Foundations of Computer Science Comp109

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Recap: Properties of relations

- **Reflexivity:**
- **Symmetry:**
- **Antisymmetry:**
- **Transitivity:**
- The **transitive closure** R^* of a realation R on a set A is

Recap: Transitivity and composition

A relation S is transitive if and only if $S \circ S \subseteq S$.

This is because

$$S \circ S = \{(a, c) \mid \text{ exists } b \text{ such that } aSb \text{ and } bSc\}.$$

Let S be a relation. Set $S^1 = S$, $S^2 = S \circ S$, $S^3 = S \circ S \circ S$, and so on.

Theorem Denote by S^* the transitive closure of S. Then xS^*y if and only if

Recap: Transitive closure in matrix form

The relation R on the set $A = \{1, 2, 3, 4, 5\}$ is represented by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Is R transitive?

Answer: Computation

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix}$$

$$R \circ R = \{(a, c) \mid \text{ exists } b \in A \text{ such that } aRb \text{ and } bRc\}.$$

Detour: Warshall's algorithm

```
def warshall(a):
    n = len(a)
    for k in range(n):
        for i in range(n):
            for j in range(n):
                a[i][j] = (a[i][j] or
                           (a[i][k] and a[k][j]))
    return a

print(warshall([[1,0,0,1,0],
                [0,1,0,0,1],
                [0,0,1,0,0],
                [1,0,1,0,0],
                [0,1,0,1,0]]))
```

Motivation: voting and ranking

Suppose we need to rank 3 items A,B,C by voters preferences. A poll shows that 2/3 prefer A to B and 2/3 prefer B to C. That means that most voters prefer A to C, right?

Transitivity

Transitivity is crucial in understanding of causality, aka the famous chicken or the egg problem.

Intransitivity contradicts our spacial intuition, as demonstrated by the Penrose staircase.

Ascending & Descending by M.Escher

Important relations: Equivalence relations

Definition A binary relation R on a set A is called an *equivalence relation* if it is reflexive, transitive, and symmetric.

Example: equivalence or not?

- The relation *has the same age* on the set of people.
- Same length on the set of cars.
- Same tax band on the set of salaries.

Examples: number line

Are the following relations on pairs (x, y) of real numbers equivalence relations?
(1) $x < y$;
(2) $x \leq y$;
(3) distance $|x - y| \leq 1$.

Example

The relation R on the non-zero integers: $xRy \iff xy > 0$.

Functions and equivalence relations

Let $f : A \to B$ be a function. Define a relation R on A by

$$a_1 R a_2 \Leftrightarrow f(a_1) = f(a_2).$$

A is a set of cars, B is the set of real numbers, and f assigns to any car in A its length. Then $a_1 R a_2$ if and only if a_1 and a_2 are of the same length.

Partition of a set

A *partition* of a set A is a collection of *non-empty* subsets $A_1, \ldots, A_n$ of A satisfying:

- $A = A_1 \cup A_2 \cup \cdots \cup A_n$;
- $A_i \cap A_j = \emptyset$ for $i \neq j$.

The A_i are called the *blocks* of the partition.

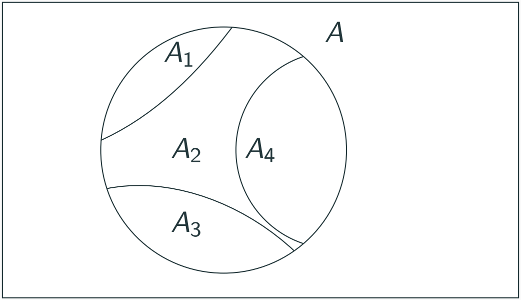

Figure 1: Partition of A

Equivalence classes

Definition The *equivalence class* E_x of any $x \in A$ is defined by

$$E_x = \{y \mid yRx\}.$$

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Example: equivalence classes for the relation ‘same tax band’ on the set of salaries

Equivalence relation vs partition

Theorem (From equivalence relation to partition) Let R be an equivalence relation on a non-empty set A. Then the equivalence classes $\{E_x \mid x \in A\}$ form a partition of A.

Theorem (From partition to equivalence relation) Suppose that $A_1, \ldots, A_n$ is a partition of A. Define a relation R on A by setting: xRy if and only if there exists i such that $1 \leq i \leq n$ and $x, y \in A_i$. Then R is an equivalence relation.

DIY task: try to prove those theorems.

Application: Rational numbers

Recall: r is rational if $r = \frac{k}{l}$, where k, l are integers and $l \neq 0$.

Evidently, $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \dots$

Consider the set $A = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid b \neq 0\}$ and relation R on A defined as:

$$(a, b)R(c, d) \Leftrightarrow ad = bc$$

R is an equivalence relation on A and the set of all equivalence classes of R is the set of rationals.

Summary

- *Equivalence relation* is a binary relation that is reflexive, transitive, and symmetric.
- A *partition* of a set A is a collection of *non-empty* subsets $A_1, \ldots, A_n$ of A:
  - $A = A_1 \cup A_2 \cup \cdots \cup A_n$;
  - $A_i \cap A_j = \emptyset$ for $i \neq j$.
- Equivalence relation partitions the set into well-defined non-overlapping equivalence classes.

DIY problem: Consider friendship relations (on Facebook or in real life): we say (A, B), if a person A knows a person B. Is it an equivalence relation?