COMP111: Artificial Intelligence

Section 6. Uniform cost search and informed (or heuristic) tree search

Frank Wolter

Recap

- Basic problem solving techniques:
  - **Breadth-first search**
    complete but expensive.
  - **Depth-first search**
    cheap but incomplete
- Variations and combinations:
  - **Limited depth search**
  - **Iterative deepening search**
  - **Avoiding repeated states**
  - **Bi-directional search**

Overview

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- Introduce heuristics: rules of thumb

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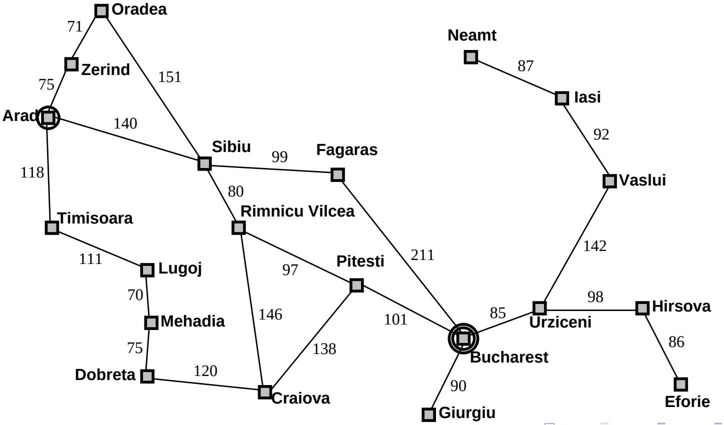
- Introduce heuristics: rules of thumb
- Introduce heuristic search:
  - greedy search
  - A* search

Search graph with costs

- A path cost function,

$$g : \text{Paths} \to \text{real numbers}$$

gives a cost to each path. We assume that the cost of a path is the sum over the costs of the steps in the path.



Uniform Cost Search

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Uniform Cost Search

- Why not expand the cheapest path first?
- Intuition: cheapest is likely to be best!
- Performance is like breadth-first search but we select (expand) the minimum cost path rather than the shortest path.
- Uniform cost search behaves in exactly the same way as breadth-first search if the cost of every step is the same.

General Algorithm for Uniform Cost Search

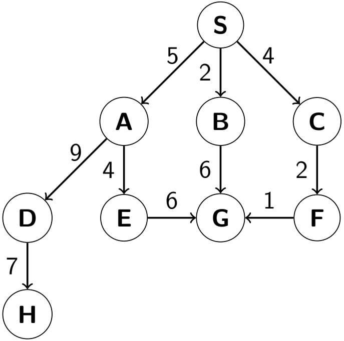
Input: a start state s_0
for each state s the successors of s
a test goal(s) checking whether s is a goal state
$g(s_0 \ldots s_k)$ for every path $s_0 \ldots s_k$

Set frontier $:= \{s_0\}$
while frontier is not empty **do**
select and remove from frontier the path $s_0 .... s_k$
with $g(s_0 \ldots s_k)$ minimal
if goal(s_k) **then**
return $s_0 \ldots s_k$ (and terminate)
else for every successor s of s_k add $s_0 \ldots s_k s$ to frontier
end if
end while

Uniform Cost Example

Reaching G from S

Exp. paths	Frontier
	$\{S : 0\}$



Uniform Cost Example

Reaching G from S

Exp. paths	Frontier
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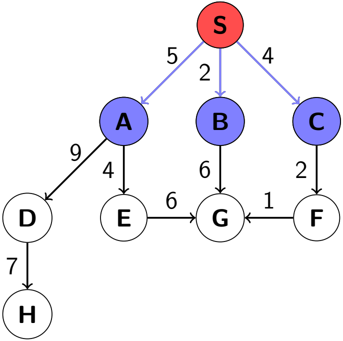
S not goal	$\{SA : 5, SB : 2, SC : 4\}$



Selected path: $S : 0$

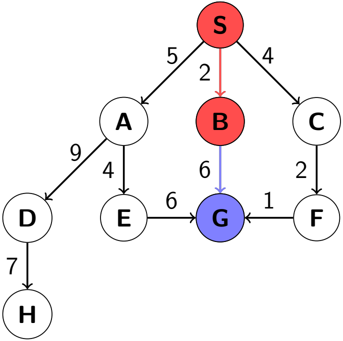
Is the last state in S_{goal}? No

Expand S: add $SA : 0 + 5, SB : 0 + 2, SC : 0 + 4$ to the frontier

Uniform Cost Example

Reaching G from S

Exp. paths	Frontier
	$\{S : 0\}$
S not goal	$\{SA : 5, SB : 2, SC : 4\}$
SB not goal	$\{SA : 5, SC : 4, SBG : 8\}$

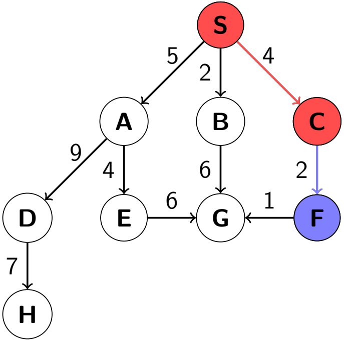


Selected path: $SB : 2$
Is the last state in S_{goal}? No
Expand SB: add $SBG : 2 + 6$ to the frontier

Uniform Cost Example

Reaching G from S

Exp. paths	Frontier
	$\{S : 0\}$
S not goal	$\{SA : 5, SB : 2, SC : 4\}$
SB not goal	$\{SA : 5, SC : 4, SBG : 8\}$
SC not goal	$\{SA : 5, SBG : 8, SCF : 6\}$



Selected path: $SC : 4$

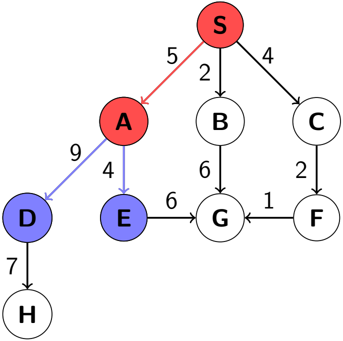
Is the last state in S_{goal}? No

Expand SC: add $SCF : 4 + 2$ to the frontier

Uniform Cost Example

Reaching G from S

Exp. paths	Frontier
	$\{S : 0\}$
S not goal	$\{SA : 5, SB : 2, SC : 4\}$
SB not goal	$\{SA : 5, SC : 4, SBG : 8\}$
SC not goal	$\{SA : 5, SBG : 8, SCF : 6\}$
SA not goal	$\{SBG : 8, SCF : 6, SAD : 14, SAE : 9\}$



Selected path: $SA : 5$

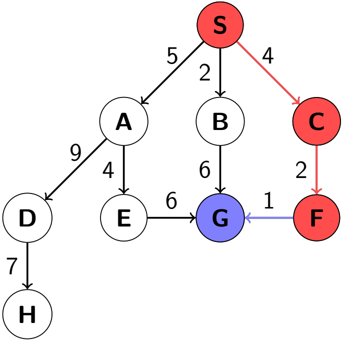
Is the last state in S_{goal}? No

Expand SA: add $SAD : 5 + 9$ and $SAE : 5 + 4$

Uniform Cost Example

Reaching G from S

Exp. paths	Frontier
	$\{S : 0\}$
S not goal	$\{SA : 5, SB : 2, SC : 4\}$
SB not goal	$\{SA : 5, SC : 4, SBG : 8\}$
SC not goal	$\{SA : 5, SBG : 8, SCF : 6\}$
SA not goal	$\{SBG : 8, SCF : 6, SAD : 14, SAE : 9\}$
SCF not goal	$\{SBG : 8, SAD : 14, SAE : 9, SCFG : 7\}$



Selected path: $SCF : 6$

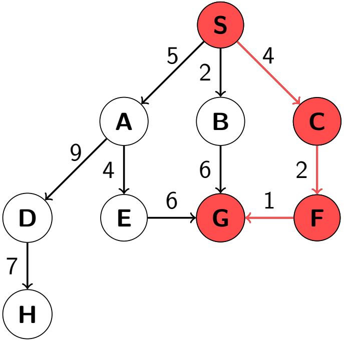
Is the last state in S_{goal}? No

Expand SCF: add $SCFG : 6 + 1$ to the frontier

Uniform Cost Example

Reaching G from S

Exp. paths	Frontier
	$\{S : 0\}$
S not goal	$\{SA : 5, SB : 2, SC : 4\}$
SB not goal	$\{SA : 5, SC : 4, SBG : 8\}$
SC not goal	$\{SA : 5, SBG : 8, SCF : 6\}$
SA not goal	$\{SBG : 8, SCF : 6, SAD : 14, SAE : 9\}$
SCF not goal	$\{SBG : 8, SAD : 14, SAE : 9, SCFG : 7\}$
$SCFG$ goal	$\{SBG : 8, SAD : 14, SAE : 9\}$



Selected path: $SCFG : 7$

Is the last state in S_{goal}? Yes!

Path found: $SCFG$ with a cost of 7

Properties of Uniform Cost Search

- Complete and optimal: Uniform cost search guaranteed to find cheapest solution assuming path costs grow monotonically, i.e. the cost of a path increases if we move along it.
- In other words, we assume that adding another step to a path makes it more costly, i.e. $g(s_0 \ldots s_k) < g(s_0 \ldots s_k s)$.
- If path costs don't grow monotonically, then exhaustive search is required.
- Time and space complexity: the same as breadth first search.

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Theorem. Uniform Cost is optimal if the cost of each action is positive and the cost of a path is the sum of the costs of its actions.

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Proof. Let $s_0 \cdots s_k$ be a cheapest path to a goal state s_k. We prove that the following always holds:

(A) When entering the while loop, the frontier contains an initial part $s_0 \cdots s_i$ of $s_0 \cdots s_k$ (in other words, $i \leq k$). (In particular that the cost of the selected path does not exceed the cost of $s_0 \cdots s_k$ (so its cost is $\leq g(s_0 \cdots s_k)$)).

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Proof by Induction over the number of times the while loop is entered.

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Now assume for a proof by induction that (A) is true for a frontier F we have reached and we enter the while loop again.

Proof of Optimality of Uniform Cost

By induction hypothesis, the frontier F contains an initial part $s_0 \cdots s_i$ of $s_0 \cdots s_k$. We show that in the next run through the while loop either a path of cost $\leq g(s_0 \cdots s_k)$ is returned or (A) holds for the updated frontier.

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If $i < k$ and so $s_0 \cdots s_i$ is shorter that $s_0 \cdots s_k$, then (A) still holds for the updated frontier after the while loop:

First, if $s_0 \cdots s_i$ itself is selected then (A) holds as $s_0 \cdots s_{i+1}$ is added to the frontier.

If another path is selected then $s_0 \cdots s_i$ is still in the updated frontier and so (A) holds.

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If another path is selected then $s_0 \cdots s_i$ is still in the updated frontier and so (A) holds.

On the other hand, if $i = k$, then we are done if $s_0 \cdots s_k$ is selected as $s_0 \cdots s_k$ is the output then.

If another path is selected then its cost is $\leq g(s_0 \cdots s_k)$.

So if it is a path to the goal it is not more expensive than $s_0 \cdots s_k$ and the output is optimal.

If it is not a path to the goal then (A) still holds as $s_0 \cdots s_k$ is in the updated frontier.

Real Life Problems

- Whatever search technique we use, exponential time complexity.

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- We need problem specific knowledge to guide the search.
- Simplest form of problem specific knowledge is heuristic.
- Standard implementation in search is via an evaluation function which indicates desirability of selecting (expanding) state.

Informed Strategies

- Use problem-specific knowledge to make the search more efficient.
- Idea: based on your knowledge, select the most promising path first.
- Rather than trying all possible search paths, you try to focus on paths that get you nearer to the goal state according to your estimate.

Heuristics

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- Example: In route finding, heuristic might be straight line distance from node to destination.

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  which estimates the cost of going from that state to the goal. h can be any function but $h(s) = 0$ if s is a goal.
- Example: In route finding, heuristic might be straight line distance from node to destination.
- Greedy search: expands the path that appears to be closest to goal.

General algorithm for greedy search

Input: a start state s_0
for each state s the successors of s
a test goal(s) checking whether s is a goal state
$h(s)$ for every state s

Set frontier $:= \{s_0\}$
while frontier is not empty **do**
  select and remove from frontier the path $s_0 .... s_k$
  with $h(s_k)$ minimal
  if goal(s_k) **then**
    return $s_0 \dots s_k$ (and terminate)
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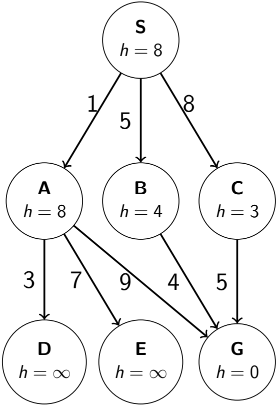
end while

Greedy Example

Reaching G from S

Exp. paths	Frontier
	$\{S : 8\}$



Greedy Example

Reaching G from S

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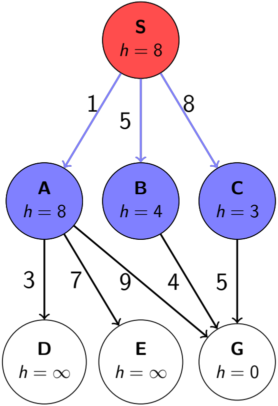
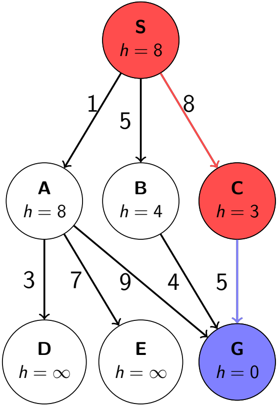


Selected path: $S : 8$
Is the last state in S_{goal}? No
Expand S: add $SA : 8, SB : 4,$ and $SC : 3$ to the frontier

Greedy Example

Reaching G from S

Exp. paths	Frontier
	$\{S : 8\}$
S not goal	$\{SA : 8, SB : 4, SC : 3\}$
SC not goal	$\{SA : 8, SB : 4, SCG : 0\}$



Selected path: $SC : 3$

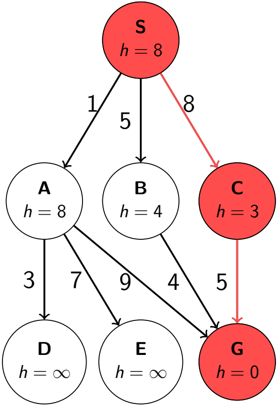
Is the last state in S_{goal}? No

Expand SC: add $SCG : 0$ to the frontier

Greedy Example

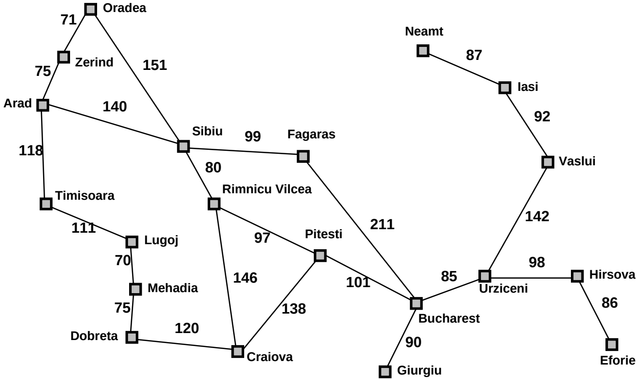
Reaching G from S

Exp. paths	Frontier
	$\{S : 8\}$
S not goal	$\{SA : 8, SB : 4, SC : 3\}$
SC not goal	$\{SA : 8, SB : 4, SCG : 0\}$
SCG goal	$\{SA : 8, SB : 4\}$



Selected path: $SCG : 0$
Is the last state in S_{goal}? Yes!
Path found: SCG with a cost of 13

Romania Example

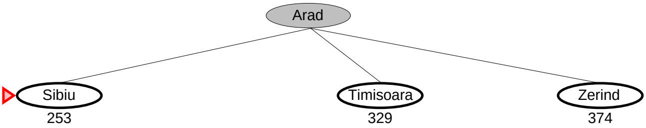


Straight−line distance
to Bucharest

Arad	366
Bucharest	0
Craiova	160
Dobreta	242
Eforie	161
Fagaras	178
Giurgiu	77
Hirsova	151
Iasi	226
Lugoj	244
Mehadia	241
Neamt	234
Oradea	380
Pitesti	98
Rimnicu Vilcea	193
Sibiu	253
Timisoara	329
Urziceni	80
Vaslui	199
Zerind	374

Greedy Search Example

Greedy Search Example



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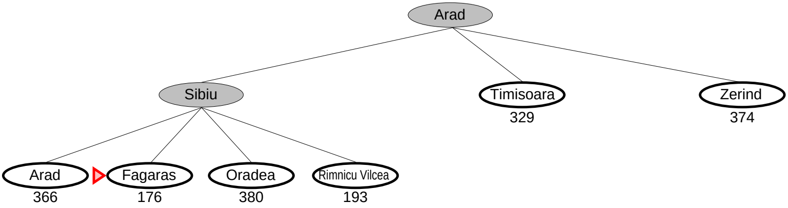

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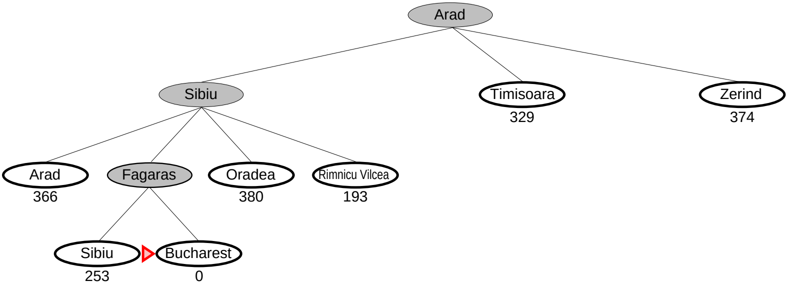
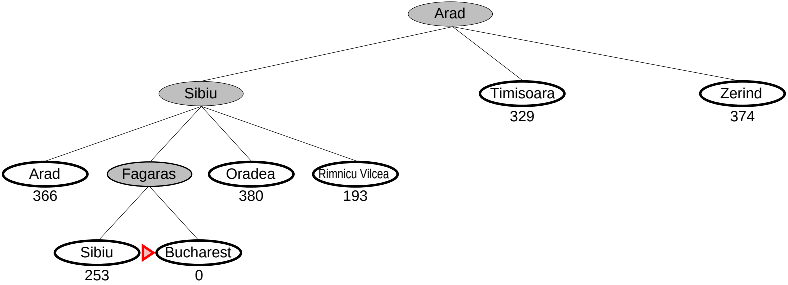


Greedy Search Example



Total distance to go: 450 km

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- May not find a solution if there is one (incomplete).
- Susceptible to false starts.
- Only looking at current state. Ignores past!

A* Search

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- Basic idea is to combine uniform cost search and greedy search.
- We look at the cost so far and the estimated cost to goal.
- Thus, we use heuristic f:

$$f(s_0 \dots s_k) = g(s_0 \dots s_k) + h(s_k)$$

where
  - $g(s_0 \dots s_k)$ is path cost of $s_0 \dots s_k$;
  - $h(s_k)$ is expected cost of cheapest solution from s_k.

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  where

  - $g(s_0 \ldots s_k)$ is path cost of $s_0 \ldots s_k$;
  - $h(s_k)$ is expected cost of cheapest solution from s_k.
- Aims to minimise overall cost.

General algorithm for A* search

Input: a start state s_0
    for each state s the successors of s
    a test goal(s) checking whether s is a goal state
    $g(s_0 \dots s_k)$ for every path $s_0 \dots s_k$
    $h(s)$ for every state s

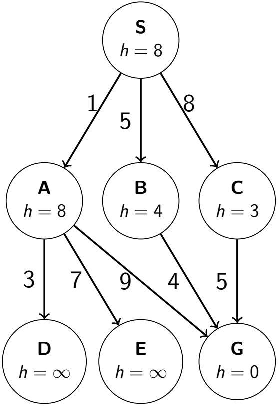
Set frontier $:= \{s_0\}$
while frontier is not empty **do**
    select and remove from frontier the path $s_0 .... s_k$
    with $g(s_0 \dots s_k) + h(s_k)$ minimal
    if goal(s_k) **then**
        return $s_0 \dots s_k$ (and terminate)
    else for every successor s of s_k add $s_0 \dots s_k s$ to frontier
    end if
end while

A* Example

Reaching G from S

Recall: $f(s_0 \dots s_k) = g(s_0 \dots s_k) + h(s_k)$

Exp. paths	Frontier
	$\{S:8\}$

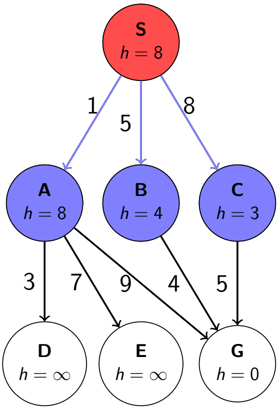


A^* Example

Reaching G from S

Recall: $f(s_0 \ldots s_k) = g(s_0 \ldots s_k) + h(s_k)$

Exp. paths	Frontier
	$\{S : 8\}$
S not goal	$\{SA : 9, SB : 9, SC : 11\}$



Selected path: $S : 8$

Is the last state in S_{goal}? No

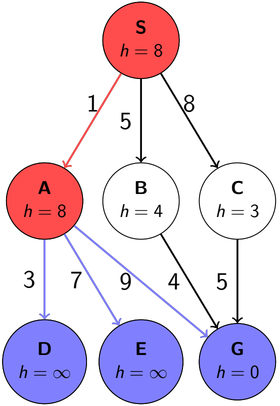
Expand S: add $SA : 1 + 8, SB : 5 + 4, SC : 8 + 3$ to the frontier

A* Example

Reaching G from S

Recall: $f(s_0 \dots s_k) = g(s_0 \dots s_k) + h(s_k)$

Exp. paths	Frontier
	$\{S : 8\}$
S not goal	$\{SA : 9, SB : 9, SC : 11\}$
SA not goal	$\{SB : 9, SC : 11, SAD : \infty, SAE : \infty, SAG : 10\}$



Selected path: $SA : 9$

Is the last state in S_{goal}? No

Expand SA: add $SAD : 4 + \infty$, $SAE : 8 + \infty$, $SAG : 10 + 0$

A^* Example

Reaching G from S

Recall: $f(s_0 \dots s_k) = g(s_0 \dots s_k) + h(s_k)$

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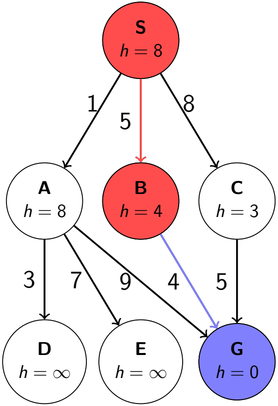
	$\{S : 8\}$
S not goal	$\{SA : 9, SB : 9, SC : 11\}$
SA not goal	$\{SB : 9, SC : 11, SAD : \infty, SAE : \infty, SAG : 10\}$
SB not goal	$\{SC : 11, SAD : \infty, SAE : \infty, SAG : 10, SBG : 9\}$



Selected path: $SB : 9$

Is the last state in S_{goal}? No

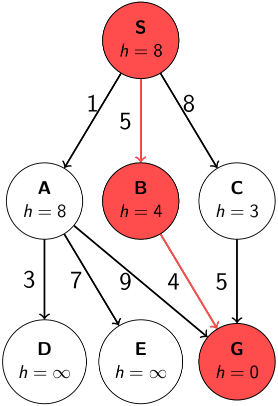
Expand SB: add $SBG : 9 + 0$ to the frontier

A* Example

Reaching G from S

Recall: $f(s_0 \dots s_k) = g(s_0 \dots s_k) + h(s_k)$

Exp. paths	Frontier
	$\{S : 8\}$
S not goal	$\{SA : 9, SB : 9, SC : 11\}$
SA not goal	$\{SB : 9, SC : 11, SAD : \infty, SAE : \infty, SAG : 10\}$
SB not goal	$\{SC : 11, SAD : \infty, SAE : \infty, SAG : 10, SBG : 9\}$
SBG goal	$\{SC : 11, SAD : \infty, SAE : \infty, SAG : 10\}$

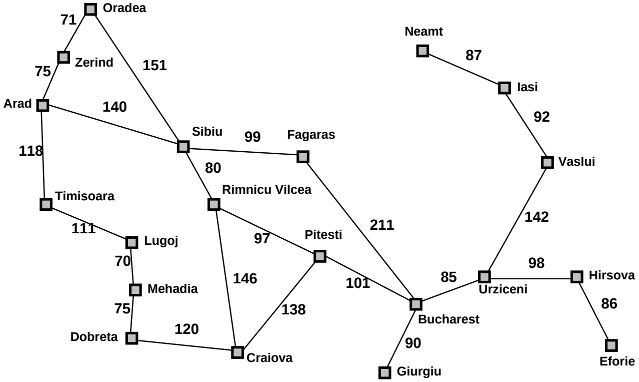


Selected path: $SBG : 9$

Is the last state in S_{goal}? Yes!

Path found: SBG with a cost of 9

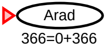
Romania Example



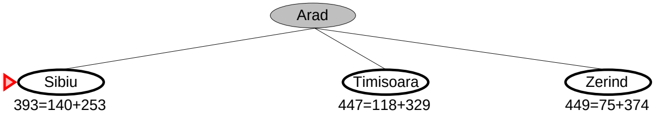
Straight−line distance to Bucharest

Arad	366
Bucharest	0
Craiova	160
Dobreta	242
Eforie	161
Fagaras	178
Giurgiu	77
Hirsova	151
Iasi	226
Lugoj	244
Mehadia	241
Neamt	234
Oradea	380
Pitesti	98
Rimnicu Vilcea	193
Sibiu	253
Timisoara	329
Urziceni	80
Vaslui	199
Zerind	374

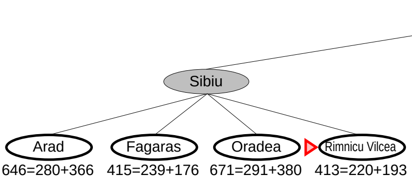
A* Search Example



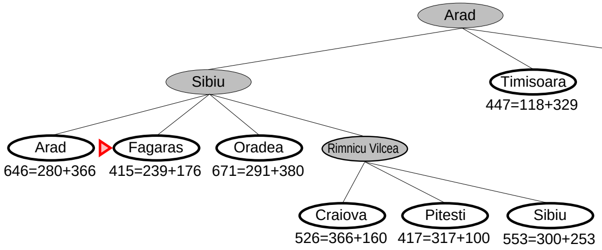
A* Search Example



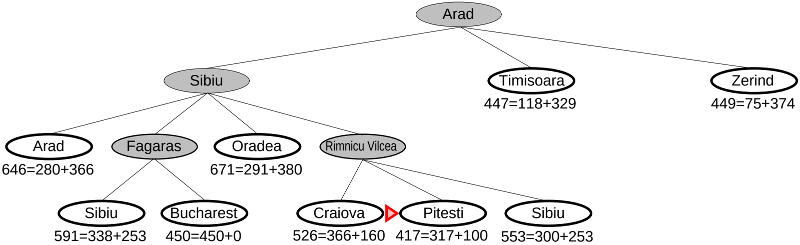
A* Search Example



A* Search Example



A* Search Example



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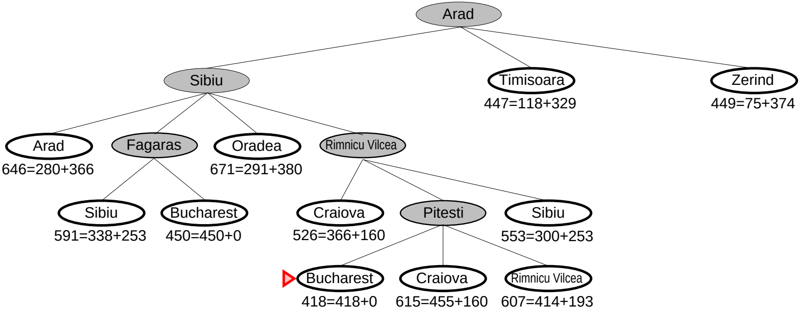
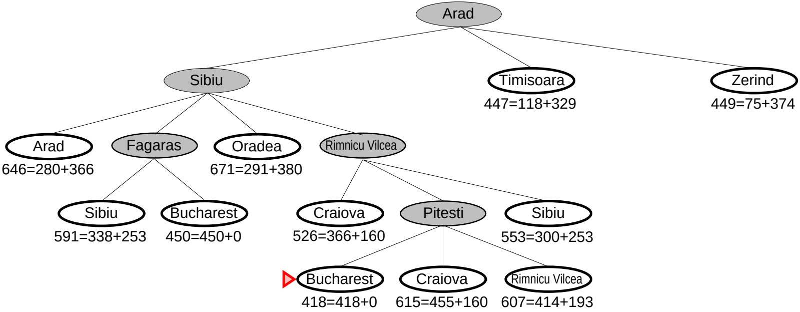


A* Search Example



Total distance to go: 418 km!

Properties of A* search

- Complete and optimal under minor conditions if
  - an admissible heuristic h is used:

$$h(s) \leq h^*(s)$$

  where h^* is the true cost from s to a goal.

Properties of A* search

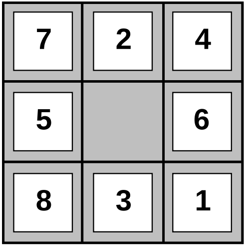
- Complete and optimal under minor conditions if
  - an admissible heuristic h is used:

    $$h(s) \leq h^*(s)$$

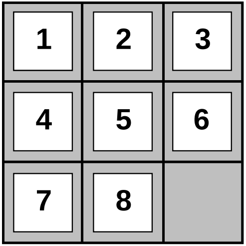
    where h^* is the true cost from s to a goal.
  - Thus, a heuristic h is admissible if it never overestimates the distance to the goal (is optimistic).

Examples of admissible heuristics for 8-Puzzle

- $h_1(s) =$ number of misplaced tiles.
- $h_2(s) =$ Manhattan distance. Take for each tile the sum over the horizontal and vertical steps from the desired location (its Manhattan distance from the desired location). Then take the sum over those distances.



Start State



Goal State

$h_1(s) = ??$
$h_2(s) = ??$

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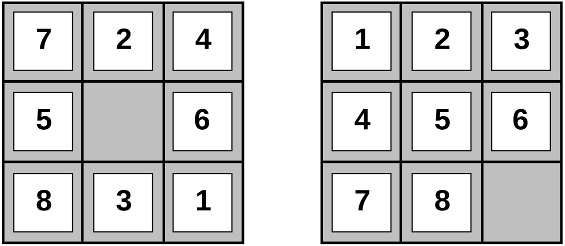

Start State **Goal State**

$h_1(s) = 6$
$h_2(s) = ??$

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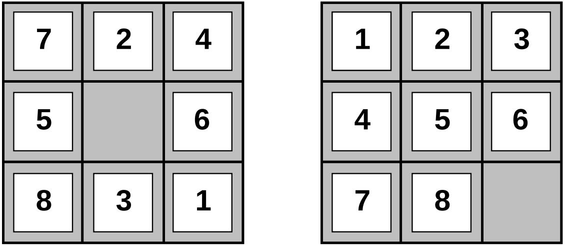

Start State **Goal State**

$h_1(s) = 6$
$h_2(s) = 4+0+3+3+1+0+2+1 = 14$

Importance of the Heuristic Choice

Typical search costs (data averaged over 100 instances of the 8-puzzle and d the length of the shortest solution path):

$d = 14$ IDS $= 3{,}473{,}941$ paths
A$^*(h_1) = 539$ paths
A$^*(h_2) = 113$ paths

$d = 24$ IDS $\approx 54{,}000{,}000{,}000$ paths
A$^*(h_1) = 39{,}135$ paths
A$^*(h_2) = 1{,}641$ paths

Summary

- Heuristic functions estimate costs of shortest paths
- Good heuristics can dramatically reduce search cost
- Greedy best-first search expands lowest h
  - incomplete and not always optimal
- A* search expands lowest $g + h$
  - complete and optimal
  - also optimally efficient