Foundations of Computer Science Comp109

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Recap: Matrix of a relation.

- A relation R is represented by the *n by m matrix* $M(i,j)$ where
  $M(i,j) = \begin{cases} 1 & \text{if} \quad (a_i, b_j) \in R \\ 0 & \text{if} \quad (a_i, b_j) \notin R \end{cases}$
- Matrix product $P = MN$ is given by
  $P(i,j) = \begin{cases} 1 & \text{if} \quad \exists l,\ 1 \leq l \leq m, \text{ such that } M(i,l) = 1 \ \& \ N(l,j) = 1 \\ 0 & \text{if} \quad \text{otherwise.} \end{cases}$
- Ordering of elements is important, for example it allows to improve the search:
  binary (*half-interval*) search is quicker, but also allows to perform other useful operations like ranking and finding the nearest neighbor.
- *Infix notation* xRy whenever $(x, y) \in R$ for a binary relation R.

Detour recap: Boolean multiplication in Python numpy

```
import numpy as np
a = np.array([[0, 0, 1],
              [1, 0, 1]], dtype=bool)
b = np.array([[1, 0],
              [0, 1],
              [0, 0]], dtype=bool)

print(1*np.dot(a,b))
```

Why is there multiplication by 1 in the last line?

Answer: relational matrices can be represented as boolean (logic true/false) types, for example:

```
print(a)
```

```
[[False False  True]
 [ True False  True]]
```

Multiplication by 1 converts it back to the 0/1 binary form.

Detour: ordering sets in Python

Matrix will depend on the order of elements in sets.

Consider relation $T \subseteq \mathbb{R} \times \{\text{Days of the week}\}$ defined by "x hours to work on the y day of the week".

What would be the order of the set that defines days of the week?

Answer: for weekdays even numeric order is not well-defined: the first day can be Sunday or Monday. In Python, order of
$\{'Monday', 'Tuesday', 'Wednesday', 'Thursday', 'Friday', 'Saturday', 'Sunday'\}$
is not the same as of $\{'M', 'T', 'W', 'T', 'F', 'S', 'S'\}$.
Can you predict auto-order for each of those sets in Python? Try it! (Hint: codes for sets were in Lecture 12.)
Then try to predict ordering of sets $\{2, 3, 1, 8\}$ and $\{1, 2, 3, 8, 9\}$.

Warning: do not rely on the auto-ordering of sets! For a relation matrix, clearly define the order.

Recap: ordering strings

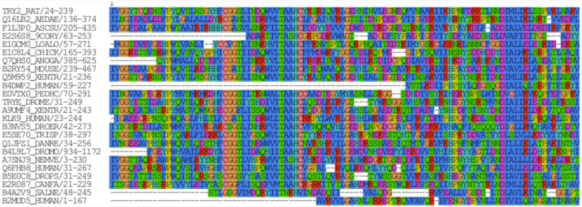
Consider relations R, S and L on the set of all strings:

- L—Lexicographic ordering:
  - first we define a *totally ordered* alphabet,
  - for two different words of the *same length*, their order follows the alphabetic order of the symbols in the first place where the two words differ,
  - for two words of *different lengths*, usually the shorter one is padded with "blanks" which are smaller than every alphabet symbol.
- uSv if, and only if, u is a Substring of v;
- uNv if, and only if, $len(u) \leq len(v)$.

Why so many relations?

Answer: Each of those relations have their own use and limitations:

1. What is the first element in the infinite sequence $b, ba, baa, baaa, \ldots$? (Compare to $2, 21, 211, 2111, \ldots$) Lexicographic ordering works well only for *same length* sequences.
2. For $len(u) \leq len(v)$ relation: $len(v) \leq len(u)$ does not mean $u = v$ (unlike for the other two mentioned relations).
3. Substrings are important in bioinformatics (for protein alignment), but it's not clear which subset to take: how would you align "banana" and "anaconda"?



Properties of binary relations

A binary relation R on a set A is

- *reflexive* when xRx for all $x \in A$.

$$\forall x\ A(x) \implies xRx$$

- *symmetric* when xRy implies yRx for all $x, y \in A$;

$$\forall x, y\ xRy \implies yRx$$

- *antisymmetric* when xRy and yRx imply $x = y$ for all $x, y \in A$;

$$\forall x, y\ xRy \text{ and } yRx \implies y = x$$

- *transitive* when xRy and yRz imply xRz for all $x, y, z \in A$.

$$\forall x, y, z\ xRy \text{ and } yRz \implies xRz$$

Example

- *reflexive* $\forall x : xRx$
- *symmetric* $\forall x, y : xRy \implies yRx$
- *antisymmetric* $\forall x, y : xRy,\ yRx \implies x = y$
- *transitive* $\forall x, y, z : xRy,\ yRz \implies xRz$

Which of the following define a relation that is reflexive, symmetric, antisymmetric or transitive?

- x divides y on the set $\mathbb{Z}^+$ of positive integers
  Answer: reflexive, transitive, antisymmetric; not symmetric.
- $x \neq y$ on the set $\mathbb{Z}$ of integers
  Answer: symmetric; not reflexive, transitive or antisymmetric.
- x *has the same age as* y on the set of people
  Answer: reflexive, symmetric, transitive; not antisymmetric.

Digraph representation

- *reflexive* $\forall x : xRx$
- *symmetric* $\forall x, y : xRy \implies yRx$
- *antisymmetric* $\forall x, y : xRy,\ yRx \implies x = y$
- *transitive* $\forall x, y, z : xRy,\ yRz \implies xRz$

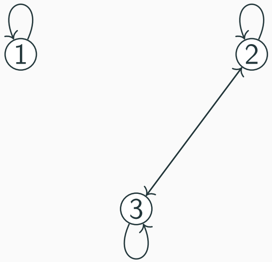
In the directed graph representation, R is

- *reflexive* if there is always an arrow from every vertex to itself;
- *symmetric* if whenever there is an arrow from x to y there is also an arrow from y to x;
- *antisymmetric* if whenever there is an arrow from x to y and $x \neq y$, then there is no arrow from y to x;
- *transitive* if whenever there is an arrow from x to y and from y to z there is also an arrow from x to z.

Example 1

- *reflexive* $\forall x : xRx$
- *symmetric* $\forall x, y : xRy \implies yRx$
- *antisymmetric* $\forall x, y : xRy,\ yRx \implies x = y$
- *transitive* $\forall x, y, z : xRy,\ yRz \implies xRz$

Let $A = \{1, 2, 3\}$, $R_1 = \{(1,1), (2,2), (3,3), (2,3), (3,2)\}$

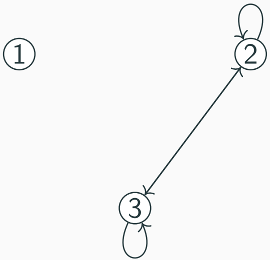


Answer: reflexive, symmetric, transitive, not antisymmetric.

Example 2

- *reflexive* $\forall x : xRx$
- *symmetric* $\forall x, y : xRy \implies yRx$
- *antisymmetric* $\forall x, y : xRy,\ yRx \implies x = y$
- *transitive* $\forall x, y, z : xRy,\ yRz \implies xRz$

Let $A = \{1, 2, 3\}$, $R_2 = \{(2, 2), (2, 3), (3, 2), (3, 3)\}$

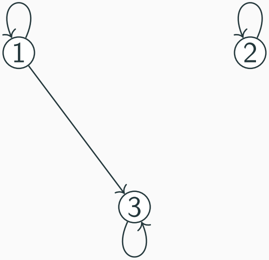


Answer: symmetric, transitive, not reflexive or antisymmetric.

Example 3

- *reflexive* $\forall x : xRx$
- *symmetric* $\forall x, y : xRy \implies yRx$
- *antisymmetric* $\forall x, y : xRy,\ yRx \implies x = y$
- *transitive* $\forall x, y, z : xRy,\ yRz \implies xRz$

Let $A = \{1, 2, 3\},\ R_3 = \{(1,1), (2,2), (3,3), (1,3)\}$

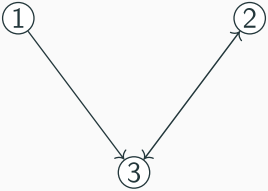


Answer: reflexive, antisymmetric, transitive, not symmetric.

Example 4

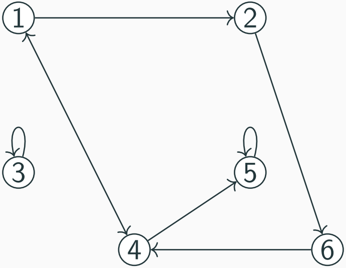
- *reflexive* $\forall x : xRx$
- *symmetric* $\forall x, y : xRy \implies yRx$
- *antisymmetric* $\forall x, y : xRy,\ yRx \implies x = y$
- *transitive* $\forall x, y, z : xRy,\ yRz \implies xRz$

Let $A = \{1, 2, 3\}$, $R_4 = \{(1, 3), (3, 2), (2, 3)\}$



Answer: none of those properties.

Example: Reachability relation



Which nodes can reach node 5?

Answer: 1, 2, 6 and 4 can reach each other, so form a looped component, all of them can reach 5.

Transitive closure

Given a binary relation R on a set A, the *transitive closure* R^* of R is the (uniquely determined) relation on A with the following properties:

- R^* is transitive;
- $R \subseteq R^*$;
- If S is a transitive relation on A and $R \subseteq S$, then $R^* \subseteq S$.

Example

Let $A = \{1, 2, 3\}$. Find the transitive closure of

$$R = \{(1,1),(1,2),(1,3),(2,3),(3,1)\}.$$

Answer: add $(3,2)$, $(2,1)$, $(3,3)$, $(2,2)$.

Transitivity and composition

A relation S is transitive if and only if $S \circ S \subseteq S$.

This is because

$$S \circ S = \{(a, c) \mid \text{ exists } b \text{ such that } aSb \text{ and } bSc\}.$$

Let S be a relation. Set $S^1 = S$, $S^2 = S \circ S$, $S^3 = S \circ S \circ S$, and so on.

Theorem Denote by S^* the transitive closure of S. Then xS^*y if and only if there exists $n > 0$ such that xS^ny.

DIY Transitive closure in matrix form

The relation R on the set $A = \{1, 2, 3, 4, 5\}$ is represented by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Is R transitive?

DIY Computation

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix}$$

$$R \circ R = \{(a, c) \mid \text{ exists } b \in A \text{ such that } aRb \text{ and } bRc\}.$$

There are pairs (a, c) that are in $R \circ R$ but not in R. Hence, R is not transitive.

DIY Detour: Warshall's algorithm

```
def warshall(a):
    n = len(a)
    for k in range(n):
        for i in range(n):
            for j in range(n):
                a[i][j] = (a[i][j] or
                          (a[i][k] and a[k][j]))
    return a

print(warshall([[1,0,0,1,0],
                [0,1,0,0,1],
                [0,0,1,0,0],
                [1,0,1,0,0],
                [0,1,0,1,0]]))
```

Summary

Attendance code: 983547

- **Reflexivity:** $\forall x \; A(x) \implies xRx$
- **Symmetry:** $\forall x, y \; xRy \implies yRx$
- **Antisymmetry:** $\forall x, y \; xRy \text{ and } yRx \implies y = x$
- **Transitivity:** $\forall x, y, z \; xRy \text{ and } yRz \implies xRz$
- The **transitive closure** R^* of a realation R on a set A is the smallest relation on A that contains R and is transitive.
- $xR^*y \iff \exists n > 0$ such that xR^ny.