Integration and Area

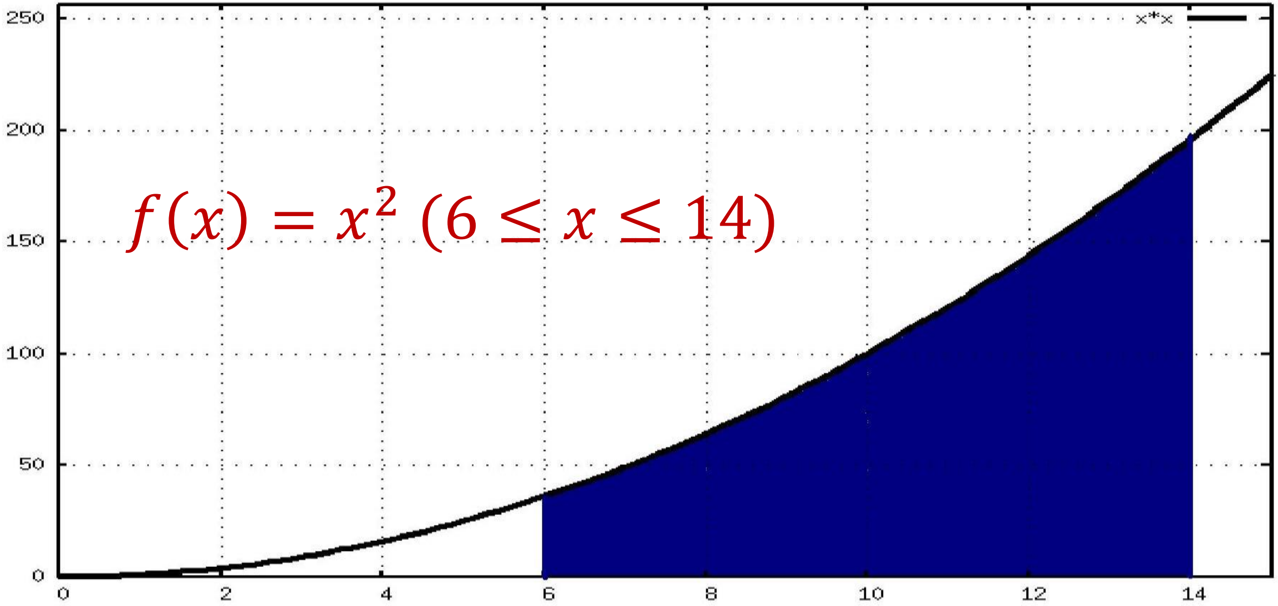
Differential Calculus – Brief Recap

- In differential calculus one of the main CS interests is in its application to *optimization problems*.
- These are considered by analysing *critical points* of functions: those for which the *first derivative* is 0.
- The "*first derivative*" defines *gradients* of a *function*.
- This is a function obtained by studying "*behaviour in the limit*": that is, as a quantity *approaches* but *never reaches* the value 0: $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

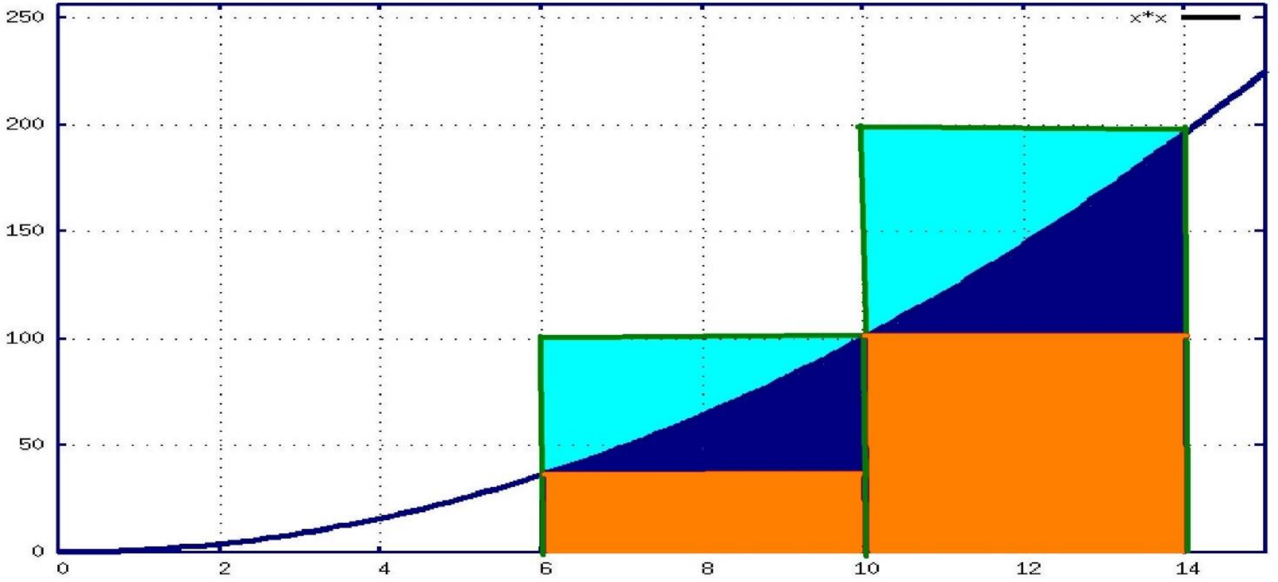
Integral Calculus – Background

- With *Integral Calculus* we return to a basic computational question: *measurement*.
- Its origins are as a means of making precise the classical "*Method of Exhaustion*" in measuring *area*.
- Integral Calculus finds a connection to Differential Calculus by the use of "*limits*" and the concept of "*anti-derivative*".
- Unlike the case of differentiation, however, a number of technical obstacles often arise.

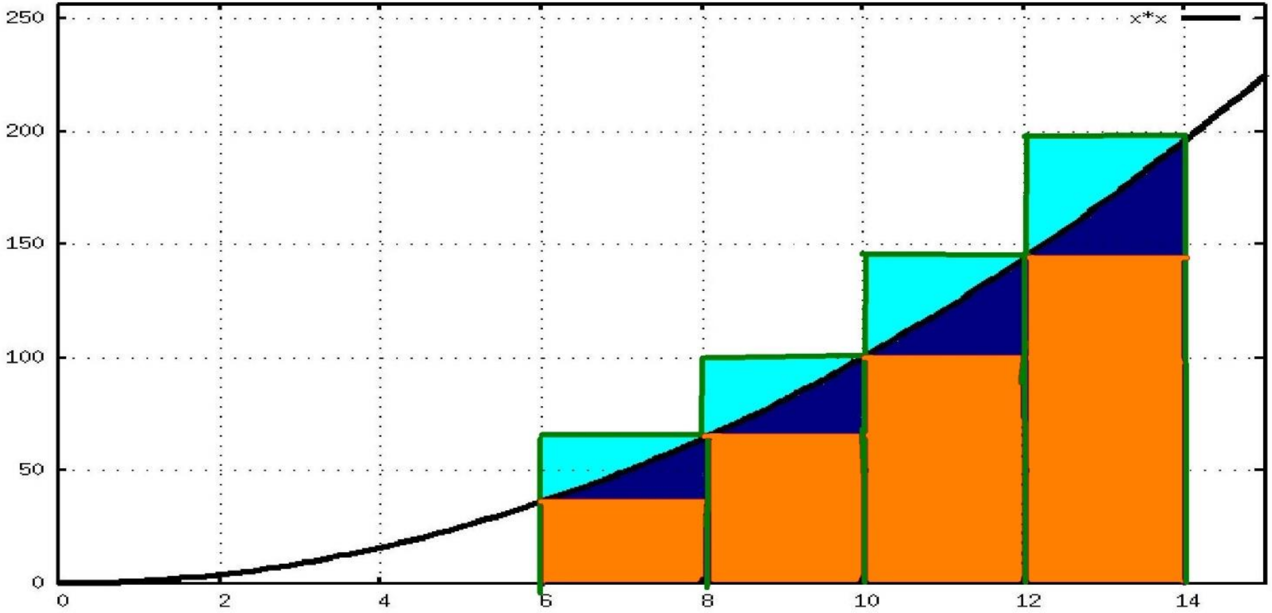
Area problem – How big is the Blue region?



Bigger than Orange; Less than Light Blue regions



A Better attempt but still: >Orange ; < Light Blue



Improving the Result

- The *Method of Exhaustion* (which dates from classical Greek mathematicians) approximates "*hard area estimation*" by using "*more and more*" *smaller* shapes of known area: *rectangles*, *triangles*, etc.
- As in the previous slides, sometimes this will *overestimate* (the Light Blue rectangles) and sometimes *underestimate* (the Orange rectangles).
- By choosing smaller rectangle bases the *upper* and *lower* estimates become *closer*.

Some statements of the “obvious”

- Suppose we are looking at an area between
$$x = a\,; x = b\,; a < b\ ?$$
- How many rectangles with a base width h can we fit in between a and b ?
$$N = \frac{b-a}{h}$$
- If the x value is, say, t what can we say about the rectangle(s) between t and $t+h$ on the function $f(x)$?
- Their area is between $hf(t)$ and $hf(t+h)$.

Consequences of the “obvious”

The area, $A(h)$, between $x = a\,; x = b\,; a < b$ for rectangles with width h is between $L(h)$ and $U(h)$:

$$L(h) = \sum_{k=0}^{N-1} hf(a + kh)\,; U(h) = \sum_{k=1}^{N} hf(a + kh)$$

- As h gets *smaller* (h *approaches* 0) these *get closer*.
- ***BUT***: we can’t let $h = 0$: $N = \frac{b-a}{h}$

Going to the "limit"

- We have just argued $A(h)$ between $x = a\,; x = b$ is
$$\lim_{h\to 0} \sum_{k=1}^{\frac{b-a}{h}} hf(a + kh)$$
- This does not seem especially helpful.
- ***BUT***
- If we *knew* $f(x)$ was *the first derivative* of $F(x)$?
$$F'(x) = f(x)$$

How does knowing $F'(x) = f(x)$ help us?

- What does $F'(x) = f(x)$ actually *mean*?

$$f(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

- So $A(h) =$

$$\lim_{h\to 0} \sum_{k=1}^{N} \lim_{h\to 0} \frac{F(a+(k+1)h) - F(a+kh)}{h}$$

- ***BUT*** we don't need *two* limits: just the *outer limit*.

Putting it together

- We have the area we want as

$$\lim_{h \to 0} \sum_{k=1}^{\frac{b-a}{h}} \frac{F(a+(k+1)h) - F(a+kh)}{h}$$

- with $f(x) = F'(x)$
- We can simplify the clutter (page 178) to get

$$F(b) - F(a)$$

- The area spanned by $f(x)$ between a and b $(a < b)$ is

$$F(b) - F(a) \text{ where } f(x) = F'(x)$$

More terminology, more notation

- The function $F(x)$ for which $f(x) = F'(x)$ is called the *anti-derivative* of $f(x)$.
- If $f(x)$ has *any* anti-derivative then it has *infinitely many*.
- If $F(x)$ is an anti-derivative of $f(x)$ then so too is $F(x) + C$ for *any constant value C*.
- The "*summation*" form is replaced by the "*integral sign*"

$$F(x) = \int f(x)dx\,;\ \ F(x) = \int_a^b f(x)dx$$

- These are "*indefinite*" and (with values) "*definite*" integrals.

An Example

- The case we started with was the function $f(x) = \ x^2$ between the values $x = 6\,; x = 14$.
- The function $\frac{x^3}{3}$ is an anti-derivative of x^2 .
- So the area we were trying to calculate is

$$\left[\frac{x^3}{3}\right]_6^{14} = \left[\frac{14^3}{3} - \frac{6^3}{3}\right] = \frac{2744-216}{3} = \frac{2528}{3}$$

- As with derivatives there are a number of *standard rules* for integrating basic functions, see textbook page 181.
- There are also many "*simple*" functions with *no usable methods*, eg $\sqrt{\sin x}$ and a number of functions in Statistics.

Summary

- **Differential Calculus**:
- An important tool in *optimization methods* despite the complexities that arise with multivariable instances.
- First and second order derivatives provide useful analytic methods for *root finding* algorithms both with *polynomial* and *general functions*.
- **Integral Calculus**:
- Deals with a basic computational issue: *area measurement*.
- Techniques can be extended to *volumes* and *line length* (pp. 182-4 of textbook).
- Has a significant role in a *specialized area* of *algorithm analysis*.