Temporary Doc
Calc 3

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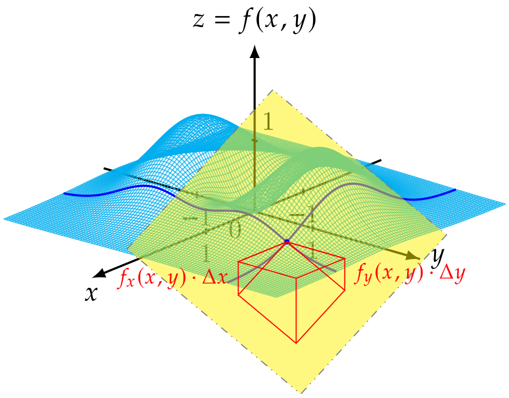
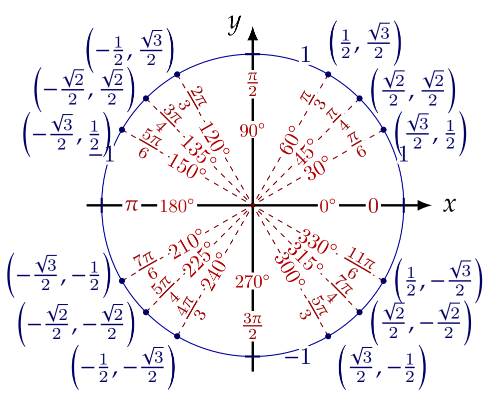
Contents

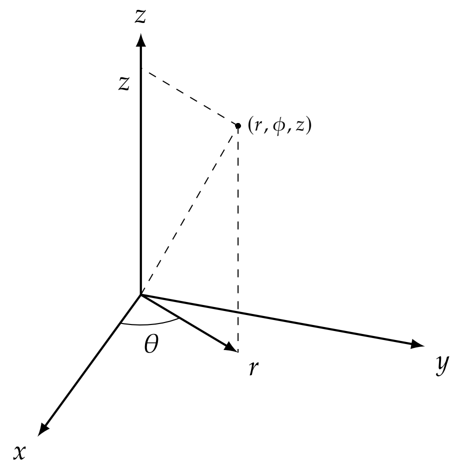
Chapter 1

Vector Valued Functions $f : \mathbb{R} \longrightarrow \mathbb{R}^n$





1.1 Cylindrical Coordinate Axes



Example 1.1.1 (Cylinders and Cones Example)

Cylinders:

$$r = 1, \quad v = 2$$

Cones:

$$z = \frac{\sqrt{3}}{3} r, \quad z = \sqrt{3} r$$

Ranges:

- r is bounded by the cylinders: $1 \leqslant r \leqslant 2$
- z is bounded by the cones: $\frac{\sqrt{3}}{3} r \leqslant z \leqslant \sqrt{3} r$
- θ is unbounded: $0 \leqslant \theta \leqslant 2\pi$

Volume:

$$V = \int_0^{2\pi} \int_1^2 \int_{\frac{\sqrt{3}}{3} r}^{\sqrt{3} r} r \, dz \, dr \, d\theta$$

(Simplify to evaluate)

1.2 Spherical Coordinate Axes

The relationships between Cartesian coordinates (x, y, z) and spherical coordinates (ρ, ϕ, θ) are as follows:

Definition 1.2.1: Polar - Spherical Relationships

$$x = \rho \cos\phi \cos\theta$$
$$y = \rho \cos\phi \sin\theta$$
$$z = \rho \sin\phi$$

Conversely, the spherical coordinates can be expressed in terms of Cartesian coordinates as:

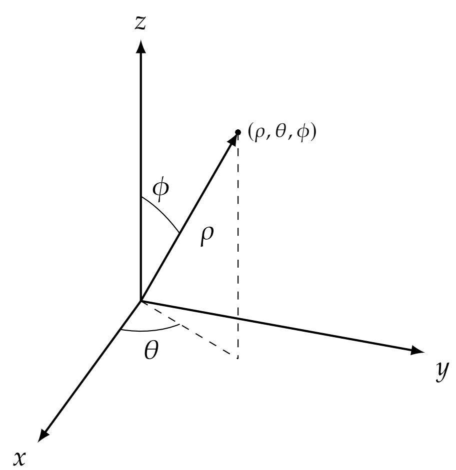
$$\rho = \sqrt{x^2 + y^2 + z^2}$$
$$\phi = \tan^{-1}\left(\frac{z}{\sqrt{x^2 + y^2}}\right)$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

The volume element in spherical coordinates is:

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

The integral of a function $f(x, y, z)$ in spherical coordinates becomes:

$$\iiint_P f(x, y, z)\, dV = \iiint_P f\big(\rho\cos\phi\cos\theta, \rho\cos\phi\sin\theta, \rho\sin\phi\big)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$



1.3 Shapes Represented in Spherical Coordinates

1.3.1 Overview of Spherical Coordinates

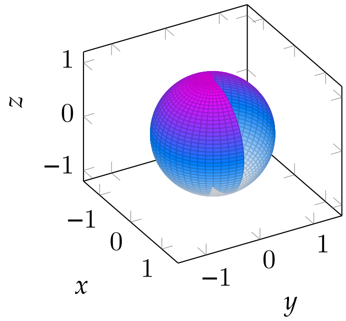
- **Definition**: Spherical coordinates (r, θ, ϕ) are defined as:
  - r: Radial distance from the origin.
  - θ: Polar angle (angle from the positive z-axis, $0 \leqslant \theta \leqslant \pi$).
  - ϕ: Azimuthal angle (angle in the xy-plane from the positive x-axis, $0 \leqslant \phi < 2\pi$).
- **Conversion to Cartesian coordinates**:

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

1.3.2 Shapes in Spherical Coordinates

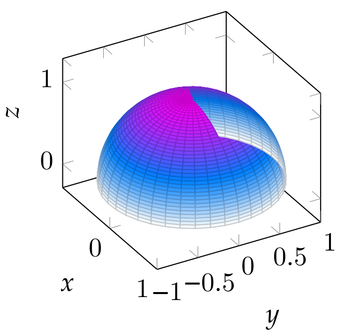
1. Sphere

Equation: $r = R$
Description: A sphere of radius R centered at the origin.
Representation: Independent of θ and ϕ.



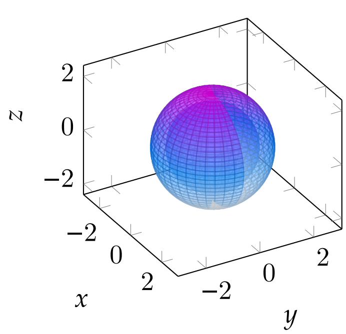
2. Half-Sphere

Equation: $r = R$, $0 \leqslant \theta \leqslant \frac{\pi}{2}$
Description: Upper half of a sphere centered at the origin, constrained by the polar angle.



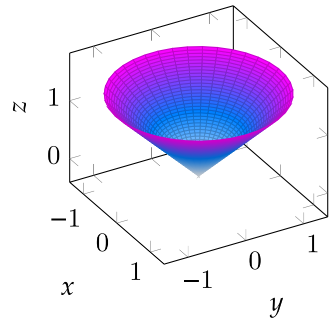
3. Spherical Shell

Equation: $R_1 \leqslant r \leqslant R_2$
Description: A hollow spherical region with inner radius R_1 and outer radius R_2.



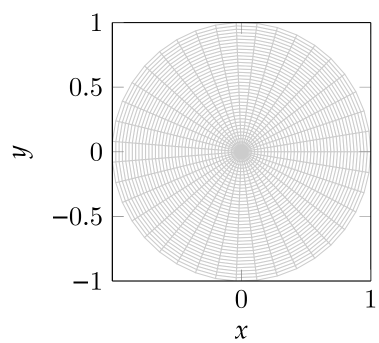
4. Cone

Equation: $\theta = \theta_0$
Description: A cone with its vertex at the origin, opening angle $2\theta_0$, symmetric around the z-axis.
Constraints: $r > 0$, ϕ varies from 0 to 2π.



5. Circular Disc (in the xy-Plane)

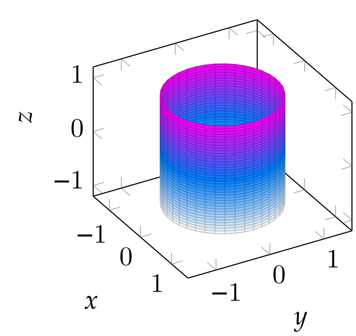
Equation: $r = R$, $\theta = \frac{\pi}{2}$
Description: A circular disc of radius R centered on the origin in the xy-plane.



6. Cylinder

Equation: $r \sin\theta = a$
Description: A cylinder of radius a around the z-axis.
Constraints:

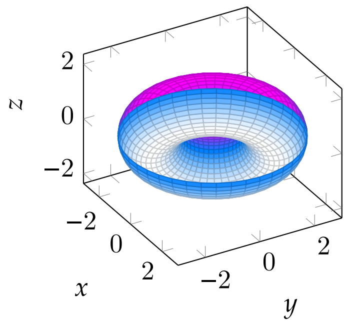
- $0 \leqslant \phi < 2\pi$,
- $r\cos\theta$ is unbounded (representing the z-coordinate).



7. Torus (Approximation)

Equation: $r = R + r_0 \sin\theta$
Description: A torus with major radius R and minor radius r_0, approximately represented in spherical coordinates.
Constraints:

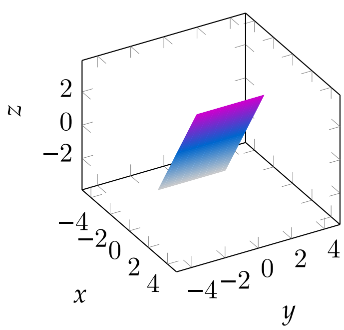
- ϕ varies from 0 to 2π,
- Specific parametric constraints apply.



8. Plane

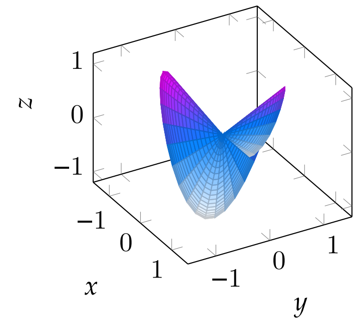
Equation: θ = constant or ϕ = constant
Description:

- θ = constant: A conical plane cutting through the origin at a fixed polar angle.
- ϕ = constant: A vertical plane slicing through the z-axis at a fixed azimuthal angle.



9. Lemniscate Shape

Equation: $r = a \sin\theta \sin 2\phi$
Description: A figure-eight or lemniscate shape in spherical coordinates.



10. Wave-like Surfaces

Equation: $r = R + A\cos(k\theta)$
Description: Oscillating surface around a spherical shell, useful in representing waves or perturbations on a sphere.