Computational Finance

Plotting Basics

- Plotting in (scientific) Python is mostly done via the `matplotlib` library (user guide), which is inspired by the plotting facilities of Matlab®.
- Its main plotting facilities reside in its `pyplot` module. It is usually imported as

In [2]:
```
import matplotlib.pyplot as plt
%matplotlib inline
```

- The second line is an ipython magic. It makes plots appear inline in the notebook.
- The `seaborn` library (user guide) provides higher-level statistical visualizations:

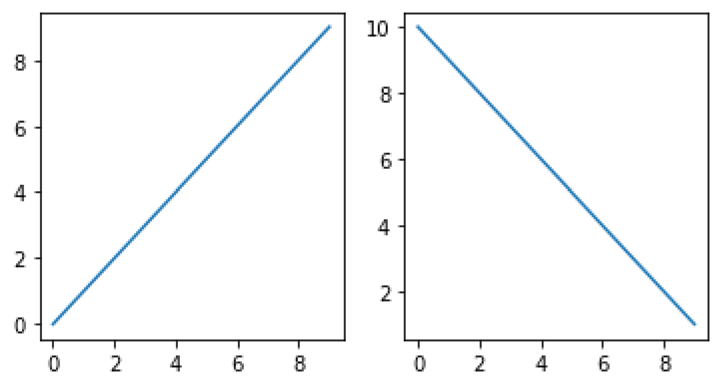
In [3]:
```
import seaborn as sns
```

- Finally, `statsmodels` is useful for QQ plots (see below):

In [4]:
```
import statsmodels.api as sm
```

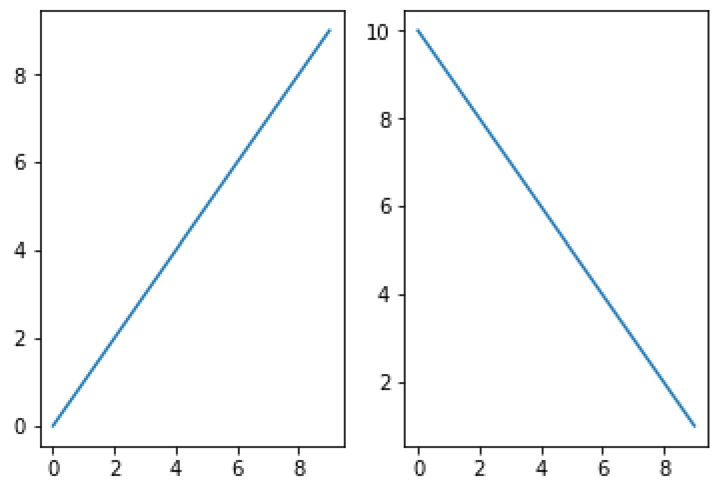
- I will only give a brief introduction to matplotlib here. However, the code for all graphs shown below is included in the notebook (though sometimes hidden in slide mode), and should be studied.
- The fundamental object in matplotlib is a `figure`, inside of which reside `subplots` (or `axes`).
- To create a new figure, add an axis, and plot to it:

In [5]:
```
#With the inline backend, these need to be in the same cell.
fig = plt.figure(figsize=(6,3))  #Create a new empty figure object. Size is optional.
ax1 = fig.add_subplot(121)  #Layout: (1x2) axes. Add one in row 1, column 1, and make it current (what plt.* c
ax2 = fig.add_subplot(122)  #Add an axes in row 1, column 2, and make it current.
ax1.plot(range(10))
ax2.plot(range(10, 0, -1));
```



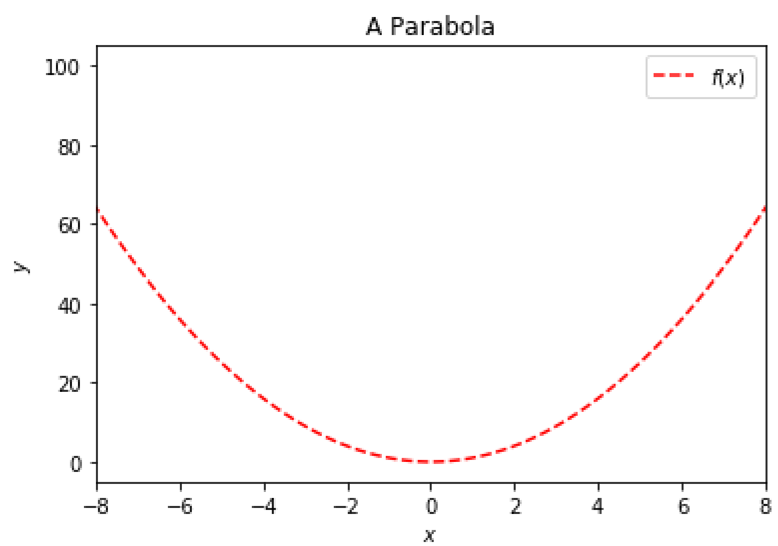
- By default, matplotlib plots into the current axis, creating one (and a figure) if needed. Using the convenience method `subplot`, this allows us to achieve the same without explicit reference to figures and axes:

In [6]:
```
plt.subplot(121)
plt.plot(range(10))
plt.subplot(122)
plt.plot(range(10, 0, -1));
```



- To plot two vectors x and y against each other:

In [7]:
```
import numpy as np
x = np.linspace(-10, 10, 100)
y = x**2
plt.plot(x,y,'r--')  #Dashed red line; see table on p. 114.
plt.xlabel('$x$')  #LaTeX equations can be included by enclosing in $$.
plt.ylabel('$y$')
plt.title('A Parabola')
plt.legend(['$f(x)$'])  #Expects a list of strings.
plt.xlim(xmin=-8, xmax=8);  #Axis limits.
#plt.savefig('filename.svg')  #Save the plot to disk.
```



Risk Measures

Introduction

- The Basel Accords mandate that financial institutions report the risk associated with their positions, so that regulators may check the adequacy of the economic capital as a buffer against market risk.
- Reporting is in the form of a *risk measure*, which condenses the risk of a position into a single number.
- Currently, the mandated measure is *Value at Risk* (VaR), but there are debates of replacing it with an alternative (*Expected Shortfall*).
- Banks are allowed to use their own, internal models for the computation of VaR, but the adequacy of these models should be *backtested*.

Value at Risk

- Consider a portfolio with value V_t and daily (simple) return R_t.
- Define the one-day loss on the portfolio as

$$\$Loss_{t+1} = -\left[V_{t+1} - V_t\right].$$

- I will distinguish between the dollar Value at Risk (an amount) and the return Value at Risk (a percentage). When unqualified, I mean the latter.
- The one-day $100p\%$ dollar Value at Risk $\$VaR^p_{t+1}$ is the loss on the portfolio that we are $100\left(1-p\right)\%$ confident will not be exceeded. The Basel committee prescribes $p = 0.01$.

- The *return Value at risk* VaR^p_{t+1} expresses $\$VaR^p_{t+1}$ as a percentage of the portfolio value:

$$VaR^p_{t+1} = \frac{\$VaR^p_{t+1}}{V_t}.$$

- Hence

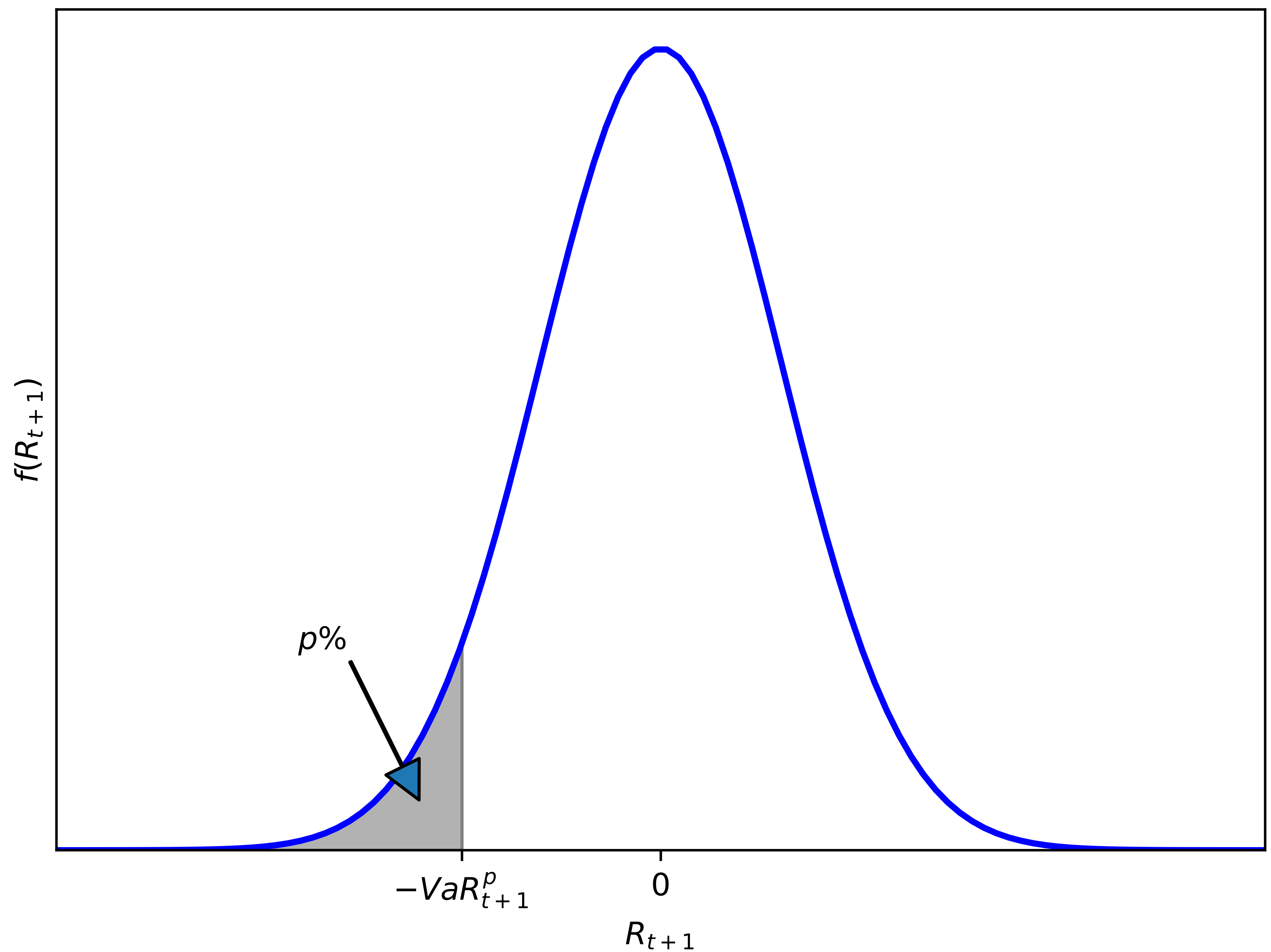
$$\Pr(R_{t+1} < -VaR^p_{t+1}) = p,$$

  because

$$R_{t+1} = -\frac{\$Loss_{t+1}}{V_t}.$$

  This holds approximately for log returns, too.

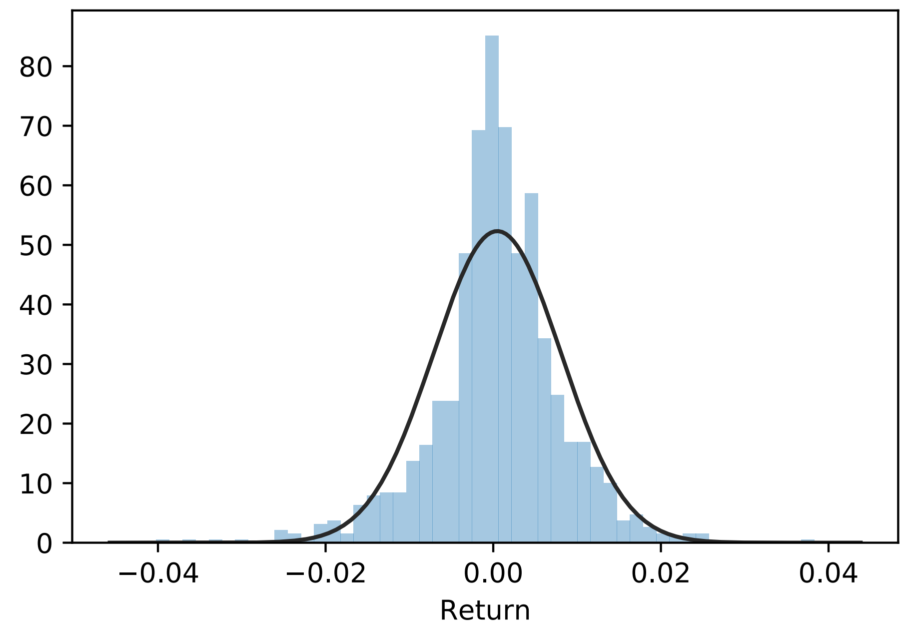
- Thus VaR^p_{t+1} is minus the $100p$th *percentile* (or minus the pth *quantile*) of the return distribution.

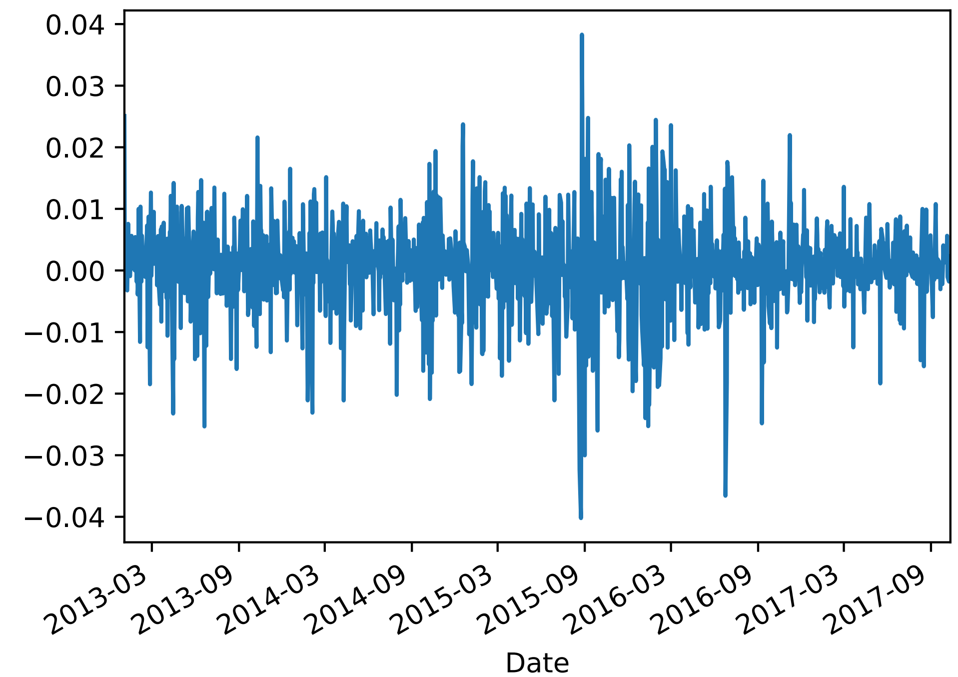

$p\%$
$-VaR^p_{t+1}$
0
R_{t+1}
$f(R_{t+1})$

Asset Returns: Stylized Facts

- Stylized facts about asset returns include
    - Lack of autocorrelation;
    - Leverage effects;
    - Heavy tails of return distribution;
    - Volatility clustering.
- These need to be taken into account when creating VaR forecasts.

In [9]:
```
import pandas as pd
import pandas_datareader.data as web
import scipy.stats as stats  #The book likes to import it as `scs`.
p = web.DataReader("^GSPC", 'yahoo', start='1/1/2013', end='10/12/2017')['Adj Close']
r = np.log(p)-np.log(p).shift(1)
r.name = 'Return'
r = r[1:]  #Remove the first observation (NaN).
plt.figure(figsize=(12, 4))
plt.subplot(121)
sns.distplot(r, kde=False, fit=stats.norm)  #Histogram overlaid with a fitted normal density.
plt.subplot(122)
r.plot()  #Note that this is a pandas method! It looks prettier than plt.plot(r).
plt.savefig('img/stylizedfacts.svg')  #Save to file.
plt.close()
```


80
70
60
50
40
30
20
10
0
−0.04
−0.02
0.00
0.02
0.04
Return


0.04
0.03
0.02
0.01
0.00
−0.01
−0.02
−0.03
−0.04
2013-03
2013-09
2014-03
2014-09
2015-03
2015-09
2016-03
2016-09
2017-03
2017-09
Date

VaR Methods: Unconditional

Non-parametric: Historical Simulation

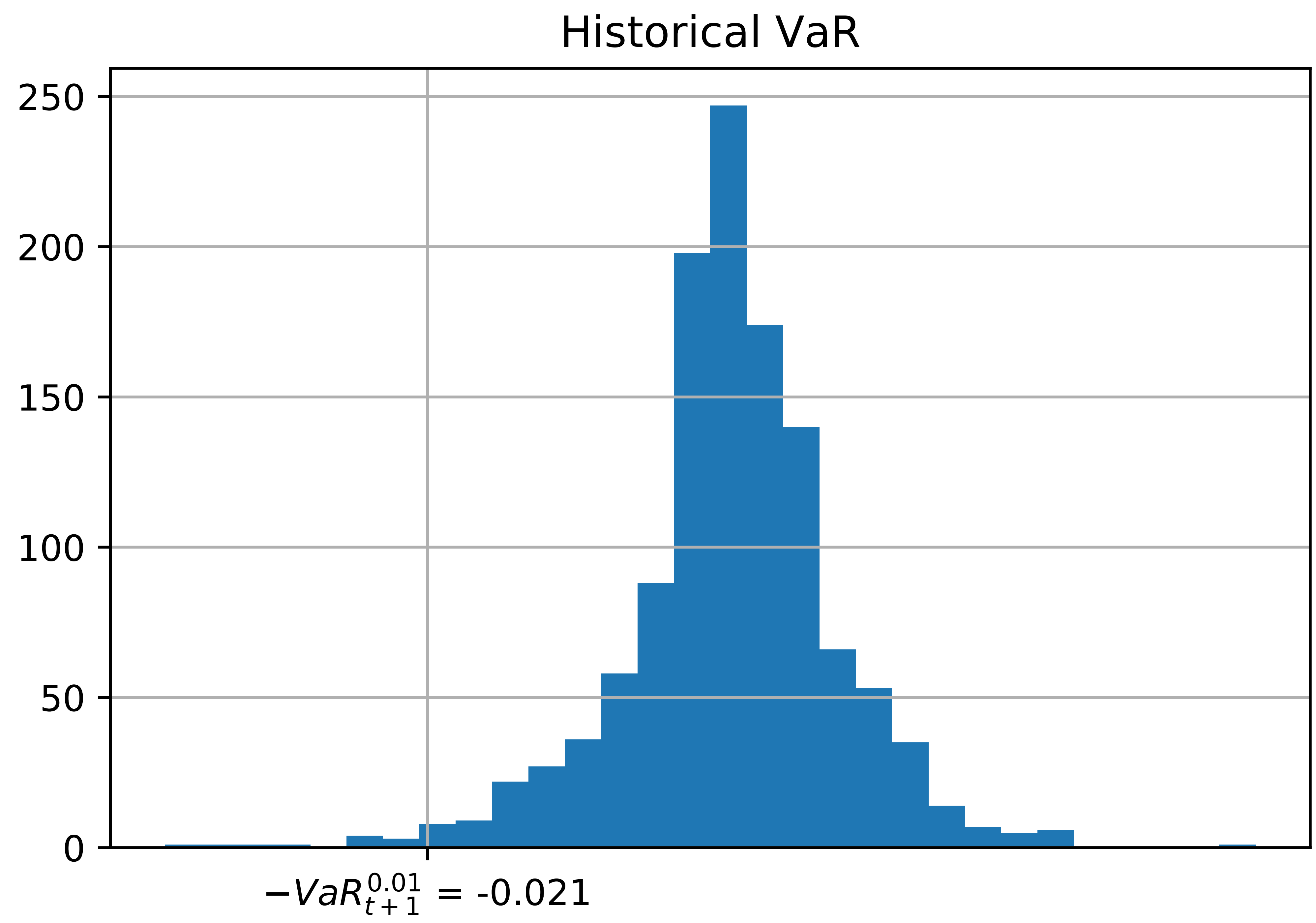
- Historical simulation assumes that the distribution of tomorrow's portfolio return is well approximated by the empirical distribution (histogram) of the past N observations $\{R_t, R_{t-1}, \ldots, R_{t+1-N}\}$.
- This is as if we draw, with replacement, from the last N returns and use this to simulate the next day's return distribution.
- The estimator of VaR is given by minus the pth *sample quantile* of the last N portfolio returns, i.e., $\widehat{VaR}^{p}_{t+1} = -R^N_p$, where R^N_p is the number such that $100p\%$ of the observations are smaller than it.

- In Python, we can use NumPy's `quantile` method, or the `percentile` function (or `nanpercentile` which ignores `NaN`s). Hilpisch uses `scoreatpercentile`, but that is is deprecated.

In [10]:
```
VaR_hist = -r.quantile(.01)  #Alternatively, VaR=np.percentile(r,1).
VaR_hist
```

Out[10]:
```
0.021317160779147 99
```

In [11]:
```
ax = r.hist(bins=30)  #Another pandas method: histogram with 30 bins.
ax.set_xticks([-VaR_hist])
ax.set_xticklabels(['$-VaR_{t+1}^{0.01}$ = -%4.3f' %VaR_hist])  #4.3f means floating point with 4 digits, of w
plt.title('Historical VaR')
plt.savefig('img/var_hist.svg')
plt.close()
```


Historical VaR
250
200
150
100
50
0
$-VaR_{t+1}^{0.01}$ = -0.021

- Problem: Last year(s) of data are not necessarily representative for the next few days (because of, e.g., volatility clustering).
- Exacerbated by the fact that a large N is required to compute the 1% VaR with any degree of precision (only 1% of the data are really used).

Parametric: Normal and t Distributions

- Another simple approach is to assume $R_{t+1} \sim N(\mu, \sigma^2)$, and to estimate μ and σ^2 from historical data (for daily data, $\mu \approx 0$). The VaR is then determined from

$$
\begin{aligned}
\Pr\left(R_{t+1} < -VaR^p_{t+1}\right) &= \Pr\left(\frac{R_{t+1} - \mu}{\sigma} < \frac{-VaR^p_{t+1} - \mu}{\sigma}\right) \\
&= \Pr\left(z_{t+1} < \frac{-VaR^p_{t+1} - \mu}{\sigma}\right) \\
&= \Phi\left(\frac{-VaR^p_{t+1} - \mu}{\sigma}\right) = p,
\end{aligned}
$$

where $\Phi(z)$ is the cumulative standard normal distribution.

- Thus,

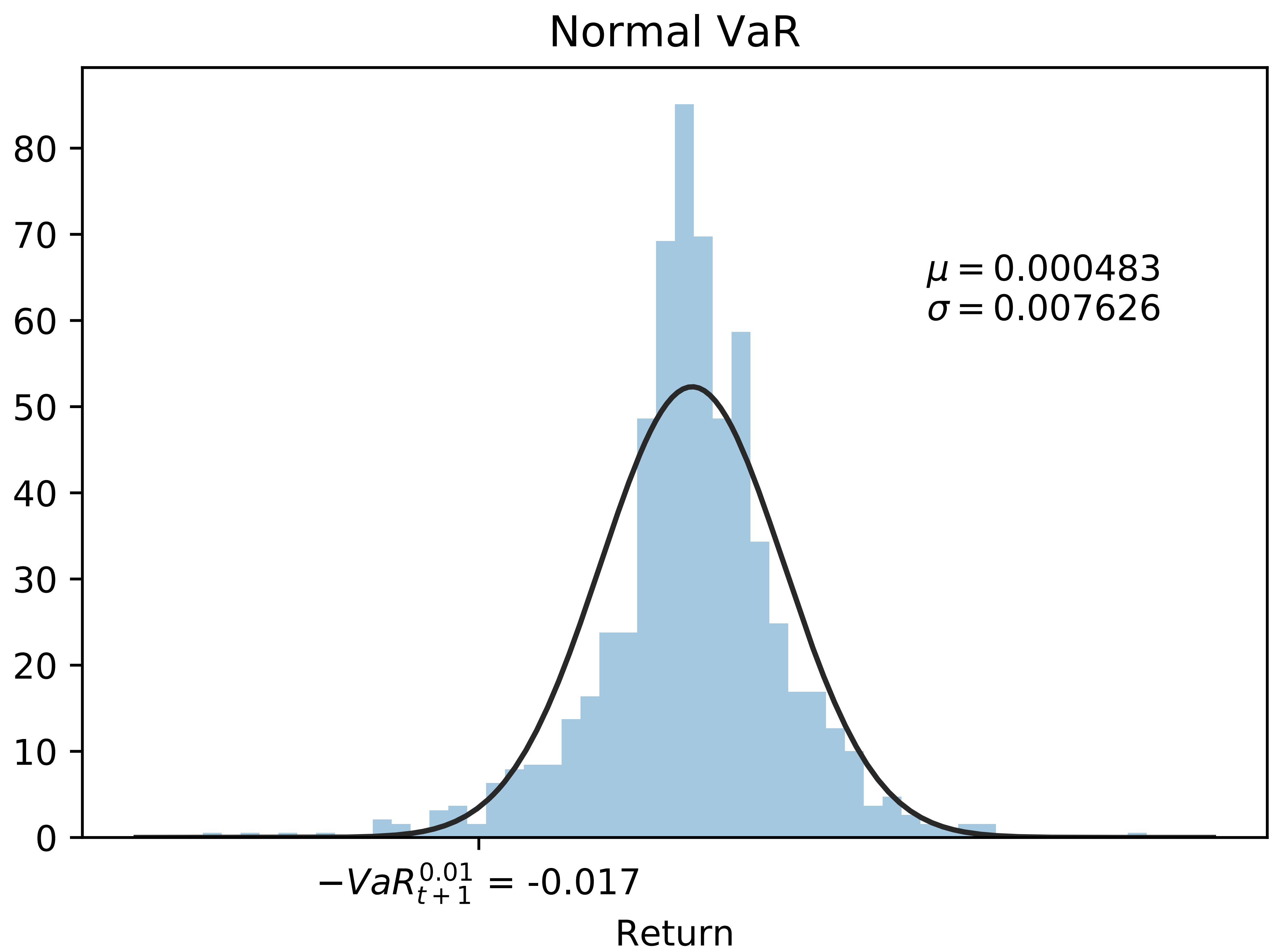
$$VaR_{t+1}^{p} = -\mu - \sigma\Phi^{-1}(p),$$

where $\Phi^{-1}(p)$ is the inverse distribution function of the standard normal, a.k.a. the *percentage point function* (ppf).

- In Python:

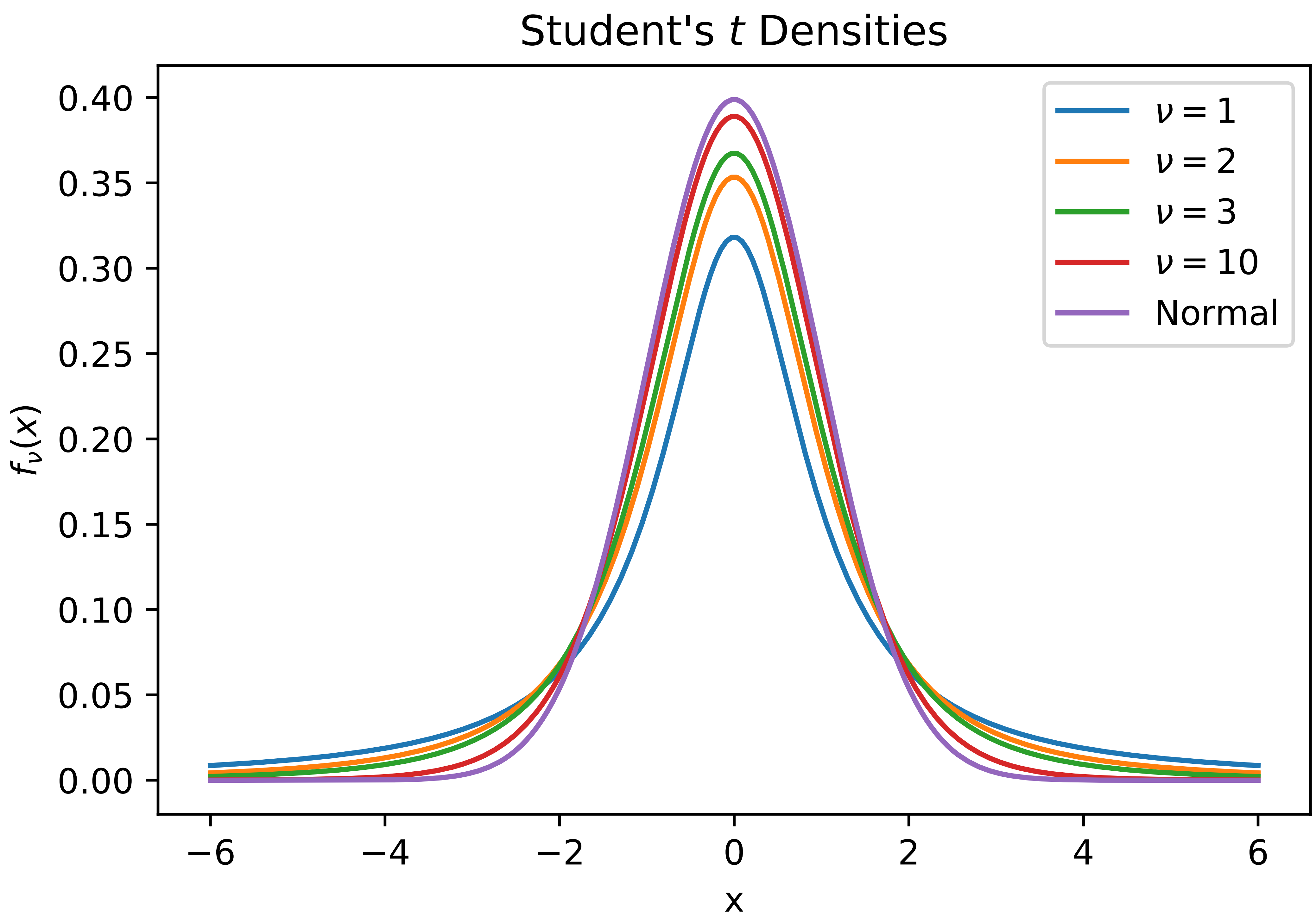
In [12]:
```
mu, sig = stats.norm.fit(r)  #Fit a normal distribution to `r`.
VaR_norm = -mu-sig*stats.norm.ppf(0.01)
VaR_norm
```

Out[12]:
```
0.01725813122968127
```


Normal VaR
$\mu = 0.000483$
$\sigma = 0.007626$
0
10
20
30
40
50
60
70
80
$-VaR^{0.01}_{t+1}$ = -0.017
Return

- Problems:
    - The variance of the past year(s) of data is not necessarily representative for the future.
    - Returns typically have heavier tails than the normal.
- The solution to the second point is to use another distribution. The Student's t distribution is a popular choice.

- The Student's t distribution with ν degrees of freedom, t_ν, is well known from linear regression as the distribution of t-statistics. In that context, $\nu = T - k$, where T is sample size and k the number of regressors.
- It can be generalized to allow $\nu \in \mathbb{R}_+$.
- Smaller values of ν correspond to heavier tails. As $\nu \to \infty$, we approach the $N(0, 1)$ distribution.
- It only has moments up to but not including ν:
  - The mean is finite and equal to zero if $\nu > 1$.
  - The variance is finite and equal to $\nu/(\nu - 2)$ if $\nu > 2$.
  - The excess kurtosis is finite and equal to $6/(\nu - 4)$ if $\nu > 4$.
- The distributions are symmetric around 0, so the mean and skewness are 0 if they exist.



Student's t Densities
$\nu = 1$
$\nu = 2$
$\nu = 3$
$\nu = 10$
Normal
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
$f_\nu(x)$
−6
−4
−2
0
2
4
6
x

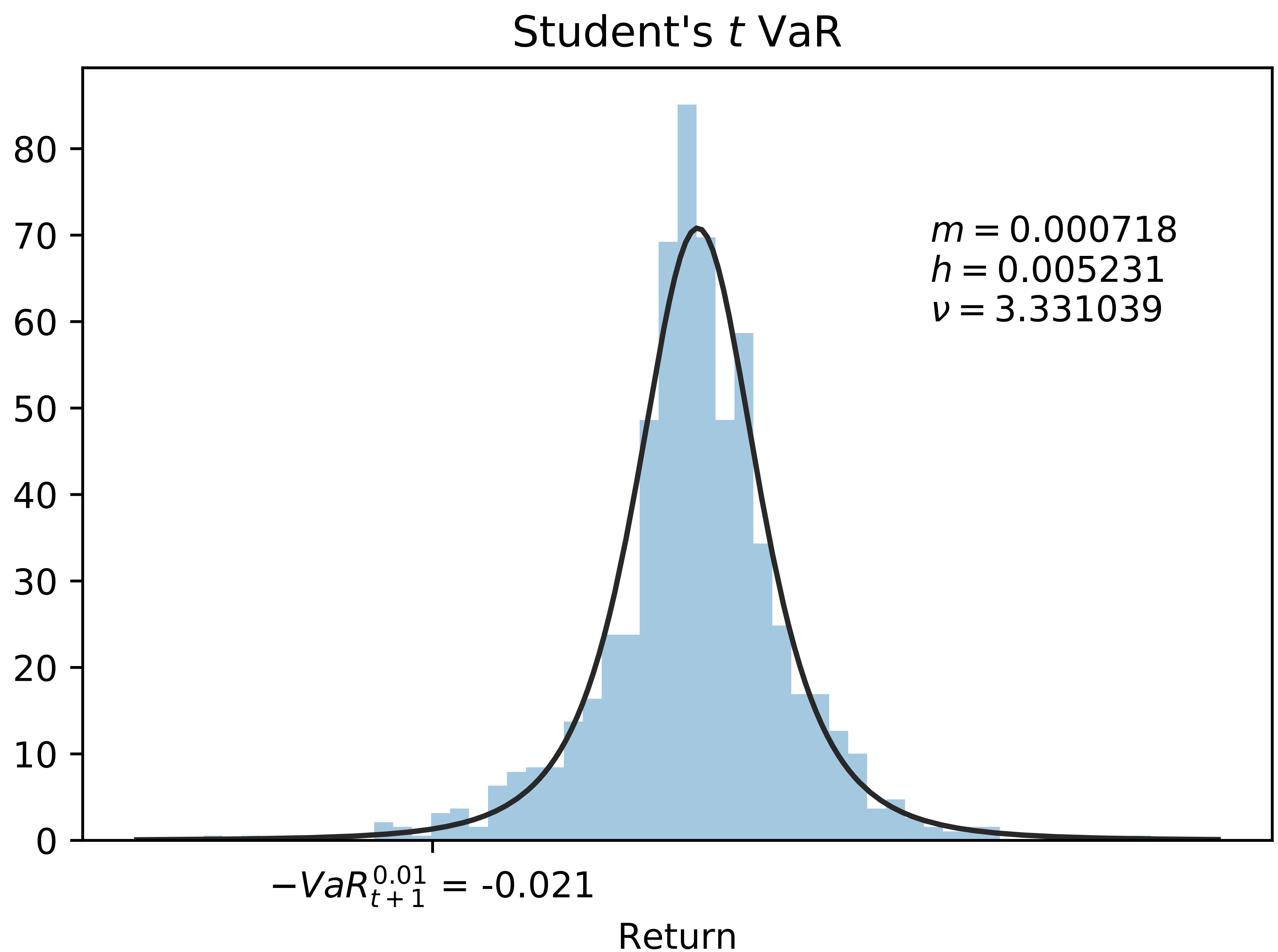
- For financial applications, we need to allow for a non-zero mean, and a variance different from $\nu/(\nu-2)$.
- This is achieved by introducing a *location parameter* m and a *scale parameter* h. We'll write $f_\nu(x;m,h)$ for the resulting density, $F_\nu(x;m,h)$ for the distribution function, and $F_\nu^{-1}(p;m,h)$ for the percentage point function.
- Note that if $x \sim t_\nu(m,h), \nu > 2$, then $\mathbb{E}[x] = m$ and $\mathrm{var}[x] = h^2\nu/(\nu-2)$.
- The VaR becomes

$$VaR_{t+1}^{p} = -F_\nu^{-1}(p;m,h).$$

- In Python:

In [15]:
```
df, m, h = stats.t.fit(r)  #Fit a location-scale t distribution to r.
VaR_t = -stats.t.ppf(0.01, df, loc=m, scale=h)
VaR_t
```

Out[15]:
```
0.021244827811891447
```


Student's t VaR
$m = 0.000718$
$h = 0.005231$
$\nu = 3.331039$
0
10
20
30
40
50
60
70
80
$-VaR_{t+1}^{0.01}$ = -0.021
Return

- There are several ways to assess whether a distributional assumption is adequate.
- One is to use a *goodness of fit test*. Many such tests exist.
- Hilpisch discusses the D'Agostino-Pearson test, available as `stats.normaltest`.
- Here we use the Jarque-Bera test. The test statistic is
$$JB = N\left(S^2/6 + (K-3)/24\right),$$
where N is the sample size, and S and K are respectively the sample skewness and kurtosis.
- Intuitively, it tests that the skewness and excess kurtosis are zero.
- It is distributed as χ^2_2 under the null of normality. The 5% critical value is

In [17]:
```
stats.chi2.ppf(0.95, 2)
```

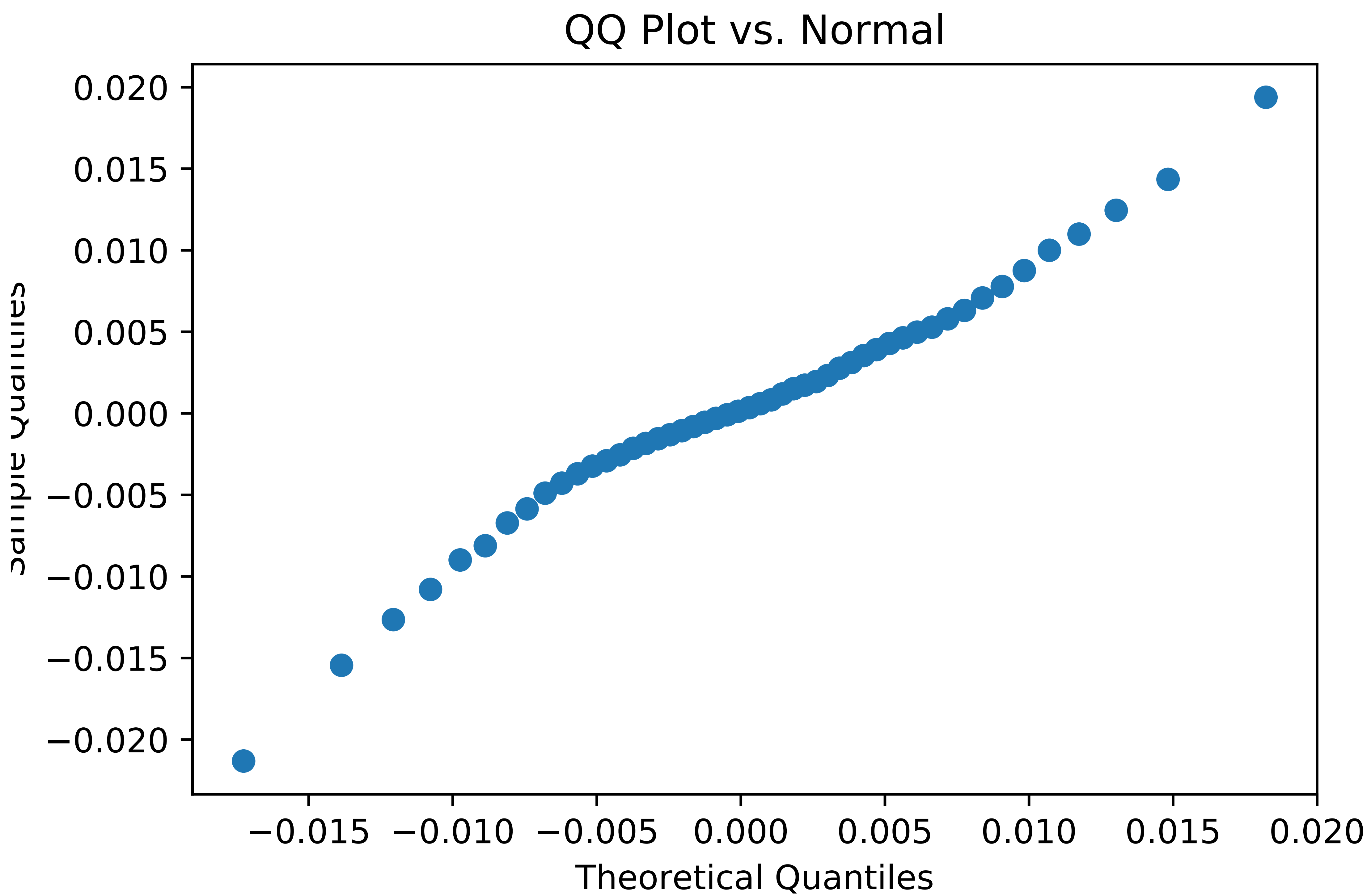
Out[17]:
```
5.9914645471079799
```

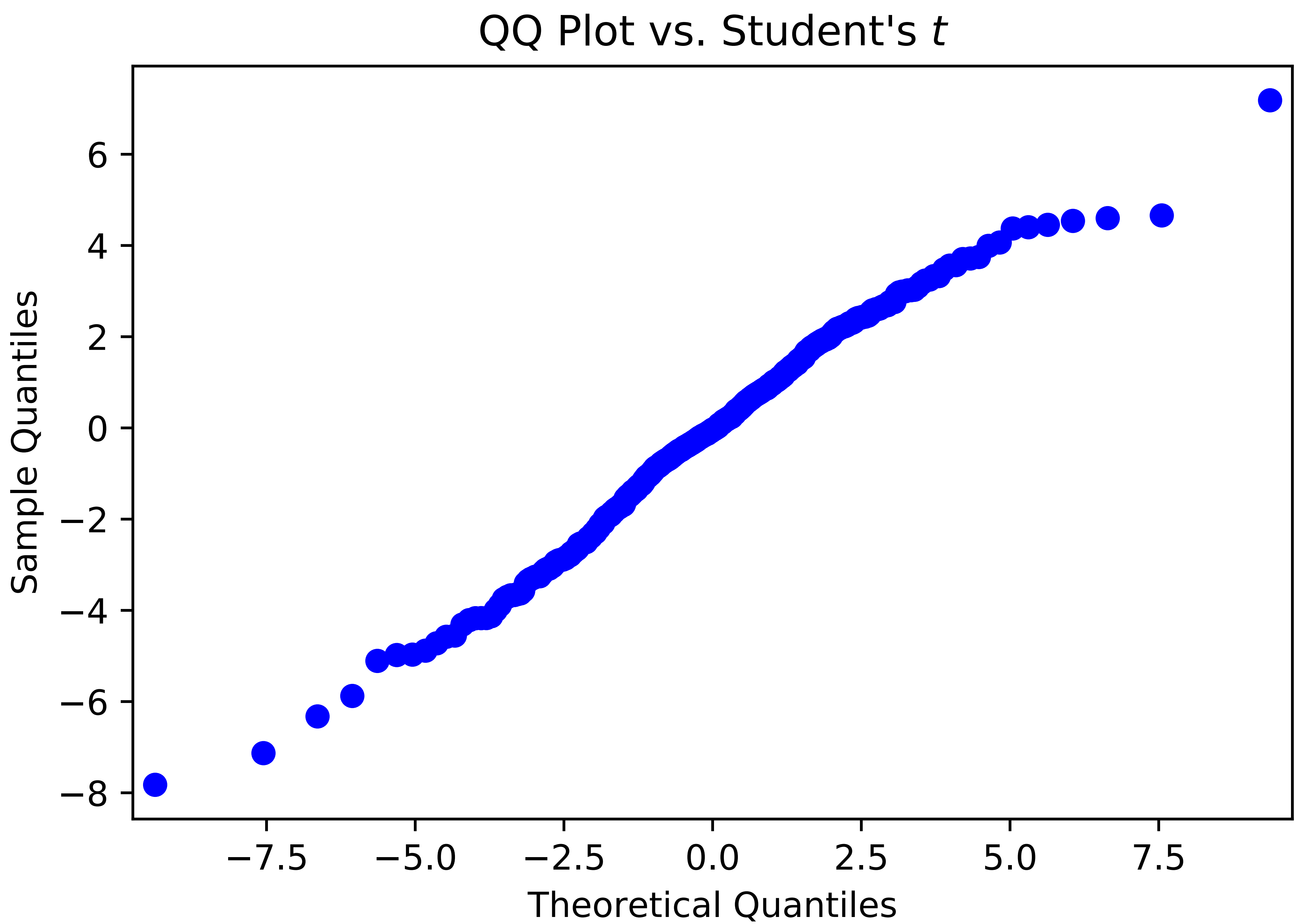
- In Python:

In [18]:
```
stats.jarque_bera(r)  #Returns (JB, p-val).
```

Out[18]:
```
(410.77889237295716, 0.0)
```

- Another option is to use a QQ-plot (quantile-quantile plot).
- It plots the empirical quantiles against the quantiles of a hypothesized distribution, e.g. $\Phi^{-1}(p)$ for the normal.
- If the distributional assumption is correct, then the plot should trace out the 45 degree line.


QQ Plot vs. Normal
0.020
0.015
0.010
0.005
0.000
−0.005
−0.010
−0.015
−0.020
−0.015
−0.010
−0.005
0.000
0.005
0.010
0.015
0.020
Theoretical Quantiles



QQ Plot vs. Student's t
6
4
2
0
−2
−4
−6
−8
Sample Quantiles
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
Theoretical Quantiles

VaR Methods: Filtered

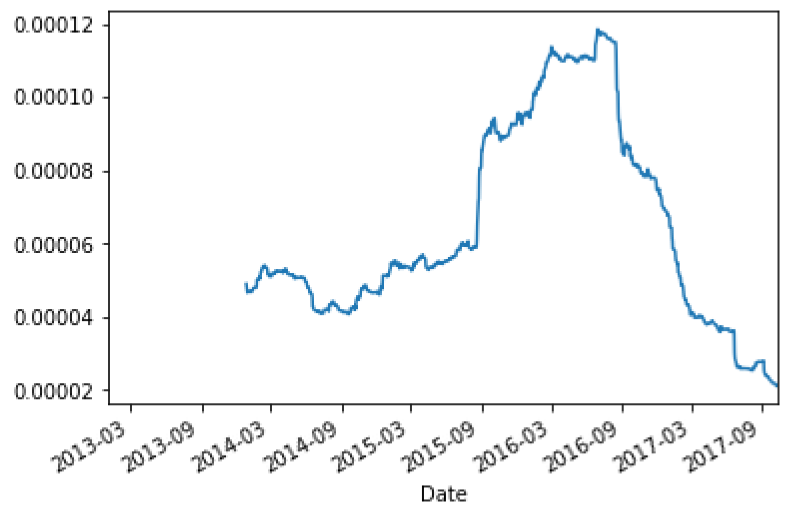
- All methods discussed so far share one drawback: they assume that the volatility is constant, at least in the estimation (and forecast) period.
- Implicitly, the Normal and Student's t method use the *historical volatility*:

$$\sigma^2_{t+1,HIST} = \frac{1}{N}\sum_{j=0}^{N-1} R^2_{t-j}.$$

  (Note: volatility usually means standard deviation, not variance. I'll be sloppy here).
- Here we assumed a zero mean, which is realistic for daily returns.
- Some adaptability is gained by choosing a smaller N such as 250 (one trading year), but there is a tradeoff because doing so decreases the sample size.
- A general solution requires a *volatility model*, which will be discussed in *Advanced Risk Management*.

- A Pandas `Series` object has a `rolling` method that can be used to construct historial volatilities for an entire series, using, at each day, the past N observations.
- The method returns a special window object that in turn has a method `var` (for variance).

In [21]:
```
sig2_hist = r.rolling(window=250).var()
sig2_hist.plot();
```



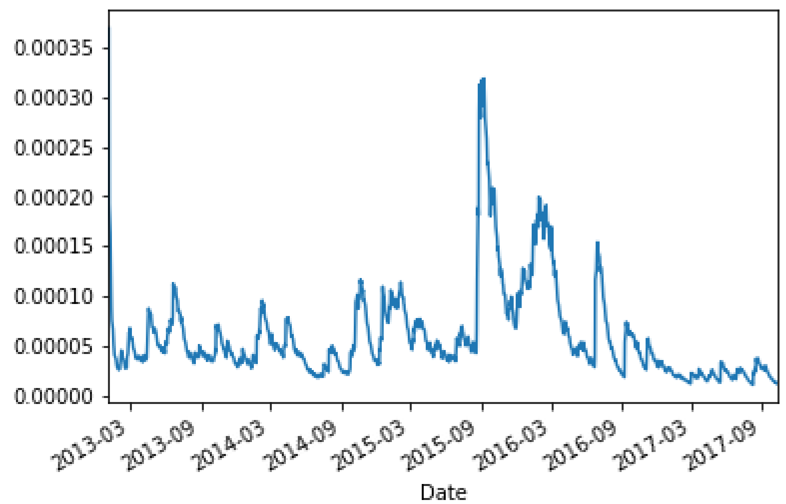
- A partial solution to the drawbacks of historical volatility is given by the RiskMetrics model, which is a special case of a more general framework known as *GARCH* models.
- The idea is to replace the equally weighted moving average used in historical volatility by an exponentially weighted moving average (EWMA):

$$\begin{aligned}\sigma^2_{t+1,EWMA} &= (1-\lambda)\sum_{j=0}^{\infty}\lambda^j R^2_{t-j}\\ &= \lambda\sigma^2_{t,EWMA} + (1-\lambda)R^2_t, \qquad 0<\lambda<1.\end{aligned}$$

- This means that observations further in the past get a smaller weight.
- Smaller λ means faster downweighting; for $\lambda \to 1$ we approach historical volatility (with an expanding window). For daily data, RiskMetrics recommends $\lambda = 0.94$.
- In practice we do not have $R_{t-\infty}$, but the second equation can be started up by an initial estimate / guess $\sigma^2_{0,EWMA}$.

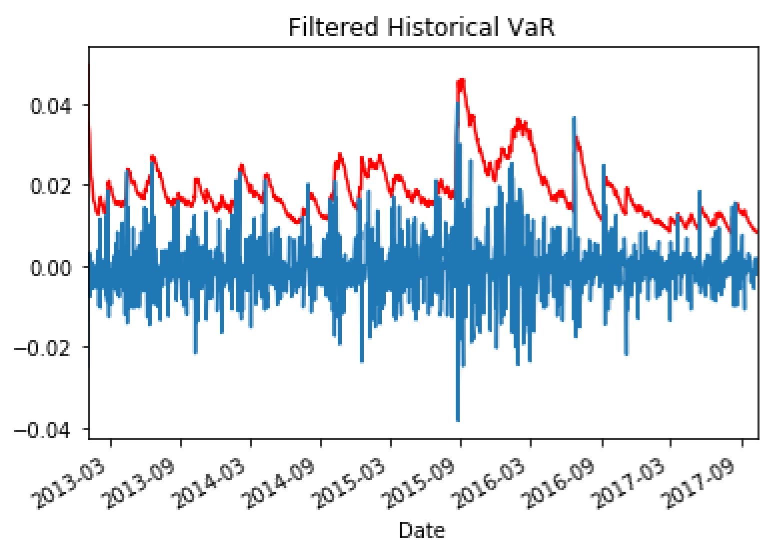
- The `ewm` (exponentially moving average) method of a Pandas `Series` can be used to achieve something similar (the exact definition is slightly different, see here).
- As before, the method returns a window object that has a `var` method.

In [22]:
```
sig2_ewma = r.ewm(alpha=0.06).var()  #alpha=(1-lambda).
sig2_ewma.plot();
```



- The idea behind a filtered VaR method is to decompose the returns as
$$R_t = \mu + \sigma_t z_t, \qquad z_t \overset{\text{i.i.d}}{\sim} (0, 1),$$
so that $\mathbb{E}[R_t] = \mu$ and $\mathrm{var}[R_t] = \sigma_t^2$. In principle, μ could be time-varying as well.
- Let z_p denote the $100p\%$ percentile of
$$z_t = \frac{R_t - \mu}{\sigma_t}.$$
It can be estimated by applying any of the VaR methods above (historical, normal, or Student's t) to the *filtered* (demeaned and devolatized) returns
$$\hat{z}_t = \frac{R_t - \hat{\mu}}{\hat{\sigma}_t}.$$
- Finally, $VaR_{t+1}^p = -\mu - \sigma_{t+1} z_p$.

In [23]:
```
sig_ewma = np.sqrt(sig2_ewma)
mu = np.mean(r)
z = (r-mu)/sig_ewma
VaR_filtered_hist = -mu-sig_ewma*z.quantile(0.01)
VaR_filtered_hist.plot(color='red')
plt.plot(-r)
plt.title('Filtered Historical VaR');
```



Backtesting

- The Basel accords require that banks' internal VaR models be *backtested*.
- They recommend constructing the 1% VaR over the last 250 trading days and counting the number of *VaR exceptions* (times that losses exceeded the day's VaR figure).
- A method is said to lie in the:
    - Green zone, in case of 0-4 exceptions;
    - Yellow zone, in case of 5-9 exceptions;
    - Red zone, in case of 10 or more exceptions.
- Being in one of the latter two incurs an extra capital charge.

- A more advanced method is the *dynamic quantile* (DQ) test by Engle and Manganelli (2004).
- It is based on the *hit series*

$$I_t = \begin{cases} 1, & \text{if } r_t < -VaR_t^p, \\ 0, & \text{if } r_t > -VaR_t^p. \end{cases}$$

- If the VaR model is correctly specified, then $\mathbb{E}[I_t] = p$ (there should be $p \cdot N$ exceptions in a sample of size N, on average). This is known as the *unconditional coverage hypothesis*.
- It can be tested by regressing $I_t - p$ on an intercept and testing that it is zero.
- In addition, it is desirable that the exceptions not be correlated. This is the *independence hypothesis*. It can be tested by including lags of I_t in the regression and testing their significance.
- Jointly testing both (with an F test) tests the *conditional coverage* hypothesis.

In [24]:

```python
import statsmodels.formula.api as smf
y = (r < -VaR_filtered_hist)*1  #Multiplication by 1 turns True/False into 1/0.
y.name='I'
data = pd.DataFrame(y)
model = smf.ols('I.subtract(0.01)~I.shift(1)', data=data)
res = model.fit()
print(res.summary2())
```

```
                  Results: Ordinary least squares
==================================================================
Model:              OLS              Adj. R-squared:     0.020
Dependent Variable: I.subtract(0.01) AIC:                -2069.9720
Date:               2017-11-23 14:41 BIC:                -2059.7852
No. Observations:   1204             Log-Likelihood:     1037.0
Df Model:           1                F-statistic:        25.67
Df Residuals:       1202             Prob (F-statistic): 4.68e-07
R-squared:          0.021            Scale:              0.010475
--------------------------------------------------------------------
              Coef.    Std.Err.      t      P>|t|     [0.025   0.975]
--------------------------------------------------------------------
Intercept    -0.0008     0.0030   -0.2576   0.7967   -0.0066   0.0051
I.shift(1)    0.1446     0.0285    5.0669   0.0000    0.0886   0.2006
------------------------------------------------------------------
Omnibus:             1816.402      Durbin-Watson:         2.014
Prob(Omnibus):       0.000         Jarque-Bera (JB):      382361.558
Skew:                9.218         Prob(JB):              0.000
Kurtosis:            88.334        Condition No.:         10
==================================================================
```

- Conclusions:
  - Unconditional coverage is not rejected. This is by construction; note that $r_t \lesseqgtr -VaR_t^p \iff z_t \lesseqgtr z_p$.
  - Independence is rejected; apparently our model is dynamically mis-specified. We may need to use a more general GARCH model instead of EWMA.
- The latter finding is likely driving the rejection of the conditional coverage test:

In [25]:
```
print(res.f_test('Intercept=0, I.shift(1)=0'))
```

```
<F test: F=array([[ 12.87315967]]), p=2.93962494772e-06, df_denom=1202, df_num=2>
```