Foundations of Computer Science Comp109

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Last lecture recap: Operations on sets.

- Sets $A = B \iff$
- $B \subseteq A \iff$
- $A \cup B =$
- $A \cap B =$
- $A - B =$
- $\sim A = \neg A =$

The symmetric difference

Definition The symmetric difference of two sets A and B is the set

$$A \Delta B = \{x \mid (x \in A \text{ and } x \notin B) \text{ or } (x \notin A \text{ and } x \in B)\}.$$

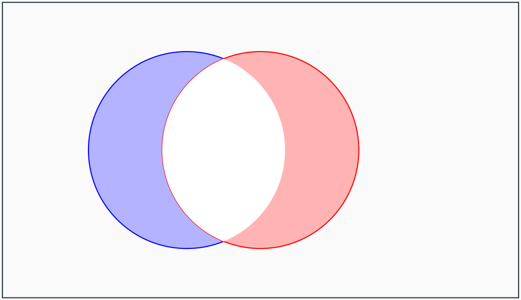


Figure 1: Venn diagram of $A \Delta B$.

Example

Suppose

$$A = \{4, 7, 8\}$$

and

$$B = \{4, 9, 10\}.$$

Then

$$A \Delta B =$$

Recap Quiz

If A and B are sets and suppose $x \in A \cup B$.

Then classify the following statements as True or False.

- $\exists x | x \in A \cap B$
- $\forall x, x \in A \cap B$
- $\exists x | x \in A - B$
- $\exists x | x \in B - A$
- $\exists x | x \in A \Delta B$
- $\exists x | (x \in A \Delta B \text{ and } x \in A - B)$
- $\exists x | (x \in A \Delta B \text{ and } x \in A \cap B)$
- $\exists x | (x \notin A \Delta B \text{ and } x \notin A \cap B)$
- $\forall x, x \in A \Delta B \text{ or } x \in A \cap B$

Proving identities: $A \Delta B = (A \cup B) - (A \cap B)$

Proof continues

The algebra of sets

The algebra of sets (1)

Suppose that A, B, C, U are sets with $A \subseteq U$, $B \subseteq U$, $C \subseteq U$

Commutative laws (a) $A \cup B = B \cup A$ and (b) $A \cap B = B \cap A$.

Associative laws (a) $A \cup (B \cup C) = (A \cup B) \cup C$ and
(b) $A \cap (B \cap C) = (A \cap B) \cap C$.

Distributive laws (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and
(b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Identity laws (a) $A \cup \emptyset = A$ and (b) $A \cap U = A$

Complement laws (a) $A \cup \sim A = U$ and (b) $A \cap \sim A = \emptyset$.

The algebra of sets (2)

Double complement law	$\sim(\sim A) = A$.
Idempotent laws	(a) $A \cup A = A$ and (b) $A \cap A = A$.
Universal bound laws	(a) $A \cup U = U$ and (b) $A \cap \emptyset = \emptyset$.
De Morgan's law	(a) $\sim(A \cup B) = \sim A \cap \sim B$ and (b) $\sim(A \cap B) = \sim A \cup \sim B$
Absorption laws	(a) $A \cup (A \cap B) = A$ and (b) $A \cap (A \cup B) = A$
Complement of U and $\emptyset$	(a) $\sim U = \emptyset$ and (b) $\sim \emptyset = U$
Set difference law	$A - B = A \cap \sim B$

Proving the commutative law $A \cup B = B \cup A$

Definition: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ and $B \cup A = \{x \mid x \in B \text{ or } x \in A\}$.

These are the same set. To see this, check all possible cases.

Case 1: Suppose $x \in A$ and $x \in B$.

Case 2: Suppose $x \in A$ and $x \notin B$.

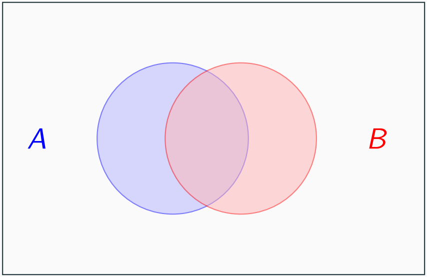
Case 3: Suppose $x \notin A$ and $x \in B$.

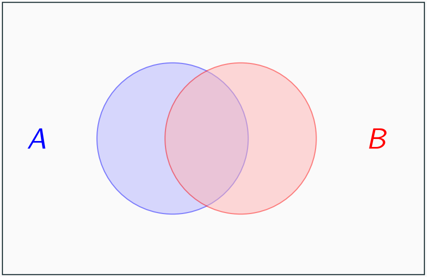
Case 4: Suppose $x \notin A$ and $x \notin B$.

Proving the commutative law $A \cup B = B \cup A$, alternative way

De Morgan's laws

$$\sim (A \cap B) = \sim A \cup \sim B.$$





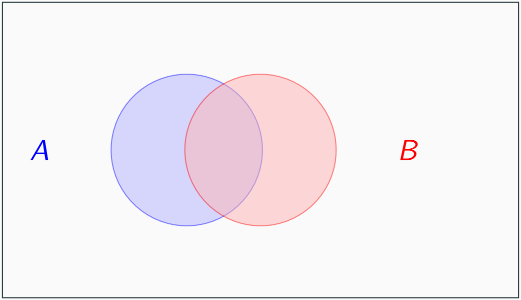
A proof of De Morgan's law $\sim (A \cap B) = \sim A \cup \sim B$

Check all possible cases.

Alternative proof of De Morgan's law $\sim(A \cap B) = \sim A \cup \sim B$

Using the algebra of sets

Prove that $(A \cap \sim B) \cup (B \cap \sim A) = (A \cup B) \cap \sim (A \cap B)$.



Algebraic proof of $(A \cap \sim B) \cup (B \cap \sim A) = (A \cup B) \cap \sim (A \cap B)$

Let's prove the statement above useng the algebraic laws:

DIY problem

Try yourself by proving the rest of the algebraic laws. Help each other to find gaps in proofs, and discuss!

Summary

Proving with sets often includes consideration of several cases, knowing the main laws can speed up the process:

Suppose that A, B, C, U are sets with $A \subseteq U$, $B \subseteq U$, $C \subseteq U$

Commutative laws	(a) $A \cup B = B \cup A$ and (b) $A \cap B = B \cap A$.
Associative laws	(a) $A \cup (B \cup C) = (A \cup B) \cup C$ and (b) $A \cap (B \cap C) = (A \cap B) \cap C$.
Distributive laws	(a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and (b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
Identity laws	(a) $A \cup \emptyset = A$ and (b) $A \cap U = A$
Complement laws	(a) $A \cup \sim A = U$ and (b) $A \cap \sim A = \emptyset$.