MA226: Differential Equations

Lecture notes for Differential Equations

Giacomo Cappelletto

Published: September 03, 2025
Last updated: September 03, 2025

Contents

List of Figures

* * *

Chapter 1: First-Order Differential Equations

First-order differential equations involve derivatives up to the first derivative only. These form the foundation for understanding more complex differential equations and are ubiquitous in mathematical modeling.

Three Approaches to Solving Differential Equations

There are three fundamental approaches to tackling differential equations, each with its own strengths:

Three Solution Approaches — Definition 1.1

1. Analytic $\longrightarrow$ Formula or equation (exact solutions)
2. Qualitative $\longrightarrow$ Sketches, describe behavior (understanding without solving)
3. Numerical $\longrightarrow$ Computing (approximate solutions using algorithms)

Choosing the Right Approach — Note 1.1

- Use analytic methods when exact solutions are needed and the equation is solvable
- Use qualitative methods to understand long-term behavior and stability
- Use numerical methods when analytic solutions are impossible or impractical

Modeling

Mathematical modeling with differential equations follows a systematic approach to translate real-world phenomena into mathematical language.

Types of Models

- Simple models: Easy to analyze; describe the dominant interactions
- Complex models: Capture behavior over a wider domain; less general

Model Building Process

Model building typically follows three steps:

1. State assumptions clearly (with units for all quantities)
2. Define variables and parameters WITH UNITS
3. Use assumptions to derive equations relating the variables

Population Modeling — Example 1.1

Target: Population of rabbits $P(t)$ as a function of time t (years).

Key Assumption: The rate of change of population is proportional to the current population size.

Mathematical Model:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP \qquad [1]$$

where k is the growth coefficient (constant parameter).

Fundamental Definitions

Solution and General Solution — Definition 1.2

A function is a solution of a differential equation on an interval if, when substituted into the equation, it satisfies the equality for every point in that interval.

A general solution contains an arbitrary constant (or constants). Determining the constant(s) from given data yields a particular solution.

Initial Value Problem (IVP) — Definition 1.3

A differential equation together with an initial condition such as $P(t_0) = P_0$.

Solving the IVP means finding the unique solution that satisfies both the equation and the initial condition on an interval.

Exponential Growth and Decay

Solving the Basic Growth Model

Consider the differential equation $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$.

Solution Strategy: Guess that $P(t) = Ce^{\{kt\}}$ for some constant C.

Verification:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(Ce^{\{kt\}}\right) = C \cdot ke^{\{kt\}} = k\left(Ce^{\{kt\}}\right) = kP \qquad [2]$$

Therefore, $P(t) = Ce^{\{kt\}}$ is indeed a solution to our differential equation.

General Solution — Note 1.2

Since C is arbitrary, $P(t) = Ce^{\{kt\}}$ represents the general solution to $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$.

The sign of k determines the behavior:

- If $k > 0$: exponential growth
- If $k < 0$: exponential decay

Finding Particular Solutions

Complete Solution Process — Example 1.2

Problem: Solve $P' = kP$ with initial conditions $P(0) = 32$ and $P(3) = 47$.

Step 1: Start with general solution $P(t) = Ce^{\{kt\}}$

Step 2: Apply first condition $P(0) = 32$

$$P(0) = Ce^{\{k\cdot 0\}} = Ce^0 = C = 32 \quad [3]$$

So $C = 32$, giving us $P(t) = 32e^{\{kt\}}$.

Step 3: Apply second condition $P(3) = 47$

$$P(3) = 32e^{\{3k\}} = 47 \quad [4]$$

$$e^{\{3k\}} = \frac{47}{32} \quad [5]$$

$$3k = \ln\left(\frac{47}{32}\right) \quad [6]$$

$$k = \frac{1}{3}\ln\left(\frac{47}{32}\right) \quad [7]$$

Final Answer: $P(t) = 32e^{\{\frac{1}{3}\ln(\frac{47}{32})\cdot t\}}$

Growth vs. Decay Analysis — Note 1.3

- If $k > 0$, then P increases exponentially
- If $k < 0$, then P decreases exponentially
- The time constant $\frac{1}{|}k|$ sets the natural timescale of change

For our example: $k = \frac{1}{3}\ln(\frac{47}{32}) \approx 0.121 > 0$, so we have exponential growth.

Equilibrium Solutions

Equilibrium Solution — Definition 1.4

A constant solution $y(t) \equiv y_*$ such that $y'(t) = 0$ for all t in an interval.

Equilibria correspond to values of y where the right-hand side of $y' = f(t, y)$ is zero for all t.

Finding Equilibrium Solutions — Example 1.3

Consider the differential equation:

$$y' = \frac{(y+2)(y-3)(t-5)}{(y+7)} \quad [8]$$

For an equilibrium solution $y(t) \equiv y_*$, we need the right-hand side to be zero for all t.

Analysis: The right-hand side equals zero when the numerator is zero (and the denominator is non-zero).

The numerator $(y+2)(y-3)(t-5) = 0$ when:

- $y + 2 = 0 \to y = -2$
- $y - 3 = 0 \to y = 3$
- $t - 5 = 0$ (but this depends on t, so doesn't give a constant solution)

Verification: Both $y = -2$ and $y = 3$ make the denominator $y + 7$ non-zero.

Answer: $y \equiv -2$ and $y \equiv 3$ are equilibrium solutions.

Important Note Attention 1.1

$y \equiv -7$ is NOT a solution because it makes the right-hand side undefined (division by zero).

Key Insights and Intuition

Why Exponential Solutions Work Note 1.4

The exponential function $e^{\{kt\}}$ has the special property that its derivative is proportional to itself:

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{\{kt\}} = k e^{\{kt\}} \quad [9]$$

This makes it the natural solution to equations of the form $y' = ky$.

Physical Interpretation Note 1.5

- Population growth: When resources are abundant, growth rate $\sim$ current population
- Radioactive decay: Decay rate $\sim$ current amount of material
- Bank interest: Continuous compounding gives exponential growth
- Cooling: Newton's law of cooling (with modifications)