Foundations of Computer Science Comp109

University of Liverpool

Boris Konev

konev@liverpool.ac.uk

Olga Anosova

O.Anosova@liverpool.ac.uk

Recap: equivalence relations

- *Equivalence relation* is
- A *partition* of a set A is
- Equivalence relation partitions the set into well-defined non-overlapping equivalence classes.

From equivalence relation to partition

Theorem Let R be an equivalence relation on a non-empty set A. Then the equivalence classes $\{E_x \mid x \in A\}$ form a partition of A.

Proof (Optional)

The proof is in four parts:

(1) We show that the equivalence classes $E_x = \{y \mid yRx\}$, $x \in A$, are non-empty subsets of A:

(2) We show that A is the union of the equivalence classes $E_x, x \in A$:

(Optional) Proof (continued)

The purpose of the last two parts is to show that distinct equivalence classes are disjoint, satisfying (ii) in the definition of partition.

(3) We show that if xRy then $E_x = E_y$:

(4) We show that any two distinct equivalence classes are disjoint:

From partition to equivalence relation

Theorem Suppose that $A_1, \ldots, A_n$ is a partition of A. Define a relation R on A by setting: xRy if and only if there exists i such that $1 \leq i \leq n$ and $x, y \in A_i$. Then R is an equivalence relation.

Proof

- Reflexivity:
- Symmetry:
- Transitivity:

Partial order and poset

Definition A binary relation R on a set A which is reflexive, transitive and antisymmetric is called a *partial order* (or *pre-order*) and is often depicted $\preceq$.

Ordered pair $(A, \preceq)$ of a set and partial order relation on this set is called *poset*.

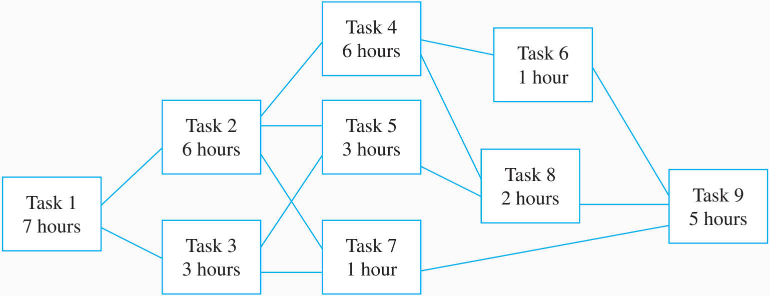
Partial orders are important in situations where we wish to characterise precedence.

Examples: Are the following relations partial orders?

- the relation $\leq$ on the the set $\mathbb{R}$ of real numbers;
- the relation $<$ on the set $\mathbb{R}$ of real numbers;
- the relation $\subseteq$ on $Pow(A)$;
- "*is a divisor of*" on the set $\mathbb{Z}^+$ of positive integers.

Example: Job scheduling

Task	Immediately Preceding Tasks
1	
2	1
3	1
4	2
5	2, 3
6	4
7	2, 3
8	4, 5
9	6, 7, 8



Predecessors in partial orders

If R is a partial order on a set A and xRy, $x \neq y$ we call x a *predecessor* of y.

If x is a predecessor of y and there is no $z \notin \{x, y\}$ for which xRz and zRy, we call x an *immediate predecessor* of y, and we say that y *covers* x.

Predecessors example

Consider the partial order "is a divisor of" on $A = \{1, 2, 3, 6, 12, 18\}$.

$R = \{$

Predecessors:

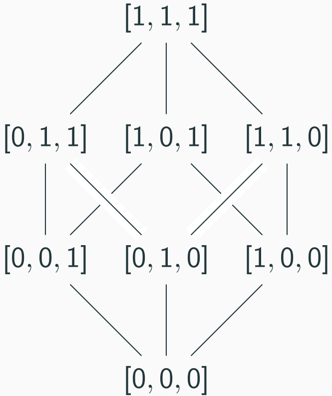
Immediate predecessors:

Hasse Diagram

The Hasse Diagram of a partial order is a digraph. The vertices of the digraph are the elements of the partial order, and the edges of the digraph are given by the "immediate predecessor" relation.

It is typical to *assume* that the arrows pointing upwards.

Example. Subsets of a set $\{a, b, c\}$, ordered by inclusion:



Example: diagram vs Hasse diagram for poset $(\{3, 4, 12, 24, 48, 72\}, /)$

Multiple diagrams, same poset

Subsets of a set $\{a, b, c, d\}$, ordered by inclusion:

Important relations: Total orders

Definition A binary relation R on a set A is a *total order* if it is a partial order such that for any $x, y \in A$, xRy or yRx.

What is the Hasse diagram of a total order?

Examples Are the following relations total orders?

- the relation $\leq$ on the set $\mathbb{R}$ of real numbers;
- the usual lexicographical ordering on the words in a dictionary;
- the relation “is a divisor of”.

n-ary relations

The Cartesian product $A_1 \times A_2 \times \cdots \times A_n$ of sets $A_1, A_2, \ldots, A_n$ is defined by

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, \ldots, a_n) \mid a_1 \in A_1, \ldots, a_n \in A_n\}.$$

Here $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$ if and only if $a_i = b_i$ for all $1 \leq i \leq n$.

An *n-ary relation* is a subset of $A_1 \times \ldots \ldots A_n$

Databases and relations

A database table $\approx$ relation

TABLE 1 **Students.**			
Student_name	***ID_number***	***Major***	***GPA***
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

Students $= \{$

Unary relations

Unary relations are just subsets of a set.

Example: The unary relation `EvenPositiveIntegers` on the set $\mathbb{Z}^+$ of positive integers is

$$\{x \in \mathbb{Z}^+ \mid x \text{ is even}\}.$$

Summary

- *Partial order* $\preceq$ is reflexive, transitive and antisymmetric.
- A *poset* is an ordered pair $(A, \preceq)$ of a set and a partial order relation on this set.
- *Total order* is a partical order that is defined between all elements.
- x is a *predecessor* of $y \iff x \preceq y$ and $x \neq y$.
- An *n-ary relation* is a subset of $A_1 \times \ldots\ldots A_n$.