Foundations of Computer Science Comp109

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Part 3. Relations

Comp109 Foundations of Computer Science

Reading

- Discrete Mathematics with Applications S. Epp, Chapter 8.
- **Discrete Mathematics and Its Applications** K. Rosen, Chapter 9

Contents

Motivation

- Intuitively, there is a "relation" between two things if there is some connection between them.
  E.g.
  - 'friend of'
  - $a < b$
  - m divides n
- Relations are used in crucial ways in many branches of mathematics
  - Equivalence
  - Ordering
- Computer Science

A database table $\approx$ relation

TABLE 1 Students.

Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

Cartesian product

The *Cartesian product* $A \times B$ of sets A and B is the *set* consisting of all **ordered** pairs (a, b) with $a \in A$ and $b \in B$, i.e.,

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Note that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

Sets $\{1, 2\} = \{2, 1\}$, but $(1, 2) \neq (2, 1)$.

Example

- Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then

$$\underline{A \times B} = \{(1, a), (2, a), (1, b), (2, b), (1, c), (2, c)\}.$$

- $B \times A = \{(a,1), (a,2), (b,1), \ldots$

- $B \times B = \{(a,a), (a,b), \ldots$

If A and B are finite, what is $|A \times B|$?

Example

- Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then

$$A \times B = \{(1, a), (2, a), (1, b), (2, b), (1, c), (2, c)\}.$$

- $B \times A =$

- $B \times B =$

If A and B are finite, what is $|A \times B|$?
$|A| \times |B|$

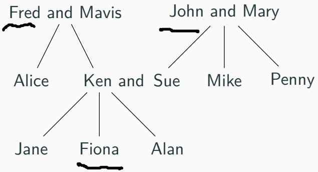
Binary relation

A *binary relation* between two sets A and B is a subset R of the Cartesian product $A \times B$, i.e. $R \subseteq A \times B$.

For each $(a, b) \in R$ we can also write aRb.

If $A = B$, then R is called *a binary relation on A*.

Example: Family tree



- $R = \{(x, y) \mid\ x \text{ is a grandfather of } y\ \} =$

{(Fred, Jane), ...

- $S = \{(x, y) \mid\ x \text{ is a sister of } y\ \} =$

{(Jane, Fiona), (Jane, Alan), (Fiona, Jane)

... (

Example 2

Write down the ordered pairs belonging to the following binary relations between $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6\}$:

- $U = \{(x, y) \in A \times B \mid x + y = 9\}$;

- $V = \{(x, y) \in A \times B \mid x < y\}$.

$= \{(1,2),(1,4),(1,6),(3,4)\ldots$

Example 3

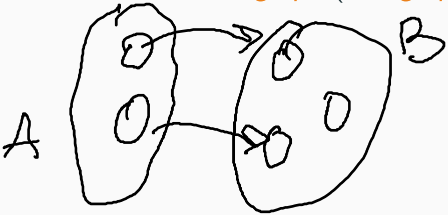
Let $A = \{1, 2, 3, 4, 5, 6\}$. Write down the ordered pairs belonging to

$$R = \{(x, y) \in A \times A \mid \ x \text{ is a divisor of } y \ \}.$$

$$= \{(1,1), (1,2) \dots$$
$$(2,4), (2,2), (2,6), (3,3), (3,6),$$
$$(4,4), (5,5), (6,6)\}$$

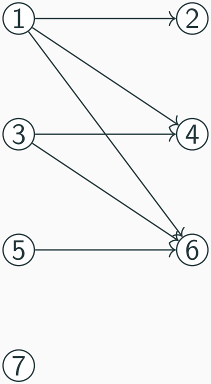
Representation of binary relations: directed graphs

- Let A and B be two finite sets and R a binary relation between these two sets (i.e., $R \subseteq A \times B$).
- We represent the *elements* of these two sets as *vertices* of a graph.
- For each $(a, b) \in R$, we draw an *arrow* linking the related elements.
- This is called the *directed graph* (or *digraph*) of R.



Example

Consider the relation V between $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6\}$ such that $V = \{(x, y) \in A \times B \mid x < y\}$.



digraph of V

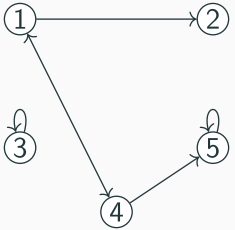
Binary relations on a single set

A binary relation between a set A and itself is called *a binary relation on A*.

To represent a binary relation on A, we use a directed graph with a single set of vertices representing the elements of A.

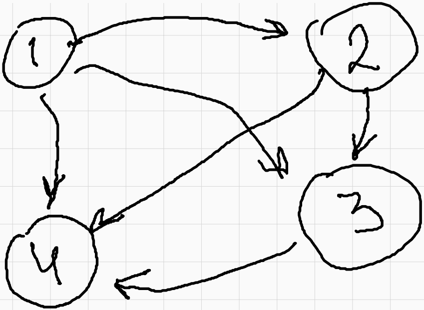
Example. Consider the relation $V \subseteq A \times A$ where $A = \{1, 2, 3, 4, 5\}$ and

$$V = \{(1,2), (3,3), (5,5), (1,4), (4,1), (4,5)\}.$$

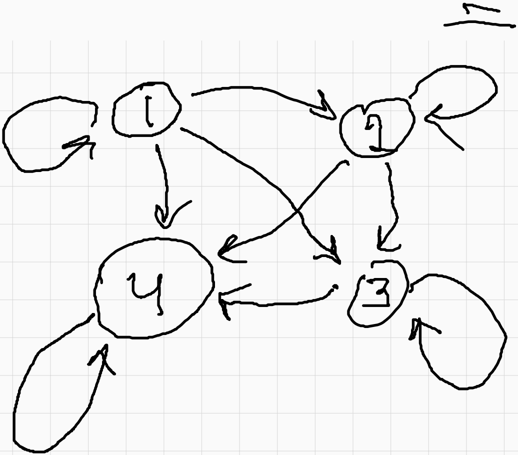


digraph of V

Example: $A = \{1, 2, 3, 4\}$, $R = \{(x, y) \in A \times A \mid x < y\}$



Example: $A = \{1, 2, 3, 4\}$, $R = \{(x, y) \in A \times A \mid x \leq y\}$



Example: functional relations

- Recall that a *function* f from a set A to a set B assigns exactly one element of B to each element of A.
  - Gives rise to the relation $R_f = \{(a, b) \in A \times B \mid b = f(a)\}$
- If a relation $S \subseteq A \times B$ is such that for every $a \in A$ there exists *at most* one $b \in B$ with $(a, b) \in S$, relation S is functional.
- Functions are sometimes introduced through functional relations:
  A functional relation is called a *partial function* (or *partial map*).
  If a functional relation is defined for *all* elements of A, then this relation is a *total function* (or simply a *function*).

Example

$A = \{i \in \mathbb{N} \mid i < 10\}$, $B = \{i \in \mathbb{N} \mid 5 < i < 15\}$,
$R = \{((x, y) \in A \times B \mid y = 2x)\}$

$A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \cancel{10}\}$

$B = \{6, 7, \ldots, 14\}$

$\{(3, 6), (4, 8), (5, 10) \ldots$

Inverse relation

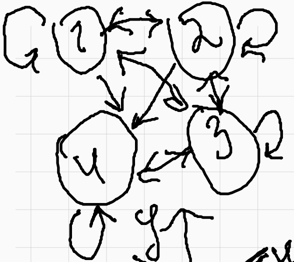
Given a relation $R \subseteq A \times B$, we define the *inverse relation* $R^{-1} \subseteq B \times A$ by

$$R^{-1} = \{(b, a) \mid (a, b) \in R\}.$$

Example. The inverse of the relation *"is a parent of"* on the set of people is the relation *"is a child of"*.

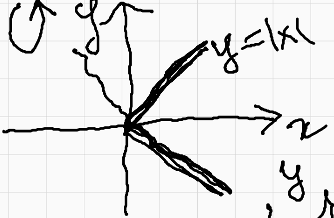
If you have a digraph representation of a relation, how do you get the inverse?
Answer: reverse the arrows.

Example: $A = \{1, 2, 3, 4\}$, $R = \{(x, y) \mid x \leq y\}$



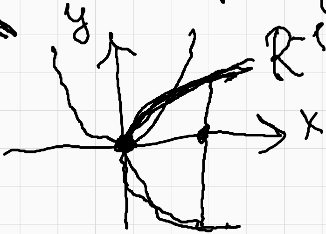
$y \leq x$

$R = \{(x, y) \in \mathbb{R} : y = |x|\}$



R^{-1}

$R = \{(x, y) \in \mathbb{R} : y = x^2\}$



$y = \pm\sqrt{x}$

Composition of relations

Let $R \subseteq A \times B$ and $S \subseteq B \times C$. The *composition* (or *relative multiplication*) of R and S, denoted by $S \circ R$, is the binary relation between A and C given by

$$S \circ R = \{(a, c) \mid \text{ exists } b \in B \text{ such that } aRb \text{ and } bSc\}.$$

Example: If R is the relation *is a sister of* and S is the relation *is a parent of*, then

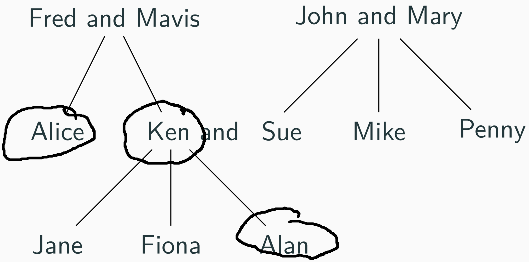
- $S \circ R$ is the relation *"is an aunt of"*;
- $S \circ S$ is the relation *"is a grandparent of"*.

Example

R : *is a sister of*

S : *is a parent of*

$S \circ R = \{(a, c) \mid$ exists $b \in B$ such that aRb and $bSc\}$.



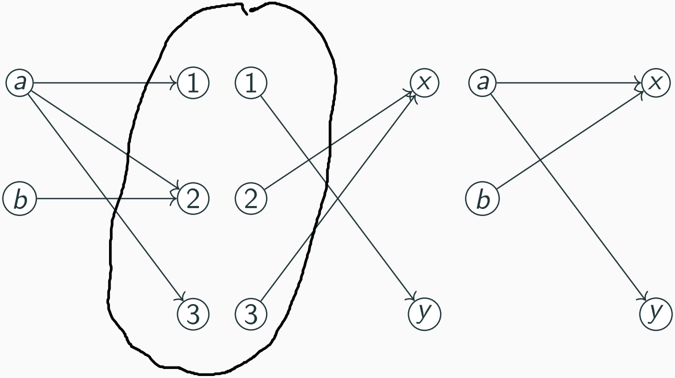
S R

Alice R Ken and Ken S Alan so Alice $S \circ R$ Alan.

Penny R Sue and Sue S Jane so Penny $S \circ R$ Jane.

Fred S Ken and Ken S Fiona so Fred $S \circ S$ Fiona.

Digraph representation of compositions



R S $S \circ R$

DIY Example

A – set of people, B – set of countries

$R \subseteq A \times A$, $R(x, y)$ represents *"x is a friend of y"*

$S \subseteq A \times B$, $S(u, v)$ represents *"u visited v"*

Create your own example of R, S and both their compositions.

Summary

Attendance code: 168020

- The *Cartesian product* $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$.
- A *binary relation* R between two sets A and B is $R \subseteq A \times B$, which can be represented by a *digraph*.
- Function = functional relation.
- $R^{-1} = \{(b, a) \mid (a, b) \in R\}$.
- $S \circ R = \{(a, c) \mid \text{ exists } b \in B \text{ such that } aRb \text{ and } bSc\}$.