Foundations of Computer Science Comp109

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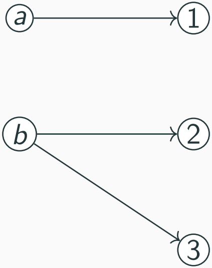
Recap: functions, function classification and pigeonhole principle.

- A **function** $f : A \to B$ is an assignment such that $\forall a \in A \exists$ one $b \in B : f(a) = b$.
- A is the **domain**, B is the **codomain**, $f(A)$ is the **range**.
- **Injective** function: $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$ for all $a_1, a_2 \in A$.
- **Surjective** function: $\forall b \in B \quad \exists a \in A$ such that $b = f(a)$.
- **Bijective** function: both injective and surjective.
- **Inverse function** $f^{-1} : Y \to X$ such that $f(a) = b \iff f^{-1}(b) = a$.

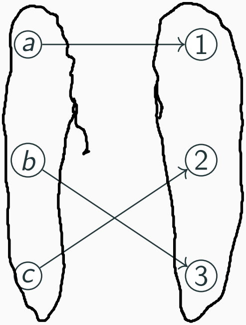
Recap quiz: Classify $f : \{a, b, c\} \to \{1, 2, 3\}$ given by



function No

inj
surj
bij

Recap quiz: Classify $g : \{a, b, c\} \to \{1, 2, 3\}$ given by



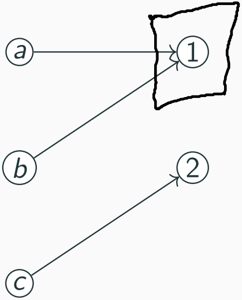
f : +

inj : +

surj : +

bij : +

Recap quiz: Classify $h : \{a, b, c\} \to \{1, 2\}$ given by



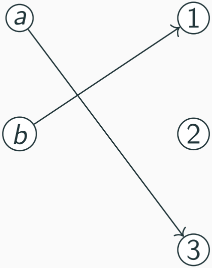
f : +

inj : −

sur : +

bij : −

Recap quiz: Classify $h_1 : \{a, b, c\} \to \{1, 2, 3\}$ given by



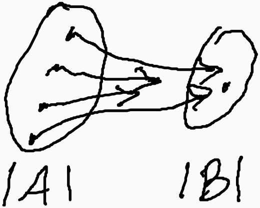
funct: +

inj: +

sur: −

bij: −

Recap: The pigeonhole principle (Dirichlet's principle).



Let $f : A \to B$ be a function where A and B are finite sets.

The *pigeonhole principle* states that if $|A| > |B|$ then $\exists a_1, a_2 \in A : a_1 \neq a2$ and $f(a_1) = f(a_2)$.

Recap Example solution: birthday problem.

Problem. There are 15 people on a bus. Show that at least two of them have a birthday in the same month of the year.

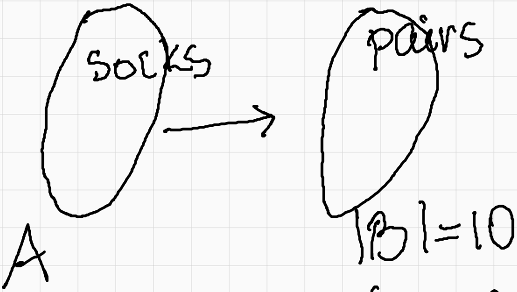
Let $A = \{$ people on the bus$\}$ and $B = \{$ the 12 months of the year$\}$.

Consider the function $f = Birthday\_month : A \to B$ which assigns each person to the month in which their birthday falls.

Then $|A| > |B|$, hence, by the pigeonhole principle, there are persons $a, b \in A$ where $a \neq b$ and $f(a) = f(b)$, that is their birth months are the same.

DIY Example solution: dark socks puzzle

Imagine you have 10 pairs of socks in a drawer. How many socks would you need to pull out blindly in a completely dark room to ensure you have at least one matching pair?



$$\exists a_1, a_2 : f(a_1) = f(a_2)$$

$$|A| > |B|$$

$$|A| \geq 10 + 1 = 11$$

Example: car number plates

How many different standard GB car number plates should be considered to guarantee that at least two of them begin with the same letter of the alphabet and end with the same letter of the alphabet?

Assume for that a standard number plate consists of 2 letters+2 numbers+3 letters: $\boxed{\text{BD51 SMR}}$

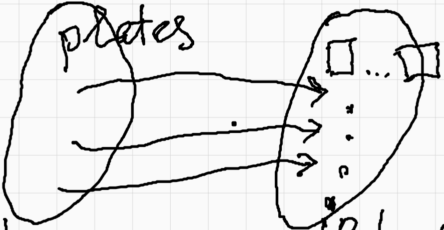
and assume for simplicity that all letters are possible.

A □ 26

x □ 26

y □ 26

z □ 26



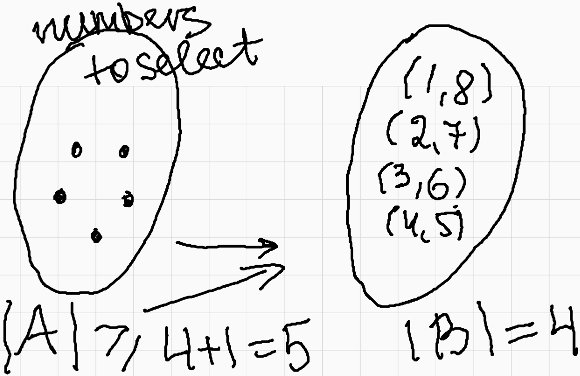
$|A| \geq 26 \times 26 + 1$

$= 677$

$|B| = 26 \times 26$

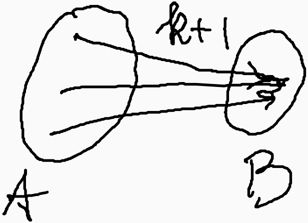
Example: ensure the total

What is the smallest number that should be selected from the numbers $1, 2, 3, 4, 5, 6, 7, 8$ to guarantee that there will always be two of the numbers that sum to 9?



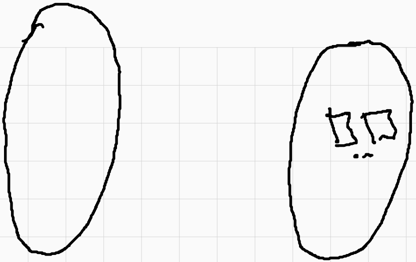
Extended pigeonhole principle

Consider a function $f : A \to B$ where A and B are finite sets and $|A| > k|B|$ for some natural number k. Then, there is a value of f which occurs at least $k + 1$ times.



Example: extended car plate coverage

In the assumptions made earlier, how many different standard GB car number plates should be considered to guarantee that at least five of them begin with the same letter of the alphabet and end with the same letter of the alphabet?



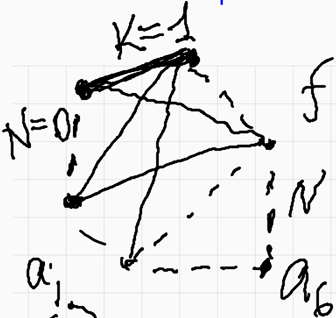
A $\qquad |B| = 26 \times 26$

$|A| > 4|B| = 4 \times 26 \times 26$

$|A| \geq 4 \times 26 \times 26 + 1$

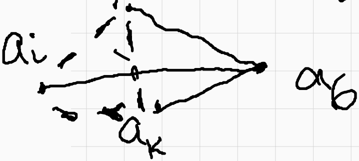
Example: Theorem of Friends and Strangers (Ramsay's theorem)

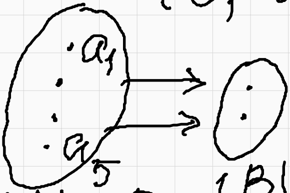
Show that in any group of six people there are either three who all know each other or three complete strangers.



$$f : \{a_1, a_2, \dots a_5\} \to \underbrace{\{0, 1\}}^{B}$$

$$f(a_i) = \begin{cases} 1, & a_i \text{ Knows } a_6 \\ 0, & a_i \text{ Not know } a_6 \end{cases}$$





$|B| = 2$

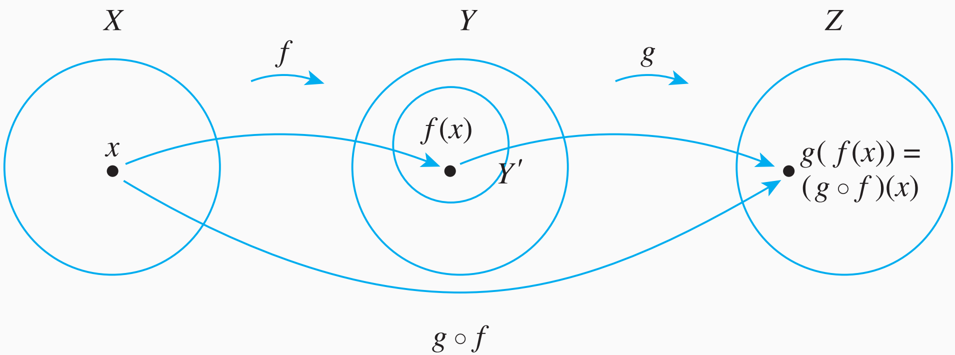
$|A| = 5 = 2|B| + 1$

$$\Rightarrow \exists\, a_i, a_j, a_k : f(a_i) = f(a_j) = f(a_k) = 1 \text{ (or } 0)$$

Composition of functions

If $f : X \to Y$ and $g : Y \to Z$ are functions, then their *composition* $g \circ f$ is a function from X to Z given by

$$\boxed{(g \circ f)(x)} = g(f(x)).$$



Example

Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ and $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = 4x + 3$.

- $g \circ f(x) =$

DIY

- $f \circ g(x) =$

- $f \circ f(x) =$

- $g \circ g(x) =$

DIY: clasification of compositions

What can you state about the composition of two

- injective functions,
- surjective functions,
- bijective functions?

Summary

Attendance code: 333043

- General properties of functions could guarantee some general properties of function outputs.
- The **extended pigeonhole principle**: if $|A| > k|B|$ for some $k \in \mathbb{N}$ and a function $f : A \to B$, then there is a value of f which occurs at least $k+1$ times.
- **Composition** $g \circ f : X \to Z$ of two functions $f : X \to Y$ and $g : Y \to Z$ is defined by $(g \circ f)(x) = g(f(x))$.