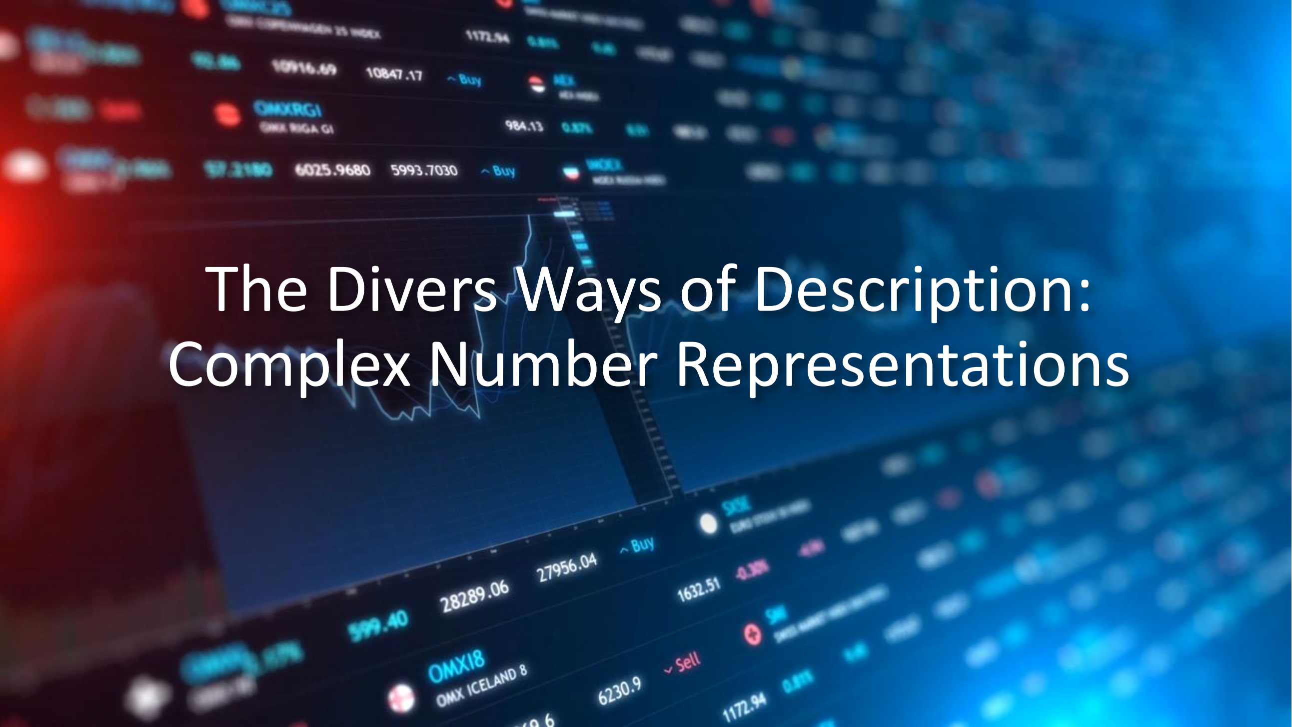


The Divers Ways of Description: Complex Number Representations

“*There are nine and sixty ways of constructing tribal lays,*
And every single one of them is right.”

In the Neolithic Age
Rudyard Kipling (1865 – 1936)

Background & Overview

- In the previous lecture we met one approach to describing the entities we shall henceforward refer to as *Complex Numbers*.
- The Complex Number, z, is given as $a + ib$ with a and b *Real*.

$$z = a + ib \quad \{a, b\} \in R$$

- By considering the *Complex Plane* and its description of $z \in C$ as a *coordinate* $(\mathrm{Re}(z)\ ,\mathrm{Im}(z))$ we obtain a second view of how to interpret $z \in C$.
- We are, however, *not limited* to these two basic forms.
- In this lecture some of the more common alternative and powerful techniques for describing $z \in C$ are reviewed.

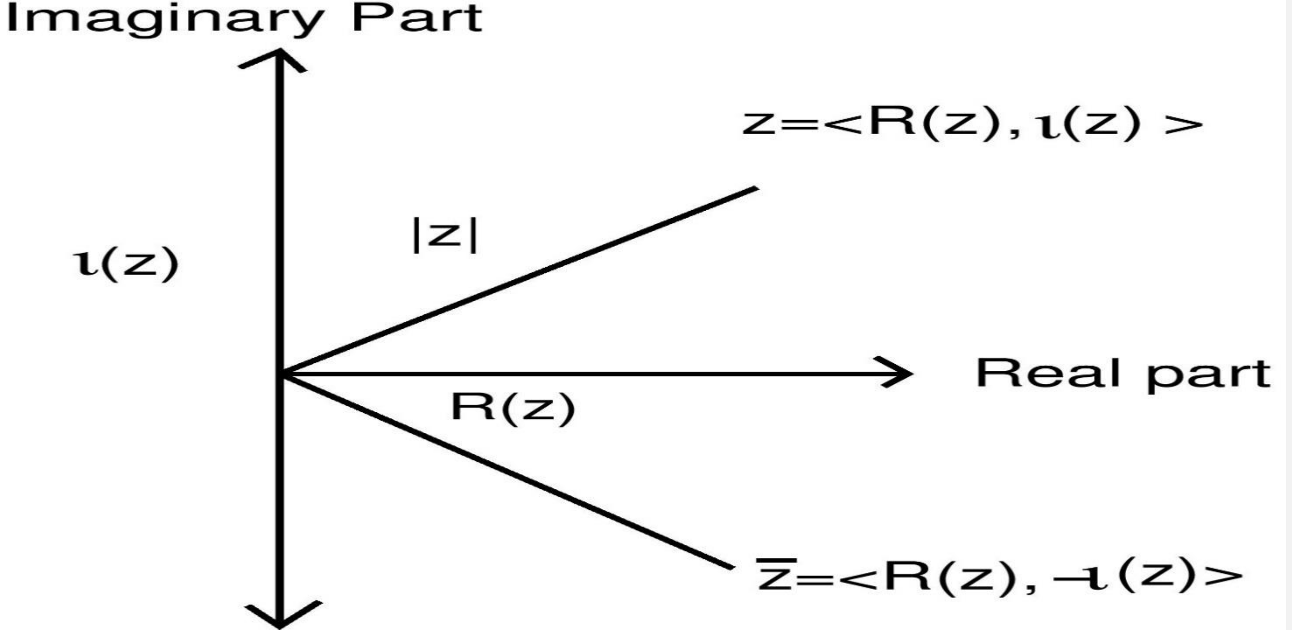
Complex Numbers as 2×2–matrices

- For $z = a + ib$.
- The Matrix Form of z is denoted $\mathbf{M}_z$ with:

$$\mathbf{M}_z = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

- The basic operations just manipulate 2×2 matrices:
- Addition: $\mathbf{M}_{u+v} = \mathbf{M}_u + \mathbf{M}_v$
- Conjugate: $\mathbf{M}_{\bar{z}} = \mathbf{M}_z^{\mathsf{T}}$ (ie the *Transpose* of $\mathbf{M}_z$)
- Modulus: $|z| = \det \mathbf{M}_z$ (ie the *determinant* of $\mathbf{M}_z$)
- Multiplication: $\mathbf{M}_{u \cdot v} = \mathbf{M}_u \cdot \mathbf{M}_v = \mathbf{M}_v \cdot \mathbf{M}_u = \mathbf{M}_{v \cdot u}$

Complex Numbers as 2–vectors: Argand Diagrams



Advantages of Argand Diagrams

- For $z = a + ib$.
- The "*philosophical*" issue of "*imaginary number*" does not arise: properties are expressed in terms of standard 2-dimensional vector systems.
- *Addition* of Complex Numbers "mimics" vector addition.
- *Conjugate* is *reflection* in the $\mathrm{Re}(z)$ axis.
- *Modulus* is standard (Euclidean) *vector size*.

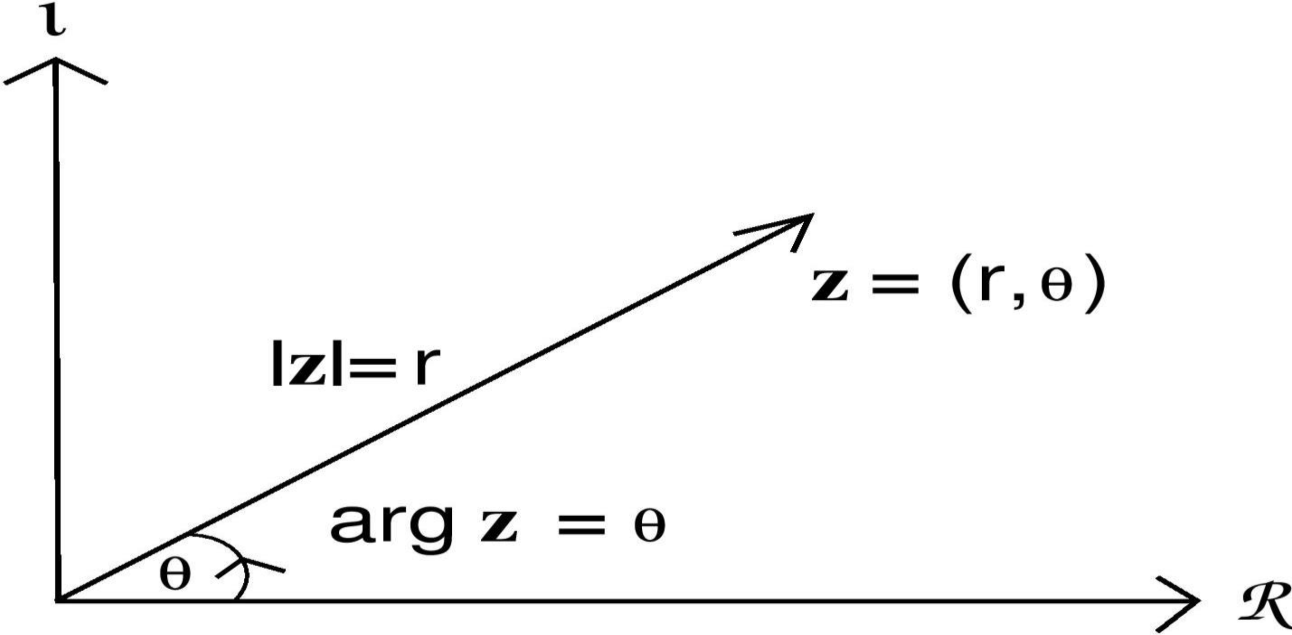
Disadvantages of Argand Diagrams

- *Multiplication* and *Division* do not seem to have a *natural* Geometric analogy.
- While useful as a *visualization aid*, Complex Number analysis has a lot of subtleties that are lost by treating this as “*just another way of dealing with 2-vectors*”.
- Some of these issues begin to be resolved with the next widely used formalism we look at: *Polar coordinates*.

Polar Coordinate Form of Complex Numbers

- For $z = a + ib$ with $|z| = r$.
- The number z is described (in *Polar form*) as
$$z = (r\,,\theta)$$
- What is θ here?
- The *angle* (in *radians*) measured *counter clockwise* from $\mathrm{Re}(z)$ to the 2-vector $\langle a, b\rangle$ in "*standard position*" (recall from earlier part of the module).
- This angle is denoted $\arg z$. It is, sometimes, referred to as the *phase* of z.
- It can be shown: $\arg z = \cos^{-1}\left(\frac{\mathrm{Re}(z)}{|z|}\right) = \sin^{-1}\left(\frac{\mathrm{Im}(z)}{|z|}\right)$

Polar Coordinates for Complex Numbers



The Euler Form for Complex Numbers

- For $z = a + ib$ with $|z| = r$ and $\arg z = \theta$.
- The number z is described (in *Euler form*) as

$$z = r \cdot e^{i\theta}$$

- This establishes a remarkable link between the *exponential* function, *trigonometric* functions and *complex numbers*:

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

- This relation is known as *Euler's Formula* and leads to:

$$\forall \alpha \in R \;:\; (\cos\theta + i\sin\theta)^{\alpha} = \cos(\alpha\theta) + i \cdot \sin(\alpha\theta)$$

Some consequences of Euler Form

- For $|u| = s$ and $\arg u = \sigma$, $|v| = t$ and $\arg v = \tau$
$$u \cdot v = se^{i\sigma} \cdot te^{i\tau} = (st)e^{i(\sigma+\tau)}$$
- Looking at Argand diagram: the *phase* resulting from the *product* of two Complex numbers is the result of *adding* the *phase of each*.
- There are *infinitely many* representations of any *single* Complex number:
$$\cos(\theta + 2\pi) = \cos\theta \ ; \ \sin(\theta + 2\pi) = \sin\theta$$
- The *principal value* of $\arg z = \theta$ is that for which $0 \leq \theta < 2\pi$

Complex Powers

- The interpretation of u^v is well understood when $u, v \in R$.
- For $u \in C$ this can be extended.
- For $v \in C$ this is *highly non-trivial*.
- We look at the case: $u^v \ : \ u \in C \, , v \ \in \ Q^+$ (positive Rationals)
- It suffices to consider $v = \frac{1}{k}$
- Writing $u = \ |u|e^{i \arg u}$ then $u^{\frac{1}{k}} = \ \left(|u|^{\frac{1}{k}}\right)e^{\frac{i \arg u}{k}}$
- There are k distinct *principal values* $\varphi_m = \frac{\arg u + 2m\pi}{k}$ so that

$$\left(e^{i\varphi_m}\right)^k = e^{i \arg u}$$

Primitive Roots of Unity

- The ideas from the previous slide prove particularly useful in the seemingly basic case $u = 1$. These, of course, are the roots of the polynomial $x^k - 1$.
- We have (for any $k \in N$) exactly k *primitive roots of unity*:

$$\left\{ e^{\frac{2\iota\pi m}{k}} : 0 \leq m < k \right\}$$

- The concept of Primitive Root of Unity underpins the important *Fourier Transform* widely used in signal processing and other electronics applications.
- The *Discrete Fourier Transform* finds a key use in *Computer Science* as the basis of the *fastest* (known) *Multiplication Algorithm*.

Summary

- Complex Numbers have a wide range of different methods that can be used as representations and it is a straightforward process to switch between different representation methods.
- There is, however, a central issue with Complex Numbers which we have not met with any of the earlier Number types:

Complex Numbers cannot be ordered

- That is to say: there is no consistent interpretation of what is to be understood by one Complex Number being "*smaller*" than another.
- While this has not proven problematic it will arise when we move on to look at the developments of *Calculus* needed for *Complex Functions*, in particular *Integral Calculus*.
- This will be (briefly) considered in the next lecture.