1. Choose the false statement in the following four statements ( ).

(A) If A is a 2×2 matrix with a zero determinant, then one column of A is a multiple of the other.

(B) If two rows of a 3×3 matrix A are the same, then $detA = 0$.

(C) If A is $n\times n$ and $detA = 2$, then $detA^3 = 8$.

(D) If A is a 2×2 matrix, then $det5A = 5detA$.

2. Let $A = (\alpha_1, \alpha_2, \alpha_3)$ be a 3×3 matrix, where $\alpha_1, \alpha_2, \alpha_3$ are column vectors of A. If $detA = 2$ and $B = (\alpha_2, \alpha_1 + \alpha_2, 2\alpha_3)$, then $detB =$( ).

(A) -4 (B) 2 (C) 4 (D) -2

3. Choose an orthogonal set in the following four sets of vectors ( ).

(A) $\begin{bmatrix}-1\\4\\-3\end{bmatrix}, \begin{bmatrix}5\\2\\1\end{bmatrix}, \begin{bmatrix}3\\-4\\-7\end{bmatrix}$ (B) $\begin{bmatrix}1\\-2\\1\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}, \begin{bmatrix}-5\\-2\\1\end{bmatrix}$

(C) $\begin{bmatrix}2\\-7\\-1\end{bmatrix}, \begin{bmatrix}-6\\-3\\9\end{bmatrix}, \begin{bmatrix}3\\1\\-1\end{bmatrix}$ (D) $\begin{bmatrix}2\\-5\\-3\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}4\\-2\\6\end{bmatrix}$

4. Let $A = \begin{bmatrix}1 & 2 & 1\\1 & 2 & 1\\1 & 0 & 3\end{bmatrix}$. The dimensions of $ColA, RowA, NulA$ are ( ).

(A) 1, 1 , 2 (B) 2, 1, 2 (C) 2, 2, 1 (D) 1, 1, 1

5. Let $A = \begin{bmatrix}1 & -2\\1 & a\end{bmatrix}$ and $B = \begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}$. If A and B are similar, then a is ( ). (A) 1 (B) 4 (C) 2 (D) 3

6. Let A be a matrix such that $A\alpha = 2\alpha$, where α is an eigenvector of 2. Then $A^2\alpha =$_______α, $A^3\alpha =$_______α, and $(A + A^3 + 2010I)\alpha =$_______$\alpha$.

7. Let the quadratic form $Q(x) = 4x_1^2 + 8x_1x_2 + 3x_2^2 + 6x_1x_3 + 2x_2x_3 + 3x_3^2$ and the symmetric matrix A satisfy $Q(x) = x^TAx$. Then $A =$_______.

8. Let $\alpha_1 = \begin{bmatrix}1\\1\\2\end{bmatrix}, \alpha_2 = \begin{bmatrix}1\\0\\a\end{bmatrix}$, and $\alpha_3 = \begin{bmatrix}1\\3\\2\end{bmatrix}$ be linearly dependent, then $a =$_______.

9. Let $u = \begin{bmatrix}1\\2\\3\end{bmatrix}$ and $v = \begin{bmatrix}4\\5\\6\end{bmatrix}$. The inner product $u{\cdot}v =$_______, the length $\|u\| =$_______, and the length $\|v\| =$_______.

10. Let $A = \begin{bmatrix}2 & 1 & 1\\1 & 2 & 1\\1 & 1 & 2\end{bmatrix}$. Then $detA =$_______.

三. Let $A = \begin{bmatrix}2 & 1 & 3\\1 & -1 & 1\\1 & 4 & -2\end{bmatrix}$.

(1) Compute the cofactors C_{21}, C_{22}, C_{23}. (2) Compute $detA$.

(3) Compute $C_{21} - C_{22} + C_{23}$ and $C_{21} + 4C_{22} - 2C_{23}$.

四. Let $A = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix}$.

(1) Please give an Echelon form of A.

(2) Please find bases for the row space $RowA$, the column space $ColA$, the null space $NulA$.

(3) Please find dimensions of $RowA$, $ColA$, $NulA$.

五. Use Cramer's rule to solve the solutions of the system:

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 4 \\ -x_1 + \quad\quad x_3 &= 2 \\ 2x_1 + 2x_2 \quad\quad &= 2 \end{aligned}$$

六. Let $P = \{f(x) : f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\}$ be the set of all the polynomials. The sum of two elements of P is defined as the sum of two polynomials. The scalar multiple $cf(x)$ is defined as the multiplication of a real number c and a polynomial $f(x)$. Then P is a vector space over R.

(1) Prove that $V = \{f(x) : f(x) = a_0 + a_1x + a_2x^2\}$ is a subspace of P.

(2) Prove that $1, x-1, (x-1)^2$ is a basis of V.

(3) Write $2x^2$ as a linear combination of the basis $1, x-1, (x-1)^2$.

七. Let $\alpha_1, \alpha_2, \alpha_3$ be three vectors in a linear space V over R. Vectors $\alpha_1, \alpha_2, \alpha_3$ are linearly independent. Let $\beta_1 = \alpha_1 + \alpha_2$, $\beta_2 = \alpha_2 + \alpha_3$, and $\beta_3 = \alpha_1 + \alpha_3$. Determine if $\beta_1, \beta_2, \beta_3$ are linearly dependent or linearly independent.

八. Let $\beta_1 = \begin{bmatrix} -1 \\ 8 \end{bmatrix}, \beta_2 = \begin{bmatrix} 1 \\ -7 \end{bmatrix}, \alpha_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ u = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

(1) Please verify that $\{\beta_1, \beta_2\}$ is a basis of R^2 and $\{\alpha_1, \alpha_2\}$ is another basis of R^2.

(2) Please write β_1 as a linear combination of α_1 and α_2; write β_2 as a linear combination of α_1 and α_2.

(3) Please find a matrix A such that $(\beta_1, \beta_2) = (\alpha_1, \alpha_2)A$; find a matrix B such that $(\alpha_1, \alpha_2) = (\beta_1, \beta_2)B$.

(4) Please find a, b such that $u = a\alpha_1 + b\alpha_2$.

九. Let $A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$.

(1) Verify that $det(A - \lambda I) = -(\lambda - 1)^2(\lambda - 5)$ and eigenvalues of A are $1, 1, 5$.

(2) Please give three linearly independent eigenvectors of A.

Diagonalize the matrix A. Please give an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

十. Let $Q(x) = 4x_1^2 + 3x_2^2 + 2x_2x_3 + 3x_3^2$.

(1) Write the symmetric matrix A such that $Q(x) = x^TAx$.

(2) Compute all the eigenvalues of A.

(3) Determine if A is positive definite or not.

(4) Please give three linearly independent eigenvectors of A.

(5) Use the Gram-Schmidt process and give an orthonormal basis from eigenvectors

in (4).