COMP111: Artificial Intelligence

Section 8. Reasoning under Uncertainty 1

Frank Wolter

Content

- Why reasoning under uncertainty?
- Basics of probability
- Conditional probability
- Independence
- Bayes' Theorem and basic applications.

Why Reasoning under Uncertainty?

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- When going to the airport by car, how early should I start? 45 minutes should be enough from Liverpool to Manchester Airport, but only under the assumption that there are no accidents, no lane closures, that my car does not break down, and so on. This uncertainty is hard to eliminate, but still an agent has to make a decision.

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- A dental patient has toothache. Does the patient have a cavity? One might want to capture the relationship between patients having a cavity and patients having toothache by the rule:

  $$\mathsf{Toothache}(x) \rightarrow \mathsf{Cavity}(x)$$

  But this does not work as not every toothache is caused by a cavity. Hard to come up with exhaustive list of reasons:

  $$\mathsf{Toothache}(x) \rightarrow \mathsf{Cavity}(x) \vee \mathsf{GumProblem}(x) \vee \mathsf{Abscess}(x) \vee \cdots$$

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- Laziness: it is too much work to list an exceptionless set of rules.
- Theoretical ignorance: Medical science has, in many cases, no strict laws connecting symptoms with diseases.
- Practical ignorance: Even if we have strict laws, we might be uncertain about a particular patient because not all the necessary tests have been or can be run.

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- We might not know for sure what disease a particular patient has, but we believe that there is, say, an 80% chance that a patient with toothache has a cavity.
  The 80% summarises those cases in which all the factors needed for a cavity to cause a toothache are present and other cases in which the patient has both toothache and cavity but the two are unconnected.
- The missing 20% summarizes all the other possible causes of toothache that we are too lazy or ignorant to confirm or deny.

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Recall that if S consists of $x_1, \ldots, x_n$, then

$$\sum_{x \in S} P(x) = P(x_1) + \cdots + P(x_n).$$

Example: Flipping a fair coin

Consider the random experiment of flipping a coin. Then the corresponding probability space (S, P) is given by

- $S = \{H, T\}$;
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Consider the random experiment of flipping a coin two times, one after the other. Then the corresponding probability space (S, P) is given as follows:

- $S = \{HH, HT, TH, TT\}$;
- $P(HH) = P(HT) = P(TH) = P(TT) = \frac{1}{4}$.

Example: Rolling a fair die

Consider the random experiment of rolling a die. Then the corresponding probability space (S, P) is given by

- $S = \{1, 2, 3, 4, 5, 6\}$;
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Consider the random experiment of rolling a die n times. Then the corresponding probability space (S, P) is given as follows:

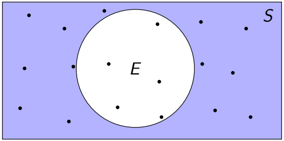
- S is the set of sequences of length n over the alphabet $\{1, \ldots, 6\}$ (sometimes denoted $\{1, \ldots, 6\}^n$).
- $P(x) = \frac{1}{6^n}$ for every elementary event x, since S has 6^n elements.

Uniform Probability Distributions

- A probability distribution is uniform if every outcome is equally likely.
- For uniform probability distributions, the probability of an outcome x is 1 divided by the number $|S|$ of outcomes in S.

$$P(x) = \frac{1}{|S|}$$

Events



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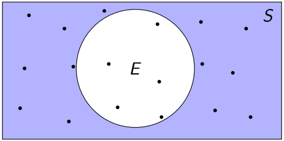

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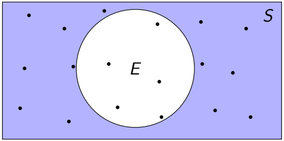

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Notice that

- $0 \leq P(E) \leq 1$ for every event E,
- $P(\emptyset) = 0$ and $P(S) = 1$.

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- the set $E = \{1, 3, 5\}$
- $P(E) = P(1) + P(3) + P(5) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$

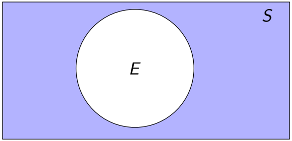
The probability of composed events

We show how the probability of

- the complement of an event can be computed from the probability of the event;
- the union of events can be computed from the probabilities of the individual events.

The probability of the complement of an event

Let $\neg E = S \setminus E$. Then $P(\neg E) = 1 - P(E)$



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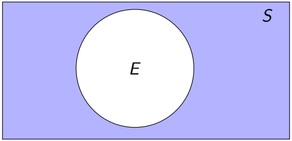

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Proof.

$$1 = \sum_{x \in S} P(x) = \sum_{x \in E} P(x) + \sum_{x \in \neg E} P(x)$$

Thus,

$$\sum_{x \in \neg E} P(x) = 1 - \sum_{x \in E} P(x)$$

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- $\neg E = \{1111111111\}$.
- $P(\neg E) = \frac{1}{2^{10}}$.
- $P(E) = 1 - \frac{1}{2^{10}}$.

The probability of the union of two events

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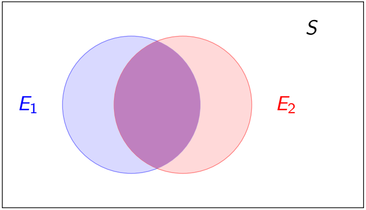

Side remark:

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$

Proof of $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

Recall:

- $P(E_1) = \sum_{x \in E_1} P(x)$;
- $P(E_2) = \sum_{x \in E_2} P(x)$;
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Thus,

$$\begin{aligned} P(E_1 \cup E_2) &= \sum_{x \in E_1 \cup E_2} P(x) \\ &= \sum_{x \in E_1} P(x) + \sum_{x \in E_2} P(x) - \sum_{x \in E_1 \cap E_2} P(x) \\ &= P(E_1) + P(E_2) - P(E_1 \cap E_2) \end{aligned}$$

Example

Suppose I have a jar of 30 sweets, as follows.

	red	blue	green
circular	2	4	3
square	6	7	8

The sample space S has 30 elements and if one chooses a sweet uniformly at random, then

$$P(x) = \frac{1}{30}$$

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Then $P(R \cup C)$ is is the probability that the sweet is red or circular. Thus

$$P(R \cup C) = P(R) + P(C) - P(R \cap C) = \frac{8}{30} + \frac{9}{30} - \frac{2}{30} = \frac{15}{30} = \frac{1}{2}$$

Disjoint events

Assume that $E_1, \ldots, E_n$ are mutually disjoint events. So $E_i \cap E_j = \emptyset$ whenever $i \neq j$.

Then

$$P(\bigcup_{1 \leq i \leq n} E_i) = \sum_{1 \leq i \leq n} P(E_i)$$

Example: three dice

Suppose that I roll a fair die three times. Then

- S is the set of sequences of length three over $\{1, \ldots, 6\}$ (or $\{1, \ldots, 6\}^3$).
- $P(x) = \frac{1}{6 \times 6 \times 6} = \frac{1}{216}$ for all $x \in S$.

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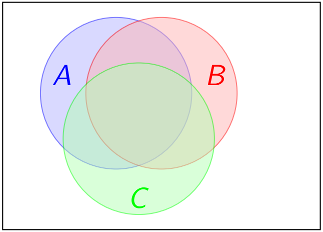
Let E_1: event that 1st roll is a 6, E_2: event that 2nd roll is a 6; E_3: event that 3rd roll is a 6.

We would like to know

$$P(E_1 \cup E_2 \cup E_3)$$

Computing the probability of a union of three sets

$$\begin{aligned} P(A \cup B \cup C) = {} & P(A) + P(B) + P(C) \\ & - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ & + P(A \cap B \cap C). \end{aligned}$$



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Three dice example continued

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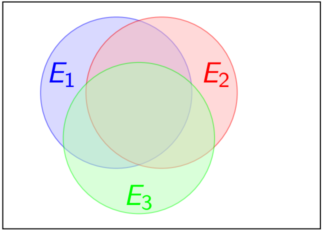

$$\begin{aligned} P(E_1 \cup E_2 \cup E_3) &= P(E_1) + P(E_2) + P(E_3) \\ &\quad - P(E_1 \cap E_2) - P(E_1 \cap E_3) - P(E_2 \cap E_3) \\ &\quad + P(E_1 \cap E_2 \cap E_3) \\ &= \frac{36}{216} + \frac{36}{216} + \frac{36}{216} \\ &\quad - \frac{6}{216} - \frac{6}{216} - \frac{6}{216} \\ &\quad + \frac{1}{216} \\ &= \frac{91}{216} \approx 0.42 \end{aligned}$$

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- Consider a family chosen at random from a set of families having two children (but not having twins).
- What is the probability that both children are boys?
- A suitable sample space $S = \{BB, GB, BG, GG\}$.
- It is reasonable to assume that $P(x) = \frac{1}{4}$ for all $x \in S$. Thus $P(BB) = \frac{1}{4}$.

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- How do we assign probabilities to the events in S'?

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- We therefore normalise by dividing the probability $P(x)$ of the outcome x in S by the probability $P(S')$ of S' in S. Thus

$$P(BB \mid \text{ at least one boy }) = P(BB \mid S') = \frac{P(BB)}{P(S')} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

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- So we want to know the probability $P(A \cap B)$ (since we know that B occurs) after the conditioning on B.
- Once again, we can't take $P(A \cap B)$ itself but have to normalise by dividing by the probability of the new sample space $P(B)$. Thus

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Conditional Probability

Let A and B be events, with $P(B) > 0$.
The conditional probability $P(A|B)$ of A given B is given by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

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$$\frac{2}{30} \div \frac{2+4+3}{30} = \frac{2}{30} \times \frac{30}{2+4+3} = \frac{2}{2+4+3} = \frac{2}{9}, \text{ which is}$$

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I choose a sweet uniformly at random from a jar of 30 sweets.

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Multiplication Rule

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This is knows as the multiplication rule.

- It can be extended to more events, for example:

$$P(A \cap B \cap C) = P(C \mid A \cap B)P(A \cap B) = P(C \mid A \cap B)P(B \mid A)P(A)$$

An Application of the Multiplication Rule

- Consider choosing a family from a set of families with just one pair of twins (and thus no other children).

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- Recall that twins are either identical (I) or fraternal (F). We know that a third of human twins are identical. Thus

$$P(I) = \frac{1}{3}, \quad P(F) = \frac{2}{3}$$

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- By multiplication rule

$$P(I \cap BB) = P(BB \mid I)P(I), \quad P(F \cap BB) = P(BB \mid F)P(F)$$

An Application of the Multiplication Rule

- The probability of being a girl or boy for fraternal twins will be the same as for any other two-child family. For the identical twins, the outcomes BG and GB are no longer possible. Thus:

$$P(BB \mid I) = \frac{1}{2}, \quad P(BB \mid F) = \frac{1}{4}$$

An Application of the Multiplication Rule

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- We obtain:

$$\begin{aligned} P(BB) &= P(I \cap BB) + P(F \cap BB) \\ &= P(BB \mid I)P(I) + P(BB \mid F)P(F) \\ &= \frac{1}{2} \times \frac{1}{3} + \frac{1}{4} \times \frac{2}{3} \\ &= \frac{1}{3} \end{aligned}$$

Independence

- In everyday language we refer to events that “have nothing to do with each other” as being independent.
- A similar notion of independence is useful in probability theory as it helps to structure probabilistic knowledge and reduce complexity.

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If $P(A) \neq 0$ and $P(B) \neq 0$, then the following are equivalent:

- A and B are independent;
- $P(B) = P(B|A)$; (recall that $P(B \mid A) = \frac{P(A \cap B)}{P(A)}$);
- $P(A) = P(A|B)$; (recall that $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$).

Example illustrating our definitions

Consider the following probability space (S, P) with

$$S = \{\text{red}, \text{green}, \text{yellow}, \text{black}\}$$

and

$$P(\text{red}) = 0.1, P(\text{green}) = 0.2, P(\text{yellow}) = 0.2, P(\text{black}) = 0.5$$

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What is $P(\neg\{\text{green}\})$?

$$P(\neg\{\text{green}\}) = P(\{\text{red}, \text{yellow}, \text{black}\}) = 0.8$$

Example illustrating our definitions continued

$$P(\text{red}) = 0.1, P(\text{green}) = 0.2, P(\text{yellow}) = 0.2, P(\text{black}) = 0.5$$

Recall that

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

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Are $\{\text{red}\}$ and $\{\text{green}\}$ independent?

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Are $\{\text{red}\}$ and $\{\text{green}\}$ independent?

A and B are independent if $P(A \mid B) = P(A)$. (Equivalently, $P(A \cap B) = P(A) \times P(B)$.)

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So they are not independent.

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A and B are independent if $P(A \mid B) = P(A)$.

So they are not independent.

Example 2: Independence

If you roll two dice, then the rolls are independent. So the event A of rolling a 1 with the first die is independent of the event B of rolling a 1 with the second die. So the probability that both of these happen is

$$P(A \cap B) = P(A) \times P(B) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

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This holds for all $ab \in \{1, 2, 3, 4, 5, 6\}^2$:

$$P(ab) = \frac{1}{36} = P(a) \times P(b)$$

Example 2 continued

- Let A be the event of rolling a 1 with the first die.
- Let C be the event of rolling a 1 with the first or second die.
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We show that A and B are independent but C and B are not.

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- As
$$A = \{11, 12, 13, 14, 15, 16\}.$$
we have $P(A) = \frac{6}{36} = \frac{1}{6}$.

Example 2 continued

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$$B = \{11, 13, 15, 22, 24, 26, 31, 33, 35, 42, 44, 46, 51, 53, 55, 62, 64, 66\}$$
we have $P(B) = \frac{18}{36} = \frac{1}{2}$.

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we have $P(B) = \frac{18}{36} = \frac{1}{2}$.
- We have $A \cap B = \{11, 13, 15\}$. So
$$P(A \cap B) = \frac{3}{36} = \frac{1}{12} = P(A)P(B)$$
and conclude that A and B are independent.

Example 2 continued

- As
$$C = \{11, 12, 13, 14, 15, 16, 21, 31, 41, 51, 61\}$$
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- As
$$B \cap C = \{11, 13, 15, 31, 51\}$$
we have $P(B \cap C) = \frac{5}{36}$.

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- As
$$B \cap C = \{11, 13, 15, 31, 51\}$$
we have $P(B \cap C) = \frac{5}{36}$.
- So
$$P(B \cap C) = \frac{5}{36} \neq \frac{5.5}{36} = P(B)P(C)$$
and we conclude that B and C are not independent.

Example 3

- Consider the random experiment of drawing two balls, one after the other, from a basket containing a red, a blue, and a green ball.
- Let R_1 be the event that the first ball is red.
- Let R_2 be the event that the second ball is blue.

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We show that R_1 and R_2 are not independent.
Let

- $S = \{RB, RG, BR, BG, GR, GB\}$;
- $P(RB) = P(RG) = P(BR) = P(BG) = P(GR) = P(GB) = \frac{1}{6}$.

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- $S = \{RB, RG, BR, BG, GR, GB\}$;
- $P(RB) = P(RG) = P(BR) = P(BG) = P(GR) = P(GB) = \frac{1}{6}$.
- Then $R_1 = \{RB, RG\}$ and $R_2 = \{RB, GB\}$.
- $P(R_1) = P(RB) + P(RG) = \frac{1}{3}$ and $P(R_2) = P(RB) + P(GB) = \frac{1}{3}$.
- Thus,

$$P(R_1 \cap R_2) = P(RB) = \frac{1}{6} \neq P(R_1) \times P(R_2) = \frac{1}{9}$$

Independence for more than two events

- Consider a finite set of events

$$A_1, \ldots, A_n$$

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- Pairwise independence is not a particularly important notion.

Pairwise independence does not imply mutual independence

- Consider the random experiment of flipping a coin two times, one after the other. Then $S = \{HH, HT, TH, TT\}$ and $P(HH) = P(HT) = P(TH) = P(TT) = \frac{1}{4}$.

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- Let $H_1 = \{HT, HH\}$. Then $P(H_1) = \frac{1}{2}$;

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- Let $H_2 = \{HH, TH\}$. Then $P(H_2) = \frac{1}{2}$;

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- Let $H^* = \{HH, TT\}$. Then $P(H^*) = \frac{1}{2}$.

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- Consider the random experiment of flipping a coin two times, one after the other. Then $S = \{HH, HT, TH, TT\}$ and $P(HH) = P(HT) = P(TH) = P(TT) = \frac{1}{4}$.
- Let $H_1 = \{HT, HH\}$. Then $P(H_1) = \frac{1}{2}$;
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$$P(H_1 \cap H_2 \cap H^*) = \frac{1}{4} \neq \frac{1}{8} = P(H_1)P(H_2)P(H^*)$$

Example

- Consider the random experiment of rolling a die n times. Recall that probability space (S, P) is given as follows:
  - S is the set of sequences of length n over the alphabet $\{1, \ldots, 6\}$ (sometimes denoted $\{1, \ldots, 6\}^n$).
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- Then $E_1, \ldots, E_n$ are mutually independent.

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By dividing by $P(A)$ we get

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Let A be the event that the patient has a stiff neck and B the event that he has meningitis. Then

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A)} = \frac{1/2 \times 1/50000}{1/20} = \frac{1}{5000}$$

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- Given this observation, we want to classify the patient as either having meningitis or not having meningitis.
- We have prior knowledge about the unconditional probability of having a stiff neck and having meningitis.
- We have causal knowledge about the number of times in which meningitis causes a stiff neck.
- We can then compute the diagnostic probabilities using

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A)}$$

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But this follows from

$$\begin{aligned} P(A) &= P((A \cap B) \cup (A \cap \neg B)) \\ &= P(A \cap B) + P(A \cap \neg B) \\ &= P(A|B) \times P(B) + P(A|\neg B) \times P(\neg B) \end{aligned}$$

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We have $P(A|B) = \frac{99}{100}$, $P(\neg A|\neg B) = \frac{99}{100}$, and $P(B) = \frac{1}{200}$.

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Thus,

$$P(B|A) = \frac{P(A|B) \times P(B)}{P(A|B) \times P(B) + P(A|\neg B) \times P(\neg B)} = 0.33$$