Calculus of Complex Valued Functions Part 1: Differential Calculus

Background

- The fact that a Complex Number is defined through *two disparate components* – the Real and Imaginary Parts – raises a number of technical issues when attempting to develop extensions of *Differential* and *Integral Calculus*.
- This is especially true of Integral Calculus where the fact that the Complex Numbers *cannot be ordered* is significant.
- The class of functions considered (in the Complex arena) are

$$f : C \to C$$

- ie which map from *Complex Numbers* to *Complex Numbers*

Real Domain & Complex Range?

- The special case $f : R \to C$ turns out to be (fairly) straightforward since $f(x)$ is simply *two Real valued* functions

$$f_{\mathrm{Re}} : R \to R \;\; ; \quad f_{\mathrm{Im}} : R \to R$$

- So that:

$$f(x) = f_{\mathrm{Re}}(x) + i f_{\mathrm{Im}}(x)$$

- From which $f'(x) = f'_{\mathrm{Re}}(x) + i\, f'_{\mathrm{Im}}(x)$

$$\int_p^q f(x)dx = \int_p^q f_{\mathrm{Re}}(x)dx + i \int_p^q f_{\mathrm{Im}}(x)dx$$

Complex Domain & Complex Range

- When $f : C \rightarrow C$ more care is needed.
- We first deal with *Differential Calculus*.
- Consider $f(z)$ with $z = p + iq$.
- We have *Real valued functions* $u(p, q)$ and $v(p, q)$.

$$f(z) = f(p + iq) \equiv f(p, q) = u(p, q) + i \cdot v(p, q)$$

- So we have a function of *two Real variables* mapping to *two Real-valued functions* of *two Real variables*.
- This suggests applying *Partial Derivatives*.

The Cauchy-Riemann Conditions

- In order for $f'(z)$ sensibly to be defined via *partial derivatives* with respect to $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ of $\mathrm{Re}\big(f(z)\big)$ and $\mathrm{Im}(f(z))$, ie the (2 variable) functions $u(p,q)$ and $v(p,q)$ of the previous slide, there are *some important preconditions* that $u(p,q)$ and $v(p,q)$ have to satisfy.
- These are called *The Cauchy-Riemann Conditions*:

$$\frac{\partial u}{\partial p} = \frac{\partial v}{\partial q} \quad \mathit{and} \quad \frac{\partial u}{\partial q} = -\frac{\partial v}{\partial p}$$

- Note that these are conditions on *functions*.

Why is it so involved?

- The original (Real-valued) development of the concept of first derivative of $f(x)$ was:

$$f'(x) \stackrel{\text{def}}{=} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- We could, in principle, adopt a similar approach with $f(z)$.
- The problem, however, is that $h \in C$ and $h \to 0$ must consider:

$$\mathrm{Re}(h) \to 0,\ \mathrm{Im}(h) \to 0$$

- Now, instead of the "one-direction" case with Real functions we have *infinitely many ways* of approaching 0.
- Very informally the Cauchy-Riemann conditions state that:

"*these are all equivalent*".

An Example

- Suppose $f(z) = z^2$.
- For $z = p + iq$ we have:

$$u(p,q) = \mathrm{Re}\big(f(z)\big) = \mathrm{Re}\big((p+iq)^2\big) = p^2 - q^2$$
$$v(p,q) = \mathrm{Im}\big(f(z)\big) = \mathrm{Im}\big((p+iq)^2\big) = 2pq$$
$$\frac{\partial u}{\partial p} = 2p = \frac{\partial v}{\partial q}$$
$$\frac{\partial u}{\partial q} = -2q = -\frac{\partial v}{\partial p}$$

- Notice: $f'(z) = 2z = 2p + 2iq = \frac{\partial u}{\partial p} + i\frac{\partial v}{\partial p} = \frac{\partial v}{\partial q} - i\frac{\partial u}{\partial q}$
- The Cauchy-Riemann conditions establish the validity of this rule.

Some More Examples

- $f(z) = \bar{z}$: The Complex Conjugate Function.
- Since $f(p+iq) = p - iq$ we obtain:

$$u(p,q) = \mathrm{Re}\big(f(z)\big) = p$$
$$v(p,q) = \mathrm{Im}\big(f(z)\big) = -q$$

- These *do not satisfy* the Cauchy-Riemann conditions:

$$\frac{\partial u}{\partial p} = 1 \neq -1 = \frac{\partial v}{\partial q}$$

- The Complex Conjugate function is *not differentiable*.

Another More Complicated Example

- $f(z) = \frac{1}{z}$: in this case we can start by rewriting

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{\bar{z}}{\left(\mathrm{Re}(z)\right)^2 + (\mathrm{Im}(z))^2}$$

- Writing $z = p + iq$

$$f(p + iq) = \frac{p - iq}{p^2 + q^2} = u(p,q) \; - iv(p,q)$$

Here:

$$u(p,q) = \frac{p}{p^2 + q^2} \;\; ; \;\; v(p,q) = \frac{-q}{p^2 + q^2}$$

- Exercise: show that the Cauchy-Riemann conditions are met and derive the expression for $f'(z)$.

Differential Calculus: Complex Functions

- When $f : C \rightarrow C$, the function $f(z),\ z \in C$ can be viewed as involving *2 Real-valued* functions of *2 Real variables*:

$$p = \mathrm{Re}(z)\,; q = \mathrm{Im}(z)$$

$$u(p,q) = \mathrm{Re}\big(f(z)\big)$$

$$v(p,q) = \mathrm{Im}\big(f(z)\big)$$

- The derivative of $f(z)$ is "*well-defined*" if $u(p,q)$ and $v(p,q)$ satisfy the *Cauchy-Riemann conditions*. In which case:

$$f'(z) = \frac{\partial u}{\partial p} + i\frac{\partial v}{\partial p} = \frac{\partial v}{\partial q} - i\frac{\partial u}{\partial q}$$

Conclusions

- Classical Real-valued Differential Calculus as reviewed in Part 3 *can* be extended to the *Complex domain*.
- For this extension to be *valid* we need the *two constraints* specified in the *Cauchy-Riemann conditions* to be met.
- These conditions, in effect, reduce *differentiation of Complex valued functions* to manipulating *partial derivatives* of *Real valued functions*.
- The situation with developing a Complex Function analogue of *Integral Calculus* is *far more challenging* but has a very powerful *application* in *Computer Science* and *Algorithmics*.
- This is the subject of the next lecture.