Foundations of Computer Science Comp109

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Part 4. Function

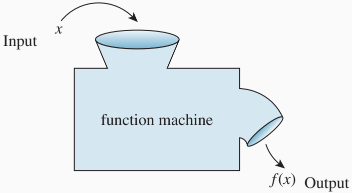
Comp109 Foundations of Computer Science

Reading

- Discrete Mathematics with Applications S. Epp, Chapter 7.
- **Discrete Mathematics and Its Applications** K. Rosen, Section 2.3.

Contents

Functions



Examples:

- $y = x^2$
- $y = \sin(x)$
- first letter of your name

Functions/methods on programming

Java

```java
public int f(int x) {
    return x+5;
}
```

C/C++

```c
int f(int x) {
    return x+5;
}
```

Python

```python
def f(int x):
    return x+5
```

Definition of a function

A *function* from a set A to a set B is an assignment of **exactly one** element of B to **each** element of A.

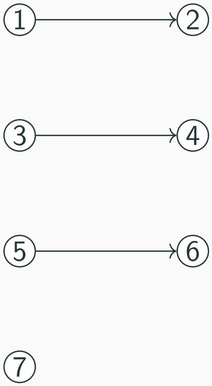
We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element of a.

If f is a function from A to B we write $f : A \to B$.

$a \mapsto b$

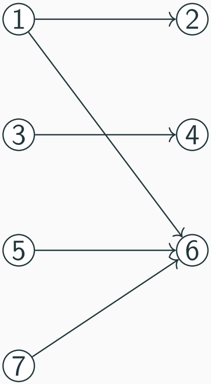
A function $f : \{1, 2, 3\} \to \{4, 5, 6\}$

Example: map, not a function



No function

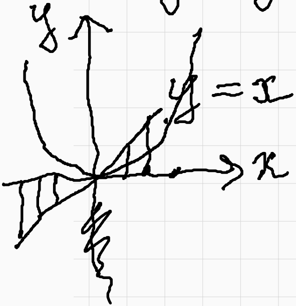
Example: map, not a function



No function

More examples

$$y = f(x) \; : \; \mathbb{R} \to \mathbb{R}$$



$$x \mapsto y = f(x)$$

$$x \mapsto x$$

$$x \mapsto x^2 \qquad y = x^2$$

$$\mathbb{R}^+ \cup \{0\} = \mathbb{R}^+_0$$

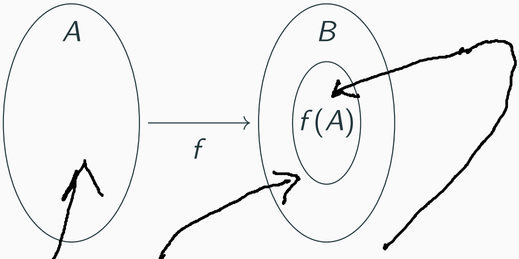
Domain, codomain, and range

Suppose the function $f : A \to B$.

- A is called the *domain* of f. B is called the *codomain* of f.
- The *range* (or *image*) $f(A)$ of f is

$$f(A) = \{f(x) \mid x \in A\}.$$

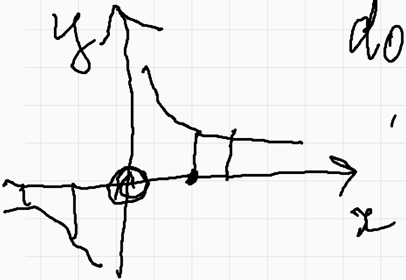
Codomain vs range



Find domain, codomain and the range of f

Example: $\frac{1}{x}$

Find domain, codomain and range of $f = \frac{1}{x}$.



domain $\neq \mathbb{R}$ >0

<0

$\mathbb{R} \setminus \{0\}$

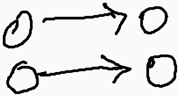
codomain $= \mathbb{R} \cancel{\setminus \{0\}}$

range $= \mathbb{R} \setminus \{0\}$

Injective (one-to-one) functions

Definition Let $f : A \to B$ be a function. We call f an *injective* function (or *one-to-one function*) if

$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2 \text{ for all } a_1, a_2 \in A.$$



This is logically equivalent to $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$, hence different inputs give different outputs.

Examples

- $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2$ is not injective.

$x = 1 \quad x = -1 \quad f(x) = 1$

- $h : \mathbb{Z} \to \mathbb{Z}$ given by $h(x) = 2x$ is

Examples

- $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2$ is not injective.

- $h : \mathbb{Z} \to \mathbb{Z}$ given by $h(x) = 2x$ is injective.

$x_1 \neq x_2$: $2x_1 = 2x_2$

$x_1 = x_2$

More examples

Boris Борис

- *first_letter* : *People* $\to$ *Char* 1st letter of the name Eng.
- *ID* : *People* $\to \mathbb{N}$

Surjective (or onto) functions

Definition Let $f : A \to B$ be a function. We call f *surjective* (or *onto*) if the range of f coincides with the codomain of f:

$$\forall b \in B \quad \exists a \in A \text{ such that } b = f(a).$$

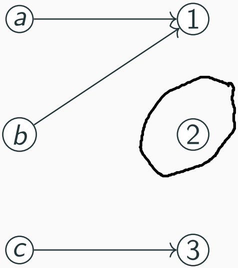
Examples

$f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2$ is not surjective.

$h : \mathbb{Z} \to \mathbb{Z}$ given by $h(x) = 2x$ is not surjective.

$h_1 : \mathbb{Q} \to \mathbb{Q}$ given by $h_1(x) = 2x$ is surjective.

Classify $f : \{a, b, c\} \to \{1, 2, 3\}$ given by

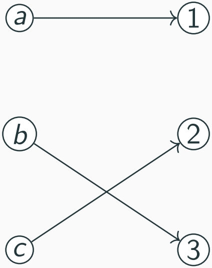


funct.

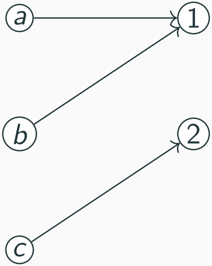
inj

surj

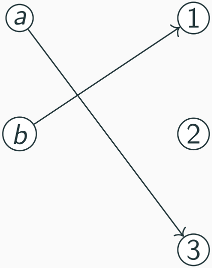
Classify $g : \{a, b, c\} \to \{1, 2, 3\}$ given by



Classify $h : \{a, b, c\} \to \{1, 2\}$ given by



Classify $h' : \{a, b, c\} \to \{1, 2, 3\}$ given by



Bijections

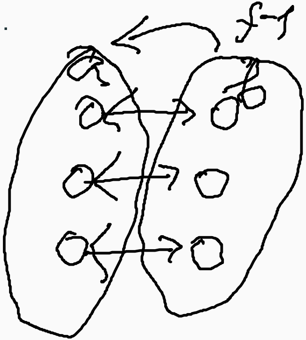
We call f *bijective* (or *one-to-one correspondence*) if f is both **injective** and **surjective**.

Examples

$f : \mathbb{Q} \to \mathbb{Q}$ given by $f(x) = 2x$ is bijective.

Inverse functions

If and only if f is a bijection from a set X to a set Y, then there exists a function f^{-1} from Y to X that "undoes" the action of f; that is, it sends each element of Y back to the element of X that it came from.



This function is called the inverse function f^{-1} for f:

$f^{-1} : Y \to X$ such that

$$f(a) = b \iff f^{-1}(b) = a.$$

Such function f is calles *invertible*.

Example: $4x + 3$

$k : \mathbb{R} \to \mathbb{R}$ given by $k(x) = 4x + 3$ is invertible and

$$k^{-1}(y) = \frac{1}{4}(y - 3).$$

x

Example: $\frac{x}{x-1}$

Let $A = \{x \mid x \in \mathbb{R}, x \neq 1\}$ and $f : A \to A$ be given by

$$f(x) = \frac{x}{x-1}.$$

Show that f is bijective and determine the inverse function.

The inverse relation is the set of pairs (y, x) with $y = x/(x-1)$ with $x \in \mathbb{R}$ and $x \neq 1$.

This means $yx - y = x \Rightarrow x(y-1) = y \Rightarrow x = y/(y-1)$.

Thus for every $y \in A$ there is exactly one such x. Also note $x \in A$.

So f is invertible. Thus, it is bijective. $f^{-1}(y) = y/(y-1)$.

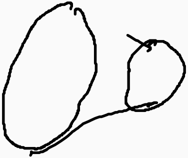
The function is its own inverse!

$$f(3) = \frac{3}{3-1} = \frac{3}{2}, \quad f^{-1}(\tfrac{3}{2}) = \frac{\frac{3}{2}}{\frac{3}{2}-1} = \frac{\frac{3}{2}}{\frac{1}{2}} = 3.$$

Cardinality of finite sets and functions

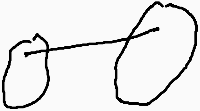
Recall: *The cardinality of a* finite *set S is the number of elements in S,*

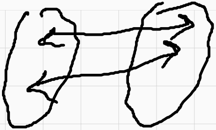
i.e. there is a bijection $f : S \to \{1, \ldots, n\}$.



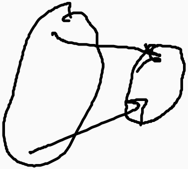
For finite sets A and B

- $|A| \geq |B|$ iff there is a surjective function from A to B.
- $|A| \leq |B|$ iff there is a injective function from A to B.
- $|A| = |B|$ iff there is a bijection from A to B.





The pigeonhole principle



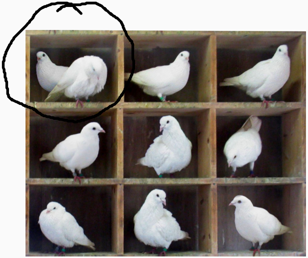
Let $f : A \to B$ be a function where A and B are finite sets.

The *pigeonhole principle* states that if $|A| > |B|$ then **at least one** value of f occurs **more than once**, i.e.

$f(a) = f(b)$ for some distinct elements a, b of A.

Pigeons and pigeonholes

If (N+1) pigeons occupy N holes, then some hole must have at least 2 pigeons.



[1]

It is also known as **Dirichlet's box principle** or **Dirichlet's drawer principle**.[2]

[1] Image by McKay from en.wikipedia

[2] Gustav Dirichlet (1805 - 1859) was a German mathematician who first introduced formal defintion of a function.

Example: birthday problem.

Problem. There are 15 people on a bus. Show that at least two of them have a birthday in the same month of the year.

DIY Example: dark socks puzzle

Imagine you have 10 pairs of socks in a drawer. How many socks would you need to pull out blindly in a completely dark room to ensure you have at least one matching pair?

Summary

Attendance code: 844739

- A **function** $f : A \to B$ is an assignment such that $\forall a \in A \exists$ one $b \in B : f(a) = b$.
- A is the **domain**, B is the **codomain**, $f(A)$ is the **range**.
- **Injective** function: $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$ for all $a_1, a_2 \in A$.
- **Surjective** function: $\forall b \in B \quad \exists a \in A$ such that $b = f(a)$.
- **Bijective** function: both injective and surjective.
- **Inverse function** $f^{-1} : Y \to X$ such that $f(a) = b \iff f^{-1}(b) = a$.
- **Pigeonhole principle:** If A and B are finite sets such that $|A| > |B|$, then for any $f : A \to B \quad \exists a, b \in A : f(a) = f(b)$.