Foundations of Computer Science Comp109

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Part 5. Combinatorics

Comp109 Foundations of Computer Science

Reading

- Discrete Mathematics with Applications, S. Epp, Chapter 9.
- Discrete Mathematics and Its Applications, K. H. Rosen, Sections 6.1, 6.3, 6.4

Contents

Developing ideas (1)

All chairs in a room are labelled with a single digit followed by a lower-case letter. What is the largest number of differently numbered chairs?

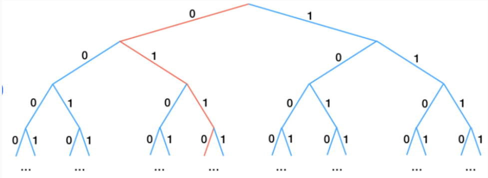
Answer: the maximum possible number of chairs will be $9 \times 26 = 234$.

Developing ideas (2)

How many different bit strings of length 8 are there?

- How many different bytes are there?

$$0000\,0000,\;\; 0000\,0001,\;\; 0000\,0010,\;\; 0000\,0011, \ldots$$



Answer: 2^8, the same as the number of subsets of the set of cardinality 8.

Developing ideas (3)

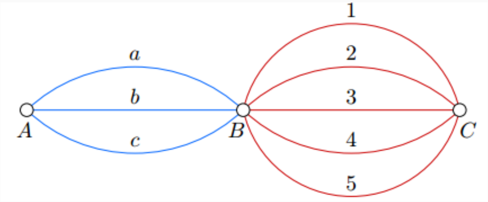
How many ways there are to select **3 students for positions** of president, vice-president and secretaly (order matters) from a group of 5?

Answer: $5 \times 4 \times 3 = 60$.

How many ways there are to select **5 students for 5 different positions** (order matters) from a group of 5?

Answer: in simpler terms, this is the number of all possible orderings of 5 people: $5 \times 4 \times 3 \times 2 \times 1 = 120$.

The product rule



How many paths there exist from A to C?

Answer: $3 \cdot 5 = 15$

The product rule: If there is a sequence of k events with $n_1, \ldots, n_k$ possible outcomes for events $1, \ldots, k$, then the total number of possible outcomes for the ordered sequence of k events is

$$n_1 \times n_2 \times \cdots \times n_k.$$

Example

How many distinct car licence plates are there consisting of six characters, the first three of which are letters and the last three of which are digits?

Answer: By the product rule, there exist $26 \times 26 \times 26 \times 10 \times 10 \times 10$ different plates.

Example

Find the total number of factors of the number 720.

$720 = 2^4 \times 3^2 \times 5$.
Hence every factor will be equal to $2^a \times 3^b \times 5^c$, where
$a \in \{0, 1, 2, 3, 4\}, b \in \{0, 1, 2\}, c \in \{0, 1\}$.
The number of factors is the same as the number of possible ordered triples (a, b, c) is $(4+1)(2+1)(1+1) = 5 \cdot 3 \cdot 2 = 30$.

The formula for counting factors: if $n = p_1^{e_1} \cdot p_2^{e_2} \cdot \dots \cdot p_k^{e_k}$, then the total number of factors is $(e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$.

Developing ideas (4)

Our group consists of **2 male and 3 female students**. How many choices of **one male and one female** students for two different posts can we make if order matters?

Answer: $3 \times 2 + 2 \times 3 = 2 \times 2 \times 3 = 12.$

Disjoint events and the sum rule

Two events are said to be *disjoint* (or *mutually exclusive*) if they can't occur simultaneously.

Example: If we have 3 pairs of blue jeans and 2 pairs of black jeans, then there are $3 + 2 = 5$ different pairs of jeans to choose and wear.

The sum rule: If A and B are disjoint events and there are n_1 possible outcomes for event A and n_2 possible outcomes for event B then there are $n_1 + n_2$ possible outcomes for the event "either A or B".

Example

How many three-digit numbers begin with 3 or 4?

Answer: There are two disjoint cases: three-digit numbers beginning with a 3 and three-digit numbers beginning with a 4.

- By the product rule there are $10 \times 10 = 100$ three-digit numbers starting with a 3.
- By the product rule there are $10 \times 10 = 100$ three-digit numbers starting with a 4.
- By the sum rules there are $100 + 100$ three-digit numbers starting with a 3 or a 4.

Example

I wish to take two pieces of fruit with me for lunch. I have 3 bananas, 4 apples and 2 pears. How many ways can I select two pieces of fruit of different type?

Answer:

- banana and apple: 3×4 selections.
- banana and pear: 3×2 selections,
- apple and pear: 4×2 selections.
- As these sets of possibilities are disjoint, by the sum rule we have $12 + 6 + 8 = 26$ different ways of selecting two pieces of fruit of different types.

Set-theoretic interpretation

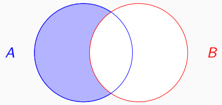
- If A and B are disjoint sets (that is, $A \cap B = \emptyset$) then $|A \cup B| = |A| + |B|$.

- Any sequence of k events can be regarded as an element of the Cartesian product $A_1 \times \cdots \times A_k$. This set has size $|A_1| \times \cdots \times |A_k|$.

Developing ideas (5)

*A computer password is a string of 8 characters, where each character is an uppercase letter or a digit. Each password must contain **at least one digit**.*

How many different passwords are there?



Answer: $(26 + 10)^8 - 26^8 = 2,612,282,842,880$

Note: lazy users

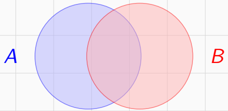
How many different 8-character passwords can be obtained by combining 3-letter word, a 4-letter word and 1 digit (for example HOT4FUZZ)? (According to `http://www.scrabblefinder.com` there are 1015 3-letter and 4030 4-letter English words.)

Answer:

$6 \times 10 \times 1015 \times 4030 = 245,427,000$ (about 0.009% of the previous number).

Developing ideas (6)

How many bit strings of length 8 start with 1 **or** finish with 00?



$$|A \cup B| = |A| + |B| - |A \cap B|$$

Answer: $2^7 + 2^6 - 2^5 = 2^5(4 + 2 - 1) = 32 \times 5 = 160.$

The subtraction rule

If there are n_1 possible outcomes for event A,
n_2 possible outcomes for event B and
n_3 of these outcomes are shared between A and B, then there are

$$n_1 + n_2 - n_3$$

possible outcomes for the event "A **or** B".

Developing ideas (7)

How many ways there are to select **2 representatives** (order is not important) from a group of 5 students?

Answer: $5 \times 4/2 = 10$.

The division rule

Given n possible outcomes, if

- some of the n outcomes are the same
- every group of indistinguishable outcomes contains exactly d elements

there are n/d different outcomes.

Summary

Four decomposition rules:

- The product rule: the total number of possible outcomes for the ordered sequence of events A **and** B is $n_1 \times n_2$.
- The sum rule: for two disjoint events A, B there are $n_1 + n_2$ possible outcomes in "A **or** B".
- The subtraction rule: in general case, there are $n_1 + n_2 - n_3$ possible outcomes for the event "A **or** B".
- The division rule: if every group of contains d indistinguishable outcomes, then there are n/d different outcomes.

Attendance code: 221804 and **DIY problems:**

- How many pairs (x, y) of positive integers satisfy the equation $xy = 2010$?
- How many line segments are formed if we place 10 points on a straight line?