# Foundations of Computer Science Comp109

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## Recap: equivalence relations

- *Equivalence relation* is a binary relation that is reflexive, transitive, and symmetric.
- A *partition* of a set $A$ is a collection of *non-empty* subsets $A_1, \dots, A_n$ of $A$:
  - $A = A_1 \cup A_2 \cup \cdots \cup A_n$;
  - $A_i \cap A_j = \emptyset$ for $i \neq j$.
- Equivalence relation partitions the set into well-defined non-overlapping equivalence classes.

# From equivalence relation to partition

**Theorem** Let $R$ be an equivalence relation on a non-empty set $A$. Then the equivalence classes $\{E_x \mid x \in A\}$ form a partition of $A$.

**Proof** (Optional)

The proof is in four parts:

(1) We show that the equivalence classes $E_x = \{y \mid yRx\}$, $x \in A$, are non-empty subsets of $A$: by definition, each $E_x$ is a subset of $A$. Since $R$ is reflexive, $xRx$. Therefore $x \in E_x$ and so $E_x$ is non-empty.

(2) We show that $A$ is the union of the equivalence classes $E_x, x \in A$: We know that $E_x \subseteq A$, for all $E_x$, $x \in A$. Therefore the union of the equivalence classes is a subset of $A$. Conversely, suppose $x \in A$. Then $x \in E_x$. So, $A$ is a subset of the union of the equivalence classes.

# (Optional) Proof (continued)

The purpose of the last two parts is to show that distinct equivalence classes are disjoint, satisfying (ii) in the definition of partition.

(3) We show that if $xRy$ then $E_x = E_y$: Suppose that $xRy$ and let $z \in E_x$. Then, $zRx$ and $xRy$. Since $R$ is a transitive relation, $zRy$. Therefore, $z \in E_y$. We have shown that $E_x \subseteq E_y$. An analogous argument shows that $E_y \subseteq E_x$. So, $E_x = E_y$.

(4) We show that any two distinct equivalence classes are disjoint: To this end we show that if two equivalence classes are not disjoint then they are identical. Suppose $E_x \cap E_y \neq \emptyset$. Take a $z \in E_x \cap E_y$. Then, $zRx$ and $zRy$. Since $R$ is symmetric, $xRz$ and $zRy$. But then, by transitivity of $R$, $xRy$. Therefore, by (3), $E_x = E_y$.

# From partition to equivalence relation

**Theorem** Suppose that $A_1, \ldots, A_n$ is a partition of $A$. Define a relation $R$ on $A$ by setting: $xRy$ if and only if there exists $i$ such that $1 \leq i \leq n$ and $x, y \in A_i$. Then $R$ is an equivalence relation.

**Proof**

# From partition to equivalence relation

**Theorem** Suppose that $A_1, \ldots, A_n$ is a partition of $A$. Define a relation $R$ on $A$ by setting: $xRy$ if and only if there exists $i$ such that $1 \leq i \leq n$ and $x, y \in A_i$. Then $R$ is an equivalence relation.

**Proof**

- Reflexivity: if $x \in A$, then $x \in A_i$ for some $i$. Therefore $xRx$.
- Symmetry: if $xRy$, then there exists $A_i$ such that $x, y \in A_i$. Therefore $yRx$.
- Transitivity: if $xRy$ and $yRz$, then there exists $A_i$ and $A_j$ such that $x, y \in A_i$ and $y, z \in A_j$. $y \in A_i \cap A_j$ implies $i = j$. Therefore $x, z \in A_i$ which implies $xRz$.

# Partial order and poset

**Definition** A binary relation $R$ on a set $A$ which is reflexive, transitive and antisymmetric is called a *partial order* (or *pre-order*) and is often depicted $\preceq$.

Ordered pair $(A, \preceq)$ of a set and partial order relation on this set is called *poset*.

Partial orders are important in situations where we wish to characterise precedence.

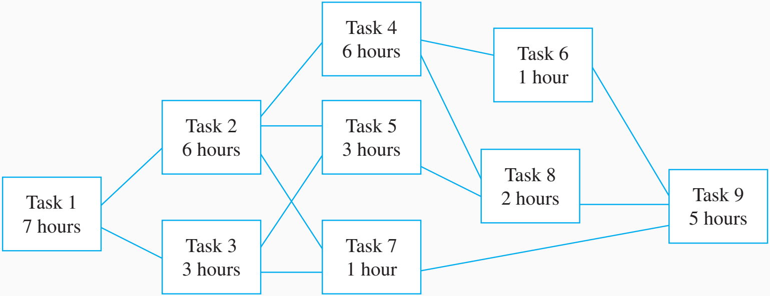
**Examples**: Are the following relations partial orders?

- the relation $\leq$ on the the set $\mathbb{R}$ of real numbers;
- the relation $<$ on the set $\mathbb{R}$ of real numbers;
- the relation $\subseteq$ on $Pow(A)$;
- "*is a divisor of*" on the set $\mathbb{Z}^+$ of positive integers.

Answer: yes, no, yes, yes.

# Example: Job scheduling

| Task | Immediately Preceding Tasks |
|---|---|
| 1 | |
| 2 | 1 |
| 3 | 1 |
| 4 | 2 |
| 5 | 2, 3 |
| 6 | 4 |
| 7 | 2, 3 |
| 8 | 4, 5 |
| 9 | 6, 7, 8 |



# Predecessors in partial orders

If $R$ is a partial order on a set $A$ and $xRy$, $x \neq y$ we call $x$ a *predecessor* of $y$.

If $x$ is a predecessor of $y$ and there is no $z \notin \{x, y\}$ for which $xRz$ and $zRy$, we call $x$ an *immediate predecessor* of $y$, and we say that $y$ *covers* $x$.

# Predecessors example

Consider the partial order “is a divisor of” on $A = \{1, 2, 3, 6, 12, 18\}$.

$R = \{(1, 1),$
$(1, 2), (2, 2), (1, 3), (3, 3),$
$(1, 6), (2, 6), (3, 6), (6, 6),$
$(1, 12), (2, 12), (3, 12), (6, 12), (12, 12),$
$(1, 18), (2, 18), (3, 18), (6, 18), (18, 18)\}$

*Predecessors:* number 1 has no predecessors and 1 is the only predecessor of 2 or of 3. 6 has predecessors $1, 2, 3$. 12 and 18 have predecessors $1, 2, 3, 6$

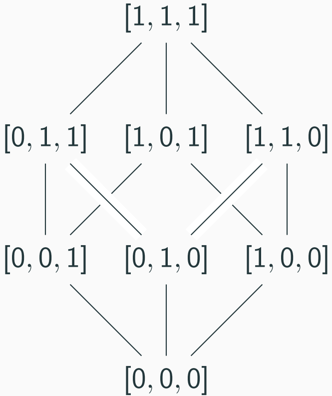
*Immediate predecessors:*
1 has no immediate predecessors, 1 is the only immediate predecessor of 2 or 3, 6 has immediate predecessors 2 and 3, 12 and 18 have immediate predecessor 6.

# Hasse Diagram

The Hasse Diagram of a partial order is a digraph. The vertices of the digraph are the elements of the partial order, and the edges of the digraph are given by the “immediate predecessor” relation.

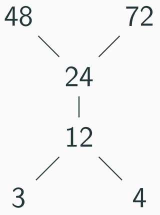
It is typical to *assume* that the arrows pointing upwards.

*Example.* Subsets of a set $\{a, b, c\}$, ordered by inclusion:



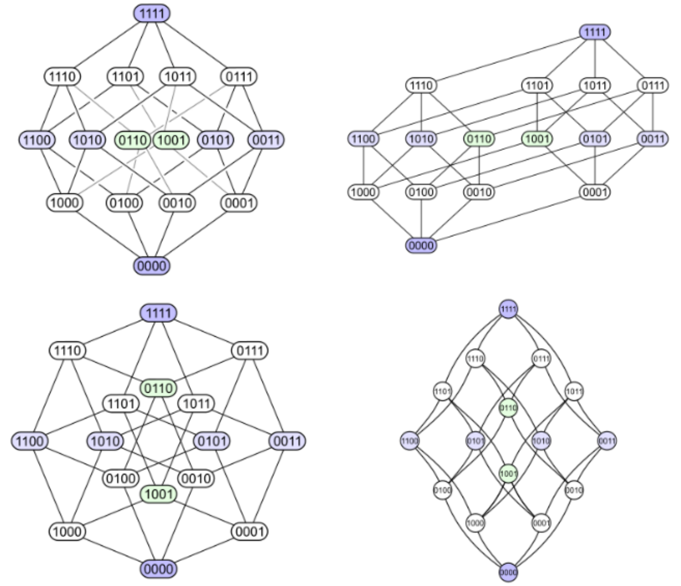
# Example: diagram vs Hasse diagram for poset $(\{3, 4, 12, 24, 48, 72\}, /)$

All the elements of the poset for the divisibility:
$A = \{(3 \preceq 12), (3 \preceq 24), (3 \preceq 48), (3 \preceq 72), (4 \preceq 12), (4 \preceq 24), (4 \preceq 48), (4 \preceq 72), (12 \preceq 24), (12 \preceq 48), (12 \preceq 72), (24 \preceq 48), (24 \preceq 72)\}$.
Hence Hasse diagram:



# Multiple diagrams, same poset

Subsets of a set $\{a, b, c, d\}$, ordered by inclusion:



# Important relations: Total orders

**Definition** A binary relation $R$ on a set $A$ is a *total order* if it is a partial order such that for any $x, y \in A$, $xRy$ or $yRx$.

What is the Hasse diagram of a total order?

**Examples** Are the following relations total orders?

- the relation $\leq$ on the set $\mathbb{R}$ of real numbers;
- the usual lexicographical ordering on the words in a dictionary;
- the relation "is a divisor of".

Answer: yes, yes, no.

# $n$-ary relations

The Cartesian product $A_1 \times A_2 \times \cdots \times A_n$ of sets $A_1, A_2, \ldots, A_n$ is defined by

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, \ldots, a_n) \mid a_1 \in A_1, \ldots, a_n \in A_n\}.$$

Here $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$ if and only if $a_i = b_i$ for all $1 \leq i \leq n$.

An *$n$-ary relation* is a subset of $A_1 \times \ldots \ldots A_n$.

# Databases and relations

A database table $\approx$ relation

**TABLE 1 Students.**

| *Student_name* | *ID_number* | *Major* | *GPA* |
|---|---|---|---|
| Ackermann | 231455 | Computer Science | 3.88 |
| Adams | 888323 | Physics | 3.45 |
| Chou | 102147 | Computer Science | 3.49 |
| Goodfriend | 453876 | Mathematics | 3.45 |
| Rao | 678543 | Mathematics | 3.90 |
| Stevens | 786576 | Psychology | 2.99 |

Students =
$\{(\text{Ackermann}, 231455, \text{Computer Science}, 3.88), (\text{Adams}, 888323, \text{Physics}, 3.45), \ldots\}$

# Unary relations

*Unary relations* are just subsets of a set.

**Example:** The unary relation `EvenPositiveIntegers` on the set $\mathbb{Z}^+$ of positive integers is

$$\{x \in \mathbb{Z}^+ \mid x \text{ is even}\}.$$

# Summary

**Attendance code: 348902**

- *Partial order* $\preceq$ is reflexive, transitive and antisymmetric.
- A *poset* is an ordered pair $(A, \preceq)$ of a set and a partial order relation on this set.
- *Total order* is a partical order that is defined between all elements.
- $x$ is a *predecessor* of $y \iff x \preceq y$ and $x \neq y$.
- An *n-ary relation* is a subset of $A_1 \times \ldots\ldots A_n$.

**DIY Question on the next topic:** Every second mathematician is a also a computer scientist, and every fifth computer scientist is also a mathematician. Based on this information, are there more mathematicians or computer scientists?