MA193
Discrete Mathematics

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Contents

Chapter 1

Fundamental Principles of Counting

1.1 Counting with Repetitions

Note:-

These notes cover the basic counting principles (often called the "Rule of Sum" and the "Rule of Product"), along with a brief discussion of permutations and combinations, including the formula for permutations of multiset objects (i.e., repeated elements).

Definition 1.1.1: Rule of Sum ("OR")

If a certain task can be done in n ways and another independent task can be done in m ways, and these tasks are mutually exclusive, then there are $n + m$ ways to do *one* of the two tasks.

Definition 1.1.2: Rule of Product ("AND")

If a procedure can be broken into two consecutive steps such that the first step can be done in n ways and the second step can be done in m ways (independently of how the first step is done), then there are $n \times m$ ways to do the entire procedure.

Note:-

In many counting problems, we break a larger procedure into a series of smaller steps and then apply either the Rule of Sum or the Rule of Product (or both) as needed.

Definition 1.1.3: Arrangements of n Distinct Objects

If you want to arrange n distinct objects in a row (i.e., an ordered list), there are $n!$ ways to do so. Order matters here, and this number is referred to as the number of permutations of n distinct items.

Definition 1.1.4: Permutations of Multisets

Suppose we have n total objects, but they are not all distinct. Instead, let there be n_1 objects of type 1, n_2 objects of type 2, ..., and n_k objects of type k. Clearly

$$n_1 + n_2 + \cdots + n_k = n.$$

Then the number of distinct ways to arrange all n objects is

$$\frac{n!}{n_1!\, n_2! \, \ldots \, n_k!}.$$

Example 1.1.1 (Examples of Counting with Repetitions)

1. **ABCD:** All letters are distinct, so the number of ways to arrange A, B, C, D is
$$4! = 24.$$
2. **AABC:** Here we have 4 total letters, with A repeated twice. The number of distinct arrangements is
$$\frac{4!}{2!} = \frac{24}{2} = 12.$$
3. **AABB:** Now we have 2 A's and 2 B's (4 letters total). The number of arrangements is
$$\frac{4!}{2!\,2!} = \frac{24}{2 \times 2} = 6.$$
4. **SUCCESS:** The word "SUCCESS" has 7 letters total: 3 S's, 2 C's, 1 U, and 1 E. The number of distinct permutations is
$$\frac{7!}{3!\,2!\,1!\,1!} = \frac{5040}{(6)(2)(1)(1)} = 420.$$

Committee-Choosing Problems

Definition 1.1.5: Combinations

A *combination* is a way to choose r objects from n distinct objects where order does *not* matter. The number of ways to do so is given by
$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}.$$
In many committee-selection problems, we use combinations because the particular order in which people are chosen does not matter.

Definition 1.1.6: When to Use Permutations vs. Combinations

- *Permutations* ($P(n,r)$ or $\frac{n!}{(n-r)!}$) are used when we care about the *order* of the chosen items (for instance, arranging people in a line for a photo).
- *Combinations* ($\binom{n}{r}$) are used when we only care about *which* items are chosen, not the order in which they appear (e.g., forming committees).

Definition 1.1.7: Basic Subset Counting

Note that the total number of distinct subsets of a set with n elements is 2^n. This comes from the fact that, for each element, we independently choose to include it or not in a given subset. For an r-element subset, specifically, we use $\binom{n}{r}$.

Example 1.1.2 (Choosing a Simple Committee)

Suppose we have 8 people in a group, and we want to choose a committee of 3 individuals. Since order does not matter, the number of ways to choose the committee is
$$\binom{8}{3} = \frac{8!}{3!(8-3)!} = \frac{8!}{3!5!} = 56.$$

Example 1.1.3 (Committee with Restrictions: Gender Balance)

Imagine we have 5 men and 6 women, and we want to form a committee of 4 people that has at least 2 women. We can break it down by the number of women selected:

$$\binom{6}{2}\binom{5}{2} + \binom{6}{3}\binom{5}{1} + \binom{6}{4}\binom{5}{0}.$$

Here, $\binom{6}{k}$ chooses the women, and $\binom{5}{4-k}$ chooses the men for each scenario. Summing these counts gives the total number of ways to form such a committee.

Definition 1.1.8: Labeled Teams

When teams themselves have distinct "names" or labels (e.g. Team A vs. Team B). A configuration where Person 1 is on Team A and Person 2 is on Team B is then different from Person 1 on Team B and Person 2 on Team A.

Definition 1.1.9: Unlabeled Teams

When the teams are treated as identical (no distinct labels). In that scenario, swapping membership between "Team A" and "Team B" would not create a new configuration.

Example 1.1.4 (Committee with Subgroups Required)

Say we have 10 people, of whom 3 are Teaching Assistants (TAs) and 7 are Professors, and we wish to form a committee of 4 people *with exactly 2 TAs*. We select 2 from the 3 TAs and 2 from the 7 Professors, yielding

$$\binom{3}{2} \times \binom{7}{2}$$

possible committees.

Note:-

Labeled vs. Unlabeled Teams. "3 people, 2 teams with a label?" Here, we wonder if putting Alice on Team A and Bob on Team B differs from Alice on Team B and Bob on Team A. If teams are unlabeled, then swapping them would not create a different outcome. If teams *are* labeled, we count each distinct assignment as different.

Example 1.1.5 (Larger Committees: Multiple Constraints)

Suppose there are 12 people divided into three categories: 4 from group A, 5 from group B, and 3 from group C. We want a committee of 5 that has at least 1 person from each group. One way to count is to enumerate possible splits of 5 among (A, B, C), ensuring each category has at least one representative. For instance:

- 1 from A, 3 from B, 1 from C,
- 1 from A, 2 from B, 2 from C,
- 2 from A, 2 from B, 1 from C,
- and so forth,

and sum the corresponding products of binomial coefficients $\binom{4}{\ldots}\binom{5}{\ldots}\binom{3}{\ldots}$.

Note:-

These examples illustrate the typical approaches to *committee choosing* problems:

- Identify if order matters (usually it does not, hence combinations).
- If there are constraints (e.g. minimum number of members from a certain group), split the problem into valid cases and sum their respective counts.

1.2 Gambling

Example 1.2.1 (Probability of Getting a Flush in Poker)

A *flush* in poker is a hand where all 5 cards are of the same suit. To calculate the probability of being dealt a flush, we proceed as follows:

- There are 4 suits in a deck (hearts, diamonds, clubs, spades).
- For each suit, there are $\binom{13}{5}$ ways to choose 5 cards from the 13 available.
- Thus, the total number of ways to get a flush is $4 \times \binom{13}{5}$.
- The total number of 5-card hands from a 52-card deck is $\binom{52}{5}$.

Therefore, the probability P of being dealt a flush is given by

$$P(\text{flush}) = \frac{\binom{4}{1} \times \binom{13}{5}}{\binom{52}{5}}.$$

So, the probability of being dealt a flush in poker is approximately 0.198%.

Example 1.2.2 (Example: 3 people into 2 teams)

Suppose we have 3 people (say Alice, Bob, and Carol). If the teams are *labeled*, we count every distinct assignment into Team A vs. Team B. If the teams are *unlabeled*, we only care about who ends up together, not which group is called "Team A." Hence the total counts differ, because labeling typically doubles the number of distinct ways (unless one team is empty).

Note:-

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ext, consider two common 5-card poker hands:

Definition 1.2.1: One Pair

A 5-card hand containing exactly two cards of the same rank and the other three cards all of different ranks (and each different from the pair's rank).

Example 1.2.3 (Counting One Pair)

To choose exactly one pair out of a standard deck, we do:

$$\binom{13}{1}\binom{4}{2}$$

to pick which rank we have a pair of (13 ways) and which 2 suits out of the 4. For a complete 5-card hand with "exactly one pair," we then choose the other 3 cards of distinct ranks, leading to the well-known formula $\binom{13}{1}\binom{4}{2} \times \binom{12}{3}$ (each chosen rank has $\binom{4}{1}$ ways).

Definition 1.2.2: Full House

A 5-card hand consisting of 3 cards of one rank plus 2 cards of another rank.

Example 1.2.4 (Counting a Full House)

We choose the rank for the three-of-a-kind in $\binom{13}{1}$ ways, and then choose 3 suits out of 4 for that rank: $\binom{4}{3}$. Next, we choose the rank for the pair out of the remaining 12 ranks: $\binom{12}{1}$, and choose 2 suits out of 4 for that pair: $\binom{4}{2}$. Hence,

$$\text{Number of Full Houses} = \binom{13}{1}\binom{4}{3} \times \binom{12}{1}\binom{4}{2}.$$

1.3 Binomial identities

Definition 1.3.1: Binomial Symmetry

$$\binom{n}{k} = \binom{n}{n-k}.$$

Note:-

The intuition is that choosing k objects out of n is equivalent to choosing which $n-k$ to *leave out*. Hence $\binom{n}{k}$ equals $\binom{n}{n-k}$.

Example 1.3.1 (Example)

For instance, $\binom{5}{2} = \binom{5}{3}$. Choosing 2 items from 5 is the same as deciding which 3 are excluded.

Definition 1.3.2: Pascal's Rule

$$\binom{n}{v} = \binom{n-1}{v-1} + \binom{n-1}{v}.$$

Note:-

A combinatorial interpretation is to imagine a set of n items where one particular "special" item can either be in your chosen subset or not. If you include the special item, then you need to pick the remaining $v-1$ items from the other $n-1$. If you exclude the special item, then you pick all v items from the other $n-1$. Summing these counts gives $\binom{n}{v}$.

Example 1.3.2 (Example)

$\binom{5}{3} = \binom{4}{2} + \binom{4}{3}$. Either include a special item (then choose 2 more from the other 4) or exclude it (then choose all 3 from the other 4).

Definition 1.3.3: Hockey-Stick Identity

$$\sum_{j=r}^{n} \binom{j}{r} = \binom{n+1}{r+1}.$$

Note:-

This identity is sometimes called the "Christmas Stocking" or "Hockey-Stick" identity because of the pattern it creates in Pascal's triangle. It can be viewed as a cumulative version of Pascal's Rule: once you fix $\binom{n+1}{r+1}$, you can "unroll" it into the sum $\binom{r}{r} + \binom{r+1}{r} + \cdots + \binom{n}{r}$.

Example 1.3.3 (Example)

$\binom{3}{3} + \binom{4}{3} + \binom{5}{3} \; = \; \binom{6}{4}$. In general, if you sum along a diagonal in Pascal's triangle, you end up at a binomial coefficient one row down and one column over.

Note:-

A useful application of these identities is to evaluate sums of products like $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + (u-2)(u-1)u$. Note that

$$k(k+1)(k+2) \; = \; 3!\binom{k+2}{3},$$

so

$$\sum_{k=1}^{u-2} k(k+1)(k+2) \; = \; 3!\sum_{k=1}^{u-2}\binom{k+2}{3} \; = \; 3!\binom{u+1}{4}.$$

Example 1.3.4 (Binomial Expansion)

$(x+y)^n \; = \; \displaystyle\sum_{k=0}^{n}\binom{n}{k} x^{n-k}\, y^k$. For instance, $(2x+3y)^9$ expands as

$$\sum_{k=0}^{9}\binom{9}{k}(2x)^{9-k}(3y)^k.$$