Complex Numbers
Properties & Applications

“*Gott Weiß wie das geschah?*”

Die Meistersinger von Nürnberg, (Act 3, Sc. 1)

Richard Wagner (1813 – 1883)

The Origins of Complex Numbers

- The notion of so-called "*Complex Number*" arises in a problem we considered earlier: roots of polynomials.
- It was claimed that "*all polynomials of degree k have k roots*".
- So what are the *two* roots of $x^2 + 1$?
- The mechanism used to resolve this question leads to the final (as regards the concerns of this module) class of numbers.

Roots Revisited

- The *two* roots of $x^2 + 1$?
- These *cannot* belong to the set R of Real numbers.
- "*Imagine*" a "*class of number*" to which these roots belong.
- In this case, let the roots of $x^2 + 1$ be dubbed $\{i, -i\}$ meaning:

$$i^2 = -1\,;\ (-i)^2 = (-1)^2 \cdot i^2 = -1$$

- Both i and $-i$ *are* roots of $x^2 + 1$.

$$i^2 + 1 = -1 + 1 = 0$$
$$(-i)^2 + 1 = -1 + 1 = 0$$

Other Cases?

- The *two* roots of $x^2 + 9$?
- The roots of $x^2 + 9$: $\{3i\ ,\ -3i\}$

$$(3i)^2 + 9 = -9 + 9 = 0$$

$$(-3i)^2 + 9 = \ 3^2(-i)^2 + 9 = -9 + 9 = 0$$

- In general: if $p_k(x)$ is a polynomial of degree k from $R[X]$ then *all* of its roots can be described in some form:

$$z = a + ib$$

- a is called the *Real Part* of z denoted by $\mathrm{Re(z)}$.
- b is called the *Imaginary Part* of z denoted by $\mathrm{Im(z)}$.
- The numbers $a + ib$ (a and b are ***both*** *Real numbers*) form the set of *Complex Numbers*: denoted by C

Complex Numbers – Overview

- A *Complex Number* takes the form:

$$z = a + ib$$

- a being the *Real Part* of z denoted by $\mathrm{Re(z)}$.
- b being the *Imaginary Part* of z denoted by $\mathrm{Im(z)}$.
- The pairs of *Real numbers* $\left(\mathrm{Re(z)}, \mathrm{Im(z)}\right)$ $z \in C$ form the *Complex Plane*.
- The Complex Plane can be viewed as the "*standard Cartesian*" system with $\mathrm{Re(z)}$ replacing the x–axis and $\mathrm{Im(z)}$ the y-axis.

Complex Numbers – Basic Operations

$u \in C, v \in C$:

- Addition: $z = u + v$

$$\mathrm{Re}(z) = \mathrm{Re}(u) + \mathrm{Re}(v)$$
$$\mathrm{Im}(z) = \mathrm{Im}(u) + \mathrm{Im}(v)$$

- *Complex Conjugate*: $\bar{z}$

$$\mathrm{Re}(\bar{z}) = \mathrm{Re}(z)$$
$$\mathrm{Im}(\bar{z}) = -\mathrm{Im}(z)$$

Notice: $\bar{\bar{z}} = z$: *the conjugate* of *the conjugate* of z is z

- *Modulus* (also called *size*): $|z|$

$$|z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}$$ (NB: *positive* root)

Examples

$u = 3 + 4i\ ; v = 1 - 2i$:

- Addition: $z = u + v$
- $\mathrm{Re}(z) = \mathrm{Re}(u) + \mathrm{Re}(v) = 3 + 1 = 4$
- $\mathrm{Im}(z) = \mathrm{Im}(u) + \mathrm{Im}(v) = 4 + (-2) = 2$

$$z = u + v = 4 + 2i$$

- *Complex Conjugate*: $\bar{z}$; if $z = 4 + 2i$
- $\mathrm{Re}(\bar{z}) = \mathrm{Re}(z) = 4$;
- $\mathrm{Im}(\bar{z}) = -\mathrm{Im}(z) = -2$

$$\overline{4 + 2i} = 4 - 2i$$

- *Modulus*: $|u|$

$$|u| = \sqrt{\mathrm{Re}(u)^2 + \mathrm{Im}(u)^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

More “complex” operations

$u \in C, v \in C, \alpha \in R$:

- *Scalar* multiplication: $z = \alpha \cdot u$

$$\mathrm{Re}(z) = \alpha \cdot \mathrm{Re}(u)\,;\,\mathrm{Im}(z) = \alpha \cdot \mathrm{Im}(u)$$

- *Complex* multiplication: $z = u \cdot v$

$$z = \big(\mathrm{Re}(u) + i \cdot \mathrm{Im}(u)\big) \cdot \big(\mathrm{Re}(v) + i \cdot \mathrm{Im}(v)\big)$$

$$z = \big(\mathrm{Re}(u) \cdot \mathrm{Re}(v) + i^2 \cdot \mathrm{Im}(u) \cdot \mathrm{Im}(v)\big) +$$

$$i \cdot (\mathrm{Re}(u) \cdot \mathrm{Im}(v) + \mathrm{Im}(u) \cdot \mathrm{Re}(v))$$

$$\mathrm{Re}(z) = \big(\mathrm{Re}(u) \cdot \mathrm{Re}(v) - \mathrm{Im}(u) \cdot \mathrm{Im}(v)\big)$$

$$\mathrm{Im}(z) = (\mathrm{Re}(u) \cdot \mathrm{Im}(v) + \mathrm{Im}(u) \cdot \mathrm{Re}(v))$$

Some More Examples

$u = 3 + 4i\,, v = 1 - 2i\,, \alpha = 2$:

- *Scalar* multiplication: $z = 2 \cdot u$

$\mathrm{Re}(z) = 2 \cdot \mathrm{Re}(u) = 6$

$\mathrm{Im}(z) = 2 \cdot \mathrm{Im}(u) = 8\,;$

$$z = 6 + 8i$$

- *Complex* Multiplication: $z = u \cdot v$

$z = (3 + 4i) \cdot (1 - 2i)$

$z = \big((3) \cdot (1) + \ i^2 \cdot (4)(-2)\big) + \ i \cdot \big((3) \cdot (-2) + (4) \cdot (1)\big)$

$\mathrm{Re}(z) = (3\ - (-8)) = 11$

$\mathrm{Im}(z) = (-6 + 4) = -2$

$$z = 11\ - 2i$$

Complex Division

$u \in C, v \in C : z = {}^{u}/_{v}$

- We know how to *multiply* two Complex Numbers.
- Therefore we only need to define the Complex Number

$$w = v^{-1} = \frac{1}{v}$$

- We need $v \neq 0$

$$\frac{1}{v} = \frac{\bar{v}}{|v|^2}$$

- To check:

$$\frac{v \cdot \bar{v}}{|v|^2} = \frac{\mathrm{Re}(v)^2 + \mathrm{Im}(v)^2}{\sqrt{\mathrm{Re}(v)^2 + \mathrm{Im}(v)^2}^{\,2}} = 1$$

Complex Division – Example

$$u = 3 + 4i$$

To compute $z = {}^{1}\!/_{u}$

$$\bar{u} = 3 - 4i$$

$$|u|^2 = 3^2 + 4^2 = 25$$

$$z = {}^{1}\!/_{u} = \frac{3 - 4i}{25}$$

Summary

- There are *many* ways of describing Complex Numbers.
- Some of these (Matrix, *Argand Diagram*, *Polar Coordinate*, *Euler Form*) will be reviewed in the next lecture.
- We also (briefly) consider the development of *Calculus* as applied to *Complex Numbers*.
- In the second part of this section, having considered the technical background, we look at important consequences of Complex Numbers in Computer Science:

Quaternions and Advanced Graphics

The Fourier Transform and Fast Arithmetic

Average Case Analysis and *Complex Integrals*

Complex Sequences and *Algorithmic Music*