Foundations of Computer Science Comp109

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Part 5. Combinatorics

Comp109 Foundations of Computer Science

Reading

- Discrete Mathematics with Applications, S. Epp, Chapter 9.
- Discrete Mathematics and Its Applications, K. H. Rosen, Sections 6.1, 6.3, 6.4

Contents

- Basics of counting
- Notation for sums and products. The factorial function.
- Counting permutations and combinations.
- Binomial coefficients.

Developing ideas (1)

All chairs in a room are labelled with a single digit followed by a lower-case letter. What is the largest number of differently numbered chairs?

Developing ideas (2)

How many different bit strings of length 8 are there?

- How many different bytes are there?

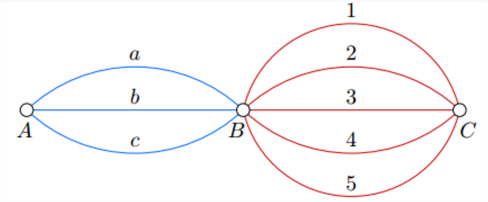
$$0000\,0000,\ 0000\,0001,\ 0000\,0010,\ 0000\,0011, \ldots$$

Developing ideas (3)

How many ways there are to select **3 students for positions** of president, vice-president and secretaly (order matters) from a group of 5?

How many ways there are to select **5 students for 5 different positions** (order matters) from a group of 5?

The product rule



How many paths there exist from A to C?

The product rule: If there is a sequence of k events with $n_1, \ldots, n_k$ possible outcomes for events $1, \ldots, k$, then the total number of possible outcomes for the ordered sequence of k events is

$$n_1 \times n_2 \times \cdots \times n_k.$$

Example

How many distinct car licence plates are there consisting of six characters, the first three of which are letters and the last three of which are digits?

Example

Find the total number of factors of the number 720.

Developing ideas (4)

Our group consists of **2 male and 3 female students**. How many choices of **one male and one female** students for two different posts can we make if order matters?

Disjoint events and the sum rule

Two events are said to be *disjoint* (or *mutually exclusive*) if they can't occur simultaneously.

Example: If we have 3 pairs of blue jeans and 2 pairs of black jeans, then there are $3 + 2 = 5$ different pairs of jeans to choose and wear.

The sum rule: If A and B are disjoint events and there are n_1 possible outcomes for event A and n_2 possible outcomes for event B then there are $n_1 + n_2$ possible outcomes for the event "either A or B".

Example

How many three-digit numbers begin with 3 or 4?

Example

I wish to take two pieces of fruit with me for lunch. I have 3 bananas, 4 apples and 2 pears. How many ways can I select two pieces of fruit of different type?

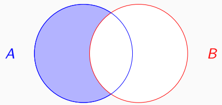
Set-theoretic interpretation

- If A and B are disjoint sets (that is, $A \cap B = \emptyset$) then $|A \cup B| = |A| + |B|$.
- Any sequence of k events can be regarded as an element of the Cartesian product $A_1 \times \cdots \times A_k$. This set has size $|A_1| \times \cdots \times |A_k|$.

Developing ideas (5)

*A computer password is a string of 8 characters, where each character is an uppercase letter or a digit. Each password must contain **at least one digit**.*

How many different passwords are there?

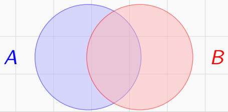


Note: lazy users

How many different 8-character passwords can be obtained by combining 3-letter word, a 4-letter word and 1 digit (for example HOT4FUZZ)? (According to `http://www.scrabblefinder.com` there are 1015 3-letter and 4030 4-letter English words.)

Developing ideas (6)

How many bit strings of length 8 start with 1 **or** finish with 00?



$$|A \cup B| = |A| + |B| - |A \cap B|$$

The subtraction rule

If there are n_1 possible outcomes for event A,
n_2 possible outcomes for event B and
n_3 of these outcomes are shared between A and B, then there are

$$n_1 + n_2 - n_3$$

possible outcomes for the event "A **or** B".

Developing ideas (7)

How many ways there are to select **2 representatives** (order is not important) from a group of 5 students?

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The division rule

Given n possible outcomes, if

- some of the n outcomes are the same
- every group of indistinguishable outcomes contains exactly d elements

there are n/d different outcomes.

Summary

Four decomposition rules:

- The product rule: the total number of possible outcomes for the ordered sequence of events A **and** B is $n_1 \times n_2$.
- The sum rule: for two disjoint events A, B there are $n_1 + n_2$ possible outcomes in "A **or** B".
- The subtraction rule: in general case, there are $n_1 + n_2 - n_3$ possible outcomes for the event "A **or** B".
- The division rule: if every group of contains d indistinguishable outcomes, then there are n/d different outcomes.

DIY problems:

- How many pairs (x, y) of positive integers satisfy the equation $xy = 2010$?
- How many line segments are formed if we place 10 points on a straight line?