Foundations of Computer Science Comp109

University of Liverpool

Boris Konev

konev@liverpool.ac.uk

Olga Anosova

O.Anosova@liverpool.ac.uk

Part 5. Combinatorics

Comp109 Foundations of Computer Science

Reading

- Discrete Mathematics with Applications, S. Epp, Chapter 9.
- Discrete Mathematics and Its Applications, K. H. Rosen, Sections 6.1, 6.3, 6.4

Contents

- Basics of counting
- Notation for sums and products. The factorial function.
- Counting permutations and combinations.
- Binomial coefficients.

Important note: **this material is the overall summary of Combinatorics topic, which does not include the whole range of problems covered in lectures. For full coverage, refer to separate lecture slides of Lectures 24, 25 and 26.**

Developing ideas (1)

All chairs in a room are labelled with a single digit followed by a lower-case letter. What is the largest number of differently numbered chairs?

Developing ideas (2)

How many different bit strings of length 8 are there?

- How many different bytes are there?

$$0000\,0000,\ \ 0000\,0001,\ \ 0000\,0010,\ \ 0000\,0011, \ldots$$

Developing ideas (3)

How many ways there are to select 3 students for a prospectus photograph (order matters) from a group of 5?

The product rule

If there is a sequence of k events with $n_1, \ldots, n_k$ possible outcomes, then the total number of outcomes for the sequence of k events is

$$n_1 \times n_2 \times \cdots \times n_k.$$

Example

How many distinct car licence plates are there consisting of six characters, the first three of which are letters and the last three of which are digits?

By the product rule, there exist

$$26 \times 26 \times 26 \times 10 \times 10 \times 10$$

different plates.

Developing ideas (4)

How many ways there are to select a male and a female student for a prospectus photograph (order matters) from a group of 2 male and 3 female students?

Disjoint events

Two events are said to be disjoint (or “mutually exclusive”) if they can’t occur simultaneously.

Example: If you have 3 pairs of blue jeans and 2 pairs of black jeans, then there are $3 + 2 = 5$ different pairs of jeans which are blue or black which you could wear.

The sum rule

If A and B are disjoint events and there are n_1 possible outcomes for event A and n_2 possible outcomes for event B then there are $n_1 + n_2$ possible outcomes for the event "either A or B".

Example

How many three-digit numbers begin with 3 or 4?

- There are two disjoint cases: three-digit numbers beginning with a 3 and three-digit numbers beginning with a 4.
- By the product rule there are $10 \times 10 = 100$ three-digit numbers starting with a 3.
- By the product rule there are $10 \times 10 = 100$ three-digit numbers starting with a 4.
- By the sum rules there are $100 + 100$ three-digit numbers starting with a 3 or a 4.

Example

I wish to take two pieces of fruit with me for lunch. I have three bananas, four apples and two pears. How many ways can I select two pieces of fruit of different type?

- If I select one of the three bananas and one of the four apples, then 3×4 selections can be made.
- If I select a banana and a pear, then 3×2 selections can be made.
- If I select an apple and a pear, then 4×2 selections can be made.
- As these sets of possibilities are disjoint

$$12 + 6 + 8 = 26$$

different ways of selecting two pieces of fruit of different types exist.

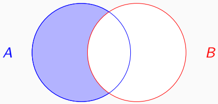
Set-theoretic interpretation

- If A and B are disjoint sets (that is, $A \cap B = \emptyset$) then $|A \cup B| = |A| + |B|$.
- Any sequence of k events can be regarded as an element of the Cartesian product $A_1 \times \cdots \times A_k$. This set has size $|A_1| \times \cdots \times |A_k|$.

Developing ideas (5)

A computer password is a string of 8 characters, where each character is an uppercase letter or a digit. Each password must contain at least one digit.

How many different passwords are there?



Answer

$$2,612,282,842,880$$

Note: lazy users

How many different 8-character passwords can be obtained by combining 3-letter word, a 4-letter word and a digit?
(According to `http://www.scrabblefinder.com` there are 1015 3-letter and 4030 4-letter English words.)

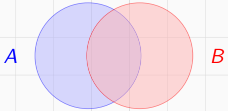
Answer

$$245,427,000 \text{ (about } 0.009\%)$$

Beware of passwords like HOT4FUZZ

Developing ideas (6)

How many bit strings of length 8 start with 1 or finish with 00?



$$|A \cup B| = |A| + |B| - |A \cap B|$$

The subtraction rule

If there are n_1 possible outcomes for event A, n_2 possible outcomes for event B and n_3 of these outcomes are shared between A and B then there are

$$n_1 + n_2 - n_3$$

possible outcomes for the event "A or B".

Developing ideas (7)

How many ways there are to select 2 representatives from a group of 5 students?

The division rule

Given n possible outcomes, if

- some of the n outcomes are the same
- every group of indistinguishable outcomes contains exactly d elements

there are n/d different outcomes.

Mini summary

Counting problems can be hard

Four decomposition rules:

- The product rule
- The sum rule
- The subtraction rule
- The division rule

To move further we need some mathematical notation

Sums and products of a sequence

Given a sequence of numbers

$$\ldots, a_1, a_2, a_3, \ldots, a_m, a_{m+1}, \ldots a_n, \ldots$$

we use the notation

- $\sum_{i=m}^{n} a_i$ to represent $a_m + a_{m+1} + \cdots + a_n$ and
- $\prod_{i=m}^{n} a_i$ to represent $a_m \times a_{m+1} \times \cdots \times a_n$

Examples

- $\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5$
- $\prod_{i=0}^{6} i^2 = 0^2 \times 1^2 \times 2^2 \times 3^2 \times 4^2 \times 5^2 \times 6^2 = 0.$

Notice that i is just an index, so

$$\sum_{i=1}^{5} i = \sum_{j=1}^{5} j = \sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5$$

Special case $m = n$

$$\sum_{i=m}^{n} = a_m, \quad \text{when } m = n$$

For example

$$\sum_{i=3}^{3} i = 3$$

Examples

We can express some equalities more neatly using this notation.

- In Part 1 of the module, we proved that

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}.$$

- In Part 2 of the module, we defined the Cartesian product $A_1 \times \cdots \times A_k$ of k sets.

$$A_1 \times \cdots \times A_k = \{(a_1, a_2, \ldots, a_k) \mid a_i \in A_i\}.$$

  The size of the Cartesian product is $\prod_{i=1}^{k} |A_i|$.

Note: Sums and products over sets of indices

Let $f : D \to \mathbb{R}$ be a function with some domain D.

Then for $S \subseteq D$,

- $\sum\limits_{i \in S} f(i)$ denotes the sum of $f(i)$ over all $i \in S$ and
- $\prod\limits_{i \in S} f(i)$ denotes the product of $f(i)$ over all $i \in S$.

The factorial function

The product $\prod_{i=1}^{n} i$ comes up so often that it has a name. It is called n factorial and is written as $n!$.

Examples:

- $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$.
- $3! = 1 \times 2 \times 3 = 6$.
- $1! = 1$.
- $0! = 1$.

Permutations

A permutation of a set is just an ordering of its elements.

Example: The permutations of the set $\{1, 2, 3\}$ are

- 1, 2, 3
- 1, 3, 2
- 2, 1, 3
- 2, 3, 1
- 3, 1, 2
- 3, 2, 1

By the product rule the number of permutations of an n-element set is

$$n! = n \times (n-1) \times \cdots \times 1$$

because there are n choices for the first element, then $n-1$ choices for the $2nd$ element, then $n-2$ choices for the 3rd element, and so on.

k-permutations

A selection of k distinct elements of a set, where order matters, is called a k-permutation of the set.

The number k-permutations of an n-element set is

$$P(n, k) = n \times (n-1) \times \cdots \times (n-(k-1)) = \frac{n!}{(n-k)!}.$$

Examples

- How many ways there are to select 3 students for a prospectus photograph (order matters) from a group of 5?

$$P(5,3) = 5 \times 4 \times 3 = 5!/2! = 60$$

- How many length-4 sequences of distinct digits are there?

$$P(10,4) = 10 \times 9 \times 8 \times 7 = 10!/6! = 5040$$

- How many four-letter words can be made with distinct letters from the list a, g, m, o, p, r?

$$P(6,4) = 6 \times 5 \times 4 \times 3 = 6!/(6-4)! = 360$$

Examples

- I have a jar with 20 different sweets. Three children come in, and each take one. How many different outcomes are there?

$$P(20, 3) = 20 \times 19 \times 18 = 20!/17! = 6840$$

- I have a jar with 3 different sweets. Three children come in, and each take one. How many different outcomes are there?

$$P(3, 3) = 3 \times 2 \times 1 = 3!/0! = 6$$

Counting k-combinations

A size-k subset is called a *k-combination*

The number of k-combinations of a set of size n is

$$C(n,k) = \frac{n!}{(n-k)!k!}.$$

Proof:

- The number of k-permutations of the set is $P(n,k) = \frac{n!}{n-k}!$
- A k-permutation is an ordering of k distinct elements of the set
- Each size-k subset has $k!$ orderings, so it corresponds to $P(k,k) = k!$ of the k-permutations
- By the division rule, $C(n,k) = \frac{P(n,k)}{P(k,k)} = \frac{n!}{(n-k)!k!}$

Examples

- The number of size-2 subsets of $\{1, 2, 3, 4, 5\}$ is

$$C(5, 2) = \frac{5!}{(5-2)!2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1) \times (2 \times 1)} = \frac{5 \times 4}{2} = 10$$

  The ten subsets are $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{1, 5\}$, $\{2, 3\}$, $\{2, 4\}$, $\{2, 5\}$, $\{3, 4\}$, $\{3, 5\}$, and $\{4, 5\}$.

- The number of size-3 subsets of $\{1, 2, 3, 4, 5\}$ is

$$C(5, 3) = \frac{5!}{(5-3)!3!} = 10$$

  (The subsets are the complements of the ones above)

Examples

- The number of size 1 subsets of $\{1, 2, 3, 4, 5\}$ is

$$C(5, 1) = \frac{5!}{(5-1)!1!} = 5,$$

  which is also the number of size 4 subsets of the set.
- The number of size-0 subsets of $\{1, 2, 3, 4, 5\}$ is

$$C(5, 0)\frac{5!}{(5-0)!0!} = 1,$$

  which is also the number of size-5 subsets $C(5, 5)$.

Example

Twelve people, including Mary and Peter, are candidates to serve on a committee of five. How many different committees are possible? Of these how many

1. *contain both Mary and Peter?*
2. *contain neither Mary and Peter?*
3. *contain either Mary or Peter (but not both)?*

Solution: There are

$$C(12, 5) = \frac{12!}{(12-5)!5!} = 792$$

possible committees.

Example continued

(1.) If Mary and Peter are already included, we have to select three more committee members from the remaining ten available people. This can be done in

$$C(10,3) = 120$$

ways.

(2.) If Mary and Peter are excluded we have to select five committee members from the remaining 10 people. This can be done in

$$C(10,5) = 252$$

ways.

Example continued

(3.) The number of committees containing Mary and not Peter is just

$$C(10, 4)$$

The same number of committees contain Peter and exclude Mary. Therefore

$$2 \times C(10, 4) = 420$$

committees contain exactly one of Peter and Mary.

Binomial coefficients

The quantity $C(n, k)$, which gives the number of k-combinations of a set of size n, is called a binomial coefficient.

It is also written as

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

(Mathematicians prefer this notation)

The binomial theorem

For every natural number n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Informally:

$$\begin{aligned}(a+b)^n &= \underbrace{(a+b)\times(a+b)\times\cdots\times(a+b)}_{n} \\ &= \sum_{k=0}^{n} \sum_{\substack{S \text{ is a} \\ k\text{-combination}}} a^k b^{n-k}\end{aligned}$$

Pascal's triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
\qquad
\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= a+b \\
(a+b)^2 &= a^2+2ab+b^2 \\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\
(a+b)^5 &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5
\end{aligned}
$$

Binomial coefficient identity

$$\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}.$$

Proof:

- $\binom{n+1}{r+1}$ is the number of ways to choose a subset of size $r+1$ from a set of size $n+1$.
- Suppose the set is $\{x_1, x_2, \ldots, x_{n+1}\}$.
- How many subsets include x_{n+1}? We just choose a size-r subset from $\{x_1, x_2, \ldots, x_n\}$, so there are $\binom{n}{r}$ ways to do it.
- How many subsets exclude x_{n+1}? We just have to choose a subset of size $r+1$ from $\{x_1, x_2, \ldots, x_n\}$, so there are $\binom{n}{r+1}$ ways to do it.
- These outcomes are disjoint, so the total number of subsets is the sum of $\binom{n}{r}$ and $\binom{n}{r+1}$.

Counting and probabilities

Counting also helps us to answer questions like

- What are the odds of winning the National Lottery?
- What payoffs should you expect?
- What is more likely in poker, full house or 4-of-a-kind?