Spectral Methods in CS

“Only big words for ordinary things on account of the sound.”

James Joyce

Ulysses

(*Lestrygonians* chapter)

Background

- The term *Spectral Method* refers to a range of techniques used to study the structure of *matrices*.
- These methods have been widely used in settings where matrix forms arise as a *basic modelling technique*, eg

Graphs (both directed and undirected)

Images (jpeg, gif, etc)

Data Science

Machine Learning

Game Theory

What this section involves

- We first return to reviewing properties of *matrices* extending from ideas introduced when considering *vectors* earlier.
- We introduce the *Eigenvalue Problem*, a key element in the study of *Spectral Methods*.
- We consider some *computational issues* involving finding *eigenvalues* and *eigenvectors*.
- Finally we illustrate *use in CS* through two examples:

Google's Page Ranking algorithm

Image Compression

Matrices Revisited

- We looked briefly at matrices in Section 2 and have seen other instances mentioned in contexts such as the *Hessian* or the *Fourier Transform*.
- There are two important concepts we did not review:

The notion of *Inverse Matrix*

The *Determinant* of a Matrix

- First recall that the *Identity Matrix*, I, has entries all 0 apart from those whose *column* and *row index* are equal (the *diagonal* elements): the diagonal elements equal 1. For any A,

$$A \cdot I = I \cdot A = A.$$

Matrix Inverse

- For an $n \times n$ matrix, A, its *inverse* is the matrix denoted A^{-1} for which:

$$A \cdot A^{-1} = A^{-1} \cdot A = I$$

- Not *every* matrix has an inverse.
- The term *singular matrix* is used for matrices with *no inverse*.
- Any matrix having an inverse is called *non-singular*.

Matrix Determinant

- For any Real-valued $n \times n$ matrix, A, its *determinant* is a Real value whose properties are key to the *existence* of an *inverse matrix* A^{-1} .
- In other words whether A is *singular* or *non-singular*.
- The determinant also is central to *The Eigenvalue Problem*.
- We refer to pages 375–379 of the module text for computational methods.
- Our principle interest is the following:

$$\det A \neq 0 \Leftrightarrow A^{-1} \text{ exists, ie } A \text{ is non-singular}$$

The Eigenvalue Problem

- Given an $n \times n$ matrix, A:

  Find *all scalar values*, λ, for which there is an n-vector, $\underline{x}_\lambda$, satisfying:

$$A \cdot \underline{x}_\lambda = \lambda \, \underline{x}_\lambda$$

  These scalar values are called the *eigenvalues* of A.

- The n-vectors, $\underline{x}_\lambda$, are called the *eigenvectors of* A (wrt λ).
- Eigenvectors of the matrix formed from a set of *linked web pages* form the basis of *Google's Page Rank* approach.
- This derives from a famous result of the early 20^{th} century:

  The Perron-Frobenius Theorem

More about Eigenvalues

- The case $\underline{x} = \underline{0}$ always produces a solution for trivial reasons.
- Therefore, in finding eigenvectors, this case is *not* considered.
- The case, $\lambda = 0$, *is* of interest.
- First note that if A is *non-singular* then only *trivial solutions* exist. (textbook, pp. 389–390).
- In general finding eigenvalues involves finding λ for which

$$(A - \lambda I)\underline{x}_{\lambda} = \underline{0}$$

- That is to say for which: $\det(A - \lambda I) = 0$.
- Notice that $(A - \lambda I)$ is the matrix A with λ subtracted from each *diagonal entry*.

Eigenvalues, Determinants, & Roots

- The form $\det(A - \lambda I) = 0$ has an interpretation involving *roots of a polynomial*.
- Since $\det A$ (with some parameter, λ, say) may be described as a polynomial of degree n in λ (with A an $n \times n$ matrix) the solutions to $\det(A - \lambda I) = 0$ are exactly *the roots* of this polynomial which is called the *Characteristic Polynomial* of A.
- This is denoted by $\chi(A)$.
- Details may be found on pages 381–382, 391–393.

Two important consequences

1. An $n \times n$ matrix has n eigenvalues (although these are *not necessarily distinct*, cf the situation with polynomial roots).
2. Eigenvalues may be *Complex Numbers*.

- In principle we can extend the *domain* of matrices to the set of *Complex Numbers* instead of only *Reals*.
- It is also possible to identify classes of *Real-valued matrix* whose eigenvalues are *guaranteed all to be Real*:

Symmetric matrices - $[a_{ij}] = [a_{ji}]$.

Ordering Eigenvalues and Dominance

- Conventionally the eigenvalues are written
$$\sigma(A) = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k, \ldots, \lambda_n)$$
- This is called the *spectrum* of A.
- The convention used is: $|\lambda_k| \geq |\lambda_{k+1}| \; \forall \; 1 \leq k < n$.
- The value λ_1, *if unique*, is called the *Dominant Eigenvalue* of A.
- The *largest* eigenvalue is referred to as the *Spectral Radius*.
- For *Complex values*, if λ is an eigenvalue of A then so too is $\overline{\lambda}$.
- We consider *Computational methods* in the next lecture.