Vector Spaces

Vector Spaces

- The notion of *vector space* provides a means of "*collecting together*" sets of vectors with "*similar characteristics*".
- The set of numbers, H, used to form a set of n-vectors is important in defining what is meant by a vector space.
- It must satisfy the algebraic property of being a "*field*". This is met by Reals and Z_m ($\{0,1,2,\ldots,m-1\}$, mod m arithmetic).
- The behaviour of vector spaces is fundamental to many basic operations supporting 2 and 3D graphics.

Vector Spaces - Basics

- We have: U – a set of n-vectors using H.
- eg – U is the set of all 2-vectors from ***R***.
- The set, U, defines a *vector space* if it has two "*closure properties*":

  C1. Adding any two vectors in U produces a vector in U.

  C2. Multiplying any vector in U by a *constant* in H produces a vector in U.

Vector Spaces – Formal Definition

- Given: U – a set of n-vectors using H.
- The set, U, defines a *vector space* if:

$$\forall v \in U, \forall w \in U : v + w \in U \text{ (C1)}$$

$$\forall v \in U, \forall \alpha \in H : \alpha v \in U \text{ (C2)}$$

- The underlying set, H, is ***very important***.

Some examples

$$EVEN = \{< x, y >: x + y = 2p \; and \; p \in Z\}$$

- If H=Z_k then $EVEN$ is a vector space.
- If H= R then $EVEN$ is *not* a vector space: consider (C2) using, for instance, $< 1,1 >, \alpha = ½$.

$$ODD = \{< x, y >: x + y = 2p - 1 \; and \; p \in Z\}$$

- For this, ODD, is a *not* a vector space eg H=Z_{16}:

$$< 1,4 > \in ODD, < 4,1 > \in ODD \; but < 5,5 > \notin ODD$$

Vector Spaces – Attributes

- Suppose U is a vector space of n-vectors from, R, say.
- Important notions are a *basis set* and its *dimension*.
- The idea of basis allows us to "*reduce*" U, even if it is *infinite*, to *at most* n *distinct* vectors.
- How do we do this?
- By expressing members of U as *linear combinations* of other vectors in U.

Linear Combinations

- Take a *set*, B, of m vectors from U:

$$B=\{t_1, t_2, \dots, t_m\}$$

- Take a *collection*, **a**, of m values *at least one* of which is *not* 0 and *all non-zero values* are from H.

$$\underline{\mathbf{a}} = \langle \alpha_1, \alpha_2, \dots, \alpha_m \rangle$$

- **w**, is a *linear combination* of B using **a** if **w** is

$$\sum_{k=1}^{m} \alpha_k t_k$$

Linear Dependence & Independence

- Now, suppose that U is a vector space and B is a subset of k n-vectors from U for which *any* w in U can be obtained as a *linear combination* of B?
- That is,

$$\forall w \in U \; \exists \langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle : w = \sum_{i=1}^{k} \alpha_i b_i$$

- Then, in principle, only B is needed to capture U.

Basis and Dimension

- A subset, B of U, for which any vector in (the vector space) U can be formed as a linear combination of B is called a *basis* of U.
- The *number* of vectors in the *smallest* basis of U is called the *dimension* of U denoted dim(U).
- A vector space, U, of n-vectors: dimension *at most* n.

Some more examples

- $EVEN = \{< x, y >: x + y = 2p \; and \; p \in H\}$
- We saw this was a vector space for $H = Z_4$.
- It contains:

$$\left\{\begin{matrix} < 0,0 >, < 0,2 >, < 1,1 >, < 1,3 >, \\ < 2,0 >, < 2,2 >, < 3,1 >, < 3,3 > \end{matrix}\right\}$$

- dim(EVEN)=2 ; $B = \{< 0,2 >, < 3,1 >\}$
- eg <1,1>=1*<0,2> + 3*<3,1> etc

Another example

- $R^4 = \{< u, v, w, z >: u, v, w, z \in R\}; H = R$
- dim(R^4)=4 : B=$\{e_1, e_2, e_3, e_4\}$
- The dimension, however, can be *smaller* than the number of components. “Trivial” example
$$One = \{< x, x, x, x >: x \in R\}$$
- dim(One)=1 : eg B={<1,1,1,1>}

Summary

- The notion of vector space is important (in CS applications) for cases such as R^2 and R^3 .
- These feature in 2D and 3D graphics.
- The concept of "*linear transformation between vector spaces*" is crucial in streamlining how complicated effects may be implemented.
- This concept and its use is what we look at next.