Homework 5: Context-Free Grammars

CSCI 161

Due 9pm, Thursday, October 22, 2020

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Overview. This week's homework gives you more practice designing and interpreting the languages of context-free grammars. Questions 1 and 2 have you design CFGs for two new languages, Question 3 touches on ambiguous strings and Chomsky normal form (3c will be easier later in the week), and Question 4 has you think about CFGs as a generalization of regular expressions. Hints are available in office hours!

Question 1. Design a CFG for the language over $\Sigma = \{1, \#\}$ whose elements consist of every pair of distinct, $\#$-separated unary values:

$$L = \{x_1 \# x_2 \mid x_1, x_2 \in 1^*, x_1 \neq x_2\}\,.$$

Note: This L is a subset of the language from your HW3 proof of nonregularity. For extra practice with the pumping lemma from Unit 1, you could show (not for a grade) that this language is nonregular. Along with the CFG you submit for this question, this classifies L as nonregular and context-free (like $\{0^n 1^n \mid n \geq 0\}$).

Question 2. Design a CFG for the language of binary strings that contain at least one 1 in their second half:

$$L = \{uv \mid u \in (0 \cup 1)^*, v \in (0 \cup 1)^* 1 (0 \cup 1)^*, |u| \geq |v|\}\,.$$

Question 3. This multi-part question concerns the following CFG:

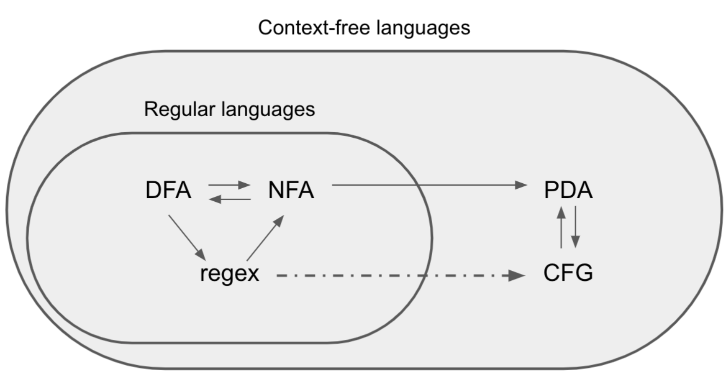
$$S \to SS \mid T$$
$$T \to aTb \mid ab$$

a) Give an English description of the language associated with the grammar.

b) Provide a string and two leftmost derivations of it to show the grammar above is ambiguous.

c) Convert the grammar above to an equivalent CFG in Chomsky normal form.

Question 4. This question will have you complete a proof that all regular languages are context-free by showing that any regular expression can be equivalently described by a context-free grammar. The core *content* of this question was discussed in class on Friday. The goal of this question is for you to review and understand how that content fits into a proof about the relationship between the two classes of languages.



The NFA$\to$PDA arrow was discussed on Wednesday as being a consequence of the PDA's memory stack generalizing NFAs (since any NFA can be written as a PDA that never touches its stack), and the PDA$\leftrightarrow$CFG equivalence will be discussed later this unit. This question gives a different illustration of how context-free languages subsume regular languages, by providing a proof of the dotted regex$\to$CFG arrow that is inspired by the regex$\to$NFA construction we saw last unit. Most of the proof is below. Fill in what's missing!

Claim: Any regular expression has an equivalent context-free grammar.

Proof overview: For any regex R, we will give rules for a CFG over the same alphabet Σ associated with the same language. The proof follows the inductive structure of the definition of a regex, constructing a CFG for each type of regex and showing that for any regex length, there is an equivalent CFG.

Base cases: We give an equivalent CFG for each regex of length 1. The regex definition gives three cases.

1. $R = a$ for some $a \in \Sigma$. This is equivalent to the CFG with a single rule $S \to a$.
2. $R = \epsilon$. This is equivalent to the CFG with a single rule $S \to \epsilon$.
3. $R = \emptyset$. This is equivalent to the trivial CFG with no rules.

Inductive step: Fix any regex length $k > 1$. We will show that if any regex of length less than k has an equivalent CFG, then we can construct an equivalent CFG for any regex of length k. By the definition of a regex and the inductive hypothesis, any regex R of length k is the union, concatenation, or star of one or two regexes R_1 and R_2 of length less than k, which by inductive hypothesis have equivalent CFGs. In the following case analysis, assume the CFG associated with R_1 is called G_1 and has start variable S_1, and assume the CFG associated with R_2 is called G_2 and has start variable S_2. Assume without that the variables of the two CFGs are distinct; this assumption is without loss of generality, because overlapping variable names can be renamed in one of the grammars. We handle each case separately:

1. $R = (R_1 \cup R_2)$. This is equivalent to the CFG that consists of all rules and variables from each of G_1 and G_2 with a new start variable S and two additional rules $\boxed{S \to S_1 \mid S_2}$.

   Equivalence is because S derives a string derived from either S_1 or S_2; the presence of each rule in the two underlying grammars ensures that any string derivable by G_1 or G_2 is still derivable in the new grammar, and there are no other rules to derive any other strings.

2. $R = (R_1 \circ R_2)$. This is equivalent to the CFG that consists of...

3. $R = (R_1^*)$. This is equivalent to the CFG that consists of...