LDOS for a coupled system of SSH chains

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Abstract: The Su-Schrieffer-Heeger (SSH) model is a one-dimensional topological insulator, exhibiting robust edge states protected by chiral symmetry. In this work, we investigate the local density of states (LDOS) for both a single SSH chain and a coupled system of two SSH chains using Green's function methods. For the single-chain case, we derive the LDOS analytically, demonstrating the presence of a zero-energy edge state localized at the boundary. For the coupled system, we employ the Dyson equation to compute the perturbed Green's function and numerically evaluate the LDOS under varying coupling strengths. Our results reveal that the interaction between the edge states of the two chains leads to the destruction of the zero-energy mode and the emergence of hybridized states, with the LDOS exhibiting symmetric peaks around the origin. The evolution of these peaks, from weak to strong coupling, highlights the transition from well-localized edge states to delocalized bulk resonances, while preserving the system's chiral symmetry. These findings provide insights into the behavior of topological edge states in coupled systems and their potential applications in designing heterostructures and quantum devices.

I. INTRODUCTION

Numerous systems in nature —and others proposed theoretically— display topological properties that manifest in distinct forms. For example, the Quantum Hall Effect [1], where the topology displays through the Quantum Hall conductance; in photonic crystals [2], as a bulk-boundary correspondence; or in topological superconductors [3], as emergent modes at the edges of the system that come from particle-hole symmetry.

The SSH model, which also shows topological behavior, was first used by J. R. Schrieffer and A. J. Heeger to theoretically model the soliton formation in polyacetylene [4], showing how topological properties emerge from alternating hopping amplitudes in a tight-binding chain. For the SSH system, topology manifests as edge states, that are highly dependent on the symmetry of the system and its preservation. This prototype is useful at the modeling of molecules and materials by the construction of heterostructures [5].

One way to further comprehend the emergent topological attributes is through the local density of states (LDOS), which gives out information about the spatial and energetical available states of the system. The LDOS can be computed using Green functions: a very powerful tool when it comes to solving differential equations. Characterizing the distribution of the states of a system is often used to design and understand devices and materials with specific interactions at their surfaces [6, 7].

In this work, we will use Green functions to compute the LDOS of a single topological SSH chain, and afterwards, the LDOS of two coupled topological SSH chains, so we can break down the nature of the interaction and the behavior of the edge states.

II. GREEN FUNCTION METHODS AND DYSON EQUATION

Consider a non-homogeneous linear differential equation of the form

$$(\lambda - \hat{\mathcal{L}})\psi = f, \tag{1}$$

where $\hat{\mathcal{L}}$ is a linear differential operator, λ is a complex parameter, f is a source term, and ψ is the unknown function to be determined. Also, the eigenfunctions ϕ of $\hat{\mathcal{L}}$, satisfying $\hat{\mathcal{L}}\phi = \lambda\phi$, form a complete set. Taking this into account the formal solution to (1) can be expressed as

$$\psi = (\lambda - \hat{\mathcal{L}})^{-1} f = \hat{G} f,$$

where we have introduced the Green's function operator $\hat{G} \equiv (\lambda - \hat{\mathcal{L}})^{-1}$. In coordinate representation, this solution takes the integral form

$$\psi(x) = \hat{G} f = \int G(x, x') f(x') \, \mathrm{d}x',$$

where $G(x, x')$ is the Green's function satisfying the differential equation

$$(\lambda - \hat{\mathcal{L}}_x) G(x, x') = \delta(x - x'),$$

with $\delta(x - x')$ being the Dirac delta function. This formulation demonstrates how the Green's function serves as a fundamental solution to the original differential equation.

Hence, several analytical techniques have been developed to determine the explicit form of $G(x, x')$, including

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the direct method [8] and the eigenfunction expansion approach. For time-dependent problems of the form

$$(-ic\partial_t - \hat{\mathcal{L}}_x)\psi(x,t) = f(x,t),$$

it is often advantageous to work in frequency space. The retarded Green's function in this representation by the eigenfunction method becomes

$$G^r(x,x',\omega) = \sum_n \frac{\psi_n(x)\psi_n^\dagger(x')}{\omega - \omega_n + i0^+},$$

where ω_n are the eigenfrequencies of the system and 0^+ denotes an infinitesimal positive imaginary part ensuring causality.

As a consequence, an important physical quantity derived from the Green's function is the local density of states (LDOS), which can be obtained via the Plemelj formula (see Appendix A) as

$$\rho(x,\omega) = -\frac{1}{\pi}\mathrm{Im}\left\{G^r(x,x,\omega)\right\}.$$

This quantity provides crucial information about the available states at position x and energy ω.

On the other hand, in quantum mechanical applications, it is frequently desirable to obtain the Green's function for a perturbed Hamiltonian. Given the unperturbed Hamiltonian $\hat{H}_0$ with known Green's function $\hat{g}$ satisfying

$$(E - \hat{H}_0)\hat{g} = \mathbb{I},$$

we seek the Green's function $\hat{G}$ for the perturbed system $\hat{H} = \hat{H}_0 + \hat{V}$ that satisfies

$$(E - \hat{H})\hat{G} = \mathbb{I}.$$

The Dyson equation provides a self-consistent relation between these quantities as

$$\hat{G} = \hat{g} + \hat{g}\hat{V}\hat{G}.$$

This recursive equation forms the basis for perturbation theory in Green's function methods and allows for systematic calculation of the perturbed Green's function in terms of the unperturbed solution and the perturbation potential $\hat{V}$.

III. GREEN FUNCTION FOR THE SSH MODEL

The SSH Hamiltonian consists of a one-dimensional lattice with two sites (atoms) per unit cell, and alternates between two amplitudes, $v = (t+\delta)$ and $w = (t-\delta)$, for intra-cell and inter-cell hopping, respectively. That is

$$H = \sum_{n=1}^{N}(t+\delta)c^\dagger_{A,n}c_{B,n} + \sum_{n=1}^{N-1}(t-\delta)c^\dagger_{A,n+1}c_{B,n} + \text{h.c.},$$

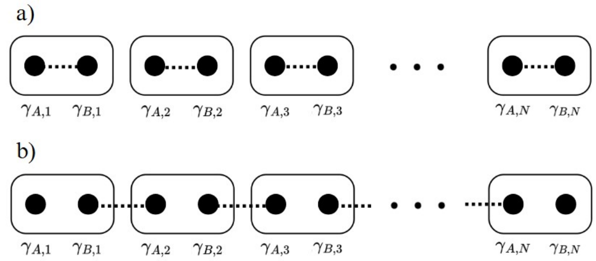


Figure 1: Esta es una imagen de prueba, reemplazar por la figura final.

where the operators, $c^\dagger$, c, create and annihilate the electrons on the respective sublattice's (A or B) site n.

Making the Fourier transform so we get into the momentum space, we get the Hamiltonian form (see Appendix B)

$$H = \sum_k \psi_k^\dagger\left[-(t-\delta)k\sigma_x + \left(2\delta t + \frac{1}{2}(t-\delta)k^2\right)\sigma_z\right]\psi_k,$$

with the spinor $\psi_k = \binom{a_k}{b_k}$ in terms of the coefficients of the transformation. This Hamiltonian has chiral symmetry and is analogous to a modified Dirac Hamiltonian, whose quadratic component $m_2 = (t-\delta)\frac{a^2}{2} = \frac{wa^2}{2}$ works as term that makes the equation topologically distinct, breaking the mass symmetry into two opposite sign mass components for $\delta < 0$.

Thus, in its topological phase, this system exhibits a zero energy edge state

$$|\psi(x)\rangle = c\left[e^{-x/d_+} - e^{-x/d_-}\right]|\phi\rangle,$$

with

$$d_\pm^{-1} = \frac{|A|}{2m^2}\left[1 \pm \sqrt{1 - \frac{4m_1m_2}{A^2}}\right]$$

and $A = wa$, $m_2 = \frac{wa^2}{2}$ and $m_1 = |v - w|$ that satisfy the boundary conditions for $x > 0$.

Recalling that the edge state has zero energy, by the eigenfunction method, the retarded Green function for the edge state of the SSH model is

$$\begin{aligned} g^r(x,x',\omega) =& \frac{|\psi(x)\rangle\langle\psi(x')|}{\omega + i0^+} \\ =& \frac{c^2}{\omega + i0^+}\left[e^{-x/d_+} - e^{-x/d_-}\right] \\ &\times \left[e^{-x'/d_+} - e^{-x'/d_-}\right]|\phi\rangle\langle\phi|. \end{aligned}$$

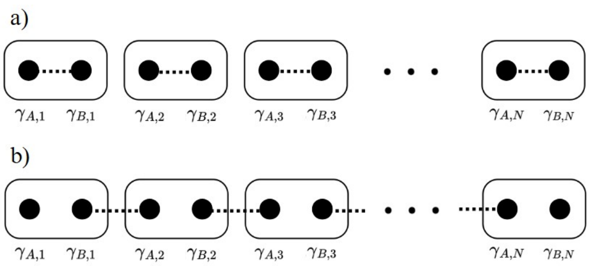


Figure 2: Esta es una imagen de prueba, reemplazar por la figura final.

It can be seen that for matrix $|\phi\rangle\langle\phi|$,

$$|\phi\rangle\langle\phi| = \begin{pmatrix} 1 \\ -i \end{pmatrix} \begin{pmatrix} 1 & i \end{pmatrix} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

and defining $\hat{A} = |\phi\rangle\langle\phi|$ the retarded Green function for the edge state is

$$\begin{aligned} g^r(x, x', \omega) =& \frac{c^2}{\omega + i0^+} \left[e^{-x/d_+} - e^{-x/d_-} \right] \\ & \times \left[e^{-x'/d_+} - e^{-x'/d_-} \right] \hat{A}. \end{aligned}$$

With this result, the LDOS for the edge state is computed as

$$\begin{aligned} \rho(x, \omega) &= -\frac{1}{\pi} \mathrm{Im}\{\mathrm{Tr}(g(x, x, \omega))\} \\ &= -\frac{1}{\pi} \mathrm{Im} \left\{ \frac{c^2}{\omega + i0^+} \left[e^{-x/d_+} - e^{-x/d_-} \right]^2 \mathrm{Tr}(\hat{A}) \right\}. \end{aligned}$$

As $\mathrm{Tr}(\hat{A}) = 2$ and by the Plemelj formula

$$\lim_{\epsilon \to 0^+} \frac{1}{\omega + i\epsilon} = \mathcal{P}\left(\frac{1}{\omega}\right) - i\pi\delta(\omega),$$

the LDOS for the edge state is

$$\rho(x, \omega) = 2c^2 \left[e^{-x/d_+} - e^{-x/d_-} \right]^2 \delta(\omega).$$

IV. COUPLING BETWEEN TOPOLOGICAL SSH CHAINS

In the non-interacting case, the edge states of two separate SSH chains can be described by

$$\begin{aligned} |\psi_R\rangle &= c_R \left(e^{-x/d_+} - e^{-x/d_-} \right) |\phi\rangle, \\ |\psi_L\rangle &= c_L \left(e^{x/\xi_+} - e^{x/\xi_-} \right) |\phi\rangle, \end{aligned}$$

where $|\psi_R\rangle$ and $|\psi_L\rangle$ represent the right and left edge states with penetration lengths $d_\pm$ and $\xi_\pm$, respectively, and c_R and c_L are normalization constants.

To study the interaction between these edge states, we introduce a tunneling potential with isotropic coupling strength t,

$$\hat{V} = \begin{pmatrix} 0 & V_{RL} \\ V_{LR} & 0 \end{pmatrix} = \begin{pmatrix} \hat{0}_{2x2} & t\mathbb{I}_{2x2} \\ t\mathbb{I}_{2x2} & \hat{0}_{2x2} \end{pmatrix},$$

where we have defined $V_{RL} = V_{LR} = t\mathbb{I}$. The retarded Green functions for the uncoupled SSH model are given by

$$\begin{aligned} g_{RR}(x, x', \omega) =& \frac{c_R^2}{\omega + i0^+} \left[e^{-x/d_+} - e^{-x/d_-} \right] \\ & \times \left[e^{-x'/d_+} - e^{-x'/d_-} \right] \hat{A}, \\ g_{LL}(x, x', \omega) =& \frac{c_L^2}{\omega + i0^+} \left[e^{x/\xi_+} - e^{x/\xi_-} \right] \\ & \times \left[e^{x'/\xi_+} - e^{x'/\xi_-} \right] \hat{A}. \end{aligned}$$

For notation convenience, we define

$$\begin{aligned} f(x, x') &= c_R^2 \left[e^{-x/d_+} - e^{-x/d_-} \right] \left[e^{-x'/d_+} - e^{-x'/d_-} \right], \\ h(x, x') &= c_L^2 \left[e^{x/\xi_+} - e^{x/\xi_-} \right] \left[e^{x'/\xi_+} - e^{x'/\xi_-} \right], \end{aligned}$$

which allows us to express the Green's functions compactly as

$$\begin{aligned} g_{RR}(x, x', \omega) &= \frac{f(x, x')}{\omega + i0^+} \hat{A}, \\ g_{LL}(x, x', \omega) &= \frac{h(x, x')}{\omega + i0^+} \hat{A}. \end{aligned}$$

Now, in the non-interacting case, the cross terms vanish, $g_{RL} = g_{LR} = 0$, and using the Dyson equation

$$\hat{G} = \hat{g} + \hat{g}\hat{V}\hat{G},$$

we obtain the coupled equations for G_{RR} and G_{LR}

$$\begin{aligned} G_{RR} &= g_{RR} + g_{RR} V_{RL} G_{LR}, \\ G_{LR} &= g_{LL} V_{LR} G_{RR}. \end{aligned}$$

Solving for G_{RR} (see Appendix C) yields

$$\begin{aligned} G_{RR}(x, x, \omega) &= \left(\mathbb{I} - t^2 g_{RR} g_{LL} \right)^{-1} g_{RR} \\ &= \frac{f(x, x')(\omega + i0^+)^3}{(\omega + i0^+)^4 - (\omega + i0^+)^2 b(x, x') + (1/2) b^2(x, x')} \hat{A}, \end{aligned} \tag{2}$$

where $b(x, x') = 4t^2 f(x, x') h(x, x')$. Note that when $t = 0$, we recover the unperturbed retarded Green's function.

Thus, using this result, we can obtain the LDOS for the right edge state after the coupling as

$$\rho(x, \omega) = -\frac{1}{\pi} \mathrm{Im}\{\mathrm{Tr}(G_{RR}(x, x, \omega))\}.$$

V. RESULTS AND DISCUSSION

Based on the result obtained in the previous section for the LDOS (2) for the right edge state after the coupling is

$$\rho(x,\omega) = -\frac{1}{\pi}\mathrm{Im}\{\mathrm{Tr}(G_{RR}(x,x,\omega))\}$$
$$= -\frac{1}{\pi}\mathrm{Im}\left\{\frac{2f(x,x)(\omega+i0^+)^3}{(\omega+i0^+)^4-(\omega+i0^+)^2 b(x,x)+(1/2)b^2(x,x)}\right\}.$$

Since the direct analytical calculation of the LDOS is computationally intractable, we adopt a numerical approach. Using `Python`, we developed a script [9] to compute the LDOS for the right edge state.

We set the parameters for the chains as follows: $a = 1$ for both chains, with $v = 0.8$ and $w = 1.0$ for the right chain, and $v' = 0.72$, $w' = 1.1$ for the left chain. These choices correspond to $A = 1.0$, $m_1 = 0.2$, and $m_2 = 0.5$ for the right chain, and $A' = 1.1$, $m_1' = 0.38$, and $m_2' = 0.55$ for the left chain. As an initial step, we select a coupling strength of $t = 1.2$ to analyze how the interaction with the left edge state influences the right edge state. For consistency with this objective, we evaluate the LDOS at $x = 0.5$, near the edge, to probe the spatial region where the edge state is localized.

These parameters yield the penetration lengths $d_+ = 0.563508$, $d_- = 4.436492$, $\xi_+ = 0.642690$, and $\xi_- = 2.252047$.

For the frequency-space discretization, we use 5000 points in the interval $\omega \in (-5, 5)$, resulting in $\Delta\omega = 0.005$. Additionally, we set $0^+ = 5\Delta\omega = 0.025$ for numerical stability. The LDOS obtained is shown in the figure 3.

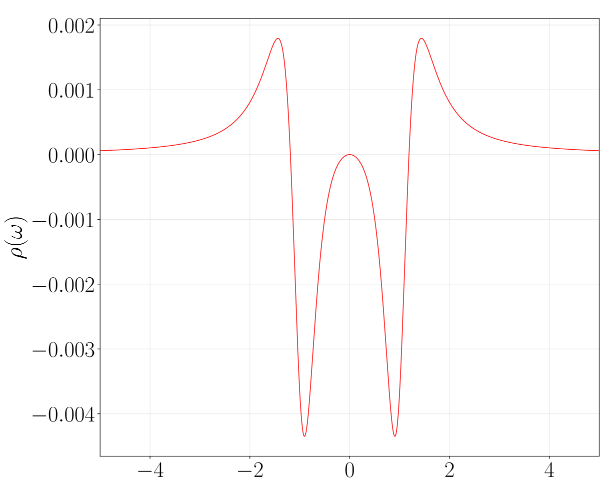
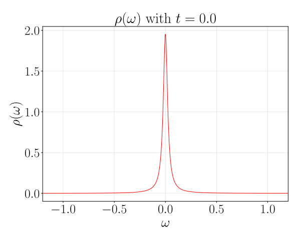


Figure 3: Local density of states LDOS for the right edge state as a function of the frequency ω for the coupling strength $t = 1.2$.

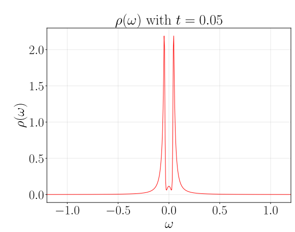
At first glance, we observe that the interaction between the two edge states has destroyed the zero-energy mode in the right chain, as no peak is observed at $\omega = 0$. Furthermore, four symmetric peaks appear around the origin, which may reflect the chiral symmetry of the SSH model and suggest hybridization between the right and left edge states, evidenced by the displacement of the energy peaks.

This interpretation is reinforced by the width and height of the peaks. Unlike the Dirac delta function appearing in the unperturbed LDOS, these peaks exhibit a small amplitude and finite width. The width could be associated with a mean lifetime, implying a weaker localization of the perturbed states compared to the unperturbed case.

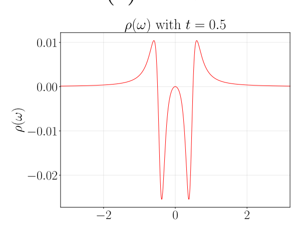
Now, to systematically analyze the effects of the coupling on the edge states, we repeated the procedure for varying coupling strengths: $t = 0$ (no coupling), $t = 0.05$ (weak coupling), $t = 0.5, 1$ (intermediate coupling), and $t = 2$ (strong coupling). This parameter range allows us to examine the complete evolution from decoupled to strongly coupled edge states. The corresponding LDOS results are presented in Fig. 4.



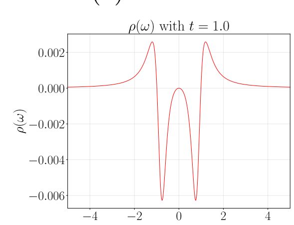
(a) $t = 0$



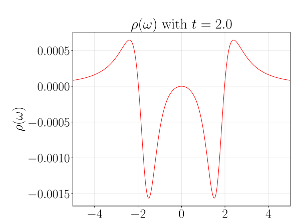
(b) $t = 0.05$



(c) $t = 0.5$



(d) $t = 1$



(e) $t = 2$

Figure 4: LDOS for the right edge state as a function of frequency ω for different coupling strengths t.

As shown in Figure 4a, the uncoupled case reproduces the unperturbed LDOS, exhibiting a Dirac delta function $\delta(\omega)$ behavior at the origin. This confirms the existence of a zero-energy edge state in the absence of inter-chain coupling.

For the weak coupling regime ($t = 0.05$), Figure 4b reveals that the edge state interaction splits the zero-energy peak into two symmetric peaks. This splitting is protected by the system's chiral symmetry. Furthermore, the finite coupling strength t hybridizes the edge state energy levels, resulting in two states with finite energies.

In the intermediate coupling regime, Figures 4d and 4c shows that the peaks shift away from $\omega = 0$, reflecting stronger hybridization between the edge states. Notably, the LDOS amplitude decreases significantly, suggesting a partial delocalization of the edge states into the bulk.

Finally, in the strong coupling regime, Figure 4e demonstrate that the coupling dominates over the original SSH gap energy scale, causing the edge states to merge with the bulk continuum. The peaks become broad and strongly suppressed, indicating complete delocalization of the edge states into bulk resonances. Remarkably, the LDOS maintains its symmetry, providing clear evidence of preserved chiral symmetry despite the strong coupling.

This systematic evolution demonstrates how increasing coupling strength drives the system from well-defined topological edge states to a trivial phase with bulk-hybridized modes, transforming the zero-energy edge state into symmetric bulk states at energies $\pm E$.

VI. CONCLUSIONS

In this work, we investigated the local density of states (LDOS) for both a single SSH chain and a coupled system of two SSH chains using Green's function methods. Our analysis revealed several key insights into the behavior of topological edge states under varying coupling strengths, shedding light on their hybridization and symmetry preservation.

For the single SSH chain, we analytically derived the LDOS and confirmed the existence of a zero-energy edge state localized at the boundary, as evidenced by the Dirac delta peak at $\omega = 0$. This result aligns with the well-known topological properties of the SSH model, where chiral symmetry ensures the robustness of edge states in the topological phase.

In the coupled system, the introduction of inter-chain tunneling significantly altered the LDOS. The zero-energy mode was destroyed, and symmetric peaks emerged around the origin, reflecting the hybridization of edge states from the two chains. These peaks, while broadened and shifted, maintained their symmetry, underscoring the preservation of chiral symmetry even under perturbation. The evolution of the LDOS with increasing coupling strength demonstrated a clear transition from localized edge states to delocalized bulk resonances. In the weak coupling regime ($t = 0.05$), the zero-energy peak split into two symmetric peaks, a hallmark of protected level repulsion due to chiral symmetry. For intermediate couplings ($t = 0.5, 1$), the peaks shifted further apart, and their amplitudes diminished, indicating partial delocalization. In the strong coupling limit ($t = 2$), the edge states merged with the bulk continuum, resulting in broad, low-amplitude resonances.

Our findings highlight the delicate interplay between topology and interactions in coupled SSH chains. The robustness of edge states against weak perturbations and their eventual dissolution into the bulk under strong coupling provide valuable insights for designing topological heterostructures and quantum devices. Future work could explore the effects of asymmetric coupling or disorder on the LDOS, as well as extensions to higher-dimensional systems or other symmetry classes. These results contribute to a deeper understanding of topological materials and their potential applications in quantum engineering and condensed matter physics.

[1] K. v. Klitzing, G. Dorda, and M. Pepper, New method for high-accuracy determination of the fine-structure constant based on quantized hall resistance, Phys. Rev. Lett. **45**, 494 (1980).

[2] L. Lu, J. D. Joannopoulos, and M. Soljačić, Topological photonics, Nature Photonics **8**, 821 (2014).

[3] A. Y. Kitaev, Unpaired majorana fermions in quantum wires, Physics-Uspekhi **44**, 131 (2001).

[4] W. P. Su, J. R. Schrieffer, and A. J. Heeger, Solitons in polyacetylene, Phys. Rev. Lett. **42**, 1698 (1979).

[5] R. Pineda and W. J. Herrera, Edge states, transport and topological properties of heterostructures in the ssh model (2022), arXiv:2205.02326 [cond-mat.mes-hall].

[6] S. Mignuzzi, S. Vezzoli, S. A. R. Horsley, W. L. Barnes, S. A. Maier, and R. Sapienza, Nanoscale design of the local density of optical states, Nano Letters **19**, 1613 (2019).

[7] T. Iffländer, S. Rolf-Pissarczyk, L. Winking, R. G. Ulbrich, A. Al-Zubi, S. Blügel, and M. Wenderoth, Local density of states at metal-semiconductor interfaces: An atomic scale study, Phys. Rev. Lett. **114**, 146804 (2015).

[8] W. J. Herrera, H. Vinck-Posada, and S. Gómez Páez,

Green's functions in quantum mechanics courses, American Journal of Physics **90**, 763 (2022), https://pubs.aip.org/aapt/ajp/article-pdf/90/10/763/20103157/763_1_5.0065733.pdf.
[9] Ldos for a coupled system of ssh chains (2025).
[10] S.-Q. Shen, *Topological Insulators: Dirac Equation in Condensed Matter*, 2nd ed., Springer Series in Solid-State Sciences (Springer Singapore, 2017) pp. XIII, 266, 53 b/w illustrations, 10 illustrations in colour.

Appendix A: Green function in the frequency space and Dyson equation

For a time dependent problem, consider the operator $[ic\partial_t - \hat{\mathcal{L}}]\psi = f(t)$, to which we can associate the Green's function, $[ic\partial_t - \hat{\mathcal{L}}]G(x,t,x',t') = \delta(x-x')\delta(t-t')$. If $\hat{\mathcal{L}}$ is explicitly independent of t, we can perform a Fourier transform

$$G(x,x',t-t') = \frac{1}{2\pi}\int \mathrm{d}\omega\, G(x,x',\omega)e^{-i\omega(t-t')},$$

so that we obtain $[c\omega - \hat{\mathcal{L}}]G(x,x',\omega) = \delta(x-x')$. If we know the solution of the equation at a given time, we can determine it at any time since it's first order. We assume

$$\psi(x,t) = i\int G(x,t,x',t')\psi(x',t)\,\mathrm{d}x',$$

and expanding the solution in the $\hat{\mathcal{L}}$ eigenfunctions

$$\psi(x,t) = \sum_n a_n(t)\varphi_n(x).$$

To find a_n,

$$\begin{aligned}[ic\partial_t - \hat{\mathcal{L}}]\psi &= [ic\partial_t - \hat{\mathcal{L}}]\sum_n a_n(t)\varphi_n(x)\\ &= \sum_n [ic\partial_t a_n(t)\varphi_n(x) - a_n(t)\omega_n\varphi_n(x)]\\ &= \sum_n [ic\partial_t a_n(t) - \omega_n a_n(t)]\varphi_n(x) = 0.\end{aligned}$$

Thus,

$$\begin{aligned}ic\partial_t a_n(t) &= \omega_n a_n(t), \quad \text{with} \quad c=1,\\ a_n(t) &= a_n(t_0)e^{-iw_n(t-t_0)},\end{aligned}$$

where

$$a_n(t_0) = \int \varphi_n^*(x)\psi(x,t_0)\,\mathrm{d}x.$$

Taking $\psi(x,t)$, $a_n(t)$, and $a_n(t_0)$,

$$\begin{aligned}\psi(x,t) &= \sum_n \int \varphi_n^*(x')\psi(x',t_0)e^{-i\omega_n(t-t_0)}\varphi_n(x)\,\mathrm{d}x'\\ &= \int \mathrm{d}x' \sum_n e^{-i\omega_n(t-t_0)}\varphi_n(x)\varphi_n^*(x')\psi(x',t_0)\\ &\equiv i\int G(x,x',t,t_0)\psi(x',t_0)\,\mathrm{d}x',\end{aligned}$$

which implies

$$G(x,x',t,t') = -i\sum_n \varphi_n(x)\varphi_n^*(x')e^{-i\omega_n(t-t')}.$$

We define the retarded Green's function

$$G^r(x,x',t,t') = \theta(t-t')G(x,x',t,t'),$$

where the Heaviside step function is

$$\theta(t-t') = \frac{i}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}p\,\frac{e^{-ip(t-t')}}{p+i\eta},$$

which equals 1 for this case where $t > t'$. Now,

$$\begin{aligned}&G^r(x,x',\omega) = \int \mathrm{d}(t-t')\,e^{i\omega(t-t')}G^r(x,x',t-t'),\\ &= (-i)\int \mathrm{d}(t-t')\,e^{i\omega(t-t')}\left(\frac{i}{2\pi}\right)\\ &\quad\times \int_{-\infty}^{\infty}\mathrm{d}p\frac{e^{ip(t-t')}}{p+i0^+}\sum_n \varphi_n(x)\varphi_n^*(x')e^{-i\omega_n(t-t')},\\ &= \sum_n \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}p}{p+i0^+}\\ &\quad\times \int \mathrm{d}(t-t')e^{i(\omega-\omega_n-p)(t-t')}\varphi_n(x)\varphi_n^*(x'),\\ &= -\sum_n \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}p}{p+i0^+}2\pi\delta(\omega-\omega_n-p)\varphi_n(x)\varphi_n^*(x').\end{aligned}$$

Then, the spectral relation for the retarded Green's function is

$$G^r(x,x',\omega) = \sum_n \frac{\varphi_n(x)\varphi_n^*(x')}{\omega-\omega_n+i0^+}.$$

By this result, using the Plemelj formula

$$\lim_{\epsilon\to 0^+}\frac{1}{\omega-\omega_n+i\epsilon} = \mathcal{P}\left(\frac{1}{\omega-\omega_n}\right) - i\pi\delta(\omega-\omega_n),$$

it follows that

$$\begin{aligned}&G^r(x,x',\omega)\\ &= \sum_n \mathcal{P}\left(\frac{\varphi_n(x)\varphi_n^*(x')}{\omega-\omega_n}\right) - i\pi\sum_n \varphi_n(x)\varphi_n^*(x')\delta(\omega-\omega_n).\end{aligned}$$

Therefore, evaluating in $x = x'$

$$\varphi_n(x)\varphi_n^*(x') \to |\varphi_n(x)|^2$$

and the imaginary part of the retarded Green function reads as

$$\mathrm{Im}\left\{G^r(x,x,\omega)\right\} = -\pi\sum_n |\varphi_n(x)|^2\delta(\omega-\omega_n).$$

By this, the local density of states is

$$\rho(x,\omega) = -\frac{1}{\pi}\mathrm{Im}\left\{G^r(x,x,\omega)\right\} = \sum_n |\varphi_n(x)|^2\delta(\omega-\omega_n).$$

Now, if for a initially unperturbed Hamiltonian problem, the retarded Green function is obtained such that

$$(E-\hat{H}_0)\,\hat{g} = \mathbb{I}.$$

To obtain the retarded Green function for the perturbed Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}$, it must satisfies the equation

$$(E-\hat{H})\hat{G} = \mathbb{I},$$

thereby, substituting the total Hamiltonian into the first equation yields

$$(E-\hat{H}_0-\hat{V})\hat{G} = \mathbb{I}.$$

Recognizing the presence of the unperturbed Green function operator, we may rewrite this expression as

$$(\hat{g}^{-1}-\hat{V})\hat{G} = \mathbb{I},$$

where we have made use of the inverse relationship $\hat{g}^{-1} = (E-\hat{H}_0)$ derived from the unperturbed system. To obtain an explicit expression for $\hat{G}$, we multiply both sides of the equation by $\hat{g}$ from the left

$$\hat{g}(\hat{g}^{-1}-\hat{V})\hat{G} = \hat{g}\mathbb{I},$$

Expanding the left-hand side and simplifying gives

$$(\mathbb{I}-\hat{g}\hat{V})\hat{G} = \hat{g}.$$

The full Green's function can now be isolated, resulting in the Dyson equation

$$\hat{G} = \hat{g} + \hat{g}\hat{V}\hat{G}.$$

Appendix B: SSH model

Applying the inverse Fourier's transform

$$C_{A,n} = \frac{1}{\sqrt{N}}\sum_k e^{-ikna}a_k, \quad C_{B,n} = \frac{1}{\sqrt{N}}\sum_k e^{-ikna}b_k,$$

to the Hamiltonian

$$H = \sum_n (t+\delta)C^\dagger_{A,n}C_B,n + \sum_n (t-\delta)C^\dagger_{A,n+1}C_{B,n} + h.c.$$

For the first term,

$$C^\dagger_{A,n}C_{B,n} = \frac{1}{N}\sum_{k'k} e^{i(k'-k)na}a^\dagger_{k'}b_k,$$

and for the second,

$$\begin{aligned} C^\dagger_{A,n+1}C_{B,n} &= \frac{1}{N}\sum_{k'k} e^{i(k'(n+1)-kn)a}a^\dagger_{k'}b_k \\ &= \frac{1}{N}\sum_{k'k} e^{ik'a}e^{i(k'-k)na}a^\dagger_{k'}b_k. \end{aligned}$$

Replacing for the first term in the hamiltonian,

$$\sum_n vC^\dagger_{A,n}C_{B,n} = \frac{1}{N}\sum_{k'k}\sum_n ve^{i(k'-k)na}a^\dagger_{k'}b_k,$$

and using the discrete orthogonality relation, $\sum_n e^{i(k'-k)na} = N\delta_{kk'}$, which states that the plane waves with different momentum are orthogonal over the lattice,

$$\sum_n vC^\dagger_{A,n}C_{B,n} = \sum_n va^\dagger_k b_k.$$

And for the second term in the Hamiltonian,

$$\begin{aligned} \sum_k wC^\dagger_{A,n+1}C_{B,n} &= \frac{1}{N}\sum_n\sum_{k'k} we^{ik'a}e^{i(k'-k)na}a^\dagger_{k'}b_k, \\ &= \sum_n we^{ika}a^\dagger_k b_k. \end{aligned}$$

Replacing,

$$\begin{aligned} H &= \sum_k va^\dagger_k b_k + \sum_k we^{ika}a^\dagger_k b_k + h.c. \\ &= \sum_k v(a^\dagger_k b_k + b^\dagger_k a_k) + \sum_k w(e^{ika}a^\dagger_k b_k + e^{-ika}b^\dagger_k a_k), \end{aligned}$$

in matrix form,

$$H = \sum_k (a^\dagger_k, b^\dagger_k)\begin{pmatrix} 0 & v+we^{ika} \\ v+we^{-ika} & 0 \end{pmatrix}\begin{pmatrix} a_k \\ b_k \end{pmatrix},$$

defining the spinor $\psi_k = \begin{pmatrix} a_k \\ b_k \end{pmatrix}$,

$$H = \sum_k \psi_k^\dagger \begin{pmatrix} 0 & v + w(\cos ka + i \sin ka) \\ v + w(\cos ka - i \sin ka) & 0 \end{pmatrix} \psi_k,$$
$$= \sum_k \psi_k^\dagger \begin{pmatrix} 0 & v + w(\cos ka + i \sin ka) \\ v + w(\cos ka - i \sin ka) & 0 \end{pmatrix} \psi_k,$$
$$= \sum_k \psi_k^\dagger \left[\begin{pmatrix} 0 & v + w \cos ka \\ v + w \cos ka & 0 \end{pmatrix} - \begin{pmatrix} 0 & -w \sin ka \\ iw \sin ka & 0 \end{pmatrix} \right] \psi_k,$$
$$= \sum_k \psi_k^\dagger \left[(v + w \cos ka)\sigma_x - w \sin ka \sigma_y \right] \psi_k.$$

Now, making an expansion around the high symmetry point $ka = \pi$, $ka = \pi + \delta ka$,

$$v + w \cos ka = v + w \cos(\pi + \delta ka),$$

using the properties of angle addition

$$\cos(\pi + \delta ka) = \cos \pi \cos \delta ka - \sin \pi \sin \delta ka = -\cos \delta ka,$$
$$v + w \cos ka = v - w \cos a\delta k,$$

and using the Taylor expansion,

$$\cos a\delta k \approx 1 - \frac{a^2(\delta k)^2}{2},$$
$$v - w \cos a\delta k = v - w + w\frac{a^2(\delta k)^2}{2},$$
$$= t + \delta - t + \delta + (t - \delta)\frac{a^2(\delta k)^2}{2},$$
$$\equiv 2\delta + (t - \delta)\frac{a^2k^2}{2},$$

with the expansion, the Hamiltonian becomes

$$H = \sum_k \psi_k^\dagger \left[\left(2\delta + (t - \delta)\frac{a^2k^2}{2} \right) \sigma_x + \sigma_y (t - \delta)ka \right] \psi_k.$$

It can be verified that $\sigma_y H \sigma_y = -H$ [10], which implies that the SSH model possesses chiral symmetry. This result reinforces its equivalence to a Dirac Hamiltonian in the low-energy limit.

To make a better analysis, we can make an unitary transformation ($\sigma_x \to \sigma_z$, $\sigma_y \to \sigma_x$, $\sigma_z \to \sigma_y$) to get a partially diagonal Hamiltonian,

$$H = \sum_k \psi_k^\dagger \left[(t - \delta)ak\sigma_x + \left(2\delta + (t - \delta)k^2\frac{a^2}{2} \right) \sigma_z \right] \psi_k.$$

Noting $m_1 = 2|\delta|$ and $m_2 = (t - \delta)\frac{a^2}{2}$,

$$H = \sum_k \psi_k^\dagger \left[(t - \delta)ak\sigma_x + (m_1 + m_2k^2)\sigma_z \right] \psi_k.$$

For simplicity, we take

$$H_k = (t - \delta)ak\sigma_x + (-m_1 + m_2k^2)\sigma_z.$$

For $\delta < 0$, and using k as $-i\partial_x$, the eigenvalue equation,

$$ua(-i\partial_x)\sigma_x\psi + (-m_1 + m_2(-\partial_{xx}))\sigma_z\psi = E\psi,$$

and to find a zero energy solution,

$$ua(-i\partial_x)\sigma_x\psi = (m_1 - m_2(-\partial_{xx}))\sigma_z\psi,$$
$$(i\sigma_x)(-i\partial_x)\sigma_x\psi = \frac{1}{ua}(m_1 - m_2(-\partial_{xx}))(i\sigma_x)\sigma_z\psi,$$
$$\partial_x\psi = \frac{1}{ua}(m_1 - m_2(-\partial_{xx}))i\sigma_x\sigma_z\psi.$$

Since $\sigma_x\sigma_z = -i\sigma_y \longrightarrow i\sigma_x\sigma_z = \sigma_y$,

$$\partial_x\psi = \frac{1}{ua}(m_1 + m_2\partial_{xx})\sigma_y\psi.$$

Now, taking $\psi(x) = \phi e^{\lambda x}$ such that $\sigma_y\phi = \eta\phi$,

$$\phi = \begin{pmatrix} 1 \\ i\eta \end{pmatrix},$$
$$\partial_x\psi = \lambda\phi e^{\lambda x}, \quad \partial_{xx}\psi = \lambda^2\phi e^{\lambda x},$$

and replacing,

$$\lambda\phi e^{\lambda x} = \frac{1}{ua}(m_1 + m_2\lambda^2)\sigma_y\phi e^{\lambda x},$$
$$\lambda\phi = \frac{1}{ua}(m_1 + m_2\lambda^2)\sigma_y\phi.$$

Using the eigenvalues of σ_y, $\eta = \pm 1$,

$$\lambda\phi = \frac{\eta}{ua}(m_1 + m_2\lambda^2)\phi,$$
$$\lambda = \frac{\eta}{ua}(m_1 + m_2\lambda^2),$$
$$\lambda^2\frac{m^2\eta^2}{ua} - \lambda + \frac{\eta}{ua}m_1 = 0,$$
$$\lambda^2 m^2 - \alpha\lambda + m_1 = 0,$$

with $A = \frac{ua}{\eta} = ua\eta$. Using the quadratic equation,

$$\lambda = \frac{A \pm \sqrt{A^2 - 4m_1m_2}}{2m^2} = \frac{A}{2m^2}\left[1 \pm \sqrt{1 - \frac{4m_1m_2}{A^2}} \right].$$

Let the boundary conditions be $\psi(0) = \psi(\infty) = 0$. So, for $\psi(x) = \phi e^{\lambda x}$, λ must satisfy $\lambda < 0$ and $\eta < 0$. We define $\lambda = -d_\pm^{-1}$, which is a penetration length,

$$d_\pm^{-1} = \frac{|A|}{2m^2}\left[1 \pm \sqrt{1 - \frac{4m_1m_2}{A^2}} \right] > 0,$$

thus, if $x > 0$, $\psi(x)$ behaves as $e^{-x/d_\pm}$. Hence, the solution that satisfies the boundary conditions for $x > 0$ are

$$\psi(x) = c\left[e^{-x/d_+} - e^{-x/d_-}\right]\begin{pmatrix}1\\-i\end{pmatrix}.$$

Analogously, for $x < 0$ the solution that satisfies the boundary conditions $\psi(0) = \psi(-\infty) = 0$ is

$$\psi(x) = c\left[e^{x/d_+} - e^{x/d_-}\right]\begin{pmatrix}1\\-i\end{pmatrix}.$$

For both cases, the normalization constant c is obtained as

$$\begin{aligned}\langle\psi|\psi\rangle &= c^2\langle\phi|\phi\rangle\int_0^\infty\left[e^{-x/d_+} - e^{-x/d_-}\right]^2 \mathrm{d}x\\ &= c^2\langle\phi|\phi\rangle\int_{-\infty}^0\left[e^{x/d_+} - e^{x/d_-}\right]^2 \mathrm{d}x = 1.\end{aligned}$$

by $\langle\phi|\phi\rangle = 1 + \eta^2 = 2$ and

$$\left[e^{-x/d_+} - e^{-x/d_-}\right]^2 = e^{-2x/d_+} - 2e^{-x(1/d_+ + 1/d_-)} + e^{-2x/d_-}.$$

The integrals are

$$\begin{aligned}\int_0^\infty e^{-2x/d_+}\,dx &= \frac{d_+}{2},\\ \int_0^\infty e^{-2x/d_-}\,dx &= \frac{d_-}{2},\\ \int_0^\infty e^{-x(1/d_+ + 1/d_-)}\,dx &= \frac{1}{\frac{1}{d_+} + \frac{1}{d_-}} = \frac{d_+ d_-}{d_+ + d_-}.\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^\infty\left[e^{-x/d_+} - e^{-x/d_-}\right]^2 \mathrm{d}x &= \frac{d_+}{2} + \frac{d_-}{2} - 2\frac{d_+ d_-}{d_+ + d_-}\\ &= \frac{d_+ + d_-}{2} - \frac{2d_+ d_-}{d_+ + d_-}\\ &= \frac{(d_+ + d_-)^2 - 4d_+ d_-}{2(d_+ + d_-)}\\ &= \frac{d_+^2 + d_-^2 - 2d_+ d_-}{2(d_+ + d_-)}\\ &= \frac{(d_+ - d_-)^2}{2(d_+ + d_-)}.\end{aligned}$$

Using this result,

$$\langle\psi|\psi\rangle = 2c^2\frac{(d_+ - d_-)^2}{2(d_+ + d_-)} = 1,$$

the value of the constant c is

$$c = \frac{\sqrt{d_+ + d_-}}{|d_+ - d_-|}$$

and the normalized edge states are

$$\begin{aligned}\psi(x) &= \frac{\sqrt{d_+ + d_-}}{|d_+ - d_-|}\left[e^{-x/d_+} - e^{-x/d_-}\right]\begin{pmatrix}1\\-i\end{pmatrix}, \quad x > 0\\ \psi(x) &= \frac{\sqrt{d_+ + d_-}}{|d_+ - d_-|}\left[e^{x/d_+} - e^{x/d_-}\right]\begin{pmatrix}1\\-i\end{pmatrix}, \quad x < 0.\end{aligned}$$

Appendix C: Solving the Dyson equation for the coupled system

Starting from the right and left edge states wavefunctions

$$\begin{aligned}|\psi_R\rangle &= c_R\left(e^{-x/d_+} - e^{-x/d_-}\right)|\phi\rangle,\\ |\psi_L\rangle &= c_L\left(e^{x/\xi_+} - e^{x/\xi_-}\right)|\phi\rangle,\\ \text{with}\ \ |\phi\rangle &= \begin{pmatrix}1\\-i\end{pmatrix}.\end{aligned}$$

Taking into account that these states are zero energy modes, we can obtain the Green function through the eigenfunction method for the right edge state as

$$\begin{aligned}g_{RR}(x,x',\omega) =& \frac{|\psi_R(x)\rangle\langle\psi_R(x')|}{\omega + i0^+}\\ =& \frac{c_R^2}{\omega + i0^+}\left[e^{-x/d_+} - e^{-x/d_-}\right]\\ &\times\left[e^{-x'/d_+} - e^{-x'/d_-}\right]|\phi\rangle\langle\phi|.\end{aligned}$$

It can be seen that for matrix $|\phi\rangle\langle\phi|$,

$$|\phi\rangle\langle\phi| = \begin{pmatrix}1\\-i\end{pmatrix}\begin{pmatrix}1 & i\end{pmatrix} = \begin{pmatrix}1 & i\\-i & 1\end{pmatrix}$$

and defining $\hat{A} = |\phi\rangle\langle\phi|$ the Green function for the right edge state is

$$\begin{aligned}g_{RR}(x,x',\omega) =& \frac{c_R^2}{\omega + i0^+}\left[e^{-x/d_+} - e^{-x/d_-}\right]\\ &\times\left[e^{-x'/d_+} - e^{-x'/d_-}\right]\hat{A}\end{aligned}$$

Analogously, for the left edge state, the Green function is obtained as

$$\begin{aligned}g_{LL}(x,x',\omega) =& \frac{|\psi_L(x)\rangle\langle\psi_L(x')|}{\omega+i0^+}\\ =& \frac{c_L^2}{\omega+i0^+}\left[e^{x/\xi_+}-e^{x/\xi_-}\right]\\ &\times\left[e^{x'/\xi_+}-e^{x'/\xi_-}\right]|\phi\rangle\langle\phi|\\ =& \frac{c_L^2}{\omega+i0^+}\left[e^{x/\xi_+}-e^{x/\xi_-}\right]\\ &\times\left[e^{x'/\xi_+}-e^{x'/\xi_-}\right]\hat{A}.\end{aligned}$$

For notation convenience, we define

$$\begin{aligned}f(x,x') &= c_R^2\left[e^{-x/d_+}-e^{-x/d_-}\right]\left[e^{-x'/d_+}-e^{-x'/d_-}\right],\\ h(x,x') &= c_L^2\left[e^{x/\xi_+}-e^{x/\xi_-}\right]\left[e^{x'/\xi_+}-e^{x'/\xi_-}\right],\end{aligned}$$

which allows us to express the Green's functions compactly as

$$\begin{aligned}g_{RR}(x,x',\omega) &= \frac{f(x,x')}{\omega+i0^+}\hat{A},\\ g_{LL}(x,x',\omega) &= \frac{h(x,x')}{\omega+i0^+}\hat{A}.\end{aligned}$$

Now, since in the unperturbed case there is no coupling between the chains, $g_{LR}=g_{RL}=0$. The unperturbed Green function for the total system is

$$\hat{g}(x,x',\omega)=\begin{pmatrix} g_{RR}(x,x',\omega) & 0\\ 0 & g_{LL}(x,x',\omega)\end{pmatrix}.$$

From now on, for all Green's functions in both the unperturbed and perturbed cases, we will assume by notation that $g_{ij}=g_{ij}(x,x',\omega)$ for $i,j=L,R$. To study the interaction between these edge states, we introduce a tunneling potential with isotropic coupling strength t,

$$\hat{V}=\begin{pmatrix}0 & V_{RL}\\ V_{LR} & 0\end{pmatrix}=\begin{pmatrix}\hat{0}_{2x2} & t\mathbb{I}_{2x2}\\ t\mathbb{I}_{2x2} & \hat{0}_{2x2}\end{pmatrix},$$

where we have defined $V_{RL}=V_{LR}=t$. For the perturbed Green function $\hat{G}$,

$$\hat{G}=\begin{pmatrix}G_{RR} & G_{RL}\\ G_{LR} & G_{LL}\end{pmatrix}.$$

The Dyson equation is

$$\begin{aligned}&\hat{G}=\hat{g}+\hat{g}\hat{V}\hat{G}\\ &=\begin{pmatrix}g_{RR} & 0\\ 0 & g_{LL}\end{pmatrix}+\begin{pmatrix}g_{RR} & 0\\ 0 & g_{LL}\end{pmatrix}\begin{pmatrix}0 & V_{RL}\\ V_{LR} & 0\end{pmatrix}\begin{pmatrix}G_{RR} & G_{RL}\\ G_{LR} & G_{LL}\end{pmatrix}\\ &=\begin{pmatrix}g_{RR} & 0\\ 0 & g_{LL}\end{pmatrix}+\begin{pmatrix}g_{RR}V_{RL}G_{LR} & g_{RR}V_{RL}G_{LL}\\ g_{LL}V_{LR}G_{RR} & g_{LL}V_{LR}G_{RL}\end{pmatrix}.\end{aligned}$$

Thus,

$$\begin{aligned}\hat{G} &= \begin{pmatrix}G_{RR} & G_{RL}\\ G_{LR} & G_{LL}\end{pmatrix}\\ &= \begin{pmatrix}g_{RR}+g_{RR}V_{RL}G_{LR} & g_{RR}V_{RL}G_{LL}\\ g_{LL}V_{LR}G_{RR} & g_{LL}+g_{LL}V_{LR}G_{RL}\end{pmatrix}.\end{aligned}$$

As we can see, the coupled equations for G_{RR} and G_{LR} are

$$\begin{aligned}G_{RR} &= g_{RR}+g_{RR}V_{RL}G_{LR},\\ G_{LR} &= g_{LL}V_{LR}G_{RR}.\end{aligned}$$

Solving for G_{RR} yields

$$\begin{aligned}G_{RR} &= g_{RR}+g_{RR}V_{RL}G_{LR}\\ &= g_{RR}+g_{RR}V_{RL}g_{LL}V_{LR}G_{RR}.\end{aligned}$$

By simple matrix algebra,

$$\begin{aligned}g_{RR} &= G_{RR}-g_{RR}V_{RL}g_{LL}V_{LR}G_{RR}\\ &= (\mathbb{I}-g_{RR}V_{RL}g_{LL}V_{LR})G_{RR},\end{aligned}$$

and taking into account that $V_{RL}=V_{LR}=t\mathbb{I}$,

$$g_{RR}=(\mathbb{I}-t^2g_{RR}g_{LL})G_{RR}.$$

By this, the perturbed Green function for the right edge state is

$$G_{RR}=(\mathbb{I}-t^2g_{RR}g_{LL})^{-1}g_{RR}.$$

To derive the explicit form of G_{RR}, we proceed to examine the following results. First, for the product $g_{RR}g_{LL}$,

$$g_{RR}g_{LL}=\frac{f(x,x')}{\omega+i0^+}\hat{A}\frac{h(x,x')}{\omega+i0^+}\hat{A}=\frac{f(x,x')h(x,x')}{(\omega+i0^+)^2}\hat{A}^2.$$

Observing that

$$\hat{A}^2=\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}=\begin{pmatrix}2 & 2i\\ -2i & 2\end{pmatrix}=2\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}=2\hat{A},$$

we obtain

$$g_{RR}g_{LL}=\frac{2f(x,x')h(x,x')}{(\omega+i0^+)^2}\hat{A}.$$

Finally, observing that $\hat{A}=\mathbb{I}-\sigma_y$, the matrix $\mathbb{I}-t^2g_{RR}g_{LL}$ is given by

$$\begin{aligned}\mathbb{I}-t^2g_{RR}g_{LL} &= \mathbb{I}-t^2\frac{2f(x,x')h(x,x')}{(\omega+i0^+)^2}\hat{A}\\ &= \mathbb{I}-\frac{2t^2f(x,x')h(x,x')}{(\omega+i0^+)^2}(\mathbb{I}-\sigma_y)\\ &= \left[1-\frac{2t^2f(x,x')h(x,x')}{(\omega+i0^+)^2}\right]\mathbb{I}+\frac{2t^2f(x,x')h(x,x')}{(\omega+i0^+)^2}\sigma_y\\ &= m(x,x',\omega)\mathbb{I}+n(x,x',\omega)\sigma_y.\end{aligned}$$

Where we defined

$$m(x,x',\omega) = 1 - \frac{2t^2 f(x,x')h(x,x')}{(\omega+i0^+)^2}$$
$$n(x,x',\omega) = \frac{2t^2 f(x,x')h(x,x')}{(\omega+i0^+)^2},$$

from it, follows that $m(x,x',\omega) = 1$ - $n(x,x',\omega)$. Now, since we wish to obtain $(\mathbb{I} - t^2 g_{RR} g_{LL})^{-1}$ through the transpose matrix property of the Pauli matrix $\sigma_y^T =$ -σ_y and its hermiticity $\sigma_y^2 = \mathbb{I}$, it can be seen that

$$\begin{aligned}
&(\mathbb{I} - t^2 g_{RR} g_{LL})(\mathbb{I} - t^2 g_{RR} g_{LL})^T \\
&= [m(x,x',\omega)\mathbb{I} + n(x,x',\omega)\sigma_y][m(x,x',\omega)\mathbb{I} + n(x,x',\omega)\sigma_y]^T \\
&= [m(x,x',\omega)\mathbb{I} + n(x,x',\omega)\sigma_y][m(x,x',\omega)\mathbb{I} - n(x,x',\omega)\sigma_y] \\
&= m^2(x,x',\omega)\mathbb{I}^2 + m(x,x',\omega)n(x,x',\omega)\sigma_y\mathbb{I} \\
&\quad - m(x,x',\omega)n(x,x',\omega)\mathbb{I}\sigma_y + n^2(x,x',\omega)\sigma_y^2 \\
&= m^2(x,x',\omega)\mathbb{I} + m(x,x',\omega)n(x,x',\omega)\sigma_y \\
&\quad - m(x,x',\omega)n(x,x',\omega)\sigma_y + n^2(x,x',\omega)\mathbb{I} \\
&= m^2(x,x',\omega)\mathbb{I} + n^2(x,x',\omega)\mathbb{I} \\
&= [m^2(x,x',\omega) + n^2(x,x',\omega)]\mathbb{I}.
\end{aligned}$$

Thus, the matrix $(\mathbb{I} - t^2 g_{RR} g_{LL})^{-1}$ is nothing but

$$\begin{aligned}
(\mathbb{I} - t^2 g_{RR} g_{LL})^{-1} &= \frac{(\mathbb{I} - t^2 g_{RR} g_{LL})^T}{m^2(x,x',\omega) + n^2(x,x',\omega)} \\
&= \frac{(\mathbb{I} - t^2 g_{RR} g_{LL})^T}{1 - 2n(x,x',\omega) + 2n^2(x,x',\omega)}.
\end{aligned}$$

As a consequence, the perturbed Green function G_{RR} can be expressed as

$$\begin{aligned}
G_{RR} &= (\mathbb{I} - t^2 g_{RR} g_{LL})^{-1} g_{RR} \\
&= \frac{(\mathbb{I} - t^2 g_{RR} g_{LL})^T}{1 - 2n(x,x',\omega) + 2n^2(x,x',\omega)} \frac{f(x,x')}{\omega + i0^+}\hat{A} \\
&= \frac{[m(x,x',\omega)\mathbb{I} + n(x,x',\omega)\sigma_y]^T}{1 - 2n(x,x',\omega) + 2n^2(x,x',\omega)} \frac{f(x,x')}{\omega + i0^+}\hat{A} \\
&= \frac{[m(x,x',\omega)\mathbb{I} - n(x,x',\omega)\sigma_y]}{1 - 2n(x,x',\omega) + 2n^2(x,x',\omega)} \frac{f(x,x')}{\omega + i0^+}\hat{A}.
\end{aligned}$$

Now, observing that

$$\begin{aligned}
&[m(x,x',\omega)\mathbb{I} - n(x,x',\omega)\sigma_y]\hat{A} \\
&= [m(x,x',\omega)\mathbb{I} - n(x,x',\omega)\sigma_y](\mathbb{I} - \sigma_y) \\
&= m(x,x',\omega)\mathbb{I}^2 - n(x,x',\omega)\sigma_y\mathbb{I} \\
&\quad - m(x,x',\omega)\mathbb{I}\sigma_y + n(x,x',\omega)\sigma_y^2 \\
&= [m(x,x',\omega) + n(x,x',\omega)]\mathbb{I} \\
&\quad - [m(x,x',\omega) + n(x,x',\omega)]\sigma_y \\
&= [m(x,x',\omega) + n(x,x',\omega)](\mathbb{I} - \sigma_y) \\
&= [1 - n(x,x',\omega) + n(x,x',\omega)](\mathbb{I} - \sigma_y) \\
&= \mathbb{I} - \sigma_y \\
&= \hat{A}.
\end{aligned}$$

The perturbed Green function is

$$G_{RR} = \frac{1}{1 - 2n(x,x',\omega) + 2n^2(x,x',\omega)} \frac{f(x,x')}{\omega + i0^+}\hat{A}.$$

And, explicitly writing $n(x,x',\omega) = \frac{2t^2 f(x,x')h(x,x')}{(\omega+i0^+)^2}$,

$$\begin{aligned}
G_{RR} &= \frac{1}{1 - \frac{4t^2 f(x,x')h(x,x')}{(\omega+i0^+)^2} + \frac{8t^4 f^2(x,x')h^2(x,x')}{(\omega+i0^+)^4}} \frac{f(x,x')}{\omega + i0^+}\hat{A} \\
&= \frac{f(x,x')(\omega+i0^+)^3\hat{A}}{(\omega+i0^+)^4 - 4(\omega+i0^+)^2 t^2 f(x,x')h(x,x') + 8t^4 f^2(x,x')h^2(x,x')} \\
&= \frac{f(x,x')(\omega+i0^+)^3\hat{A}}{(\omega+i0^+)^4 - (\omega+i0^+)^2 b(x,x') + (1/2)b^2(x,x')},
\end{aligned}$$

where $b(x,x') = 4t^2 f(x,x')h(x,x')$. Finally, the trace of G_{RR} at $x = x'$ is

$$\begin{aligned}
&\mathrm{Tr}(G_{RR}(x,x',\omega)) \\
&= \frac{f(x,x)(\omega+i0^+)^3\mathrm{Tr}(\hat{A})}{(\omega+i0^+)^4 - (\omega+i0^+)^2 b(x,x) + (1/2)b^2(x,x)} \\
&= \frac{f(x,x)(\omega+i0^+)^3}{(\omega+i0^+)^4 - (\omega+i0^+)^2 b(x,x) + (1/2)b^2(x,x)} \mathrm{Tr}\left(\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\right) \\
&= \frac{2f(x,x)(\omega+i0^+)^3}{(\omega+i0^+)^4 - (\omega+i0^+)^2 b(x,x) + (1/2)b^2(x,x)}.
\end{aligned}$$

Which allows us to compute the LDOS for the right edge state after the coupling as

$$\begin{aligned}
&\rho(x,\omega) = -\frac{1}{\pi}\mathrm{Im}\{\mathrm{Tr}(G_{RR}(x,x,\omega))\} \\
&= -\frac{1}{\pi}\mathrm{Im}\left\{\frac{2f(x,x)(\omega+i0^+)^3}{(\omega+i0^+)^4 - (\omega+i0^+)^2 b(x,x) + (1/2)b^2(x,x)}\right\}.
\end{aligned}$$