Estimation the half-logistic distribution based on multiply Type-II hybrid censoring

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1. Abstract

◈ We derive some estimators of the scale parameter of half-logistic distribution based on multiply Type-II hybrid censoring.

◈ We obtain the maximum likelihood estimator (MLE) of the scale parameter, and approximate maximum likelihood estimator (AMLE). We also obtain the Bayesian estimator by proposed prior distribution. The confidence intervals (CIs) are also obtained by the proposed estimators.

◈ We compare the proposed estimators in the sense of the bias and the mean squared error (MSE) through Monte Carlo simulation.

◈ The Bayesian estimator is generally more efficient than MLE and AMLE. But the CI based on the Bayesian estimator is wider than the corresponding of CI based on MLE and AMLE.

2. Introduction

◈ The probability density function (pdf) $f(x)$ and cumulative distribution function $F(x)$ of half-logistic distribution are

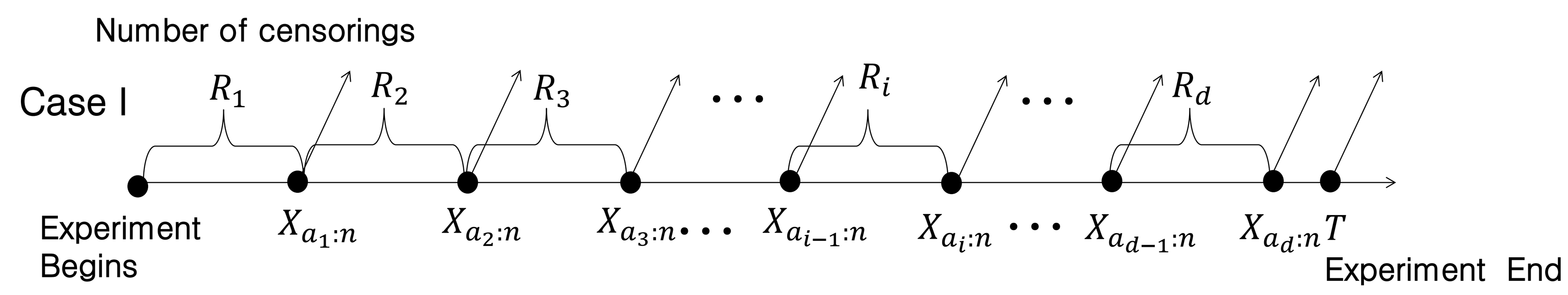
$$f(x;\theta)=\frac{2e^{-\frac{x}{\theta}}}{\theta\left(1+e^{\frac{x}{\theta}}\right)^2},\qquad F(x;\theta)=\frac{1-e^{\frac{x}{\theta}}}{1+e^{\frac{x}{\theta}}}.\ x>0,\theta>0 \tag{2.1}$$

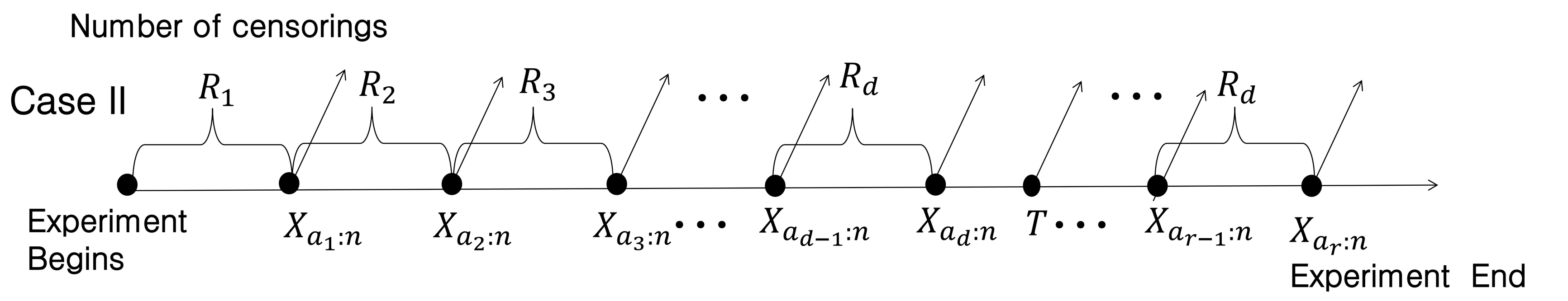
◈ Multiply Type-II hybrid censoring

$X_{a_i:n}$ be a_i-th observed failure time, T be the predetermined experiment end time. Let, r be the predetermined observation number and R_i be the number of censoring between $X_{a_i:n}$ and $X_{a_{i+1}:n}$. The multiply Type-II hybrid censoring have two cases.

- case I : $\{X_{a_1:n}<X_{a_2:n}<\cdots<X_{a_d:n}<T,\quad if\quad X_{a_d:n}<T$
- case II : $\{X_{a_1:n}<X_{a_2:n}<\cdots<X_{a_d:n}<T<X_{a_{d+1}:n}<\cdots<X_{a_r:n},$ $\quad if\quad d<r\quad and\quad X_{a_d:n}<T<X_{a_r:n}$

where d is the observation number until T.





The likelihood function of the censored sample is

$$L\propto\prod_{i=1}^{m}f(X_{a_i:n})\prod_{i=1}^{m-1}[F(X_{a_{i+1}:n})-F(X_{a_i:n})]^{R_{i+1}}F(X_{a_1:n})^{R_1}[1-F(X_{a_m:n})]^{n-a_m} \tag{2.2}$$

where m is d or r as the observation number until the experiment end time, $R_i=a_i-a_{i-1}-1$, $a_0=0$

3. Maximum likelihood estimator

The random variable $Z=\frac{X}{\theta}$ has a standard half-logistic distribution. Then we rewrite (2.2) as

$$L\propto\theta^{-m}\prod_{i=1}^{m}f(Z_{a_i:n})\prod_{i=1}^{m-1}[F(Z_{a_{i+1}:n})-F(Z_{a_i:n})]^{R_{i+1}}F(Z_{a_1:n})^{R_1}[1-F(Z_{a_m:n})]^{n-a_m} \tag{3.1}$$

On differentiating the function of (3.1) with respect to θ and equation to zero, we obtain the estimating equation as

$$\frac{\partial\ln L}{\partial\theta}=-\frac{1}{2\theta}\left[2m-2\sum_{i=1}^{m}F(z_{a_i:n})z_{a_i:n}+2R_1\frac{f(z_{a_1:n})}{F(z_{a_1:n})}-(n-a_m)\left(1+F(z_{a_m:n})f(z_{a_m:n})\right)z_{a_m:n}\right.$$
$$\left.+2\sum_{i=1}^{m-1}R_{i+1}\frac{f(z_{a_{i+1}:n})z_{a_{i+1}:n}-f(z_{a_i:n})z_{a_i:n}}{F(z_{a_{i+1}:n})-F(z_{a_i:n})}\right]=0 \tag{3.2}$$

with properties $f'(z_{a_i:n})=-F(z_{a_i:n})f(z_{a_i:n})$, $f(z_{a_i:n})=\frac{[1-F(z_{a_i:n})][1+F(z_{a_i:n})]}{2}$

We can find the MLE $\tilde{\theta}$ of θ. However the equation (3.2) is a nonlinear, we can't explicitly solved for θ. So we obtain $\tilde{\theta}$ using the Newton-Raphson method.

To construct the CI, we can use the asymptotic normality of the MLE with $VAR(\tilde{\theta})$ estimated from the inverse of observed Fisher information. The asymptotic normality of the MLE denotes $\tilde{\theta}\approx N(\theta,I^{-1}(\theta))$. In practice, we usually estimate $I^{-1}(\theta)$ by $I^{-1}(\tilde{\theta})$. Then, $\tilde{\theta}\approx N(\theta,I^{-1}(\tilde{\theta}))$. Therefore, $100(1-\alpha)\%$ CI for θ is

$$\tilde{\theta}\pm z_{\alpha/2}\sqrt{I^{-1}(\tilde{\theta})}\quad\text{where}\quad Var(\tilde{\theta})=I^{-1}(\tilde{\theta})\simeq-\left(\frac{\partial^2\ln L}{\partial\theta^2}\right)^{-1}\Bigg|_{\theta=\tilde{\theta}}$$

where $z_{\alpha/2}$ denotes the percentile of the standard normal distribution with right-tail probability $\alpha/2$.

4. Approximate maximum likelihood estimator

Since the equation (3.2) can't solve explicitly with closed form, We can approximate it by using the Taylor series expansion. Let, $\xi_{a_i:n}=F^{-1}(p_{a_i:n})=-\ln\frac{1-p_{a_i:n}}{1+p_{a_i:n}}$, where $p_{a_i:n}=\frac{a_i}{n+1}$, $q_{a_i:n}=1-p_{a_i:n}$, $i=1,2,\ldots m$. We expand the functions $F(z_{a_i:n})z_{a_i:n}$, $\frac{f(z_{a_1:n})}{F(z_{a_1:n})}z_{a_1:n}$, $\frac{f(z_{a_{i+1}:n})z_{a_{i+1}:n}}{F(z_{a_{i+1}:n})-F(z_{a_i:n})}$, $\frac{f(z_{a_i:n})z_{a_i:n}}{F(z_{a_{i+1}:n})-F(z_{a_i:n})}$ in Taylor series around the points ξ. We approximate the functions by;

$$F(z_{a_i:n})z_{a_i:n}\approx\alpha_{1i}+\beta_{1i}z_{a_i:n},\qquad\frac{f(z_{a_1:n})}{F(z_{a_1:n})}z_{a_1:n}\approx\alpha_1+\beta_1 z_{a_1:n} \tag{4.1}$$

$$\frac{f(z_{a_{i+1}:n})z_{a_{i+1}:n}}{F(z_{a_{i+1}:n})-F(z_{a_i:n})}\approx\gamma_{1i}+\delta_{1i}z_{a_i:n}+\iota_{1i}z_{a_{i+1}:n},\quad\frac{f(z_{a_i:n})z_{a_i:n}}{F(z_{a_{i+1}:n})-F(z_{a_i:n})}\approx\kappa_{1i}+\eta_{1i}z_{a_i:n}+\tau_{1i}z_{a_{i+1}:n} \tag{4.2}$$

By substituting the equations (4.1) and (4.2) into the equation (3.2), We can derive AMLE as follows;

$$\hat{\theta}=-\frac{B}{A}$$

where

$$A=2m-2\sum_{i=1}^{m}\alpha_{1i}+2R_1\alpha_1-(n-a_m)\alpha_{1m}+2\sum_{i=1}^{m-1}R_{i+1}(\gamma_{1i}-\kappa_{1i})$$

$$B=-2\sum_{i=1}^{m}\beta_{1i}x_{a_i:n}+2R_1\beta_1x_{a_1:n}-(n-a_m)(x_{a_m:n}+\beta_{1m}x_{a_m:n})$$
$$+2\sum_{i=1}^{m-1}R_{i+1}(\delta_{1i}x_{a_i:n}+\iota_{1i}x_{a_{i+1}:n}-\eta_{1i}x_{a_i:n}-\tau_{1i}x_{a_{i+1}:n})$$

5. Bayes estimation

We consider the natural conjugate family of prior distribution for θ used as

$$\pi(\theta)\propto\theta^{-\alpha-1}e^{-\frac{\beta}{\theta}},\quad\theta>0 \tag{5.1}$$

where $\alpha>0,\beta>0$. For $\beta=0$, $\pi(\theta)$ reduces to a general class of improper priors and the results of this paper will hold for this non-informative prior as $\alpha=2$ and $\beta=1$.

Using the squared error loss function, we obtain the Bayesian estimator by posterior distribution mean of θ. Since the Bayesian estimator can't explicitly be obtained, we may approximate using Tierney and Kadane (1986).

$$\hat{\theta}_B=\left(\frac{|\psi^*|}{|\psi|}\right)^{\frac{1}{2}}\left(\frac{\hat{\theta}^{\alpha+m+1}}{\hat{\theta}^{*\alpha+m}}\right)e^{\left(\frac{1}{\hat{\theta}}-\frac{1}{\hat{\theta}^*}\right)\left(\beta+\sum_{i=1}^{m}x_{a_i:n}+(n-a_m)x_{a_m:n}\right)}$$
$$\times\prod_{i=1}^{m-1}\left(\frac{1+e^{-\frac{x_{a_i:n}}{\hat{\theta}}}}{1+e^{-\frac{x_{a_i:n}}{\hat{\theta}^*}}}\right)^2\left[\frac{\left(e^{-\frac{x_{a_i:n}}{\hat{\theta}^*}}-e^{-\frac{x_{a_{i+1}:n}}{\hat{\theta}^*}}\right)\left(1+e^{-\frac{x_{a_{i+1}:n}}{\hat{\theta}}}\right)\left(1+e^{-\frac{x_{a_i:n}}{\hat{\theta}}}\right)}{\left(e^{-\frac{x_{a_i:n}}{\hat{\theta}}}-e^{-\frac{x_{a_{i+1}:n}}{\hat{\theta}}}\right)\left(1+e^{-\frac{x_{a_{i+1}:n}}{\hat{\theta}^*}}\right)\left(1+e^{-\frac{x_{a_i:n}}{\hat{\theta}^*}}\right)}\right]^{R_{i+1}} \tag{5.2}$$

6. Illustration example and simulation results

◈ Illustration example

We use the following real data suggested by Lawless (1982).

12.3 21.8 24.4 28.6 43.2 46.9 70.7 75.3 95.5 98.1 138.6 151.9

We assume that this data is the half-logistic distribution based on multiply Type-II hybrid censoring scheme. When we apply for $T=100, r=8, a_i=(1\sim3,6\sim12)$. the results are as follows.

$\tilde{\theta}$	$\hat{\theta}$	$\hat{\theta}_B$
47.65407	47.50923	44.80874

◈ Simulation results

To compare the performance of the proposed estimators for the scale parameter θ, we obtain the MSE and biases through Monte Carlo simulation based on 10,000 times. Also, we obtain the average length of CI (CL) and corresponding coverage probability.

n	T	R_i	a_r	$\tilde{\theta}$ MSE(bias)	$\tilde{\theta}$ CL(CP)	$\hat{\theta}$ MSE(bias)	$\hat{\theta}$ CL(CP)	$\hat{\theta}_B$ MSE(bias)	$\hat{\theta}_B$ CL(CP)
20	1	$R_4=2,R_6=1$, $R_i=0,i\neq4,6$	10	0.0536 (−0.0088)	0.8936 (0.9141)	0.0532 (−0.0139)	0.8952 (0.9131)	0.0488 (−0.0039)	0.9283 (0.9356)
			12	0.0473 (−0.0074)	0.8285 (0.9149)	0.0469 (−0.0128)	0.8283 (0.9128)	0.0435 (−0.0039)	0.8546 (0.9345)
			15	0.0390 (−0.0044)	0.7599 (0.9243)	0.0387 (−0.0094)	0.7569 (0.9215)	0.0363 (−0.0028)	0.7773 (0.9394)
	2	$R_4=2,R_6=1$, $R_i=0,i\neq4,6$	10	0.0475 (−0.0270)	0.8149 (0.8979)	0.0473 (−0.0323)	0.8159 (0.8956)	0.0438 (−0.0234)	0.8404 (0.9178)
			12	0.0454 (−0.0210)	0.7988 (0.9005)	0.0452 (−0.0263)	0.7990 (0.8979)	0.0421 (−0.0177)	0.8222 (0.9172)
			15	0.0374 (−0.0060)	0.7581 (0.9228)	0.0371 (−0.0111)	0.7552 (0.9213)	0.0348 (−0.0044)	0.7755 (0.9363)
40	1	$R_5=1,R_8=3$, $R_i=0,i\neq5,8$	29	0.0212 (−0.0008)	0.5627 (0.9385)	0.0211 (−0.0037)	0.5620 (0.9356)	0.0204 (0.0011)	0.5709 (0.9462)
			32	0.0196 (−0.0021)	0.5392 (0.9343)	0.0195 (−0.0048)	0.5380 (0.9328)	0.0189 (−0.0008)	0.5460 (0.9423)
			35	0.0181 (−0.0030)	0.5209 (0.9346)	0.0180 (−0.0051)	0.5193 (0.9325)	0.0175 (−0.0021)	0.5266 (0.9412)
	2	$R_5=1,R_8=3$, $R_i=0,i\neq5,8$	29	0.0204 (−0.0035)	0.5597 (0.9302)	0.0204 (−0.0064)	0.5591 (0.9282)	0.0197 (−0.0016)	0.5678 (0.9392)
			32	0.0185 (−0.0034)	0.5384 (0.9418)	0.0184 (−0.0061)	0.5376 (0.9403)	0.0178 (−0.0021)	0.5452 (0.9496)
			35	0.0170 (−0.0024)	0.5213 (0.9415)	0.0170 (−0.0045)	0.5196 (0.9398)	0.0165 (−0.0015)	0.5270 (0.9473)

7.Conclusion

◈ We propose three estimators of the scale parameter of the half-logistic distribution based on multiply Type-II hybrid censoring.

◈ To compare the performance of the proposed estimators for the scale parameter θ, we obtain the MSE and biases through Monte Carlo simulation based on 10,000 times.

◈ The Bayesian estimator is generally more efficient than MLE in sense of MSE. But the CL based on the Bayesian estimator is wider than the corresponding CL based on the MLE and AMLE.

◈ Although we don't report it here, we obtain the other AMLEs using different Taylor series expansion types. Theses AMLEs also usually have good performance.