Foundations of Computer Science Comp109

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Recap: four decomposition rules

- The product rule: the total number of possible outcomes for the ordered sequence of events A **and** B is $n_1 \times n_2$.
- The sum rule: for two disjoint events A, B there are $n_1 + n_2$ possible outcomes in "A **or** B".
- The subtraction rule: in general case, there are $n_1 + n_2 - n_3$ possible outcomes for the event "A **or** B".
- The division rule: if every group of contains d indistinguishable outcomes, then there are n/d different outcomes.

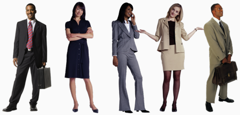
DIY problem 1 solution

How many pairs (x, y) of positive integers satisfy the equation $xy = 2010$?

Answer: each y in such pair is uniquely defined by x from formula $y = 2010/x$. Hence we just need all possible values of x, i.e. the number of factors of 2010. $2010 = 2 \times 3 \times 5 \times 67$, hence the number of factors is $2 \times 2 \times 2 \times 2 = 16$.

Example: using the division rule

How many ways there are to select **3 representatives** (order is not important) from a group of 5 students?



Answer: $(5 \times 4 \times 3)/(3 \times 2) = 10$.

Maths notation: Sums and products of a sequence

Given a sequence of numbers

$$\ldots, a_1, a_2, a_3, \ldots, a_m, a_{m+1}, \ldots a_n, \ldots$$

we use the notation

- $\sum_{i=m}^{n} a_i$ to represent $a_m + a_{m+1} + \cdots + a_n$ and
- $\prod_{i=m}^{n} a_i$ to represent $a_m \times a_{m+1} \times \cdots \times a_n$

Examples

- $\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5$
- $\prod_{i=0}^{6} i^2 = 0^2 \times 1^2 \times 2^2 \times 3^2 \times 4^2 \times 5^2 \times 6^2 = 0.$

Notice that i is just an index, so

$$\sum_{i=1}^{5} i = \sum_{j=1}^{5} j = \sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5$$

Special case $m = n$

$$\text{when } m = n, \sum_{i=m}^{n} a_m = \sum_{i=m}^{m} a_m = a_m$$

For example

$$\sum_{i=3}^{3} i = 3$$

Examples

We can express some equalities more neatly using this notation.

- In Part 1 of the module, we proved that
$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}.$$
- In Part 2 of the module, we defined the Cartesian product $A_1 \times \cdots \times A_k$ of k sets.
$$A_1 \times \cdots \times A_k = \{(a_1, a_2, \ldots, a_k) \mid a_i \in A_i\}.$$
The size of the Cartesian product is $\prod_{i=1}^{k} |A_i|$.

Note: Sums and products over sets of indices

Let $f : D \to \mathbb{R}$ be a function with some domain D.

Then for $S \subseteq D$,

- $\sum_{i \in S} f(i)$ denotes the sum of $f(i)$ over all $i \in S$ and
- $\prod_{i \in S} f(i)$ denotes the product of $f(i)$ over all $i \in S$.

The factorial function

The product $\prod_{i=1}^{n} i$ comes up so often that it has a name.
It is called *n factorial* and is written as $n!$

Examples:

- $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120.$
- $3! = 1 \times 2 \times 3 = 6.$
- $1! = 1.$
- $0! = 1.$

Permutations

A *permutation* of a set is just an ordering of its elements.

Example. The permutations of the set $\{1, 2, 3\}$ are

- 1, 2, 3
- 1, 3, 2
- 2, 1, 3
- 2, 3, 1
- 3, 1, 2
- 3, 2, 1

By the product rule the number of permutations of an n-element set is

$$n! = n \times (n-1) \times \cdots \times 1$$

because there are n choices for the first element, then $n-1$ choices for the $2nd$ element, then $n-2$ choices for the 3rd element, and so on.

Example. How many permutations are there of a 4-element set?
Answer: $4! = 24$.

k-permutations

A *selection* of k distinct elements of a set, where *order matters*, is called a *k-permutation* of the set.

The number of k-permutations of an n-element set is

$$P(n, k) = n \times (n-1) \times \cdots \times (n-(k-1)) = \frac{n!}{(n-k)!}$$

Examples

- How many ways there are to select 3 students for 3 different tasks (order matters) from a group of 5?

$$P(5,3) = 5 \times 4 \times 3 = 5!/2! = 60$$

- How many length-4 sequences of distinct digits are there?

$$P(10,4) = 10 \times 9 \times 8 \times 7 = 10!/6! = 5040$$

- How many four-letter words can be made with distinct letters from the list a, g, m, o, p, r?

$$P(6,4) = 6 \times 5 \times 4 \times 3 = 6!/(6-4)! = 360$$

Examples

- I have a jar with 20 different sweets. Three children come in, and each take one. How many different outcomes are there?

$$P(20,3) = 20 \times 19 \times 18 = 20!/17! = 6840$$

- I have a jar with 3 different sweets. Three children come in, and each take one. How many different outcomes are there?

$$P(3,3) = 3 \times 2 \times 1 = 3!/0! = 6$$

A size-k *unordered* subset is called a k-combination

The number of k-combinations of a set of size n is

$$C(n, k) = \frac{n!}{(n-k)!k!}.$$

Proof:

- The number of k-permutations of the set is $P(n, k) = \frac{n!}{(n-k)!}$
- A k-permutation is an ordering of k distinct elements of the set
- Each size-k subset has $k!$ orderings, so it corresponds to $P(k, k) = k!$ of the k-permutations
- By the division rule, $C(n, k) = \frac{P(n,k)}{P(k,k)} = \frac{n!}{(n-k)!k!}$

Examples

- The number of size-2 subsets of $\{1, 2, 3, 4, 5\}$ is

$$C(5,2) = \frac{5!}{(5-2)!2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1) \times (2 \times 1)} = \frac{5 \times 4}{2} = 10$$

  The ten subsets are $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{1, 5\}$, $\{2, 3\}$, $\{2, 4\}$, $\{2, 5\}$, $\{3, 4\}$, $\{3, 5\}$, and $\{4, 5\}$.

- The number of size-3 subsets of $\{1, 2, 3, 4, 5\}$ is

$$C(5,3) = \frac{5!}{(5-3)!3!} = 10$$

  (The subsets are the complements of the ones above)

Examples

- The number of size 1 subsets of $\{1, 2, 3, 4, 5\}$ is

$$C(5,1) = \frac{5!}{(5-1)!1!} = 5,$$

which is also the number of size 4 subsets of the set.

- The number of size-0 subsets of $\{1, 2, 3, 4, 5\}$ is

$$C(5,0)\frac{5!}{(5-0)!0!} = 1,$$

which is also the number of size-5 subsets $C(5,5)$.

Example

Twelve people, including Mary and Peter, are candidates to serve on a committee of five. How many different committees are possible? Of these how many

1. *contain both Mary and Peter?*
2. *contain neither Mary and Peter?*
3. *contain either Mary or Peter (but not both)?*

Solution. The number of possible committees is

$$C(12,5) = \frac{12!}{(12-5)!5!} = 792.$$

Example continued

(1. Both:) If Mary and Peter are already included, we have to select three more committee members from the remaining ten available people. This can be done in

$$C(10, 3) = 120$$

ways.

(2. Neither:) If Mary and Peter are excluded we have to select five committee members from the remaining 10 people. This can be done in

$$C(10, 5) = 252$$

ways.

(3. Only one:) The number of committees containing Mary and not Peter is

$$C(10, 4) = 210.$$

The same number of committees contain Peter and exclude Mary. Hence, $2 \times C(10, 4) = 420$ committees contain exactly one of Peter and Mary.

Example: previous DIY question

How many line segments are formed if we place n points on a straight line?

Answer:

$$C(n,2) = \frac{n!}{(n-2)!2!} = \frac{n(n-1)}{2}$$

Alternatively, we can add together the number of shortest line segments ($=n-1$), the number of line segments with 1 point inside ($=n-2$), etc., up to a single largest line segment with $n-2$ points inside:
$(n-1)+(n-2)+\cdots+1$, and see that the answer is the same.

Binomial coefficients

The quantity $C(n, k)$, which gives the number of k-combinations of a set of size n, is called a *binomial coefficient*.

It is also written as

$$\binom{n}{k} = C_n^k = \frac{n!}{(n-k)!k!}$$

The binomial theorem

For every natural number n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Informally:

$$\begin{aligned}
(a+b)^n &= \underbrace{(a+b) \times (a+b) \times \cdots \times (a+b)}_{n} \\
&= \sum_{k=0}^{n} \sum_{\substack{k\text{-combination} \\ \text{to choose } a}} a^k b^{n-k}.
\end{aligned}$$

Pascal's triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
\qquad
\begin{aligned}
(a+b)^0 &= 1\\
(a+b)^1 &= a+b\\
(a+b)^2 &= a^2+2ab+b^2\\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3\\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4\\
(a+b)^5 &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5
\end{aligned}
$$

Binomial coefficient identity

$$\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}.$$

Proof (try to DIY, we'll cover it at the next lecture).

Summary

- Maths notations:
  - $\sum_{i=m}^{n} a_i$ to represent $a_m + a_{m+1} + \cdots + a_n$ and
  - $\prod_{i=m}^{n} a_i$ to represent $a_m \times a_{m+1} \times \cdots \times a_n$
  - $n! = \prod_{i=1}^{n} i$
- *k*-permutations (ordered) of an *n*-element set: $P(n, k) = n \times (n-1) \times \cdots \times (n-(k-1)) = \frac{n!}{(n-k)!}$
- *k*-combinations (unordered) from an *n*-element set: $C(n, k) = \frac{n!}{(n-k)!k!}$

DIY problems: 1. What is the smallest number of people required in a group to ensure that the probability of at least two of them sharing the same birthday exceeds 50%?

2. How many distinct ways can 10 identical apples be distributed among 5 boys in such a way that each boy gets at least one apple? What if the restriction of at least one apple is removed?

Reminder: maths notations for famous sets

$$\begin{aligned}
\mathbb{N} &= \text{all natural numbers} = \{0, 1, 2, 3, \dots\}.\\
\mathbb{Z} &= \text{all integer numbers} = \{0, 1, -1, 2, -2, \dots\}.\\
\mathbb{Z}^+ &= \text{all positive integer numbers.}\\
\mathbb{Z}^- &= \text{all negative integer numbers.}\\
\mathbb{Q} &= \text{all rational numbers} = \{p/q | p, q \in \mathbb{Z}, q \neq 0\}.\\
\mathbb{Q}^+ &= \text{all positive rational numbers.}\\
\mathbb{Q}^- &= \text{all negative rational numbers.}\\
\mathbb{R} &= \text{all real numbers.}\\
\mathbb{R}^+ &= \text{all positive real numbers.}\\
\mathbb{R}^- &= \text{all negative real numbers.}\\
\emptyset &= \text{the empty set} = \text{the set that contains no elements.}
\end{aligned}$$

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