Foundations of Computer Science Comp109

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Part 4. Function

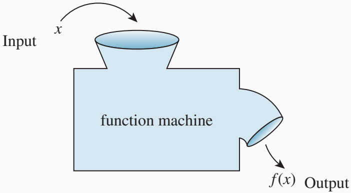
Comp109 Foundations of Computer Science

Reading

- Discrete Mathematics with Applications S. Epp, Chapter 7.
- **Discrete Mathematics and Its Applications** K. Rosen, Section 2.3.

Contents

Functions



Examples:

- $y = x^2$
- $y = \sin(x)$
- first letter of your name

Functions/methods on programming

Java

```java
public int f(int x) {
    return x+5;
}
```

C/C++

```c
int f(int x) {
    return x+5;
}
```

Python

```python
def f(int x):
    return x+5
```

Definition

A **function** from a set A to a set B is an assignment of exactly one element of B to each element of A.

We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element of a.

If f is a function from A to B we write $f\colon A \to B$.

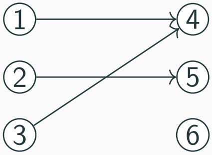


Figure 1: A function $f\colon \{1, 2, 3\} \to \{4, 5, 6\}$

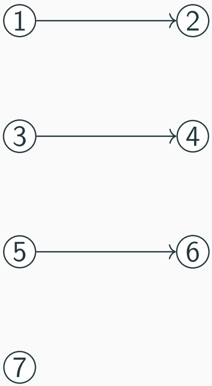


Figure 2: No function

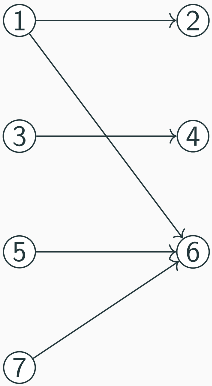


Figure 3: No function

Domain, codomain, and range

Suppose $f\colon A \to B$.

- A is called the *domain* of f. B is called the *codomain* of f.
- The *range* $f(A)$ of f is
$$f(A) = \{f(x) \mid x \in A\}.$$

Codomain vs range

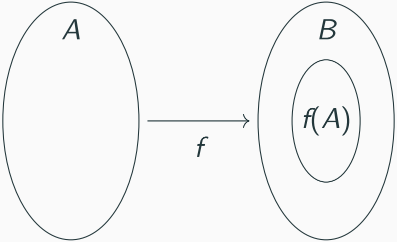
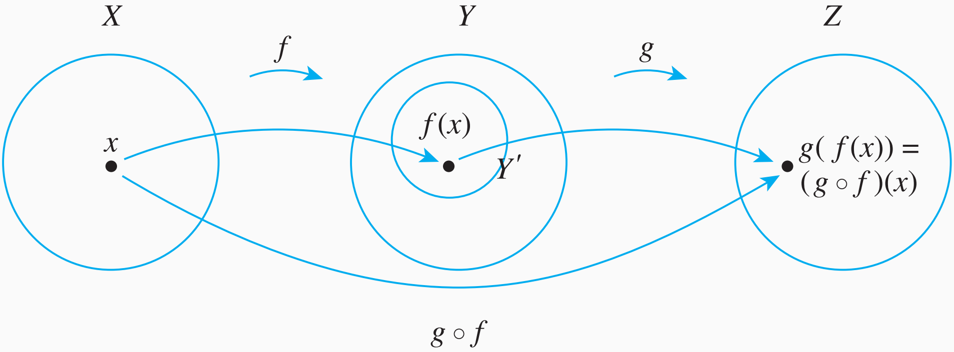


Figure 4: the range of f

Composition of functions

If $f : X \to Y$ and $g : Y \to Z$ are functions, then their **composition** $g \circ f$ is a function from X to Z given by

$$(g \circ f)(x) = g(f(x)).$$



Example

Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ and $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = 4x + 3$.

- $g \circ f(x) =$

  $= g(f(x)) = 4(x^2) + 4$

- $f \circ g(x) =$

  $= f(g(x)) = (4x + 3)^2$

- $f \circ f(x) =$

  $= f(f(x)) = (x^2)^2 = x^4$

- $g \circ g(x) =$

  $g(g(x)) = 4(4x + 3) + 3 = 16x + 12 + 3 = 16x + 15.$

Injective (one-to-one) functions

Definition Let $f : A \to B$ be a function. We call f an *injective* (or *one-to-one*) function if

$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2 \text{ for all } a_1, a_2 \in A.$$

This is logically equivalent to $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$ and so injective functions never repeat values. In other words, different inputs give different outputs.

Examples

$f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2$ is not injective.

$h : \mathbb{Z} \to \mathbb{Z}$ given by $h(x) = 2x$ is injective.

Prove that $f(x)$ is not injective and $h(x) = 2x$ is injective

- $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2$ is not injective.

  $f(-1) = (-1)^2 = 1 = 1^2 = f(1)$

- $h : \mathbb{Z} \to \mathbb{Z}$ given by $h(x) = 2x$ is injective.

  Suppose for a proof by contradiction that there exist a_1 and a_2 such that $f(a_1) = f(a_2)$ but $a_1 \neq a_2$. As $f(a_1) = f(a_2)$, $2a_1 = 2a_2$ but then $a_1 = a_2$ a contradiction.

More examples

- $first\_letter : People \to Char$

  Not injective: $first\_letter(Ann) = A = first\_letter(Alice)$

- $ID : People \to \mathbb{N}$

  injective

Surjective (or onto) functions

Definition $f : A \to B$ is *surjective* (or onto) if the range of f coincides with the codomain of f. This means that for every $b \in B$ there exists $a \in A$ with $b = f(a)$.

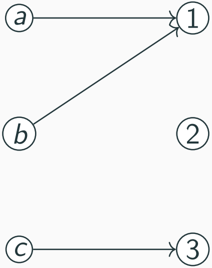
Examples

$f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2$ is not surjective.

$h : \mathbb{Z} \to \mathbb{Z}$ given by $h(x) = 2x$ is not surjective.

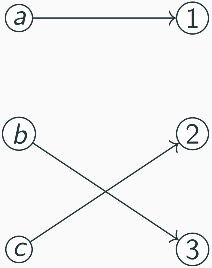
$h' : \mathbb{Q} \to \mathbb{Q}$ given by $h'(x) = 2x$ is surjective.

Classify $f\colon \{a, b, c\} \to \{1, 2, 3\}$ given by



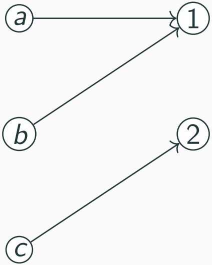
- f is a function
- f is not injective: $f(a) = f(b)$
- f is not surjective: for no x, $f(x) = 2$.

Classify $g : \{a, b, c\} \to \{1, 2, 3\}$ given by



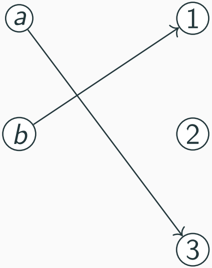
- g is a function
- g is injective
- g is surjective

Classify $h : \{a, b, c\} \to \{1, 2\}$ given by



- h is a function
- h is not injective: $h(a) = h(b)$
- h is surjective

Classify $h' : \{a, b, c\} \to \{1, 2, 3\}$ given by



- h' is a function
- h' is injective
- h' is not surjective: for no x, $h'(x) = 2$.

Bijections

We call f *bijective* if f is both injective and surjective.

Examples

$f : \mathbb{Q} \to \mathbb{Q}$ given by $f(x) = 2x$ is bijective.

Inverse functions

If f is a bijection from a set X to a set Y, then there is a function f^{-1} from Y to X that "undoes" the action of f; that is, it sends each element of Y back to the element of X that it came from. This function is called the inverse function for f.

Then $f(a) = b$ if, and only if, $f^{-1}(b) = a$.

Example

$k : \mathbb{R} \to \mathbb{R}$ given by $k(x) = 4x + 3$ is invertible and

$$k^{-1}(y) = \frac{1}{4}(y - 3).$$

Example

Let $A = \{x \mid x \in \mathbb{R}, x \neq 1\}$ and $f\colon A \to A$ be given by

$$f(x) = \frac{x}{x-1}.$$

Show that f is bijective and determine the inverse function.

Cardinality of finite sets and functions

Recall: *The cardinality of a* finite *set S is the number of elements in S*

A bijection $f\colon S \to \{1, \dots, n\}$.

For finite sets A and B

- $|A| \geq |B|$ iff there is a surjective function from A to B.
- $|A| \leq |B|$ iff there is a injective function from A to B.
- $|A| = |B|$ iff there is a bijection from A to B.

The pigeonhole principle

Let $f: A \to B$ be a function where A and B are finite sets.

The *pigeonhole principle* states that if $|A| > |B|$ then at least one value of f occurs more than once.

In other words, we have $f(a) = f(b)$ for some distinct elements a, b of A.

Pigeons and pigeonholes

If (N+1) pigeons occupy N holes, then some hole must have at least 2 pigeons.

 Image by McKay from en.wikipedia

Example

Problem. There are 15 people on a bus. Show that at least two of them have a birthday in the same month of the year.

Let A be the set of people on the bus and B be the set consisting of the 12 months of the year. Consider the function $f: A \to B$ which assigns each person to the month in which their birthday falls. Then $|A| > |B|$. By the pigeonhole principle, there are persons $a, b \in A$ where $a \neq b$ and $f(a) = f(b)$.

Example

Problem. How many different surnames must appear in a telephone directory to guarantee that at least two of the surnames begin with the same letter of the alphabet and end with the same letter of the alphabet?

Let A be the set of surnames and B be the set of pairs of letters drawn from the standard alphabet of 26 letters. Consider the function $f : A \to B$ which assigns each surname to the pair of letters corresponding to the first and last letters of the surname. For example, $f(boris) = (b, s)$. B contains $26 \times 26 = 676$ elements. The pigeonhole principle guarantees that at least two surnames begin and end with the same letter provided $|A| > 676$.

Example

Problem. Five numbers are selected from the numbers $1, 2, 3, 4, 5, 6, 7$ and 8. Show that there will always be two of the numbers that sum to 9.

Let A be the set of five numbers selected. Let B be the set $\{(1,8),(2,7),(3,6),(4,5)\}$. Consider the function $f : A \to B$ which assigns each of the five numbers to the element of B to which it belongs.
For example, $f(2) = (2,7)$.
By the pigeonhole principle, at least two of the numbers in A go to the same pair. So they sum to 9.

Extended pigeonhole principle

Consider a function $f\colon A \to B$ where A and B are finite sets and $|A| > k|B|$ for some natural number k. Then, there is a value of f which occurs at least $k+1$ times.

Example

Problem. How many different surnames must appear in a telephone directory to guarantee that at least five of the surnames begin with the same letter of the alphabet and end with the same letter of the alphabet?

Once again, let A be the set of surnames and B be the set of pairs of letters . Consider the function $f : A \to B$ which assigns each surname to the pair of letters corresponding to the first and last letters of the surname. For example, $f(boris) = (b, s)$.
B contains $26 \times 26 = 676$ elements.
To ensure that at least five of the surnames begin with the same letter and end with the same letter we require that $|A| > 4|B| = 2704$. Therefore, the telephone directory must contain at least 2705 surnames.

Example

Problem. Show that in any group of six people there are either three who all know each other or three complete strangers.

Let x be one the six people, A be the set of remaining five people and $B = \{0, 1\}$. Consider the function $f : A \to B$ given by $f(a) = 0$ if a does not know x and $f(a) = 1$ if a knows x. $|A| > 2|B|$. Therefore there are three people who know x or there are three who do not.

In the former case, suppose a, b and c all know x. If any two of a, b and c know each other, then those two and x all know each other. Otherwise, a, b, and c are mutual strangers.

Otherwise, suppose a, b and c all don't know x. If any two of them don't know each other, then those two and x all don't know each other. Otherwise, a, b and c all know each other.

Bijections and cardinality

Recall that the cardinality of a finite set is the number of elements in the set.

Sets A and B have the same cardinality iff there is a bijection from A to B.

Recall: the powerset and bit vectors

Let $S = \{1, 2, \ldots, n\}$ and let B^n be the set of bit strings of length n. The function

$$f\colon Pow(S) \to B^n$$

which assigns each subset A of S to its characteristic vector is a bijection.

We used this to compute the cardinality of the powerset.

Infinite sets

Sets A and B have the same cardinality iff there is a bijection from A to B.

- $\mathbb{Z}$ and even integers

consider $f(n) = 2n$

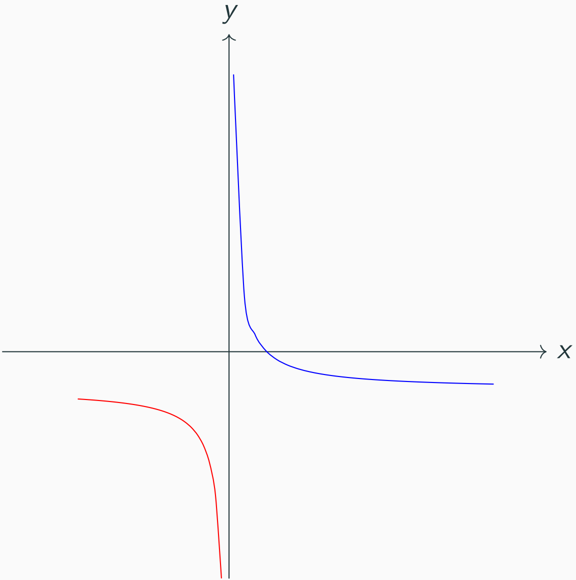
Hilbert's infinite hotel

- $\mathbb{N}$ and $\mathbb{Z}$

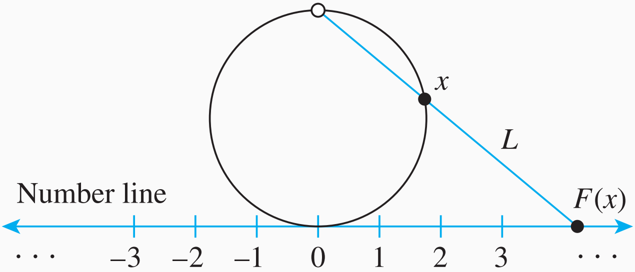
map 0 to 0, odd natural numbers to positive integers and even natural numbers except zero to negative integers.

Real numbers: $\{x \in \mathbb{R} \mid 0 < x < 1\}$ and $\mathbb{R}^+$

- consider $g(x) = \frac{1}{x} - 1$



$\{x \in \mathbb{R} \mid 0 < x < 1\}$ and $\mathbb{R}$



Countable sets

A set that is either finite or has the same cardinality as $\mathbb{N}$ is called **countable**.

- $\mathbb{Z}$

Countable Sets: $\mathbb{Q}$

$$
\begin{array}{ccccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \frac{2}{6} & \cdots \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \frac{3}{6} & \cdots \\
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & \frac{4}{6} & \cdots \\
\frac{5}{1} & \frac{5}{2} & \frac{5}{3} & \frac{5}{4} & \frac{5}{5} & \frac{5}{6} & \cdots \\
\frac{6}{1} & \frac{6}{2} & \frac{6}{3} & \frac{6}{4} & \frac{6}{5} & \frac{6}{6} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots
\end{array}
$$

Uncountable sets

- A set that is not countable is called **uncountable**.
  - $S = \{x \in \mathbb{R} \mid 0 < x < 1\}$ is uncountable

Cantor's diagonal argument

Suppose for a proof by contradiction that there exists a bijection $f : \mathbb{N}^+ \to S$. Consider decimal representations of $f(n)$, for $n \in \mathbb{N}^+$:

$$
\begin{array}{l}
f(1) = 0.a_{11}\ a_{12}\ a_{13} \ldots\ a_{1n} \ldots \\
f(2) = 0.a_{21}\ a_{22}\ a_{23} \ldots\ a_{2n} \ldots \\
f(3) = 0.a_{31}\ a_{32}\ a_{33} \ldots\ a_{3n} \ldots \\
\vdots \qquad\qquad\qquad\qquad \ldots \\
f(n) = 0.a_{n1}\ a_{n2}\ a_{n3} \ldots\ a_{nn} \ldots \\
\vdots
\end{array}
$$

We show that there exists $d \in S$ such that for no $i \in \mathbb{N}^+$ we have $f(i) = d$.

Let $d = 0.d_1\ d_2\ d_3 \ldots d_n \ldots$ where $\qquad d_i = \begin{cases} 2, & \text{if } a_{ii} = 1 \\ 1, & \text{if } a_{ii} \neq 1 \end{cases}$

Then for every $i \in \mathbb{N}^+$ d is different at position i from $f(i)$. So, for no $i \in \mathbb{N}^+$ we have $f(i) = d$, so f is not surjective. A contradiction