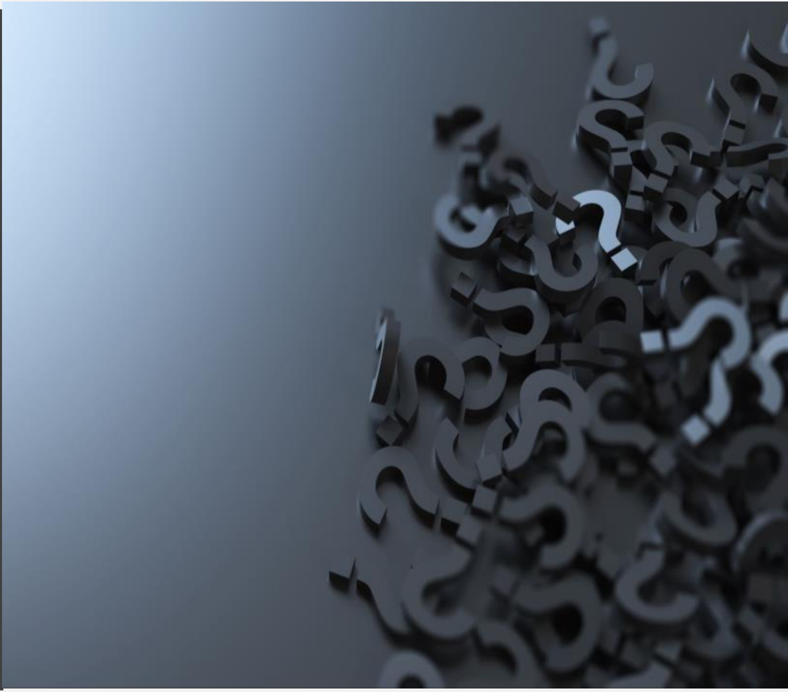
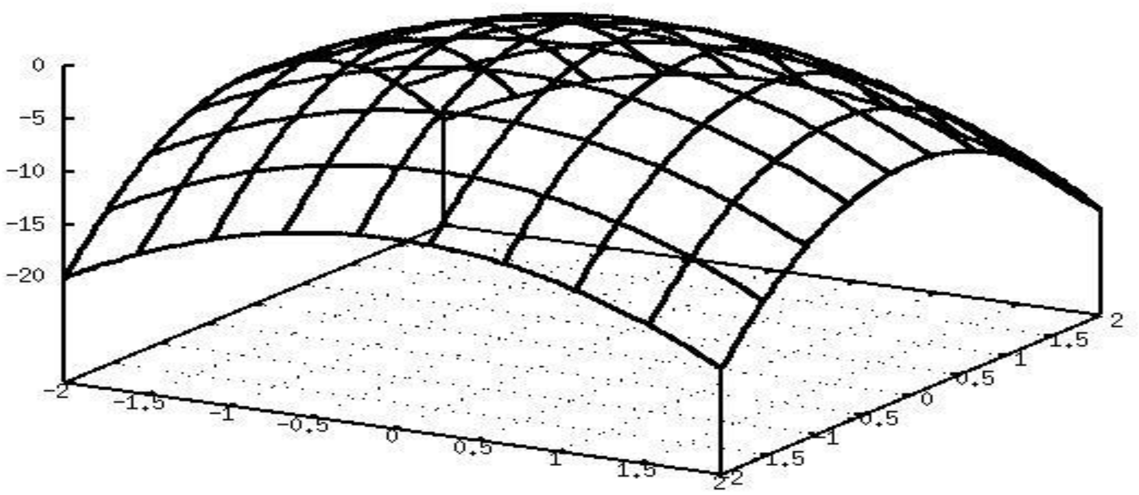
Partial Derivatives & Multivariable Functions



A 2-Variable Function



Some questions

- The surface shown on the preceding slide depicts a function of *two* variables – $f(x, y)$ or $z = f(x, y)$.
- Some choice of (x, y) will *maximize* $z = f(x, y)$.
- How can we *find* this value?
- Can we use a *similar approach* to the case $y = f(x)$?
- What if 3 or 4 or, in general, n variables are involved?

Some Problems with Answers

- Before the notions of "*maximal*" and "*minimal*" were supported by considering the "*gradients of lines touching a point*".
- When this gradient was 0 we could justify the point as *critical* since the line concerned was "*horizontal*": small changes would result in positive (*increasing*) lines and negative (*decreasing*) lines.
- For two or more variables we can no longer, safely, use this analogy.

A Different Approach

- Consider a function such as $z = -(3x^2 + 2y^2 + xy)$.
- What are the value(s) of x and y which *maximize* z?
- Are there, in fact, any such values?
- and, if so, *how* do we find them?
- We use the concept of *partial derivative* to do this.

Partial Derivatives

Consider again the function $z = -(3x^2 + 2y^2 + xy)$.

1. We need value(s) of x and y which *maximize* z.
2. We *know* x to maximize z when y is *fixed*
3. We *know* y to maximize z when x is *fixed*
4. But we want to do things *simultaneously*.
5. Assume that y *is* fixed and find the "*right*" x.
6. Assume that x *is* fixed and find the "*right*" y.
7. (2) and (3) yield *simultaneous equations.*

Deriving the system of equations

Consider again the function $z = -(3x^2 + 2y^2 + xy)$.

1. Assume that y *is* fixed and find the "*right*" x.

Meaning: *differentiate* $f(x,y)$ "*as if* y *was a constant*"

$$f_x(x,y) = -(6x + y)$$

2. Assume that x *is* fixed and find the "*right*" y.

Meaning: *differentiate* $f(x,y)$ "*as if* x *was a constant*"

$$f_y(x,y) = -(4y + x)$$

3. Now find the values of x, y for which $f_x(x,y) = 0$ ***AND*** $f_y(x,y)$=0: that is, solve the *simultaneous equations.*

A Bit of notation

- For functions $z = f(x, y)$ we have:

$f_x(x, y)$: the (first) *partial derivative* of $f(x, y)$ wrt x

$f_y(x, y)$: the (first) *partial derivative* of $f(x, y)$ wrt y

Also used are:

$$\frac{\partial z}{\partial x} ; \frac{\partial z}{\partial y}$$

- $\mathrm{z} = f(x_1, x_2, \dots, x_k, \dots, x_n)$

$$\frac{\partial z}{\partial x_k}$$

For $z = -(3x^2 + 2y^2 + xy)$

- $f_x(x, y) = -(6x + y)$
- $f_y(x, y) = -(4y + x)$

$$-6x - y = 0$$
$$-x - 4y = 0$$

- Only solution is: $x = 0\,;\, y = 0$
- $-(3x^2 + 2y^2 + xy)$ is *maximized* at the point $(0,0)$.
- Q: How do we know this is a *maximum*?
- A: We need an analogue of *The Second Derivative Test*.

The Second Derivative Test with 2 variables

- This is similar, but rather more involved.
- Problem 1: With *2 variables* there are *4 possible* forms of "*second order*" partial derivative.
- Problem 2: with n variables there are n^2
- Problem 3: how do we "*combine*" these?
- The 4 Forms: $\left(f_{xx}, f_{xy}, f_{yx}, f_{yy}\right)$ or

$$\left(\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y \partial x}, \frac{\partial^2 z}{\partial y^2}\right)$$

Interpretation

- f_{xx} means "*the (partial) derivative of* f_x *wrt* x"
- f_{xy} means "*the (partial) derivative of* f_y *wrt* x"
- f_{yx} means "*the (partial) derivative of* f_x *wrt* y"
- f_{yy} means "*the (partial) derivative of* f_y *wrt* y"
- For $f(x,y) = -(3x^2 + 2y^2 + xy)$:
  $f_{xx} = -6\,;\; f_{xy} = -1\,;\; f_{yx} = -1\,;\; f_{yy} = -4$
- In general: f_{xy} and f_{yx} are *identical functions*.

Using the Test

- .There is a ***precondition***:

$$\left(f_{xx} f_{yy} - \left(f_{xy}\right)^2\right)(\alpha, \beta) > 0$$

- if $f_{xx}(\alpha, \beta) > 0$ the point is a *minimum*.
- if $f_{xx}(\alpha, \beta) < 0$ the point is a *maximum*
- if $f_{xx}(\alpha, \beta) = 0$ *no conclusion can be made*.
- There is a sophisticated extension for n variables called *The Hessian*. (see textbook pages 168-9)
- A more detailed two variable case may be found on pages 169 – 173 of the module text.