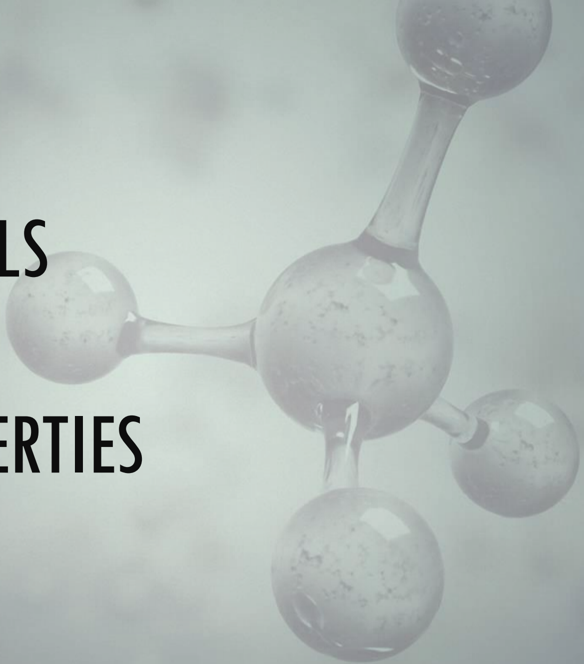
# POLYNOMIALS AND THEIR PROPERTIES



# IRRATIONALS REVISITED

- Quantities such as $\sqrt{2}=2^{1/2}$ or $\sqrt[3]{20}=20^{1/3}$ can be seen as *solutions* to particular *identities*.
- That is values of $x$ for which some *function*, $p(x)$, satisfies $p(x)=0$ .
- For example,

$$2^{1/2}: p(x) = x^2 - 2$$

$$20^{1/3}: p(x) = x^3 - 20$$

# POLYNOMIALS & THEIR ROOTS

- In general, a *polynomial* of *degree* $k$ in $x$ is defined through $k+1$ *coefficients*,

$$(c_0, c_1, \dots, c_k)$$

$$p(x) = \sum_{n=0}^{k} c_n x^n$$

- The coefficient, $c_t$ , multiplies $x^t$ when $p(x)$ is evaluated at some $x \coloneqq \alpha$. This is written as $p(\alpha)$.

# MORE ABOUT COEFFICIENTS

- The number types used to choose coefficients can be from any of the sets we have seen so far.
- Coefficients can also be limited to finite subsets of these, e.g. $Z_m$ the integers $\{0,1,2, \dots, m-1\}$ with modulo $m$ arithmetic.
- If $H$ is such a set, $H_k[X]$ or $H[X]$ , is used for the set of (degree $k$) polynomials with coefficients in $H$.

# ROOTS OF POLYNOMIALS

- A *root* of $p(x)$ is a value, $r$, of $x$ for which $p(r) = 0$
- A *polynomial* of *degree* $k$ in $x$ has $k$ roots,
$$(r_1, r_2, \dots, r_k)$$
- These are *not*, necessarily, distinct.
- It is also *possible* that some are not Real numbers.
- We consider this possibility later in the module.

# SOME EXAMPLES

- $x^2 - 1$ (degree $2$, roots $< -1,1 >$ )
- $x^2 - 2x + 1$ (degree $2$, roots $< 1,1 >$ )
- $x^6 - x^4 + x^3 + x^2 - 10$ (degree $6$)
- $x^4 + x^3 + x^2 + x + 1$ (degree $4$)
- $x^2 + 9$ (degree $2$, roots $??$)

# SOME OPERATIONS

$p(x)$ and $q(x)$ have degrees $m$ and $n$

$p(x) + q(x)$ : add matching coefficients

$\alpha p(x)$ : (scalar multiplication) multiply each $c_k$ by $\alpha$

$p(x) \times q(x)$ : has degree $m + n$ (book pp. 30-32)

$p(x)/q(x)$ (book pp. 33-37)

# FINDING ROOTS

- In many applications the problem of finding values, $\alpha$, for which $p(\alpha) = 0$ arises: Differential calculus and optimization, using spectral methods for Image Compression and web-page analysis.
- For degree 2, 3, and 4 can do “directly” (pp. 43-45).
- In general this can be quite taxing. (pp. 37-43)
- Methods use techniques from Calculus (pp 152-161)

# ANOTHER RATIONAL(E): REFINING $R$

Consider $Q[X]$ polynomials with rational coefficients.

The values $\sqrt{2}$ and $\sqrt[3]{20}$ are not in $Q$ but are the roots of polynomial(s) in $Q[X]$.

Since any rational can be described in *finite length* we could, in principle, describe $\sqrt{2}$ by the coefficients of a "minimal" polynomial in $Q[X]$ with $\sqrt{2}$ as a root.

This description would be *finite*.

# Q: CAN WE DO THIS FOR ALL $\alpha \in R$?

- Given any $\alpha \in R$, find $p(x)$ in $Q[X]$, with $\alpha$ a root.
- If this *was* possible we could give a *finite description* for *every* $\alpha \in R$ by giving the coefficients of $p(x)$.
- It's *not* possible: there are Real numbers not definable as the root of a polynomial in $Q[X]$.
- Example $\pi$ : this argument was only found in 1882.

# SUMMARY

- Polynomial structures occur repeatedly in many distinct later parts of the module.
- Among these are

Calculus and Optimization methods

Complex Numbers

Analysis of matrix properties

- Root finding is an important part of these.