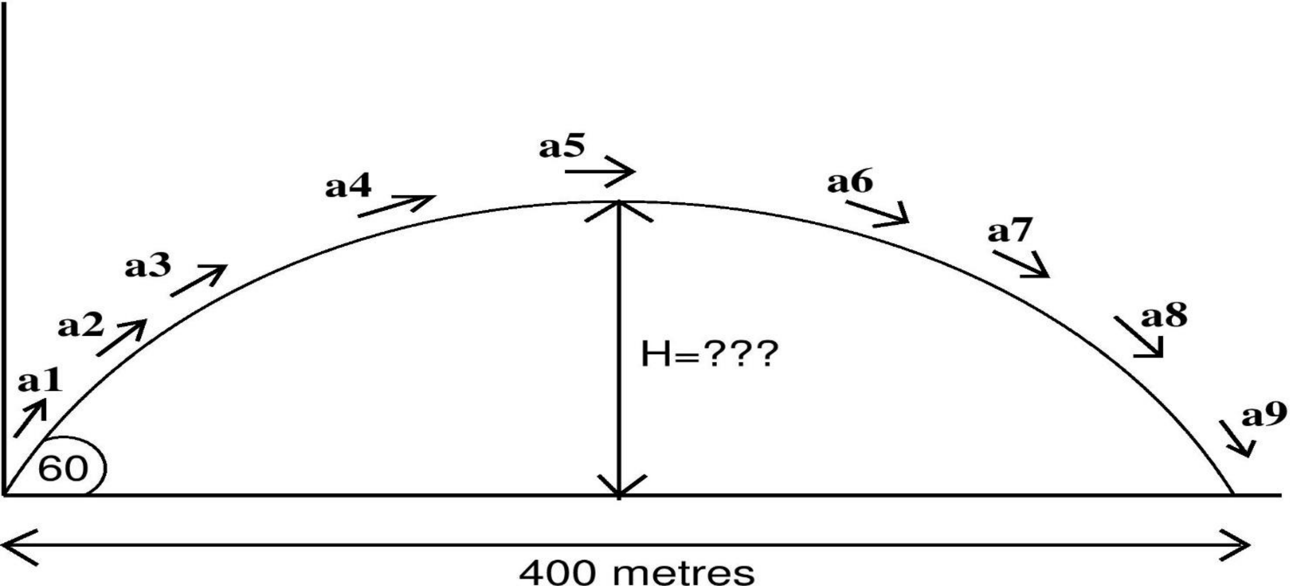
The First Derivative

What it is and how to find it

An Example Problem



What's the Problem?

- An *object* (eg an arrow) is launched at an angle of $60°$ landing 400 metres (measured horizontally) from its starting position. Suppose we know that the *height* of the arrow after it has travelled x metres is given by some function $f(x)$, ie after x metres a height of $y = f(x)$ metres is reached.

What is the *highest* height achieved and *when* (x)?

- Notice that $f(0) = f(400) = 0$.

A "crude" analysis

- Think of the path followed by the arrow as "*two lines*": $\{(0,0),(x,H)\}$ and $\{(x,H),(400,0)\}$;
- Then as "*four lines*":

$$\{(0,0),(x_1,f(x_1))\},$$
$$\{(x_1,f(x_1)),(x_2,H)\},$$
$$\{(x_2,H),(x_3,f(x_3))\},$$
$$\{(x_3,f(x_3)),(400,0)\}$$

- Then as 8, 16, 32, … , lines.
- The gradient of these lines "*mimics*" the arrow's trajectory.
- When this gradient is 0 the height is *maximal*. (Qn: *Why*?)

Summary

- We need a value $\{t\}$ for which $f(t) = H$. This will be when the gradient of the line "*touching*" $(t, f(t))$ is 0.
- How do we find "*the (function defining) gradients of lines touching*" $(x, f(x))$?
- We know *one point* on these lines: $(t, f(t))$
- If we have a point that is "*very close*" to $(t, f(t))$ say the point $(t + h, f(t + h))$ then the gradient of the line between these "*ought to be very close*" to that for the line touching $(t, f(t))$.

Stating the "obvious"

- What is the "*closest point*" to $(t, f(t))$?
- "*Obviously*" it's the point $(t, f(t))$ itself.
- What value of h does this correspond to?
- Again "*obviously*" the value $h = 0$.
- So the gradient of the line touching $(t, f(t))$ is the gradient of the line connecting $(t, f(t))$ and $(t, f(t))$.
- Which, of course, is $\frac{f(t)-f(t)}{t-t} = \frac{0}{0}$.
- Formalists view such "*derivations*" as rather flawed!!!

An alternative to the "obvious"

- Instead of using a treatment that leads to division by 0, let's consider how this gradient function behaves using *arbitrarily small values* of h.
- In other words,

$$\frac{f(t+h)-f(t)}{t+h-t} = \frac{f(t+h)-f(t)}{h}$$

- To capture "*as h gets arbitrarily small*", we write

$$\lim_{h \to 0} \frac{f(t+h)-f(t)}{h}$$

How does this help? A Worked Example:

$$f(x) = x^2$$

$$\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2 + 2hx + h^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{2hx+h^2}{h}$$

$= \lim_{h\to 0} 2x + h$ [division by h $(\neq 0)$]

$$= 2x$$

- *The function describing the gradient of lines touching the point (x, x^2) when $f(x) = x^2$ is $2x$.*
- This is *The First Derivative* of x^2 written $f'(x) = 2x$

Some Notation

- The phrasing "*The function describing the gradient of lines touching the point* $(x, f(x))$" is a little verbose.
- This function is called the *first derivative* of $f(x)$.
- The two (most common) notations are:

$$f'(x)$$

and (using the convention $y = f(x)$)

$$\frac{dy}{dx}$$

- The form $\dot{x}$ is seen in (older) Physics books (originally Newton).

Summary and Questions

- In general if we wish to look at how a function, $f(x)$, behaves we can study its *first derivative* $f'(x)$.
- The first derivative is a *function* and (from "*first principles*") is

$$f'(x) \stackrel{\text{def}}{=} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- Q1: Do we *always* need to analyse $f'(x)$ by working through this formation?
- Q2: *Can* we *always* analyse $f'(x)$ by working through this?

Some Answers

- Q1: Do we *always* need to analyse $f'(x)$ by working through this formation?
- A1: No.
- We can apply *standard rules* (Table 4.4, p. 137, textbook)
- Q2: *Can* we *always* analyse $f'(x)$ by working through this?
- A2: No.
- Some functions may be "*ill-behaved*" at (at least) one point.

  For example, $f(x) = {}^{1}/_{x}$, $(x = 0)$; $f(x) = \log x$ $(x \leq 0)$.

Eight Simple Rules for Finding First Derivatives

1. $f(x) = C \;;\; f'(x) = 0$ (*Constant* Rule)
2. $f(x) = g(x) + h(x) \;;\; f'(x) = g'(x) + h'(x)$ (*Sum* Rule)
3. $f(x) = g(x) \cdot h(x) \;;\; f'(x) = g'(x) \cdot h(x) + g(x) \cdot h'(x)$ (*Product* Rule)
4. $f(x) = g\big(h(x)\big) \;;\; f'(x) = g'(h(x)) \cdot h'(x)$ (*Chain*/*composition* Rule)
5. $f(x) = x^t \;;\; f'(x) = t \cdot x^{t-1}$ (*Power* Rule, valid for any $t \in R$)
6. $f(x) = \ln x \;;\; g(x) = \exp x \;;\; f'(x) = {}^1\!/_x \;;\; g'(x) = \exp x$
7. $f(x) = \sin x \;;\; g(x) = \cos x \;;\; f'(x) = \cos x \;;\; g'(x) = -\sin x$
8. $f(x) = \dfrac{g(x)}{h(x)} \;;\; f'(x) = \dfrac{g'(x)h(x) - g(x)h'(x)}{h(x) \cdot h(x)}$ (*Quotient*)

Examples

- $f(x) = x^2 + 2x + 5$
- $f'(x) = 2x + 2$ (*power* and *constant* rules)
- $f(x) = \tan x$; using the fact that $\tan x = \frac{\sin x}{\cos x}$
- $f'(x) = \frac{(\cos x)^2 + (\sin x)^2}{(\cos x)^2} = \frac{1}{(\cos x)^2}$ (*quotient* and *trig*. rules)
- $f(x) = (1 + 5x^2)^3$
- $f'(x) = 3(1 + 5x^2)^2(10x) = 30x(1 + 5x^2)^2$ (*chain* rule)

Where next?

- Our starting motivation was couched in terms of solving an *optimization problem*.
- In order fully to do so we need to look at first derivatives in a little more detail.
- Specifically the notion of “*critical points*” and how these may be classified.
- With the tests provided this will give some very basic (but useful) techniques for approaching optimization issues.
- These are the subject of the next part of this section.