Linear Transformations

Background

- *Linear Transformations* "*map*" between vector spaces.
- When we use some method, T, to produce a vector, **v**, in some vector space V from a vector, **u**, belonging to a vector space U.
- Formally, $T : U \rightarrow V$
- U and V can be the *same* vector space, eg $U = V = R^2$
- These may also be *different* vector spaces: $U = R^3$; $V = R^2$
- An example of this is "*projecting 3D objects to 2D displays*".

Why are these “interesting”?

- *Graphical effects* (*scaling* an object, *rotation*) may be implemented by *matrix and vector multiplication*.
- The vectors are “screen positions”, ie coordinates.
- A “graphic effect” is, eg “*double the length of a line*”.
- If an “effect” is a *linear transformation* then it is a *matrix product*.
- If an “effect” is a *matrix product* then it is a *linear transformation*.

Quick Review - Matrices

- 2×2 matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

- Matrix-vector product: (2×2 case)

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{pmatrix}$$

3×3 Matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

- 3×3 matrix-vector

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \\ a_{31}x + a_{32}y + a_{33}z \end{pmatrix}$$

$n \times n$ Matrices

$$A = [a_{ij}]$$

- $n \times n$ matrix product : $C = A \cdot B$

$$[c_{ij}] = \sum_{k=1}^{n} a_{ik}\, b_{kj}$$

- The numbers of rows and columns may differ.
- Note: in general $A \cdot B \neq B \cdot A$

Linear Transformation – Definition

- Vector spaces U and V (but may have U=V).
- T a mapping from U to V ($T : U \to V$).
- T is a *linear transformation* if

$$\forall\, p, q \in U : T(p+q) = T(p) + T(q)$$
$$\forall \alpha \in H, \forall p \in U : T(\alpha p) = \alpha T(p)$$

Some examples of 2×2 effects

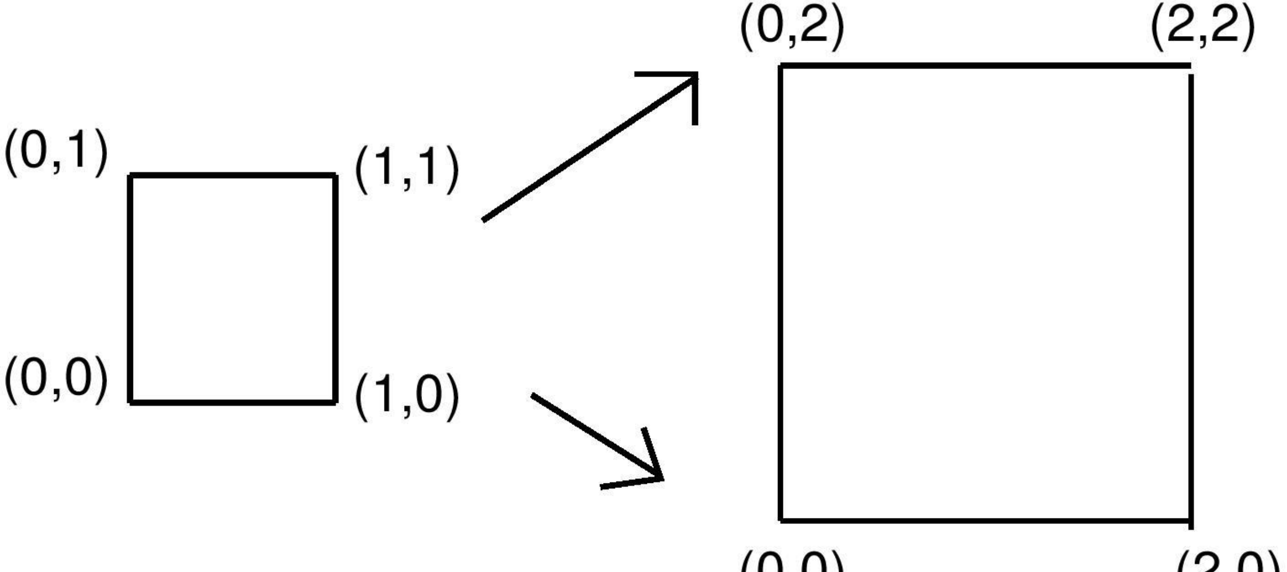
- Scaling by a factor $s \in R$:

$$\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} sx \\ sy \end{pmatrix} = s \begin{pmatrix} x \\ y \end{pmatrix}$$

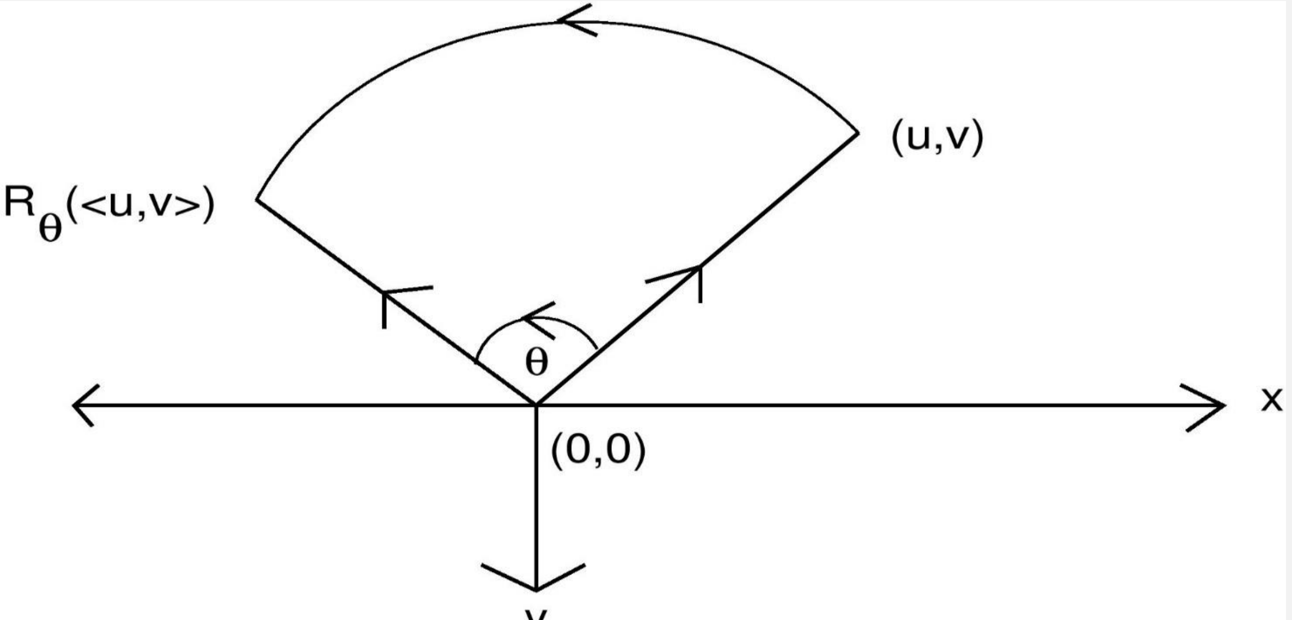
- Rotating (anti-clockwise) around origin by $\theta°$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

Scaling in 2-dimensions



Rotation in 2-dimensions



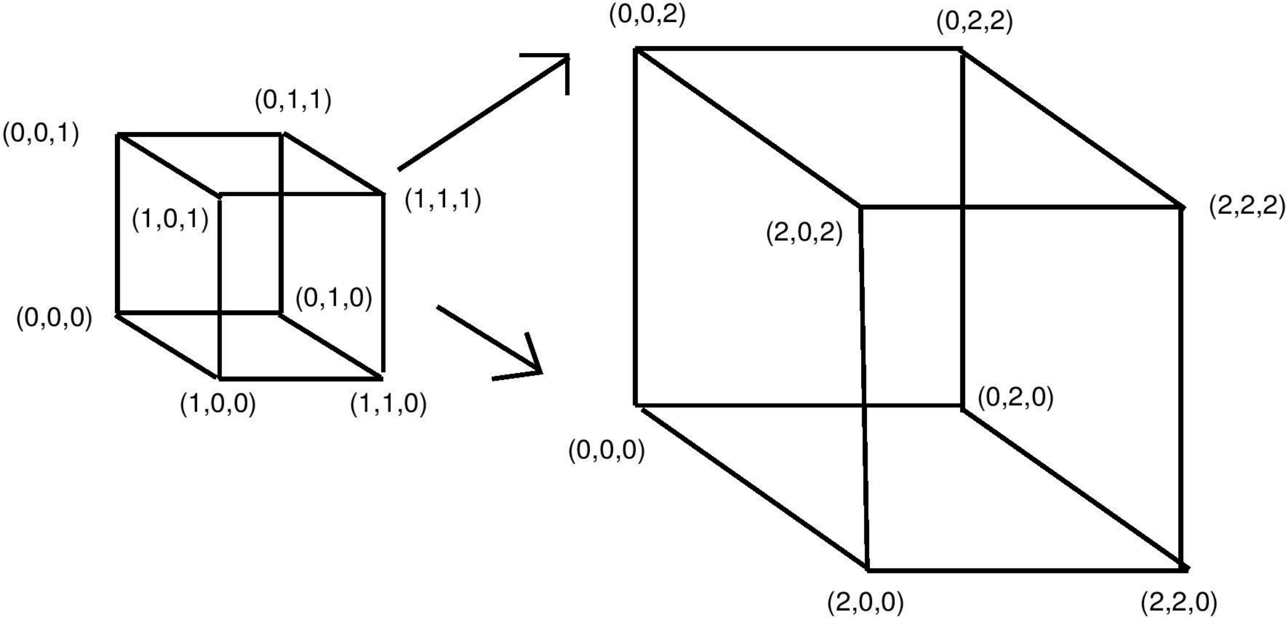
Scaling in 3-D

- Scaling by a factor $s \in R$:

$$\begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} sx \\ sy \\ sz \end{pmatrix} = s \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

- Rotation a little more complicated (page 108)

Scaling in 3-dimensions



Minor inconvenience - Translation

- A very basic effect is "*move* an object at (x,y) to a position (x+p,y+q)"
- Similarly with three dimensions.
- The mapping $T: R^2 \to R^2$ defined through

$$T(x,y) = (x+p, y+q)$$

is *not* a linear transformation.

- nor that, in 3D: $T(x,y,z) = (x+p, y+q, z+r)$.

Homogenous Coordinates I

- Using "*homogenous coordinate systems*" we can implement translations by matrix-vector products.
- Instead of using (x,y) use (x,y,1).

$$\begin{pmatrix} 1 & 0 & p \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x+p \\ y+q \\ 1 \end{pmatrix}$$

Homogenous Coordinates II

- By adding an extra row and column to the 2×2 matrices for scaling and rotation we can implement "most" effects via 3×3 matrix-vector products.

$$\begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} sx \\ sy \\ 1 \end{pmatrix}$$

Combining Effects

- Using 3×3 matrices - $S_t, R_\theta, M_{(p,q)}$ - to scale, rotate and translate, combinations of effects can be achieved with a single 3×3 matrix, eg rotate, then scale and move.
- Care is needed, however, since the outcome of move-then-scale is not the same as that of scale-then-move.
- 3-dimensional rotations raises additional challenges.