# LDOS for a coupled system of SSH chains

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**Abstract:** This work studies the local density of states (LDOS) for a coupled system of two Su-Schrieffer-Heeger (SSH) chains using Green's function methods. The SSH model, a prototypical example of a topological insulator, exhibits edge states that are robust against perturbations preserving chiral symmetry. We first derive the LDOS for a single SSH chain, demonstrating how the zero-energy edge state manifests as a Dirac delta peak in the LDOS. Subsequently, we analyze the coupling between two topological SSH chains via a tunneling potential, which hybridizes their edge states. Using the Dyson equation, we compute the perturbed Green's function and show that the original zero-energy state splits into two finite-energy states, reflecting the preservation of chiral symmetry. Numerical calculations reveal the spatial and energetic distribution of these states for varying coupling strengths, confirming their topological robustness. Our results highlight the interplay between symmetry-protected edge states and inter-chain interactions, providing insights into the design of topological heterostructures.

## I. INTRODUCTION

Numerous systems in nature, and others proposed theoretically, display topological properties that manifest in distinct forms. For example, the Quantum Hall Effect [1], where the topology displays through the Quantum Hall conductance; in photonic crystals [2], as a bulk-boundary correspondence; or in topological superconductors [3], as emergent modes at the edges of the system that come from particle-hole symmetry.

The SSH model, which also shows topological behavior, was first used by J. R. Schrieffer and A. J. Heeger to theoretically model the soliton formation in polyacetylene [4], showing how topological properties emerge from alternating hopping amplitudes in a tight-binding chain. For the SSH system, topology manifests as edge states, that are highly dependent on the symmetry of the system and its preservation. This prototype is useful at the modeling of molecules and materials by the construction of heterostructures [5].

One way to further comprehend the emergent topological attributes is through the local density of states (LDOS), which gives out information about the spatial and energetical available states of the system. The LDOS can be computed using Green functions: a very powerful tool when it comes to solving differential equations. Characterizing the distribution of the states of a system is often used to design and understand devices and materials with specific interactions at their surfaces [6, 7].

In this work, we will use Green functions to compute the LDOS of a single topological SSH chain, and afterwards, the LDOS of two coupled topological SSH chains, so we can break down the nature of the interaction and the behavior of the edge states.

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## II. GREEN FUNCTION METHODS AND DYSON EQUATION

Consider a non-homogeneous linear differential equation of the form

$$(\lambda - \hat{\mathcal{L}})\psi = f, \tag{1}$$

where $\hat{\mathcal{L}}$ is a linear differential operator, $\lambda$ is a complex parameter, $f$ is a source term, and $\psi$ is the unknown function to be determined. Also, the eigenfunctions $\phi$ of $\hat{\mathcal{L}}$, satisfying $\hat{\mathcal{L}}\phi = \lambda\phi$, form a complete set. Taking this into account, the formal solution to (1) can be expressed as

$$\psi = (\lambda - \hat{\mathcal{L}})^{-1} f = \hat{G} f,$$

where we have introduced the Green's function operator $\hat{G} \equiv (\lambda - \hat{\mathcal{L}})^{-1}$. In coordinate representation, this solution takes the integral form

$$\psi(x) = \hat{G} f = \int G(x, x') f(x')\, \mathrm{d}x',$$

where $G(x, x')$ is the Green's function satisfying the differential equation

$$(\lambda - \hat{\mathcal{L}}_x) G(x, x') = \delta(x - x'),$$

with $\delta(x - x')$ being the Dirac delta function. This formulation demonstrates how the Green's function serves as a fundamental solution to the original differential equation.

Hence, several analytical techniques have been developed to determine the explicit form of $G(x, x')$, including the direct method [8] and the eigenfunction expansion approach. For time-dependent problems of the form

$$(-ic\partial_t - \hat{\mathcal{L}}_x)\psi(x, t) = f(x, t),$$

it is often advantageous to work in frequency space. The retarded Green's function in this representation by the eigenfunction method becomes

$$G^r(x,x',\omega)=\sum_n \frac{\psi_n(x)\psi_n^\dagger(x')}{\omega-\omega_n+i0^+},$$

where $\omega_n$ are the eigenfrequencies of the system and $0^+$ denotes an infinitesimal positive imaginary part ensuring causality.

As a consequence, an important physical quantity derived from the Green's function is the local density of states (LDOS), which can be obtained via the Plemelj formula (see Appendix A) as

$$\rho(x,\omega)=-\frac{1}{\pi}\mathrm{Im}\left\{G^r(x,x,\omega)\right\}.$$

This quantity provides crucial information about the available states at position $x$ and energy $\omega$.

On the other hand, in quantum mechanical applications, it is frequently desirable to obtain the Green's function for a perturbed Hamiltonian. Given the unperturbed Hamiltonian $\hat{H}_0$ with known Green's function $\hat{g}$ satisfying

$$(E-\hat{H}_0)\hat{g}=\mathbb{I},$$

we seek the Green's function $\hat{G}$ for the perturbed system $\hat{H}=\hat{H}_0+\hat{V}$ that satisfies

$$(E-\hat{H})\hat{G}=\mathbb{I}.$$

The Dyson equation provides a self-consistent relation between these quantities as

$$\hat{G}=\hat{g}+\hat{g}\hat{V}\hat{G}.$$

This recursive equation forms the basis for perturbation theory in Green's function methods and allows for systematic calculation of the perturbed Green's function in terms of the unperturbed solution and the perturbative potential $\hat{V}$.

## III. GREEN FUNCTION FOR THE SSH MODEL

The SSH Hamiltonian consists of a one-dimensional lattice with two sites (atoms) per unit cell, and alternates between two amplitudes, $v=(t+\delta)$ and $w=(t-\delta)$, for intra-cell and inter-cell hopping, respectively. That is

$$H=\sum_{n=1}^{N}(t+\delta)c^\dagger_{A,n}c_{B,n}+\sum_{n=1}^{N-1}(t-\delta)c^\dagger_{A,n+1}c_{B,n}+\text{h.c.},$$

where the operators, $c^\dagger$, $c$, create and annihilate the electrons on the respective sublattice's ($A$ or $B$) site $n$, as seen in the figure 1.

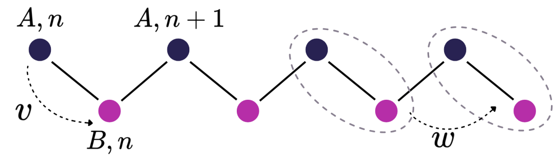



Figure 1: Visual representation of the SSH system.

Making the Fourier transform so we get into the momentum space, we get the Hamiltonian form (see Appendix B)

$$H=\sum_k \psi_k^\dagger\left[-(t-\delta)k\sigma_x+\left(2\delta t+\frac{1}{2}(t-\delta)k^2\right)\sigma_z\right]\psi_k,$$

with the spinor $\psi_k=\binom{a_k}{b_k}$ in terms of the coefficients of the transformation. This Hamiltonian has chiral symmetry and is analogous to a modified Dirac Hamiltonian, whose quadratic component $m_2=(t-\delta)\frac{a^2}{2}=\frac{wa^2}{2}$ works as term that makes the equation topologically distinct, breaking the mass symmetry into two opposite sign mass components for $\delta<0$.

Thus, in its topological phase, this system exhibits a zero energy edge state

$$|\psi(x)\rangle=c\left[e^{-x/d_+}-e^{-x/d_-}\right]|\phi\rangle,$$

with

$$d_\pm^{-1}=\frac{|A|}{2m^2}\left[1\pm\sqrt{1-\frac{4m_1m_2}{A^2}}\right]$$

and $A=wa$, $m_2=\frac{wa^2}{2}$ and $m_1=|v-w|$ that satisfy the boundary conditions for $x>0$.

Recalling that the edge state has zero energy, by the eigenfunction method, the retarded Green function for the edge state of the SSH model is

$$\begin{aligned}g^r(x,x',\omega)=&\frac{|\psi(x)\rangle\langle\psi(x')|}{\omega+i0^+}\\=&\frac{c^2}{\omega+i0^+}\left[e^{-x/d_+}-e^{-x/d_-}\right]\\&\times\left[e^{-x'/d_+}-e^{-x'/d_-}\right]|\phi\rangle\langle\phi|.\end{aligned}$$

It can be seen that for the matrix $|\phi\rangle\langle\phi|$,

$$|\phi\rangle\langle\phi|=\begin{pmatrix}1\\-i\end{pmatrix}\begin{pmatrix}1 & i\end{pmatrix}=\begin{pmatrix}1 & i\\-i & 1\end{pmatrix}$$

and defining $\hat{A} = |\phi\rangle\langle\phi|$ the retarded Green function for the edge state is

$$\begin{aligned} g^r(x,x',\omega) =& \frac{c^2}{\omega + i0^+}\left[e^{-x/d_+} - e^{-x/d_-}\right] \\ &\times \left[e^{-x'/d_+} - e^{-x'/d_-}\right]\hat{A}. \end{aligned}$$

With this result, the LDOS for the edge state is computed as

$$\begin{aligned} \rho(x,\omega) &= -\frac{1}{\pi}\mathrm{Im}\{\mathrm{Tr}(g^r(x,x,\omega))\} \\ &= -\frac{1}{\pi}\mathrm{Im}\left\{\frac{c^2}{\omega + i0^+}\left[e^{-x/d_+} - e^{-x/d_-}\right]^2 \mathrm{Tr}(\hat{A})\right\}. \end{aligned}$$

As $\mathrm{Tr}(\hat{A}) = 2$ and by the Plemelj formula

$$\lim_{\epsilon\to 0^+}\frac{1}{\omega + i\epsilon} = \mathcal{P}\left(\frac{1}{\omega}\right) - i\pi\delta(\omega),$$

the LDOS for the edge state is

$$\rho(x,\omega) = 2c^2\left[e^{-x/d_+} - e^{-x/d_-}\right]^2\delta(\omega).$$

## IV. COUPLING BETWEEN TOPOLOGICAL SSH CHAINS

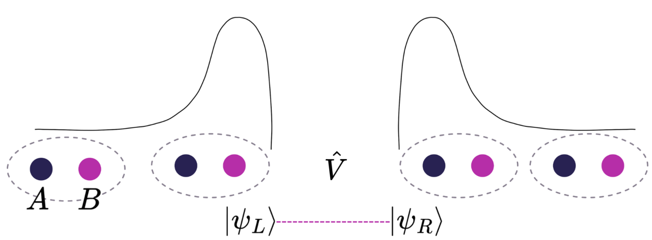



Figure 2: Visual representation of the edge states for the coupling.

In the non-interacting case, the edge states of two separate SSH chains can be described by

$$\begin{aligned} |\psi_R\rangle &= c_R\left(e^{-x/d_+} - e^{-x/d_-}\right)|\phi\rangle, \\ |\psi_L\rangle &= c_L\left(e^{x/\xi_+} - e^{x/\xi_-}\right)|\phi\rangle, \end{aligned}$$

where $|\psi_R\rangle$ and $|\psi_L\rangle$ represent the right and left edge states with penetration lengths $d_\pm$ and $\xi_\pm$, respectively, and $c_R$ and $c_L$ are normalization constants.

To study the interaction between these edge states, we introduce a tunneling potential with isotropic coupling strength $t$ as seen in Figure 2,

$$\hat{V} = \begin{pmatrix} 0 & V_{RL} \\ V_{LR} & 0 \end{pmatrix} = \begin{pmatrix} \hat{0}_{2x2} & t\mathbb{I}_{2x2} \\ t\mathbb{I}_{2x2} & \hat{0}_{2x2} \end{pmatrix},$$

where we have defined $V_{RL} = V_{LR} = t\mathbb{I}$. The retarded Green functions for the uncoupled SSH model are given by

$$\begin{aligned} g_{RR}(x,x',\omega) =& \frac{c_R^2}{\omega + i0^+}\left[e^{-x/d_+} - e^{-x/d_-}\right] \\ &\times \left[e^{-x'/d_+} - e^{-x'/d_-}\right]\hat{A}, \\ g_{LL}(x,x',\omega) =& \frac{c_L^2}{\omega + i0^+}\left[e^{x/\xi_+} - e^{x/\xi_-}\right] \\ &\times \left[e^{x'/\xi_+} - e^{x'/\xi_-}\right]\hat{A}. \end{aligned}$$

For notation convenience, we define

$$\begin{aligned} f(x,x') &= c_R^2\left[e^{-x/d_+} - e^{-x/d_-}\right]\left[e^{-x'/d_+} - e^{-x'/d_-}\right], \\ h(x,x') &= c_L^2\left[e^{x/\xi_+} - e^{x/\xi_-}\right]\left[e^{x'/\xi_+} - e^{x'/\xi_-}\right], \end{aligned}$$

which allows us to express the Green's functions compactly as

$$\begin{aligned} g_{RR}(x,x',\omega) &= \frac{f(x,x')}{\omega + i0^+}\hat{A}, \\ g_{LL}(x,x',\omega) &= \frac{h(x,x')}{\omega + i0^+}\hat{A}. \end{aligned}$$

Now, in the non-interacting case, the cross terms vanish, $g_{RL} = g_{LR} = 0$, and using the Dyson equation

$$\hat{G} = \hat{g} + \hat{g}\hat{V}\hat{G},$$

we obtain the coupled equations for $G_{RR}$ and $G_{LR}$

$$\begin{aligned} G_{RR} &= g_{RR} + g_{RR}V_{RL}G_{LR}, \\ G_{LR} &= g_{LL}V_{LR}G_{RR}. \end{aligned}$$

Solving for $G_{RR}$ (see Appendix C) yields

$$\begin{aligned} G_{RR}(x,x,\omega) &= \left(\mathbb{I} - t^2 g_{RR}g_{LL}\right)^{-1} g_{RR} \\ &= \frac{f(x,x')(\omega + i0^+)}{(\omega + i0^+)^2 - b^2(x,x')}\hat{A}, \end{aligned} \tag{2}$$

where $b^2(x,x') = 4t^2 f(x,x')h(x,x')$. Note that when $t = 0$, we recover the unperturbed retarded Green's function.

Thus, using this result, we can obtain the LDOS for the right edge state after the coupling as

$$\rho(x,\omega) = -\frac{1}{\pi}\mathrm{Im}\{\mathrm{Tr}(G_{RR}(x,x,\omega))\}.$$

Finally, using the Plemelj formula, the LDOS for the right edge state after the coupling is

$$\rho(x,\omega) = f(x,x)\left[\delta(\omega-b)+\delta(\omega+b)\right], \tag{3}$$

where the original zero-energy edge state has split into two new states at energies $\pm b = \pm 2t\sqrt{f(x,x)h(x,x)}$. As we wanted, for $t=0$,

$$\rho(x,\omega) = 2f(x,x)\delta(\omega) = 2c^2\left[e^{-x/d_+} - e^{-x/d_-}\right]^2\delta(\omega),$$

recovering the unperturbed LDOS for the right edge state.

## V. RESULTS AND DISCUSSION

Based on the result obtained in the previous section of the LDOS (2) for the right edge state after the coupling is

$$\rho(x,\omega) = -\frac{1}{\pi}\mathrm{Im}\left\{\frac{2f(x,x')(\omega+i0^+)}{(\omega+i0^+)^2 - b^2(x,x')}\right\}. \tag{4}$$

Since a direct computational calculation of the Dirac delta terms can be tedious, we adopt an alternative numerical approach. Using `Python`, we developed a script [9] to compute the LDOS for the right edge state with the original expression (4) taking advantage of the Plemelj formula.

We set the parameters for the chains as follows: $a=1$ for both chains, with $v=0.8$ and $w=1.0$ for the right chain, and $v'=0.72$, $w'=1.1$ for the left chain. These choices correspond to $A=1.0$, $m_1=0.2$, and $m_2=0.5$ for the right chain, and $A'=1.1$, $m_1'=0.38$, and $m_2'=0.55$ for the left chain, thus, both chains are in their topological phase. As an initial step, we select a coupling strength of $t=0.5$ to analyze how the interaction with the left edge state influences the right edge state. For consistency with this objective, we evaluate the LDOS at $x=0.5$ near the edge, to probe the spatial region where the edge state is localized.

These parameters yield the penetration lengths $d_+=0.563508$, $d_-=4.436492$, $\xi_+=0.642690$, and $\xi_-=2.252047$.

For the frequency-space discretization, we use 5000 points in the interval $\omega\in(-4,4)$, resulting in $\Delta\omega=0.001$. Additionally, we set $0^+ = 2\Delta\omega = 0.002$ for numerical stability. The LDOS obtained is shown in the figure 3.

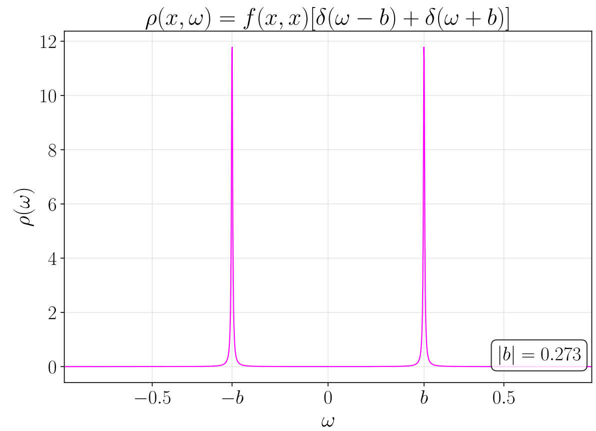
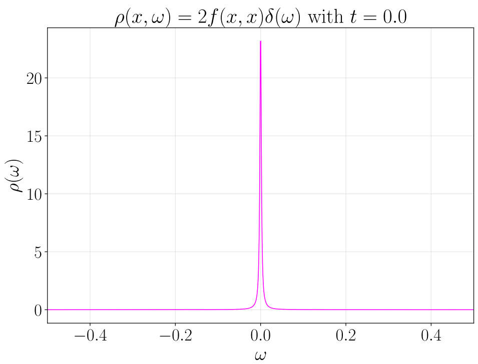



Figure 3: Local density of states LDOS for the right edge state as a function of the frequency $\omega$ for the coupling strength $t=0.5$.

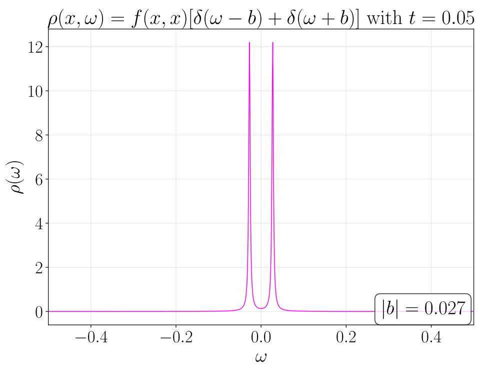
At first glance, we observe that the interaction between the two edge states has destroyed the zero-energy mode in the right chain, as no peak is observed at $\omega=0$. Furthermore, two symmetric peaks appear around the origin at $\omega=\pm b=\pm 0.273$ , which may reflect the chiral symmetry of the SSH model and suggest hybridization between the right and left edge states, evidenced by the displacement of the energy peaks and the emergence of finite energy states.

Furthermore, the function $f(x,x)$ describes the spatial decay of the edge state amplitude away from the right boundary. Although the graph does not explicitly display spatial details, the presence of sharp peaks in the local density of states (LDOS) at energies $\omega=\pm b$ and $x=0.5$, near the edge, confirms that the state remains localized near the right edge. The persistence of these LDOS peaks under isotropic coupling suggests that the edge state's spatial profile remains intact, even as the hybridization between left and right edge states lifts their degeneracy, shifting their energies to finite values.

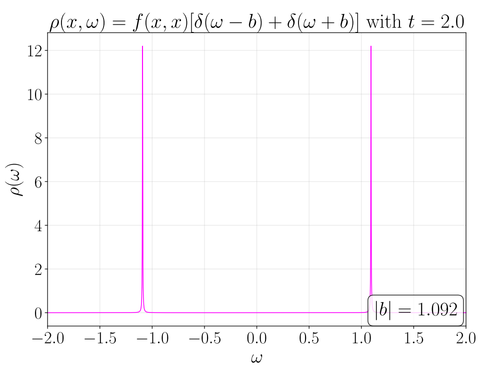
Now, to systematically analyze the effects of the coupling on the edge states, we repeated the procedure for varying coupling strengths: $t=0$ (no coupling), $t=0.05$ (weak coupling), $t=2$ (intermediate coupling), and $t=5$ (strong coupling). This parameter range allows us to examine the complete evolution from decoupled to strongly coupled edge states. The corresponding LDOS results are presented in Fig. 4.



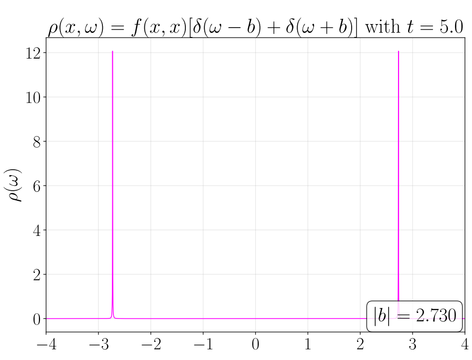

(a) $t = 0$




(b) $t = 0.05$




(c) $t = 2$




(d) $t = 5$

Figure 4: Local density of states (LDOS) for the right edge state as a function of frequency $\omega$ for different inter-chain coupling strengths $t$. The Dirac delta peaks (or their splitting) highlight the evolution of the topological edge state under increasing hybridization.

As shown in Figure 4a, the uncoupled case reproduces the unperturbed local density of states (LDOS), exhibiting a Dirac delta function $\delta(\omega)$ behavior at the origin. This characteristic confirms the existence of a zero-energy edge state in the absence of inter-chain coupling.

In the weak coupling regime ($t = 0.05$), Figure 4b reveals that the edge state interaction splits the zero-energy peak into two symmetric peaks, a feature protected by the system's chiral symmetry. The finite coupling strength $t$ induces hybridization of the edge state energy levels, resulting in two states with finite energies.

For intermediate and strong coupling regimes, Figures 4d and 4c demonstrate that the peaks shift away from $\omega = 0$, indicating stronger hybridization between the edge states. While these states retain their localized character at the edge, their energies now reflect significant overlap. Remarkably, the system may remain in a topologically non-trivial regime, as the edge states persist despite hybridization.

This systematic evolution reveals that for all coupling strengths, the splitting $2|b|$ scales with $t$, confirming that $b$ corresponds to the hybridization energy between edge states. As consistently shown across all figures, the energy splitting increases with coupling strength while the edge state localization within the gap remains invariant. This behavior demonstrates the topological robustness of the edge states against perturbations. Since the coupling potential $\hat{V}$ respects the chiral symmetry of the system, the edge states remain well-defined throughout the coupling range.

## VI. CONCLUSIONS

In this study, we analyzed the local density of states (LDOS) for a coupled system of two SSH chains, focusing on the behavior of their topological edge states under inter-chain interactions. Using Green's function methods and the Dyson equation, we demonstrated that the zero-energy edge state of an isolated SSH chain splits into two finite-energy states when coupled to a second chain. This splitting, governed by the coupling strength $t$, preserves the chiral symmetry of the system, ensuring the robustness of the edge states.

Numerical calculations revealed that the LDOS exhibits symmetric peaks at energies $\omega = \pm b$, where $b$ scales with the coupling strength. For weak coupling, the peaks remain close to the original zero-energy state, while stronger coupling leads to significant hybridization and energy shifts. Notably, the spatial localization of the edge states persists even under strong coupling, underscoring their topological protection.

These findings illustrate the delicate balance between symmetry preservation and inter-chain interactions in topological systems. The results provide a framework for understanding and engineering coupled topological systems, with potential applications in quantum devices and materials design. Future work could explore the effects of disorder or asymmetric coupling on the LDOS, further enriching our understanding of topological heterostructures.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## Appendix A: Green function in the frequency space and Dyson equation

For a time dependent problem, consider the operator $[ic\partial_t - \hat{\mathcal{L}}]\psi = f(t)$, to which we can associate the Green's function, $[ic\partial_t - \hat{\mathcal{L}}]G(x,t,x',t') = \delta(x-x')\delta(t-t')$. If $\hat{\mathcal{L}}$ is explicitly independent of $t$, we can perform a Fourier transform

$$G(x,x',t-t') = \frac{1}{2\pi}\int \mathrm{d}\omega\, G(x,x',\omega)e^{-i\omega(t-t')},$$

so that we obtain $[c\omega - \hat{\mathcal{L}}]G(x,x',\omega) = \delta(x-x')$. If we know the solution of the equation at a given time, we can determine it at any time since it's first order. We assume

$$\psi(x,t) = i\int G(x,t,x',t')\psi(x',t)\,\mathrm{d}x',$$

and expanding the solution in the $\hat{\mathcal{L}}$ eigenfunctions

$$\psi(x,t) = \sum_n a_n(t)\varphi_n(x).$$

To find $a_n$,

$$\begin{aligned}[ic\partial_t - \hat{\mathcal{L}}]\psi &= [ic\partial_t - \hat{\mathcal{L}}]\sum_n a_n(t)\varphi_n(x)\\ &= \sum_n [ic\partial_t a_n(t)\varphi_n(x) - a_n(t)\omega_n\varphi_n(x)]\\ &= \sum_n [ic\partial_t a_n(t) - \omega_n a_n(t)]\varphi_n(x) = 0.\end{aligned}$$

Thus,

$$\begin{gathered}ic\partial_t a_n(t) = \omega_n a_n(t), \quad \text{with} \quad c=1,\\ a_n(t) = a_n(t_0)e^{-iw_n(t-t_0)},\end{gathered}$$

where

$$a_n(t_0) = \int \varphi_n^*(x)\psi(x,t_0)\,\mathrm{d}x.$$

Taking $\psi(x,t)$, $a_n(t)$, and $a_n(t_0)$,

$$\begin{aligned}\psi(x,t) &= \sum_n \int \varphi_n^*(x')\psi(x',t_0)e^{-i\omega_n(t-t_0)}\varphi_n(x)\,\mathrm{d}x'\\ &= \int \mathrm{d}x' \sum_n e^{-i\omega_n(t-t_0)}\varphi_n(x)\varphi_n^*(x')\psi(x',t_0)\\ &\equiv i\int G(x,x',t,t_0)\psi(x',t_0)\,\mathrm{d}x',\end{aligned}$$

which implies

$$G(x,x',t,t') = -i\sum_n \varphi_n(x)\varphi_n^*(x')e^{-i\omega_n(t-t')}.$$

We define the retarded Green's function

$$G^r(x,x',t,t') = \theta(t-t')G(x,x',t,t'),$$

where the Heaviside step function is

$$\theta(t-t') = \frac{i}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}p\,\frac{e^{-ip(t-t')}}{p+i\eta},$$

which equals 1 for this case where $t > t'$. Now,

$$
\begin{aligned}
G^r(x,x',\omega) &= \int \mathrm{d}(t-t')\, e^{i\omega(t-t')} G^r(x,x',t-t'),\\
&= (-i)\int \mathrm{d}(t-t')\, e^{i\omega(t-t')}\left(\frac{i}{2\pi}\right)\\
&\quad\times \int_{-\infty}^{\infty} \mathrm{d}p \frac{e^{ip(t-t')}}{p+i0^+}\sum_n \varphi_n(x)\varphi_n^*(x')e^{-i\omega_n(t-t')},\\
&= \sum_n \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}p}{p+i0^+}\\
&\quad\times \int \mathrm{d}(t-t')e^{i(\omega-\omega_n-p)(t-t')}\varphi_n(x)\varphi_n^*(x'),\\
&= -\sum_n \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}p}{p+i0^+}2\pi\delta(\omega-\omega_n-p)\varphi_n(x)\varphi_n^*(x').
\end{aligned}
$$

Then, the spectral relation for the retarded Green's function is

$$
G^r(x,x',\omega) = \sum_n \frac{\varphi_n(x)\varphi_n^*(x')}{\omega-\omega_n+i0^+}.
$$

By this result, using the Plemelj formula

$$
\lim_{\epsilon\to 0^+}\frac{1}{\omega-\omega_n+i\epsilon} = \mathcal{P}\left(\frac{1}{\omega-\omega_n}\right) - i\pi\delta(\omega-\omega_n),
$$

it follows that

$$
\begin{aligned}
&G^r(x,x',\omega)\\
&= \sum_n \mathcal{P}\left(\frac{\varphi_n(x)\varphi_n^*(x')}{\omega-\omega_n}\right) - i\pi\sum_n \varphi_n(x)\varphi_n^*(x')\delta(\omega-\omega_n).
\end{aligned}
$$

Therefore, evaluating in $x = x'$

$$
\varphi_n(x)\varphi_n^*(x') \to |\varphi_n(x)|^2
$$

and the imaginary part of the retarded Green function reads as

$$
\mathrm{Im}\left\{G^r(x,x,\omega)\right\} = -\pi\sum_n |\varphi_n(x)|^2\delta(\omega-\omega_n).
$$

By this, the local density of states is

$$
\rho(x,\omega) = -\frac{1}{\pi}\mathrm{Im}\left\{G^r(x,x,\omega)\right\} = \sum_n |\varphi_n(x)|^2\delta(\omega-\omega_n).
$$

Now, if for a initially unperturbed Hamiltonian problem, the retarded Green function is obtained such that

$$
(E-\hat{H}_0)\,\hat{g} = \mathbb{I}.
$$

To obtain the retarded Green function for the perturbed Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}$, it must satisfy the equation

$$
(E-\hat{H})\hat{G} = \mathbb{I}.
$$

Thereby, substituting the total Hamiltonian into the first equation yields

$$
(E-\hat{H}_0-\hat{V})\hat{G} = \mathbb{I}.
$$

Recognizing the presence of the unperturbed Green function operator, we may rewrite this expression as

$$
(\hat{g}^{-1}-\hat{V})\hat{G} = \mathbb{I},
$$

where we have made use of the inverse relationship $\hat{g}^{-1} = (E-\hat{H}_0)$, derived from the unperturbed system. To obtain an explicit expression for $\hat{G}$, we multiply both sides of the equation by $\hat{g}$ from the left

$$
\hat{g}(\hat{g}^{-1}-\hat{V})\hat{G} = \hat{g}\mathbb{I},
$$

Expanding the left-hand side and simplifying gives

$$
(\mathbb{I}-\hat{g}\hat{V})\hat{G} = \hat{g}.
$$

The full Green's function can now be isolated, resulting in the Dyson equation

$$
\hat{G} = \hat{g} + \hat{g}\hat{V}\hat{G}.
$$

## Appendix B: SSH model

Applying the inverse Fourier's transform

$$
C_{A,n} = \frac{1}{\sqrt{N}}\sum_k e^{-ikna}a_k, \quad C_{B,n} = \frac{1}{\sqrt{N}}\sum_k e^{-ikna}b_k,
$$

to the Hamiltonian

$$
H = \sum_n (t+\delta)C^\dagger_{A,n}C_B,n + \sum_n (t-\delta)C^\dagger_{A,n+1}C_{B,n} + h.c.
$$

For the first term,

$$
C^\dagger_{A,n}C_{B,n} = \frac{1}{N}\sum_{k'k} e^{i(k'-k)na}a^\dagger_{k'}b_k,
$$

and for the second,

$$
\begin{aligned}
C^\dagger_{A,n+1}C_{B,n} &= \frac{1}{N}\sum_{k'k} e^{i(k'(n+1)-kn)a}a^\dagger_{k'}b_k\\
&= \frac{1}{N}\sum_{k'k} e^{ik'a}e^{i(k'-k)na}a^\dagger_{k'}b_k.
\end{aligned}
$$

Replacing for the first term in the hamiltonian,

$$\sum_n vC^\dagger_{A,n}C_{B,n} = \frac{1}{N}\sum_{k'k}\sum_n ve^{i(k'-k)na}a^\dagger_{k'}b_k,$$

and using the discrete orthogonality relation, $\sum_n e^{i(k'-k)na} = N\delta_{kk'}$, which states that the plane waves with different momentum are orthogonal over the lattice,

$$\sum_n vC^\dagger_{A,n}C_{B,n} = \sum_n va^\dagger_k b_k.$$

And for the second term in the Hamiltonian,

$$\sum_k wC^\dagger_{A,n+1}C_{B,n} = \frac{1}{N}\sum_n\sum_{k'k} we^{ik'a}e^{i(k'-k)na}a^\dagger_{k'}b_k,$$
$$= \sum_n we^{ika}a^\dagger_k b_k.$$

Replacing,

$$H = \sum_k va^\dagger_k b_k + \sum_k we^{ika}a^\dagger_k b_k + h.c.$$
$$= \sum_k v(a^\dagger_k b_k + b^\dagger_k a_k) + \sum_k w(e^{ika}a^\dagger_k b_k + e^{-ika}b^\dagger_k a_k),$$

in matrix form,

$$H = \sum_k (a^\dagger_k, b^\dagger_k)\begin{pmatrix} 0 & v+we^{ika} \\ v+we^{-ika} & 0\end{pmatrix}\begin{pmatrix} a_k \\ b_k\end{pmatrix},$$

defining the spinor $\psi_k = \begin{pmatrix} a_k \\ b_k\end{pmatrix}$,

$$H = \sum_k \psi^\dagger_k \begin{pmatrix} 0 & v+w(\cos ka + i\sin ka) \\ v+w(\cos ka - i\sin ka) & 0\end{pmatrix}\psi_k,$$
$$= \sum_k \psi^\dagger_k \begin{pmatrix} 0 & v+w(\cos ka + i\sin ka) \\ v+w(\cos ka - i\sin ka) & 0\end{pmatrix}\psi_k,$$
$$= \sum_k \psi^\dagger_k\left[\begin{pmatrix} 0 & v+w\cos ka \\ v+w\cos ka & 0\end{pmatrix} - \begin{pmatrix} 0 & -w\sin ka \\ iw\sin ka & 0\end{pmatrix}\right]\psi_k,$$
$$= \sum_k \psi^\dagger_k\left[(v+w\cos ka)\sigma_x - w\sin ka\sigma_y\right]\psi_k.$$

Now, making an expansion around the high symmetry point $ka = \pi$, $ka = \pi + \delta ka$,

$$v + w\cos ka = v + w\cos(\pi + \delta ka),$$

using the properties of angle addition

$$\cos(\pi+\delta ka) = \cos\pi\cos\delta ka - \sin\pi\sin\delta ka = -\cos\delta ka,$$
$$v + w\cos ka = v - w\cos a\delta k,$$

and using the Taylor expansion,

$$\cos a\delta k \approx 1 - \frac{a^2(\delta k)^2}{2},$$
$$v - w\cos a\delta k = v - w + w\frac{a^2(\delta k)^2}{2},$$
$$= t + \delta - t + \delta + (t-\delta)\frac{a^2(\delta k)^2}{2},$$
$$\equiv 2\delta + (t-\delta)\frac{a^2k^2}{2},$$

with the expansion, the Hamiltonian becomes

$$H = \sum_k \psi^\dagger_k\left[\left(2\delta + (t-\delta)\frac{a^2k^2}{2}\right)\sigma_x + \sigma_y(t-\delta)ka\right]\psi_k.$$

It can be verified that $\sigma_y H\sigma_y = -H$ [10], which implies that the SSH model possesses chiral symmetry. This result reinforces its equivalence to a Dirac Hamiltonian in the low-energy limit.

To make a better analysis, we can make an unitary transformation ($\sigma_x \to \sigma_z$, $\sigma_y \to \sigma_x$, $\sigma_z \to \sigma_y$) to get a partially diagonal Hamiltonian,

$$H = \sum_k \psi^\dagger_k\left[(t-\delta)ak\sigma_x + \left(2\delta + (t-\delta)k^2\frac{a^2}{2}\right)\sigma_z\right]\psi_k.$$

Noting $m_1 = 2|\delta|$ and $m_2 = (t-\delta)\frac{a^2}{2}$,

$$H = \sum_k \psi^\dagger_k\left[(t-\delta)ak\sigma_x + (m_1 + m_2k^2)\sigma_z\right]\psi_k.$$

For simplicity, we take

$$H_k = (t-\delta)ak\sigma_x + (-m_1 + m_2k^2)\sigma_z.$$

For $\delta < 0$, and using $k$ as $-i\partial_x$, the eigenvalues equation is,

$$ua(-i\partial_x)\sigma_x\psi + (-m_1 + m_2(-\partial_{xx}))\sigma_z\psi = E\psi,$$

and to find a zero energy solution,

$$ua(-i\partial_x)\sigma_x\psi = (m_1 - m_2(-\partial_{xx}))\sigma_z\psi,$$
$$(i\sigma_x)(-i\partial_x)\sigma_x\psi = \frac{1}{ua}(m_1 - m_2(-\partial_{xx}))(i\sigma_x)\sigma_z\psi,$$
$$\partial_x\psi = \frac{1}{ua}(m_1 - m_2(-\partial_{xx}))i\sigma_x\sigma_z\psi.$$

Since $\sigma_x\sigma_z = -i\sigma_y \longrightarrow i\sigma_x\sigma_z = \sigma_y$,

$$\partial_x\psi = \frac{1}{ua}(m_1 + m_2\partial_{xx})\sigma_y\psi.$$

Now, taking $\psi(x) = \phi e^{\lambda x}$ such that $\sigma_y\phi = \eta\phi$,

$$\phi = \begin{pmatrix} 1 \\ i\eta\end{pmatrix},$$
$$\partial_x\psi = \lambda\phi e^{\lambda x}, \quad \partial_{xx}\psi = \lambda^2\phi e^{\lambda x},$$

and replacing,

$$\lambda\phi e^{\lambda x} = \frac{1}{ua}(m_1 + m_2\lambda^2)\sigma_y \phi e^{\lambda x},$$
$$\lambda\phi = \frac{1}{ua}(m_1 + m_2\lambda^2)\sigma_y \phi.$$

Using the eigenvalues of $\sigma_y$, $\eta = \pm 1$,

$$\begin{aligned}\lambda\phi &= \frac{\eta}{ua}(m_1 + m_2\lambda^2)\phi,\\ \lambda &= \frac{\eta}{ua}(m_1 + m_2\lambda^2),\\ \lambda^2\frac{m^2\eta^2}{ua} - \lambda &+ \frac{\eta}{ua}m_1 = 0,\\ \lambda^2 m^2 - \alpha\lambda &+ m_1 = 0,\end{aligned}$$

with $A = \frac{ua}{\eta} = ua\eta$. Using the quadratic equation,

$$\lambda = \frac{A \pm \sqrt{A^2 - 4m_1m_2}}{2m^2} = \frac{A}{2m^2}\left[1 \pm \sqrt{1 - \frac{4m_1m_2}{A^2}}\right].$$

Let the boundary conditions be $\psi(0) = \psi(\infty) = 0$. So, for $\psi(x) = \phi e^{\lambda x}$, $\lambda$ must satisfy $\lambda < 0$ and $\eta < 0$. We define $\lambda = -d_\pm^{-1}$, which is a penetration length,

$$d_\pm^{-1} = \frac{|A|}{2m^2}\left[1 \pm \sqrt{1 - \frac{4m_1m_2}{A^2}}\right] > 0,$$

thus, if $x > 0$, $\psi(x)$ behaves as $e^{-x/d_\pm}$. Hence, the solution that satisfies the boundary conditions for $x > 0$ are

$$\psi(x) = c\left[e^{-x/d_+} - e^{-x/d_-}\right]\begin{pmatrix}1\\ -i\end{pmatrix}.$$

Analogously, for $x < 0$ the solution that satisfies the boundary conditions $\psi(0) = \psi(-\infty) = 0$ is

$$\psi(x) = c\left[e^{x/d_+} - e^{x/d_-}\right]\begin{pmatrix}1\\ -i\end{pmatrix}.$$

For both cases, the normalization constant $c$ is obtained as

$$\begin{aligned}\langle\psi|\psi\rangle &= c^2\langle\phi|\phi\rangle\int_0^\infty \left[e^{-x/d_+} - e^{-x/d_-}\right]^2 \mathrm{d}x\\ &= c^2\langle\phi|\phi\rangle\int_{-\infty}^0 \left[e^{x/d_+} - e^{x/d_-}\right]^2 \mathrm{d}x = 1.\end{aligned}$$

by $\langle\phi|\phi\rangle = 1 + \eta^2 = 2$ and

$$\left[e^{-x/d_+} - e^{-x/d_-}\right]^2 = e^{-2x/d_+} - 2e^{-x(1/d_+ + 1/d_-)} + e^{-2x/d_-}.$$

The integrals are

$$\begin{aligned}\int_0^\infty e^{-2x/d_+}\, dx &= \frac{d_+}{2},\\ \int_0^\infty e^{-2x/d_-}\, dx &= \frac{d_-}{2},\\ \int_0^\infty e^{-x(1/d_+ + 1/d_-)}\, dx &= \frac{1}{\frac{1}{d_+} + \frac{1}{d_-}} = \frac{d_+d_-}{d_+ + d_-}.\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^\infty \left[e^{-x/d_+} - e^{-x/d_-}\right]^2 \mathrm{d}x &= \frac{d_+}{2} + \frac{d_-}{2} - 2\frac{d_+d_-}{d_+ + d_-}\\ &= \frac{d_+ + d_-}{2} - \frac{2d_+d_-}{d_+ + d_-}\\ &= \frac{(d_+ + d_-)^2 - 4d_+d_-}{2(d_+ + d_-)}\\ &= \frac{d_+^2 + d_-^2 - 2d_+d_-}{2(d_+ + d_-)}\\ &= \frac{(d_+ - d_-)^2}{2(d_+ + d_-)}.\end{aligned}$$

Using this result,

$$\langle\psi|\psi\rangle = 2c^2\frac{(d_+ - d_-)^2}{2(d_+ + d_-)} = 1,$$

the value of the constant $c$ is

$$c = \frac{\sqrt{d_+ + d_-}}{|d_+ - d_-|}$$

and the normalized edge states are

$$\begin{aligned}\psi(x) &= \frac{\sqrt{d_+ + d_-}}{|d_+ - d_-|}\left[e^{-x/d_+} - e^{-x/d_-}\right]\begin{pmatrix}1\\ -i\end{pmatrix}, \quad x > 0\\ \psi(x) &= \frac{\sqrt{d_+ + d_-}}{|d_+ - d_-|}\left[e^{x/d_+} - e^{x/d_-}\right]\begin{pmatrix}1\\ -i\end{pmatrix}, \quad x < 0.\end{aligned}$$

## Appendix C: Solving the Dyson equation for the coupled system

Starting from the right and left edge states wavefunctions

$$\begin{aligned}|\psi_R\rangle &= c_R\left(e^{-x/d_+} - e^{-x/d_-}\right)|\phi\rangle,\\ |\psi_L\rangle &= c_L\left(e^{x/\xi_+} - e^{x/\xi_-}\right)|\phi\rangle,\\ \text{with } \ |\phi\rangle &= \begin{pmatrix}1\\ -i\end{pmatrix}.\end{aligned}$$

Taking into account that these states are zero energy modes, we can obtain the Green function through the eigenfunction method for the right edge state as

$$
\begin{aligned}
g_{RR}(x,x',\omega) =& \frac{|\psi_R(x)\rangle\langle\psi_R(x')|}{\omega+i0^+} \\
=& \frac{c_R^2}{\omega+i0^+}\left[e^{-x/d_+}-e^{-x/d_-}\right] \\
& \times\left[e^{-x'/d_+}-e^{-x'/d_-}\right]|\phi\rangle\langle\phi|.
\end{aligned}
$$

It can be seen that for matrix $|\phi\rangle\langle\phi|$,

$$
|\phi\rangle\langle\phi| = \begin{pmatrix}1\\-i\end{pmatrix}\begin{pmatrix}1 & i\end{pmatrix} = \begin{pmatrix}1 & i\\-i & 1\end{pmatrix}
$$

and defining $\hat{A} = |\phi\rangle\langle\phi|$ the Green function for the right edge state is

$$
\begin{aligned}
g_{RR}(x,x',\omega) =& \frac{c_R^2}{\omega+i0^+}\left[e^{-x/d_+}-e^{-x/d_-}\right] \\
& \times\left[e^{-x'/d_+}-e^{-x'/d_-}\right]\hat{A}
\end{aligned}
$$

Analogously, for the left edge state, the Green function is obtained as

$$
\begin{aligned}
g_{LL}(x,x',\omega) =& \frac{|\psi_L(x)\rangle\langle\psi_L(x')|}{\omega+i0^+} \\
=& \frac{c_L^2}{\omega+i0^+}\left[e^{x/\xi_+}-e^{x/\xi_-}\right] \\
& \times\left[e^{x'/\xi_+}-e^{x'/\xi_-}\right]|\phi\rangle\langle\phi| \\
=& \frac{c_L^2}{\omega+i0^+}\left[e^{x/\xi_+}-e^{x/\xi_-}\right] \\
& \times\left[e^{x'/\xi_+}-e^{x'/\xi_-}\right]\hat{A}.
\end{aligned}
$$

For notation convenience, we define

$$
\begin{aligned}
f(x,x') &= c_R^2\left[e^{-x/d_+}-e^{-x/d_-}\right]\left[e^{-x'/d_+}-e^{-x'/d_-}\right], \\
h(x,x') &= c_L^2\left[e^{x/\xi_+}-e^{x/\xi_-}\right]\left[e^{x'/\xi_+}-e^{x'/\xi_-}\right],
\end{aligned}
$$

which allows us to express the Green's functions compactly as

$$
\begin{aligned}
g_{RR}(x,x',\omega) &= \frac{f(x,x')}{\omega+i0^+}\hat{A}, \\
g_{LL}(x,x',\omega) &= \frac{h(x,x')}{\omega+i0^+}\hat{A}.
\end{aligned}
$$

Now, since in the unperturbed case there is no coupling between the chains, $g_{LR} = g_{RL} = 0$. The unperturbed Green function for the total system is

$$
\hat{g}(x,x',\omega) = \begin{pmatrix} g_{RR}(x,x',\omega) & 0 \\ 0 & g_{LL}(x,x',\omega)\end{pmatrix}.
$$

From now on, for all Green's functions in both the unperturbed and perturbed cases, we will assume by notation that $g_{ij} = g_{ij}(x,x',\omega)$ with $i,j = L,R$. To study the interaction between these edge states, we introduce a tunneling potential with isotropic coupling strength $t$,

$$
\hat{V} = \begin{pmatrix} 0 & V_{RL} \\ V_{LR} & 0 \end{pmatrix} = \begin{pmatrix} \hat{0}_{2x2} & t\mathbb{I}_{2x2} \\ t\mathbb{I}_{2x2} & \hat{0}_{2x2}\end{pmatrix},
$$

where we have defined $V_{RL} = V_{LR} = t$. This perturbation maintains the chiral symmetry of the two chains with the chiral operator $\Gamma = \sigma_y \otimes \sigma_y$,

$$
\begin{aligned}
\Gamma = \sigma_y\otimes\sigma_y &= \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\otimes\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\hat{V}\Gamma &= \begin{pmatrix} 0 & t\mathbb{I} \\ t\mathbb{I} & 0\end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\
&= t\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\Gamma(\hat{V}\Gamma) &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} t \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
&= t\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = -\hat{V}.
\end{aligned}
$$

Thus, $\Gamma\hat{V}\Gamma$ = -$\hat{V}$ and the chiral symmetry for the whole Hamiltonian $H_L\otimes\mathbb{I}+\mathbb{I}\otimes H_R$ is preserved.

Now, for the perturbed Green function $\hat{G}$,

$$
\hat{G} = \begin{pmatrix} G_{RR} & G_{RL} \\ G_{LR} & G_{LL}\end{pmatrix}.
$$

The Dyson equation is

$$
\begin{aligned}
&\hat{G} = \hat{g} + \hat{g}\hat{V}\hat{G} \\
&= \begin{pmatrix} g_{RR} & 0 \\ 0 & g_{LL}\end{pmatrix} + \begin{pmatrix} g_{RR} & 0 \\ 0 & g_{LL}\end{pmatrix}\begin{pmatrix} 0 & V_{RL} \\ V_{LR} & 0\end{pmatrix}\begin{pmatrix} G_{RR} & G_{RL} \\ G_{LR} & G_{LL}\end{pmatrix} \\
&= \begin{pmatrix} g_{RR} & 0 \\ 0 & g_{LL}\end{pmatrix} + \begin{pmatrix} g_{RR}V_{RL}G_{LR} & g_{RR}V_{RL}G_{LL} \\ g_{LL}V_{LR}G_{RR} & g_{LL}V_{LR}G_{RL}\end{pmatrix}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\hat{G} &= \begin{pmatrix} G_{RR} & G_{RL} \\ G_{LR} & G_{LL}\end{pmatrix} \\
&= \begin{pmatrix} g_{RR}+g_{RR}V_{RL}G_{LR} & g_{RR}V_{RL}G_{LL} \\ g_{LL}V_{LR}G_{RR} & g_{LL}+g_{LL}V_{LR}G_{RL}\end{pmatrix}.
\end{aligned}
$$

As we can see, the coupled equations for $G_{RR}$ and $G_{LR}$ are

$$\begin{aligned}G_{RR} &= g_{RR} + g_{RR}V_{RL}G_{LR},\\ G_{LR} &= g_{LL}V_{LR}G_{RR}.\end{aligned}$$

Solving for $G_{RR}$ yields

$$\begin{aligned}G_{RR} &= g_{RR} + g_{RR}V_{RL}G_{LR}\\ &= g_{RR} + g_{RR}V_{RL}g_{LL}V_{LR}G_{RR}.\end{aligned}$$

By simple matrix algebra,

$$\begin{aligned}g_{RR} &= G_{RR} - g_{RR}V_{RL}g_{LL}V_{LR}G_{RR}\\ &= (\mathbb{I} - g_{RR}V_{RL}g_{LL}V_{LR})G_{RR},\end{aligned}$$

and taking into account that $V_{RL} = V_{LR} = t\mathbb{I}$,

$$g_{RR} = (\mathbb{I} - t^2 g_{RR}g_{LL})G_{RR}.$$

By this, the perturbed Green function for the right edge state is

$$G_{RR} = (\mathbb{I} - t^2 g_{RR}g_{LL})^{-1} g_{RR}.$$

To derive the explicit form of $G_{RR}$, we proceed to examine the following results. First, for the product $g_{RR}g_{LL}$,

$$g_{RR}g_{LL} = \frac{f(x,x')}{\omega + i0^+}\hat{A}\frac{h(x,x')}{\omega + i0^+}\hat{A} = \frac{f(x,x')h(x,x')}{(\omega + i0^+)^2}\hat{A}^2.$$

Observing that

$$\hat{A}^2 = \begin{pmatrix} 1 & i \\ -i & 1\end{pmatrix}\begin{pmatrix} 1 & i \\ -i & 1\end{pmatrix} = \begin{pmatrix} 2 & 2i \\ -2i & 2\end{pmatrix} = 2\begin{pmatrix} 1 & i \\ -i & 1\end{pmatrix} = 2\hat{A},$$

we obtain

$$g_{RR}g_{LL} = \frac{2f(x,x')h(x,x')}{(\omega + i0^+)^2}\hat{A}.$$

Finally, observing that $\hat{A} = \mathbb{I} - \sigma_y$, the matrix $\mathbb{I} - t^2 g_{RR}g_{LL}$ is given by

$$\begin{aligned}\mathbb{I} - t^2 g_{RR}g_{LL} &= \mathbb{I} - t^2\frac{2f(x,x')h(x,x')}{(\omega + i0^+)^2}\hat{A}\\ &= \mathbb{I} - \frac{2t^2 f(x,x')h(x,x')}{(\omega + i0^+)^2}(\mathbb{I} - \sigma_y)\\ &= \left[1 - \frac{2t^2 f(x,x')h(x,x')}{(\omega + i0^+)^2}\right]\mathbb{I} + \frac{2t^2 f(x,x')h(x,x')}{(\omega + i0^+)^2}\sigma_y\\ &= m(x,x',\omega)\mathbb{I} + n(x,x',\omega)\sigma_y.\end{aligned}$$

Where we defined

$$\begin{aligned}m(x,x',\omega) &= 1 - \frac{2t^2 f(x,x')h(x,x')}{(\omega + i0^+)^2}\\ n(x,x',\omega) &= \frac{2t^2 f(x,x')h(x,x')}{(\omega + i0^+)^2},\end{aligned}$$

from it, follows that $m(x,x',\omega) = 1 - n(x,x',\omega)$. Now, since we wish to obtain $(\mathbb{I} - t^2 g_{RR}g_{LL})^{-1}$ through the transpose matrix property of the Pauli matrix $\sigma_y^T = -\sigma_y$ and its hermiticity $\sigma_y^2 = \mathbb{I}$, it can be seen that

$$\begin{aligned}&(\mathbb{I} - t^2 g_{RR}g_{LL})(\mathbb{I} - t^2 g_{RR}g_{LL})^T\\ &= [m(x,x',\omega)\mathbb{I} + n(x,x',\omega)\sigma_y][m(x,x',\omega)\mathbb{I} + n(x,x',\omega)\sigma_y]^T\\ &= [m(x,x',\omega)\mathbb{I} + n(x,x',\omega)\sigma_y][m(x,x',\omega)\mathbb{I} - n(x,x',\omega)\sigma_y]\\ &= m^2(x,x',\omega)\mathbb{I}^2 + m(x,x',\omega)n(x,x',\omega)\sigma_y\mathbb{I}\\ &\quad - m(x,x',\omega)n(x,x',\omega)\mathbb{I}\sigma_y - n^2(x,x',\omega)\sigma_y^2\\ &= m^2(x,x',\omega)\mathbb{I} + m(x,x',\omega)n(x,x',\omega)\sigma_y\\ &\quad - m(x,x',\omega)n(x,x',\omega)\sigma_y - n^2(x,x',\omega)\mathbb{I}\\ &= m^2(x,x',\omega)\mathbb{I} - n^2(x,x',\omega)\mathbb{I}\\ &= [m^2(x,x',\omega) - n^2(x,x',\omega)]\mathbb{I}.\end{aligned}$$

Thus, the matrix $(\mathbb{I} - t^2 g_{RR}g_{LL})^{-1}$ is nothing but

$$\begin{aligned}(\mathbb{I} - t^2 g_{RR}g_{LL})^{-1} &= \frac{(\mathbb{I} - t^2 g_{RR}g_{LL})^T}{m^2(x,x',\omega) - n^2(x,x',\omega)}\\ &= \frac{(\mathbb{I} - t^2 g_{RR}g_{LL})^T}{1 - 2n(x,x',\omega)}.\end{aligned}$$

As a consequence, the perturbed Green function $G_{RR}$ can be expressed as

$$\begin{aligned}G_{RR} &= (\mathbb{I} - t^2 g_{RR}g_{LL})^{-1} g_{RR}\\ &= \frac{(\mathbb{I} - t^2 g_{RR}g_{LL})^T}{1 - 2n(x,x',\omega)}\frac{f(x,x')}{\omega + i0^+}\hat{A}\\ &= \frac{[m(x,x',\omega)\mathbb{I} + n(x,x',\omega)\sigma_y]^T}{1 - 2n(x,x',\omega)}\frac{f(x,x')}{\omega + i0^+}\hat{A}\\ &= \frac{[m(x,x',\omega)\mathbb{I} - n(x,x',\omega)\sigma_y]}{1 - 2n(x,x',\omega)}\frac{f(x,x')}{\omega + i0^+}\hat{A}.\end{aligned}$$

Now, observing that

$$\begin{aligned}&[m(x,x',\omega)\mathbb{I} - n(x,x',\omega)\sigma_y]\hat{A}\\ &= [m(x,x',\omega)\mathbb{I} - n(x,x',\omega)\sigma_y](\mathbb{I} - \sigma_y)\\ &= m(x,x',\omega)\mathbb{I}^2 - n(x,x',\omega)\sigma_y\mathbb{I}\\ &\quad - m(x,x',\omega)\mathbb{I}\sigma_y + n(x,x',\omega)\sigma_y^2\\ &= [m(x,x',\omega) + n(x,x',\omega)]\mathbb{I}\\ &\quad - [m(x,x',\omega) + n(x,x',\omega)]\sigma_y\\ &= [m(x,x',\omega) + n(x,x',\omega)](\mathbb{I} - \sigma_y)\\ &= [1 - n(x,x',\omega) + n(x,x',\omega)](\mathbb{I} - \sigma_y)\\ &= \mathbb{I} - \sigma_y\\ &= \hat{A}.\end{aligned}$$

The perturbed Green function is

$$G_{RR} = \frac{1}{1-2n(x,x',\omega)}\frac{f(x,x')}{\omega+i0^+}\hat{A}.$$

And, explicitly writing $n(x,x',\omega) = \frac{2t^2 f(x,x')h(x,x')}{(\omega+i0^+)^2}$,

$$\begin{aligned} G_{RR} &= \frac{1}{1-\frac{4t^2 f(x,x')h(x,x')}{(\omega+i0^+)^2}}\frac{f(x,x')}{\omega+i0^+}\hat{A} \\ &= \frac{f(x,x')(\omega+i0^+)\hat{A}}{(\omega+i0^+)^2 - 4t^2 f(x,x')h(x,x')} \\ &= \frac{f(x,x')(\omega+i0^+)\hat{A}}{(\omega+i0^+)^2 - b^2(x,x')}, \end{aligned}$$

where $b^2(x,x') = 4t^2 f(x,x')h(x,x')$. Finally, the trace of $G_{RR}$ at $x = x'$ is

$$\begin{aligned} &\mathrm{Tr}(G_{RR}(x,x',\omega)) \\ &= \frac{f(x,x)(\omega+i0^+)\mathrm{Tr}(\hat{A})}{(\omega+i0^+)^2 - b^2(x,x)} \\ &= \frac{f(x,x)(\omega+i0^+)}{(\omega+i0^+)^2 - b^2(x,x)}\mathrm{Tr}\left(\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\right) \\ &= \frac{2f(x,x)(\omega+i0^+)}{(\omega+i0^+)^2 - b^2(x,x)}. \end{aligned}$$

Which allows us to compute the LDOS for the right edge state after the coupling as

$$\begin{aligned} \rho(x,\omega) &= -\frac{1}{\pi}\mathrm{Im}\{\mathrm{Tr}(G_{RR}(x,x,\omega))\} \\ &= -\frac{1}{\pi}\mathrm{Im}\left\{\frac{2f(x,x)(\omega+i0^+)}{(\omega+i0^+)^2 - b^2(x,x)}\right\}. \end{aligned}$$

Looking at $G^R(x,x)$, it can be written as

$$\begin{aligned} &\frac{2f(x,x)(\omega+i0^+)}{(\omega+i0^+)^2 - b^2(x,x)} \\ &= f(x,x)\left(\frac{1}{(\omega+i0^+) - b(x,x)} + \frac{1}{(\omega+i0^+) + b(x,x)}\right). \end{aligned}$$

Explicitly,

$$\begin{aligned} &f(x,x)\left(\frac{1}{(\omega+i0^+) - b(x,x)} + \frac{1}{(\omega+i0^+) + b(x,x)}\right) \\ &= f(x,x)\left(\frac{(\omega+i0^+) - b(x,x) + (\omega+i0^+) + b(x,x)}{[(\omega+i0^+) - b(x,x)][(\omega+i0^+) + b(x,x)]}\right) \\ &= f(x,x)\left(\frac{2(\omega+i0^+)}{(\omega+i0^+)^2 - b^2(x,x)}\right), \end{aligned}$$

as we wanted. As a result, by the Plemelj formula,

$$\lim_{\epsilon\to 0^+}\frac{1}{\omega - b + i\epsilon} = \mathcal{P}\left(\frac{1}{\omega - b}\right) - i\pi\delta(\omega - b),$$

the LDOS for the right edge state after the coupling is

$$\rho(x,\omega) = f(x,x)\left[\delta(\omega-b) + \delta(\omega+b)\right],$$

where the original zero-energy edge state has split into two new states at energies $\pm b = \pm 2t\sqrt{f(x,x)h(x,x)}$.