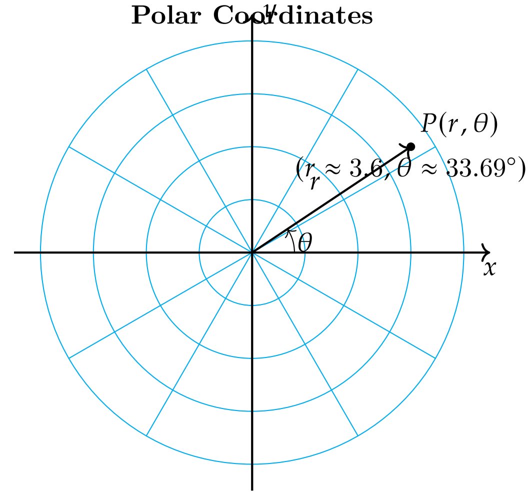
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2}$$
$$\theta = \arctan\left(\frac{y}{x}\right)$$

These equations allow for the translation of a point's position between Cartesian and polar forms, showing the adaptability of the polar system for various types of analyses, especially when working with circular or rotational symmetry.

Cartesian Coordinates



Polar Coordinates



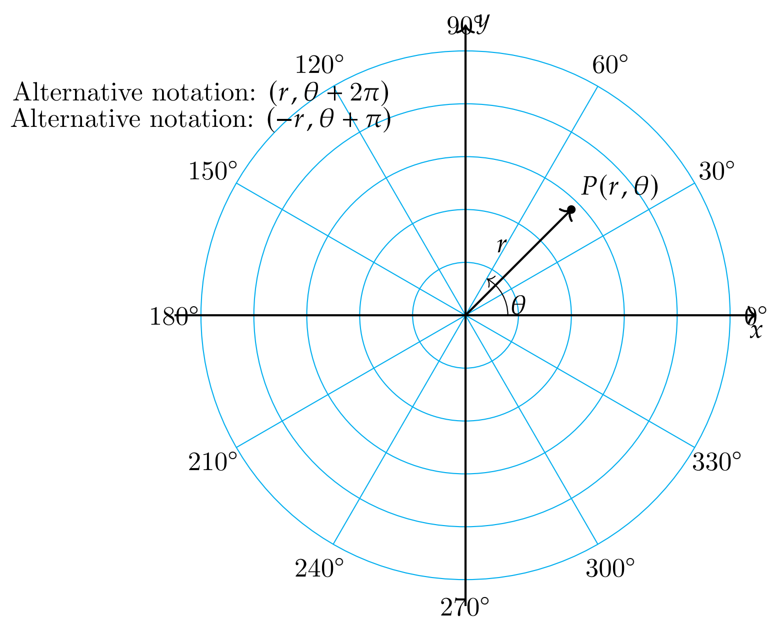


Diagram illustrating the polar coordinate system with concentric circles and angular lines, showing a point P represented as (r, θ) and alternative notations.

Note:-

An interesting feature of polar coordinates is that a single point P can have multiple equivalent representations. For instance, the point (r, θ) can also be expressed as $(-r, \theta+\pi)$ by reversing the radial direction and adjusting the angle. Additionally, due to the periodic nature of angles, adding any integer multiple of 2π to θ results in the same point, i.e., $(r, \theta + 2\pi k)$ for integer k.

Example 1.1.1 (Applications of Polar Coordinates)

Polar coordinates are particularly useful in problems involving symmetry around a central point, such as in physics for modeling circular motion, waves, and fields. They simplify equations and visualizations in cases where Cartesian coordinates might be cumbersome.

1.2 Basic Equations

1.2.1 Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

For parametric representation:

$$x = a\cos\theta, \quad y = b\sin\theta$$

1.2.2 Line

$$r(\theta) = \frac{b}{\sin\theta - \cos\theta}$$

where $y = mx + b$.

1.2.3 Cardioid

The equation for a cardioid:

$$r(\theta) = 1 + \cos\theta$$

Transformations:

$$r = 1 + \cos\theta \Rightarrow r = x + \cos\theta$$

$$r = r\cos\theta \Rightarrow x = \cos\theta$$

1.3 Slope of a Curve

For a function $r = f(\theta)$:

$$\text{slope} = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

Using parametric form $r(\theta) = f(\theta)$, then

$$x(\theta) = f(\theta)\cos\theta, \quad y(\theta) = f(\theta)\sin\theta$$

Therefore,

$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$$

Definition 1.3.1: Slope of a curve in polar coordinates

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta$$

Example 1.3.1 (Example: Slope of Cardioid)

For the cardioid $r = 1 + \cos\theta$,

$$f(\theta) = 1 + \cos\theta, \quad f'(\theta) = -\sin\theta$$

Then,

$$\text{slope} = \frac{-\sin\theta(\sin\theta) + (1+\cos\theta)\cos\theta}{-\sin\theta\cos\theta - (1+\cos\theta)\sin\theta}$$

Simplifying,

$$\text{slope} = -\frac{\cos^2\theta}{\sin\theta(2\cos\theta + 1)}$$

To determine where the slope is undefined, we examine the denominator:

$$\text{slope} = -\frac{\cos^2\theta}{\sin\theta(2\cos\theta + 1)}$$

The slope is undefined when the denominator is zero.

At $\theta = \pi$:

$$\sin\pi = 0 \quad \text{and} \quad \cos\pi = -1$$

$$\sin\pi \cdot (2\cos\pi + 1) = 0 \cdot (2 \cdot (-1) + 1) = 0$$

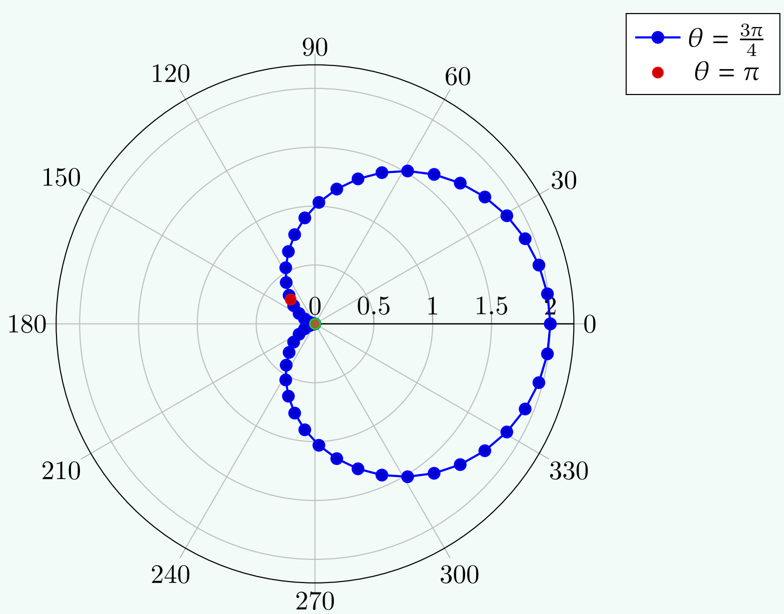
Thus, the slope is undefined at $\theta = \pi$.

At $\theta = \frac{3\pi}{4}$:

$$\sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2} \quad \text{and} \quad \cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$$

$$\sin\frac{3\pi}{4} \cdot \left(2\cos\frac{3\pi}{4} + 1\right) = \frac{\sqrt{2}}{2} \cdot \left(-\sqrt{2} + 1\right) = 0$$

Thus, the slope is also undefined at $\theta = \frac{3\pi}{4}$.



The slope at $\theta = \frac{3\pi}{4}$ does not exist in the usual Cartesian sense. Here's why:

1. Radial Line: At $\theta = \frac{3\pi}{4}$, the line connecting the point $\left(1+\cos\left(\frac{3\pi}{4}\right), \frac{3\pi}{4}\right)$ to the origin is a radial line. Radial lines in polar coordinates point directly toward or away from the origin, meaning they lack a well-defined Cartesian slope.

2. Slope in Polar Coordinates: In polar coordinates, the concept of "slope" differs from Cartesian coordinates. We can calculate $\frac{dr}{d\theta}$ at $\theta = \frac{3\pi}{4}$, but this does not correspond to a Cartesian slope.

3. Alternative Interpretation: Converting $r = 1 + \cos(\theta)$ to Cartesian form and calculating derivatives would show a vertical tangent at $\theta = \frac{3\pi}{4}$, indicating an undefined slope.

1.4 Deriving the Integral for Area Under a Curve in Polar Coordinates

To find the area under a curve defined in polar coordinates using Riemann sums, we follow these steps:

In Cartesian coordinates, the area under a curve from $x = a$ to $x = b$ is given by:

$$\int_a^b f(x)\, dx$$

In polar coordinates, a point is represented by (r, θ), where r is the radial distance from the origin, and θ is the angle from the positive x-axis. To calculate the area under a curve $r = f(\theta)$ between two angles $\theta = \alpha$ and $\theta = \beta$, we use sectors of circles.

For a small angle $d\theta$, the area of a sector with radius r is approximately:

$$dA \approx \frac{1}{2} r^2\, d\theta$$

This comes from the area formula for a sector, $\frac{1}{2}r^2\theta$, with a small change in angle $d\theta$.

To find the total area under the curve $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$, we can divide the interval $[\alpha, \beta]$ into n small subintervals, each of width $\Delta\theta = \frac{\beta - \alpha}{n}$.

In each subinterval $[\theta_i, \theta_{i+1}]$, we approximate the radius r by $f(\theta_i)$, where θ_i is a sample point in that interval. The area of each small sector is then approximately:

$$\Delta A_i \approx \frac{1}{2} f(\theta_i)^2 \Delta\theta$$

The total area A is the sum of the areas of these small sectors:

$$A \approx \sum_{i=0}^{n-1} \frac{1}{2} f(\theta_i)^2 \Delta\theta$$

As we take the limit where $n \to \infty$ and $\Delta\theta \to 0$, this Riemann sum becomes an integral:

$$A = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{2} f(\theta_i)^2 \Delta\theta = \int_\alpha^\beta \frac{1}{2} f(\theta)^2\, d\theta$$

Thus, the area under the curve $r = f(\theta)$ in polar coordinates from $\theta = \alpha$ to $\theta = \beta$ is:

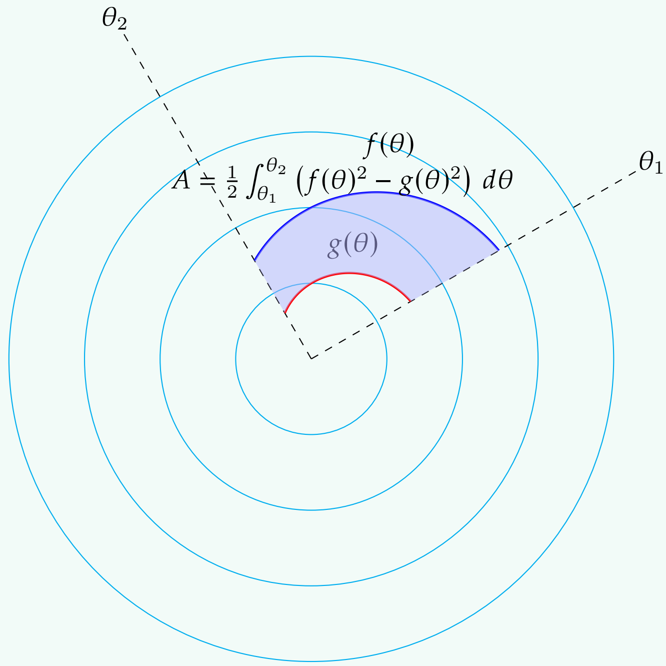
$$\boxed{A = \int_\alpha^\beta \frac{1}{2} f(\theta)^2\, d\theta}$$

This is the formula for the area in polar coordinates, derived using the concept of Riemann sums.

Example 1.4.1 (Example: Circle $r = a$)

$$A = \frac{1}{2}\int_0^{2\pi} a^2\,d\theta = \pi a^2$$

Example 1.4.2 (Two curves)



For two curves $r = f(\theta)$ and $r = g(\theta)$:

$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2} f(\theta)^2\,d\theta - \int_{\theta_1}^{\theta_2} \frac{1}{2} g(\theta)^2\,d\theta$$

which simplifies to

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} (f(\theta)^2 - g(\theta)^2)\,d\theta$$