# Foundations of Computer Science Comp109

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# Part 3. Relations

Comp109 Foundations of Computer Science

## Reading

- Discrete Mathematics with Applications S. Epp, Chapter 8.
- **Discrete Mathematics and Its Applications** K. Rosen, Chapter 9

# Contents


- The Cartesian product
- Definition and examples
- Representation of binary relations by directed graphs
- Representation of binary relations by matrices
- Properties of binary relations
- Transitive closure
- Equivalence relations and partitions
- Partial orders and total orders.
- Unary relations

# Motivation

- Intuitively, there is a "relation" between two things if there is some connection between them.
  E.g.
  - 'friend of'
  - $a < b$
  - $m$ divides $n$
- Relations are used in crucial ways in many branches of mathematics
  - Equivalence
  - Ordering
- Computer Science

# Databases and relations

A database table $\approx$ relation

**TABLE 1 Students.**

| *Student_name* | *ID_number* | *Major* | *GPA* |
|---|---|---|---|
| Ackermann | 231455 | Computer Science | 3.88 |
| Adams | 888323 | Physics | 3.45 |
| Chou | 102147 | Computer Science | 3.49 |
| Goodfriend | 453876 | Mathematics | 3.45 |
| Rao | 678543 | Mathematics | 3.90 |
| Stevens | 786576 | Psychology | 2.99 |

# Cartesian product

The *Cartesian product* $A \times B$ of sets $A$ and $B$ is the *set* consisting of all **ordered** pairs $(a, b)$ with $a \in A$ and $b \in B$, i.e.,

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Note that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

Sets $\{1, 2\} = \{2, 1\}$, but $(1, 2) \neq (2, 1)$.

# Example

- Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then

$$A \times B = \{(1, a), (2, a), (1, b), (2, b), (1, c), (2, c)\}.$$

- $B \times A =$

- $B \times B =$

If $A$ and $B$ are finite, what is $|A \times B|$?

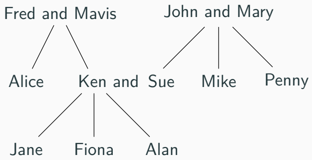
# Binary relation

A *binary relation* between two sets $A$ and $B$ is a subset $R$ of the Cartesian product $A \times B$, i.e. $R \subseteq A \times B$.

For each $(a, b) \in R$ we can also write $aRb$.

If $A = B$, then $R$ is called *a binary relation on $A$*.

# Example: Family tree




- $R = \{(x, y) \mid\ x \text{ is a grandfather of } y\ \} =$

- $S = \{(x, y) \mid\ x \text{ is a sister of } y\ \} =$

# Example 2

Write down the ordered pairs belonging to the following binary relations between $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6\}$:

- $U = \{(x, y) \in A \times B \mid x + y = 9\}$;

- $V = \{(x, y) \in A \times B \mid x < y\}$.

# Example 3

Let $A = \{1, 2, 3, 4, 5, 6\}$. Write down the ordered pairs belonging to

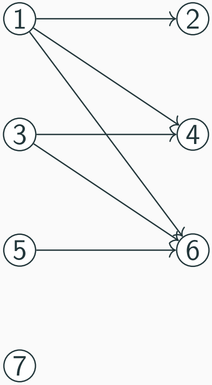
$$R = \{(x, y) \in A \times A \mid \ x \text{ is a divisor of } y\ \}.$$

# Representation of binary relations: directed graphs

- Let $A$ and $B$ be two finite sets and $R$ a binary relation between these two sets (i.e., $R \subseteq A \times B$).
- We represent the *elements* of these two sets as *vertices* of a graph.
- For each $(a, b) \in R$, we draw an *arrow* linking the related elements.
- This is called the *directed graph* (or *digraph*) of $R$.

## Example

Consider the relation $V$ between $A = \{1,3,5,7\}$ and $B = \{2,4,6\}$ such that $V = \{(x,y) \in A \times B \mid x < y\}$.



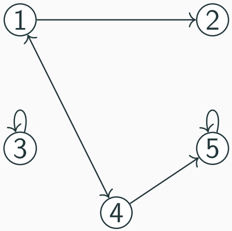
digraph of $V$

A binary relation between a set $A$ and itself is called *a binary relation on $A$*.

To represent a binary relation on $A$, we use a directed graph with a single set of vertices representing the elements of $A$.

**Example.** Consider the relation $V \subseteq A \times A$ where $A = \{1, 2, 3, 4, 5\}$ and

$$V = \{(1, 2), (3, 3), (5, 5), (1, 4), (4, 1), (4, 5)\}.$$




digraph of $V$

## **Example:** $A = \{1, 2, 3, 4\}$, $R = \{(x, y) \in A \times A \mid x < y\}$

## **Example:** $A = \{1, 2, 3, 4\}$, $R = \{(x, y) \in A \times A \mid x \leq y\}$

# Example: functional relations

- Recall that a *function* $f$ from a set $A$ to a set $B$ assigns exactly one element of $B$ to each element of $A$.
  - Gives rise to the relation $R_f = \{(a, b) \in A \times B \mid b = f(a)\}$

- If a relation $S \subseteq A \times B$ is such that for every $a \in A$ there exists at most one $b \in B$ with $(a, b) \in S$, relation $S$ is functional.

- Functions are sometimes introduced through functional relations.

# Example

$A = \{i \in \mathbb{N} \mid i < 10\}$, $B = \{i \in \mathbb{N} \mid 5 < i < 15\}$,
$R = \{((x, y) \in A \times B \mid y = 2x)\}$

# Inverse relation

Given a relation $R \subseteq A \times B$, we define the *inverse relation* $R^{-1} \subseteq B \times A$ by

$$R^{-1} = \{(b, a) \mid (a, b) \in R\}.$$

**Example.** The inverse of the relation *"is a parent of"* on the set of people is the relation "                    "

If you have a digraph representation of a relation, how do you get the inverse?
Answer:

## **Example:** $A = \{1, 2, 3, 4\}$, $R = \{(x, y) \mid x \leq y\}$

# Composition of relations

Let $R \subseteq A \times B$ and $S \subseteq B \times C$. The *composition* (or *relative multiplication*) of $R$ and $S$, denoted by $S \circ R$, is the binary relation between $A$ and $C$ given by

$$S \circ R = \{(a, c) \mid \text{ exists } b \in B \text{ such that } aRb \text{ and } bSc\}.$$

Note: be careful with the order!

**Example:** If $R$ is the relation *is a sister of* and $S$ is the relation *is a parent of*, then

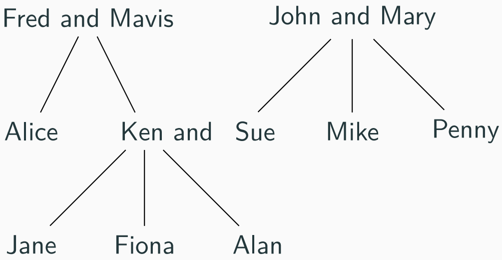
- $S \circ R$ is the relation "                              "
- $S \circ S$ is the relation "                              "

# Example

*R* : *is a sister of*

*S* : *is a parent of*

$S \circ R = \{(a, c) \mid$ exists $b \in B$ such that $aRb$ and $bSc\}$.



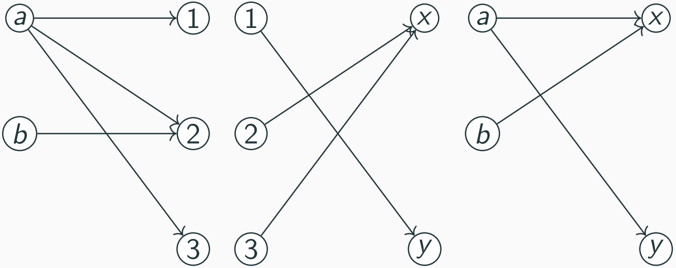

Find:

Alice $S \circ R$ ?

Penny $S \circ R$ ?

Fred $S \circ S$ ?

# Digraph representation of compositions




$R$ $S$ $S \circ R$

# DIY Example

$A$ – set of people, $B$ – set of countries

$R \subseteq A \times A$, $R(x, y)$ represents *"x is a friend of y"*

$S \subseteq A \times B$, $S(u, v)$ represents *"u visited v"*

Create your own example of $R$, $S$ and both their compositions.

# Summary

- The *Cartesian product* $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$.
- A *binary relation* $R$ between two sets $A$ and $B$ is $R \subseteq A \times B$, which can be represented by a *digraph*.
- Function $=$ functional relation.
- $R^{-1} = \{(b, a) \mid (a, b) \in R\}$.
- $S \circ R = \{(a, c) \mid \text{ exists } b \in B \text{ such that } aRb \text{ and } bSc\}$.