Finding the “right” Curve

Recap : Parameterized Curves

- We have a function $f : \mathbb{C} \rightarrow \mathbb{C}$ that we wish to integrate between two points

$$p = x + \imath y$$
$$q = a + \imath b$$

- $f(r + \imath s) = u(r, s) + \imath\, v(r, s)$
- That is, we wish to evaluate

$$\int_p^q f(z)dz$$

- BUT : the Complex Numbers are *unordered*.

Solution – "reducing" to Reals

- Suppose we take *ANY* function $c : \mathbb{R} \to \mathbb{C}$ which has the property:

$$c(x) = f(x + \imath y)$$

AND

$$c(a) = f(a + \imath b)$$

- Then, we can use a "standard" substitution trick to rewrite our integral as

$$\int_x^a f\big(c(t)\big)c'(t)dt$$

Why does this help?

- Because now the range for our integral is between two *Real numbers*.
- This means that

$$\int_x^a f(z) = \int_x^a \Re(f(z))dz + \imath \int_x^a \Im(f(z))dz$$

- and now we are dealing with two REAL valued functions over the Reals.
- So, we can use *standard* methods.

How do we find a suitable $c : \mathbb{R} \to \mathbb{C}$*?*

A. The "*easy*" case.

$$f(r + \imath s) = u(r, s) + \imath\, v(r, s)$$

- Suppose that $u(r, s)$ *only* depends on r?
- Can use $c(t) = u(t) + \imath\, g(t)$
- What is $g(t)$?
- Any Real valued function with $g(x) = v(x, y)$ AND $g(a) = v(a, b)$.
- We can, in this case, treat y and b as *CONSTANTS*.

So, what happens?

- We need to have $g : \mathbb{R} \to \mathbb{R}$ with
$$g(x) = v(x,y) \;\; ; \;\; g(a) = v(a,b)$$
- So, we have *TWO* points lying on the *same* line
$$\big(x, v(x,y)\big) \;\; ; \;\; (a, v(a,b))$$
- This line has gradient $m(x,a) = \frac{v(a,b)-v(x,y)}{a-x}$ and some offset $K(x,a)$. [Recall line functions from Calculus earlier.]
- The curve we can use is, therefore
$$c(t) = u(t) + \imath(\, t \cdot m(x,a) + K(x,a))$$

Okay, but what if $u(r,s)$ depends on s?

- We can use the same "trick" to ensure that the Imaginary part of $c(t)$ behaves properly, ie so that
$$\Im\big(c(x)\big) = v(x,y)\ ;\ \ \Im\big(c(a)\big) = v(a,b)$$
- For the Real part, consider a function $\alpha : \mathbb{R} \to \mathbb{R}$ with
$$\alpha(x) = u(x,y)\ ;\ \ \alpha(a) = u(a,b)$$
- That is, we get rid of the "*two variable*" dependence.
- How do we do this?
- In general, this depends on the exact form of $u(r,s)$.

An Example

$$u(r, s) = r^k s^n$$

- Use: $\alpha(t) = t^k (d(t, b, y))^n$
- Where $d(x, b, y) = y$; $d(a, b, y) = b$
- There are methods to find a suitable d.

Summary

- Typically, the problem is less that of finding *any* parameterization $c(t)$ but more that of a finding a parameterization which will have *suitable properties*.
- For so-called "*contour integrals*" involving *closed paths* it is often the case that a contour which behaves in an "*extreme*" manner at selected points is the most useful to analyze.