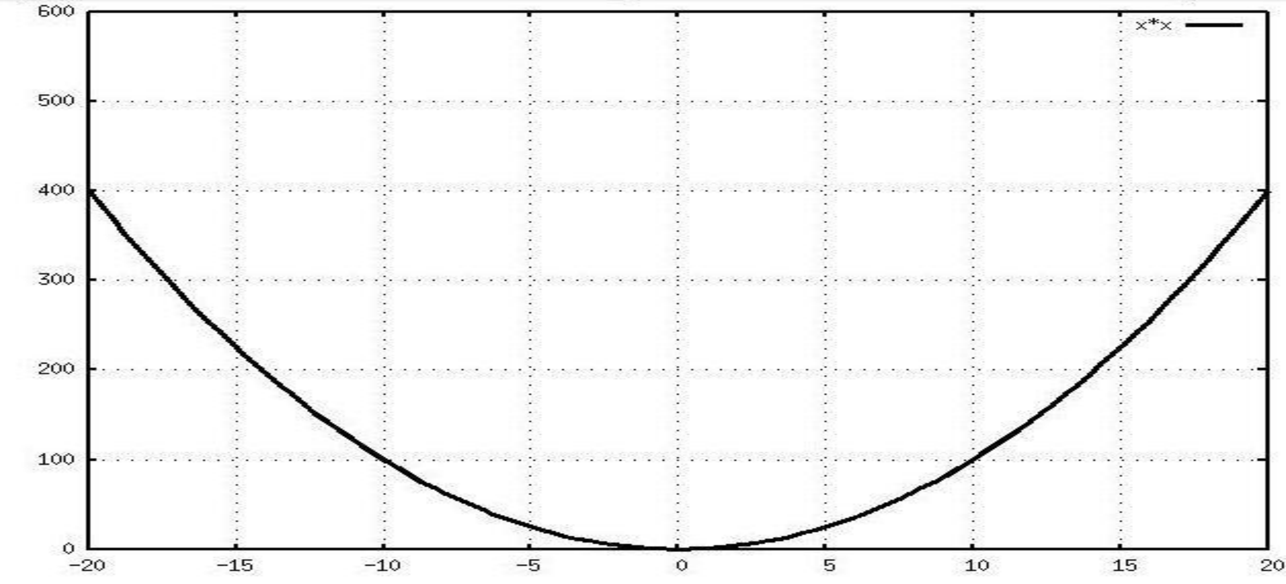
Critical Points
and
The Second Derivative Test

Basic Optimization Methods

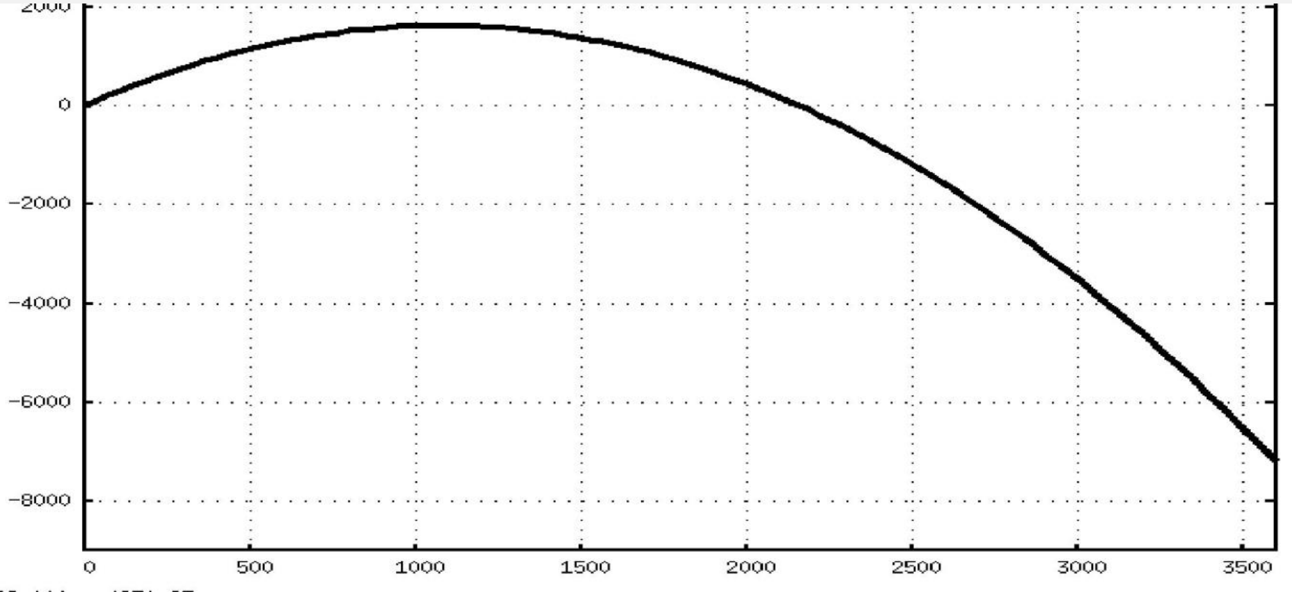
Critical Points

- In the introductory problem - determining the maximum height attained by an object whose height after t metres is $f(t)$ - it was argued that the maximizing point was the value of t at which "*the gradient of the line touching the point* $(t, f(t))$ *was* 0".
- In other words, that value of t for which $f'(t) = 0$.
- In general, a point, $(x, f(x))$ at which $f'(x) = 0$ is called a *critical point* of f.
- Finding and analysing critical points is an important tool in solving *optimization problems*.

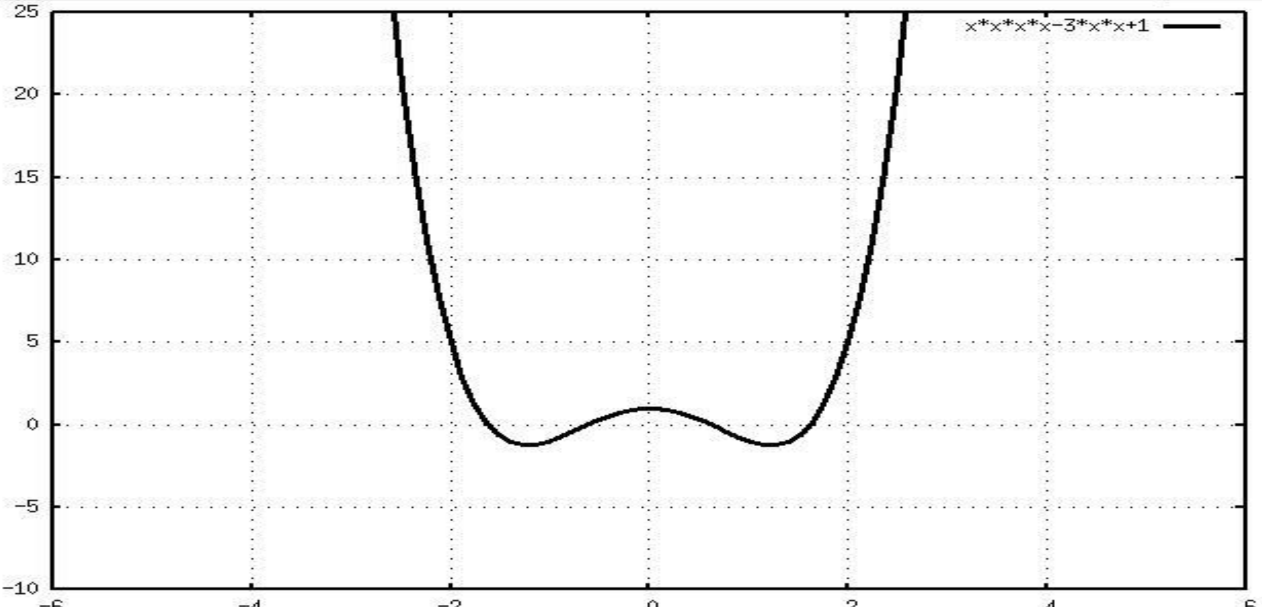
Critical Points - Minimum



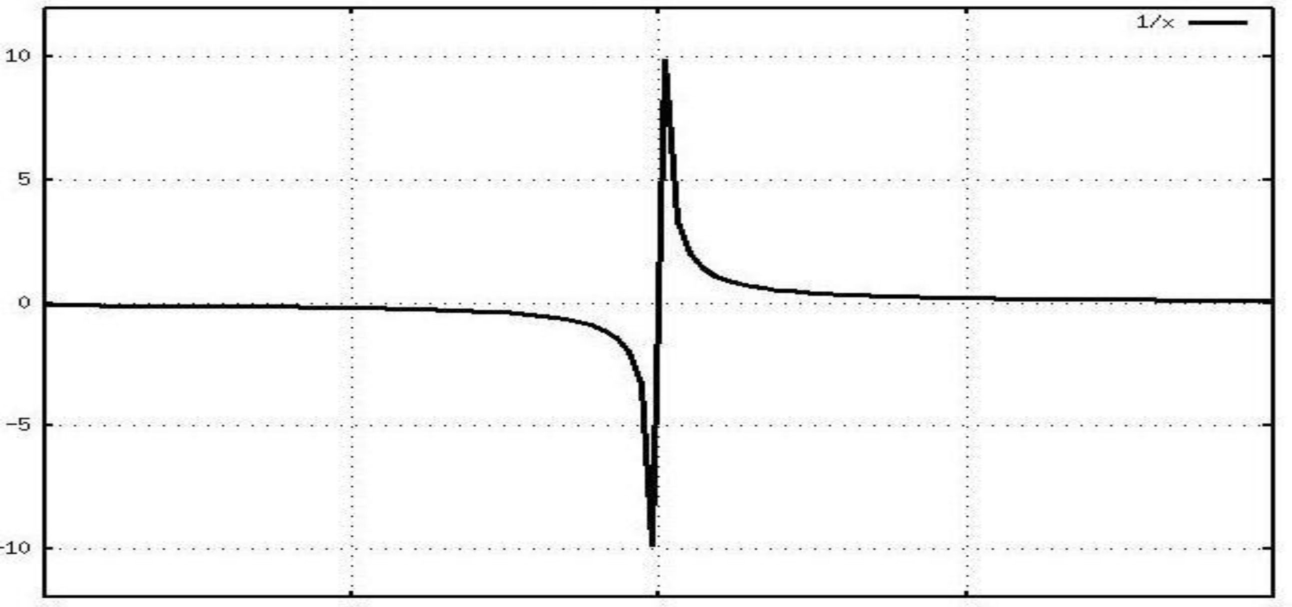
Critical Points – Maximum



Critical Points – Minimum and Maximum



Critical Points – ????



Critical Points – Analysis I

- From these cases we see that a function may have
    1. A single minimal critical point
    2. A single maximal critical point
    3. A mix of minima and maxima.
    4. *No* critical points at all.
- How do we tell these apart?
- We *can* identify those functions falling into the fourth class.

  Q: How?

  A: $f'(x) = 0$ has *no solutions*.
- And those in the third class: $f'(x) = 0$ has *at least two solutions*.

Critical Points – Analysis II

- How do we tell if a critical point is a “local” minimum or maximum point? (ie the smallest or largest value in a range).
- We use

The Second Derivative Test

- $f(x)$ is a *function*.
- $f'(x)$ (if “well-defined”) is *also* a *function*.
- Therefore there is (in principle) an object $f''(x)$ which is (also) a *function*. [and, similarly, objects: $f'''(x)$, $f''''(x)$ …].
- Other notation: $\frac{d^2y}{dx^2}$, … , $\frac{d^ky}{dx^k}$

The Second Derivative Test

- Given the function $f(x)$:

1. Construct $f'(x)$ its *first* derivative.
2. Find all solutions, α, for which $f'(\alpha) = 0$.
3. Construct the *second* derivative: $f''(x)$ from $f'(x)$.
4. For each solution, α, found in (2), compute $f''(\alpha)$.

   If $f''(\alpha) < 0$ then α is a (local) *maximum*.

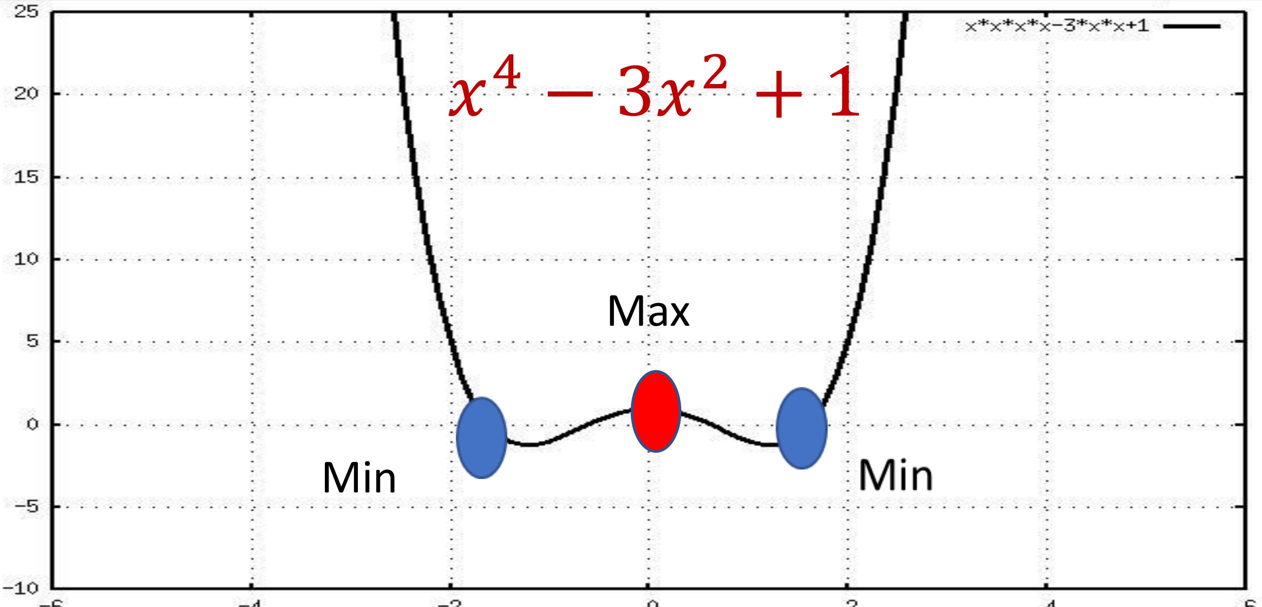
   If $f''(\alpha) > 0$ then α is a (local) *minimum*.

   If $f''(\alpha) = 0$ then *no conclusion* can be made.

Examples

- $f(x) = x^4 - 3x^2 + 1$.
- $f'(x) = 4x^3 - 6x$.
- There are *three* critical points: $x = 0\,;\, x = \pm\sqrt{\frac{3}{2}}$.
- $f''(x) = 12x^2 - 6$.
- $f''(0) < 0$: (local) *maximum*.
- $f''\left(\sqrt{\frac{3}{2}}\right) = 12 > 0\,;\, f''\left(-\sqrt{\frac{3}{2}}\right) = 12 > 0$: *minima*.

The Example on Preceding Slide



Some more examples

- $f(x) = 25(\log x)^3 - 15(\log x)^4$
- $f'(x) = \frac{75(\log x)^2}{x} - \frac{60(\log x)^3}{x} = \frac{75(\log x)^2 - 60(\log x)^3}{x}$
- $f'(x) = 0$ when $15(\log x)^2(5 - 4\log x) = 0$
- That is when, $x = 1$ or $x = \exp\left(\frac{5}{4}\right)$
- $f''(1) = 0$ (*Why*?) ; $f''\left(\exp\left(\frac{5}{4}\right)\right) \neq 0$ (is actually a *maximum*).
- pages 144 – 152 of the module text give several worked examples of using *first derivatives* to solve *optimization problems* and the *second derivative test*.

Summary

- The concept of "*critical points*" and their analysis is an important part of the subject of *Optimization Methods*.
- This is one of the specialist studies dealt with in more advanced CS: for example the module **COMP331**.
- The techniques covered in this lecture have, however, a drawback in practice: only a *single parameter* is involved.
- To adapt to "more realistic" settings approaches to "*multivariable functions*" are needed.
- The extension of "basic differential calculus" to functions of *several variables* is the topic of the next part.