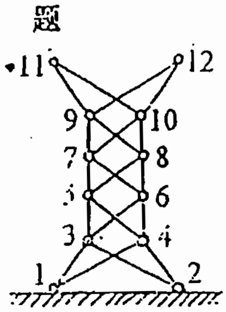
第二章　平面体系的机动分析

思考题（选答）

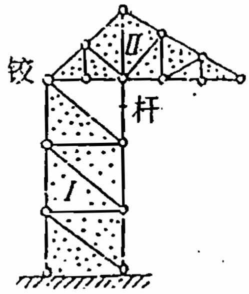
7. 不正确。原体系显然为几何不变无多余联系的。去二元体时每次都只能从暴露在最外面的两杆铰结点开始去，而不能从中间任意抽取。

8. 不正确。原体系少一联系因而是常变的。在机动分析过程中，每根杆件或作刚片或作链杆，都必须也只能使用一次，不得遗漏也不得重复。原题分析中，杆 2 使用了两次，故错。

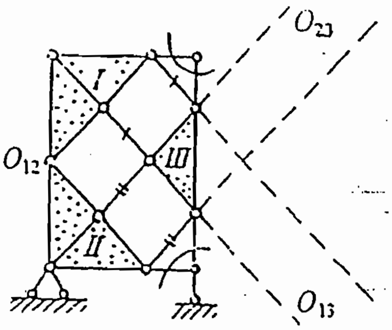
习　题



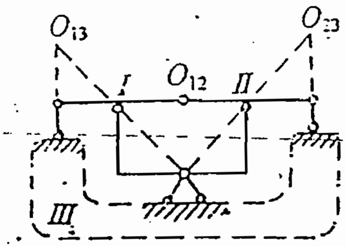
2-1　自下而上加二元体，几何不变无多余联系。



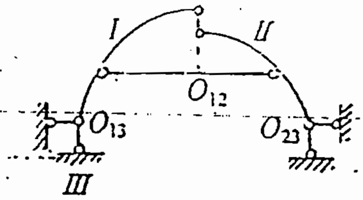
2-2　两刚片一铰一杆相联，几何不变无多余联系。



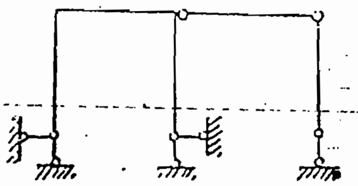
2-3　支杆 3 根只查本身。去二元体，再按三刚片分析，几何不变无多余联系。



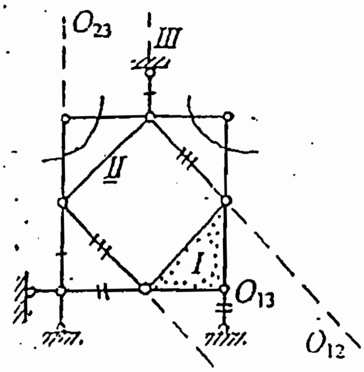
2-4　按三刚片规则，几何不变无多余联系。



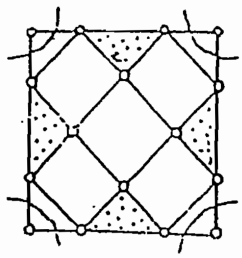
2-5　符合三刚片规则，几何不变无多余联系。



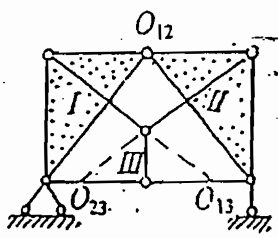
2-6　$W=0$. 左部有一多余联系，右部少一联系，常变。



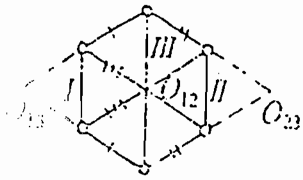
2-7　去二元体，再按三刚片分析，三铰不共线，几何不变无多余联系。



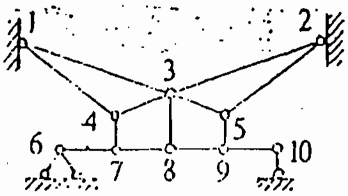
2-8　去四角之二元体后，$W=3\times4-8=4>3$，常变。



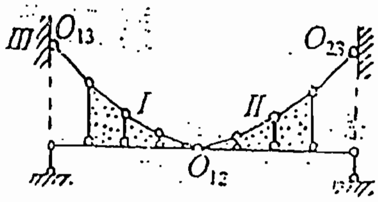
2-9　支杆 3 根只查本身。符合三刚片规则，几何不变无多余联系。



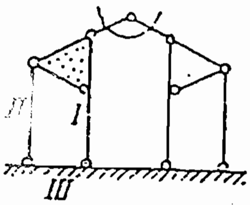
三刚片两两铰联，三铰共线，瞬变。



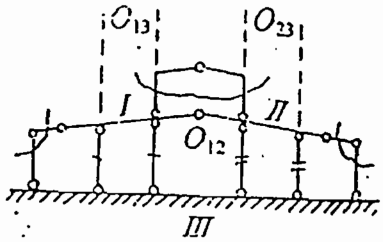
2-11　按1，2，3，…顺序加二元体，几何不变无多余联系。



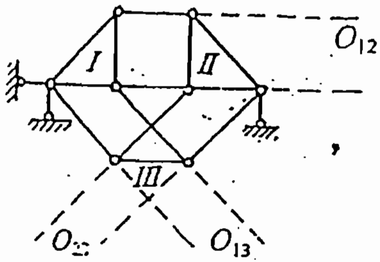
2-12　按三刚片规则，几何不变无多余联系。



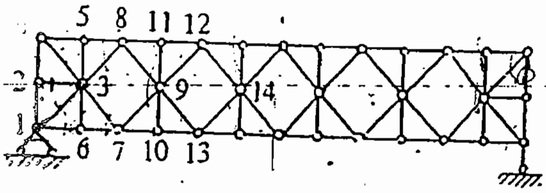
2-13　去二元体，左边按三刚片规则组成，右边同理，几何不变无多余联系。



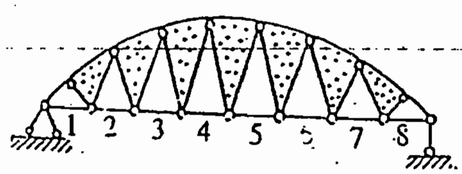
2-14　去二元体，再按三刚片，两铰无穷远，四杆皆平行，瞬变。



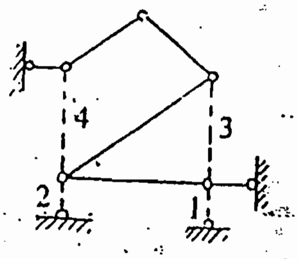
2-15　支杆3根只查本身。三刚片，三铰均无穷远，三对平行链杆各自等长，常变。



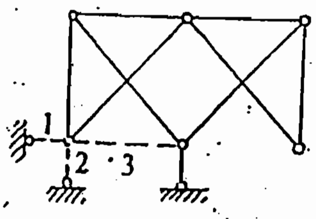
2-16　支杆3根只查本身。由左至右依次加二元体，几何不变，有一个多余联系。



2-17　支杆3根只查本身。曲杆为刚片，加二元体，几何不变，且有8根个多余链杆。



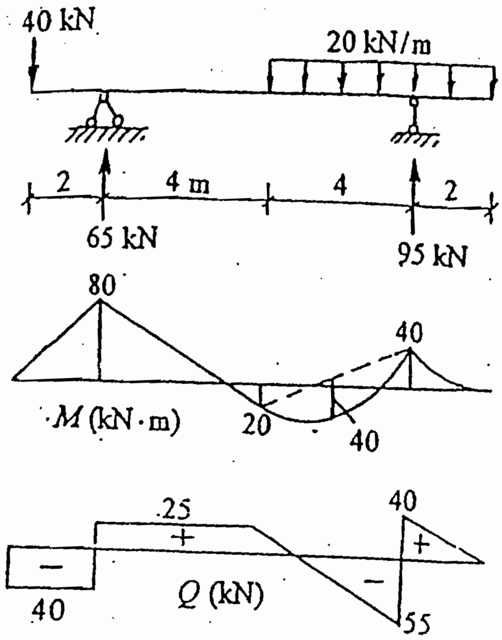
2-18　方案之一。



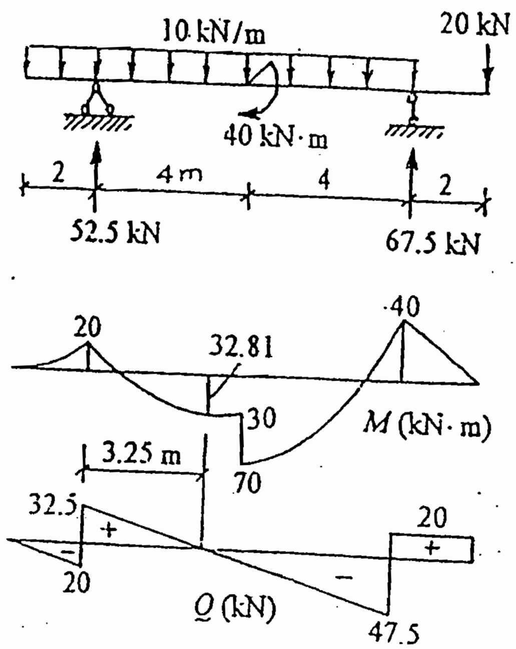
2-19　方案之一。

第三章　静定梁与静定刚架

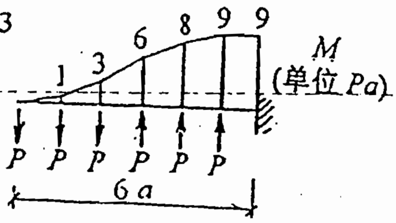
3-1



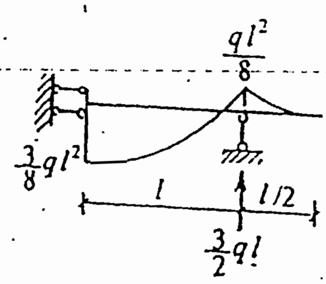
3-2



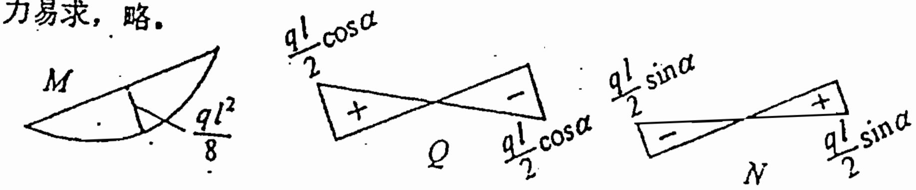
3-3



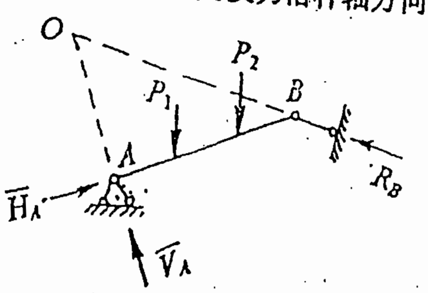
3-4



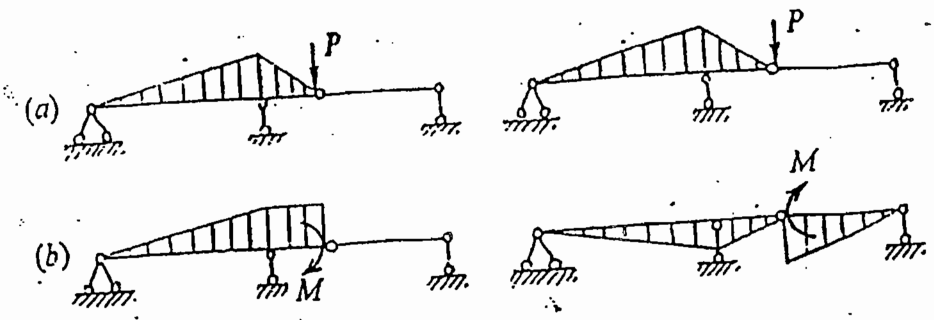
3-5　反力易求，略。



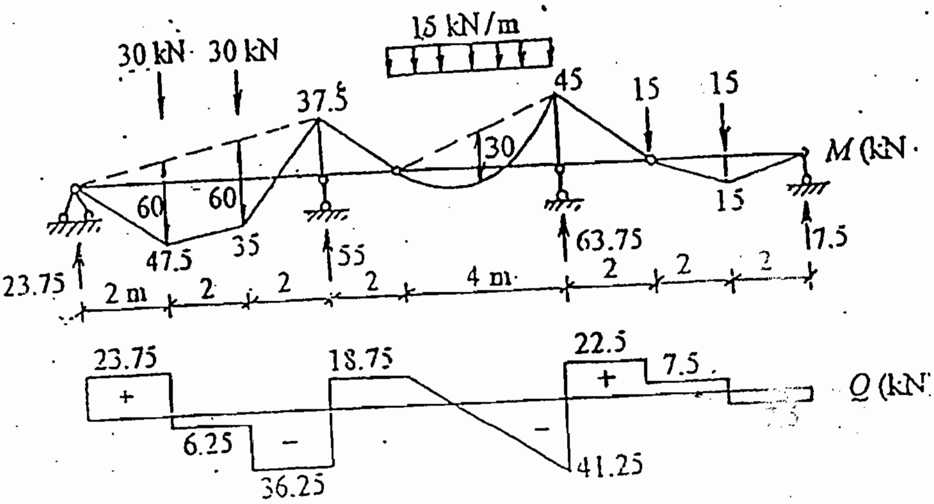
*3-6　将支座 A 的反力沿杆轴方向和横截面方向分解为 $\overline{H}_A$ 和 $\overline{V}_A$，则不论支杆 B 的方向如何改变，由 $\Sigma M_B=0$ 知 $\overline{V}_A$ 不变，故任一截面的弯矩和剪力不变（由该截面以左来计算）；但支杆 B 方向改变时，O 点位置改变，由 $\Sigma M_O=0$ 知 $\overline{H}_A$ 大小改变，故各截面的轴力大小亦随之而变。



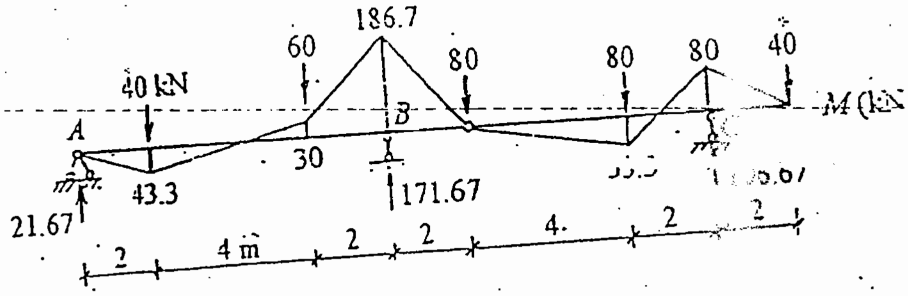
3-7



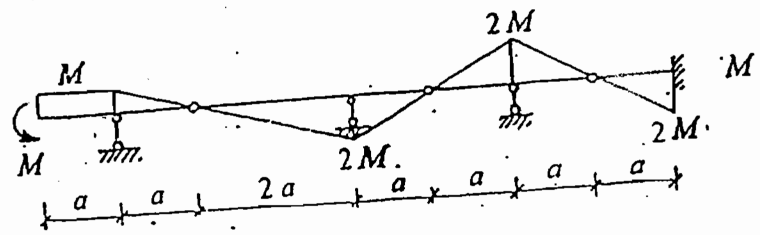
3-8



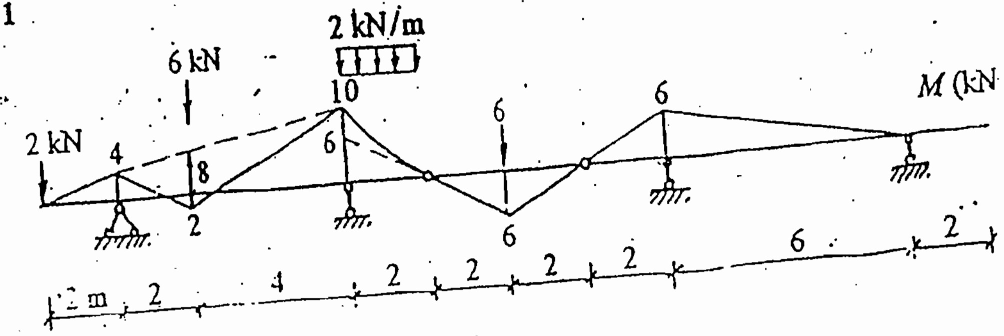
3-9



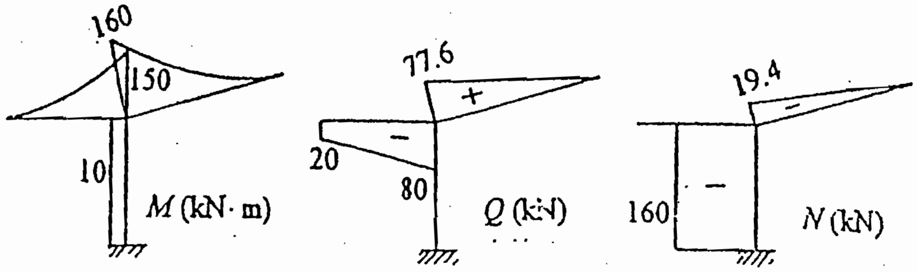
3-10



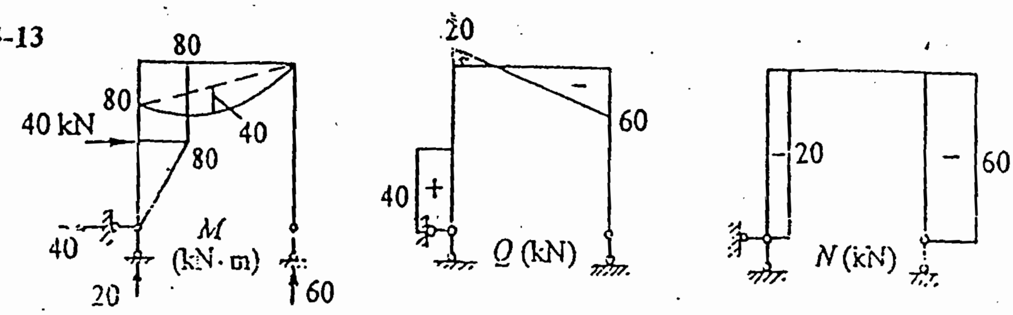
3-11



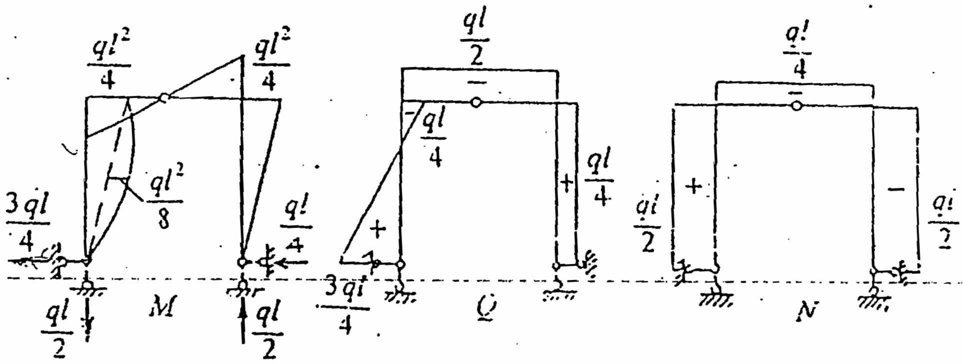
3-12



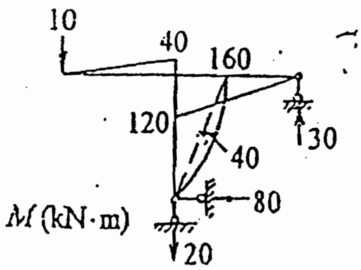
3-13



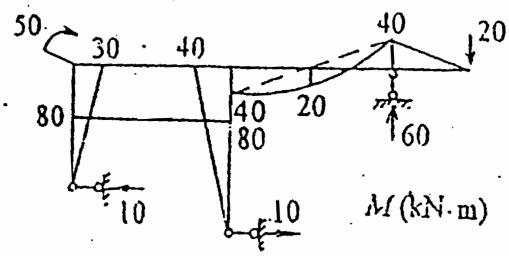
3-14



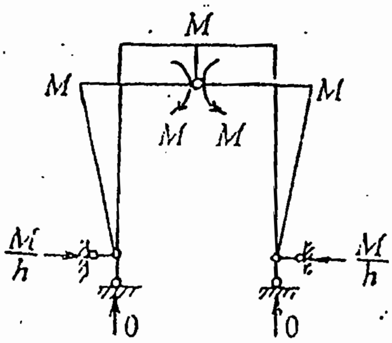
3-15



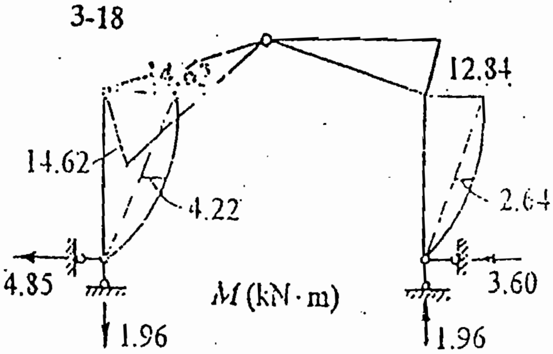
3-16



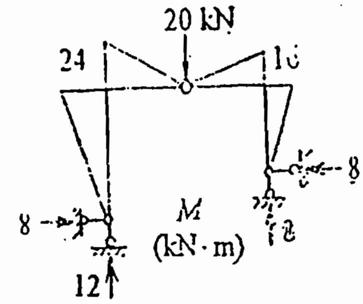
3-17



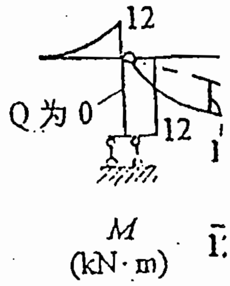
3-18



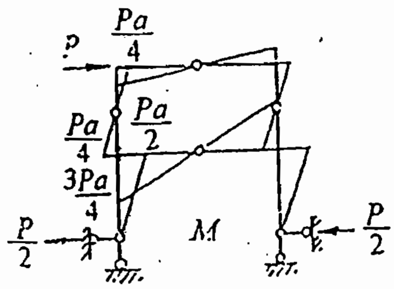
3-19



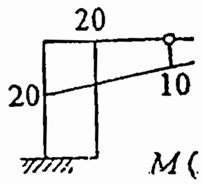
3-20



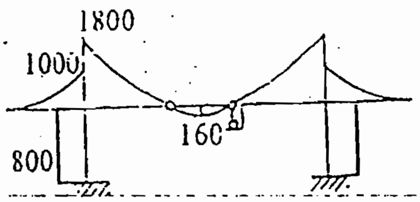
3-21



3-22

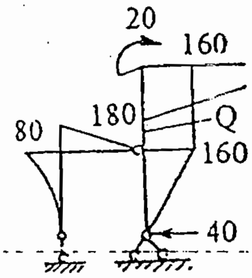


3-23

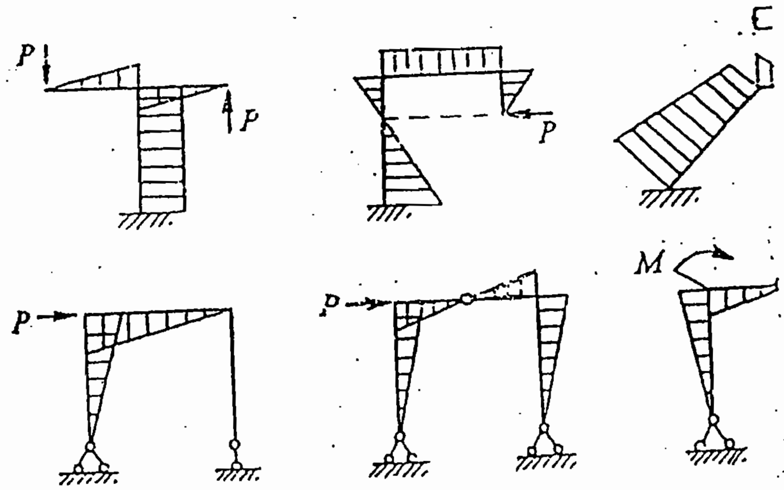


M (kN·m)

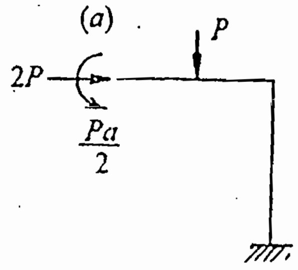
3-24

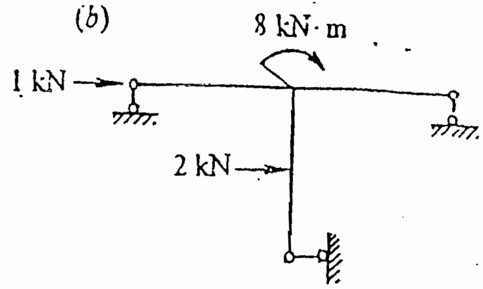


3-25 均有错，正确的如下：

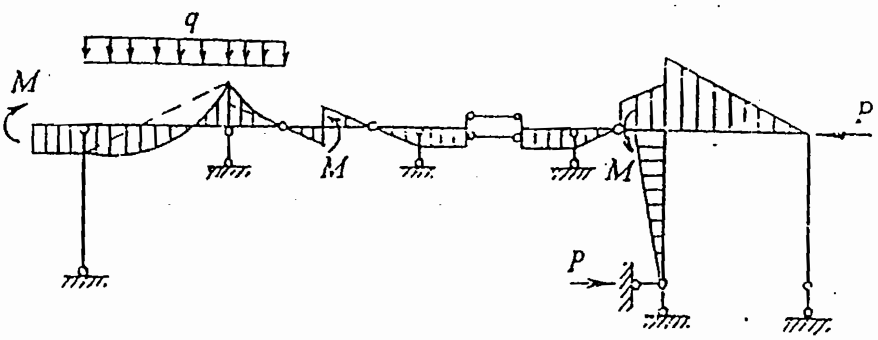


3-26

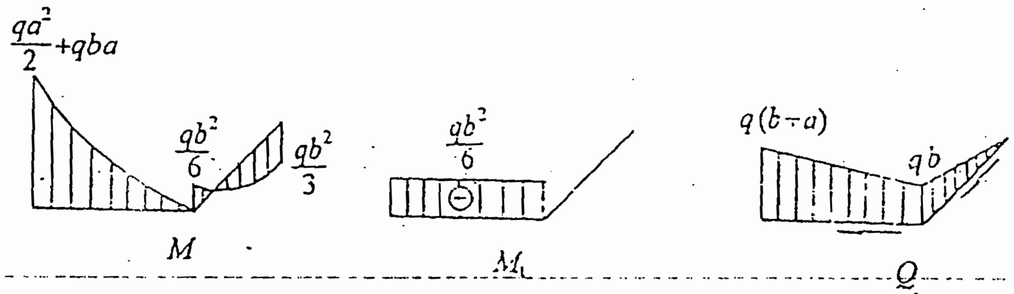




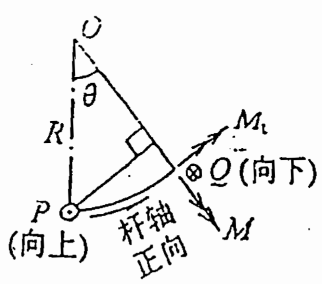
3-27



*3-28



*3-29



杆段隔离体(俯视图)

$Q=P$ （正面上剪力向下为正）

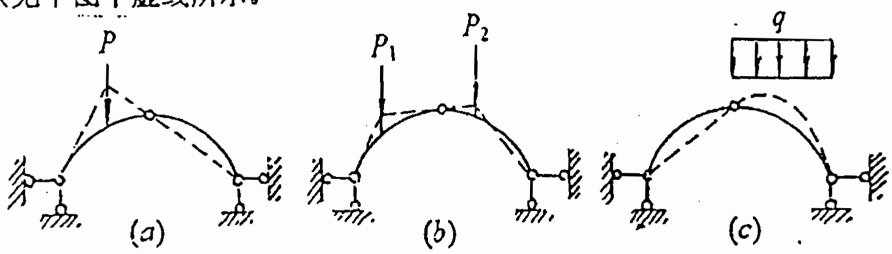
$M=PR\sin\theta$ （下边受拉为正）

$M_t=-PR(1-\cos\theta)$ （力偶矢量与截面外法线指向一致为正）

第四章　静定拱

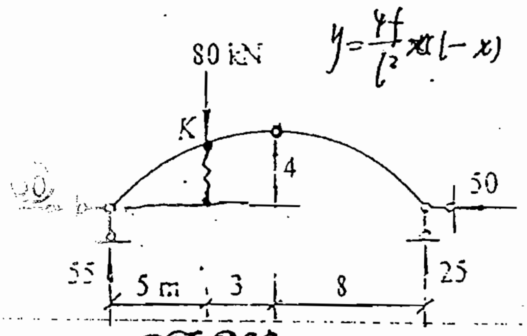
思考题（选答）

3. 拱上所有截面的弯矩均为零时的拱轴线称为合理拱轴线。原题各图合理拱轴线的形状见下图中虚线所示。



习　题

4-1　反力易求，如图



$y=\frac{4f}{l^2}x(l-x)=x(1-\frac{x}{16})$，　$y'=1-\frac{x}{8}$

$y_K=55/16=3.4375$，　$y'=3/8=0.375$

$\sin\varphi_K=0.3511$，　$\cos\varphi_K=0.9363$

$M_K=55\times5-50\times3.4375=103.1$ kN m

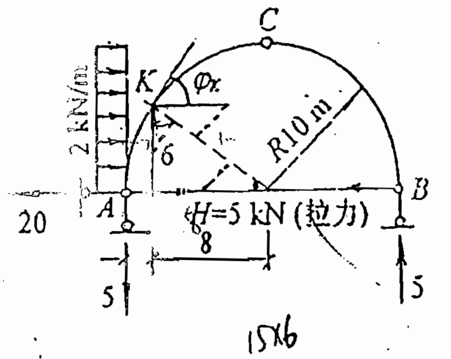
$Q_{K左}=55\times0.9363-50\times0.3511=33.9$ kN

$Q_{K右}=(55-80)0.9363-50\times0.3511=-41.0$ kN

$N_{K左}=55\times0.3511+50\times0.9636=66.1$ kN

$N_{K右}=(55-80)0.3511+50\times0.9636=38.0$ kN

4-2　反力及拉杆轴力 H 见图。



$M_K=-5\times2+(20-5)6-2\times6\times3=44$ kN m

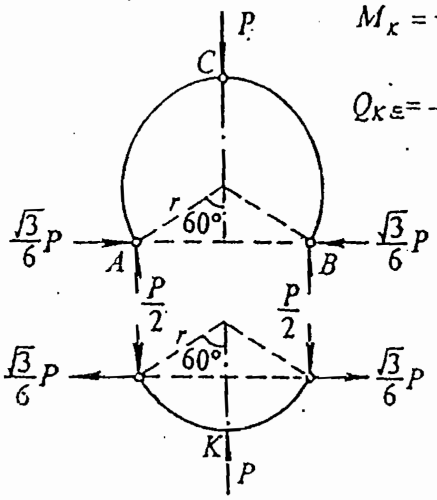
$Q_K=-5\times0.6+(20-5-2\times6)0.8=-0.6$ kN

$N_K=-5\times0.8-(20-5-2\times6)0.6=-5.8$ kN (拉力)

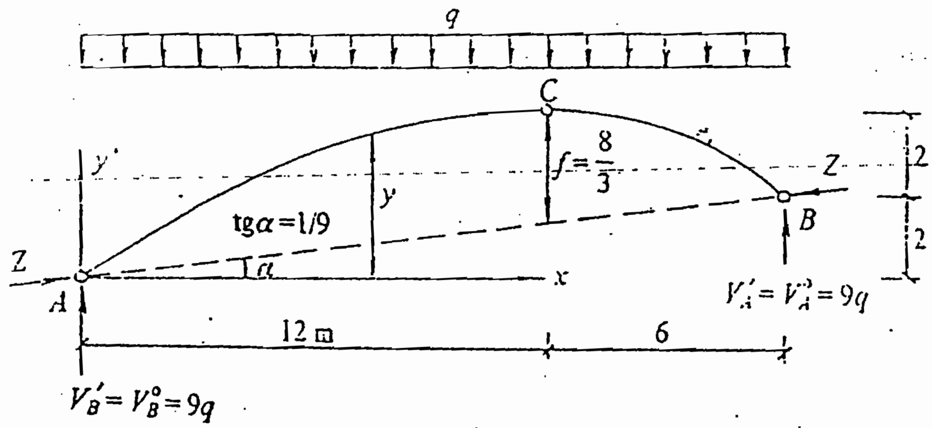
4-3　取上部，可得铰 A, B 处反力如图。再取下部，得

$$M_K=\frac{P}{2}\frac{\sqrt{3}}{2}r+\frac{\sqrt{3}}{6}Pr(1-\cos 60^\circ)=\frac{\sqrt{3}}{3}Pr \quad \text{（内侧受拉）}$$

$$Q_{K左}=-P/2,\quad Q_{K右}=P/2,\quad N_K=\frac{\sqrt{3}}{6}P \quad \text{(拉力)}$$



4-4



$$H=Z\cos\alpha=\frac{M_C^0}{f}=\frac{9q\times 12-\frac{q}{2}12^2}{\frac{8}{3}}=\frac{27}{2}q$$

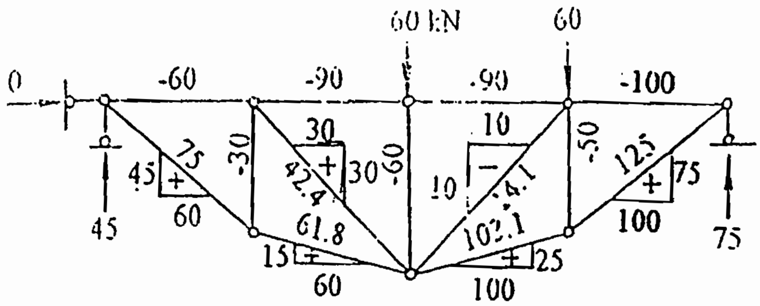
$$V_A=V_A^0+H\,\mathrm{tg}\,\alpha=9q+\frac{27}{2}q\frac{1}{9}=\frac{21}{2}q$$

$$M=V_Ax-\frac{qx^2}{2}-Hy=q\left(\frac{21x}{2}-\frac{x^2}{2}-\frac{27}{2}y\right)=0$$

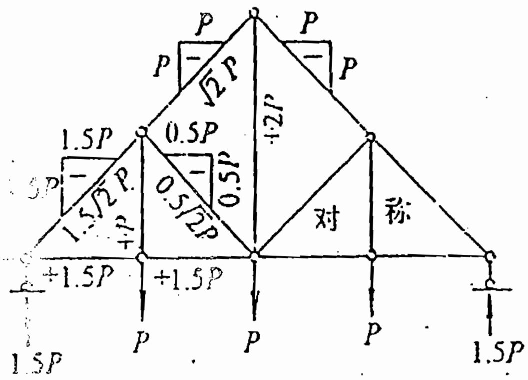
$$y=\frac{x}{27}(21-x)$$

第五章　静定平面桁架

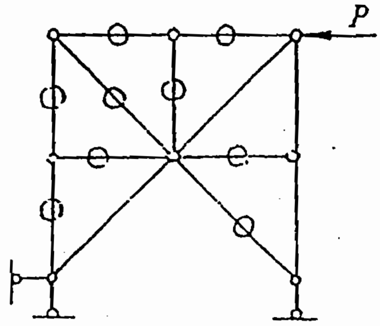
5-1



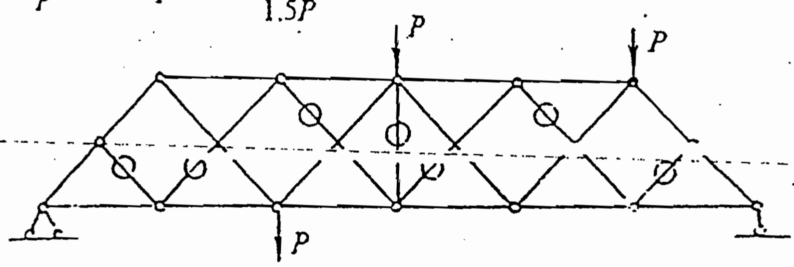
5-2



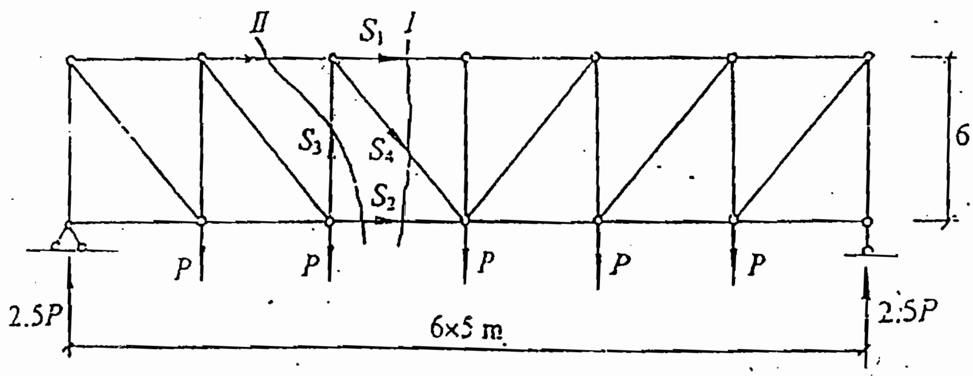
5-4



5-3



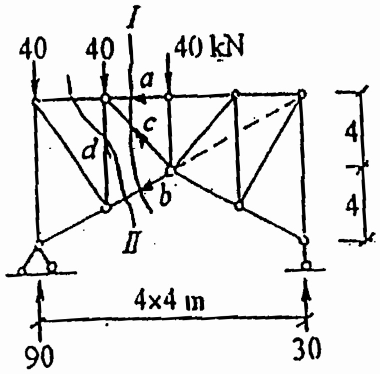
5-5



$$S_1 = -\frac{2.5\times15 - 1\times10 - 1\times5}{6}P = -3.75P \qquad S_4 = \frac{\sqrt{61}}{6}(2.5P - P - P) = 0.65P$$

$$S_2 = \frac{2.5\times10 - 1\times5}{6}P = 3.33P \qquad S_3 = -(2.5P - P - P) = -0.5P$$

5-6



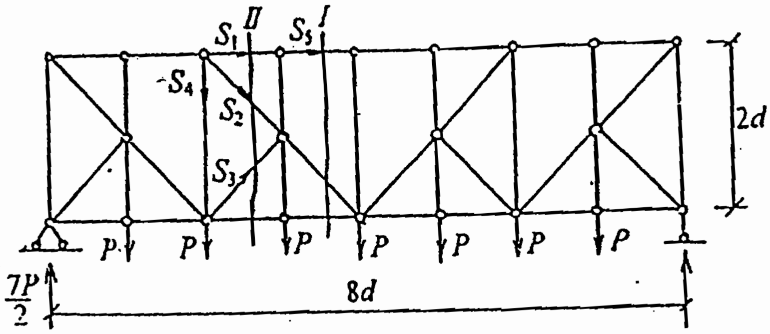
$$S_a = -\frac{30\times 8}{4} = -60\ \text{kN}$$

$$S_b = \left(\frac{30\times 12 - 40\times 4}{6}\right)\frac{\sqrt{5}}{2} \doteq 33.33\times\frac{\sqrt{5}}{2} = 37.27\ \text{kN}$$

$$S_c = \left(\frac{40\times 8}{12}\right)\sqrt{2} = 26.67\sqrt{2} = 37.71\ \text{kN}$$

$$S_d = -\frac{(90-40)16}{12} = -66.67\ \text{kN}$$

5-7



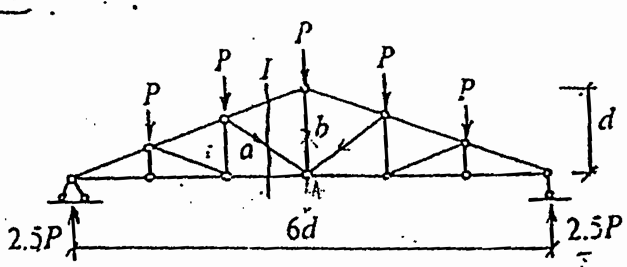
$$S_1 = S_5 = -\frac{\frac{7}{2}\times 4 - 1\times 3 - 1\times 2 - 1\times 1}{2}P = -4P$$

$$S_2 = \left(-\frac{\frac{7}{2}\times 2 - 1\times 1 - 4\times 2}{2}\right)P\sqrt{2} = \sqrt{2}P$$

$S_3 = -\frac{\sqrt{2}}{2}P$（结点法）

$S_4 = -P$（结点法）

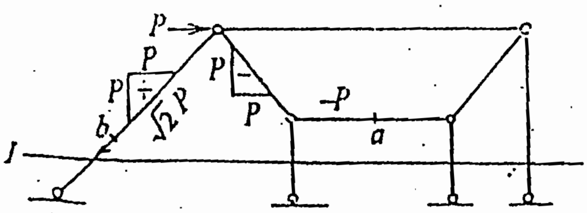
5-8



$$Y_a = -\frac{Pd + P\times 2d}{3d} = -P,\qquad S_a = -\frac{\sqrt{13}}{2}P = -1.803P$$

$S_b = -2Y_a = 2P$（结点法）

5-9

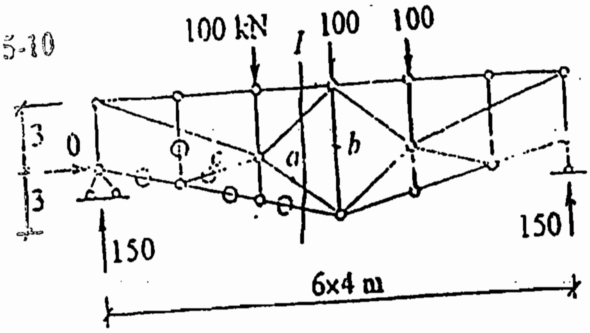


先由截面 I，$\Sigma X = 0$ 得

$$X_b = P$$

再依次用结点法求。

5-10



先判零杆，知 $S_c=0$.

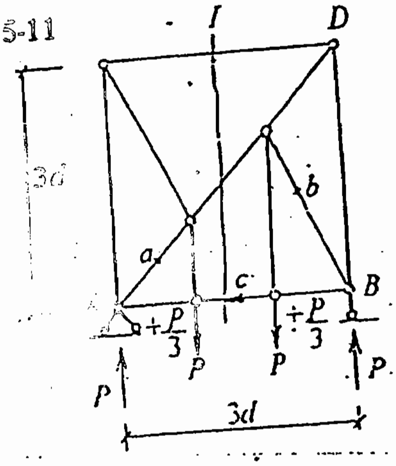
$X_a=(150\times12-100\times4)/6=233.33$ kN

$S_a=233.33\ (5/4)=291.67$ kN

$Y_a=233.33\ (3/4)=175$ kN

$S_b=-2Y_a=-350$ kN

5-11



截面 I 右，$\Sigma M_D=0$，$S_c=\dfrac{Pd}{3d}=\dfrac{P}{3}$

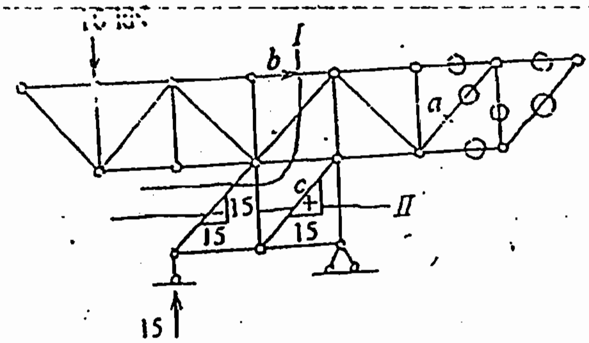
结点 A，$S_a=-\dfrac{\sqrt{2}}{3}P$

结点 B，$S_b=(-\dfrac{P}{3})\dfrac{\sqrt{5}}{1}=-\dfrac{\sqrt{5}}{3}P$

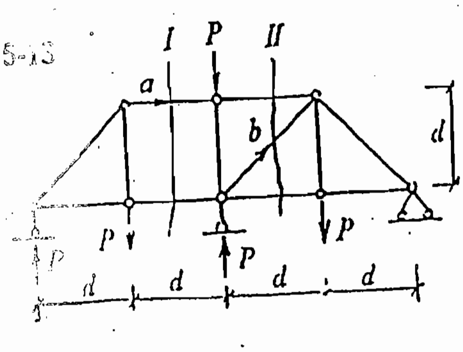
5-12

I 左，$S_b=\dfrac{10\times2d}{d}=20$ kN

II 上，$S_c=15\sqrt{2}=21.21$ kN



5-13



截面 I 左，$S_a=-P$

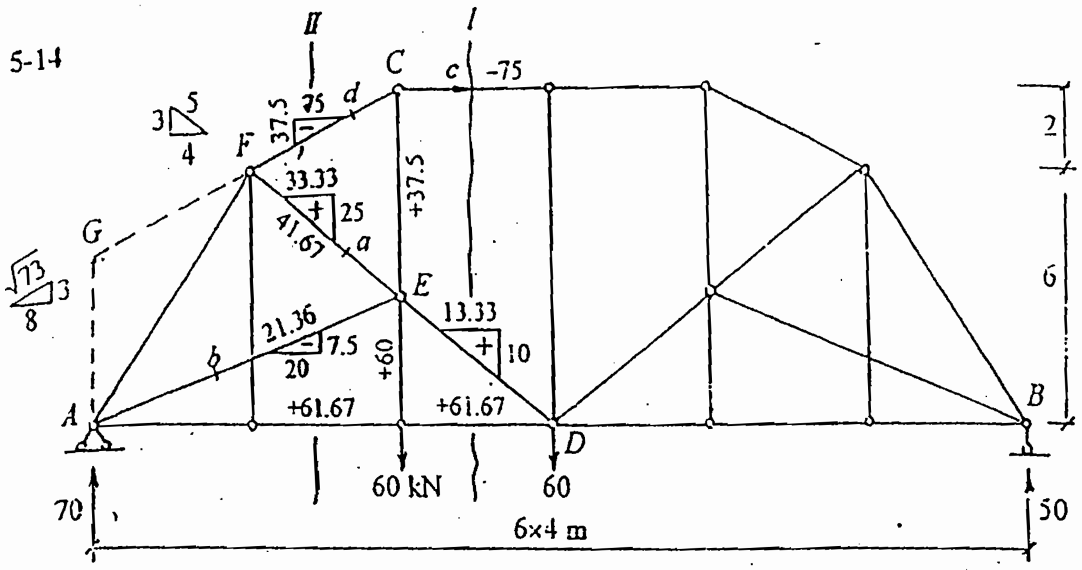
截面 II 左，$Y_b=2P-2P=0$

$S_b=0$

5-14

5-15

5-14



可有不同方法，较简之一为：

截面 I 左，$\Sigma M_D=0$ 得

$$S_C=\frac{-(70\times12-60\times4)}{8}=-75$$ （单位 kN，下同）

结点 C 有

$$X_d=-75$$

截面 II 左，$\Sigma M_A=0$，且将 S_a 在 D 分解，S_d 在 G 分解，有

$$Y_a=\frac{75\times4}{12}=25$$

$$S_a=25\,(5/3)=41.67$$

截面 II 左，$\Sigma Y=0$，有

$$Y_b=-(70-37.5-25)=-7.5$$

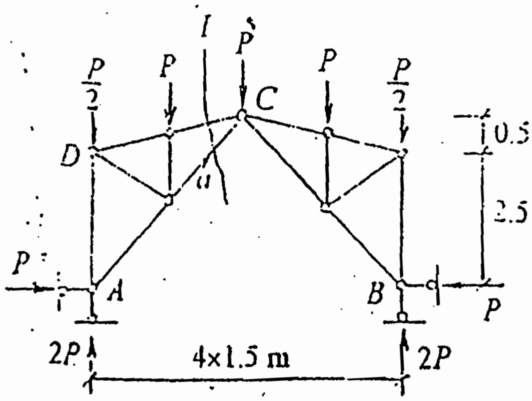
$$S_b=-7.5\frac{\sqrt{73}}{3}=-21.36$$

5-15 先求得反力如图，再由截面 I 左：

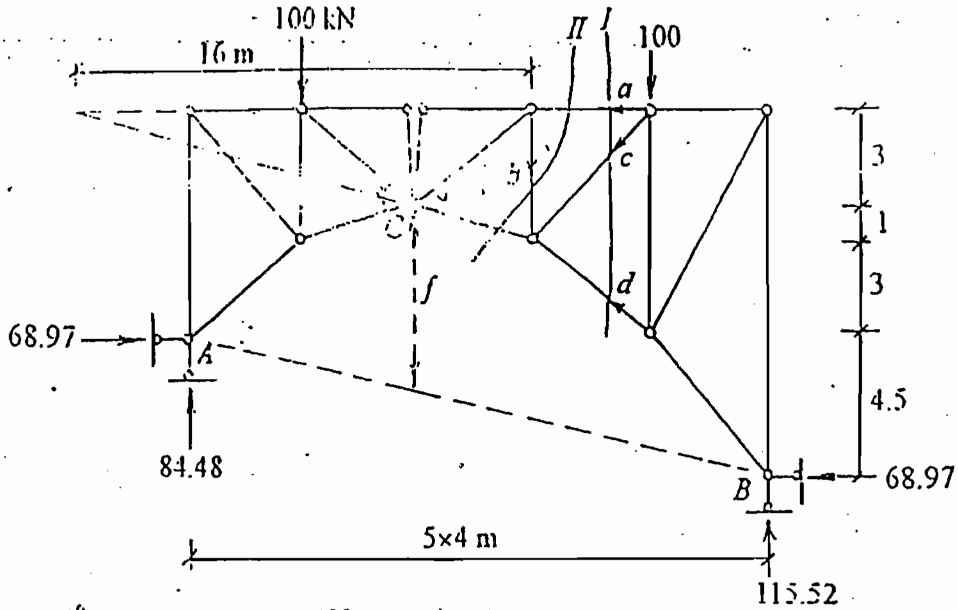
$\Sigma M_D=0$ 得

$$X_a=\frac{-P\times2.5-P\times1.5}{2.5}=-0.4P$$

$$S_a=-0.4\sqrt{2}\,P=-0.5657P$$



5-16



$$H=\frac{M_C^0}{f}=\frac{100\times 4}{4+4.5(8/20)}=\frac{400}{5.8}=68.97$$ (单位 kN，下同)

$$V_A=100-\frac{4.5}{20}H=84.48,\qquad V_B=100+\frac{4.5}{20}H=115.52$$

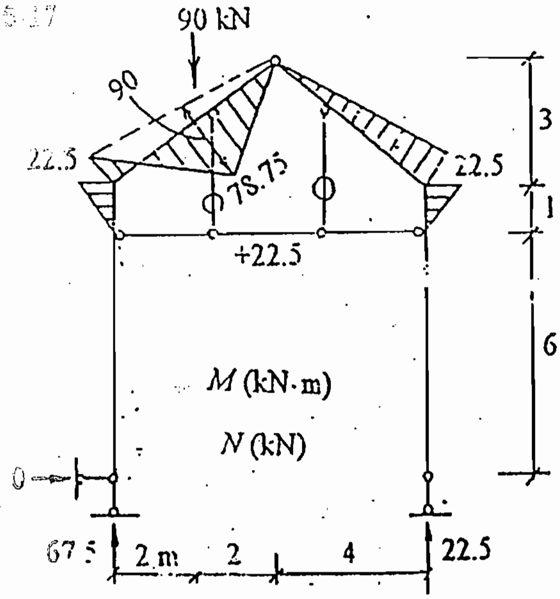
$$S_c=\frac{-115.52\times 8+68.97\times 7.5+100\times 4}{4}=-1.72$$

$$X_d=\frac{115.52\times 4-68.97\times 11.5}{7}=-47.29,\quad Y_d=-35.47,\quad S_d=-59.11$$

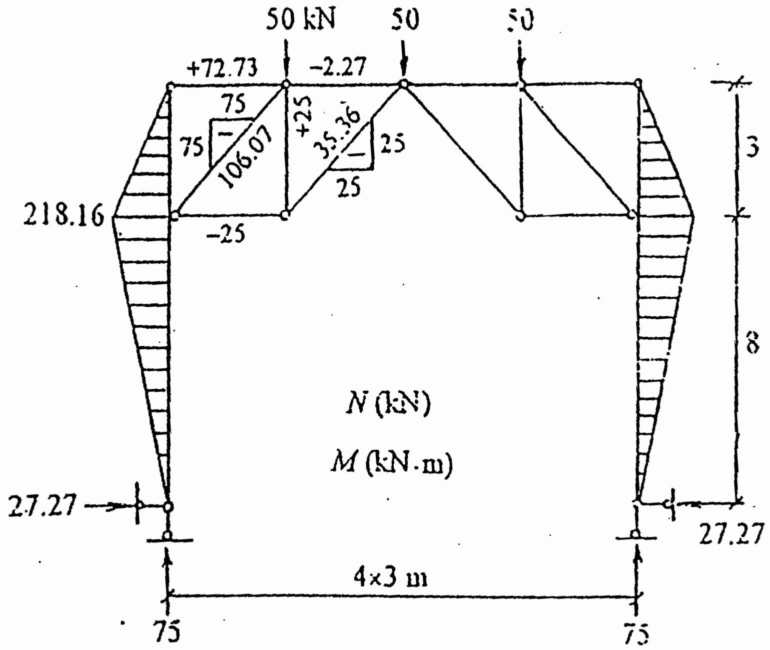
$$Y_c=115.52-100-35.47=-19.95,\qquad S_c=-28.21$$

$$S_b=\frac{84.48\times 4+68.97\times 7-100\times 8}{16}=1.29$$

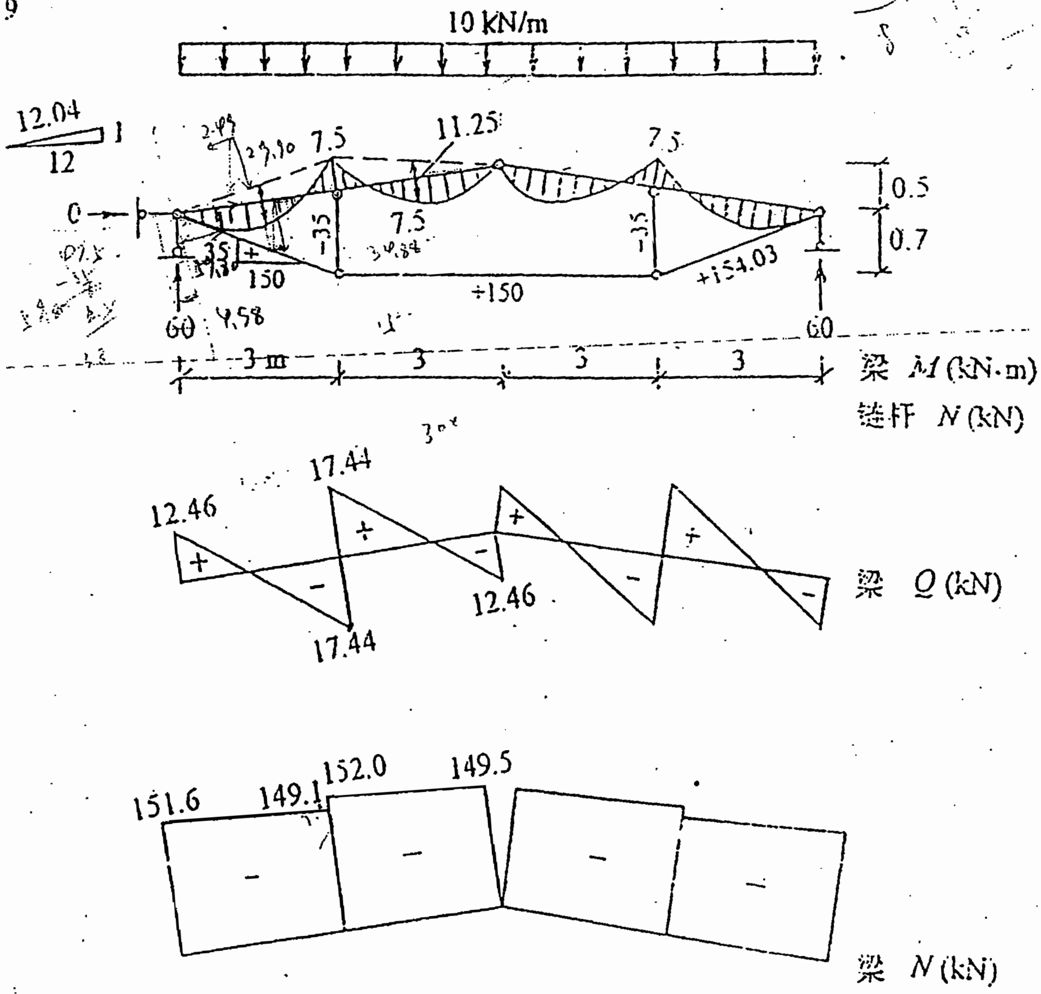
5-17



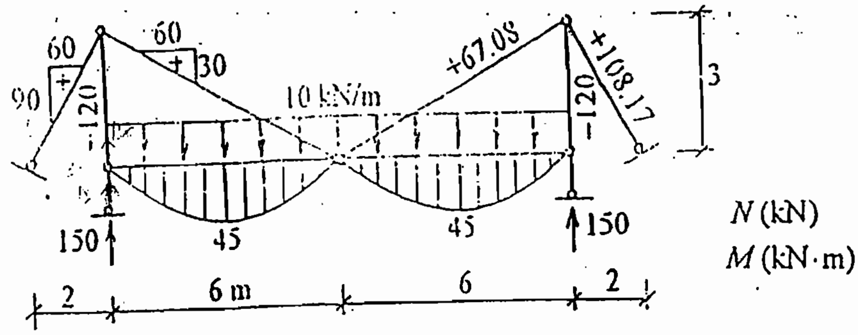
5-18


50 kN
50
50
+72.73
−2.27
75
75
106.07
+25
35.36
25
25
218.16
−25
3
8
N (kN)
M (kN·m)
27.27
27.27
4×3 m
75
75

5-19

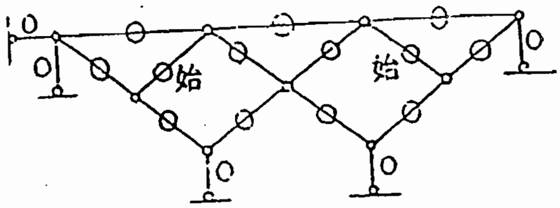

10 kN/m
12.04
12
1
7.5
11.25
7.5
0.5
0.7
0
−35
7.5
−35
150
+150
+154.03
60
60
3 m
3
3
3
梁 M (kN·m)
链杆 N (kN)
17.44
12.46
12.46
17.44
梁 Q (kN)
151.6
149.1
152.0
149.5
梁 N (kN)

5-20

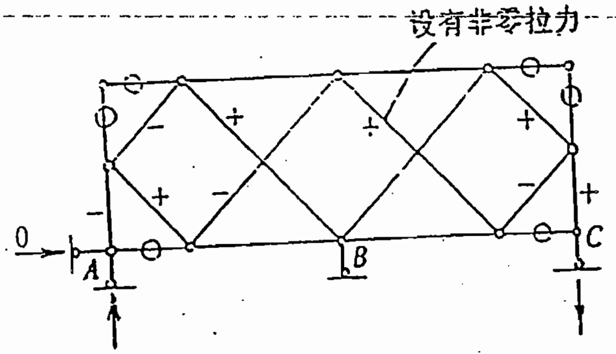


$$H=\frac{M_C^0}{f}=\frac{10\times 12^2}{8\times 3}=60\,\text{kN}$$

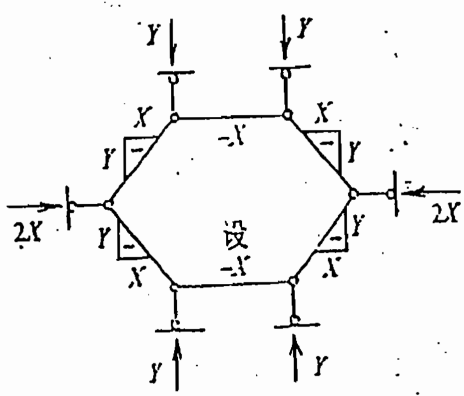
*5-21 $W=2\times 9-(13+5)=0$，零载时，必全为零杆，几何不变。



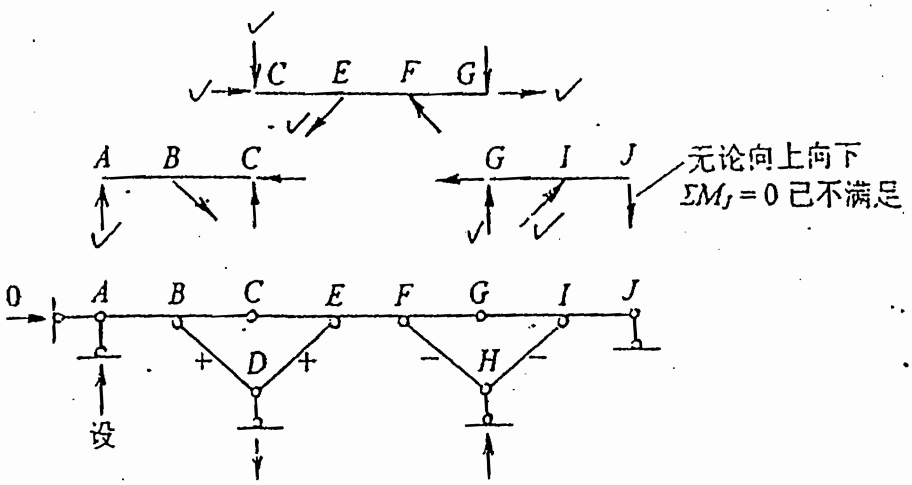
*5-22 $W=2\times 12-(20+4)=0$，零载时，设有图示非零内力，得两端竖向反力方向相反，已不能满足 $\Sigma M_B=0$，故只能全为零杆，几何不变。



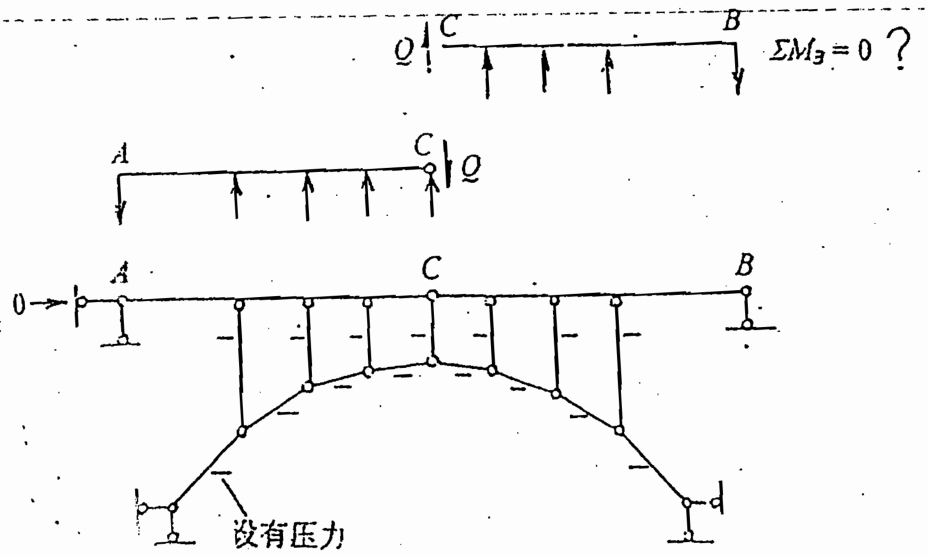
*5-23 $W=2\times 6-(6+6)=0$，零载时，非零内力可平衡，故为非几何不变。



*5-24 $W=3\times7-(2\times8+5)=0$．零载时，设 A 处有向上的非零反力，自左至右依次取隔离体判定各力所需方向（打 ✓ 者为已判定方向），最后取 GJ 杆时，已无法满足 $\Sigma M_J=0$。故 A 处反力只能为零，从而所有反力内力均必为零，体系几何不变。



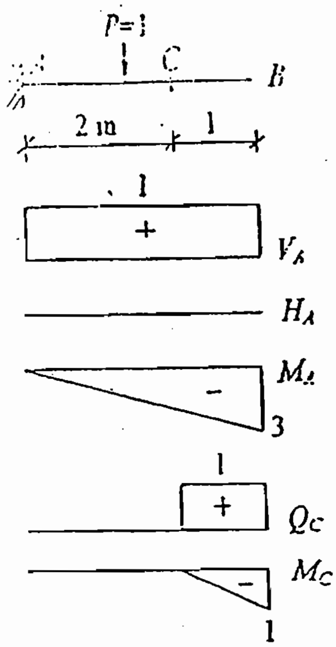
5-25 $W=3\times17-(2\times22+7)=0$．零载时，设拱部有非零压力，由结点平衡可推知所有竖杆均为压力；若从铰 C 右侧切开，取 AC 为隔离体，由 $\Sigma M_A=0$ 知 Q 必向下；再取 CB 为隔离体，Q 则向上，但这已不能满足 $\Sigma M_B=0$。因此只能是所有内力反力均为零，体系几何不变。



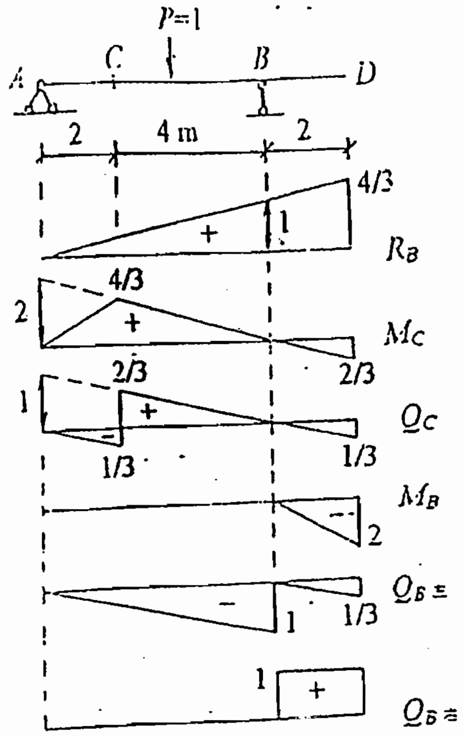
第六章　影响线及其应用

6-1 略。

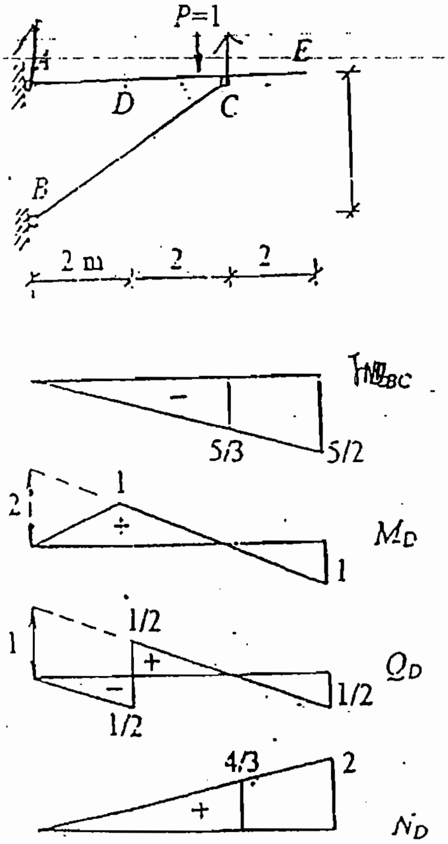
6-2



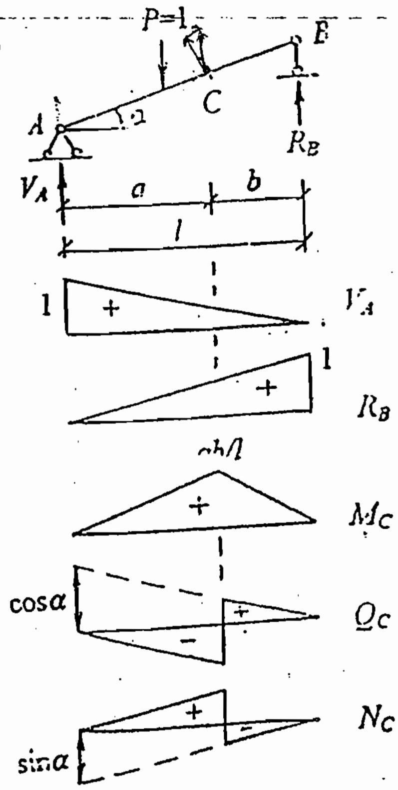
6-3



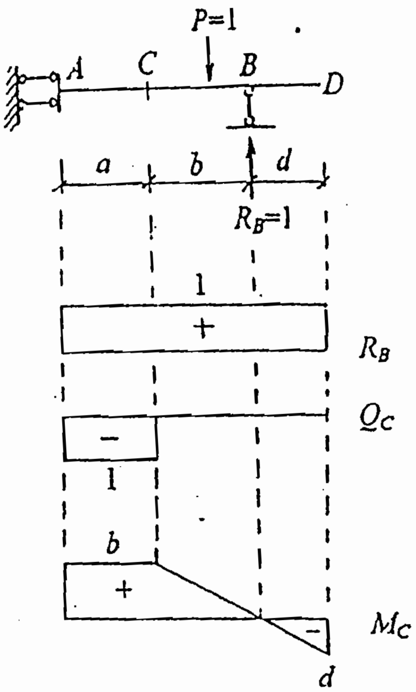
6-4



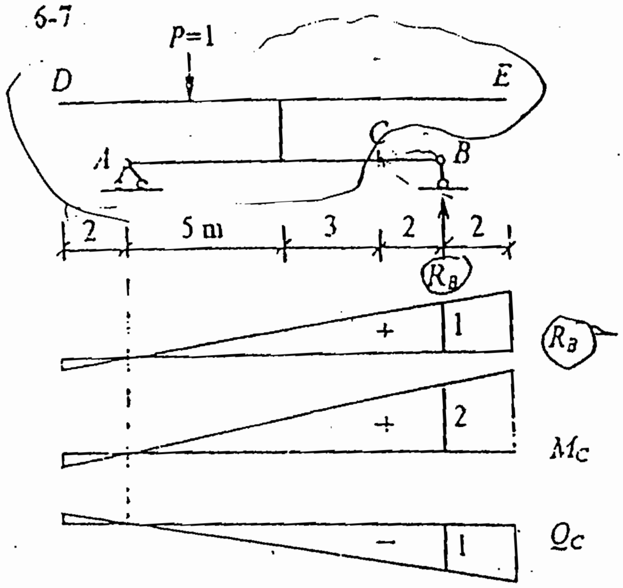
6-5



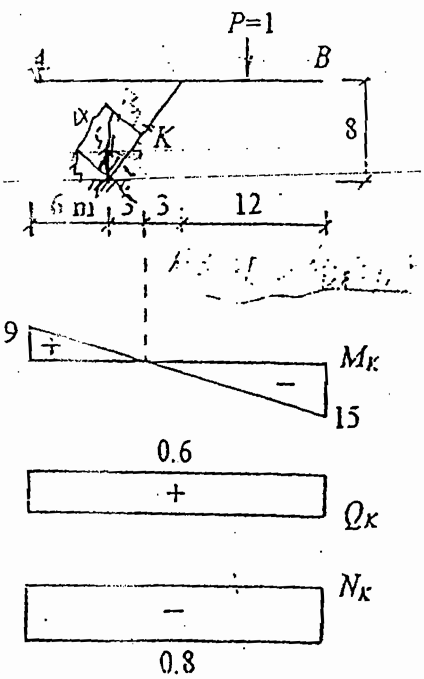
6-6


P=1
A
C
B
D
a
b
d
R_B=1
1
+
R_B
Q_C
−
1
b
+
−
M_C
d

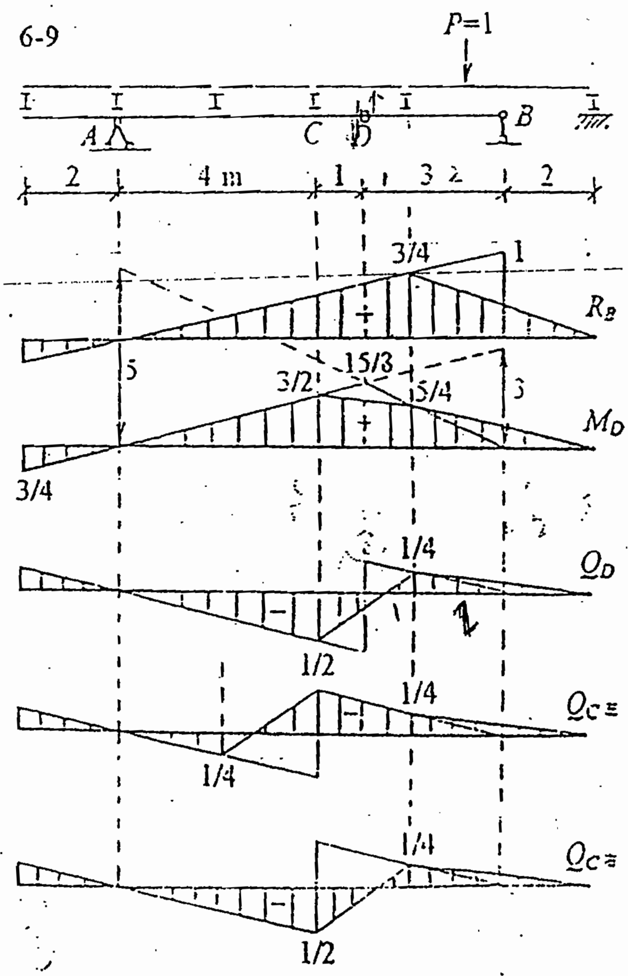
6-7


P=1
D
E
C
A
B
2
5 m
3
2
2
R_B
+
1
R_B
+
2
M_C
−
1
Q_C

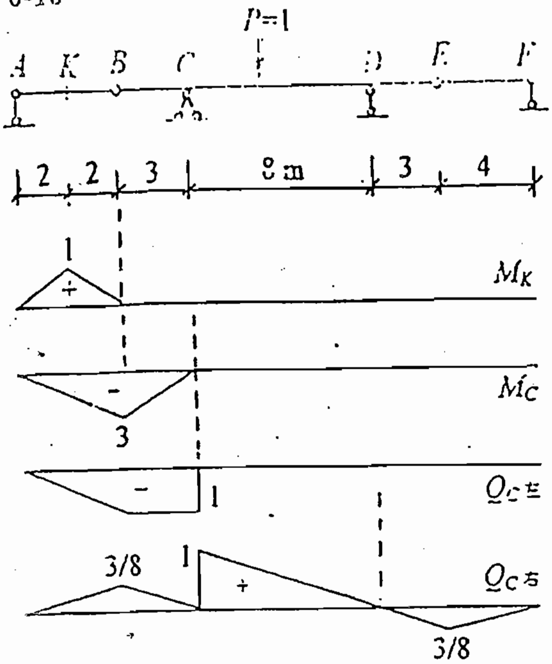
6-8


P=1
A
B
K
8
6 m
3
3
12
9
+
−
M_K
15
0.6
+
Q_K
N_K
−
0.8

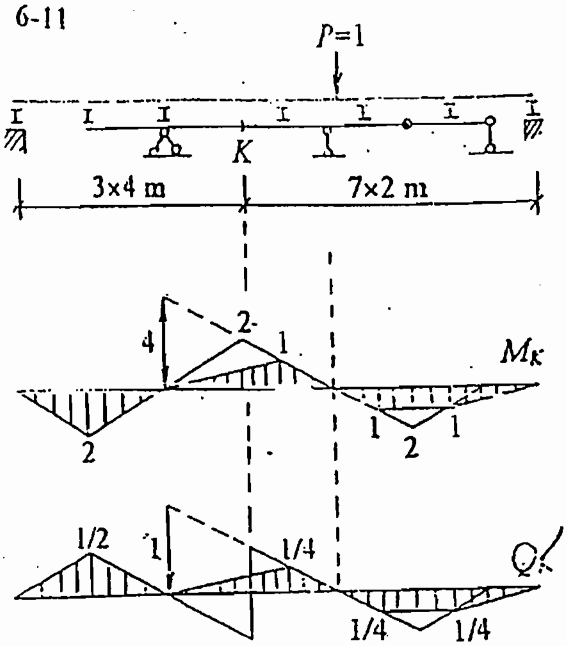
6-9


P=1
A
C
D
B
2
4 m
1
3
2
3/4
1
+
R_B
5
3/2
15/8
5/4
3
+
M_D
3/4
1/4
Q_D
−
1/2
1/4
Q_C左
1/4
1/4
Q_C右
−
1/2

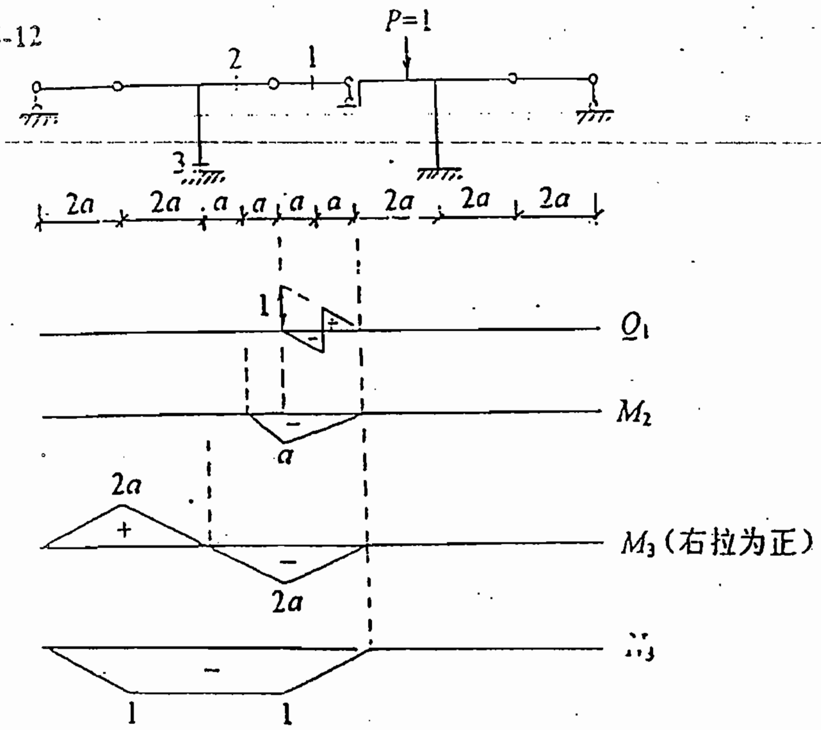
6-10


P=1
A K B C D E F
2 2 3 8 m 3 4
1
+
M_K
-
3
M_C
-
1
$Q_{C左}$
3/8
1
+
$Q_{C右}$
3/8

6-11


P=1
K
3×4 m
7×2 m
4
2
1
M_K
2
1
2
1
1/2
1
1/4
1/4
1/4

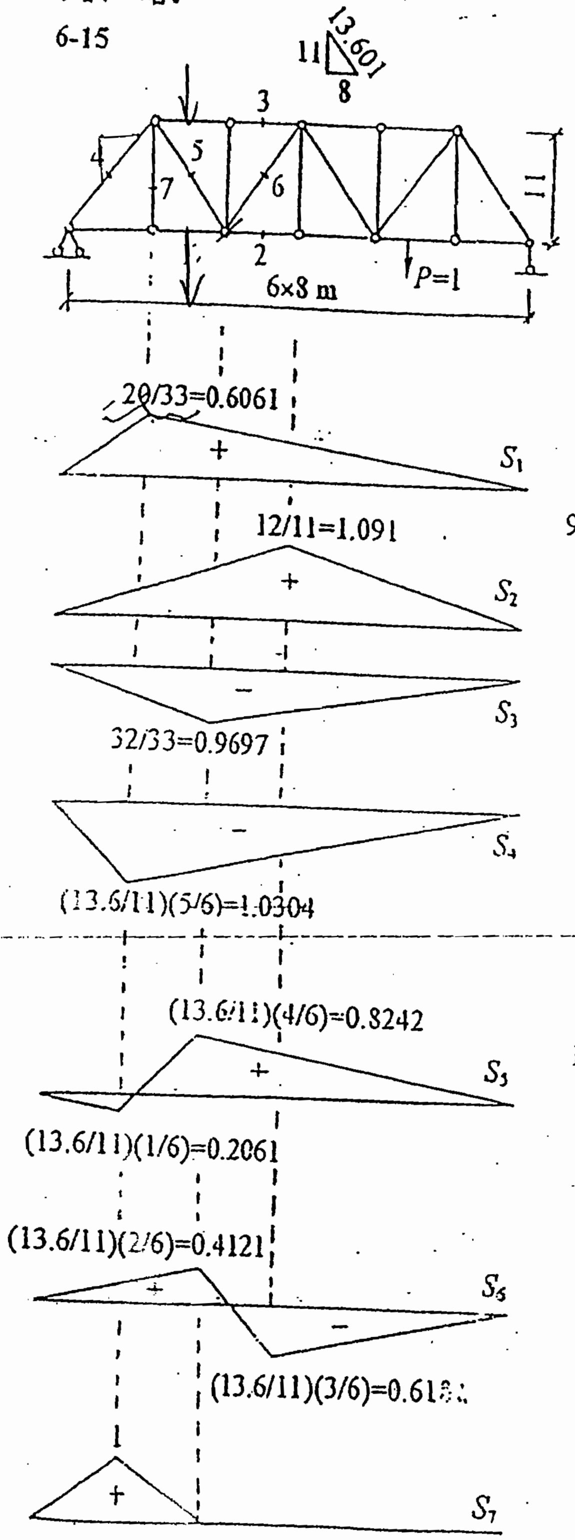
6-12


P=1
2
1
3
2a 2a a a a a 2a 2a 2a
1
+
-
Q_1
-
a
M_2
2a
+
-
2a
M_3（右拉为正）
-
1
1

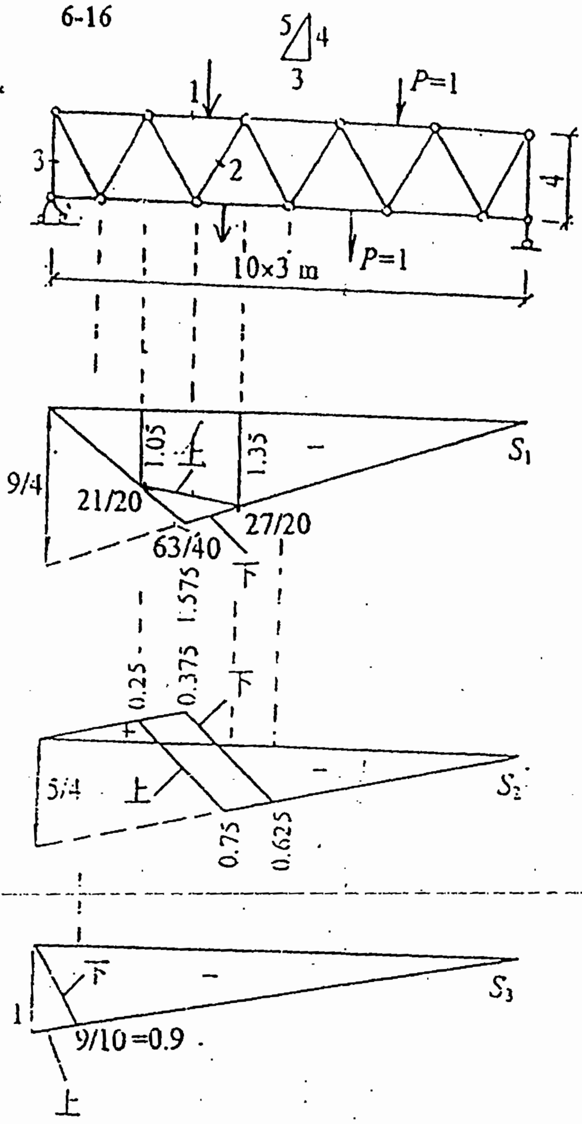
6-13 略。

6-14 略。

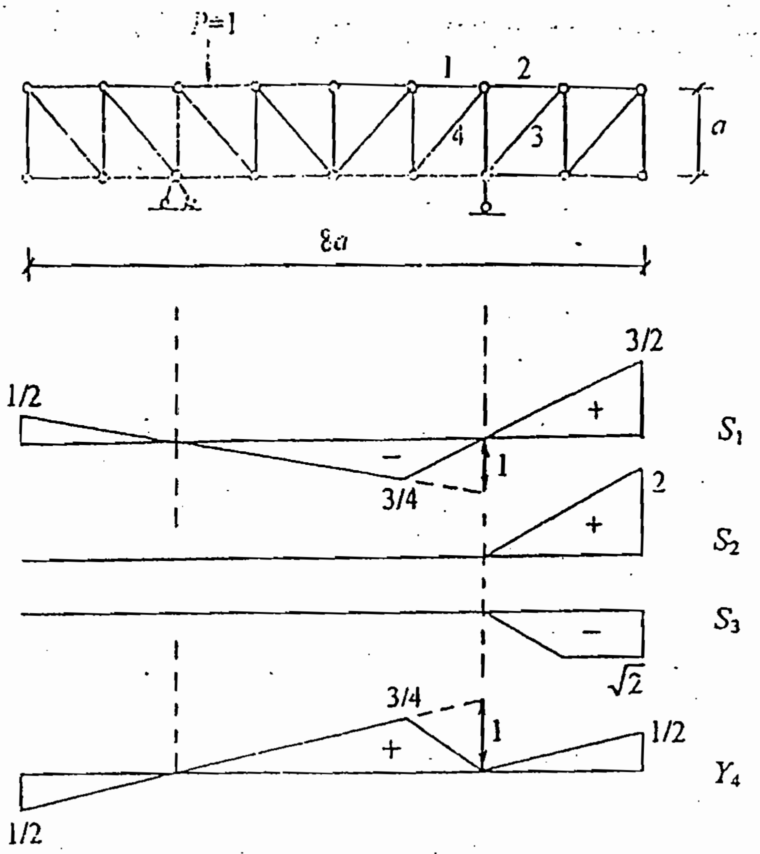
6-15



6-16



6-17



6-18 S_a、Y_b易作。

由对称，可得 Y_d。

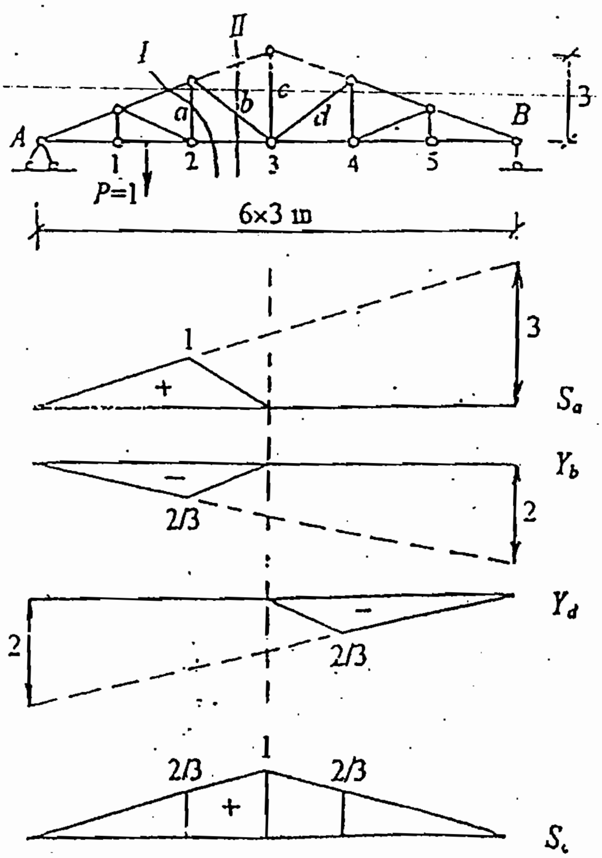
S_c作法：结点法，取结点 3，

当 $P=1$ 不在 3 时，有

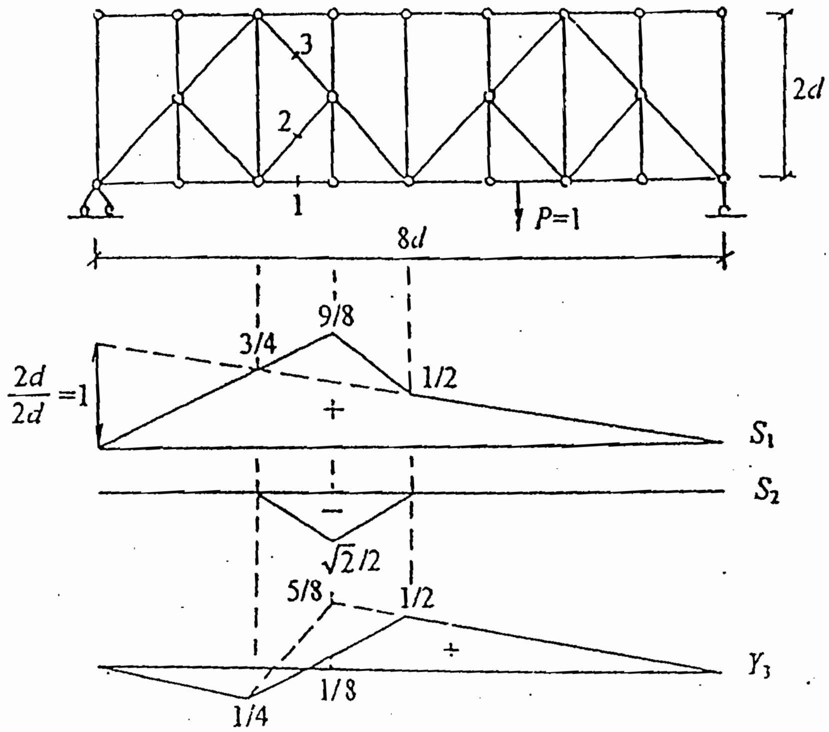
$$S_C=-(Y_b+Y_d)$$

当 $P=1$ 在 3 时，有

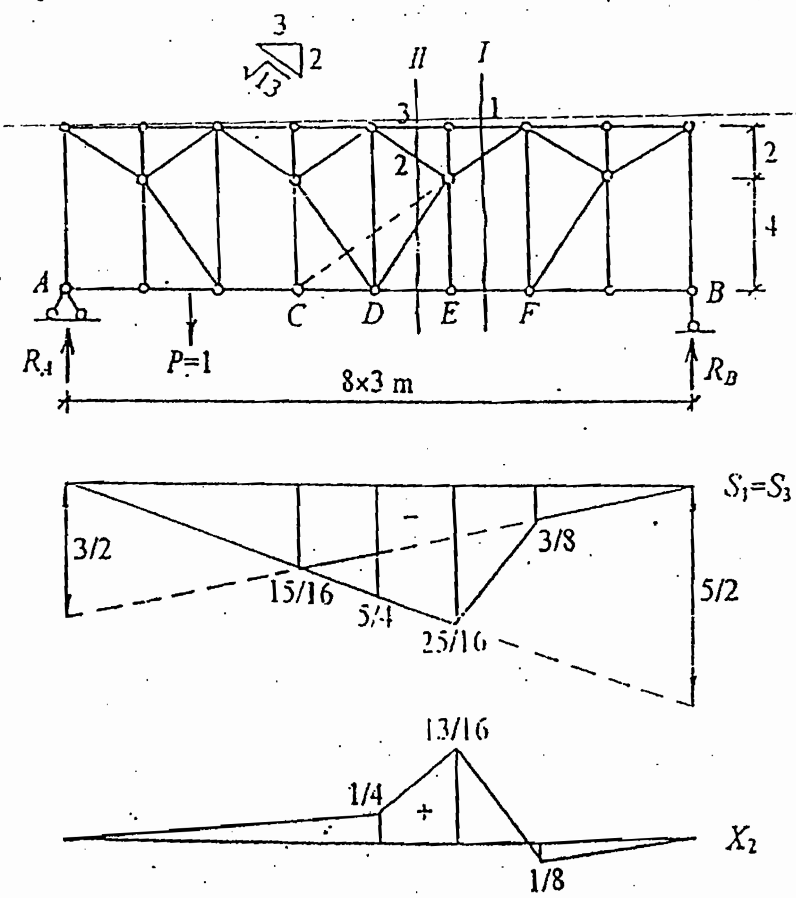
$$S_C=-(Y_b+Y_d)+1$$
$$=-(0+0)+1$$
$$=+1$$



6-19



6-20



S_2作法：

当 $P=1$ 在 AD 段，

取 II 右，$\Sigma M_D=0$，

$X_2=-2R_B-S_3$

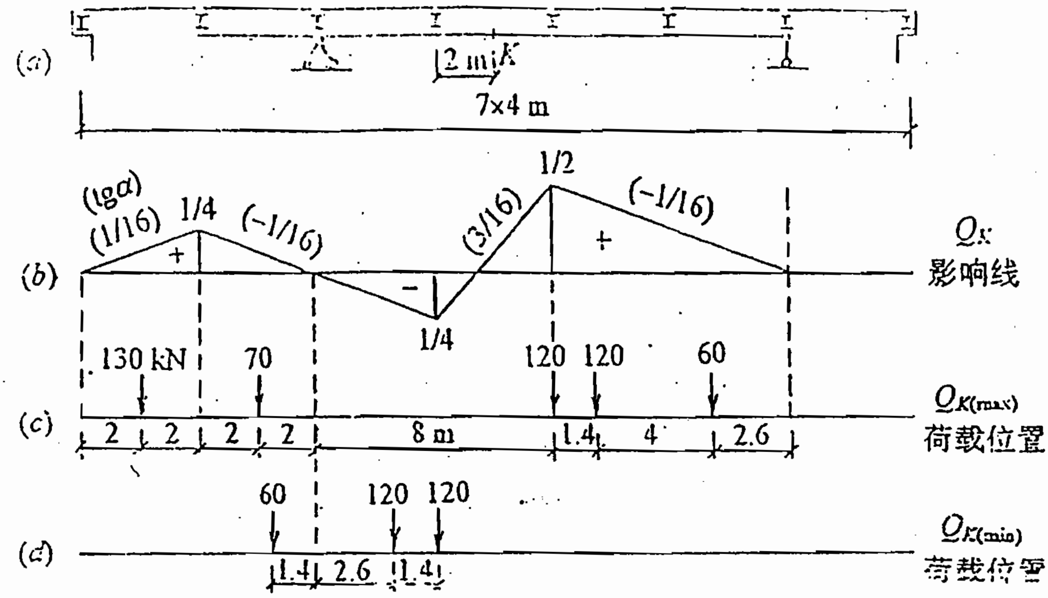
当 $P=1$ 在 EB 段，

取 II 左，$\Sigma M_D=0$，

$X_2=-2R_A-S_3$

由此可计算各控制点的竖标。

6-21



求 $Q_{K(\max)}$：判断知，应以车队向右行驶，重车第二后轮位于 $y_{\max}$ 处（图 c）试算：

左移　$\Sigma R\mathrm{tg}\alpha = (1/16)(130-70+3\times120-120-60) > 0$

右移　$\Sigma R\mathrm{tg}\alpha = (1/16)(130-70-120-120-60) < 0$

变号，判别式满足。得

$$Q_{K(\max)} = 130(1/8)+70(1/8)+120(1/2)+120(1/2)(6.6/8)+60(1/2)(2.6/8) = 144.25\ \mathrm{kN}$$

求 $Q_{K(\min)}$：车队向左行驶，重车第二后轮位于 $y_{\min}$ 处（图 d）：

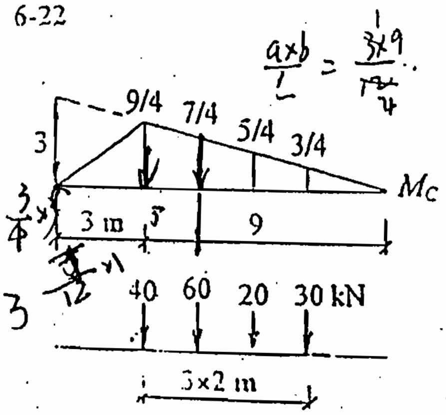
左移　$\Sigma R\mathrm{tg}\alpha = (1/16)(-60-120-120) < 0$

右移　$\Sigma R\mathrm{tg}\alpha = (1/16)(-60-120+3\times120) > 0$

变号，判别式满足。得

$$Q_{K(\min)} = 60(1/4)(1.4/4)-120(1/4)(2.6/4)-120(1/4) = -44.25\ \mathrm{kN}$$

6-22



荷载在图示位置有

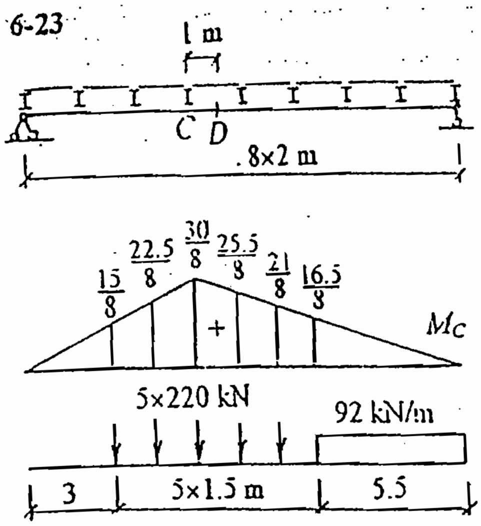
$$\frac{40}{3} > \frac{110}{9}$$

$$\frac{0}{3} < \frac{40+110}{9}$$

判别式满足，得

$$M_{C(\max)} = 40(9/4)+60(7/4)+20(5/4)+30(3/4) = 242.5\ \mathrm{kN\cdot m}$$

6-23



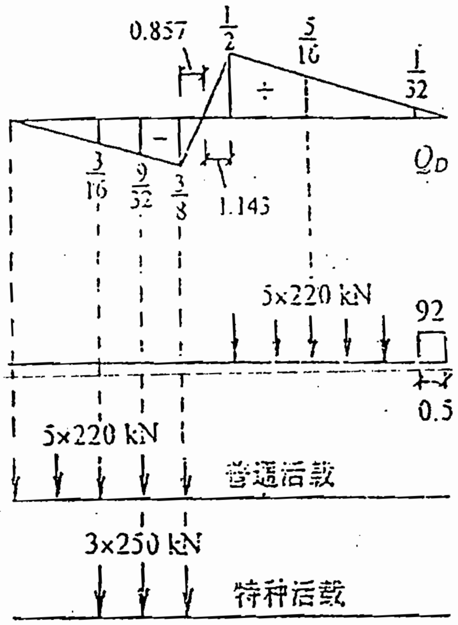
求 $M_{C(max)}$：列车向左开行，第三轮位于顶点时，有

$$\frac{440+220}{6} > \frac{440+92\times5.5}{10}$$

$$\frac{440}{6} < \frac{220+440+92\times5.5}{10}$$

故

$$\begin{aligned} M_{C(max)} &= 220(1/8)(15+22.5+30+25.5+21) \\ &\quad +92(1/2)(16.5/8)5.5 \\ &= 3657\ \text{kN·m} \end{aligned}$$



求 $Q_{D(max)}$：荷载在图示位置有

$$\frac{220}{1.143} > \frac{880+92\times0.5}{8}$$

$$\frac{0}{1.143} < \frac{220+880+92\times0.5}{8}$$

$$\begin{aligned} Q_{D(max)} &= 1100(5/16)+92(1/2)0.5(1/32) \\ &= 344.5\ \text{kN} \end{aligned}$$

求 $Q_{D(min)}$：荷载位置可直观判定如图。

普通活载 $Q_{D(min)} = 1100(-3/16) = -206.2\ \text{kN}$

特种活载 $Q_{D(min)} = 750(-9/32) = -210.9\ \text{kN}$

故取 $Q_{D(min)} = -210.9\ \text{kN}$

6-25 M_C：l=16 m, α=3/8, K=121.9 kN/m

$$M_{C(max)} = 121.9(1/2)16(30/8) = 3657\ \text{kN·m}$$

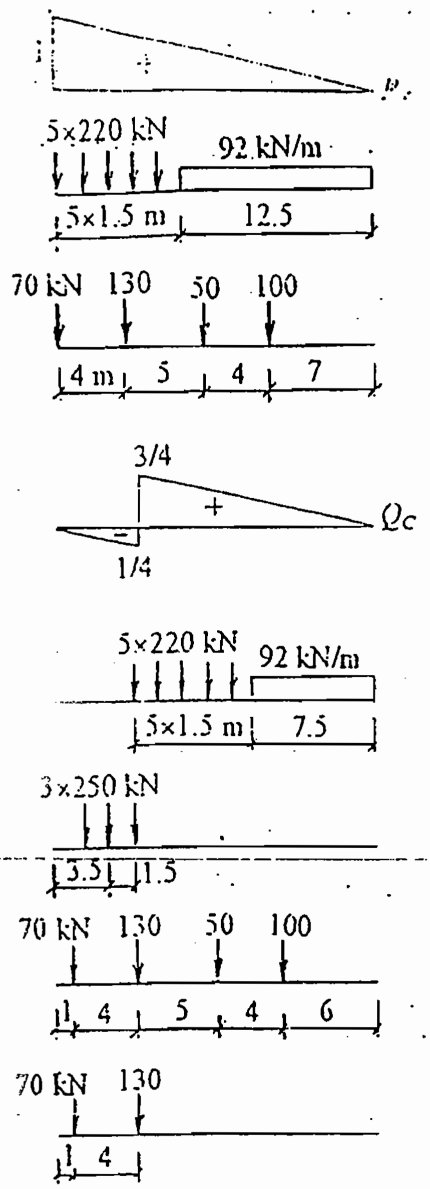
最大 Q_D：l=9.143 m, α=1/8, K=150.7 kN/m

$$Q_{D(max)} = 150.7(1/2)9.143(1/2) = 344.5\ \text{kN}$$

最大 Q_D：l=6.857 m, α=1/8, K=164.2 kN/m

$$Q_{D(min)} = 164.2(1/2)6.857(-3/8) = -211.1\ \text{kN}$$

6-24



中-活载：

$$R_{c(max)} = 1100(17/20)+92(1/2)12.5(12.5/20)$$
$$= 1294 \text{ kN}$$

$$Q_{C(max)} = 1100(12/20)+92(1/2)7.5(7.5/20)$$
$$= 789 \text{ kN}$$

$$Q_{C(min)} = 750(-1/4)(3.5/5) = -131 \text{ kN}$$

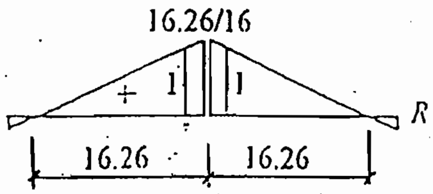
汽车-15：

$$R_{A(max)} = 70\times1+130(16/20)+50(11/20)+100(7/20)$$
$$=236.5 \text{ kN}$$

$$Q_{C(max)} = 70(-1/20)+130(3/4)+50(10/20)+100(6/20)$$
$$= 149 \text{ kN}$$

$$Q_{C(min)} = 130(-1/4)+70(-1/20) = -36 \text{ kN}$$

6-26



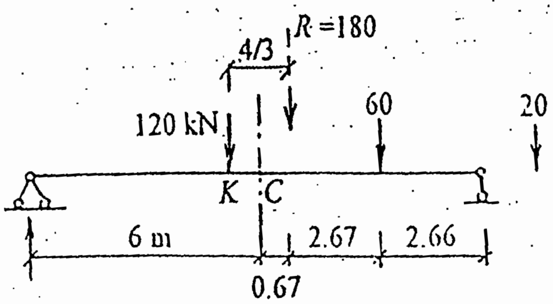
一孔有车：

$$R = 137.1(1/2)16.26(16.26/16) = 1133 \text{ kN}$$

两孔有车：

$$R = 98.2(1/2)32.52(16.26/16) = 1623 \text{ kN}$$

6-27



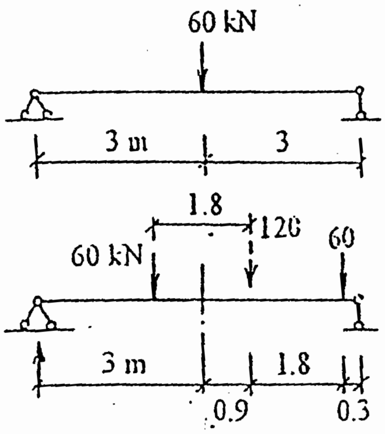
二力在梁上:

$$M_{max} = \frac{R}{l}\left(\frac{l}{2}-\frac{a}{2}\right)^2 - M_K$$

$$= \frac{180}{12}\left(6-\frac{2}{3}\right)^2 - 0$$

$$= 426.7\ \text{kN}\cdot\text{m}$$

6-28



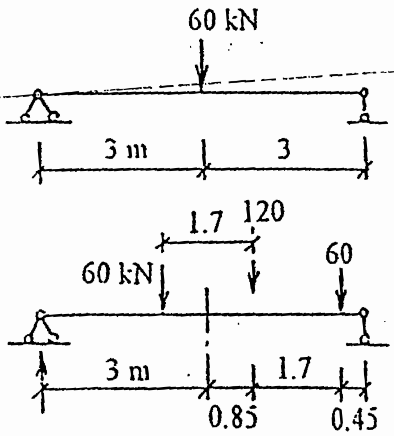
(a) 一力在梁上

$$M_{(1)} = \frac{60\times 6}{4} = 90\ \text{kN}\cdot\text{m}$$

二力在梁上

$$M_{(2)} = \frac{120}{6}\left(\frac{6}{2}-\frac{1.8}{2}\right)^2 = 88.2\ \text{kN}\cdot\text{m}$$

故取　$M_{max} = 90\ \text{kN}\cdot\text{m}$



(b) 一力在梁上

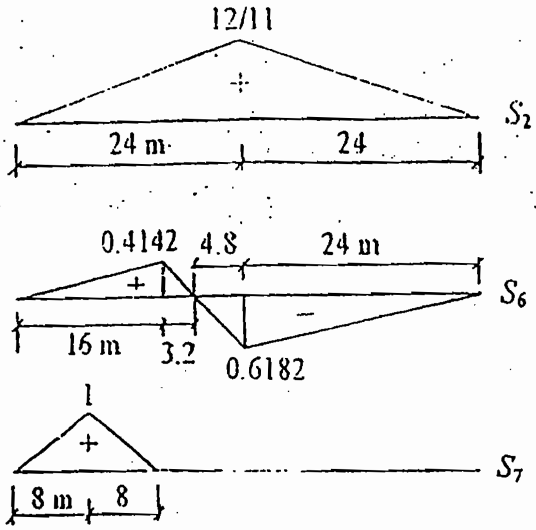
$$M_{(1)} = \frac{60\times 6}{4} = 90\ \text{kN}\cdot\text{m}$$

二力在梁上

$$M_{(2)} = \frac{120}{6}\left(\frac{6}{2}-\frac{1.7}{2}\right)^2 = 92.45\ \text{kN}\cdot\text{m}$$

故取　$M_{max} = 92.45\ \text{kN}\cdot\text{m}$

6-29



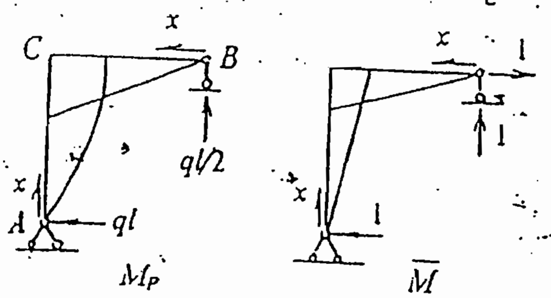
杆件	影响线 加载 l	α	ω	$\Sigma\omega$	$S_q=16\,\Sigma\omega$	K	$S_K=K\omega/2$	$1+\mu$	$(1+\mu)S_K$	S_{max} S_{min}
单位	m		m	m	kN	kN/m	kN		kN	kN
2	48	0.5	+26.18	+26.18	+419	94.5	+1237	1.318	+1630	+2049 +419
6	19.2 28.8	0.1667 0.1667	+3.956 −8.902	−4.946	−79	120.4 110.2	+238 −490	[illegible]	+314 −646	+235 −725
7	16	0.5	+8	+8	+128	19.4	+478	1.5	+717	+845 +128

6 杆 K 值内插计算

加载 l	$K_{0.125}$	$(K_{0.1667})$	$K_{0.25}$
18	122.8	(122.0)	120.3
(19.2)	(121.3)	120.4	(118.6)
20	120.3	(119.3)	117.4
25	114.7	[illegible]	[illegible]
(28.8)	(111.4)	110.2	(107.7)
30	110.3	(109.1)	106.6

第七章　结构位移计算

7-1

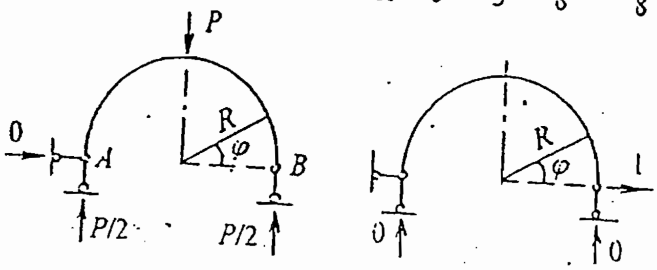


$$\Delta_{Bx}=\sum\int\frac{\overline{M}M_P ds}{EI}$$

$$=\int_0^l x\frac{qlx}{2}\frac{dx}{EI}+\int_0^l x\left(qlx-\frac{qx^2}{2}\right)\frac{dx}{EI}$$

$$=\frac{q}{EI}\left(\frac{l^4}{6}+\frac{l^4}{3}-\frac{l^4}{8}\right)=\frac{3}{8}\frac{ql^4}{EI}\ \rightarrow$$

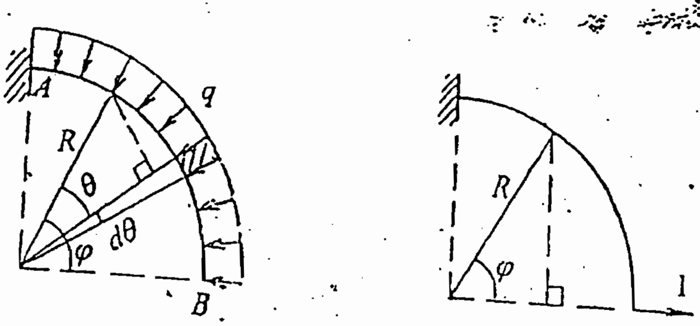
7-2



$$\Delta_{Bx}=\frac{2}{EI}\int_0^{\pi/2}R\sin\varphi\frac{P}{2}R(1-\cos\varphi)Rd\varphi=\frac{PR^3}{EI}\int_0^{\pi/2}(\sin\varphi-\sin\varphi\cos\varphi)d\varphi$$

$$=\frac{PR^3}{EI}\left[-\cos\varphi-\frac{\sin^2\varphi}{2}\right]_0^{\pi/2}=\frac{PR^3}{EI}\left[1-\frac{1}{2}\right]=\frac{PR^3}{2EI}\ \rightarrow$$

7-3

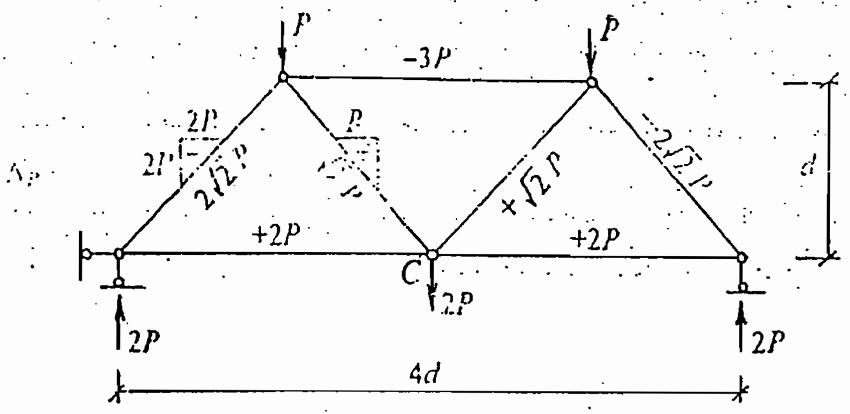


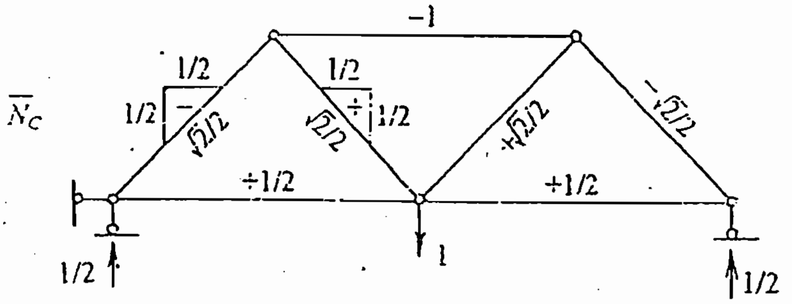
$$M_P=\int_0^{\varphi}-qRd\theta\,R\sin\theta=-qR^2\int_0^{\varphi}\sin\theta\,d\theta=-qR^2(1-\cos\varphi)$$

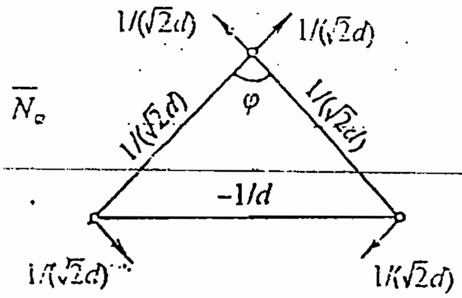
$$\overline{M}=R\sin\varphi$$

$$\Delta_{Bx}=\frac{1}{EI}\int_0^{\pi/2}R\sin\varphi[-qR^2(1-\cos\varphi)]Rd\varphi=-\frac{qR^4}{2EI}\ \leftarrow$$

7-4







以队为单位

$$\Delta_C = \frac{1}{EA}\left[(-1)(-3P)2d + 2(1/2)(2P)2d + 2(-\sqrt{2}/2)(-2\sqrt{2}P)\sqrt{2}d + 2(\sqrt{2}/2)(\sqrt{2}P)\sqrt{2}d\right]$$

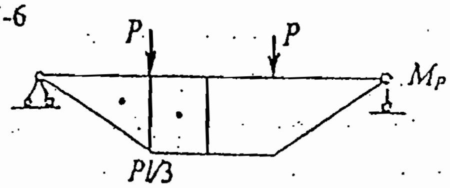
$$= (10+6\sqrt{2})\frac{Pd}{EA} = (10+6\sqrt{2})\frac{40\times10^3\times2}{210\times10^9\times2\times10^{-3}} = 3.52\times10^{-3}\,\text{m}$$

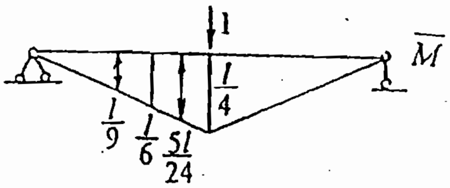
$$\Delta\varphi = \frac{1}{EA}\left[(-\frac{1}{d})(2P)2d + \frac{1}{\sqrt{2}d}(-2\sqrt{2}P)\sqrt{2}d + \frac{1}{\sqrt{2}d}(\sqrt{2}P)\sqrt{2}d\right]$$

$$= -(4+\sqrt{2})\frac{P}{EA} = -(4+\sqrt{2})\frac{40\times10^3}{210\times10^9\times2\times10^{-3}} = -0.516\times10^{-3}\ \text{rad}\ （增大）$$

7-5 略。

7-6



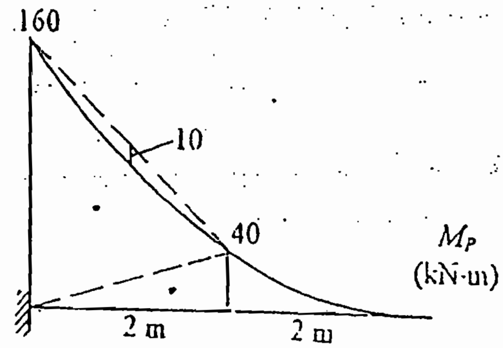


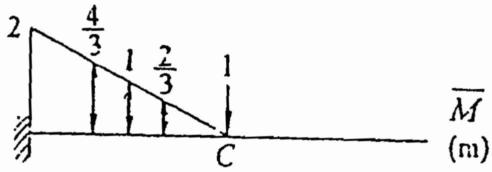
$$y_{max}=\frac{2}{EI}\left[\left(\frac{1}{2}\frac{Pl}{3}\frac{l}{3}\right)\frac{l}{9}+\left(\frac{Pl}{3}\frac{l}{6}\right)\frac{5l}{24}\right]$$

$$=\frac{2}{EI}\left[\frac{1}{162}+\frac{5}{432}\right]Pl^3$$

$$=\frac{23}{648}\frac{Pl^3}{EI}\ \downarrow$$

7-7



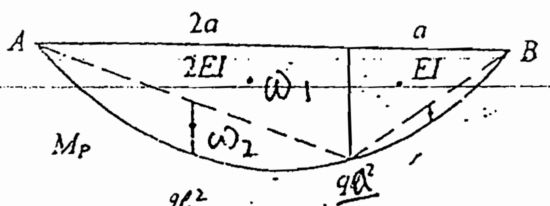


$$\Delta_{Cy}=\frac{1}{EI}\left(\frac{160\times2}{2}\frac{4}{3}+\frac{40\times2}{2}\frac{2}{3}-\frac{2\times10\times2}{3}\times1\right)$$

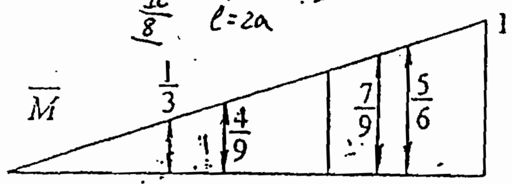
$$=\frac{1}{EI}\left(\frac{640}{3}+\frac{80}{3}-\frac{40}{3}\right)$$

$-2a^2q+\frac{3}{2}qa\cdot2a$ $=\frac{680}{3EI}\ \downarrow$

7-8



$\frac{ql^2}{8}$ $l=2a$



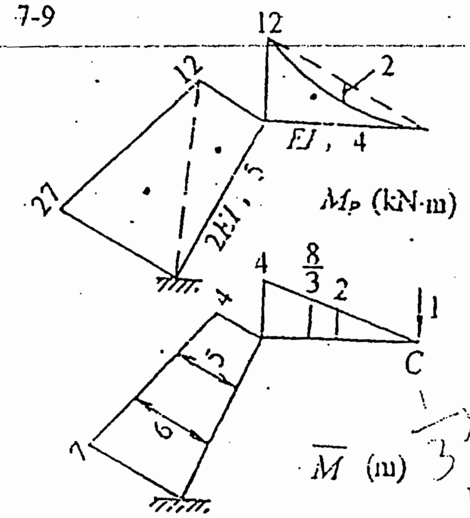
$$\varphi_B=-\frac{1}{2EI}\left[\frac{qa^2\times2a}{2}\frac{4}{9}+\frac{2}{3}\frac{q(2a)^2}{8}2a\frac{1}{3}\right]$$

$$-\frac{1}{EI}\left[\frac{qa^2a}{2}\frac{7}{9}+\frac{2}{3}\frac{qa^2}{8}a\frac{5}{6}\right]$$

$$=-\frac{qa^3}{EI}\left[\frac{2}{9}+\frac{1}{9}+\frac{7}{18}+\frac{5}{72}\right]$$

$$=-\frac{19qa^3}{24EI}\quad(\text{反时针})$$

7-9

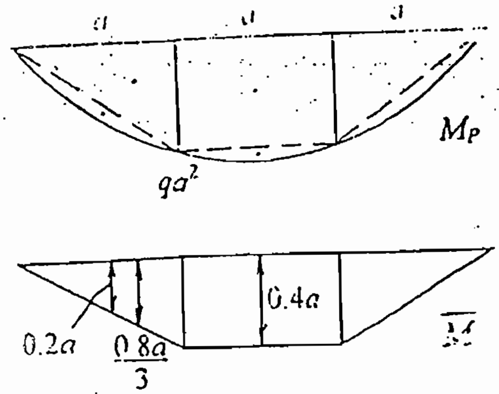


$$\Delta_{Cy}=\frac{1}{EI}\left(\frac{12\times4}{2}\frac{8}{3}-\frac{2\times2\times4}{3}\times2\right)$$

$$+\frac{1}{2EI}\left(\frac{12\times5}{2}\times5+\frac{27\times5}{2}\times6\right)$$

$$=\frac{1985}{6EI}=\frac{330.8}{EI}\ \downarrow$$

7-10

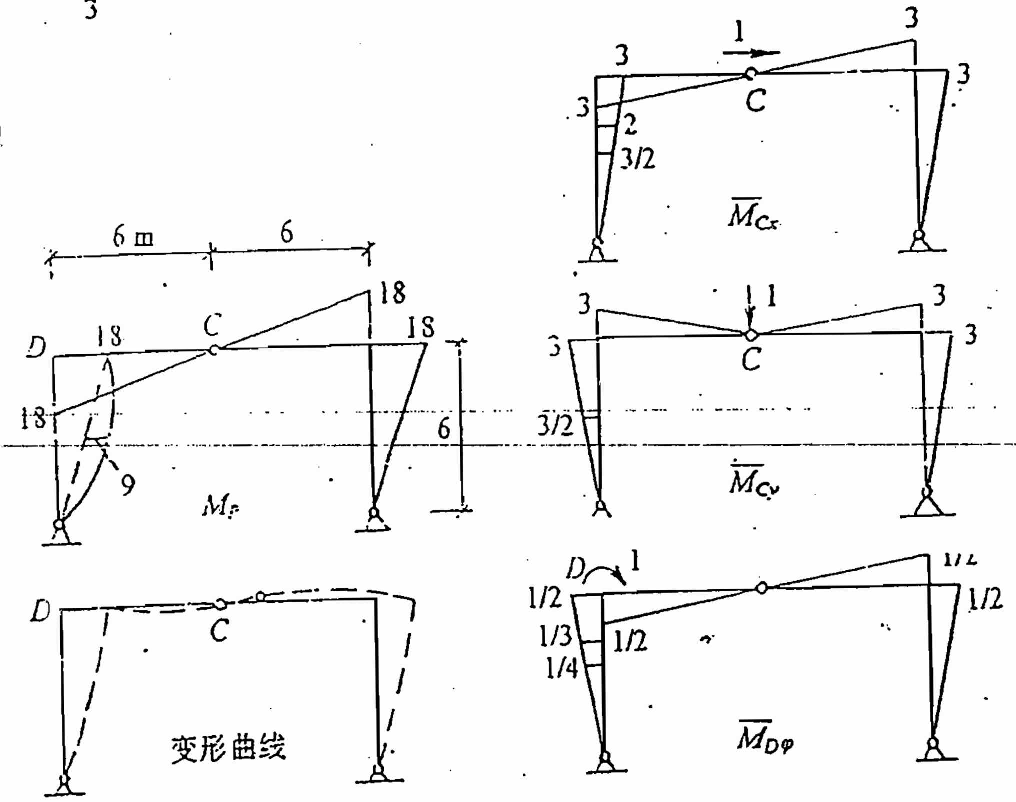


$$\Delta_{CD}=\frac{1}{EI}\left[2\left(\frac{qa^2a}{2}\frac{0.8a}{3}+\frac{2}{3}\frac{qa^2}{8}a\times 0.2a\right)+\left(qa^2a+\frac{2}{3}\frac{qa^2}{8}a\right)0.4a\right]$$

$$=\frac{qa^4}{EI}\left[2\left(\frac{0.8}{6}+\frac{0.1}{6}\right)+\left(1+\frac{1}{12}\right)0.4\right]$$

$$=\frac{qa^4}{EI}\left[\frac{9}{30}+\frac{13}{30}\right]=\frac{11}{15}\frac{qa^4}{EI}\text{（离开）}$$

7-11

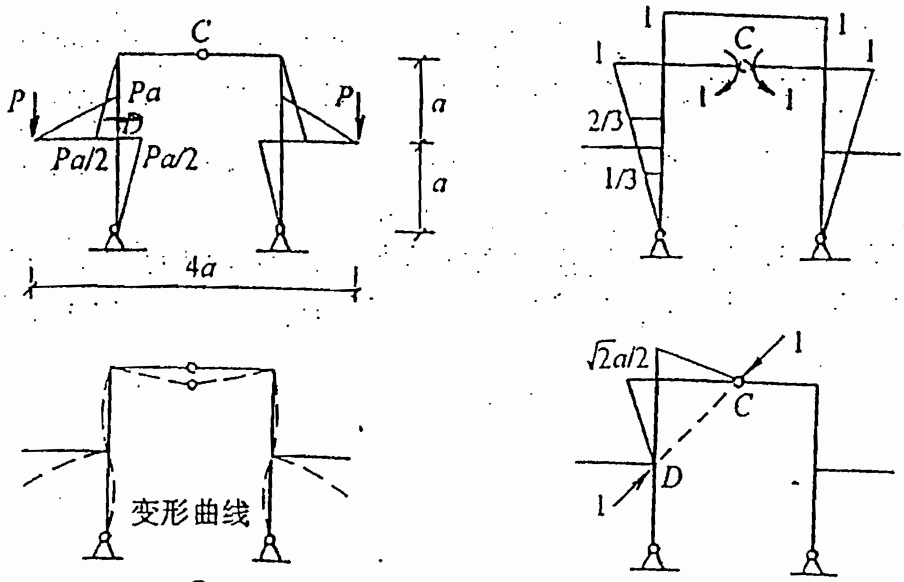


$$\Delta_{Cx}=\frac{1}{EI}\left[\frac{2}{3}\times 9\times 6\times\frac{3}{2}+4\left(\frac{18\times 6}{2}\times 2\right)\right]=\frac{486}{EI}\ \rightarrow$$

$$\Delta_{Cy}=\frac{1}{EI}\left[-\frac{2}{3}\times 9\times 6\times\frac{3}{2}+0\right]=-\frac{54}{EI}\ \uparrow$$

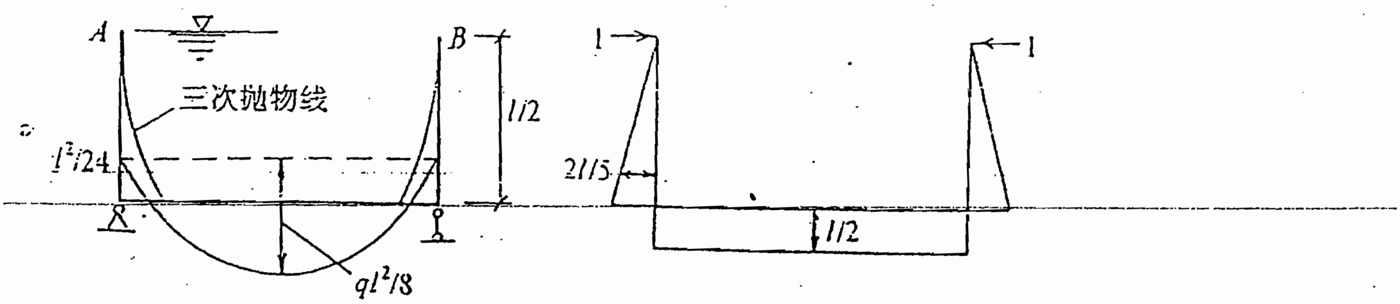
$$\varphi_D=\frac{1}{EI}\left[-\frac{2}{3}\times 9\times 6\times\frac{1}{4}+2\left(\frac{18\times 6}{2}\times\frac{1}{3}\right)\right]=\frac{27}{EI}\text{（顺时针）}$$

.2



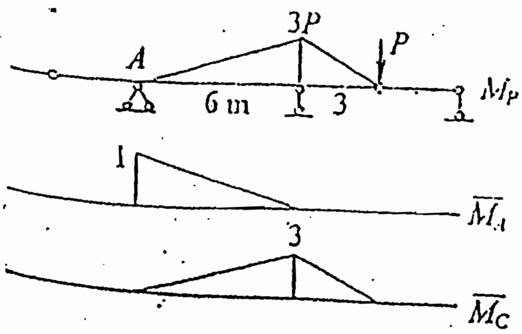
$$\Delta\varphi_C=\frac{1}{EI}\left[2\left(\frac{1}{2}\frac{Pa}{2}a\right)\left(\frac{2}{3}-\frac{1}{3}\right)\right]=\frac{Pa^2}{6EI}$$（下边角度增大）

$$\Delta_{CD}=\frac{1}{EI}\left(\frac{1}{2}\frac{Pa}{2}a\right)\frac{1}{3}\frac{\sqrt{2}}{2}a=\frac{\sqrt{2}}{24}\frac{Pa^3}{EI}$$（靠拢）



$$\Delta_{AB}=\frac{1}{EI}\left[-2\left(\frac{1}{4}\frac{ql^2}{24}\frac{l}{2}\right)\frac{2l}{5}+\left(\frac{2}{3}\frac{ql^2}{8}l-\frac{ql^2}{24}l\right)\frac{l}{2}\right]$$

$$=\frac{ql^4}{EI}\left[-\frac{1}{240}+\frac{1}{48}\right]=\frac{ql^4}{60EI}$$（靠拢）

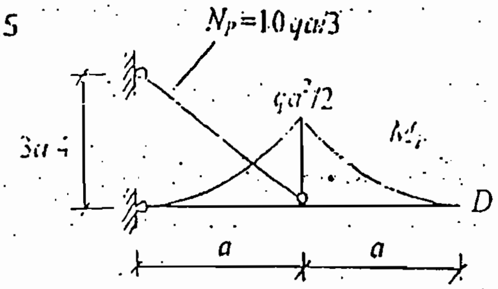


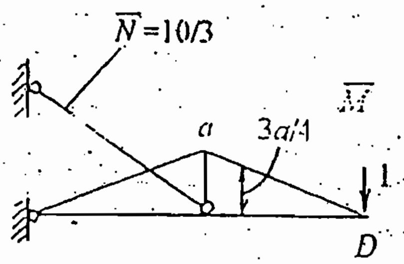
$$\varphi_A=\frac{1}{EI}\frac{3P\times6}{2}\frac{1}{3}=\frac{3P}{EI}=0.001$$

$$\Delta_{Cy}=\frac{1}{EI}\left(\frac{3P\times3}{2}\times2+\frac{3P\times6}{2}\times2\right)$$

$$=\frac{27P}{EI}=0.009\text{ m}\downarrow$$

7-15



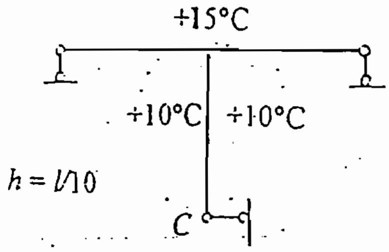


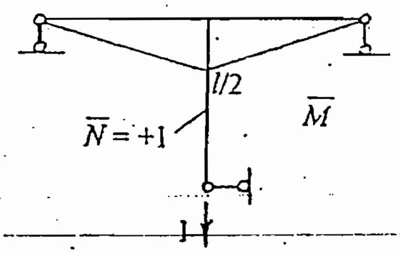
$$\Delta_{Dy}=\frac{2}{EI}\frac{1}{3}\frac{qa^2}{2}a\frac{3a}{4}+\frac{1}{EA}\frac{10}{3}\frac{10qa}{3}\frac{5}{4}a=\frac{qa^4}{4EI}+\frac{125}{9}\frac{qa^2}{EA}$$

$$=\frac{5\times10^3\times2^4}{4\times210\times10^9\times2500\times10^{-8}}+\frac{125}{9}\frac{5\times10^3\times2^2}{210\times10^9\times\frac{\pi}{4}\times20^2\times10^{-6}}$$

$$=3.81\times10^{-3}+4.21\times10^{-3}=8.02\times10^{-3}\ \text{m}$$

7-16





$$\Delta_{Cy}=\Sigma\alpha tNl+\Sigma\frac{\alpha\Delta t}{h}\int M ds$$

$$=\alpha\times10\times1\times l+\frac{\alpha}{h}(15-10)(-2\times\frac{1}{2}\frac{l}{2}l)$$

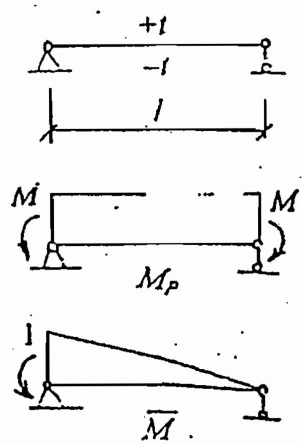
$$=10\alpha-25\alpha l=-15\alpha l\ \uparrow$$

7-17

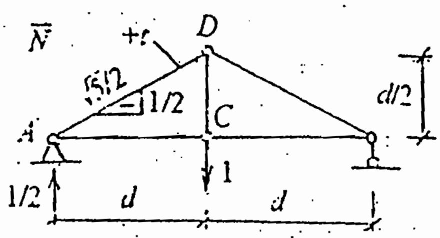
由 $\varphi_t+\varphi_M=0$

$$\frac{\alpha}{h}(2t)(-\frac{l}{2})+\frac{1}{EI}Ml\times\frac{1}{2}=0$$

$$M=\frac{2\alpha tEI}{h}$$

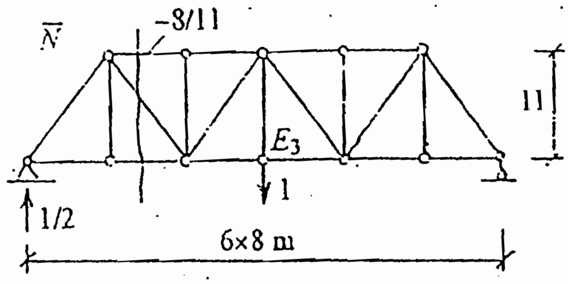


7-18 $\Delta_{Cy} = (-\frac{\sqrt{5}}{2})\alpha t\frac{\sqrt{5}d}{2} = -\frac{5\alpha td}{4}$ ↑



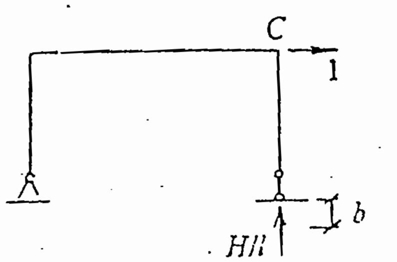
7-19 $\Delta_{E3} = \Sigma\bar{N}\Delta l = 2(-8/11)16$

$= -23.27$ mm ↑



7-20 $\Delta_{Cx} = -\Sigma\bar{R}c = -(-\frac{H}{l}b)$

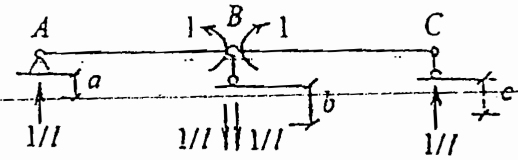
$= \frac{Hb}{l}$ →



7-21 $\varphi = -\Sigma\bar{R}c = -\left(-\frac{a}{l}+\frac{2b}{l}-\frac{c}{l}\right)$

$= \frac{a-2b+c}{l} = \frac{40-2\times100+80}{16\times1000}$

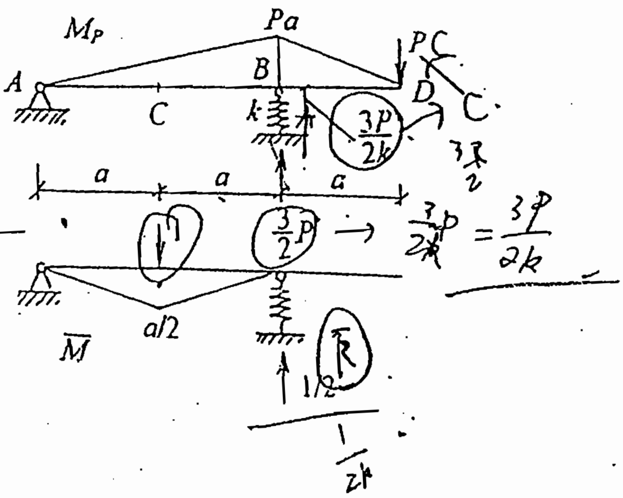
$= -0.005$ rad（上边角度减小）



7-22 $\Delta_{Cy} = \Sigma\int\frac{\bar{M}M_P ds}{EI} - \Sigma\bar{R}c$

$= -\frac{Pa^3}{4EI} - \left(-\frac{1}{2}\frac{3P}{2k}\right) = \frac{Pa^3}{2EI}$ ↓

$-\Sigma RC$



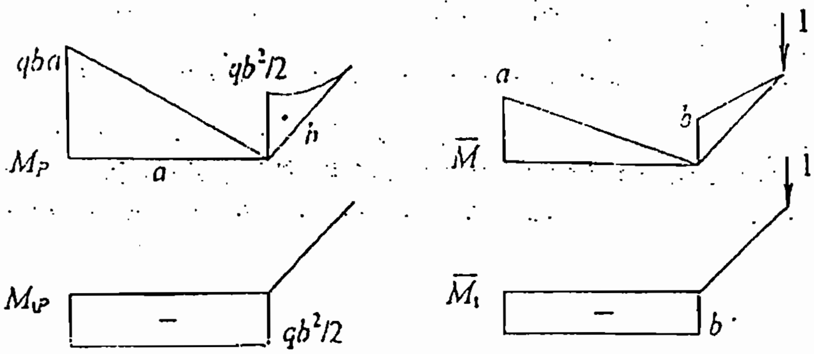
7-23 由 $W_{12} = W_{21}$ 有

$10\Delta_H = \frac{100}{15}(2\times0.5+3\times0.6+2\times1+1.2)$

得 $\Delta_H = 4$ mm ↓

7-24 略。

*7-25



$$\Delta_{Cz}=\frac{1}{EI}\left[\frac{1}{3}\frac{qb^2}{2}b\frac{3b}{4}+\frac{qbaa}{2}\frac{2a}{3}\right]+\frac{1}{GI_t}\frac{qb^2a}{2}b$$

$$=\frac{qb}{EI}\left[\frac{b^3}{8}+\frac{a^3}{3}\right]+\frac{qb^3a}{2GI_t}$$

$$=\frac{2\times10^3\times0.4}{210\times10^9\times\frac{\pi}{64}\times0.03^4}\left(\frac{0.4^3}{8}+\frac{0.6^3}{3}\right)+\frac{2\times10^3\times0.4^3\times0.6}{2\times80\times10^9\times\frac{\pi}{32}\times0.03^4}$$

$$=7.66\times10^{-3}+6.04\times10^{-3}=13.7\times10^{-3}\ \mathrm{m}\downarrow$$

*7-26 $M_P=-PR\sin\theta$，$M_{tP}=-PR(1-\cos\theta)$，令 $P=1$ 有

$\bar{M}=-R\sin\theta$，$\bar{M}_t=-R(1-\cos\theta)$

$$\Delta_{Bz}=\frac{1}{EI}\int_0^{\pi/2}(-R\sin\theta)(-PR\sin\theta)Rd\theta+\frac{1}{GI_t}\int_0^{\pi/2}R(1-\cos\theta)PR(1-\cos\theta)Rd\theta$$

$$=\frac{PR^3}{EI}\int_0^{\pi/2}\sin^2\theta d\theta+\frac{PR^3}{GI_t}\int_0^{\pi/2}(1-\cos\theta)^2d\theta$$

$$=\frac{PR^3}{2EI}\int_0^{\pi/2}(1-\cos2\theta)d\theta+\frac{PR^3}{GI_t}\int_0^{\pi/2}\left(1-2\cos\theta+\frac{1}{2}+\frac{\cos2\theta}{2}\right)d\theta$$

$$=\frac{PR^3}{2EI}\left[\theta-\frac{\sin2\theta}{2}\right]_0^{\pi/2}+\frac{PR^3}{GI_t}\left[\theta-2\sin\theta+\frac{\theta}{2}+\frac{\sin2\theta}{4}\right]_0^{\pi/2}$$

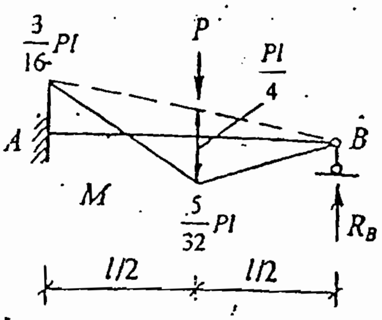
$$=\frac{PR^3}{2EI}\left[\frac{\pi}{2}-0\right]+\frac{PR^3}{GI_t}\left[\frac{\pi}{2}-2+\frac{\pi}{4}+0\right]$$

$$=\frac{\pi PR^3}{4EI}+\frac{3\pi-8}{4}\frac{PR^3}{GI_t}\ \downarrow$$

第八章　力　法

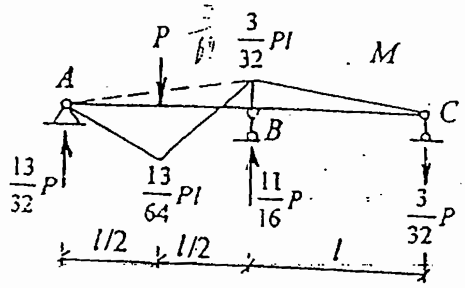
8-1　略。

8-2



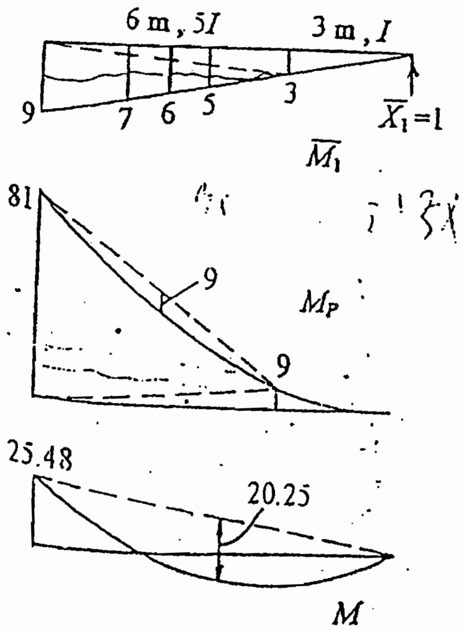
多余未知力 X_1	$EI\delta_{11}$	$EI\Delta_{1P}$	X_1 值
R_B	$l^3/3$	$-5Pl^3/48$	$5P/16$
M_A	$l/3$	$Pl^2/16$	$-3Pl/16$

8-3



多余未知力 X_1	$EI\delta_{11}$	$EI\Delta_{1P}$	X_1 值
M_B	$2l/3$	$Pl^2/16$	$-3Pl/32$
R_B	$l^3/6$	$-11Pl^3/96$	$11P/16$
R_C	$2l^3/3$	$Pl^3/16$	$-3P/32$

8-4



$$EI\delta_{11}=\frac{3^3}{3}+\frac{1}{5}\left(\frac{3\times6}{5}\times5+\frac{9\times6}{2}\times7\right)=55.8$$

$$EI\Delta_{1P}=-\frac{9\times3}{3}\frac{3\times3}{4}+\frac{1}{5}\left(-\frac{9\times6}{2}\times5-\frac{81\times6}{2}\times7+\frac{2\times9\times6}{3}\times6\right)=-20.25-324$$

$$=-344.25$$

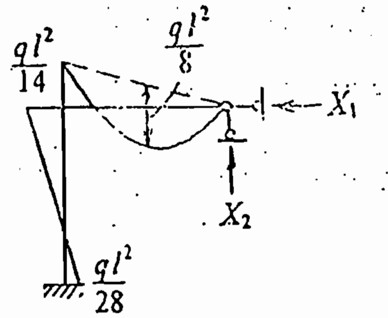
$$X_1=\frac{344.25}{55.8}=6.169\text{ kN}\ \uparrow$$

$\frac{1}{EI}(\frac{1}{2}\times3\times3\times2)+\frac{1}{5EI}(6\times3\times6+\frac{1}{2}\times6\times6\times7)$

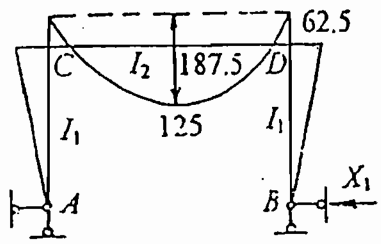
8-5

$$\begin{cases} \dfrac{l^3}{3EI}X_1 + \dfrac{l^3}{2EI}X_2 - \dfrac{ql^4}{4EI} = 0 \\ \dfrac{l^3}{2EI}X_1 + \dfrac{4l^3}{3EI}X_2 - \dfrac{5ql^4}{8EI} = 0 \end{cases}$$

$$X_1 = \frac{3}{28}ql \qquad X_2 = \frac{3}{7}ql$$



8-6 $I_2 = nI_1$, $n = 5/2$



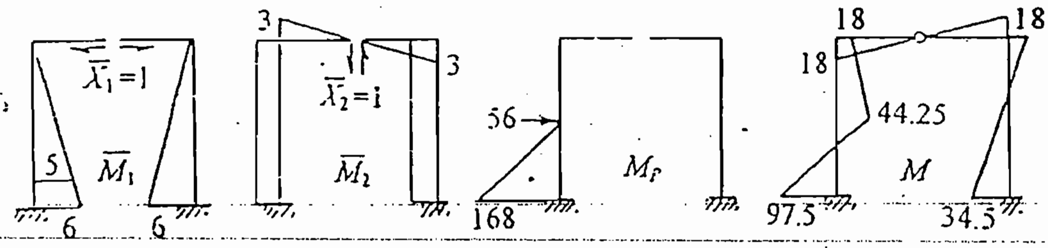
$$\frac{288}{EI_1}X_1 - \frac{3000}{EI_1} = 0$$

$$X_1 = \frac{125}{12} = 10.4167\ \text{kN}$$

$n \to 0$, $M_D = 125\ \text{kN}\cdot\text{m}$

$n \to \infty$, $M_D = 0$

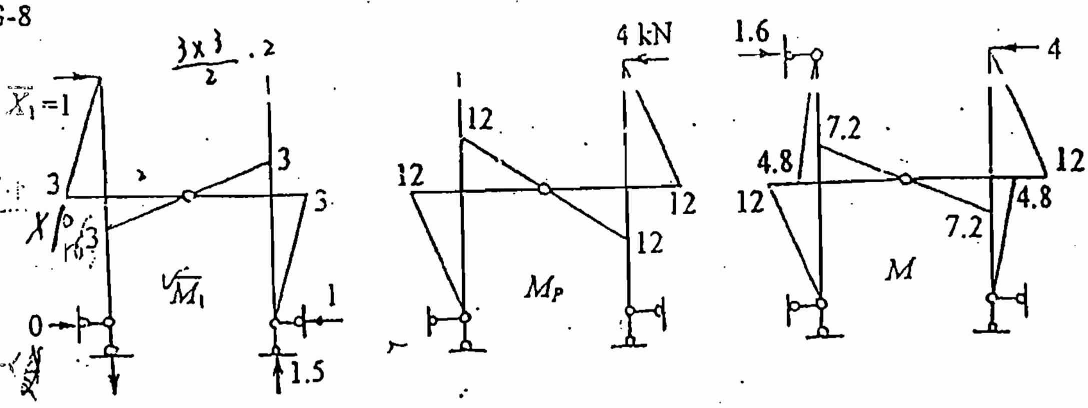
8-7



$$\delta_{12} = \delta_{21} = 0$$

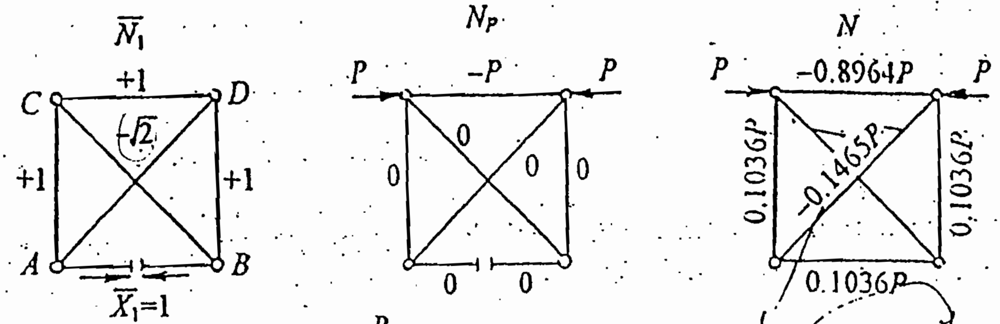
$$\begin{cases} 144X_1 - 1260 = 0 \\ 126X_2 + 756 = 0 \end{cases} \qquad \begin{aligned} &X_1 = 8.75\ \text{kN}（压力） \\ &X_2 = -6\ \text{kN}（负剪力） \end{aligned}$$

8-8



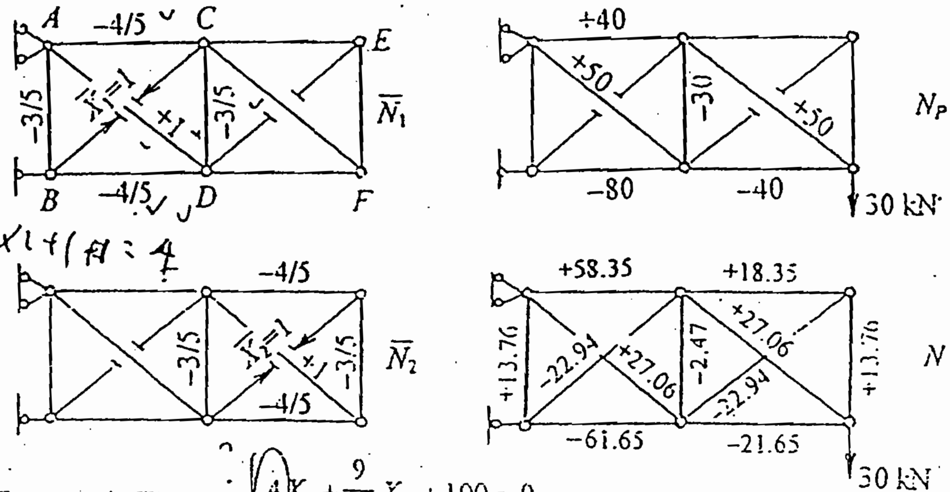
$EI\delta_{11} = 30$ $\qquad EI\delta\Delta_{1P} = -48$ $\qquad X_1 = 1.6\ \text{kN}\ \rightarrow$

8-9



$$X_1=\frac{P}{4+4\sqrt{2}}=0.1036P$$

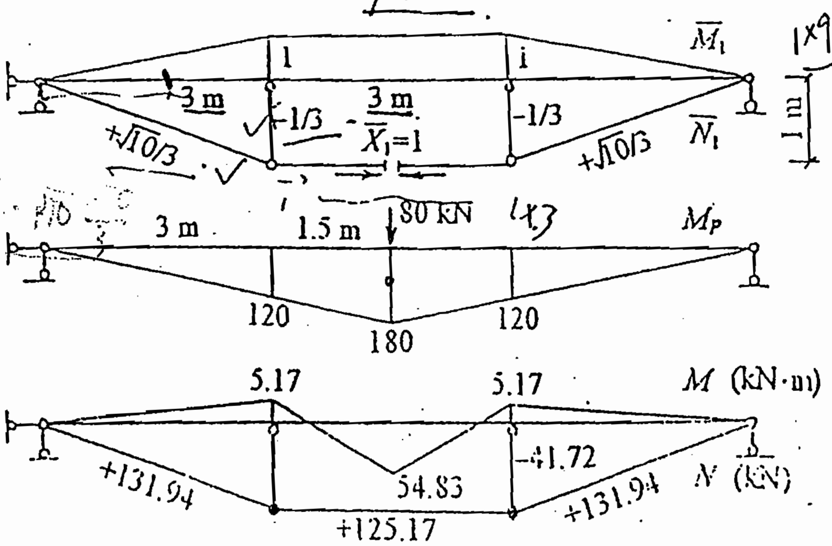
8-10



各杆 $l/(EA)$ 相同。

$$\begin{cases}4X_1+\dfrac{9}{25}X_2+100=0\\[2mm]\dfrac{9}{25}X_1+4X_2+100=0\end{cases}\qquad X_1=X_2=-\frac{2500}{109}=-22.94\ \text{kN}$$

8-11



$$X_1=\frac{\dfrac{2}{EI}\left(\dfrac{120\times3}{2}\cdot\dfrac{2}{3}+\dfrac{120+180}{2}\cdot\dfrac{3}{2}\times1\right)}{\dfrac{1}{EI}\left(2\times\dfrac{1\times3}{2}\cdot\dfrac{2}{3}+3\right)+\dfrac{1}{EA}\left(3+2\times\dfrac{10}{9}\sqrt{10}+2\times\dfrac{1}{9}\right)}=\frac{690}{5+\dfrac{I}{A}\times10.25}=125.17\ \text{kN}$$

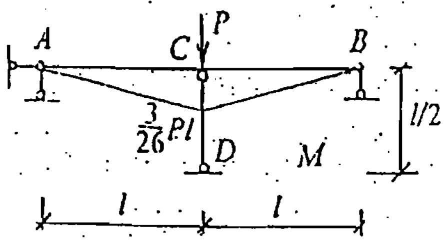
8-12 截断 CD 杆，设 X_1 为拉力，有

$$\left(\frac{l^3}{6EI}+\frac{l}{2EA}\right)X_1+\frac{Pl^3}{6EI}=0$$

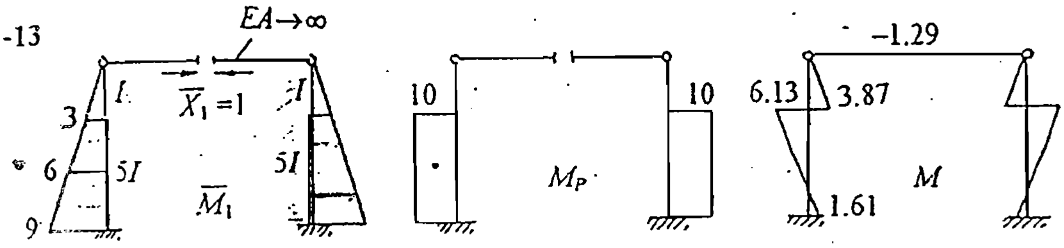
$$X_1=-\frac{10}{13}P\ (\text{压力})$$

$$A\to 0,\ X_1=0,\ M_C=\frac{Pl}{2}$$

$$A\to\infty,\ X_1=-P,\ M_C=0$$

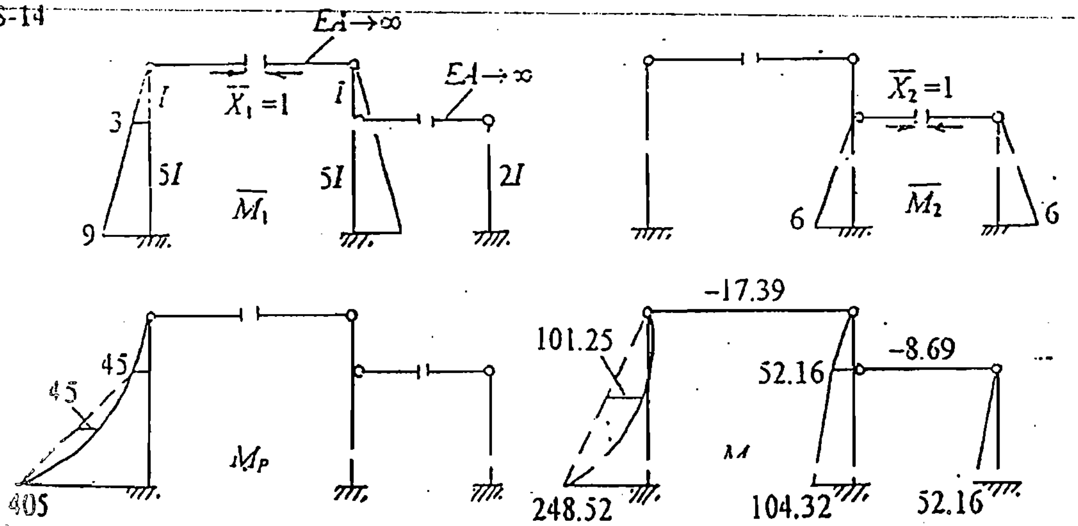


8-13



$$X_1=-\frac{2\times 72}{2\times 55.8}=-\frac{144}{111.6}=-1.29\text{ kN}$$

8-14



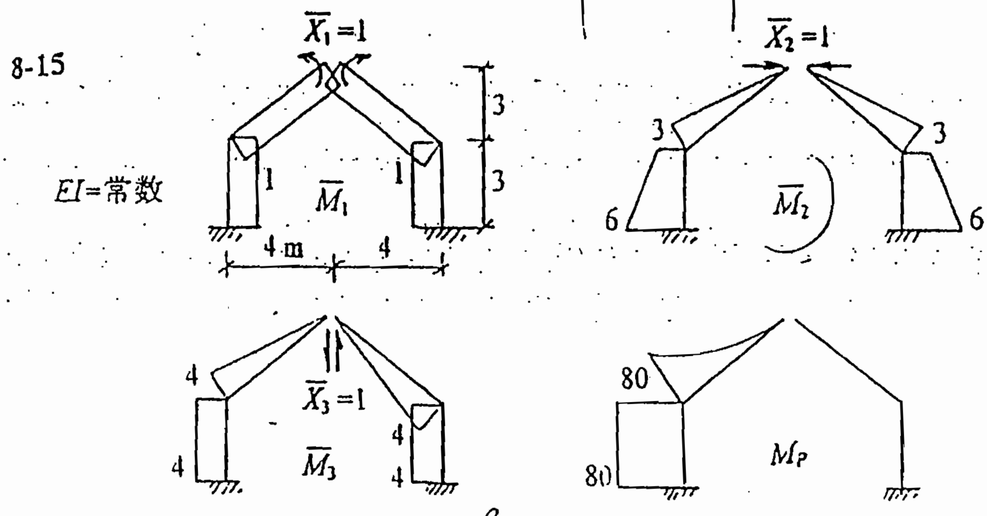
$$111.6X_1-25.2X_2+1721.25=0$$

$$-25.2X_1+50.4X_2+0=0$$

$$X_1=-17.39\text{ kN}$$

$$X_2=-8.69\text{ kN}$$

8-15



δ_{13} $\delta_{23}=0$ $\delta_{12}\neq 0$

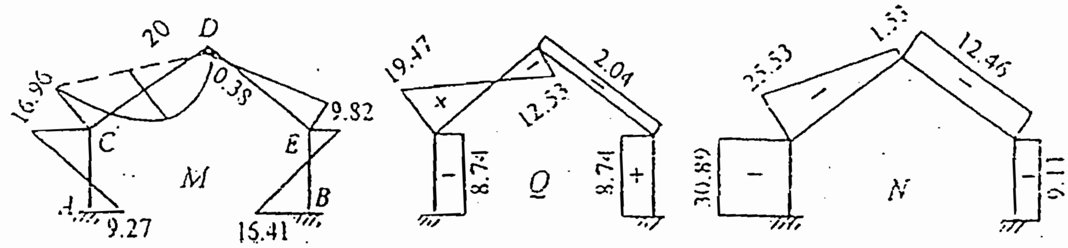
$$\begin{cases} 16X_1-42X_2-373.33=0 \\ -42X_1+156X_2+1380=0 \\ 149.333X_3+1360=0 \end{cases}$$

$\delta_{1P}\neq 0$

$X_1=0.3825\ \text{kN}$

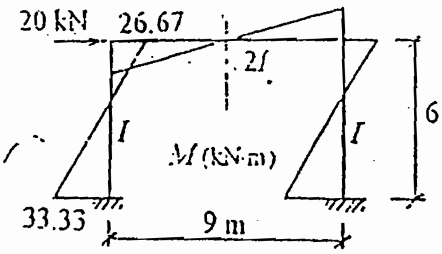
$X_2=-8.734\ \text{kN}$

$X_3=-9.107\ \text{kN}$

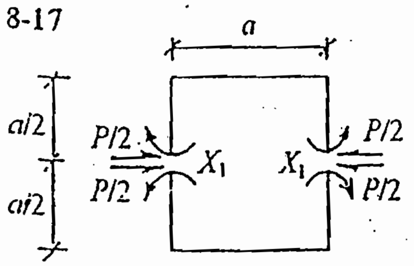


8-16　跨中剪力为

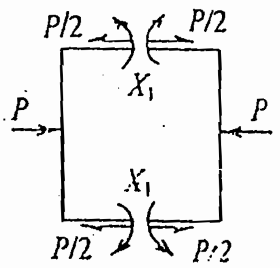
$$X_1=\frac{-\dfrac{60\times 6}{2}\times 4.5}{\dfrac{4.5^3}{2\times 3}+4.5^2\times 6}=\frac{-810}{136.8875}=-5.926\ \text{kN}$$



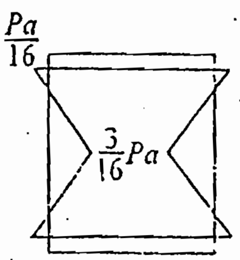
8-17



基本机构 1



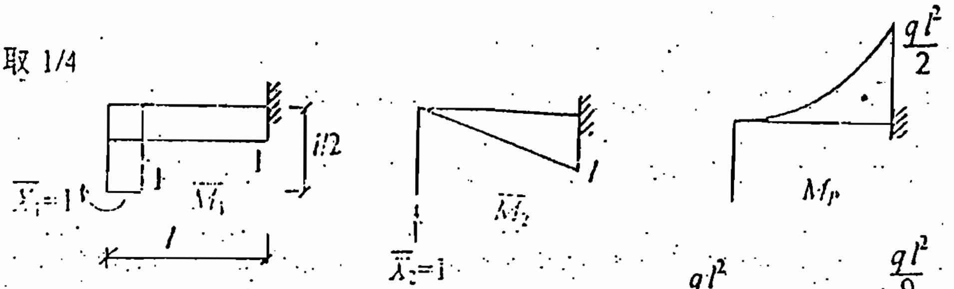
基本机构 2



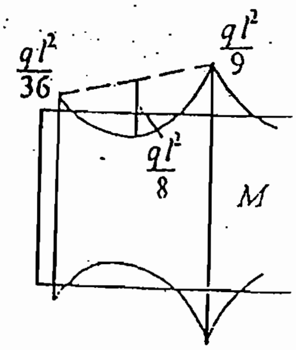
M

$$X_1=\frac{\dfrac{3}{4}Pa^2}{4a}=\frac{3}{16}Pa \qquad X_1=-\frac{\dfrac{Pa^2}{4}}{4a}=-\frac{Pa}{16}$$

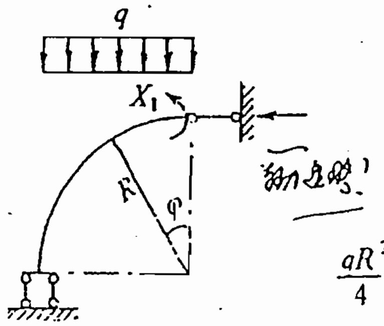
8-18　取 1/4



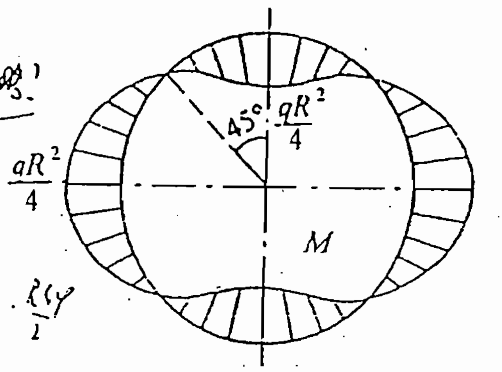
$$\begin{cases}\frac{3l}{2}X_1+\frac{l^2}{2}X_2-\frac{ql^3}{6}=0\\ \frac{l^2}{2}X_1+\frac{l^3}{3}X_2-\frac{ql^4}{8}=0\end{cases}\qquad X_1=-\frac{ql^2}{36}\qquad X_2=\frac{5ql}{12}$$



8-19　取 1/4



$M=\frac{qR^2}{4}-q\cdot R\sin\varphi\cdot\frac{R\sin\varphi}{2}$



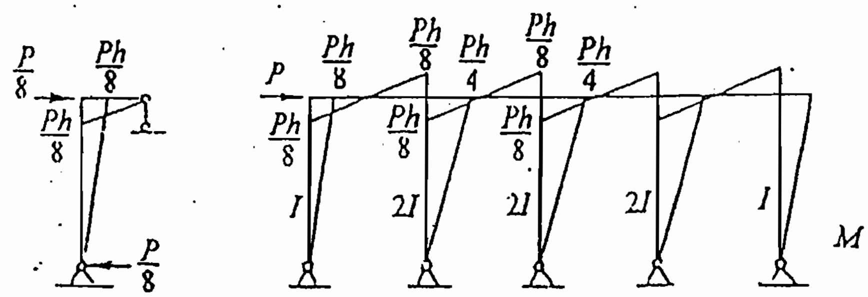
$$EI\delta_{11}=\frac{\pi R}{2}$$

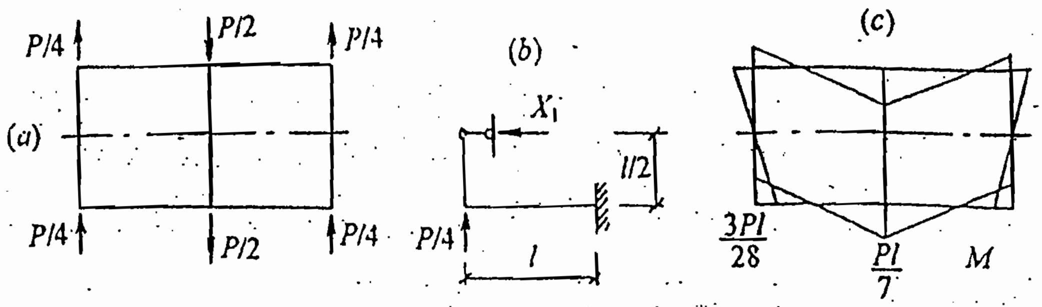
$$EI\Delta_{1P}=-\int_0^{\pi/2}\frac{q(R\sin\varphi)^2}{2}Rd\varphi=-\frac{qR^3}{2}\int_0^{\pi/2}\sin^2\varphi d\varphi=-\frac{\pi qR^3}{8}$$

$$X_1=\frac{qR^2}{4}$$

$$M=\frac{qR^2}{4}-\frac{qR^2}{2}\sin^2\varphi=\frac{qR^2}{4}(1-2\sin^2\varphi)$$

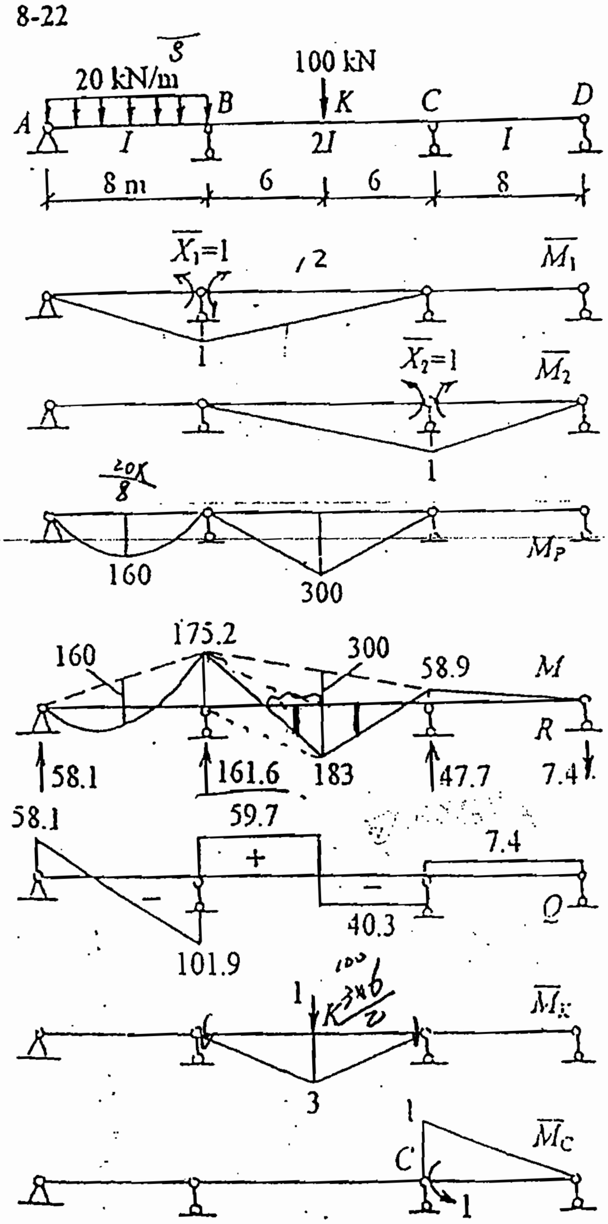
8-20





8-21 6 次超静定，但反力静定。去掉支座代以反力，则外力对竖直中轴正对称。将外力分解为对水平中轴为正、反对称的两组，则正对称时弯矩为零，反对称时（图 *a*），可取 1/4（图 *b*），仅为 1 次超静定，解得 $X_1=3P/14$，M 如图 *c*。

8-22



$\frac{12\times1}{2}\cdot\frac{1}{3}$

$$\begin{cases} \dfrac{14}{3}X_1+X_2+\dfrac{2630}{3}=0 \\ X_1+\dfrac{14}{3}X_2+\dfrac{1350}{3}=0 \end{cases}$$

$166X$

$\frac{X}{6-X}=\frac{183}{175.2}$ 102.6

$X_1=-175.2\ \text{kN}\cdot\text{m}$

$X_2=-58.9\ \text{kN}\cdot\text{m}$

63.6 ~~6488~~ 6482

$\frac{x}{6-x}=\frac{183}{58.9}$

5833

$$\Delta_{Ky}=\frac{1}{EI}\left[-\frac{3\times12}{2}\,\frac{175.2+58.9}{2}+\frac{100\times12^3}{48}\right]$$

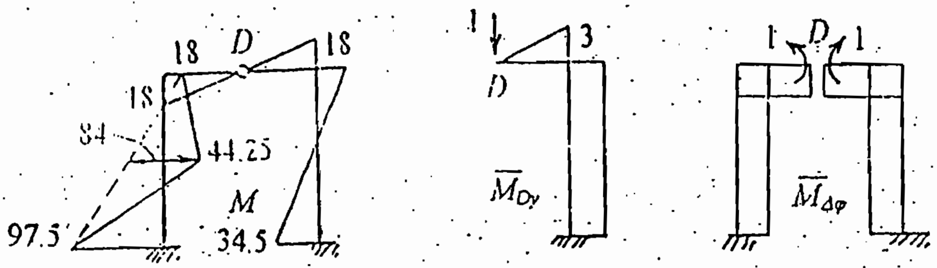
$$=\frac{747}{EI}\downarrow$$

$$\varphi_C=\frac{1}{EI}\,\frac{58.9\times8}{2}\,\frac{2}{3}$$

$$=\frac{157}{EI}\ \text{（反时针）}$$

$20\times8\times4.$

8-23



$$\Delta_{Dy}=\frac{1}{EI}\left[\frac{18\times3}{2}\times2+3\times6\left(\frac{18-34.5}{2}\right)\right]=-\frac{94.5}{EI}\ \uparrow$$

$$\Delta\varphi_D=\frac{1}{EI}\left[\frac{34.5\times6}{2}-\frac{97.5\times6}{2}+\frac{84\times6}{2}\right]=\frac{63.0}{EI}\ \text{（上边角度增大）}$$

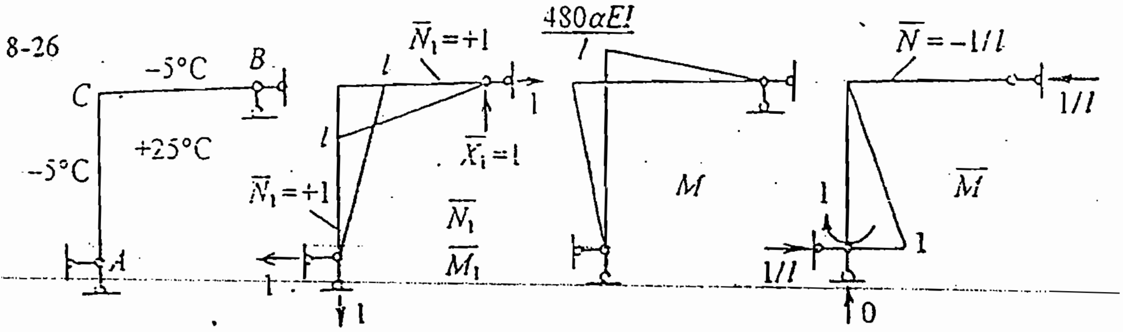
8-25 (*a*) 左、中结点 $\Sigma M\neq0$；截断各柱取上部为隔离体，三柱剪力均同向，$\Sigma X\neq0$。

(*b*) 柱有剪力，截断柱取上部为隔离体，$\Sigma X\neq0$。

(*c*) 左起第二支座处相对转角为零的条件不满足。

(*d*) 左支座处竖向位移为零的条件不满足。

8-26

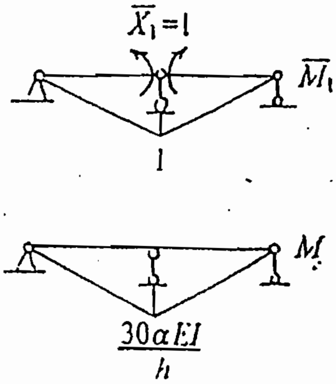


$$\delta_{11}=\frac{2}{3}\frac{l^3}{EI}\qquad \Delta_{1t}=\alpha\frac{25-5}{2}\times2l+\alpha\frac{25-(-5)}{h}\times2\frac{l^2}{2}=20\alpha l+300\alpha l=320\alpha l$$

$$X_1=-\frac{480\alpha EI}{l^2}\qquad \varphi_A=-\frac{1}{EI}\frac{480\alpha EI}{2l}l\times\frac{1}{3}+\alpha\frac{25-5}{2}\left(-\frac{1}{l}\right)l+\alpha\frac{25-(-5)}{h}\frac{l}{2}$$

$$=-80\alpha-10\alpha+160\alpha\ =60\alpha\ \text{（顺时针）}$$

8-27



$$\delta_{11}=\frac{2}{3}\frac{l}{EI}$$

$$\Delta_{1t}=\frac{\alpha(t_2-t_1)}{h}2\frac{1\times l}{2}=\frac{\alpha(0-20)}{h}l=-20\frac{\alpha l}{h}$$

$$X_1=30\frac{\alpha EI}{h}$$

$$\sigma_{max}=\frac{M_{max}}{W}=\frac{30\alpha EI/h}{\dfrac{I}{h/2}}=15\alpha E$$

$$=15\times10^{-5}\times210\times10^{9}=31.5\times10^{6}\ \text{Pa}$$

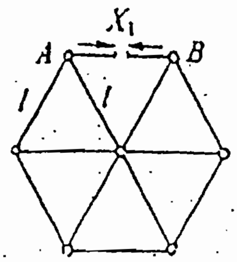
可见，σ_{max} 与 I/h 无关，加大工字钢号码 σ_{max} 仍不变。

8-28 $\delta_{11}X_1+\Delta_{1\Delta}=0$

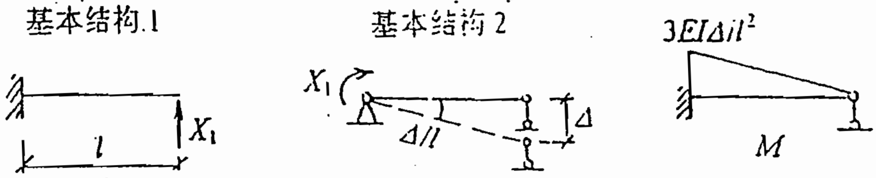
$\delta_{11}=12l/(EA)$ $\quad\Delta_{1\Delta}=\Sigma N_1\Delta l=-\Delta$

$X_1=\dfrac{EA\Delta}{12l}$ （拉力）

$l_{AB}=l-\Delta+\dfrac{X_1 l}{EA}=l-\Delta+\dfrac{EA\Delta}{12l}\dfrac{l}{EA}=l-\dfrac{11}{12}\Delta$



8-29 基本结构 1 　　基本结构 2

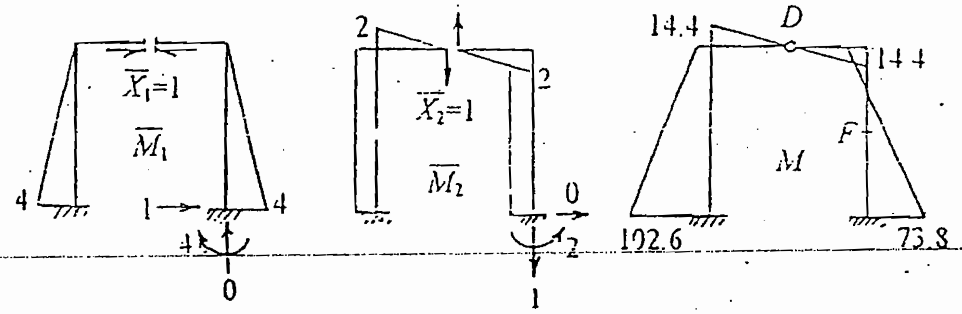


$\delta_{11}X_1+\Delta_{1\Delta}=-\Delta$ 　　 $\delta_{11}X_1+\Delta_{1\Delta}=0$

$(l^3/3EI)X_1+0=-\Delta$ 　　 $(l/3EI)X_1+\Delta/l=0$

$X_1=-3EI\Delta/l^3\downarrow$ 　　 $X_1=-3EI\Delta/l^2$（上拉）

8-30 $EI=210\times10^9\times10^{-3}\times6400\times10^{-8}=13440\text{ kN}\cdot\text{m}^2$



$\delta_{11}=\dfrac{128}{3EI}$ $\quad\delta_{22}=\dfrac{112}{3EI}$ $\quad\delta_{12}=\delta_{21}=0$

$\Delta_{1\Delta}=-(1\times0.03+4\times0.01)=-0.07\text{ m}$

$\Delta_{2\Delta}=-(1\times0.04-2\times0.01)=-0.02\text{ m}$

$X_1=\dfrac{0.07\times3\times13440}{128}=22.05\text{ kN}$

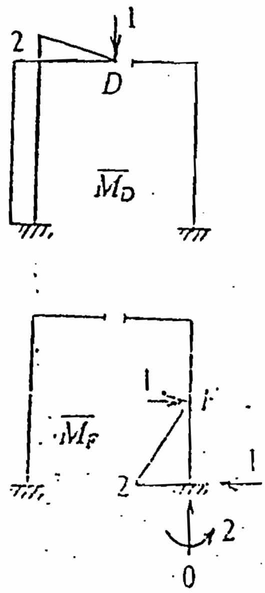
$X_2=\dfrac{0.02\times3\times13440}{112}=7.20\text{ kN}$

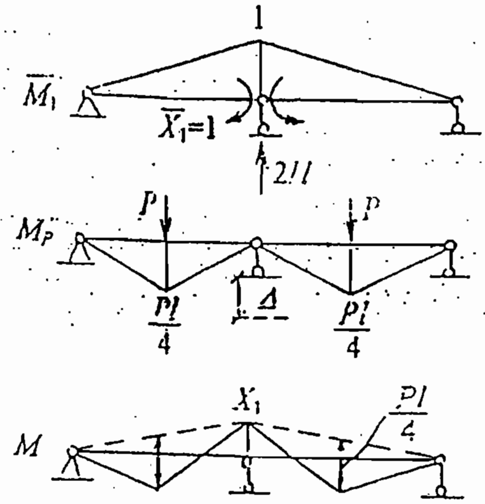
$\Delta_{Dy}=\dfrac{1}{13440}\left[\dfrac{14.4\times2}{2}\times\dfrac{2\times2}{3}+\dfrac{(14.4+192.6)\times4}{2}\times2\right]$

$=36.25\times10^{-3}\text{ m}\downarrow$

$\Delta_{Fx}=\dfrac{1}{13440}\left[\dfrac{2\times2}{2}\left(-\dfrac{5}{6}\times73.8+\dfrac{14.4}{6}\right)\right]$

$-(-1\times0.03-2\times0.01)=41.2\times10^{-3}\text{ m}\rightarrow$





8-31 $\delta_{11}X_1+\Delta_{1P}+\Delta_{1\Delta}=0$，有

$$\frac{2l}{3EI}X_1-\frac{Pl^2}{8EI}+\frac{2\Delta}{l}=0$$

由题意 $X_1=\frac{Pl}{4}-\frac{X_1}{2}$

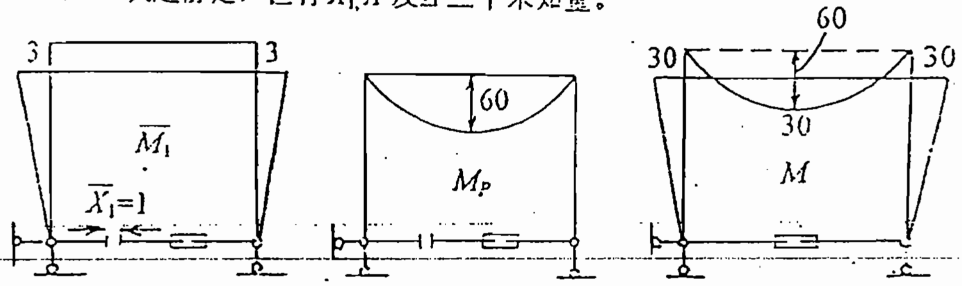
得 $X_1=\frac{Pl}{6}$

代入力法典型方程有

$$\frac{2l}{3EI}X_1-\frac{Pl^2}{8EI}+\frac{2\Delta}{l}=0$$

$$\Delta=\frac{Pl^3}{144EI}=\frac{50\times10^3\times10^3}{144\times210\times10^9\times7144\times10^{-8}}$$
$$=23.2\times10^{-3}\ \text{m}\downarrow$$

*8-32 虽为一次超静定，但有 X_1, A 及 Δ 三个未知量。



由正负弯矩绝对值相等，有 $3X_1=30$，得

$$X_1=10\ \text{kN}$$

由 $A\geq\frac{X_1}{[\sigma]}=\frac{10\times10^3}{160\times10^6}=62.5\times10^{-6}\ \text{m}^2=62.5\text{mm}^2$

可选拉杆直径 $d=10$ mm, $A=78.54$ mm^2。

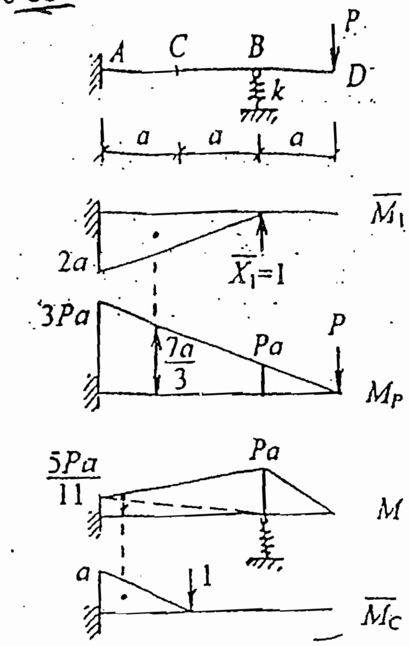
由 $\delta_{11}X_1+\Delta_{1P}+\Delta=0$

$$\left(\frac{54}{EI}+\frac{4}{EA}\right)X_1-\frac{480\times10^3}{EI}+\Delta=0$$

$$\Delta=\frac{1}{EI}\left[480-(54+4\frac{I}{A})X_1\right]$$

$$=\frac{480-\left(54+4\times\frac{2500\times10^{-8}}{7854\times10^{-6}}\right)10\times10^3}{210\times10^9\times2500\times10^{-8}}=-0.01385\text{m}\ \text{（缩短）}$$

8-33



$$k=\frac{EI}{a^3}$$

$$\delta_{11}X_1+\Delta_{1P}=-\frac{X_1}{k}$$

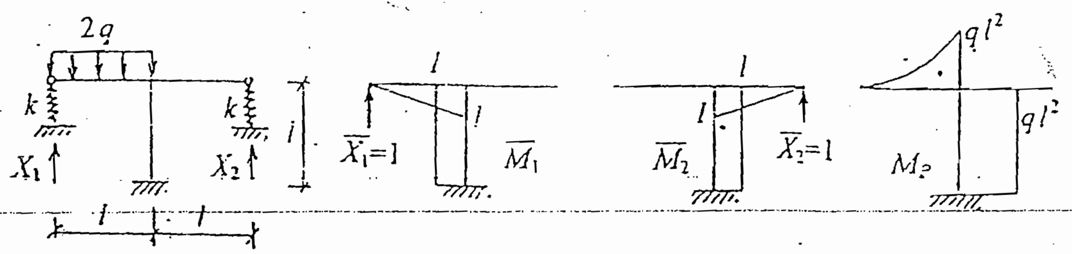
$$\delta_{11}=\frac{8a^3}{3EI}\qquad \Delta_{1P}=-\frac{14}{3}\frac{Pa^3}{EI}$$

$$X_1=-\frac{\Delta_{1P}}{\delta_{11}+\frac{1}{k}}=\frac{14}{11}P$$

$$\Delta_{Cy}=\frac{1}{EI}\frac{a^2}{2}\left(\frac{5}{6}\frac{5Pa}{11}+\frac{Pa}{6}\right)$$

$$=\frac{3}{11}\frac{Pa^3}{EI}\downarrow$$

8-34



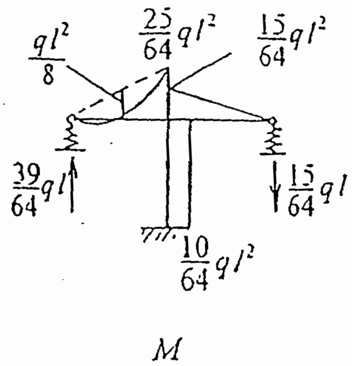
$$k=\frac{3EI}{a^3}$$

$$\delta_{11}=\delta_{22}=\frac{4}{3}\frac{l^3}{EI}\qquad \delta_{12}=\delta_{21}=-\frac{l^3}{EI}$$

$$\Delta_{1P}=-\frac{5}{4}\frac{ql^4}{EI}\qquad \Delta_{2P}=\frac{ql^4}{EI}$$

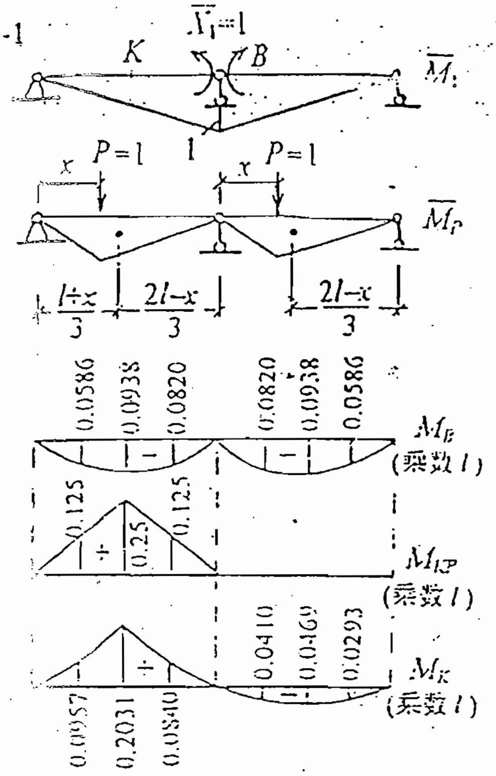
$$\begin{cases}\frac{4}{3}X_1-X_2-\frac{5}{4}ql=-\frac{X_1}{3}\\ -X_1+\frac{4}{3}X_2+ql=-\frac{X_2}{3}\end{cases}$$

$$X_1=\frac{39}{64}ql\uparrow\qquad X_2=-\frac{15}{64}ql\downarrow$$



第九章　力法应用

-1



$$\delta_{11}=\frac{2l}{3EI}$$

第1跨 $\delta_{P1}=\frac{1}{EI}\frac{1}{2}\frac{x(l-x)}{l}\frac{l+x}{3l}=\frac{x(l^2-x^2)}{6EIl}$

第2跨 $\delta_{P1}=\frac{1}{EI}\frac{1}{2}\frac{x(l-x)}{l}\frac{2l-x}{3l}=\frac{x(l-x)(2l-x)}{6EIl}$

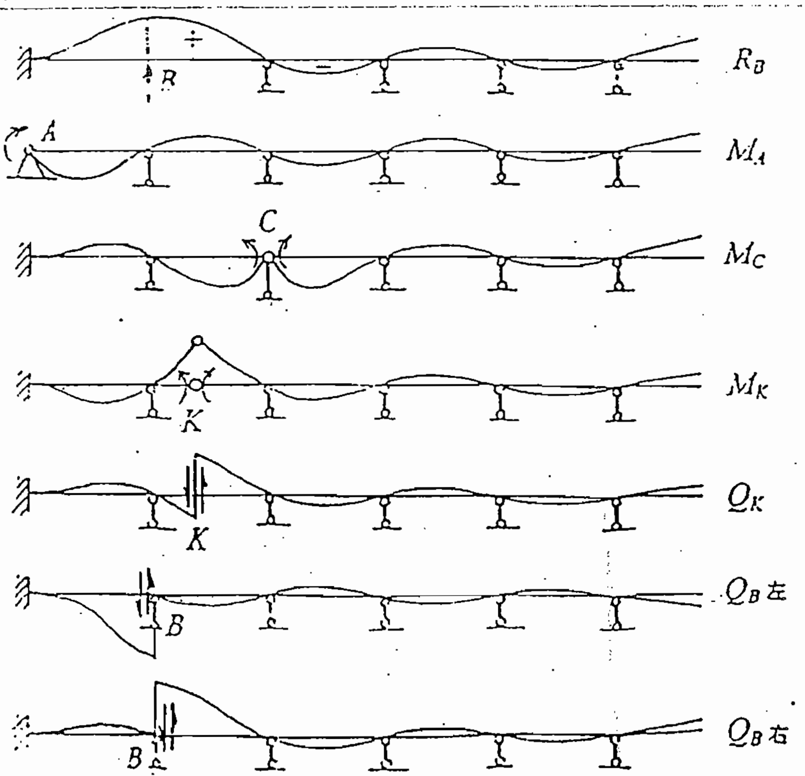
$X_1=-\frac{\delta_{P1}}{\delta_{11}}$，而 X_1 即 M_B，故有

第1跨 $M_B=-\frac{l}{4}\frac{x}{l}\left(1-\frac{x^2}{l^2}\right)$

第2跨 $M_B=-\frac{l}{4}\frac{x}{l}\left(1-\frac{x}{l}\right)\left(2-\frac{x}{l}\right)$

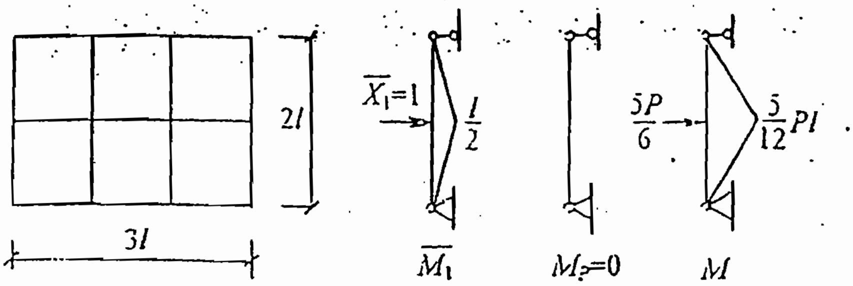
$$M_K=\overline{M}_{K1}X_1+M_{KP}=\frac{1}{2}M_B+M_{KP}$$

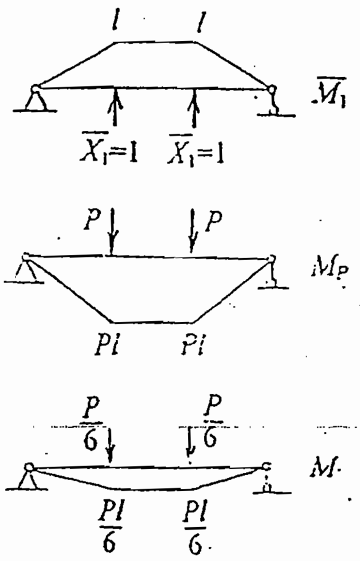
-2



9-3 略。

*9-4 取基本结构时， 设长梁在上层，短梁在下层。



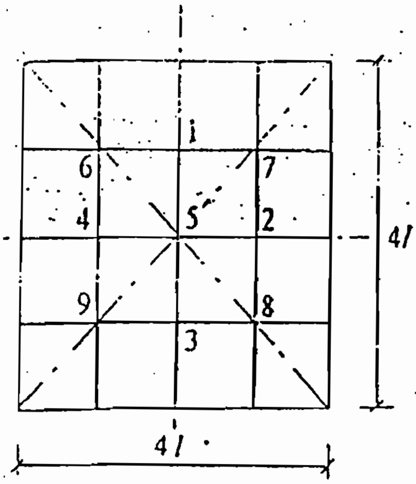


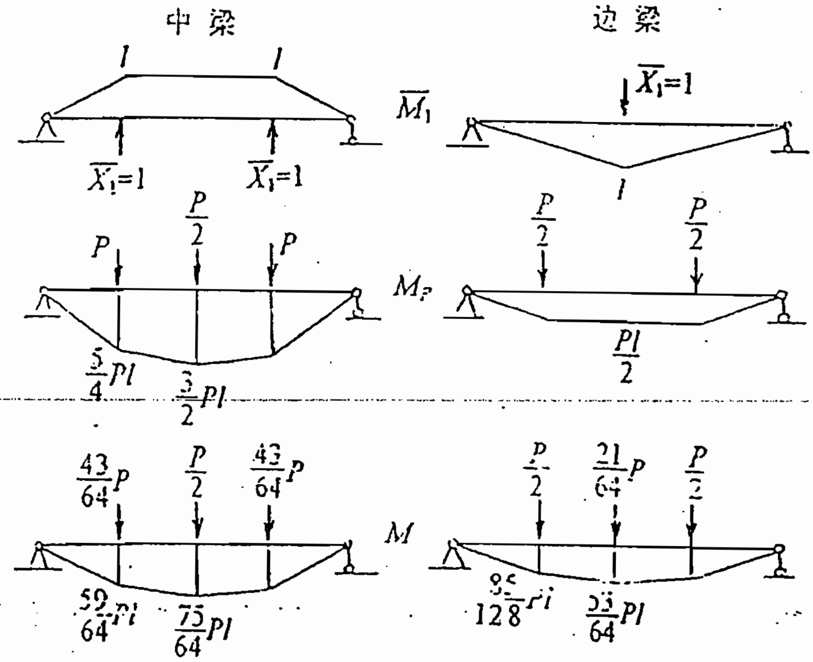
$$EI\delta_{11}=2\frac{l^2}{2}\frac{2l}{3}+l^2l+2\frac{l\frac{l}{2}}{2}\frac{2}{3}\frac{l}{2}\times 2=2l^3$$

$$EI\Delta_{1P}=-\frac{5}{3}Pl^3$$

$$X_1=-\frac{\Delta_{1P}}{\delta_{11}}=\frac{5}{6}P$$

*9-5 此结构对水平中轴、竖直中轴和两对角线均对称，故知在结点 5,6,7,8,9 处交叉二梁各承担 $P/2$；取基本结构时，设在结点 1,2,3,4 处中梁均在下层而边梁均在上层，这样对水平中轴、竖直中轴和两对角线均保持对称，因而此四结点之多余未知力均 X_1。





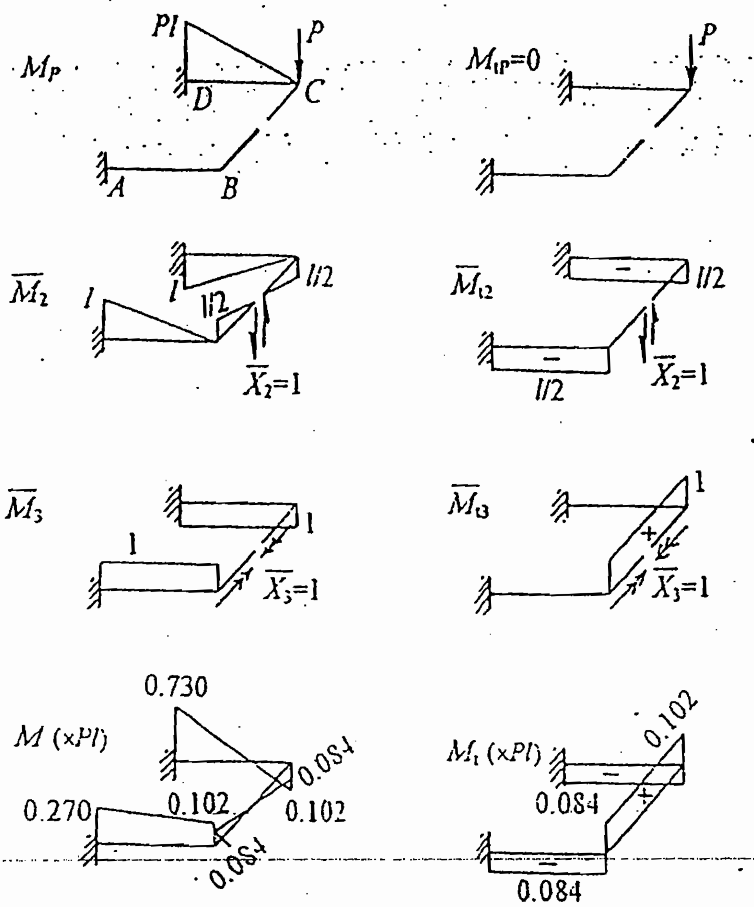
$$EI\delta_{11}=2\left[\left(\frac{l^2}{2}\frac{2l}{3}\right)2+2l^2l\right]+4\left[\left(\frac{2l\cdot l}{2}\frac{2l}{3}\right)2\right]=\frac{32}{3}l^3$$

$$EI\Delta_{1P}=-2\left[\left(\frac{1}{2}\frac{5Pl}{4}l\frac{2l}{3}\right)2+\frac{1}{2}\left(\frac{5Pl}{4}+\frac{3Pl}{2}\right)l\cdot l\times 2\right]+4\left[\left(\frac{1}{2}\frac{Pl}{2}l\frac{l}{3}\right)2+\left(\frac{Pl\cdot l}{2}\frac{3l}{4}\right)2\right]$$

$$=\left(-\frac{43}{6}+\frac{22}{6}\right)Pl^3=-\frac{7}{2}Pl^3$$

$$X_1=\frac{21}{64}P$$

*9-6　从杆 BC 中点截开，可判断只有剪力 X_2 和扭矩 X_3，而弯矩 $X_1=0$（因由图乘法可知 $\delta_{12}=\delta_{13}=\Delta_{1P}=0$）。

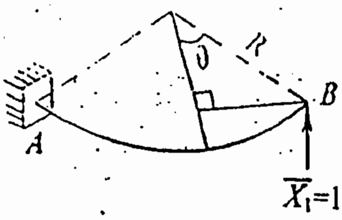


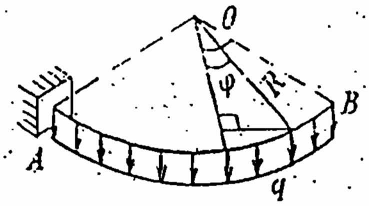
$$\begin{cases} 1.375\,l^3X_2 + l^2X_3 - Pl^3/3 = 0 \\ l^2X_2 + 3.25\,lX_3 - Pl^2/2 = 0 \end{cases}$$

$X_2 = 0.168P$

$X_3 = 0.102Pl$

*9-7





$$\overline{M}_{\mathrm{I}} = R\sin\theta$$

$$\overline{M}_{\mathrm{II}} = R(1-\cos\theta)$$

$$\delta_{11} = \frac{R^3}{EI}\int_0^{\pi/2}\sin^2\theta d\theta + \frac{R^3}{GI_t}\int_0^{\pi/2}(1-\cos\theta)^2 d\theta = \frac{\pi R^3}{4EI} + \frac{3\pi-8}{4}\frac{R^3}{GI_t}$$

（积分过程可参见36页习题 * 7－26）

$$M_{\mathrm{I}P} = \int_0^{\theta} -qRd\varphi\cdot R\sin\varphi = -qR^2\int_0^{\theta}\sin\varphi d\varphi = -qR^2(1-\cos\theta)$$

$$M_{\mathrm{II}P} = \int_0^{\theta} -qRd\varphi\cdot R(1-\cos\varphi) = -qR^2\int_0^{\theta}(1-\cos\varphi)d\varphi = -qR^2(\theta-\sin\theta)$$

$$\Delta_{1P} = -\frac{qR^4}{EI}\int_0^{\pi/2}\sin\theta(1-\cos\theta)d\theta - \frac{qR^4}{GI_t}\int_0^{\pi/2}(1-\cos\theta)(\theta-\sin\theta)d\theta$$

$$= -\frac{qR^4}{EI}\left[-\cos\theta+\frac{\cos^2\theta}{2}\right]_0^{\pi/2} - \frac{qR^4}{GI_t}\left[\frac{\theta^2}{2}+\cos\theta-\cos\theta-\theta\sin\theta+\frac{\sin^2\theta}{2}\right]_0^{\pi/2}$$

$$= -\frac{qR^4}{EI}\left[1-\frac{1}{2}\right] - \frac{qR^4}{GI_t}\left[\frac{\pi^2}{8}-\frac{\pi}{2}+\frac{1}{2}\right]$$

$$= -\frac{qR^4}{2EI} - \frac{qR^4}{2GI_t}\left(\frac{\pi}{2}-1\right)^2$$

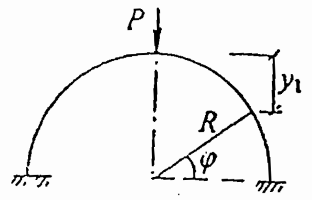
$$X_1 = \frac{\dfrac{1}{2EI}+\dfrac{\left(\dfrac{\pi}{2}-1\right)^2}{2GI_t}}{\dfrac{\pi}{4EI}+\dfrac{3\pi-8}{4GI_t}}qR \quad \uparrow$$

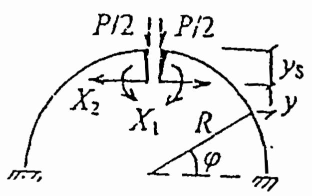
9-8.

$$y_S=\frac{\int y_1\frac{ds}{EI}}{\int\frac{ds}{EI}}=\frac{2\int_0^{\pi/2}R(1-\sin\varphi)Rd\varphi}{\int ds}=\frac{2R^2\left(\frac{\pi}{2}-1\right)}{\pi R}=(1-\frac{2}{\pi})R$$

$$y=y_1-y_S=R(1-\sin\varphi)-(1-\frac{2}{\pi})R=(\frac{2}{\pi}-\sin\varphi)R$$

$$M_P=-\frac{P}{2}R\cos\varphi$$





$$EI\delta_{11}=\int ds=\pi R$$

$$EI\delta_{22}=\int y^2ds=2\int_0^{\pi/2}\left(\frac{2}{\pi}-\sin\varphi\right)^2R^2Rd\varphi$$

$$=2R^3\int_0^{\pi/2}\left(\frac{4}{\pi^2}-\frac{4}{\pi}\sin\varphi+\sin^2\varphi\right)d\varphi$$

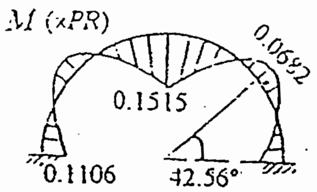
$$=2R^3\left(\frac{2}{\pi}-\frac{4}{\pi}+\frac{\pi}{4}\right)=\left(\frac{\pi}{2}-\frac{4}{\pi}\right)R^3$$

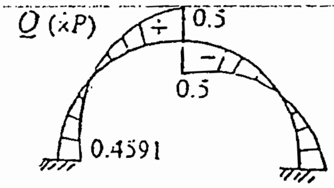
$$EI\Delta_{1P}=\int M_Pds=2\int_0^{\pi/2}-\frac{P}{2}R\cos\varphi Rd\varphi=-PR^2$$

$$EI\Delta_{2P}=\int yM_Pds=2\int_0^{\pi/2}\left(\frac{2}{\pi}-\sin\varphi\right)R\left(-\frac{P}{2}R\cos\varphi\right)Rd\varphi$$

$$=-PR^3\int_0^{\pi/2}\left(\frac{2}{\pi}\cos\varphi-\sin\varphi\cos\varphi\right)d\varphi$$

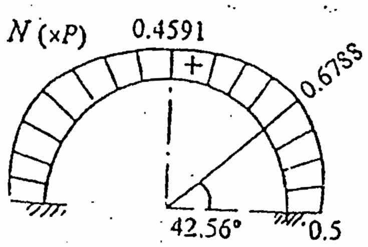
$$=-PR^3\left(\frac{2}{\pi}-\frac{1}{2}\right)$$





$$X_1=\frac{PR}{\pi}=0.3183\,PR$$

$$X_2=\frac{4-\pi}{\pi^2-8}P=0.4591\,P$$



$$M=\frac{PR}{\pi}+\frac{4-\pi}{\pi^2-8}P\left(\frac{2}{\pi}-\sin\varphi\right)R-\frac{PR}{2}\cos\varphi$$

$$=PR(0.6106-0.4591\sin\varphi-0.5\cos\varphi)$$

当 $\frac{dM}{d\varphi}=PR(-0.4591\cos\varphi+0.5\sin\varphi)=0$

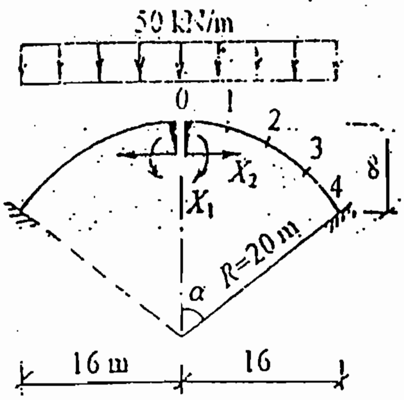
有 $\operatorname{tg}\varphi=\frac{0.4591}{0.5}=0.9182,\quad \varphi=42.56°$

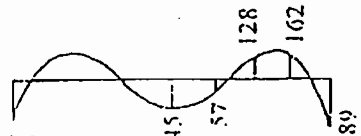
此时 $M_{min}=-0.0682\,PR$

$Q=0.4591P\cos\varphi-0.5P\sin\varphi$（右半拱）

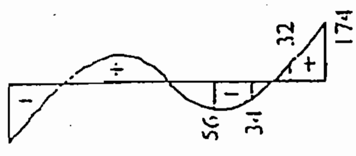
$N=0.4591P\sin\varphi+0.5P\cos\varphi$

9-9

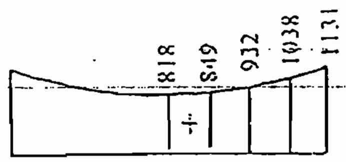




M(kN·m)



Q(kN)



N(kN)

$$\sin\alpha = \frac{16}{20} = 0.8$$

$$\alpha = 53.13^\circ$$

$$x = R\sin\varphi$$

$$y_1 = R(1-\cos\varphi)$$

$$y_S = \frac{\sum \frac{y_1}{EI}\Delta S}{\sum \frac{1}{EI}\Delta S} = \frac{\sum \frac{y_1}{I}}{\sum \frac{1}{I}} = \frac{94.77}{39.40} = 2.41\ \text{m}$$

$$X_1 = -\frac{\sum \frac{M_P}{I}}{\sum \frac{1}{I}} = -\frac{-83372}{39.40} = 2116\ \text{kN}\cdot\text{m}$$

$$X_2 = -\frac{\sum \frac{yM_P}{I}}{\sum \frac{y^2}{I}} = -\frac{-185029}{226.21} = 818.0\ \text{kN}$$

$$M = X_1 + X_2 y + M_P = 2116 + 818y - 25x^2$$

$$Q = X_2\sin\varphi + Q_P = 818\sin\varphi - 50x\cos\varphi$$

$$N = X_2\cos\varphi + N_P = 818\cos\varphi + 50x\sin\varphi$$

截面	φ	x	y_1	h	$1/I=\frac{12}{h^3}$	$\frac{y_1}{I}$	$y=y_1-2.41$	$\frac{y^2}{I}$	$M_P=-25x^2$	$\frac{M_P}{I}$	$\frac{yM_P}{I}$
0	0	0	0	1.00	12.00	0	−2.41	69.70	0	0	0
1	13.28°	4.60	0.54	1.02	11.31	6.11	−1.87	39.55	−529	−5983	11188
2	26.56°	8.94	2.11	1.06	10.08	21.27	−0.30	0.91	−1999	−20150	6045
3	39.85°	12.81	4.64	1.12	8.54	39.63	2.23	42.47	−4102	−35031	−78119
4	53.13°	16.00	8.00	1.20	6.94	55.52	5.59	216.86	−6400	−44416	−248285
				梯形总和 Σ	39.44	94.77		226.21		−83372	−185029

内力列表计算略。

9-10

$$\mu=\frac{45}{16}\frac{I_C l}{A_C f^3}\text{arc tg}\left(\frac{4f}{l}\right)=\frac{45}{16}\frac{\frac{0.6^3}{12}\times 12}{0.6\times 4^3}\text{arc tg}\left(\frac{4\times 4}{12}\right)=0.01467$$

$$X_1=\frac{45\alpha tEI_C}{4(1+\mu)f^2}=\frac{45\times 10^{-5}(-20)20\times 10^9\times\frac{0.6^3}{12}}{4(1+0.01467)4^2}\times 10^{-3}=-49.9\text{ kN}\text{（拉力）}$$

拱顶 $M_C=49.9\times 1.33=66.4\text{ kN}\cdot\text{m}$

拱趾 $M_K=-49.9\times 2.67=-133.2\text{ kN}\cdot\text{m}$

9-11 以左支座为坐标原点，拱轴方程为 $y=\frac{4f}{l^2}x(l-x)$。

$$\delta_{11}=\int y^2\frac{ds}{EI}+\frac{l}{E_1A_1}=\frac{1}{EI_C}\int_0^l\left[\frac{4f}{l^2}x(l-x)\right]^2dx+\frac{l}{E_1A_1}$$

$$=\frac{8}{15}\frac{f^2l}{EI_C}+\frac{l}{E_1A_1}=\frac{8}{15}\frac{f^2l}{EI_C}\left(1+\frac{15}{8}\frac{EI_C}{E_1A_1f^2}\right)$$

$$=\frac{8}{15}\frac{f^2l}{EI_C}\left(1+\frac{15}{8}\frac{5000}{2\times 10^5\times 5^2}\right)=\frac{8}{15}\frac{f^2l}{EI_C}(1.001875)$$

$$\Delta_{1P}=\int -yM_P\frac{ds}{EI}$$

$$=-\frac{1}{EI_C}\int_0^{l/2}\frac{4f}{l^2}(lx-x^2)\left(\frac{3ql}{8}x-\frac{qx^2}{2}\right)dx-\frac{1}{EI_C}\int_{l/2}^{l}\frac{4f}{l^2}(lx-x^2)\frac{ql}{8}(l-x)dx$$

$$=-\frac{fq}{2l^2EI_C}\int_0^{l/2}(3l^2x^2-7lx^3+4x^4)dx-\frac{fq}{2l^2EI_C}\int_{l/2}^{l}(l^3x-2l^2x^2+lx^3)dx$$

$$=-\frac{fq}{2l^2EI_C}\left[\frac{1}{8}-\frac{7}{64}+\frac{1}{40}+\frac{3}{8}-\frac{7}{12}+\frac{15}{64}\right]=-\frac{qfl^3}{30EI_C}$$

$$X_1=\frac{ql^2}{16f\times 1.001875}=\frac{20\times 20^2}{16\times 5\times 1.001875}=99.81\text{ kN}$$

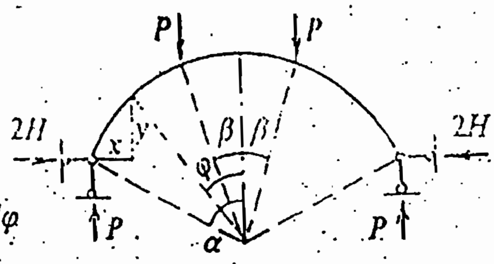
$y_K=3.75\text{ m}$ $\quad$ $\text{tg}\varphi_K=0.5$ $\quad$ $\sin\varphi_K=0.4472$ $\quad$ $\cos\varphi_K=0.8944$

$$M_K=150\times 5-\frac{20\times 5^2}{2}-99.81\times 3.75=125.7\text{ kN}\cdot\text{m}$$

$$Q_K=(150-20\times 5)0.8944-99.81\times 0.4472=0.085\text{ kN}$$

$$N_K=(150-20\times 5)0.4472+99.81\times 0.8944=111.63\text{ kN}$$

9-12



$$EI\delta_{11}=\int y^2 ds=\int R^2(\cos\varphi-\cos\alpha)^2 Rd\varphi$$

$$=2\int_0^{\alpha} R^3(\cos^2\varphi-2\cos\alpha\cos\varphi+\cos^2\alpha)d\varphi$$

$$=2R^3\int_0^{\alpha}(\frac{1}{2}+\frac{1}{2}\cos 2\varphi-2\cos\alpha\cos\varphi+\cos^2\alpha)d\varphi$$

$$=2R^3\left(\frac{\alpha}{2}+\frac{1}{4}\sin 2\alpha-2\cos\alpha\sin\alpha+\alpha\cos^2\alpha\right)$$

当 $\alpha=\dfrac{\pi}{2}$ 时，有 $\qquad EI\delta_{11}=\dfrac{\pi R^3}{2}$

$$-EI\Delta_{1P}=\int M_P y ds=\int_{\beta}^{\alpha} PR(\sin\alpha-\sin\varphi)R(\cos\varphi-\cos\alpha)Rd\varphi$$

$$+\int_0^{\beta} PR(\sin\alpha-\sin\beta)R(\cos\varphi-\cos\alpha)Rd\varphi$$

$$=PR^3\left[\sin\alpha(\sin\alpha-\sin\beta)-(\alpha-\beta)\sin\alpha\cos\alpha+\frac{\cos^2\alpha-\cos^2\beta}{2}-\right.$$

$$\left.-(\cos\alpha-\cos\beta)\cos\alpha\right]+PR^3(\sin\alpha-\sin\beta)(\sin\beta-\beta\cos\alpha)$$

$$=PR^3\left[(\sin\alpha-\sin\beta)(\sin\alpha+\sin\beta-\beta\cos\alpha)-(\alpha-\beta)\sin\alpha\cos\alpha+\right.$$

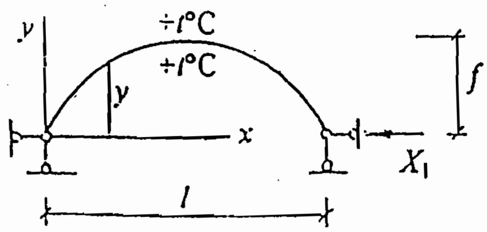
$$\left.+\frac{\cos^2\alpha-\cos^2\beta}{2}-(\cos\alpha-\cos\beta)\cos\alpha\right]$$

当 $\alpha=\dfrac{\pi}{2}$，有

$$-EI\Delta_P=PR^3\left[(1-\sin\beta)(1+\sin\beta)-\frac{\cos^2\beta}{2}\right]=PR^3\left[1-\sin^2\beta-\frac{\cos^2\beta}{2}\right]=PR^3\frac{\cos^2\beta}{2}$$

此时有 $\qquad H=\dfrac{-\Delta_P}{\delta_{11}}=\dfrac{P\cos^2\beta}{2}$

9-13



$$\delta_{11}=\sum\int\frac{\overline{M}_1^2 ds}{EI}+\sum\int\frac{\overline{N}_1^2 ds}{EA}$$

$$=\int_0^l\frac{y^2 ds}{EI}+\int_0^l\cos^2\varphi\frac{ds}{EA}$$

$$\Delta_{1t}=\sum\alpha t\int\overline{N}_1 ds=\alpha t\int_0^l-\cos\varphi ds$$

$$=-\alpha t\int_0^l dx=-\alpha t l$$

$$X_1=\frac{\alpha t l}{\int_0^l\frac{y^2}{EI}+\int_0^l\cos^2\varphi\frac{ds}{EA}},\qquad M=-X_1 y=-\frac{\alpha t l y}{\int_0^l\frac{y^2 ds}{EI}+\int_0^l\cos^2\varphi\frac{ds}{EA}}$$

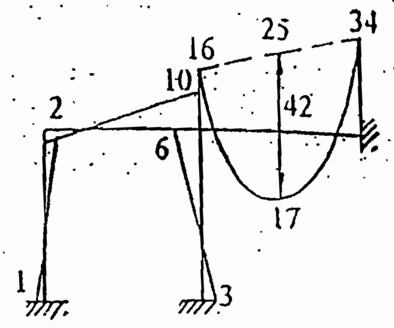
第十章　位移法

10-1　略。

10-2　$\overline{M_1}, \overline{M_2}, M_P$ 图略（下同）。

$$\begin{cases} 12iZ_1+4iZ_2+0=0 \\ 4iZ_1+20iZ_2-\dfrac{ql^2}{12}=0 \end{cases}$$

$$Z_1=-\frac{ql^2}{672i}\qquad Z_2=\frac{3ql^2}{672i}$$

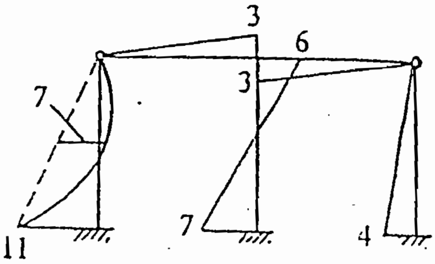


$M\,(\times ql^2/336)$

10-3

$$\begin{cases} 16iZ_1-\dfrac{6i}{l}Z_2+0=0 \\ -\dfrac{6i}{l}Z_1+\dfrac{18i}{l^2}Z_2-\dfrac{3ql}{8}=0 \end{cases}$$

$$Z_1=\frac{ql^2}{112i}\qquad Z_2=\frac{ql^3}{42i}$$

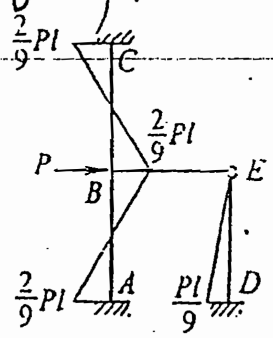


$M\,(\times ql^2/56)$

10-4

$$\begin{cases} 14iZ_1+0+0=0 \\ 0+\dfrac{27i}{l^2}Z_2-P=0 \end{cases}$$

$$Z_1=0\qquad Z_2=\frac{Pl^2}{27i}$$

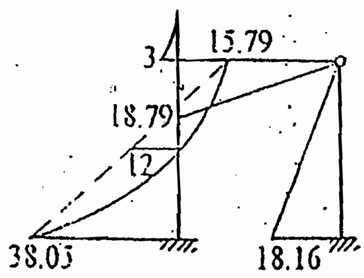


M

10-5

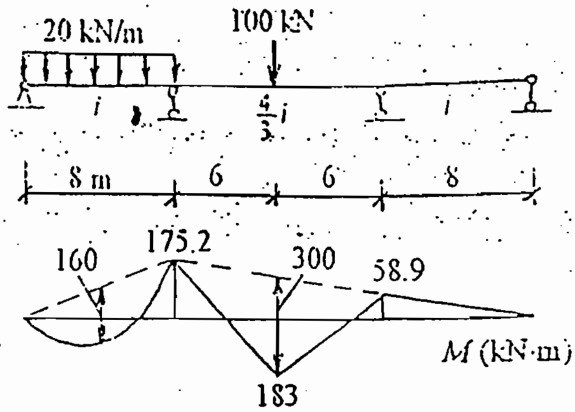
$$\begin{cases} 10iZ_1-1.5iZ_2+5=0 \\ -1.5iZ_1+\dfrac{15i}{16}Z_2-18=0 \end{cases}$$

$$Z_1=\frac{3.1316}{i}\qquad Z_2=\frac{24.2105}{i}$$



$M\,(\text{kN}\cdot\text{m})$

10-6 令 $i=\dfrac{EI}{8}$



$$\begin{cases}\dfrac{25}{3}iZ_1+\dfrac{8}{3}iZ_2+10=0\\[2mm]\dfrac{8}{3}iZ_1+\dfrac{25}{3}iZ_2+150=0\end{cases}$$

$$Z_1=\frac{5.08}{i}\qquad Z_2=\frac{-19.63}{i}$$

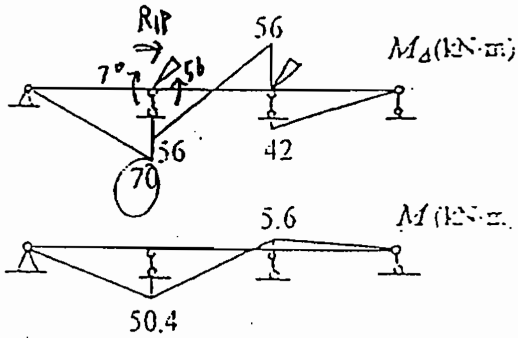
10-7

$EI=42000\ \mathrm{kN\cdot m^2}$, $i=7000\ \mathrm{kN\cdot m}$

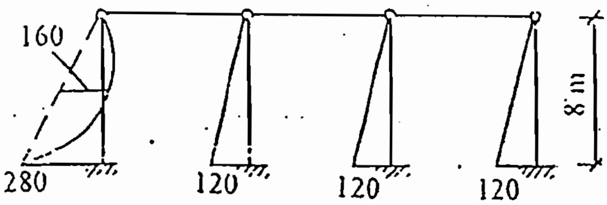
$$\begin{cases}7iZ_1+2iZ_2-14=0\\2iZ_1+7iZ_2+98=0\end{cases}$$

$$Z_1=\frac{98}{15i}=\frac{6.533}{i}=0.9333\times10^{-3}\ \mathrm{rad}$$

$$Z_2=-\frac{238}{15i}=-\frac{15.867}{i}=-2.2667\times10^{-3}\ \mathrm{rad}$$



10-8



M (kN·m)

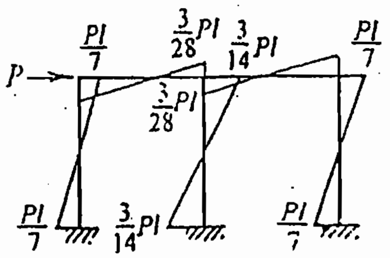
$$\frac{12i}{l^2}Z_1-\frac{3ql}{8}=0$$

$$Z_1=\frac{ql^3}{32i}$$

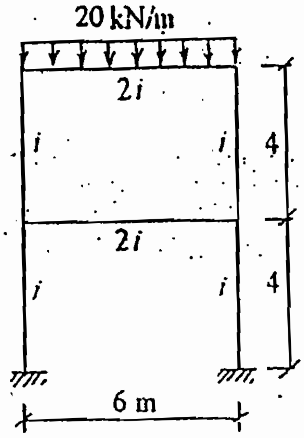
10-9 三刚结点均无转角，只有线位移Z_1：

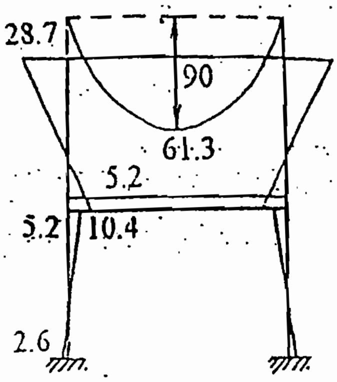
$$\frac{42EI}{l^3}Z_1-P=0;\quad Z_1=\frac{Pl^3}{42EI}$$

先绘各柱弯矩图，再由结点平衡推定横梁弯矩，因反对称，故中间结点处两横梁杆端弯矩各为$\dfrac{3}{28}Pl$.



10-10





M (kN·m)

令 $i=\dfrac{EI}{4}$

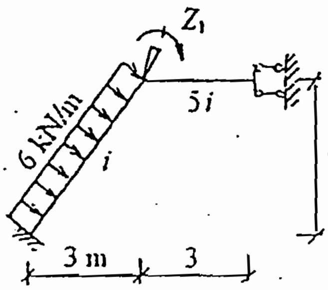
取一半（图略）有

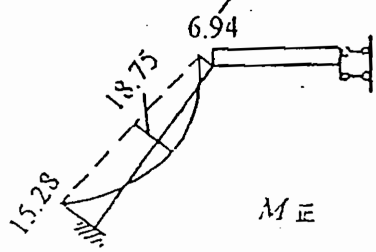
$$\begin{cases}8iZ_1+2iZ_2-60=0\\2iZ_1+12iZ_2+0=0\end{cases}$$

$$Z_1=\frac{180}{23i}=\frac{7.826}{i}$$

$$Z_2=-\frac{30}{23i}=-\frac{1.3043}{i}$$

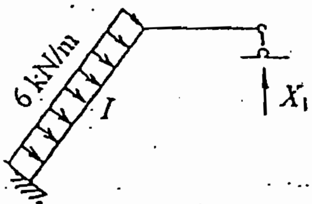
10-11

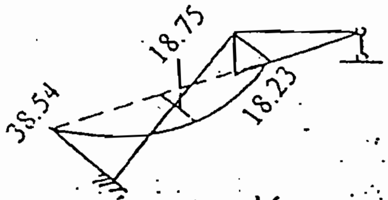




$M_{正}$

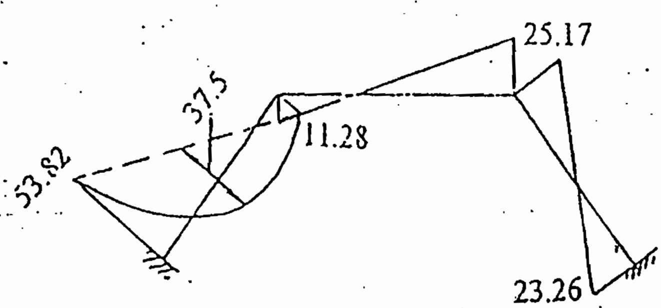
正对称 $Z_1=-\dfrac{R_{1P}}{r_{11}}=-\dfrac{12.5}{9i}=-\dfrac{1.3889}{i}$





$M_{反}$

反对称 $X_1=-\dfrac{\Delta_{1P}}{\delta_{11}}=-\dfrac{-656.25}{108}=6.0764\text{ kN}$

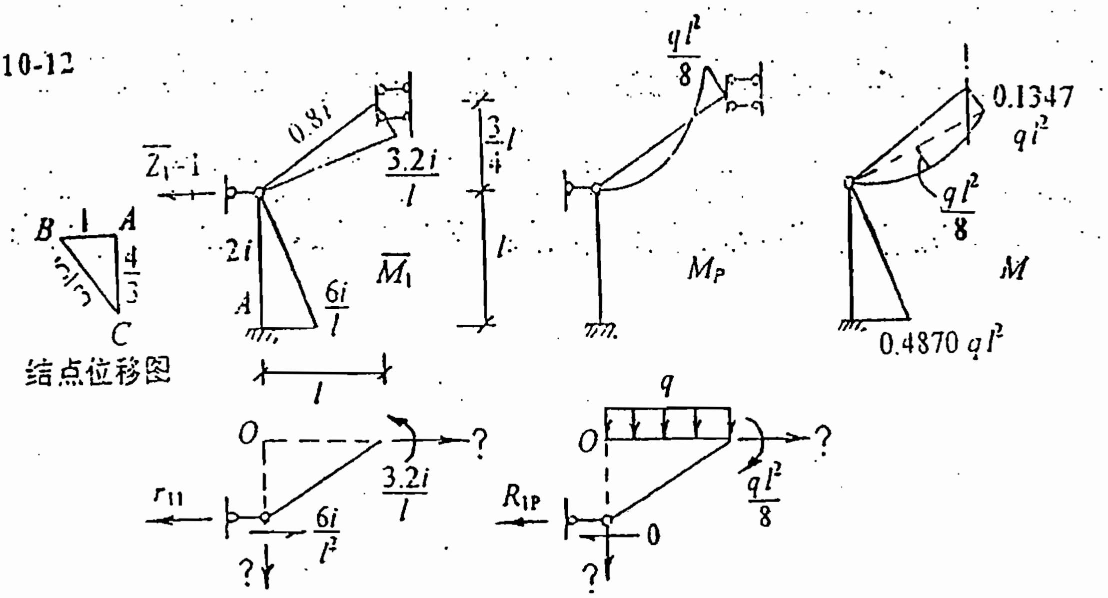


M (kN·m)

$\frac{12i}{7}$

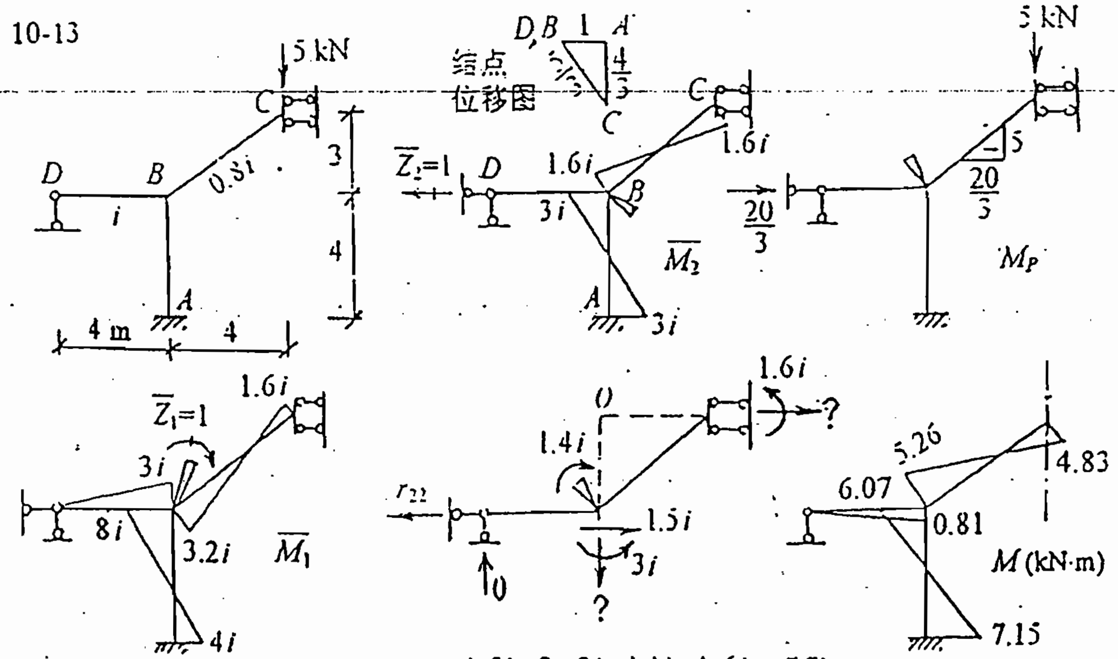
$\frac{8\times6\times8}{8}$ $\frac{EI}{8}$ $\frac{100\times1\times3}{8\ 42}$

10-12



$$r_{11}=\frac{\frac{6i}{l^2}\cdot\frac{3l}{4}+\frac{3.2i}{l}}{\frac{3l}{4}}=\frac{30.8i}{3l^2}\qquad R_{1P}=-\frac{ql\frac{l}{2}+\frac{ql^2}{8}}{\frac{3l}{4}}=-\frac{5ql}{6}\qquad Z_1=\frac{25ql^3}{308i}=0.081169\frac{ql^3}{i}$$

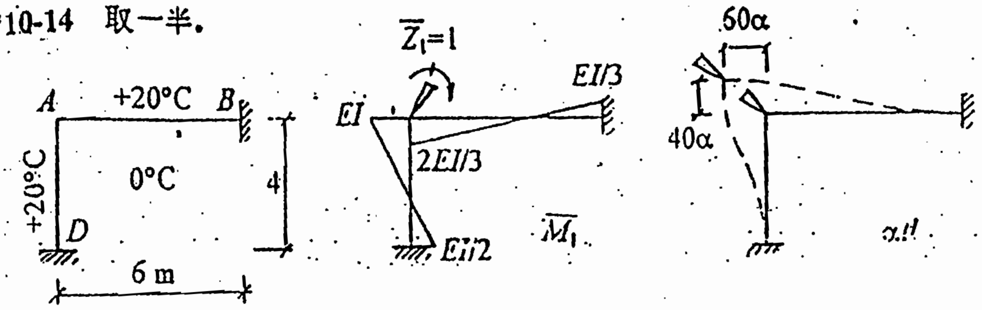
10-13

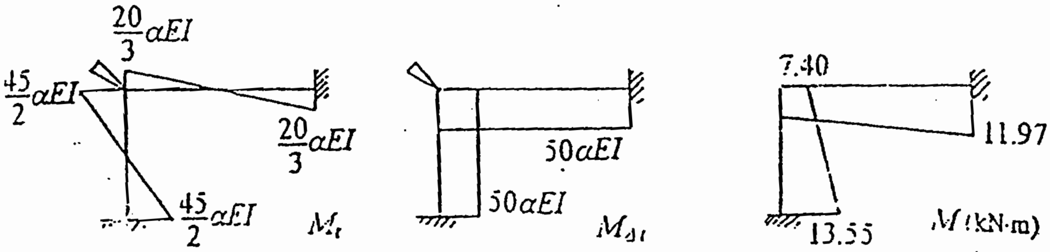


$$r_{22}=1.5i+\frac{1.5i\times3+3i-1.4i+1.6i}{3}=\frac{7.7i}{3}$$

$$\begin{cases}14.2iZ_1+1.4iZ_2+0=0\\ 1.4iZ_1+\frac{7.7i}{3}Z_2-\frac{20}{3}=0\end{cases}\qquad Z_1=-\frac{0.2706}{i}\qquad Z_2=\frac{2.745}{i}$$

*10-14 取一半。

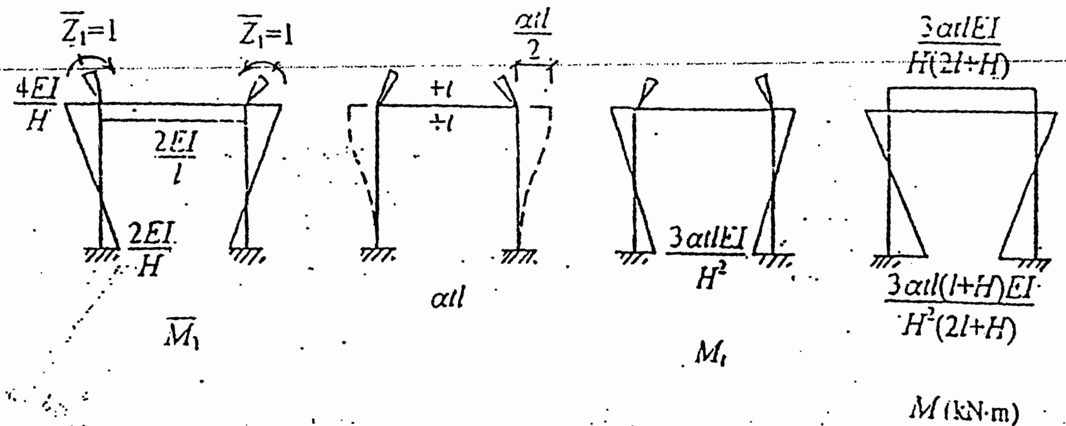




$t=+10°\mathrm{C}$，$\Delta t=-20°\mathrm{C}$，$\alpha EI=0.2\ \mathrm{kN\cdot m^2}$.

$$Z_1=-\frac{R_{1t}}{r_{11}}=-\frac{\dfrac{95\alpha EI}{6}}{\dfrac{5EI}{3}}=-9.5\alpha$$

*10-15



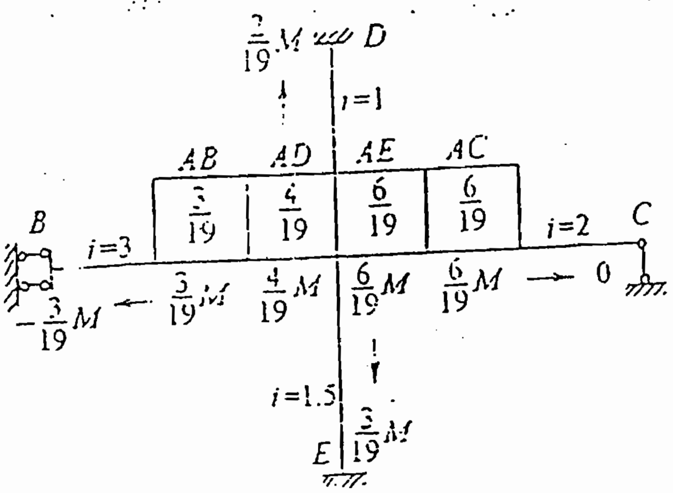
$$Z_1=-\frac{R_{1t}}{r_{11}}=-\frac{3\alpha tlEI}{H^2}\cdot\frac{1}{\dfrac{4EI}{H}+\dfrac{2EI}{l}}=-\frac{3\alpha tl^2}{2H(2l+H)}$$

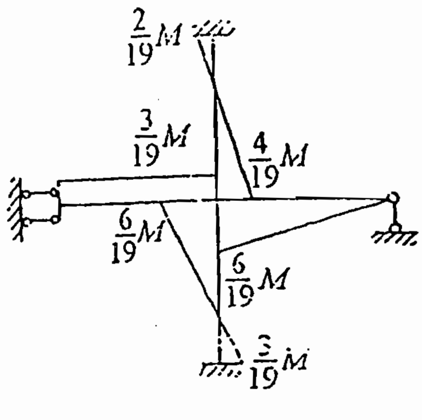
$$M_{BC}=\frac{2EI}{l}\left(-\frac{3\alpha tl^2}{2H(2l+H)}\right)=-\frac{3\alpha tlEI}{H(2l+H)}$$

$$M_{AB}=\frac{2EI}{H}\left(-\frac{3\alpha tl^2}{2H(2l+H)}\right)+\frac{3\alpha tlEI}{H^2}=\frac{3\alpha tlEI}{H^2}\left(1-\frac{l}{2l+H}\right)=\frac{3\alpha tl(l+H)EI}{H^2(2l+H)}$$

第十一　渐近法

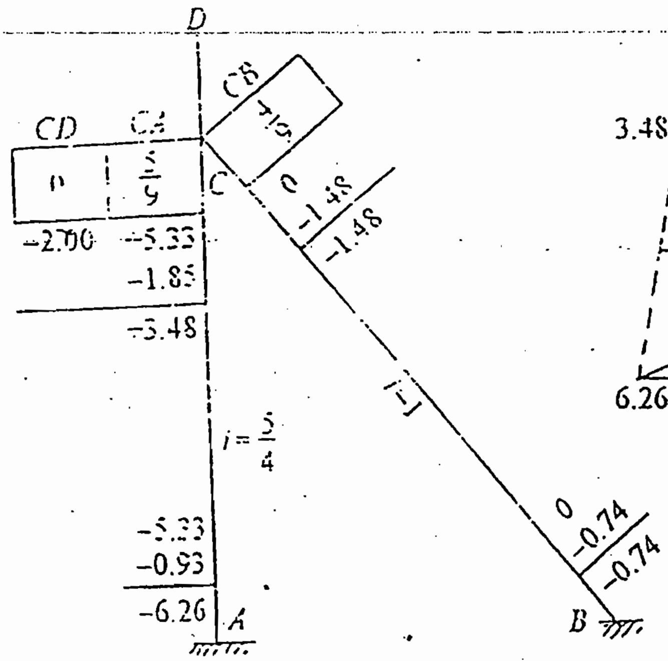
11-1 不[illegible]力[illegible]上的反[illegible]为 $-M$

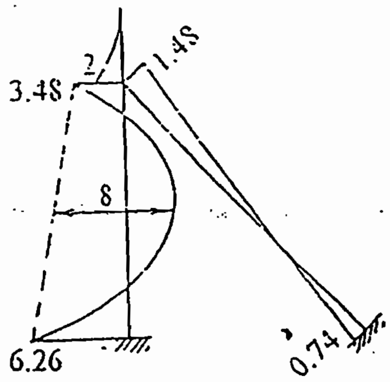




M

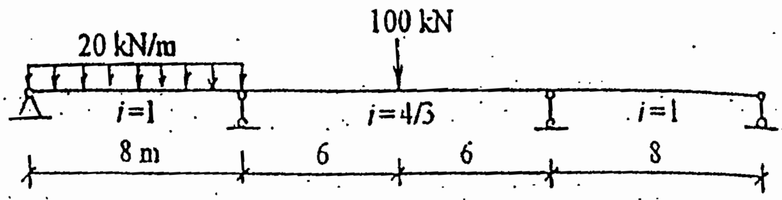
11-2



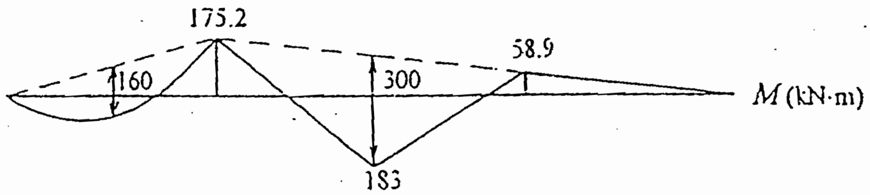


M (kN·m)

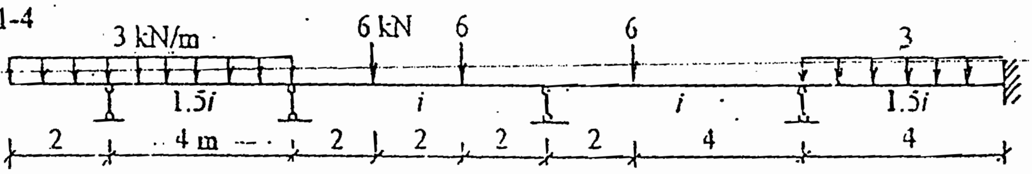
11-3



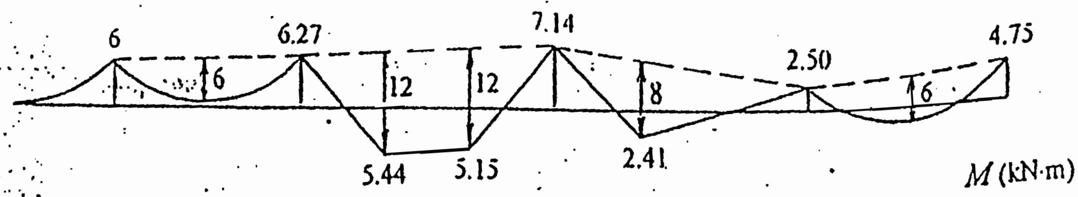
μ		0.36	0.64		0.64	0.36	
M^F	0	+160.0	−150.0		+150.0		
分			−48.0	←	−96.0	−54.0	0
	0	+13.7	+24.3	→	+12.2		
			−3.9	←	−7.8	−4.4	
传		+1.4	+2.5	→	+1.2		
			−0.4	←	−0.8	−0.4	
		+0.1	+0.3	→	+0.2		
					−0.1	−0.1	
M	0	+175.2	−175.2		+58.9	−58.9	0



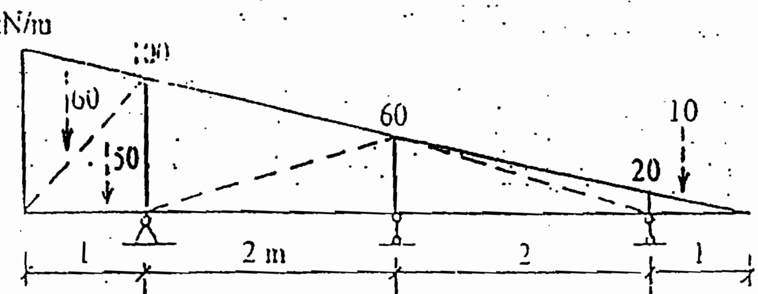
11-4



μ		0.5294	0.4706		0.5	0.5		0.4	0.6		
M^F	−6	+3	−8		+8	−5.33		+2.67	−4		+4
	0	+2.65	+2.35	→	+1.18	+0.27	←	+0.53	+0.80	→	+0.40
分			−1.03	←	−2.06	−2.06	→	−1.03			
		+0.55	+0.48	→	+0.24	+0.21	←	+0.41	+0.62	→	+0.31
			−0.11	←	−0.23	−0.22	→	−0.11			
传		+0.06	+0.05	→	+0.03	+0.02	←	+0.04	+0.07	→	+0.03
			−0.01	←	−0.02	−0.03	→	−0.01			
		+0.01	0					0	+0.01		
M	−6	+6.27	−6.27		+7.14	−7.14		+2.50	−2.50		+4.75

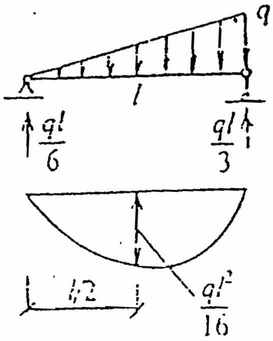


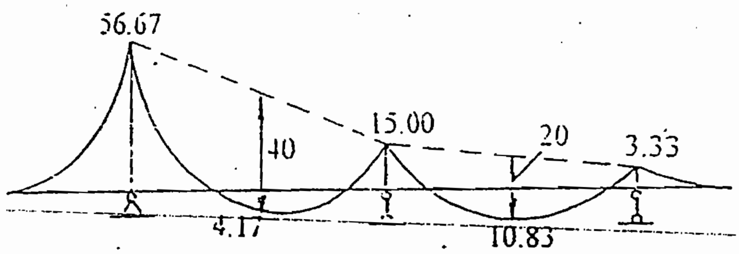
11-5



μ		1/2	1/2	
悬臂荷载	−56.67 →	−28.33	+ 1.67 ←	+3.33
△荷载		+16.00	−16.00	
◺◿荷载		+23.33	− 4.67	
总计 M^F		+11.00	−19.00	
分配		+ 4.00	+ 4.00	
M	−56.67	+15.00	−15.00	+3.33

附：简支梁在三角形
荷载下的弯矩图





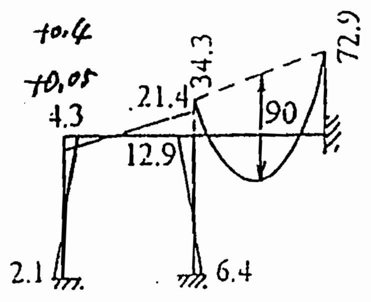
M(kN·m)

11-6

A		B			C
0.333	0.667	0.4	0.2	0.4	
				−60.0	+60.0
	+12.0 ←	+24.0	+12.0	+24.0 →	+12.0
−4.0	− 8.0 ←	− 4.0			
	+0.8 ←	+ 1.6	+ 0.8	+ 1.6 →	+ 0.8
−0.3	−0.5 →	− 0.3			
−4.3	+4.3	+ 0.1	+ 0.1	+ 0.1 →	+ 0.1
		+21.4	+12.9	−34.3	+72.9

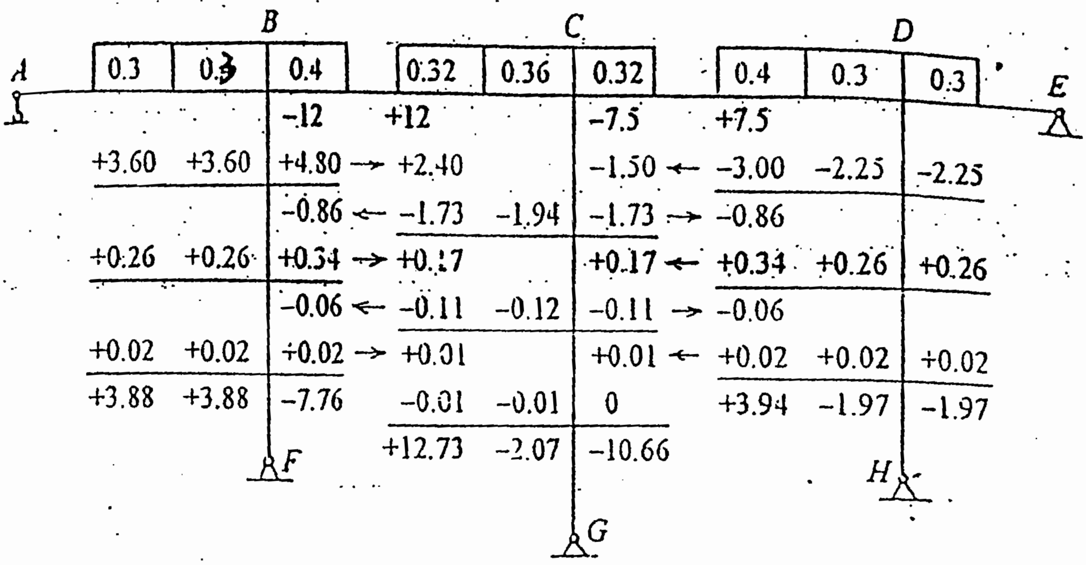
+6
+0.4
+0.05

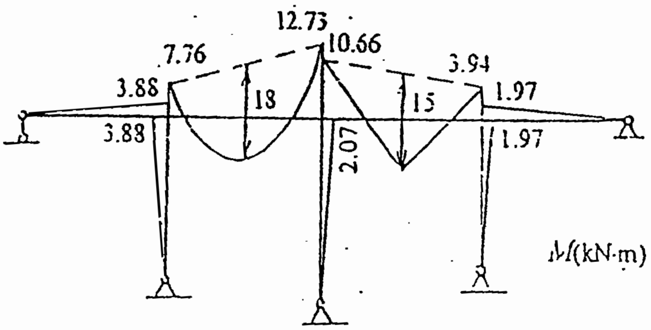
−2.1 D　　　　+6.4 E



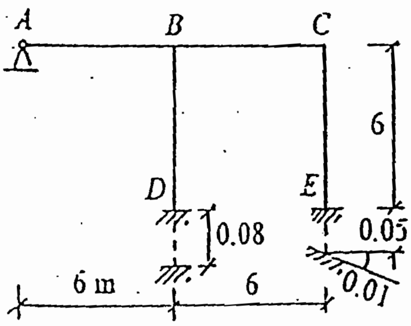
M(kN·m)

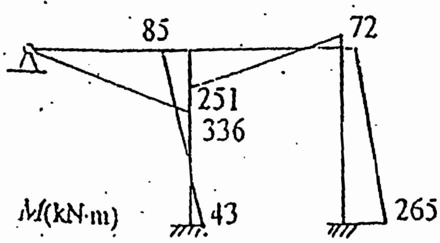
11-7

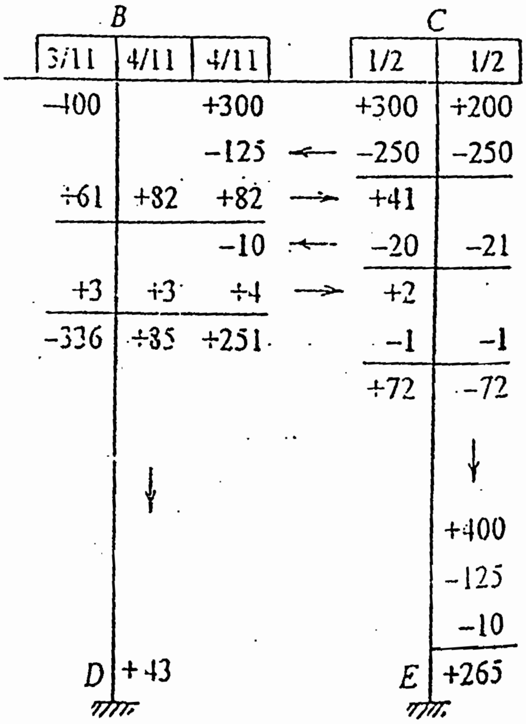




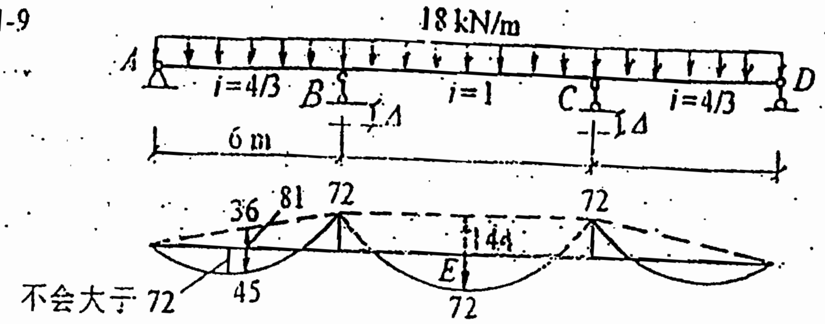
11-8 各杆 $EI=6\times10^4\,\text{kN·m}^2$.







11-9



	μ	2/3	1/3
荷载弯矩	M^F	+81	−96
	分配	+10	+5
	M_P	+91	−91
$\Delta=1$ 时之弯矩	$\overline{M}_A^F$	$-\frac{3EI}{l^2}$	
	分配	$+\frac{2EI}{l^2}$	$+\frac{EI}{l^2}$
	$\overline{M}_A$	$-\frac{EI}{l^2}$	$+\frac{EI}{l^2}$

最大正负弯矩发生在何处：最大负弯矩显然在 B,C 处。若令中跨中点 E 处之弯矩与 B,C 处的相等，则有

$$M_E = 144/2 = 72\ \text{kN·m}$$

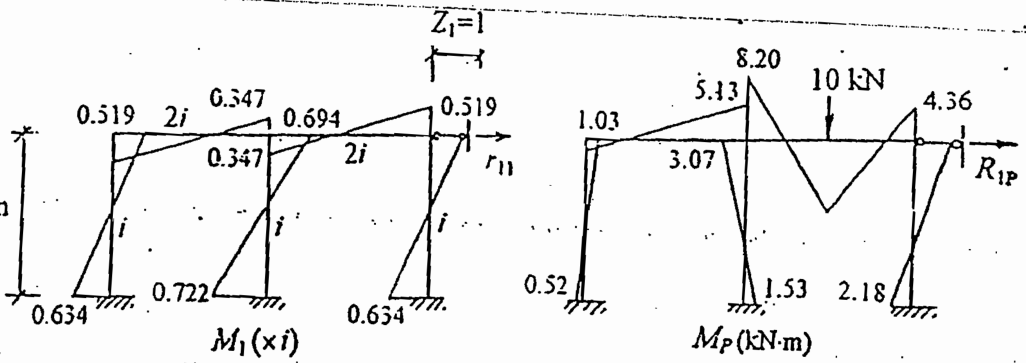
此时边跨中点之弯矩为 45 kN·m，而其最大正弯矩仅稍大于此，显然不会超过 72 kN·m。故全梁最大正弯矩即在 E 处。

叠加：$M_{BA} = 91 - \frac{EI}{l^2}\Delta = 72$

得 $\Delta = \frac{l^2}{EI}(91-72)$

$$= \frac{36}{36000}\times 19 = 0.019\ \text{m}\downarrow$$

11-10



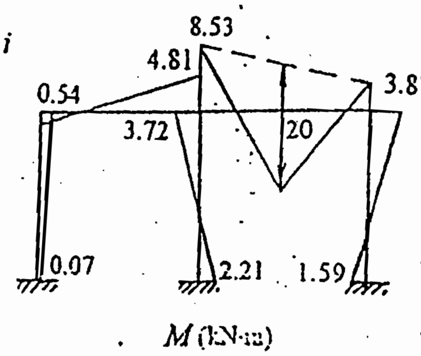
用力矩分配法分别算出 M_1 及 M_P（见图，详细计算略），有

$$r_{11} = \frac{(0.519+0.634)2+0.694+0.722}{8}i = 0.46525\,i$$

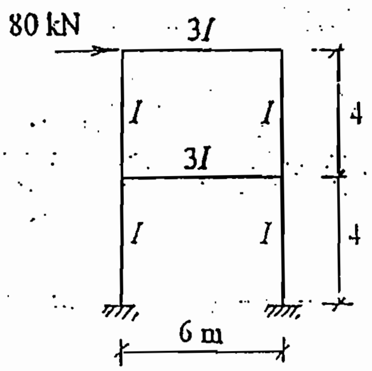
$$R_{1P} = \frac{1.03+0.52-3.07-1.53+4.36+2.18}{8}$$

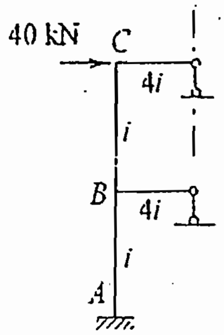
$$= 0.43625\ \text{kN}$$

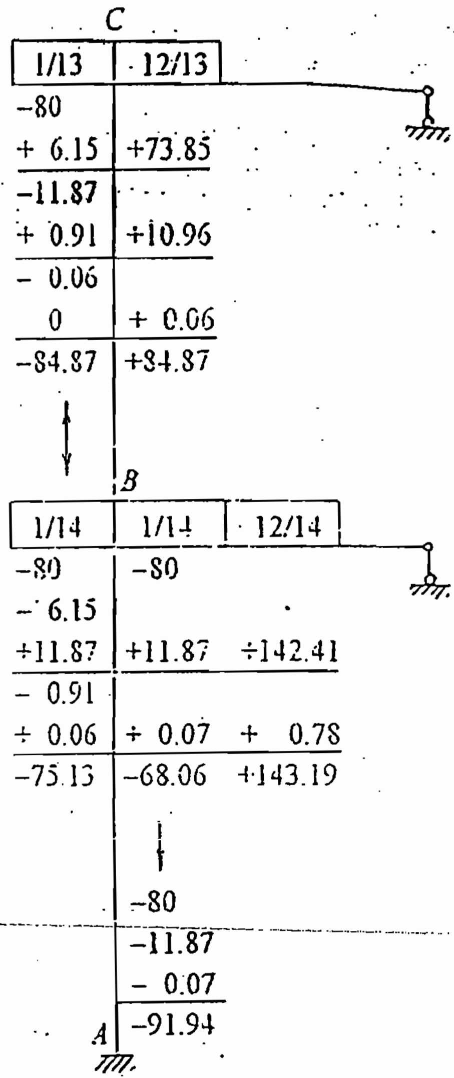
$$Z_1 = -\frac{R_{1P}}{r_{11}} = -\frac{0.9377}{i}$$



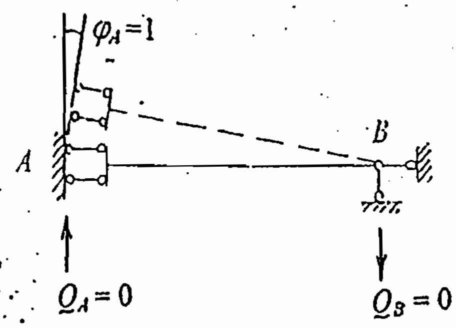
11-11



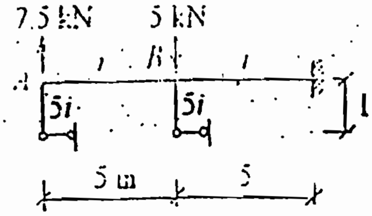




11-12 在（f）中，求 AB 杆 A 端之转动刚度 S_{AB} 时，注意到 $Q_A=0$，由 $\Sigma Y=0$ 有 $Q_B=0$，而 B 端又为铰支，于是全杆弯矩均为零（不受力），故 $S_{AB}=0$。



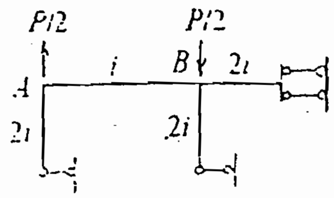
11-13 左右正对称，上下反对称时，取 1/4。



15/16	1/16		1/17	15/17	1/17	
	−18.75		−18.75		−6.25	−6.25
	−1.47	←	+1.47	+22.06	+1.47 →	−1.47
+18.96	+1.26	→	−1.26			
	−0.07	←	+0.07	+1.11	+0.08 →	−0.08
+0.07	0		−18.47	+23.17	−4.70	−7.80
+19.03	−19.03					

M (kN·m)

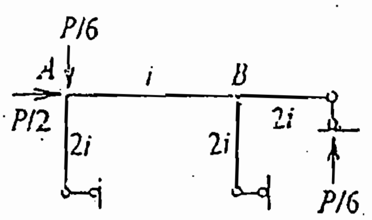
11-14 左右正对称，上下反对称时，取 1/4。



6/7	1/7		1/9	6/9	2/9
	−2500		−2500		
	−278	←	+278	+1667	+555
+2381	+397	→	−397		
	−44	←	+44	+265	+88
+38	+6	→	−6		
	−1	←	+1	+4	+1
+1	0		−2580	+1936	+644
+2420	−2420				

$M\ (\times 10^{-4}Pl)$

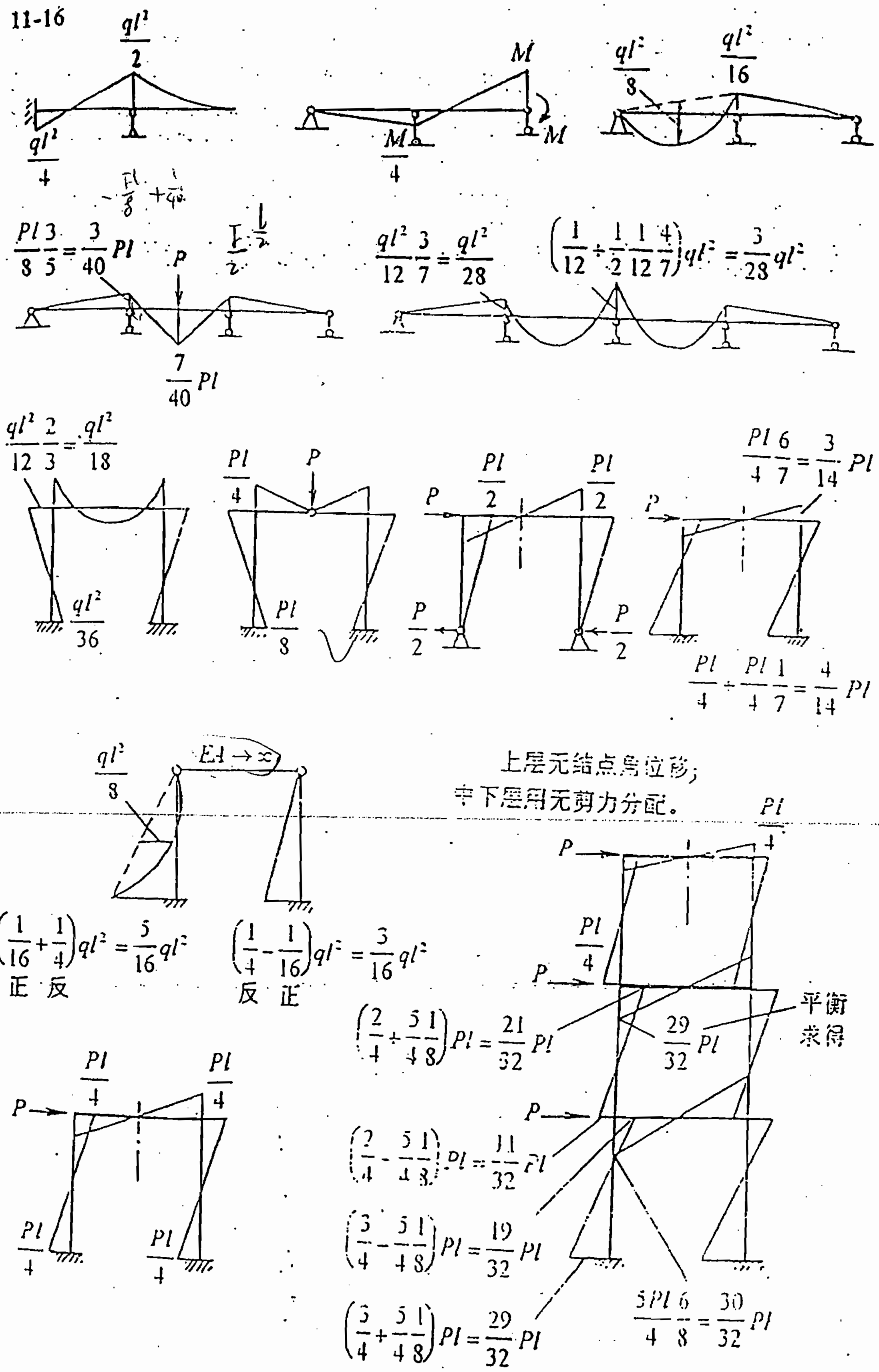
11-15 左右上下均反对称时，取 1/4。



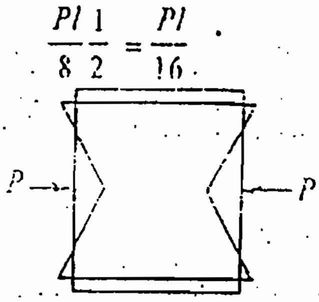
6/7	1/7		1/7	6/7	0
	+833		+833		+833
	+238	←	−238	−1428	
−918	−153	→	+153		
	+23	←	−23	−131	
−20	−3	→	+3		
−938	+938	←		−3	
			+728	−1562	+833

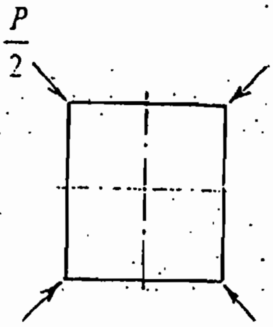
$M\ (\times 10^{-4}Pl)$

11-16

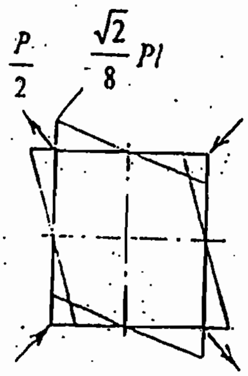


$\frac{ql^2}{2}$
$\frac{ql^2}{4}$
M
$\frac{M}{4}$
$\frac{ql^2}{8}$
$\frac{ql^2}{16}$
$\frac{Pl}{8}\frac{3}{5}=\frac{3}{40}Pl$
P
$\frac{7}{40}Pl$
$\frac{ql^2}{12}\frac{3}{7}=\frac{ql^2}{28}$
$\left(\frac{1}{12}+\frac{1}{2}\frac{1}{12}\frac{4}{7}\right)ql^2=\frac{3}{28}ql^2$
$\frac{ql^2}{12}\frac{2}{3}=\frac{ql^2}{18}$
$\frac{ql^2}{36}$
$\frac{Pl}{4}$
$\frac{Pl}{8}$
$\frac{Pl}{2}$
$\frac{P}{2}$
$\frac{Pl}{4}\frac{6}{7}=\frac{3}{14}Pl$
$\frac{Pl}{4}-\frac{Pl}{4}\frac{1}{7}=\frac{4}{14}Pl$
$EI\to\infty$
$\frac{ql^2}{8}$
$\left(\frac{1}{16}+\frac{1}{4}\right)ql^2=\frac{5}{16}ql^2$
正 反
$\left(\frac{1}{4}-\frac{1}{16}\right)ql^2=\frac{3}{16}ql^2$
反 正
上层无结点角位移;
中下层用无剪力分配。
$\left(\frac{2}{4}+\frac{5}{4}\frac{1}{8}\right)Pl=\frac{21}{32}Pl$
$\frac{29}{32}Pl$
平衡求得
$\left(\frac{2}{4}-\frac{5}{4}\frac{1}{8}\right)Pl=\frac{11}{32}Pl$
$\left(\frac{3}{4}-\frac{5}{4}\frac{1}{8}\right)Pl=\frac{19}{32}Pl$
$\left(\frac{3}{4}+\frac{5}{4}\frac{1}{8}\right)Pl=\frac{29}{32}Pl$
$\frac{5Pl}{4}\frac{6}{8}=\frac{30}{32}Pl$

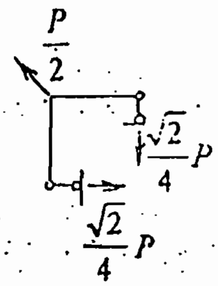




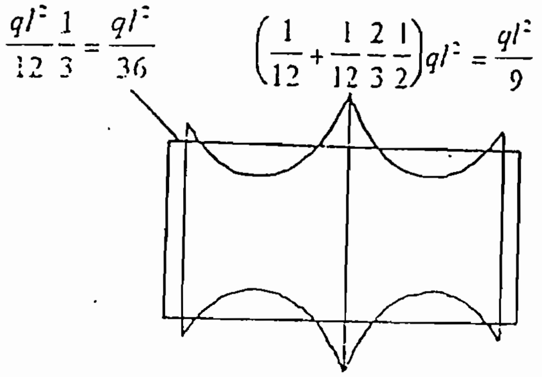
正对称 M 为零

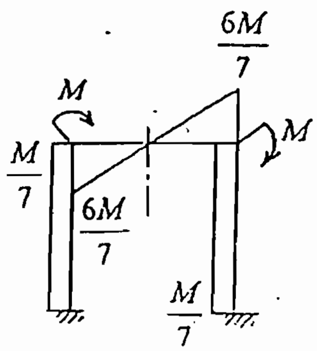


反对称 M 图



取 1/4，
已静定。

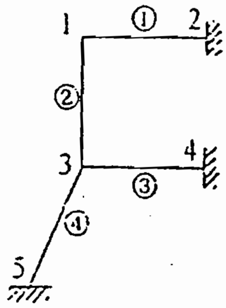




第十二章　矩阵位移法

12-1　略。

12-2　随结点、单元编号不同而异，例如对于图示编号，有

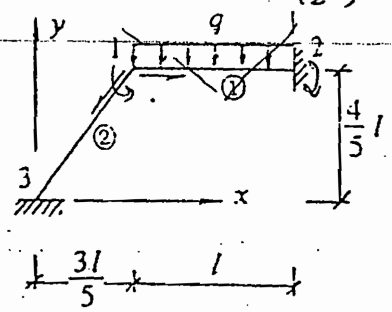


$$K=\left[\begin{array}{c|c|c|c|c} k_{11}^{①}+k_{11}^{②} & k_{12}^{①} & k_{13}^{②} & 0 & 0 \\ \hline k_{21}^{①} & k_{22}^{①} & 0 & 0 & 0 \\ \hline k_{31}^{②} & 0 & k_{33}^{②}+k_{33}^{③}+k_{33}^{④} & k_{34}^{③} & k_{35}^{④} \\ \hline 0 & 0 & k_{43}^{③} & k_{44}^{③} & 0 \\ \hline 0 & 0 & k_{53}^{④} & 0 & k_{55}^{④} \end{array}\right]$$

12-3　略。

12-4　$A=1000\,I/l^2$，暂设 $E=I=l=q=1$.

$$\begin{bmatrix} 0 \\ +\frac{ql}{2} \\ -\frac{ql^2}{12} \end{bmatrix}$$



单元①：$\alpha=0$

$$k^{①}=\begin{array}{c} \begin{array}{cc} \quad\quad 1 \quad\quad\quad & \quad\quad\quad 2 \end{array} \\ \left[\begin{array}{ccc|ccc} 1000 & 0 & 0 & -1000 & 0 & 0 \\ 0 & 12 & 6 & 0 & -12 & 6 \\ 0 & 6 & 4 & 0 & -6 & 2 \\ \hline -1000 & 0 & 0 & 1000 & 0 & 0 \\ 0 & -12 & -6 & 0 & 12 & -6 \\ 0 & 6 & 2 & 0 & -6 & 4 \end{array}\right] \begin{array}{c} \\ 1 \\ \\ \\ 2 \\ \\ \end{array} \end{array}$$

单元②：$\cos\alpha=-0.6$，$\sin\alpha=-0.8$

$$k^{②}=\begin{array}{c} \begin{array}{cc} \quad\quad 1 \quad\quad\quad & \quad\quad\quad 3 \end{array} \\ \left[\begin{array}{ccc|ccc} 367.68 & 474.24 & 4.8 & -367.68 & -474.24 & 4.8 \\ 474.24 & 644.32 & -3.6 & -474.24 & -644.32 & -3.6 \\ 4.8 & -3.6 & 4 & -4.8 & 3.6 & 2 \\ \hline -367.68 & -474.24 & -4.8 & 367.68 & 474.24 & -4.8 \\ -474.24 & -644.32 & 3.6 & 474.24 & 644.32 & 3.6 \\ 4.8 & -3.6 & 2 & -4.8 & 3.6 & 4 \end{array}\right] \begin{array}{c} \\ 1 \\ \\ \\ 3 \\ \\ \end{array} \end{array}$$

原始总刚

$$
K=\begin{array}{c}
\begin{array}{ccc}\quad\quad\quad 1 & \quad\quad\quad\quad\quad\quad\quad\quad 2 & \quad\quad\quad\quad\quad\quad\quad\quad 3\end{array}\\
\left[\begin{array}{ccc|ccc|ccc}
1367.68 & 474.24 & 4.8 & -1000 & 0 & 0 & -367.68 & -474.24 & 4.8\\
474.24 & 656.32 & 2.4 & 0 & -12 & 6 & -474.24 & -644.32 & -3.6\\
4.8 & 2.4 & 8 & 0 & -6 & 2 & -4.8 & 3.6 & 2\\
\hline
-1000 & 0 & 0 & 1000 & 0 & 0 & 0 & 0 & 0\\
0 & -12 & -6 & 0 & 12 & -6 & 0 & 0 & 0\\
0 & 6 & 2 & 0 & -6 & 4 & 0 & 0 & 0\\
\hline
-367.68 & -474.24 & -4.8 & 0 & 0 & 0 & 367.68 & 474.24 & -4.8\\
-474.24 & -644.32 & 3.6 & 0 & 0 & 0 & 474.24 & 644.32 & 3.6\\
4.8 & -3.6 & 2 & 0 & 0 & 0 & -4.8 & 3.6 & 4
\end{array}\right]
\end{array}
\begin{array}{c} \\ \\ 1\\ \\ \\ 2\\ \\ \\ 3\\ \\ \end{array}
$$

固端力

$$
\overline{F}_F^{①}=\begin{Bmatrix}\overline{F}_{F1}^{①}\\ \overline{F}_{F2}^{①}\end{Bmatrix}=\begin{Bmatrix}0\\1/2\\1/12\\ \hline 0\\1/2\\-1/12\end{Bmatrix}\begin{array}{c}\\1\\ \\ \\2\\ \\ \end{array},\quad
F_F^{①}=\begin{Bmatrix}F_{F1}^{①}\\ F_{F2}^{①}\end{Bmatrix}=\begin{Bmatrix}0\\1/2\\1/12\\ \hline 0\\1/2\\-1/12\end{Bmatrix}\begin{array}{c}\\1\\ \\ \\2\\ \\ \end{array};\quad
\overline{F}_F^{②}=\begin{Bmatrix}\overline{F}_{F1}^{②}\\ \overline{F}_{F3}^{②}\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\ \hline 0\\0\\0\end{Bmatrix}\begin{array}{c}\\1\\ \\ \\3\\ \\ \end{array},\quad
F_F^{②}=\begin{Bmatrix}F_{F1}^{②}\\ F_{F3}^{②}\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\ \hline 0\\0\\0\end{Bmatrix}\begin{array}{c}\\1\\ \\ \\3\\ \\ \end{array}
$$

结点1上的综合结点荷载

$$
P_1=P_{D1}+P_{E1}=P_{D1}-\sum F'_{F1}=\begin{Bmatrix}0\\0\\0\end{Bmatrix}-\begin{Bmatrix}0\\1/2\\1/12\end{Bmatrix}=\begin{Bmatrix}0\\-1/2\\-1/12\end{Bmatrix}
$$

支承条件为

$$
u_2=0,\ v_2=0,\ \varphi_2=0,\ u_3=0,\ v_3=0,\ \varphi_3=0
$$

删去对应的行和列后，得整体刚度方程为

$$
\begin{Bmatrix}0\\-1/2\\-1/12\end{Bmatrix}=\begin{bmatrix}1367.68 & 474.24 & 4.8\\474.24 & 656.32 & 2.4\\4.8 & 2.4 & 8\end{bmatrix}\begin{Bmatrix}u_1\\v_1\\\varphi_1\end{Bmatrix}
$$

解得

$$
\begin{Bmatrix}u_1\\v_1\\\varphi_1\end{Bmatrix}=10^{-3}\begin{Bmatrix}0.38342\ (\times ql^4/(EI))\\-1.00104\ (\times ql^4/(EI))\\-10.34641\ (\times ql^3/(EI))\end{Bmatrix}
$$

$$\overline{F}^{①}=\left\{\begin{array}{c}0\\1/2\\1/12\\\hline 0\\1/2\\-1/12\end{array}\right\}+\left[\begin{array}{ccc|ccc}1000&0&0&-1000&0&0\\0&12&6&0&-12&6\\0&6&4&0&-6&2\\\hline -1000&0&0&1000&0&0\\0&-12&-6&0&12&-6\\0&6&2&0&-6&4\end{array}\right]10^{-3}\left\{\begin{array}{c}0.38342\\-1.00104\\-10.34641\\\hline 0\\0\\0\end{array}\right\}=10^{-3}\left\{\begin{array}{c}383.42\ (\times ql)\\425.91\ (\times ql)\\35.94\ (\times ql^2)\\\hline -383.42\ (\times ql)\\574.09\ (\times ql)\\-110.03\ (\times ql^2)\end{array}\right\}\begin{array}{c}1\\ \\ \\ \\2\\ \\ \end{array}$$

$$\overline{F}^{②}=\left\{\begin{array}{c}0\\0\\0\\\hline 0\\0\\0\end{array}\right\}+T\left[\begin{array}{ccc|ccc}367.68&474.24&4.8&-367.68&-474.24&4.8\\474.24&644.32&-3.6&-474.24&-644.32&-3.6\\4.8&-3.6&4&-4.8&3.6&2\\\hline -367.68&-474.24&-4.8&367.68&474.24&-4.8\\-474.24&-644.32&3.6&474.24&644.32&3.6\\4.8&-3.6&2&-4.8&3.6&4\end{array}\right]10^{-3}\left\{\begin{array}{c}0.38342\\-1.00104\\-10.34641\\\hline 0\\0\\0\end{array}\right\}$$

$$=\left[\begin{array}{ccc|ccc}-0.6&-0.8&0&0&0&0\\0.8&-0.6&0&0&0&0\\0&0&1&0&0&0\\\hline 0&0&0&-0.6&-0.8&0\\0&0&0&0.8&-0.6&0\\0&0&0&0&0&1\end{array}\right]10^{-3}\left\{\begin{array}{c}-383.42\\-425.91\\-35.94\\\hline 383.42\\425.91\\-15.25\end{array}\right\}=10^{-3}\left\{\begin{array}{c}570.78\ (\times ql)\\-51.19\ (\times ql)\\-35.94\ (\times ql^2)\\\hline -570.78\ (\times ql)\\51.19\ (\times ql)\\-15.25\ (\times ql^2)\end{array}\right\}\begin{array}{c}1\\ \\ \\ \\3\\ \\ \end{array}$$

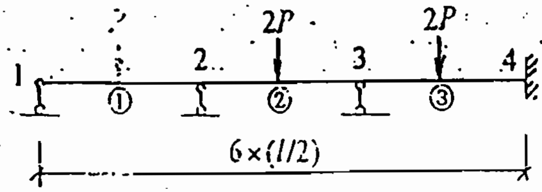
12-5 各单元单刚 k^e 均同式(12-5)(见教材上册 241 页)，略。

考虑支承条件并不计轴向变形有

$$u_1=v_1=u_2=v_2=u_3=v_3=u_4=v_4=\varphi_4=0$$

删去相应行列，总刚为

$$K=\frac{EI}{l}\begin{array}{c}\begin{array}{ccc}\varphi_1 & \varphi_2 & \varphi_3\end{array}\\ \begin{bmatrix}4 & 2 & 0\\ 2 & 8 & 2\\ 0 & 2 & 8\end{bmatrix}\end{array}\begin{array}{c}\\ P_1\\ P_2\\ P_3\end{array}$$



杆端力只计算杆端弯矩，固端弯矩为

$$\overline{F}_F^{①}=Pl\begin{Bmatrix}1/8\\-1/8\end{Bmatrix}\begin{matrix}1\\2\end{matrix}\qquad \overline{F}_F^{②}=Pl\begin{Bmatrix}1/4\\-1/4\end{Bmatrix}\begin{matrix}2\\3\end{matrix}\qquad \overline{F}_F^{③}=Pl\begin{Bmatrix}1/4\\-1/4\end{Bmatrix}\begin{matrix}3\\4\end{matrix}$$

等效结点荷载只计算力偶荷载，有

$$\begin{Bmatrix}P_1\\P_2\\P_3\end{Bmatrix}=Pl\begin{Bmatrix}-1/8\\-1/8\\0\end{Bmatrix}$$

总刚度方程为

$$Pl\begin{Bmatrix}-1/8\\-1/8\\0\end{Bmatrix}=\frac{EI}{l}\begin{bmatrix}4 & 2 & 0\\ 2 & 8 & 2\\ 0 & 2 & 8\end{bmatrix}\begin{Bmatrix}\varphi_1\\\varphi_2\\\varphi_3\end{Bmatrix}$$

解得

$$\begin{Bmatrix}\varphi_1\\\varphi_2\\\varphi_3\end{Bmatrix}=\frac{Pl^2}{416EI}\begin{Bmatrix}-11\\-4\\1\end{Bmatrix}$$

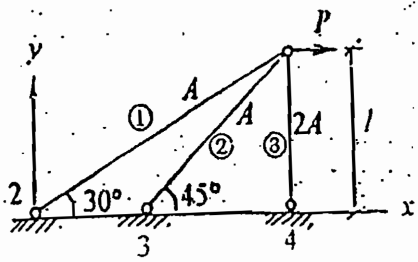
杆端弯矩为

$$\overline{F}^{①}=Pl\begin{Bmatrix}1/8\\-1/8\end{Bmatrix}+\frac{EI}{l}\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix}\frac{Pl^2}{416EI}\begin{Bmatrix}-11\\-4\end{Bmatrix}=\frac{Pl}{416}\begin{Bmatrix}52\\-52\end{Bmatrix}+\frac{Pl}{416}\begin{Bmatrix}-52\\-38\end{Bmatrix}=\frac{Pl}{208}\begin{Bmatrix}0\\-45\end{Bmatrix}$$

$$\overline{F}^{②}=Pl\begin{Bmatrix}1/4\\-1/4\end{Bmatrix}+\frac{EI}{l}\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix}\frac{Pl^2}{416EI}\begin{Bmatrix}-4\\1\end{Bmatrix}=\frac{Pl}{416}\begin{Bmatrix}104\\-104\end{Bmatrix}+\frac{Pl}{416}\begin{Bmatrix}-14\\-4\end{Bmatrix}=\frac{Pl}{208}\begin{Bmatrix}45\\-54\end{Bmatrix}$$

$$\overline{F}^{③}=Pl\begin{Bmatrix}1/4\\-1/4\end{Bmatrix}+\frac{EI}{l}\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix}\frac{Pl^2}{416EI}\begin{Bmatrix}1\\0\end{Bmatrix}=\frac{Pl}{416}\begin{Bmatrix}104\\-104\end{Bmatrix}+\frac{Pl}{416}\begin{Bmatrix}4\\2\end{Bmatrix}=\frac{Pl}{208}\begin{Bmatrix}54\\-51\end{Bmatrix}$$

12-6



$$k^{①}=\frac{EA}{l}\begin{array}{c}\begin{array}{cc}\quad 2 & \qquad\qquad\qquad 1\end{array}\\ \left[\begin{array}{cc|cc}3/8 & \sqrt{3}/8 & -3/8 & -\sqrt{3}/8\\ \sqrt{3}/8 & 1/8 & -\sqrt{3}/8 & -1/8\\ \hline -3/8 & -\sqrt{3}/8 & 3/8 & \sqrt{3}/8\\ -\sqrt{3}/8 & -1/8 & \sqrt{3}/8 & 1/8\end{array}\right]\end{array}\begin{array}{c}2\\ \\ 1\end{array}$$

$$k^{②}=\frac{EA}{l}\begin{array}{c}\begin{array}{cc}\quad 3 & \qquad\qquad\qquad 1\end{array}\\ \left[\begin{array}{cc|cc}\sqrt{2}/4 & \sqrt{2}/4 & -\sqrt{2}/4 & -\sqrt{2}/4\\ \sqrt{3}/8 & 1/8 & -\sqrt{2}/4 & -\sqrt{2}/4\\ \hline -\sqrt{2}/4 & -\sqrt{2}/4 & \sqrt{2}/4 & \sqrt{2}/4\\ -\sqrt{2}/4 & -\sqrt{2}/4 & \sqrt{2}/4 & \sqrt{2}/4\end{array}\right]\end{array}\begin{array}{c}3\\ \\ 1\end{array}\qquad k^{③}=\frac{EA}{l}\begin{array}{c}\begin{array}{cc}\quad 4 & \qquad\quad 1\end{array}\\ \left[\begin{array}{cc|cc}0 & 0 & 0 & 0\\ 0 & 2 & 0 & -2\\ \hline 0 & 0 & 0 & 0\\ 0 & -2 & 0 & 2\end{array}\right]\end{array}\begin{array}{c}4\\ \\ 1\end{array}$$

引入支承条件后有

$$\begin{Bmatrix}P\\0\end{Bmatrix}=\frac{EA}{l}\begin{bmatrix}0.72855 & 0.57006\\ 0.57006 & 2.47855\end{bmatrix}\begin{Bmatrix}u_1\\v_1\end{Bmatrix}$$

解得

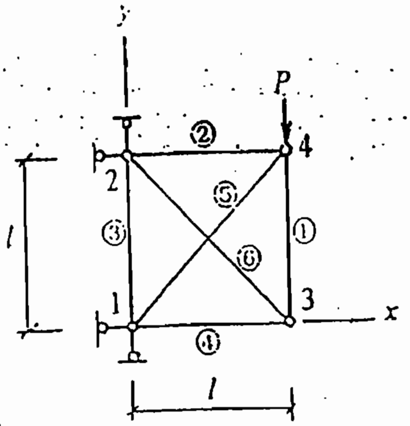
$$\begin{Bmatrix}u_1\\v_1\end{Bmatrix}=\frac{Pl}{EA}\begin{Bmatrix}1.67381\\-0.38497\end{Bmatrix}$$

$$\overline{F}^{①}=Tk^{①}\Delta^{①}=\left[\begin{array}{cc|cc}\sqrt{3}/2 & 1/2 & 0 & 0\\ -1/2 & \sqrt{3}/2 & 0 & 0\\ \hline 0 & 0 & \sqrt{3}/2 & 1/2\\ 0 & 0 & -1/2 & \sqrt{3}/2\end{array}\right]k^{①}\frac{Pl}{EA}\begin{Bmatrix}0\\0\\ \hline 1.67381\\-0.38497\end{Bmatrix}=P\begin{Bmatrix}-0.6285\\0\\ \hline 0.6285\\0\end{Bmatrix}\text{(拉)}$$

$$\overline{F}^{②}=Tk^{②}\Delta^{②}=\left[\begin{array}{cc|cc}\sqrt{2}/2 & \sqrt{2}/2 & 0 & 0\\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 & 0\\ \hline 0 & 0 & \sqrt{2}/2 & \sqrt{2}/2\\ 0 & 0 & -\sqrt{2}/2 & \sqrt{2}/2\end{array}\right]k^{②}\frac{Pl}{EA}\begin{Bmatrix}0\\0\\ \hline 1.67381\\-0.38497\end{Bmatrix}=P\begin{Bmatrix}-0.6442\\0\\ \hline 0.6442\\0\end{Bmatrix}\text{(拉)}$$

$$\overline{F}^{③}=Tk^{③}\Delta^{③}=\left[\begin{array}{cc|cc}0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{array}\right]k^{③}\frac{Pl}{EA}\begin{Bmatrix}0\\0\\ \hline 1.67381\\-0.38497\end{Bmatrix}=P\begin{Bmatrix}0.7699\\0\\ \hline -0.7699\\0\end{Bmatrix}\text{(压)}$$

12-7



$$k^{①}=k^{③}=\frac{EA}{l}\begin{array}{c}\begin{array}{cc}1 & 2\\ 3 & 4\end{array}\\ \left[\begin{array}{cc|cc}0 & 0 & 0 & 0\\ 0 & 1 & 0 & -1\\ \hline 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 1\end{array}\right]\end{array}\begin{array}{cc}3 & 1\\ \\ 4 & 2\end{array}$$

$$k^{②}=k^{④}=\frac{EA}{l}\begin{array}{c}\begin{array}{cc}1 & 3\\ 2 & 4\end{array}\\ \left[\begin{array}{cc|cc}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ \hline -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right]\end{array}\begin{array}{cc}2 & 1\\ \\ 4 & 3\end{array}$$

$$k^{⑤}=\frac{EA}{l}\frac{\sqrt{2}}{4}\begin{array}{c}\begin{array}{cc}1 & 4\end{array}\\ \left[\begin{array}{cc|cc}1 & 1 & -1 & -1\\ 1 & 1 & -1 & -1\\ \hline -1 & -1 & 1 & 1\\ -1 & -1 & 1 & 1\end{array}\right]\end{array}\begin{array}{c}1\\ \\ 4\end{array}\qquad k^{⑥}=\frac{EA}{l}\frac{\sqrt{2}}{4}\begin{array}{c}\begin{array}{cc}2 & \end{array}\\ \left[\begin{array}{cc|cc}1 & -1 & -1 & 1\\ -1 & 1 & 1 & -1\\ \hline -1 & 1 & 1 & -1\\ 1 & -1 & -1 & 1\end{array}\right]\end{array}\begin{array}{c}2\\ \\ 3\end{array}$$

引入支承条件后有

$$\left\{\begin{array}{c}0\\ 0\\ \hline 0\\ -P\end{array}\right\}=\frac{EA}{l}\left[\begin{array}{cc|cc}1.35355 & -0.35355 & 0 & 0\\ -0.35355 & 1.35355 & 0 & -1\\ \hline 0 & 0 & 1.35355 & 0.35355\\ 0 & -1 & 0.35355 & 1.35355\end{array}\right]\left\{\begin{array}{c}u_3\\ v_3\\ \hline u_4\\ v_4\end{array}\right\}$$

解得

$$\left\{\begin{array}{c}u_3\\ v_3\\ \hline u_4\\ v_4\end{array}\right\}=\frac{Pl}{EA}\left\{\begin{array}{c}-0.4422\\ -1.6931\\ \hline 0.5578\\ -2.1353\end{array}\right\}$$

$$\overline{F}^{①}=\left[\begin{array}{cc|cc}0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{array}\right]\frac{EA}{l}\left[\begin{array}{cc|cc}0 & 0 & 0 & 0\\ 0 & 1 & 0 & -1\\ \hline 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 1\end{array}\right]\frac{Pl}{EA}\left\{\begin{array}{c}-0.4422\\ -1.6931\\ \hline 0.5578\\ -2.1353\end{array}\right\}=P\left\{\begin{array}{c}0.4422\\ 0\\ \hline -0.4422\\ 0\end{array}\right\}\text{(压)}$$

$$\overline{F}^{②}=\left[\begin{array}{cc|cc}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ \hline 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right]\frac{EA}{l}\left[\begin{array}{cc|cc}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ \hline -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right]\frac{Pl}{EA}\left\{\begin{array}{c}0\\ 0\\ \hline 0.5578\\ -2.1353\end{array}\right\}=P\left\{\begin{array}{c}-0.5578\\ 0\\ \hline 0.5578\\ 0\end{array}\right\}\text{(拉)}$$

$$\bar{F}^{③}=\begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix}\frac{EA}{l}\begin{bmatrix}0&0&0&0\\0&1&0&-1\\0&0&0&0\\0&-1&0&1\end{bmatrix}\frac{Pl}{EA}\begin{Bmatrix}0\\0\\0\\0\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\0\end{Bmatrix}$$

$$\bar{F}^{④}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\frac{EA}{l}\begin{bmatrix}1&0&-1&0\\0&0&0&0\\-1&0&1&0\\0&0&0&0\end{bmatrix}\frac{Pl}{EA}\begin{Bmatrix}0\\0\\-0.4422\\-1.6931\end{Bmatrix}=P\begin{Bmatrix}0.4422\\0\\-0.4422\\0\end{Bmatrix}\text{（压）}$$

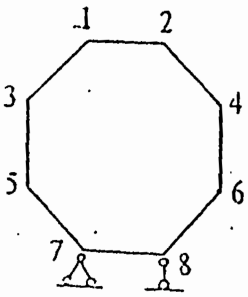
$$\bar{F}^{⑤}=\begin{bmatrix}\sqrt{2}/2&\sqrt{2}/2&0&0\\-\sqrt{2}/2&\sqrt{2}/2&0&0\\0&0&\sqrt{2}/2&\sqrt{2}/2\\0&0&-\sqrt{2}/2&\sqrt{2}/2\end{bmatrix}\frac{EA}{l}\frac{\sqrt{2}}{4}\begin{bmatrix}1&1&-1&-1\\1&1&-1&-1\\-1&-1&1&1\\-1&-1&1&1\end{bmatrix}\frac{Pl}{EA}\begin{Bmatrix}0\\0\\0.5578\\-2.1353\end{Bmatrix}$$

$$=P\begin{Bmatrix}0.7888\\0\\-0.7888\\0\end{Bmatrix}\text{（压）}$$

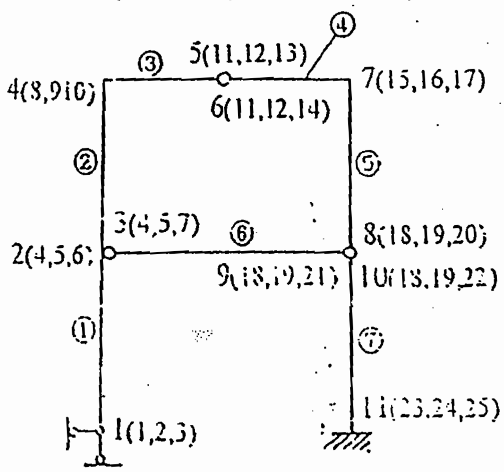
$$\bar{F}^{⑥}=\begin{bmatrix}\sqrt{2}/2&-\sqrt{2}/2&0&0\\\sqrt{2}/2&\sqrt{2}/2&0&0\\0&0&\sqrt{2}/2&-\sqrt{2}/2\\0&0&\sqrt{2}/2&\sqrt{2}/2\end{bmatrix}\frac{EA}{l}\frac{\sqrt{2}}{4}\begin{bmatrix}1&-1&-1&1\\-1&1&1&-1\\-1&1&1&-1\\1&-1&-1&1\end{bmatrix}\frac{Pl}{EA}\begin{Bmatrix}0\\0\\-0.4422\\-1.6931\end{Bmatrix}$$

$$=P\begin{Bmatrix}-0.6254\\0\\0.6254\\0\end{Bmatrix}\text{（拉）}$$

12-8

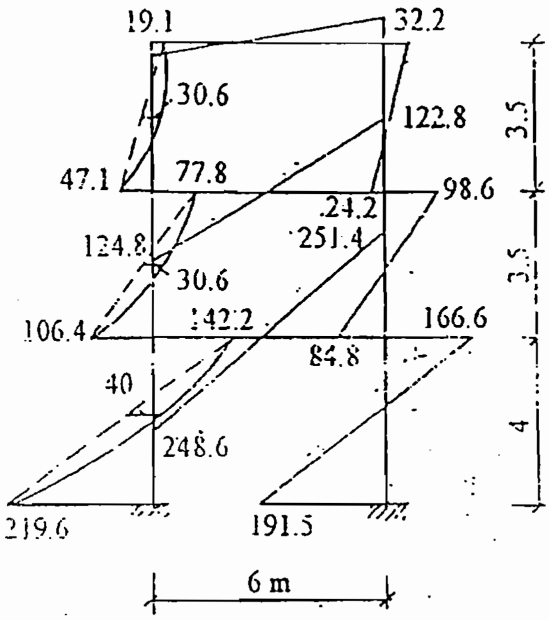


12-9

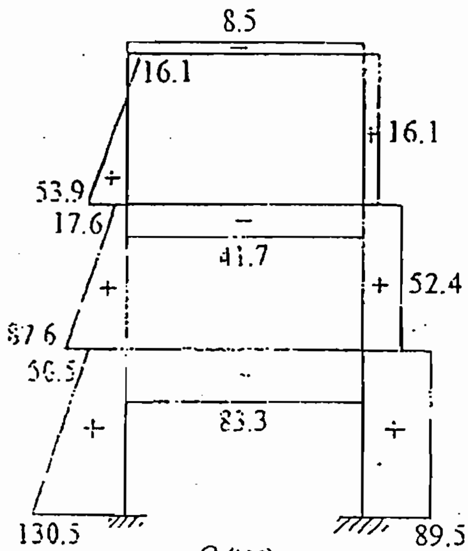


第十三章　平面刚架静力分析程序

13-1

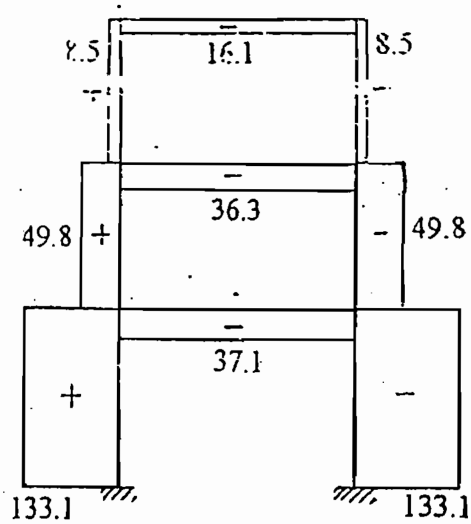


M (kN·m)



Q (kN)

M 图绘在受拉边；绘 Q、N 图时正负号得按传统规定。



N (kN)

13-2　略。

*13-3　略。

*13-4　略。

第十四章　结构的极限荷载

14-1

(a) $M_u=\dfrac{bh^2}{4}\sigma_s=\dfrac{5\times10^2\times10^{-6}}{4}\times240\times10^6\times10^{-3}=30\ \text{kN}\cdot\text{m}$

(b) 查型钢表有 $I_x=2370$，$I_x/S_x=17.2$，故

$S_x=2370/17.2=137.8\ \text{cm}^3$.

$W_s=2S_x=275.6\ \text{cm}^3$

$M_u=275.6\times10^{-6}\times240\times10^6\times10^{-3}$

$=66.14\ \text{kN}\cdot\text{m}$

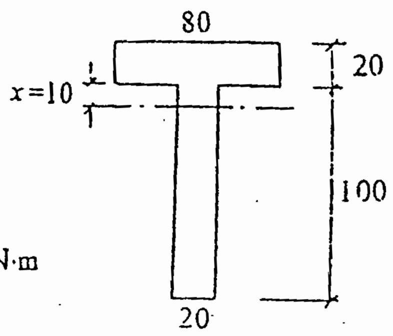
(c) $80\times20+20x=20(100-x)$

$x=10\ \text{mm}$

$W_s=80\times20\times20+20\times10\times5+20\times90\times45$

$=114000\ \text{mm}^3$

$M_u=114000\times10^{-9}\times240\times10^6\times10^{-3}=27.36\ \text{kN}\cdot\text{m}$



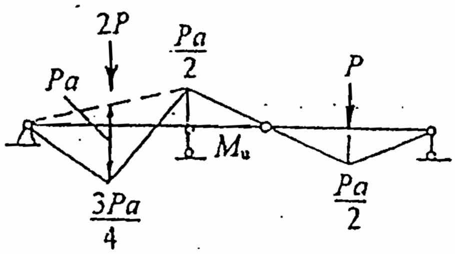
(单位: mm)

14-2

(a) $M_u=\sigma_s\times2\times\dfrac{1}{2}\dfrac{\pi D^2}{4}\times\dfrac{2D}{3\pi}=\sigma_s\dfrac{D^3}{6}$

(b) $M_u=\sigma_s\left[\dfrac{D^3}{6}-\dfrac{(D-2t)^3}{6}\right]=\sigma_s\dfrac{D^3}{6}\left[1-\left(1-\dfrac{2t}{D}\right)^3\right]$

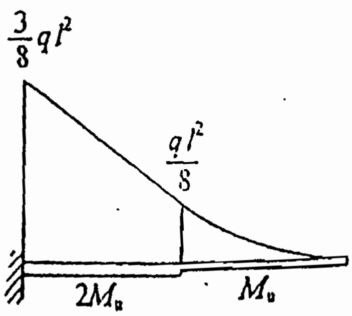
14-3



$$\frac{3Pa}{4}=M_u$$

$$P_u=\frac{4M_u}{3a}=\frac{4\times300}{3\times2}=200\ \text{kN}$$

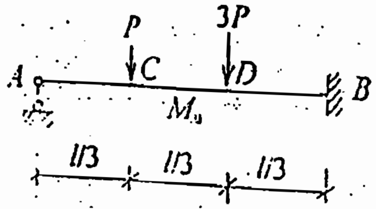
14-4



$$\frac{3}{8}ql^2=2M_u$$

$$q_u=\frac{16}{3}\frac{M_u}{l^2}$$

14-5



穷举法　此梁为1次超静定，出现两个塑性铰即成破坏机构，而塑性铰可能出现在 B, C, D 三处，故共有3种可能的机构。

机构1　$P\dfrac{l}{3}\theta+3P\dfrac{2l}{3}\theta=M_u(3\theta+2\theta)$

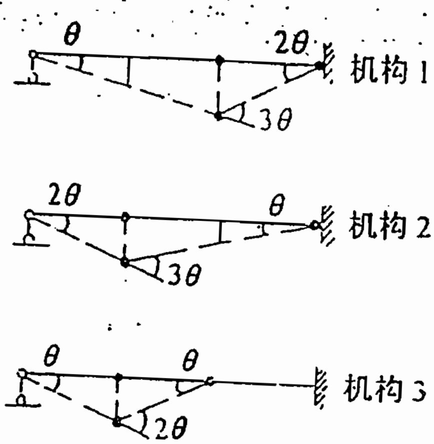
$$P=\frac{15}{7}\frac{M_u}{l}$$

机构2　$P\dfrac{2l}{3}\theta+3P\dfrac{l}{3}\theta=M_u(3\theta+\theta)$

$$P=\frac{12}{5}\frac{M_u}{l}$$

机构3　$P\dfrac{l}{3}\theta=M_u(2\theta+\theta)$

$$P=9\frac{M_u}{l}$$



取最小得

$$P_u=\frac{15}{7}\frac{M_u}{l}\qquad（机构1）$$

试算法　试选机构1，求得可破坏荷载（方法同上）为

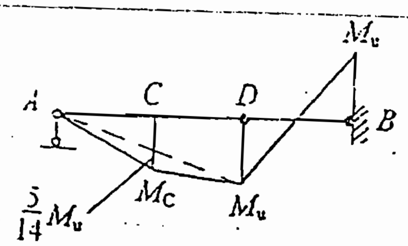
$$P=\frac{15}{7}\frac{M_u}{l}$$

作 M 图，截面 C 之弯矩由迭加法有

$$M_C=\frac{M_u}{2}+\frac{1}{4}\frac{15}{7}\frac{M_u}{l}\frac{2l}{3}$$

$$=\frac{M_u}{2}+\frac{5}{14}M_u=\frac{6}{7}M_u<M_u$$

可接受。故　$P_u=\dfrac{15}{7}\dfrac{M_u}{l}$



机构1　M 图

14-6　用静力法。最大正弯矩在 x 处，有

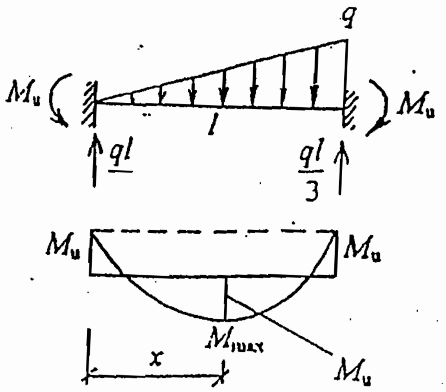
$$M_x=\frac{ql}{6}x-\frac{x}{2}\frac{xq}{l}\frac{x}{3}-M_u=\frac{qlx}{6}-\frac{qx^3}{6l}-M_u$$

由　$Q_x=\dfrac{ql}{6}-\dfrac{qx^2}{2l}=0$　得　$x=\dfrac{\sqrt{3}}{3}l=0.577l$

$$M_{max}=\frac{ql}{6}\frac{\sqrt{3}l}{3}-\frac{q}{6l}\frac{3\sqrt{3}}{27}l^3-M_u=\frac{\sqrt{3}}{27}ql^2-M_u$$

由　$M_{max}=M_u$　有　$\dfrac{\sqrt{3}}{27}ql^2-M_u=M_u$

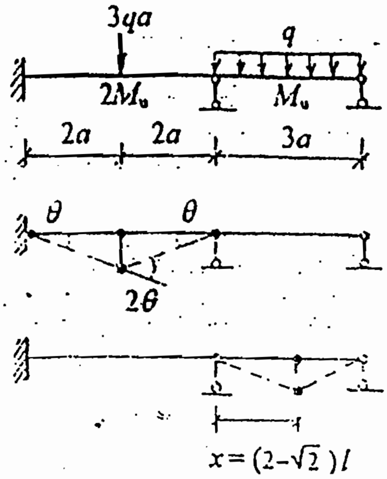
得　$q_u=18\sqrt{3}\dfrac{M_u}{l^2}$



14-7　左跨机构：

$$3qa\times 2a\theta = 2M_u\theta + 2M_u\times 2\theta + M_u\theta$$

$$q=\frac{7}{6}\frac{M_u}{a^2}=1.167\frac{M_u}{a^2}$$



右跨机构：引用例 14-2（见教材下册 6 页）之结果，有

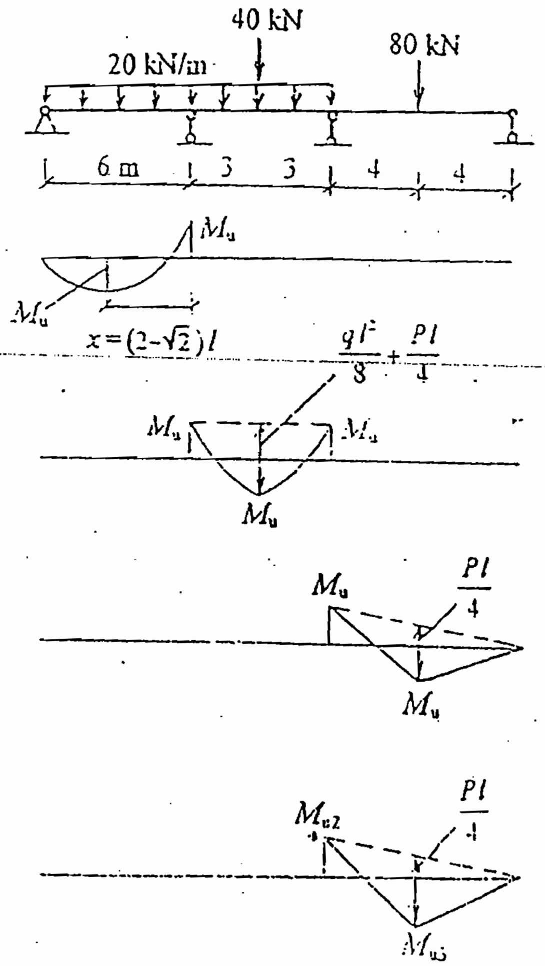
$$q=\frac{11.66M_u}{(3a)^2}=1.296\frac{M_u}{a^2}$$

取小者　$q_u=1.167\dfrac{M_u}{a^2}$

14-8　第 3 跨机构（图略），虚功法有

$$q\frac{1}{2}l_3\frac{l_3}{2}\theta=M_u(\theta+2\theta+\theta)\quad 得\ q_u=16\frac{M_u}{l_3^2}$$

14-9



(a)　第1跨　$q=\dfrac{11.66M_u}{l^2}$

$$M_u=\frac{ql^2}{11.66}=\frac{20\times 6^2}{11.66}=61.75$$

第2跨　$\dfrac{ql^2}{8}+\dfrac{Pl}{4}=2M_u$

$$M_u=\frac{ql^2}{16}+\frac{Pl}{8}=\frac{20\times 6^2}{16}+\frac{40\times 6}{8}=75$$

第3跨　$\dfrac{Pl}{4}=\dfrac{M_u}{2}+M_u$

$$M_u=\frac{Pl}{6}=\frac{80\times 8}{6}=106.7$$

选最大，并考虑安全系数有

$$M_u=1.7\times 106.7=181.4\ \text{kN}\cdot\text{m}$$

(b)　设第1,2跨截面小于第3跨。

第1,2跨　由上可知

$$M_{u1,2}=1.7\times 75=127.5\ \text{kN}\cdot\text{m}$$

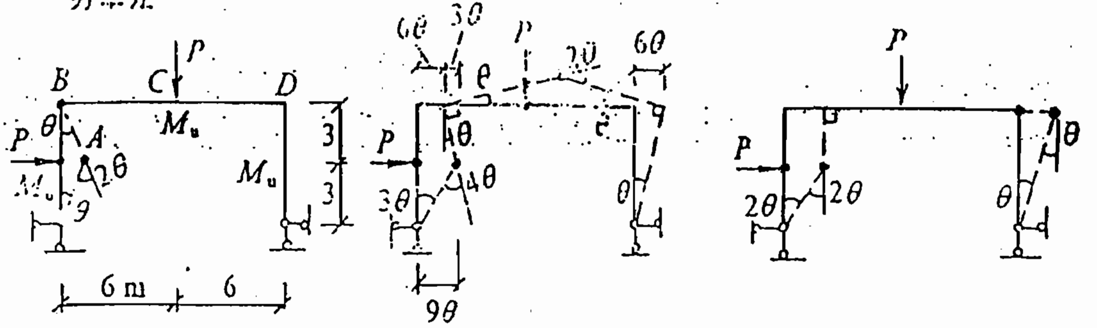
第3跨　$\dfrac{Pl}{4}=\dfrac{M_{u2}}{2}+M_{u3}$

$$M_{u3}=\frac{80\times 8}{4}-\frac{75}{2}=160-37.5=122.5$$

故　$M_{u3}=1.7\times 122.5=208.2\ \text{kN}\cdot\text{m}$

14-10　1次超静定，出现两个塑性铰即成为破坏机构，而塑性铰可能出现在A,B,C,D处，故共需考虑6种可能的机构。

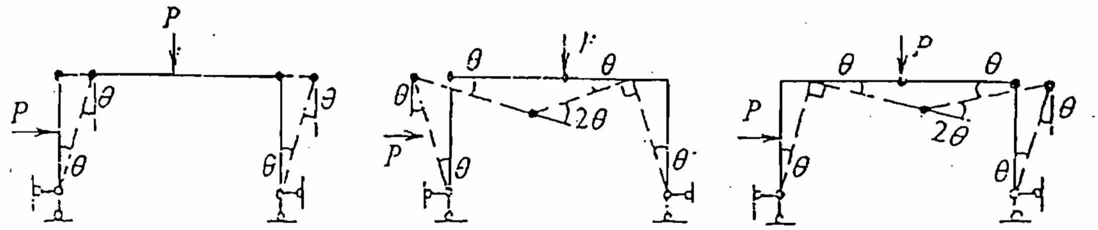
穷举法



机构1　$P\times3\theta=M_u(2\theta+\theta)$　$P=M_u=90$ kN

机构2　$P\times9\theta-P\times6\theta=M_u(4\theta+2\theta)$　$P=2M_u=180$ kN

机构3　$P\times3\times2\theta=M_u(2\theta+\theta)$　$P=M_u/2=45$ kN



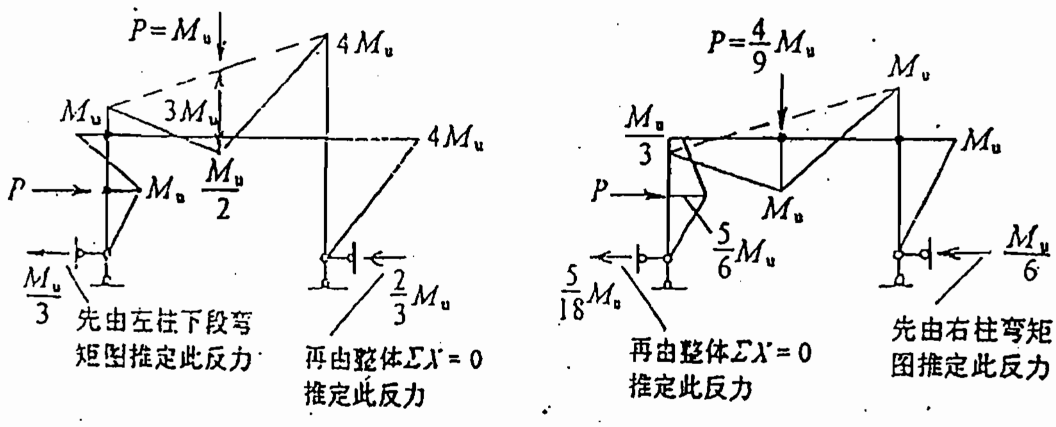
机构4　$P\times3\theta=M_u(\theta+\theta)$　$P=2M_u/3=60$ kN

机构5　$-P\times3\theta+P\times6\theta=M_u(2\theta+2\theta)$　$P=4M_u/3=120$ kN

机构6　$P\times3\theta+P\times6\theta=M_u(2\theta+2\theta)$　$P=4M_u/9=40$ kN

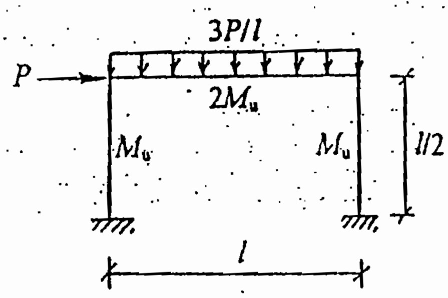
选最小得　$P_u=4M_u/9=40$ kN　(机构6)

试算法　试选机构1，有$P=M_u$（求法同上），作其M图，不可接受。再试选机构6，$P=4M_u/9$，作M图，可接受，故得　$P_u=4M_u/9=40$ kN。

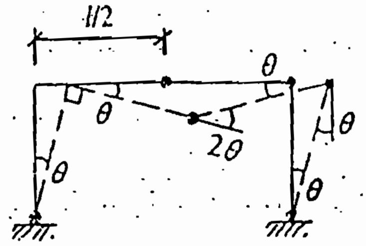


机构1　M图　　　机构6　M图

14-11



(a)



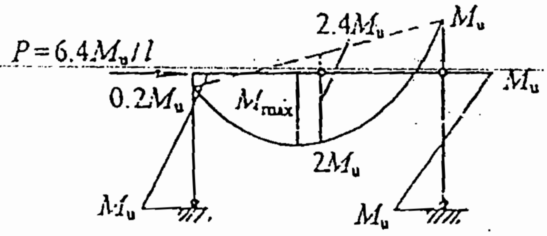
(b)

3 次超静定。两柱杆身无荷载，故不会各自出现 3 个塑性铰而形成“梁机构”，于是只剩下横梁上出现 3 个塑性铰的“梁机构”和出现 4 个塑性铰的“侧移机构”及“联合机构”。今用试算法，取图 b 所示 4 个塑性铰的“联合机构”，横梁中部之塑性铰位置近似取在跨度中点，有

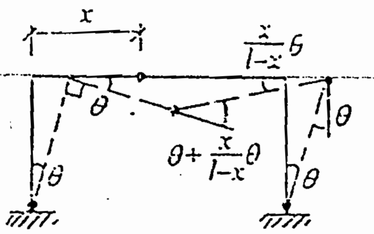
$$P\frac{l}{2}\theta+\frac{3P}{l}\frac{1}{2}l\frac{l}{2}\theta=M_u\theta+2M_u\times2\theta+M_u\times2\theta+M_u\theta$$

得
$$P=\frac{32}{5}\frac{M_u}{l}=6.4\frac{M_u}{l}$$

作 M 图（见图 c），可见横梁中点稍左处有最大正弯矩，它仅略大于 $2M_u$，故可近似认为 $P_u\approx6.4M_u/l$。



(c)



(d)

精确解（*注）：设横梁中部塑性铰在 x 处（图 a），有

$$P\frac{l}{2}\theta+\frac{3P}{l}\frac{l}{2}x\theta=M_u\theta+2M_u\left(\theta+\frac{x}{l-x}\theta\right)+M_u\left(\theta+\frac{x}{l-x}\theta\right)+M_u\theta$$

得
$$P=\frac{4+\dfrac{6l}{l-x}}{l+3x}M_u$$

由
$$\frac{dP}{dx}=\frac{1}{(l+3x)^2}\left[(l+3x)\frac{-6l(-1)}{(l-x)^2}-\left(4+\frac{6l}{l-x}\right)3\right]M_u=0$$

化简有 $x^2-5xl+2l^2=0$ 　　解得 $x=0.43845l$

代入得
$$P_u=6.342M_u/l$$

*注：这里所谓“精确”仍未计轴力和剪力对极限弯矩的影响，故仍是近似的。

第十五章 结构弹性稳定

思考题（选答）

5. 在各弹性支座的弹簧刚度能较容易地单独确定时才宜于简化。这通常要具备两个条件：一是除所选的一根压杆外，其余用以组成弹性支座的各杆件中无压杆（含对称结构正、反对称失稳取一半后无压杆），否则在计算弹簧刚度时将须考虑压杆上之纵向荷载影响，这使分析复杂（多为非线性函数）。图 15-33a 即属其余杆件中无压杆的情况，图 b 则属有压杆。二是当弹性支座不止一个时，组成各弹性支座的杆件应互不重复，这样各弹性支座的弹簧刚度才便于单独确定，否则它们将相互影响而不能各自独立，处理较为麻烦。具体例子略，可参阅有关资料*。

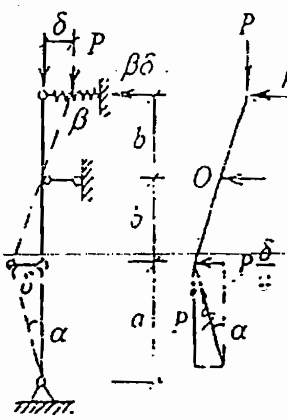
8. 公式 (15-61) 只适用于$\alpha<\pi$ 的情况。

习 题

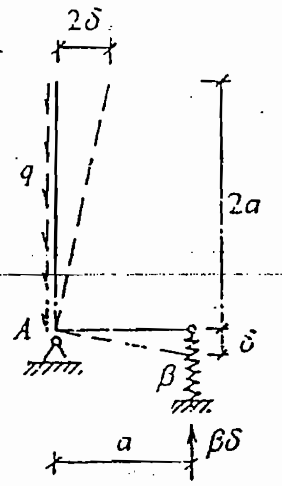
15-1 取上段杆 $\Sigma M_O=0$，有

$$P\delta-\beta\delta b-P\delta+P\frac{\delta}{a}b=0$$

得 $P_{cr}=\dfrac{b\beta}{2+\dfrac{b}{a}}=\dfrac{ab}{2a+b}\beta$



题 15-1 图



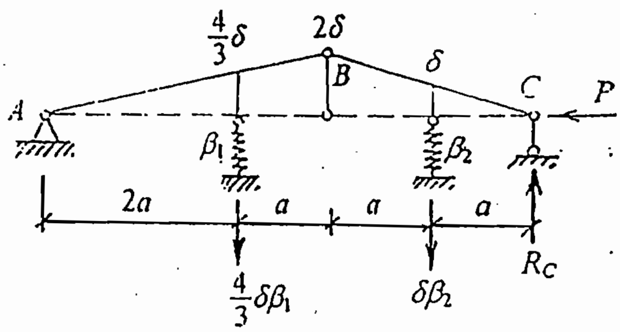
题 15-2 图

15-2 $\Sigma M_A=0$，$2qa\delta-\beta\delta a=0$，得 $q_{cr}=\beta/2$．

15-3 单自由度，只需设一个独立参数δ．偏离后，取 BC 为隔离体，由 $\Sigma M_B=0$，有

$$P\times2\delta-R_C\times2a+\delta\beta_2a=0$$

得 $R_C=P\dfrac{\delta}{a}+\dfrac{\beta_2\delta}{2}$



再取整体，$\Sigma M_A=0$，有

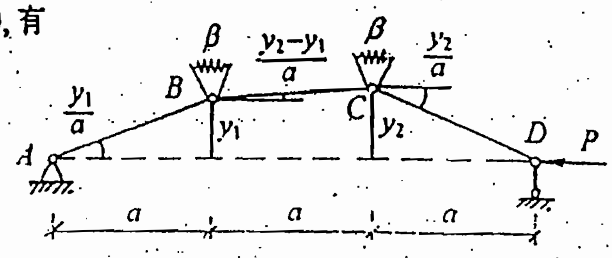
$$\left(P\frac{\delta}{a}+\frac{\beta_2\delta}{2}\right)5a-\delta\beta_2\times4a-\frac{4}{3}\delta\beta_1\times2a=0 \qquad 得 \quad P_{cr}=\frac{8}{15}\beta_1a+\frac{3}{10}\beta_2a$$

---

*缪加玉．结构力学的若干问题．成都科技大学出版社．1993.146~151 页．

15-4 自由度为2．取BD，$\Sigma M_B=0$，有

$$Py_1-\beta\left(\frac{y_1}{a}-\frac{y_2-y_1}{a}\right)=0$$



再取CD，$\Sigma M_C=0$，有

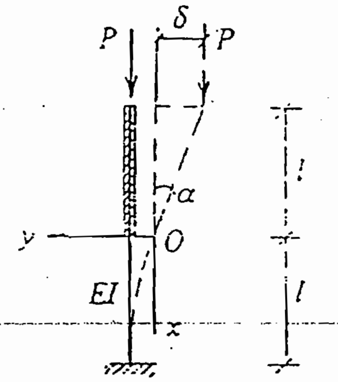
$$Py_2-\beta\left(\frac{y_2}{a}+\frac{y_2-y_1}{a}\right)=0$$

即 $\begin{cases}\left(P-\dfrac{2\beta}{a}\right)y_1+\dfrac{\beta}{a}y_2=0\\ \dfrac{\beta}{a}y_1+\left(P-\dfrac{2\beta}{a}\right)y_2=0\end{cases}$ 应有 $\begin{vmatrix}P-\dfrac{2\beta}{a} & \dfrac{\beta}{a}\\ \dfrac{\beta}{a} & P-\dfrac{2\beta}{a}\end{vmatrix}=0$，展开 $\left(P-\dfrac{2\beta}{a}\right)^2-\left(\dfrac{\beta}{a}\right)^2=0$

解得 $P=\dfrac{2\beta}{a}\pm\dfrac{\beta}{a}=\begin{cases}\dfrac{3\beta}{a} & \text{较大，不取，此时有 } y_1=-y_2\\ \dfrac{\beta}{a}=P_{cr} & \text{此即临界荷载，此时有 } y_1=y_2\end{cases}$

15-5 无限自由度。为方便，取偏离后杆中点为原点。

$EIy''=-P(y+\delta)\qquad(0\le x\le l)$



$EIy''+Py=-P\delta$　　令 $n^2=\dfrac{P}{EI}$

解　$y=A\cos nx+B\sin nx-\delta$

边界条件　$x=0$，$y=0$，　得　$A=\delta$

$x=0$，$y'=\alpha=\dfrac{\delta}{l}$，有 $B=\dfrac{\delta}{nl}$

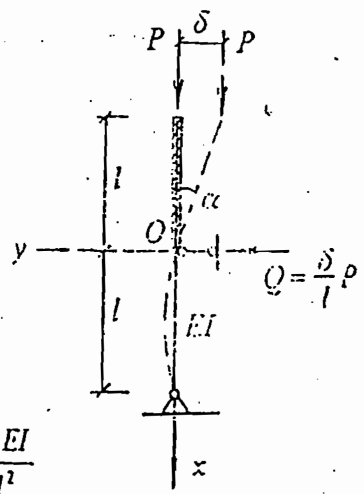
$x=l$，$y'=0$，有

$-\delta n\sin nl+\dfrac{\delta}{nl}n\cos nl=0$　即 $\delta\left(-n\sin nl+\dfrac{1}{l}\cos nl\right)=0$

稳定方程　$\mathrm{tg}\,nl=\dfrac{1}{nl}$　　最小正根　$nl=0.8603$　　$P_{cr}=\dfrac{0.7401EI}{l^2}$

15-6 $EIy''=-P(y+\delta)+Qx$

$EIy''+Py=-P\delta+\dfrac{\delta}{l}Px$



解　$y=A\cos nx+B\sin nx-\delta+\dfrac{\delta}{l}x$

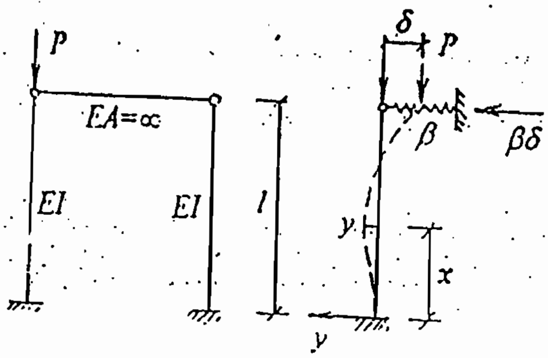
边界条件　$x=0$，$y=0$，　得　$A=\delta$

$x=0$，$y'=\dfrac{\delta}{l}$，　$nB+\dfrac{\delta}{l}=\dfrac{\delta}{l}$，　得　$B=0$

$x=l$，$y=0$，　$\delta\cos nl-\delta+\dfrac{\delta}{l}l=0$，　即　$\delta\cos nl=0$

稳定方程　$\cos nl=0$　　$(nl)_{min}=\dfrac{\pi}{2}$　　$P_{cr}=\dfrac{\pi^2EI}{4l^2}$

15-7



$$EIy'' = -P(y+\delta)+\beta\delta(l-x)$$
$$EIy''+Py = \beta\delta(l-x)-P\delta$$

解为

$$y = A\cos nx + B\sin nx + \frac{\beta\delta}{P}(l-x)-\delta$$

边界条件

$$x=0,\ y=0,\qquad A+(\frac{\beta l}{P}-1)\delta=0$$
$$x=0,\ y'=0,\qquad Bn-\frac{\beta}{P}\delta=0$$
$$x=l,\ y=-\delta,\qquad A\cos nl+B\sin nl=0$$

应有
$$\begin{vmatrix} 1 & 0 & \frac{\beta l}{P}-1 \\ 0 & n & -\frac{\beta}{P} \\ \cos nl & \sin nl & 0 \end{vmatrix}=0$$

展开
$$\frac{\beta}{P}\sin nl+n\left(1-\frac{\beta l}{P}\right)\cos nl=0$$

$$\mathrm{tg}\,nl=\frac{P}{\beta}n\left(\frac{\beta l}{P}-1\right)=\frac{n^2EI}{\beta}n\left(\frac{\beta l}{n^2EI}-1\right)=nl-\frac{n^3EI}{\beta}$$

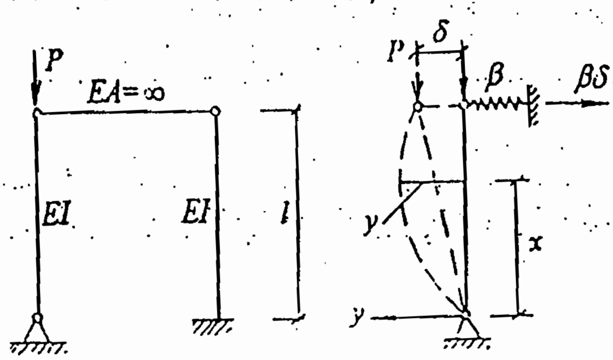
将 $\beta=\frac{3EI}{l^3}$ 代入有

$$\mathrm{tg}\,nl=nl-\frac{(nl)^3}{3}$$

最小正根为（可用试算法求得） $nl=2.204$

得
$$P_{cr}=n^2EI=\frac{4.858EI}{l^2}$$

15-8 简化为单根压杆，上端弹性支座刚度$\beta=3EI/l^3$.



本题可有不同解法，以下为解法之一。假设失稳时上端偏离δ同时压杆弯曲。

$$EIy''=-P(y-\delta)-\beta\delta(l-x)$$
$$EIy''+Py=P\delta-\beta\delta(l-x)$$

解 $$y=A\cos nx+B\sin nx+\delta-\frac{\beta\delta}{P}(l-x) \qquad (a)$$

边界条件 $x=0$，$y''=0$，有 $A=0$ 式(a)成为

$$y=B\sin nx+\delta-\frac{\beta\delta}{P}(l-x) \qquad (b)$$

$$x=0,\ y=0,\quad \delta\left(1-\frac{\beta l}{P}\right)=0 \qquad (c)$$

$$x=l,\ y=\delta,\quad B\sin nl=0 \qquad (d)$$

应有 $\begin{vmatrix} 1-\dfrac{\beta l}{P} & 0 \\ 0 & \sin nl \end{vmatrix}=0$. 展开 $\left(1-\dfrac{\beta l}{P}\right)\sin nl=0$ 此即稳定方程。

1. 当 $\left(1-\dfrac{\beta l}{P}\right)=0$ 有 $P_1=\beta l$

此时 $\sin nl=\sin\sqrt{\dfrac{P_1}{EI}}l=\sin\sqrt{\dfrac{\beta l^3}{EI}}\neq 0$（即一般不等于零），故由式$(d)$应有

$B=0$，从而式(b)成为 $y=\delta-\dfrac{\beta\delta}{\beta l}(l-x)=\dfrac{\delta}{l}x$，压杆只偏不弯。

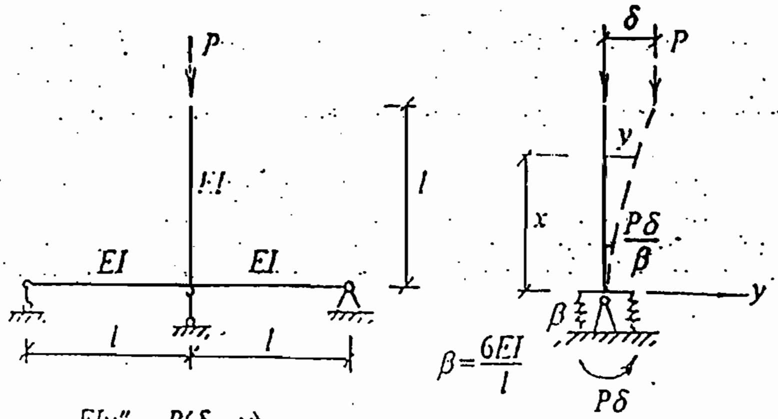
2. 当 $\sin nl=0$，$(nl)_{\min}=\pi$ 有 $P_2=\dfrac{\pi^2EI}{l^2}$

此时 $\left(1-\dfrac{\beta l}{P_2}\right)=1-\dfrac{\beta l^3}{\pi^2EI}\neq 0$（即一般不等于零），故由式$(c)$应有 $\delta=0$，

从而式(b)成为 $y=B\sin nx$，压杆只弯不偏。

3. 取小者： 令 $\beta=\dfrac{3EI}{l^3}$，有 $P_1=\dfrac{3EI}{l^2}<P_2$，故 $P_{cr}=P_1=\dfrac{3EI}{l^2}$ （只偏不弯）

15-9



$$EIy''=P(\delta-y)$$

解 $$y=A\cos nx+B\sin nx+\delta$$

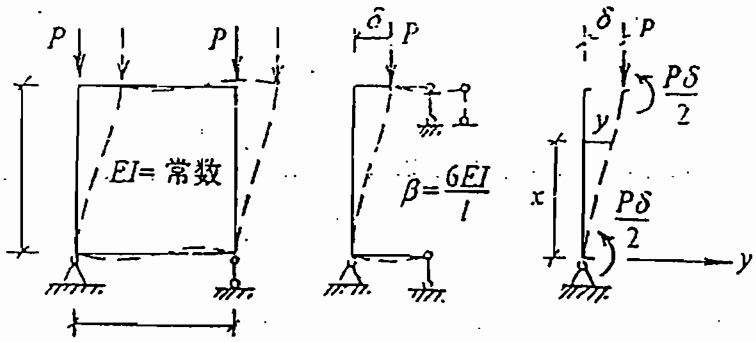
边界条件 $x=0$，$y=0$ 得 $A=-\delta$ 从而 $y=-\delta\cos nx+B\sin nx+\delta$

$x=0$，$y'=\dfrac{P\delta}{\beta}$ 有 $B=\dfrac{P\delta}{n\beta}$ 从而 $y=-\delta\cos nx+\dfrac{P\delta}{n\beta}\sin nx+\delta$

$x=l$，$y=\delta$ 有 $-\delta\cos nl+\dfrac{P\delta}{n\beta}\sin nl=0$

稳定方程 $\operatorname{tg} nl=\dfrac{n\beta}{P}=\dfrac{n\dfrac{6EI}{l}}{n^2EI}=\dfrac{6}{nl}$，得 $(nl)_{\min}=1.3496$，$P_{cr}=n^2EI=\dfrac{1.8214EI}{l^2}$

15-10 判断可知（略）反对称失稳临界荷载较小，取一半，并简化为单根压杆，上下抗转弹簧刚度均为 $\beta=6EI/l$。



解法 1 因上下变形亦反对称，故端弯矩各为 $P\delta/2$，有

$$EIy''=P(\delta-y)-\frac{P\delta}{2}=-Py+\frac{P\delta}{2}$$

解 $$y=A\cos nx+B\sin nx+\frac{\delta}{2}$$

边界条件 $x=0$，$y=0$ 得 $A=-\dfrac{\delta}{2}$

$x=0$，$y'=\dfrac{P\delta}{2\beta}$ 得 $B=\dfrac{P\delta}{2n\beta}$

从而
$$y=-\frac{\delta}{2}\cos nx+\frac{P\delta}{2n\beta}\sin nx+\frac{\delta}{2}$$

$x=l$，$y=\delta$ 有 $-\frac{\delta}{2}\cos nl+\frac{P\delta}{2n\beta}\sin nl+\frac{\delta}{2}=\delta$

$$1+\cos nl=\frac{P}{n\beta}\sin nl=\frac{nEI}{\beta}\sin nl$$

将 $\beta=\frac{6EI}{l}$ 代入，有

$$1+\cos nl=\frac{nl}{6}\sin nl \qquad (nl)_{\min}=2.385 \qquad P_{cr}=n^2EI=\frac{5.688EI}{l^2}$$

解法 2　由于上下变形反对称，第三个边界条件改用

$x=\frac{l}{2}$，$y=\frac{\delta}{2}$　有　$-\frac{\delta}{2}\cos\frac{nl}{2}+\frac{P\delta}{2n\beta}\sin\frac{nl}{2}+\frac{\delta}{2}=\frac{\delta}{2}$

$$\mathrm{tg}\frac{nl}{2}=\frac{n\beta}{P}=\frac{\beta}{nEI}=\frac{6EI}{nEIl}=\frac{6}{nl}$$

或 $\frac{nl}{2}\mathrm{tg}\frac{nl}{2}=3 \qquad \left(\frac{nl}{2}\right)_{\min}=1.1925 \qquad P_{cr}=n^2EI=\frac{5.688EI}{l^2}$

解法 3　利用一般情况之结果即公式(15-6)（见教材下册 22 页）：

$$\begin{vmatrix} 1 & 0 & \left(1-\frac{\beta_3 l}{P}\right) & \frac{\beta_2}{P} \\ \cos nl & \sin nl & 0 & \frac{\beta_2}{P} \\ 0 & n & \left(\frac{\beta_3}{P}-\frac{\beta_3}{\beta_1}l-\frac{P}{\beta_1}\right) & -\frac{\beta_2}{\beta_1} \\ -n\sin nl & n\cos nl & \frac{\beta_3}{P} & 1 \end{vmatrix}=0$$

今有 $\beta_1=\beta_2=\beta$，$\beta_3=0$　代入并逐步化简降阶（过程略）可得

$$\begin{vmatrix} \frac{P}{\beta} & n \\ -n\sin nl-\frac{P}{\beta}\cos nl & n\cos nl-\frac{P}{\beta}\sin nl \end{vmatrix}=0$$

展开
$$\frac{P}{\beta}\left(n\cos nl-\frac{P}{\beta}\sin nl\right)+n\left(n\sin nl+\frac{P}{\beta}\cos nl\right)=0$$

$$2\frac{nP}{\beta}\cos nl+\left(n^2-\frac{P^2}{\beta^2}\right)\sin nl=0, \qquad 2\cos nl+\left(\frac{n\beta}{P}-\frac{P}{n\beta}\right)\sin nl=0$$

$$\mathrm{tg}\,nl=\frac{2}{\frac{nEI}{\beta}-\frac{\beta}{nEI}}=\frac{2}{\frac{nl}{6}-\frac{6}{nl}} \qquad (nl)_{\min}=2.385 \qquad P_{cr}=\frac{5.688EI}{l^2}$$

15-11 $EIy'' = -Py$ $(0 \le x \le l)$

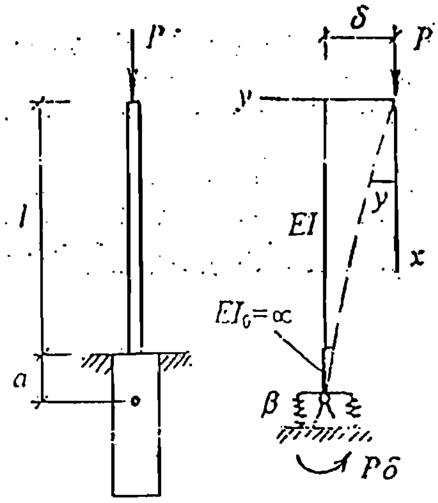
$$y = A\cos nx + B\sin nx$$

边界条件 $x=0$，$y=0$，得 $A=0$

$x=l$，$y=\delta - a\dfrac{P\delta}{\beta}$，有 $B\sin nl - \delta\left(1-\dfrac{aP}{\delta}\right)=0$

$x=l$，$y'=\dfrac{P\delta}{\beta}$，有 $Bn\cos nl - \dfrac{P\delta}{\beta}=0$

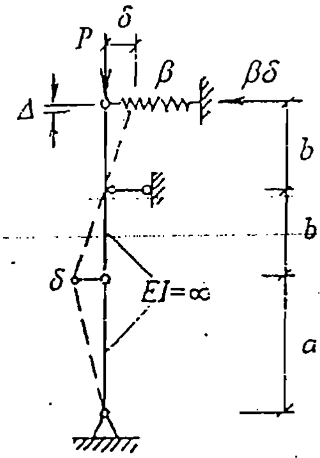
故应有
$$\begin{vmatrix} \sin nl & -\left(1-\dfrac{aP}{\beta}\right) \\ n\cos nl & -\dfrac{P}{\beta} \end{vmatrix} = 0$$



展开 $\dfrac{P}{\beta}\sin nl - \left(1-\dfrac{aP}{\beta}\right)n\cos nl = 0$，$\dfrac{\operatorname{tg} nl}{n} = \dfrac{\beta}{P} - a = \dfrac{\beta}{n^2 EI} - a$，$\dfrac{\operatorname{tg} nl}{nl} = \dfrac{\beta l}{(nl)^2 EI} - \dfrac{a}{l}$

要给出 a 对 l 之相对值及 β 对 $\dfrac{EI}{l}$ 之相对值时方可具体求解。

15-12 用能量法作题 15-1 ~ 15-4。



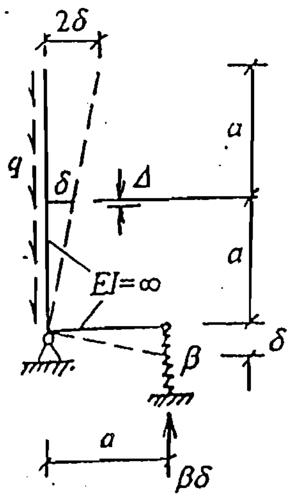
15-1 $U = \dfrac{1}{2}\beta\delta^2$

$$V = -P \times \frac{1}{2}\left(\frac{\delta^2}{a} + \frac{\delta^2}{b} + \frac{\delta^2}{b}\right)$$

$$\Pi = U + V = \frac{\beta}{2}\delta^2 - \frac{P}{2}\left(\frac{1}{a} + \frac{2}{b}\right)\delta^2$$

$$\frac{d\Pi}{d\delta} = \left[\beta - P\left(\frac{1}{a} + \frac{2}{b}\right)\right]\delta = 0$$

因 $\delta \ne 0$ 故 $\beta - P\left(\dfrac{1}{a} + \dfrac{2}{b}\right) = 0$ 得 $P_{cr} = \dfrac{ab\beta}{b+2a}$



15-2 $U = \dfrac{1}{2}\beta\delta^2$

均布荷载可用其合力代替，故

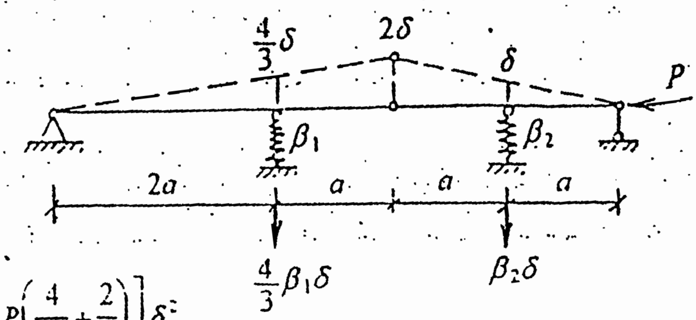
$$V = -2qa \times \frac{1}{2}\frac{\delta^2}{a} = -q\delta^2$$

$$\Pi = U + V = \frac{\beta}{2}\delta^2 - q\delta^2$$

$$\frac{d\Pi}{d\delta} = (\beta - 2q)\delta = 0$$

因 $\delta \ne 0$ 故 $\beta - 2q = 0$ 得 $q_{cr} = \dfrac{\beta}{2}$

15-3



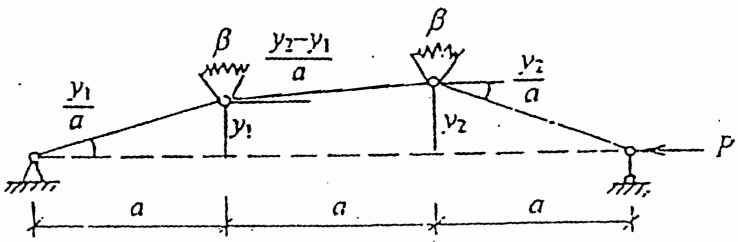
$$U=\frac{1}{2}\beta_1\left(\frac{4}{3}\delta\right)^2+\frac{1}{2}\beta_2\delta^2$$

$$V=-P\times\frac{1}{2}\left[\frac{(2\delta)^2}{3a}+\frac{(2\delta)^2}{2a}\right]$$

$$\Pi=U+V=\frac{1}{2}\left[\frac{16}{9}\beta_1+\beta_2-P\left(\frac{4}{3a}+\frac{2}{a}\right)\right]\delta^2$$

$$\frac{d\Pi}{d\delta}=\left[\frac{16}{9}\beta_1+\beta_2-\frac{10}{3a}P\right]\delta=0$$ 而 $\delta\neq0$ 故 $P_{cr}=\left(\frac{8}{15}\beta_1+\frac{3}{10}\beta_2\right)a$

15-4



$$U=\frac{1}{2}\beta\left(\frac{y_1}{a}-\frac{y_2-y_1}{a}\right)^2+\frac{1}{2}\beta\left(\frac{y_2}{a}+\frac{y_2-y_1}{a}\right)^2=\frac{\beta}{2a^2}\left[(2y_1-y_2)^2+(2y_2-y_1)^2\right]$$

$$V=-P\times\frac{1}{2}\left[\frac{y_1^2}{a}+\frac{(y_2-y_1)^2}{a}+\frac{y_2^2}{a}\right]=-\frac{P}{a}(y_1^2-y_1y_2+y_2^2)$$

$$\Pi=U+V=\frac{\beta}{2a^2}(5y_1^2-8y_1y_2+5y_2^2)-\frac{P}{a}(y_1^2-y_1y_2+y_2^2)$$

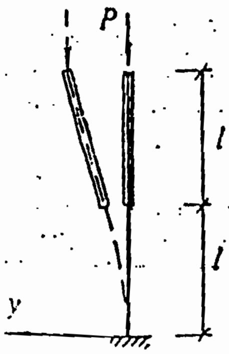
$$=\frac{1}{a}\left[\left(\frac{5\beta}{2a}-P\right)y_1^2-\left(\frac{4\beta}{a}-P\right)y_1y_2+\left(\frac{5\beta}{2a}-P\right)y_2^2\right]$$

由 $\frac{\partial\Pi}{\partial y_1}=0,\quad \left(\frac{5\beta}{a}-2P\right)y_1-\left(\frac{4\beta}{a}-P\right)y_2=0$

$\frac{\partial\Pi}{\partial y_2}=0,\quad -\left(\frac{4\beta}{a}-P\right)y_1+\left(\frac{5\beta}{a}-2P\right)y_2=0$

由以上两式系数行列式等于零并展开有

$$\left(\frac{5\beta}{a}-2P\right)^2-\left(\frac{4\beta}{a}-P\right)^2=0$$ 得 $P=\begin{cases}\frac{3\beta}{a} & \text{（较大、不取）}\\ \frac{\beta}{a}=P_{cr}\end{cases}$



15-13　弹性部分取 $y=\dfrac{ax^2}{l^2}$

$$y'=\frac{2ax}{l^2}\qquad y'_{x=l}=\frac{2a}{l}$$

$$y''=\frac{2a}{l^2}$$

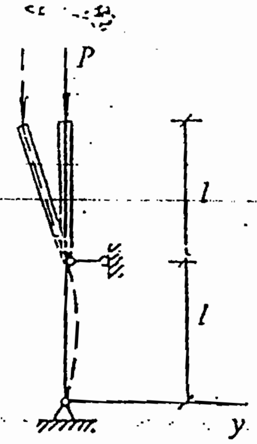
$$U=\frac{1}{2}\int_0^l EI(y'')^2dx=\frac{EI}{2}\int_0^l\left(\frac{2a}{l^2}\right)^2dx=\frac{2EI}{l^3}a^2$$

$$V=-\frac{P}{2}\left[\int_0^l(y')^2dx+(y'_{x=l})^2l\right]=-\frac{P}{2}\left[\int_0^l\left(\frac{2ax}{l^2}\right)^2dx+\left(\frac{2a}{l}\right)^2l\right]=-\frac{8P}{3l}a^2$$

$$\Pi=\left(\frac{2EI}{l^3}-\frac{8P}{3l}\right)a^2$$

$\dfrac{d\Pi}{da}=0$　而　$a\neq0$　有　$\dfrac{2EI}{l^3}-\dfrac{8P}{3l}=0$　得　$P_{cr}=\dfrac{3EI}{4l^2}$

比精确解 $\dfrac{0.7401EI}{l^2}$（见题15－5）大1.3％.



15-14　弹性部分取 $y=ax\left(1-\dfrac{x^2}{l^2}\right)$

$$y'=\left(1-\frac{3x^2}{l^2}\right)a\qquad y'_{x=l}=-2a$$

$$y''=-\frac{6x}{l^2}a$$

$$U=\frac{1}{2}\int_0^l EI(y'')^2dx=\frac{EI}{2}\int_0^l\left(-\frac{6x}{l^2}a\right)^2dx=\frac{6EI}{l}a^2$$

$$V=-\frac{P}{2}\left[\int_0^l(y')^2dx+(y'_{x=l})^2l\right]=-\frac{P}{2}\left[\int_0^l\left(1-\frac{3x^2}{l^2}\right)^2a^2dx+(-2a)^2l\right]$$

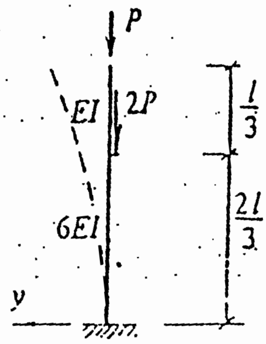
$$=-\frac{P}{2}\left[l-2l+\frac{9}{5}l+4l\right]a^2=-\frac{12l}{5}Pa^2$$

$$\Pi=\left(\frac{6EI}{l}-\frac{12l}{5}P\right)a^2$$

$\dfrac{d\Pi}{da}=0$　而　$a\neq0$　有　$\dfrac{6EI}{l}-\dfrac{12l}{5}P=0$　得　$P_{cr}=\dfrac{5EI}{2l^2}$

比精确解 $\dfrac{\pi^2EI}{4l^2}$（见题15－6）大1.3％.

15-15



$$y = a\left(1-\cos\frac{\pi x}{2l}\right)$$

$$y' = \frac{\pi a}{2l}\sin\frac{\pi x}{2l}$$

$$y'' = \frac{\pi^2 a}{4l^2}\cos\frac{\pi x}{2l}$$

$$U = \frac{6EI}{2}\int_0^{\frac{2l}{3}}\left(\frac{\pi^2 a}{4l^2}\cos\frac{\pi x}{2l}\right)^2 dx + \frac{EI}{2}\int_{\frac{2l}{3}}^{l}\left(\frac{\pi^2 a}{4l^2}\cos\frac{\pi x}{2l}\right)^2 dx$$

$$= \frac{\pi^4 EIa^2}{32l^4}\left[6\int_0^{\frac{2l}{3}}\cos^2\frac{\pi x}{2l}dx + \int_{\frac{2l}{3}}^{l}\cos^2\frac{\pi x}{2l}dx\right]$$

$$= \frac{\pi^4 EIa^2}{32l^4}\left[6\left(\frac{l}{3}+\frac{l}{2\pi}\sin\frac{2\pi}{3}\right)+\left(\frac{l}{2}-\frac{l}{3}-\frac{l}{2\pi}\sin\frac{2\pi}{3}\right)\right]$$

$$= \frac{\pi^4 EIa^2}{32l^4}\left[\frac{13}{6}l+\frac{5l}{2\pi}\sin\frac{2\pi}{3}\right]$$

$$= \frac{\pi^4 EIa^2}{32l^3}\left(\frac{13}{6}+\frac{5\sqrt{3}}{4\pi}\right)$$

$$V = -\frac{P}{2}\int_0^{l}\left(\frac{\pi a}{2l}\sin\frac{\pi x}{2l}\right)^2 dx - \frac{2P}{2}\int_0^{\frac{2l}{3}}\left(\frac{\pi a}{2l}\sin\frac{\pi x}{2l}\right)^2 dx$$

$$= -\frac{P\pi^2 a^2}{8l^2}\left[\int_0^{l}\sin^2\frac{\pi x}{2l}dx + 2\int_0^{\frac{2l}{3}}\sin^2\frac{\pi x}{2l}dx\right]$$

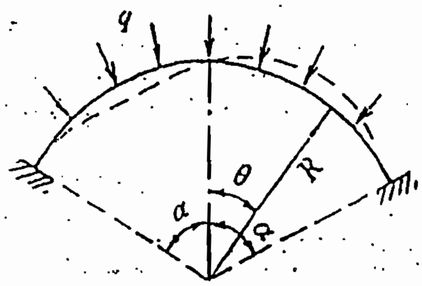
$$= -\frac{P\pi^2 a^2}{8l^2}\left[\frac{l}{2}+2\left(\frac{l}{3}-\frac{l}{2\pi}\sin\frac{2\pi}{3}\right)\right]$$

$$= -\frac{P\pi^2 a^2}{8l^2}\left[\frac{7l}{6}-\frac{l}{\pi}\sin\frac{2\pi}{3}\right]$$

$$= -\frac{P\pi^2 a^2}{8l}\left(\frac{7}{6}-\frac{\sqrt{3}}{2\pi}\right)$$

$$\Pi = U+V = \left[\frac{\pi^4 EI}{32l^3}\left(\frac{13}{6}+\frac{5\sqrt{3}}{4\pi}\right)-\frac{P\pi^2}{8l}\left(\frac{7}{6}-\frac{\sqrt{3}}{2\pi}\right)\right]a^2$$

$\frac{d\Pi}{da}=0$ 而 $a\neq 0$ 得 $P_{cr} = \frac{\pi^2 EI}{4l^2}\cdot\frac{\left(\frac{13}{6}+\frac{5\sqrt{3}}{4\pi}\right)}{\left(\frac{7}{6}-\frac{\sqrt{3}}{2\pi}\right)} = \frac{7.908\,EI}{l^2}$



*15-16 w 应为 θ 的奇函数，故由公式（15-57）有

$$w = A_3\sin\theta + A_4\sin n\theta \qquad 其中 \quad n^2 = 1+\frac{qR^3}{EI}$$

边界条件 $\theta=\alpha$，有 $w=0$ 及 $w'=0$

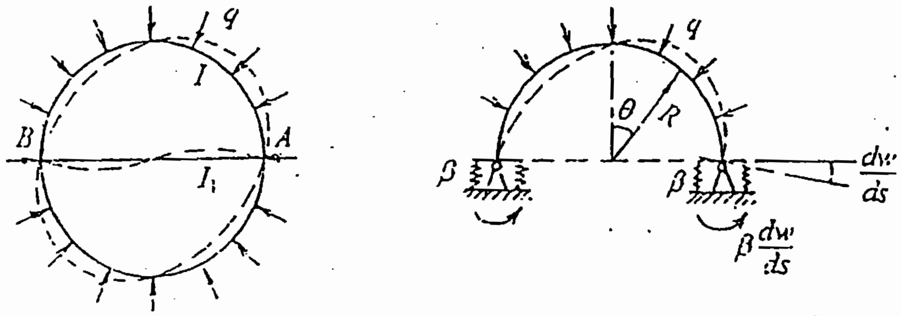
$$A_3\sin\alpha + A_4\sin n\alpha = 0$$

$$A_3\cos\alpha + A_4 n\cos n\alpha = 0$$

A_3、A_4 不全为零，由其系数行列式为零并展开有

$$n\sin\alpha\cos n\alpha - \sin n\alpha\cos\alpha = 0 \qquad 即 \quad n=\frac{\operatorname{tg} n\alpha}{\operatorname{tg}\alpha} \qquad 或 \quad \frac{\operatorname{tg} n\alpha}{n\alpha} = \frac{\operatorname{tg}\alpha}{\alpha}$$

*15-17



反对称失稳临界力较小。当结点 A、B 转动单位角时，横撑端力矩为 $\frac{3EI_1}{R}$。此时拱端力矩即抗转弹簧刚度由结点力矩平衡知应为 $\beta = \frac{3EI_1}{2R}$。反对称变形时 w、M 均为 θ 的奇函数，故由公式（15-57）有

$$w = A_3\sin\theta + A_4\sin n\theta$$

及
$$\frac{dw}{ds} = \frac{1}{R}\frac{dw}{d\theta} = \frac{1}{R}(A_3\cos\theta - A_4 n\cos n\theta) \qquad 和 \quad M = -\frac{EI}{R^2}A_4(1-n^2)\sin n\theta$$

边界条件 $\theta=\frac{\pi}{2}$，$w=0$

$$\theta=\frac{\pi}{2},\quad M = +\beta\frac{dw}{ds} = \frac{3EI_1}{2R}\frac{dw}{ds}$$

有
$$A_3\sin\frac{\pi}{2} + A_4 n\sin\frac{n\pi}{2} = 0$$

$$-\frac{EI}{R^2}A_4(1-n^2)\sin\frac{\pi n}{2} = \frac{3EI_1}{2R^2}\left(A_3\cos\frac{\pi}{2} + A_4 n\cos\frac{\pi n}{2}\right)$$

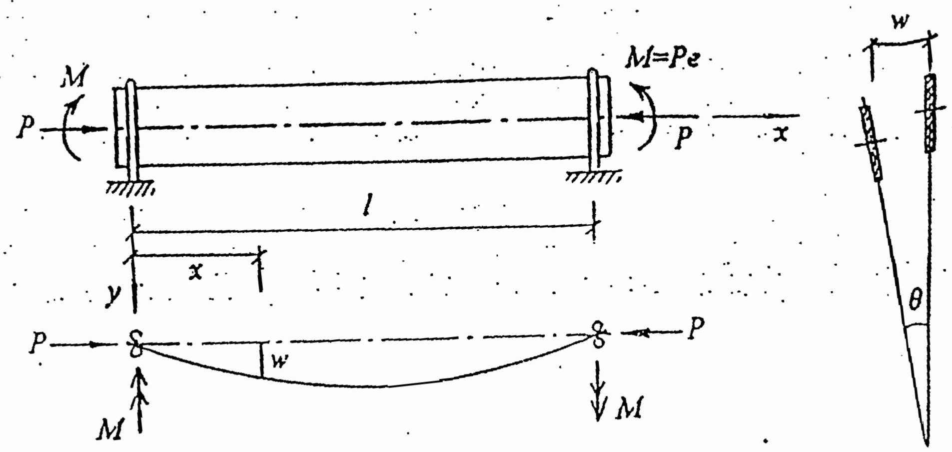
即
$$A_3 + A_4 n\sin\frac{n\pi}{2} = 0$$

$$0 + A_4\left[I(1-n^2)\sin\frac{\pi n}{2} + \frac{3I_1}{2}n\cos\frac{\pi n}{2}\right] = 0$$

A_3、A_4 不全为零，由上二式系数行列式等于零并展开得

$$I(1-n^2)\sin\frac{\pi n}{2} + \frac{3I_1}{2}n\cos\frac{\pi n}{2} = 0 \qquad 即 \quad \operatorname{tg}\frac{\pi n}{2} = \frac{3I_1}{2I}\frac{n}{n^2-1}$$

*15-18



侧向弯曲和扭转微分方程为

$$EI_y\frac{d^2w}{dx^2}=-M\theta-Pw \tag{a}$$

$$GI_t\frac{d\theta}{dx}=M\frac{dw}{dx} \tag{b}$$

由(b) $\qquad \theta=\dfrac{M}{GI_t}w+C$

边界条件 $x=0$, $w=0$, $\theta=0$ 得 $C=0$, 代入式(a)有

$$EI_y\frac{d^2w}{dx^2}=-\frac{M^2}{GI_t}w-Pw$$

即

$$\frac{d^2w}{dx^2}+\left(\frac{M^2+GI_tP}{EI_yGI_t}\right)w=0$$

令

$$n^2=\frac{M^2+GI_tP}{EI_yGI_t} \tag{c}$$

有

$$\frac{d^2w}{dx^2}+n^2w=0$$

解 $\qquad w=A\sin nx+B\cos nx$

边界条件 $x=0$, $w=0$ 得 $B=0$

$x=l$, $w=0$ 有 $A\sin nl=0$

因 $A\neq0$, 故 $\sin nl=0$, $(nl)_{min}=\pi$, 代入式(c)得

$$M^2+GI_tP=\frac{\pi^2EI_yGI_t}{l^2}$$

讨论: (1) 当 $e=0$, 则 $M=0$, 有 $P_{cr}=\dfrac{\pi^2EI_y}{l^2}$

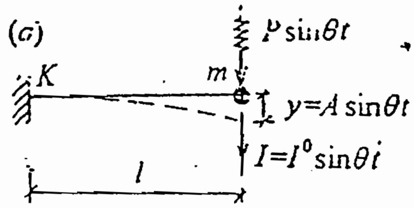
(2) 当 $P=0$ 且设 $M\neq0$, 则有 $M_{cr}=\dfrac{\pi}{l}\sqrt{EI_yGI_t}$

第十六章　结构动力学

思考题（选答）

6. 单自由度结构在动力荷载作用下，质点处之最大动力位移与静力位移（即将动力荷载之最大值当作静力荷载作用时产生之位移）之比，称为位移动力系数，亦常简称动力系数。简谐荷载作用下动力系数 μ 的数值与干扰力频率和自频振率之比 θ/ω 及阻尼比 ξ 有关。当干扰力与惯性力作用点重合（即动力荷载作用在质点上）时，内力动力系数与位移动力系数相同；否则将不相同。对此举简例说明如下。

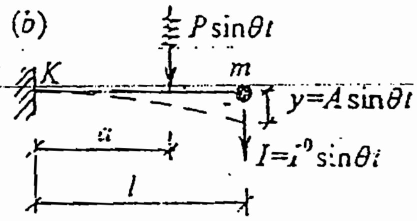
图 a 为动力荷载作用在质点上。为简明，设不计阻尼，则动力荷载、位移 y 及惯性力 I 均同时达到最大值 P、A 及 I^0（注意，$\theta>\omega$ 时前者与后两者反向），位移动力系数为



$$\mu=\frac{A}{y_{st}}=\frac{(P+I^0)\delta_{11}}{P\delta_{11}}=\frac{P+I^0}{P}$$

求内力动力系数时，以截面 K 之弯矩为例，有

$$\mu_M=\frac{M_{K(\max)}}{M_{K(st)}}=\frac{(P+I^0)l}{Pl}=\frac{P+I^0}{P}=\mu$$



不难看出，其他任一截面之内力动力系数亦均等于 μ。实际上，将动力荷载与惯性力之和看作一个力，则结构将只受到一个力作用，不论此力如何变化，结构的全部位移和内力均按同一比例增减。

图 b 则为动力荷载不作用在质点上，此时位移动力系数为

$$\mu=\frac{A}{y_{st}}=\frac{P\delta_{1P}+I^0\delta_{11}}{P\delta_{1P}}=\frac{P+I^0\dfrac{\delta_{11}}{\delta_{1P}}}{P}$$

求内力动力系数时，仍以截面 K 之弯矩为例，有

$$\mu_M=\frac{M_{K(\max)}}{M_{K(st)}}=\frac{Pa+I^0l}{Pa}=\frac{P+I^0\dfrac{l}{a}}{P}$$

显然，一般 $\dfrac{l}{a}\neq\dfrac{\delta_{11}}{\delta_{1P}}$，故 $\mu_M\neq\mu$。此外，不同截面的内力，其内力动力系数亦将各不相同，它们当然一般不会等于位移动力系数。

至于有阻尼的情形，上述结论依然适用，只不过此时质点的位移和惯性力将比荷载落后一个相位，前两者与后者以及某截面的某内力均不是同时达到最大值，某内力之最大值及动力系数可由极值条件确定，兹从略。

习 题 $I_1\cdot\frac{l}{3}+I_2 l+K y_B\cdot\frac{2l}{3}=0$

16-1 略。

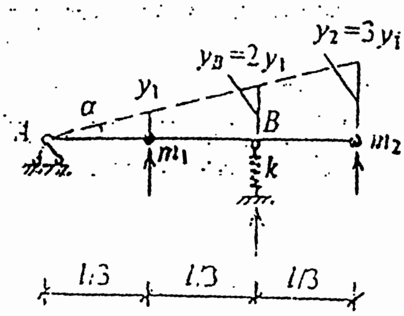
16-2 (a) $\Sigma M_A=0$

$$-m_1\ddot{y}_1\frac{l}{3}-m_2\ddot{y}_2 l-ky_B\frac{2l}{3}=0$$

$$-m_1\ddot{y}_1\frac{l}{3}-m_2(3\ddot{y}_1)l-k(2y_1)\frac{2l}{3}=0$$

$$(m_1+9m_2)\ddot{y}_1+4ky_1=0$$

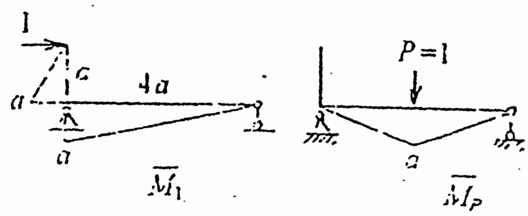
$$\ddot{y}_1+\frac{4k}{m_1+9m_2}y_1=0$$



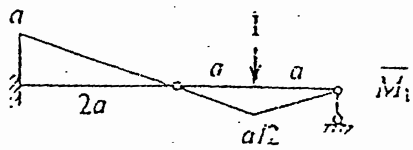
(b) $\delta_{11}=\frac{5}{3}\frac{a^3}{EI}$ $\delta_{1P}=\frac{a^3}{EI}$

$y_1=I_1\delta_{11}+P(t)\cdot\delta_{1P}$ $y=\frac{5a^3}{3EI}(-m\ddot{y})+\frac{a^3}{EI}P(t)$

$$\ddot{y}+\frac{3EI}{5a^3m}y=\frac{3}{5}\frac{P(t)}{m}$$

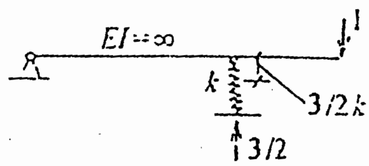


16-3 (a) $\delta_{11}=\frac{5}{6}\frac{a^3}{EI}$ $\omega=\sqrt{\frac{6EI}{5a^3m}}$

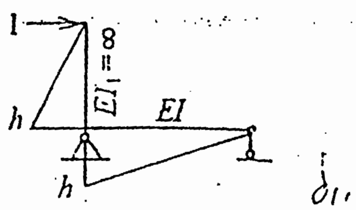


(b) $\delta_{11}=-\Sigma\overline{R}c=-\left(-\frac{3}{2}\frac{3}{2k}\right)=\frac{9}{4k}$

$$\omega=\sqrt{\frac{4k}{9m}}=\frac{2}{3}\sqrt{\frac{k}{m}}$$



(c) $\delta_{11}=\frac{1}{EI}\frac{lh}{2}\frac{2h}{3}=\frac{lh^2}{3EI}$ $\omega=\sqrt{\frac{3EI}{lh^2m}}$

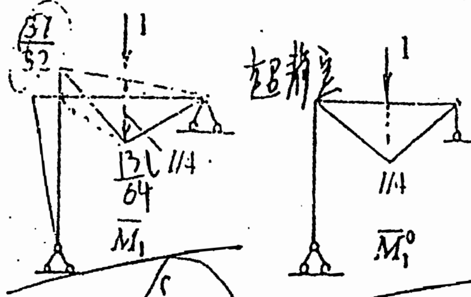


(d) $\delta_{11}=\frac{1}{EI}\left(\frac{l^3}{48}-\frac{1}{2}l\frac{l}{4}\frac{3l}{64}\right)$

$$=\frac{l^3}{EI}\left(\frac{1}{48}-\frac{3}{512}\right)$$

$$=\frac{23}{1536}\frac{l^3}{EI}=0.014974\frac{l^3}{EI}$$

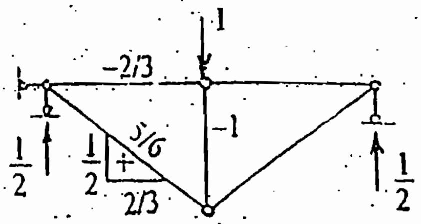
$$\omega=\sqrt{\frac{1536}{23}\frac{EI}{l^3m}}=\sqrt{66.78\frac{EI}{l^3m}}=8.172\sqrt{\frac{EI}{l^3m}}$$



16-4 略，见题 16-2 (a)。

16-5 $\delta_{11} = \dfrac{1}{EA}\left[2\left(\dfrac{5}{6}\right)^2 5 + 2\left(-\dfrac{2}{3}\right)^2 4 + (-1)^2 3\right]$

$= \dfrac{243}{18EA} = \dfrac{13.5}{EA}$



$$\omega = \sqrt{\frac{9.81\times 210\times 10^9\times 2\times 10^{-3}}{40\times 10^3\times 13.5}} = 87.3 \quad 1/\mathrm{s}$$

16-6 $k_{11} = \dfrac{12EI}{l_1^3} + \dfrac{12EI}{l_2^3} = 12\times 5\times 10^4\left(\dfrac{1}{10^3} + \dfrac{1}{8^3}\right) = 1771.9\ \mathrm{kN/m}$

$\omega = \sqrt{\dfrac{1771.9\times 9.81}{200}} = 9.32 \quad 1/\mathrm{s} \qquad T = \dfrac{2\pi}{\omega} = 0.674\ \mathrm{s}$

16-7 $a = \sqrt{0.01^2 + \left(\dfrac{0.1}{9.32}\right)^2} = 0.01467\ \mathrm{m} = 14.67\ \mathrm{mm}$, $\varphi = \mathrm{arc\,tg}\left(\dfrac{0.01}{0.1/9.32}\right) = 0.75\ \mathrm{rad}$

$y_{t=1} = 14.67\sin(9.32\times 1 + 0.75) = -8.82\ \mathrm{mm}$

16-8 $\omega' = 9.32\sqrt{1-0.05^2} = 9.30 \quad 1/\mathrm{s} \qquad T = \dfrac{2\pi}{\omega'} = 0.676\ \mathrm{s}$

$k = \xi\omega = 0.05\times 9.32 = 0.466 \quad 1/\mathrm{s}$

$$b = \sqrt{0.01^2 + \left(\frac{0.1 + 0.466\times 0.01}{9.30}\right)^2} = 0.01505\ \mathrm{m} = 15.05\ \mathrm{mm}$$

$$\varphi' = \mathrm{arc\,tg}\left(\frac{9.30\times 0.01}{0.1 + 0.466\times 0.01}\right) = 0.726 \quad \mathrm{rad}$$

$y_{t=1} = 15.05\,e^{-0.466\times 1}\sin(9.30\times 1 + 0.726) = -5.34\ \mathrm{mm}$

16-9 $\omega = \sqrt{\dfrac{3EIg}{Gl^3}} = \sqrt{\dfrac{3\times 210\times 10^9\times 3.4\times 10^{-5}\times 9.81}{12\times 10^3\times 2^3}} = \sqrt{2189} = 46.79 \quad 1/\mathrm{s}$

(1) $\theta = \dfrac{300\times 2\pi}{60} = 10\pi$

$\mu = \dfrac{1}{1 - \dfrac{(10\pi)^2}{2189}} = 1.821$

$\Delta_{\max} = \dfrac{(12 + 5\times 1.821)10^3\times 2^3}{3\times 210\times 10^9\times 3.4\times 10^{-5}}$

$= 0.00788\ \mathrm{m}$

$M_A = -(12 + 5\times 1.821)\times 2 = -42.2\ \mathrm{kN\,m}$

(2) $\theta = \dfrac{600\times 2\pi}{60} = 20\pi$

$\mu = \dfrac{1}{1 - \dfrac{(20\pi)^2}{2189}} = -1.244$

$\Delta_{\max} = \dfrac{(12 + 5\times 1.244)10^3\times 2^3}{3\times 210\times 10^9\times 3.4\times 10^{-5}}$

$= 0.00681\ \mathrm{m}$

$M_A = -(12 + 5\times 1.244)\times 2 = -36.4\ \mathrm{kN\,m}$

16-10 $\ln\frac{1}{0.05}\approx 2\pi\times 10\xi$ $\quad\xi\approx\frac{\ln 20}{20\pi}=0.0477$ $\quad$ 共振时 $\mu\approx\frac{1}{2\xi}=10.5$

16-11 当 $t\le t_1$ 时

$$y=\frac{1}{m\omega}\int_0^t P(1-\frac{\tau}{t_1})\sin\omega(t-\tau)d\tau$$

$$=\frac{P}{m\omega}\int_0^t\left[\sin\omega(t-\tau)-\frac{\tau}{t_1}\sin\omega(t-\tau)\right]d\tau$$

积分得（其中第二项采用分部积分法）：

$$y=\frac{P}{m\omega^2}\left[\cos\omega(t-\tau)-\frac{\tau}{t_1}\cos\omega(t-\tau)-\frac{1}{t_1\omega}\sin\omega(t-\tau)\right]_0^t$$

$$=y_{st}(1-\cos\omega t+\frac{\sin\omega t}{\omega t_1}-\frac{t}{t_1})$$

当 $t\ge t_1$ 时

$$y=\frac{1}{m\omega}\int_0^{t_1}P(1-\frac{\tau}{t_1})\sin\omega(t-\tau)d\tau$$

$$=\frac{P}{m\omega^2}\left[\cos\omega(t-\tau)-\frac{\tau}{t_1}\cos\omega(t-\tau)-\frac{1}{t_1\omega}\sin\omega(t-\tau)\right]_0^{t_1}$$

$$=\frac{P}{m\omega^2}\left[\cos\omega(t-t_1)-\cos\omega t-\cos\omega(t-t_1)-\frac{\sin\omega(t-t_1)}{t_1\omega}+\frac{\sin\omega t}{t_1\omega}\right]$$

$$=y_{st}\left[-\cos\omega t+\frac{\sin\omega t-\sin\omega(t-t_1)}{\omega t_1}\right]$$

$\frac{m}{k_{11}}$ $\quad\frac{k_{11}}{m}$ $\quad\frac{1}{m_1\delta_{11}}$

$y=I\delta_{11}+P(t)\delta_{12}$

$=-m\ddot{y}\delta_{11}+P(t)\delta_{12}$

$\lambda_1=\frac{(\delta_{11}m_1+\delta_{22}m_2)+\sqrt{(\delta_{11}m_1+\delta_{22}m_2)^2-4(\delta_{11}\delta_{22}-\delta_{12}^2)m_1m_2}}{2}$

$\lambda_2=$

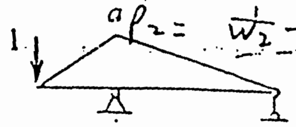
$w_1=\frac{1}{\sqrt{\lambda_1}}$

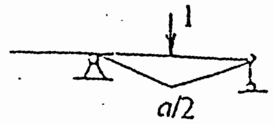
$w_2=\frac{1}{\sqrt{\lambda_2}}$

$\rho_1=\frac{\frac{1}{w_1^2}-\delta_{11}m_1}{\delta_{12}m_2}$

$\rho_2=\frac{\frac{1}{w_2^2}-\delta_{11}m_1}{\delta_{12}m_2}$

16-12 $\delta_{11}=\frac{a^3}{EI}$ $\quad\delta_{22}=\frac{a^3}{6EI}$ $\quad\delta_{12}=-\frac{a^3}{4EI}$

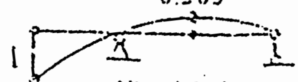




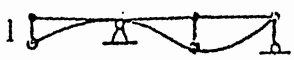
$$\left(\frac{1}{\omega^2}\right)_{1,2}=\frac{1}{2}\left(\frac{4}{3}\pm\sqrt{\frac{51}{54}}\right)\frac{ma^3}{EI}=\begin{Bmatrix}1.1526\\0.18075\end{Bmatrix}\frac{ma^3}{EI}$$

$$\omega_{1,2}=\begin{Bmatrix}0.931\\2.352\end{Bmatrix}\sqrt{\frac{EI}{ma^3}}$$

$$\rho_1=\frac{1.1526-1}{-\frac{1}{4}\times 2}=-0.305,\quad\rho_2=\frac{0.18075-1}{-\frac{1}{4}\times 2}=1.638$$



第一振型



第二振型

[illegible]

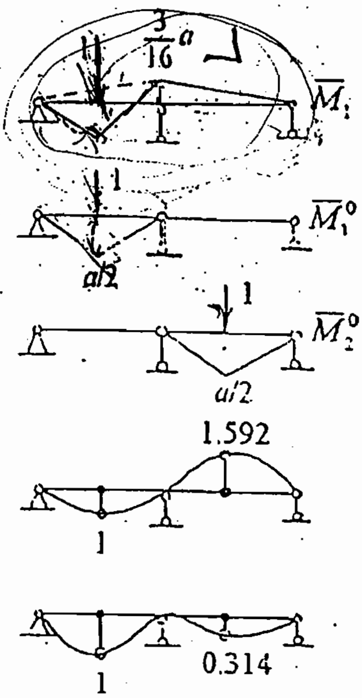
16-13 $\delta_{11}=\delta_{22}=\frac{23}{192}\frac{a^3}{EI}$ $\delta_{12}=-\frac{3}{64}\frac{a^3}{EI}$

$$\lambda_{1,2}=\frac{1}{2}(0.359375\pm0.178685)\frac{ma^3}{EI}=\begin{Bmatrix}0.2690\\0.09034\end{Bmatrix}\frac{ma^3}{EI}$$

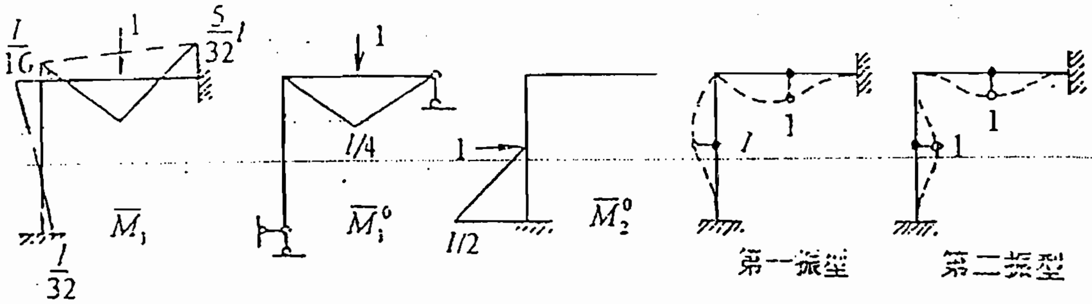
$$\omega_{1,2}=\begin{Bmatrix}1.928\\3.327\end{Bmatrix}\sqrt{\frac{EI}{ma^3}}$$

$$\rho_1=\frac{0.2690-\frac{23}{192}\times1}{-\frac{3}{64}\times2}=-1.592$$

$$\rho_2=\frac{0.09034-\frac{23}{192}\times1}{-\frac{3}{64}\times2}=0.314$$



16-14 $\delta_{11}=\delta_{22}=\frac{11}{1536}\frac{l^3}{EI}$ $\delta_{12}=-\frac{3}{1536}\frac{l^3}{EI}$



$$\lambda_1=(\delta_{11}-\delta_{12})m=\frac{14}{1536}\frac{ml^3}{EI}\qquad \omega_1=10.47\sqrt{\frac{EI}{ml^3}}\qquad \rho_1=\frac{(\delta_{11}-\delta_{12})m-\delta_{11}m}{\delta_{12}m}=-1$$

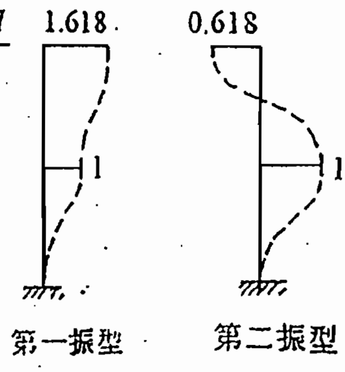
$$\lambda_2=(\delta_{11}+\delta_{12})m=\frac{8}{1536}\frac{ml^3}{EI}\qquad \omega_2=13.86\sqrt{\frac{EI}{ml^3}}\qquad \rho_2=\frac{(\delta_{11}+\delta_{12})m-\delta_{11}m}{\delta_{12}m}=1$$

16-15 $k_{11}=\frac{48EI}{l^3}$ $k_{22}=\frac{24EI}{l^3}$ $k_{12}=k_{21}=-\frac{24EI}{l^3}$

$$\omega_{1,2}^2=\frac{24EI}{ml^3}\left(1.5\mp\sqrt{\frac{5}{4}}\right)=\frac{24EI}{ml^3}\begin{Bmatrix}0.382\\2.618\end{Bmatrix}=\begin{Bmatrix}9.617\\62.83\end{Bmatrix}\frac{EI}{ml^3}$$

$$\omega_{1,2}=\begin{Bmatrix}3.028\\7.927\end{Bmatrix}\sqrt{\frac{EI}{ml^3}}$$

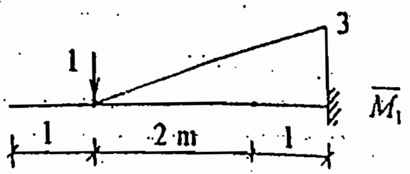
$$\rho_1=\frac{0.382-1}{-1}=1.618\qquad \rho_2=\frac{2.618-1}{-1}=-0.618$$

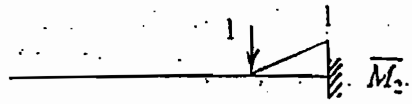


16-16

$$\delta_{11}=\frac{9}{EI}\qquad \delta_{22}=\frac{1}{3EI}\qquad \delta_{12}=\frac{4}{3EI}$$

$$\Delta_{1P}=\frac{9P}{EI}\qquad \Delta_{2P}=\frac{4P}{3EI}$$



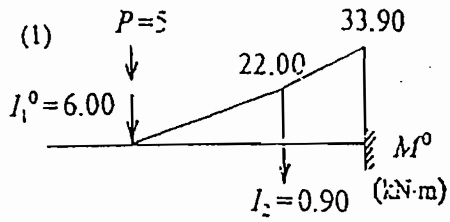


(1) $n=300$ 次/分，$\theta=10\pi$

代入式(16－66)并乘以 EI 有

$$\begin{cases}-7.6985\,I_1^0+1.3333\,I_2^0+45=0\\ 1.3333\,I_1^0-16.3652\,I_2^0+6.6667=0\end{cases}$$

$I_1^0=6.00\ \text{kN}\qquad I_2^0=0.90\ \text{kN}$

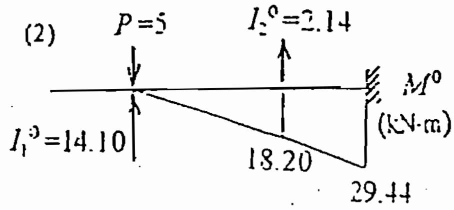


(2) $n=500$ 次/分，$\theta=\dfrac{50}{3}\pi=16.6667\pi$

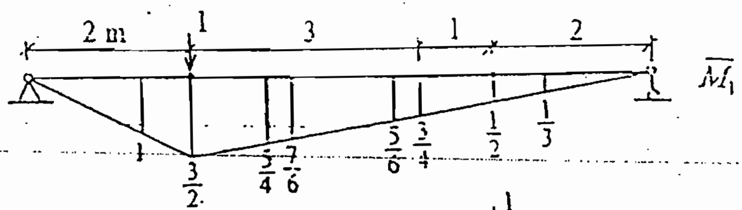
代入式(16－66)并乘以 EI 有

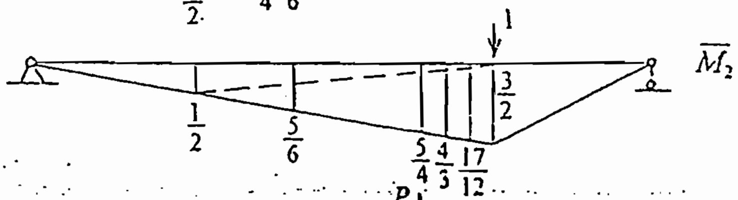
$$\begin{cases}2.98853\,I_1^0+1.3333\,I_2^0+45=0\\ 1.3333\,I_1^0-5.67814\,I_2^0+6.6667=0\end{cases}$$

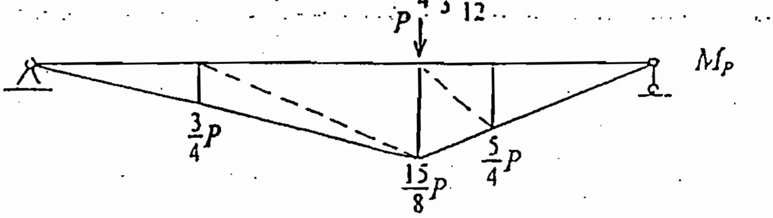
$I_1^0=-14.10\ \text{kN}\qquad I_2^0=-2.14\ \text{kN}$



16-17







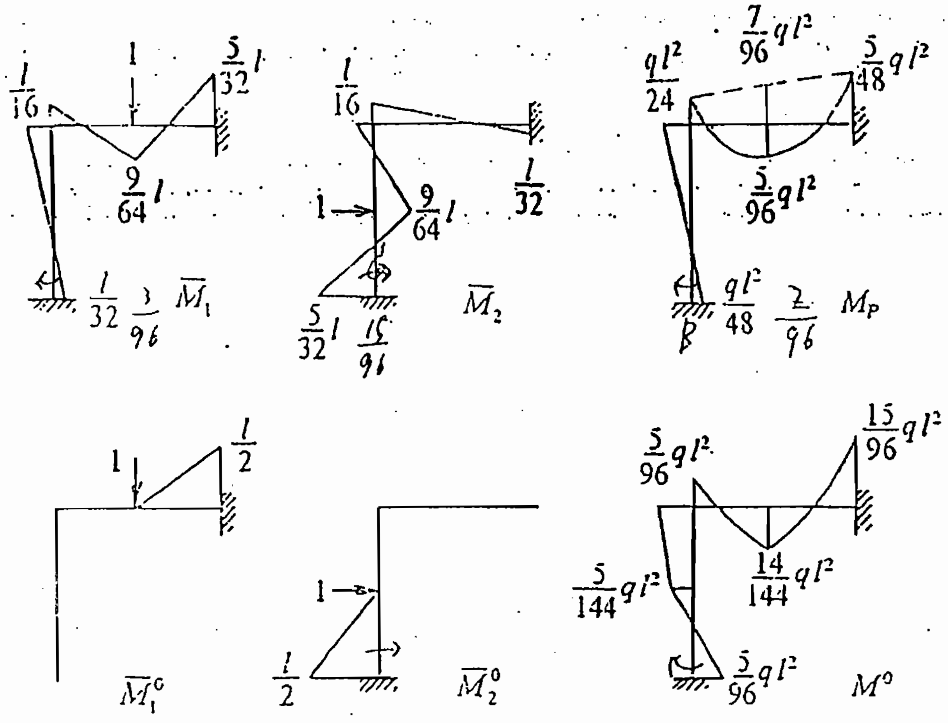
$$\delta_{11}=\delta_{22}=\frac{6}{EI}\qquad \delta_{12}=\frac{14}{3EI}\qquad \Delta_{1P}=\frac{51P}{8EI}\qquad \Delta_{2P}=\frac{175P}{24EI}$$

$\theta=30\ 1/\text{s}$，代入式(16－64)并乘以 EI 有

$$\begin{cases}-25.1009\,y_1^0+9.5141\,y_2^0+0.034=0\\ 9.5141\,y_1^0-25.1009\,y_2^0+0.038889=0\end{cases}$$

$y_1^0=2.268\times10^{-3}\ \text{m}\qquad y_2^0=2.409\times10^{-3}\ \text{m}$

16-18



$$\delta_{11}=\delta_{22}=\frac{11}{1536}\frac{l^3}{EI},\quad \delta_{12}=\delta_{21}=-\frac{3}{1536}\frac{l^3}{EI},\quad \Delta_{1P}=\frac{1}{256}\frac{ql^4}{EI},\quad \Delta_{2P}=-\frac{1}{768}\frac{ql^4}{EI}$$

$$\frac{1}{\theta^2}=\frac{ml^3}{48EI}$$

代入公式 (16－66) 并约去 $\frac{l^3}{EI}$ 后有

$$\begin{cases}\left(\frac{11}{1536}-\frac{1}{48}\right)I_1^0-\frac{3}{1536}I_2^0+\frac{ql}{256}=0\\ -\frac{3}{1536}I_1^0+\left(\frac{11}{1536}-\frac{1}{48}\right)I_2^0-\frac{ql}{768}=0\end{cases}$$

$$\begin{cases}-21I_1^0-3I_2^0+6ql=0\\ -3I_1^0-21I_2^0-21ql=0\end{cases}$$

解得　　$I_1^0=\frac{11}{36}ql,\quad I_2^0=-\frac{5}{36}ql$

动力弯矩幅值图可按迭加法

$$M^0=I_1^0\overline{M}_1+I_2^0\overline{M}_2+M_P$$

绘得如图。

B: $\frac{ql^2}{48}+\frac{l}{32}\cdot\frac{11}{36}ql-\frac{5}{32}l\cdot\frac{5}{36}ql$

$=\frac{ql^2}{48}\cdot\theta-\frac{14}{32\times36}$

16-19 用刚度法，由公式 (16-68)

$$(K-\theta^2 M)Y^0 = P$$

求解。今有

$$K=\frac{24EI}{l^3}\begin{bmatrix}6 & -2 & 0\\ -2 & 3 & -1\\ 0 & -1 & 1\end{bmatrix}$$ （见教材下册84页例16-5）

$$\frac{24EI}{l^3}=\frac{24\times5\times10^5}{5^3}=96\times10^3\ \mathrm{kN/m},\qquad \theta=\frac{2\pi\times240}{60}=8\pi,\qquad \theta^2=64\pi^2$$

$$M=100\begin{bmatrix}2 & 0 & 0\\ 0 & 1.5 & 0\\ 0 & 0 & 1\end{bmatrix}$$ （单位 t，即10^3 kg）

代入式（16-68）有

$$10^3\begin{bmatrix}449.669 & -192 & 0\\ -192 & 193.252 & -96\\ 0 & -96 & 32.835\end{bmatrix}\begin{Bmatrix}y_1^0\\ y_2^0\\ y_3^0\end{Bmatrix}=\begin{Bmatrix}0\\ 30\\ 0\end{Bmatrix}$$

解得

$$y_1^0=-0.0756\times10^{-3}\ \mathrm{m}$$
$$y_2^0=-0.1771\times10^{-3}\ \mathrm{m}$$
$$y_3^0=-0.5178\times10^{-3}\ \mathrm{m}$$

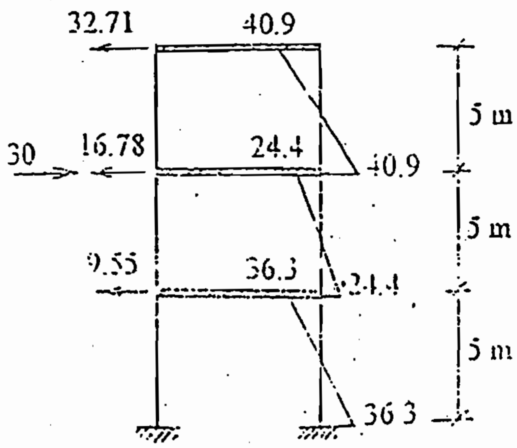
惯性力幅值为

$$I_1^0=200\times64\pi^2(-0.0756\times10^{-3})=-9.55\ \mathrm{kN}$$
$$I_2^0=150\times64\pi^2(-0.1771\times10^{-3})=-16.78\ \mathrm{kN}$$
$$I_3^0=100\times64\pi^2(-0.5178\times10^{-3})=-32.71\ \mathrm{kN}$$

本题横梁刚度为∞，每层只有两根柱且截面及高度相等，故每根柱之柱端弯矩为

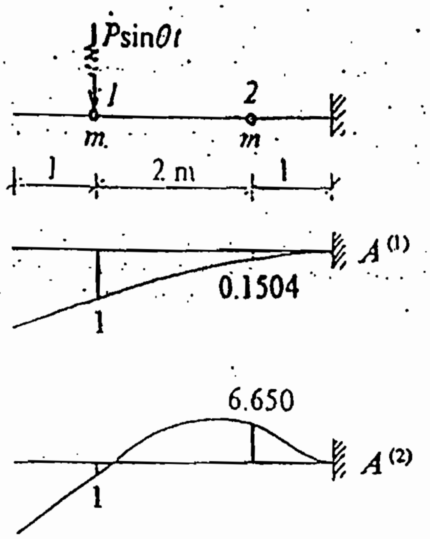
$$M_i=\frac{Q_i h}{4}$$

Q_i为该层之总剪力，等于该层以上水平外力（包括惯性力）之代数和；h为该层柱高。横梁之杆端弯矩可由刚结点力矩平衡推求。



一根柱的 M^0 图 (kN·m)

16-20 $EI\delta_{11}=9$ $EI\delta_{22}=1/3$ $EI\delta_{12}=4/3$



$$\lambda_{1,2}=\frac{1}{2}\left(\frac{28}{3}\pm\sqrt{\frac{740}{3}}\right)\frac{m}{EI}=\begin{Bmatrix}9.2005\\0.132843\end{Bmatrix}\frac{m}{EI}$$

$$\omega_{1,2}=\begin{Bmatrix}0.32968\\2.74366\end{Bmatrix}\sqrt{\frac{EI}{m}}=\begin{Bmatrix}42.32\\352.22\end{Bmatrix}1/s$$

$$\rho_1=\frac{9.2005-9}{4/3}=0.1504$$

$$\rho_2=\frac{0.13284-9}{4/3}=-6.650$$

即 $$A=\left[A^{(1)}\quad A^{(2)}\right]=\begin{bmatrix}1&1\\0.1504&-6.650\end{bmatrix}$$

$$\overline{M}_1=\begin{bmatrix}1&0.1504\end{bmatrix}\begin{bmatrix}m&0\\0&m\end{bmatrix}\begin{Bmatrix}1\\0.1504\end{Bmatrix}=1.0226m,\quad \overline{P}_1=\begin{bmatrix}1&0.1504\end{bmatrix}\begin{Bmatrix}P\sin\theta t\\0\end{Bmatrix}=P\sin\theta t$$

$$\overline{M}_2=\begin{bmatrix}1&-6.650\end{bmatrix}\begin{bmatrix}m&0\\0&m\end{bmatrix}\begin{Bmatrix}1\\-6.650\end{Bmatrix}=45.22m,\quad \overline{P}_2=\begin{bmatrix}1&-6.650\end{bmatrix}\begin{Bmatrix}P\sin\theta t\\0\end{Bmatrix}=P\sin\theta t$$

(1) $n=300$ 次/分，$\theta=10\pi$，因系简谐荷载，不计阻尼，参照式(16-20)可

$$\alpha_1=\frac{\overline{P}_1^0}{\overline{M}_1(\omega_1^2-\theta^2)}\sin\theta t=\frac{P}{1.0226m(42.32^2-100\pi^2)}\sin\theta t=1.989\times10^{-3}\sin\theta t$$

$$\alpha_2=\frac{\overline{P}_2^0}{\overline{M}_2(\omega_2^2-\theta^2)}\sin\theta t=\frac{P}{45.22m(352.22^2-100\pi^2)}\sin\theta t=0.294\times10^{-6}\sin\theta t$$

$$y_1=\alpha_1+\alpha_2=1.989\times10^{-3}\sin\theta t$$

$$y_2=0.1504\alpha_1-6.650\alpha_2=0.297\times10^{-3}\sin\theta t$$

惯性力幅值为

$$I_1^0=m_1\theta^2y_1^0=\frac{30}{9.81}\times100\pi^2\times1.989\times10^{-3}=6.00\text{ kN}$$

$$I_2^0=m_2\theta^2y_2^0=\frac{30}{9.81}\times100\pi^2\times0.297\times10^{-3}=0.90\text{ kN}$$

M^0图同题16-16。

(2) $n=500$ 次/分，$\theta=\frac{50}{3}\pi$，$\theta^2=277.78\pi^2$，代入按同上步骤可得

$\alpha_1=-1.682\times10^{-3}\sin\theta t$ $\alpha_2=0.298\times10^{-6}\sin\theta t$

$y_1=-1.682\times10^{-3}\sin\theta t$ $y_2=-0.255\times10^{-3}\sin\theta t$

$I_1^0=-14.10$ kN $I_2^0=-2.14$ kN

M^0图同题16-16。

16-21　由例 16-5（见教材下册 85 页）已有

$$\omega_1 = 19.40\ 1/s \qquad A^{(1)} = [1 \quad 2.608 \quad 4.290]^T$$

$$\omega_2 = 41.27\ 1/s \qquad A^{(2)} = [1 \quad 1.226 \quad -1.584]^T$$

$$\omega_3 = 60.67\ 1/s \qquad A^{(3)} = [1 \quad -0.834 \quad 0.294]^T$$

$$M = m\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{今有} \quad P = \begin{Bmatrix} 0 \\ P\sin\theta t \\ 0 \end{Bmatrix} \qquad \theta = 8\pi\ 1/s$$

可得

$$\overline{M}_1 = A^{(1)T}MA^{(1)} = 30.607m \qquad \overline{P}_1 = A^{(1)T}P = 2.608P\sin\theta t$$

$$\overline{M}_2 = A^{(2)T}MA^{(2)} = 6.7637m \qquad \overline{P}_2 = A^{(2)T}P = 1.226P\sin\theta t$$

$$\overline{M}_3 = A^{(3)T}MA^{(3)} = 3.1298m \qquad \overline{P}_3 = A^{(3)T}P = -0.834P\sin\theta t$$

参照式（16－20）可算出各正则坐标之幅值为

$$\alpha_1^0 = \frac{2.608P}{30.607m(19.40^2 - 64\pi^2)} = -0.10013 \times 10^{-3}\ \text{m}$$

$$\alpha_2^0 = \frac{1.226P}{6.7637m(41.27^2 - 64\pi^2)} = 0.050747 \times 10^{-3}\ \text{m}$$

$$\alpha_3^0 = \frac{-0.834P}{3.1298m(60.67^2 - 64\pi^2)} = -0.026217 \times 10^{-3}\ \text{m}$$

各质点位移幅值为

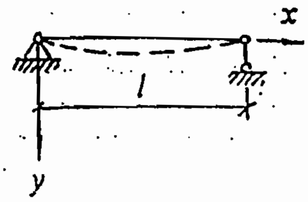
$$y_1^0 = \left[-0.10013 + 0.050747 + (-0.026217)\right]10^{-3}$$
$$= -0.0756 \times 10^{-3}\ \text{m}$$

$$y_2^0 = \left[2.608(-0.10013) + 1.266 \times 0.050747 - 0.834(-0.026217)\right]10^{-3}$$
$$= -0.1771 \times 10^{-3}\ \text{m}$$

$$y_3^0 = \left[4.290(-0.10013) - 1.584 \times 0.050747 + 0.294(-0.026217)\right]10^{-3}$$
$$= -0.5176 \times 10^{-3}\ \text{m}$$

以下计算同题16-19，略。

*16-22



$$y_0=0$$
$$M_0=0$$
$$y_l=0,\quad 有\quad EIy_0'\frac{1}{k}B_{kl}+Q_0\frac{1}{k^3}D_{kl}=0$$
$$M_l=0;\quad 有\quad EIy_0'kD_{kl}+Q_0\frac{1}{k}B_{kl}=0$$

y_0'、Q_0不全为零，应有

$$\begin{vmatrix}\frac{EI}{k}B_{kl} & \frac{1}{k^3}D_{kl}\\ EIkD_{kl} & \frac{1}{k}B_{kl}\end{vmatrix}=0$$

即
$$B_{kl}^2-D_{kl}^2=0$$
$$(B_{kl}+D_{kl})(B_{kl}-D_{kl})=0$$
$$\operatorname{sh}kl\sin kl=0$$

而 $\operatorname{sh}kl\neq0$，故

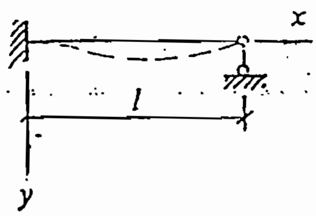
$$\sin kl=0$$

有 $$kl=i\pi\qquad(i=1,2,3,\cdots)$$

得 $$\omega_i=\frac{i^2\pi^2}{l^2}\sqrt{\frac{EIg}{q}}\qquad(i=1,2,3,\cdots)$$

振型 $$y^{(i)}=C\sin\frac{i\pi x}{l}\qquad(i=1,2,3,\cdots)$$

*16-23



$$y_0=0$$
$$y_0'=0$$
$$y_l=0,\quad 有\quad M_0\frac{1}{k^2}C_{kl}+Q_0\frac{1}{k^3}D_{kl}=0$$
$$M_l=0,\quad 有\quad M_0A_{kl}+Q_0\frac{1}{k}B_{kl}=0$$

M_0、Q_0不全为零，则上二式中它们的系数行列式等于零并展开得

$$C_{kl}B_{kl}=A_{kl}D_{kl}$$

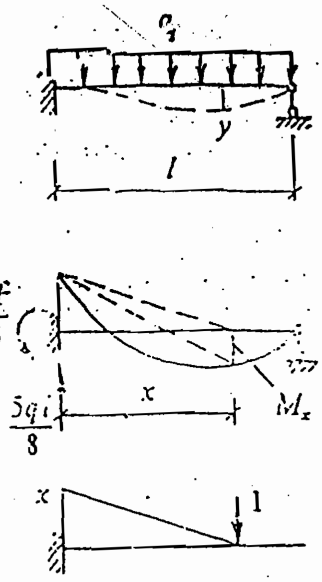
用试算法可解得非零最小正根为 $(kl)_1=3.927$，次小正根为 $(kl)_2=7.069$，由

$$\omega=\frac{(kl)^2}{l^2}\sqrt{\frac{EIg}{q}}$$

得 $$\omega_1=\frac{15.42}{l^2}\sqrt{\frac{EIg}{q}}\qquad\omega_2=\frac{49.97}{l^2}\sqrt{\frac{EIg}{q}}$$

16-24　均布自重下的弯矩方程为

$$M_x=-\frac{ql^2}{8}+\frac{5qlx}{8}-\frac{qx^2}{2}$$



由图乘法求挠曲线方程：

$$y=\frac{1}{EI}\left[\frac{ql^2}{8}\cdot\frac{x}{2}\cdot\frac{2x}{3}-\left(-\frac{ql^2}{8}+\frac{5qlx}{8}-\frac{qx^2}{2}\right)\frac{x}{2}\cdot\frac{x}{3}-\frac{2}{3}\cdot\frac{qx^2}{8}\cdot x\cdot\frac{x}{2}\right]$$

$$=\frac{q}{48EI}[3l^2x^2-5lx^3+2x^4]$$

$$\int_0^l qy\,dx=\frac{q^2}{48EI}\int_0^l(3l^2x^2-5lx^3+2x^4)\,dx$$

$$=\frac{q^2l^5}{48EI}\left(\frac{3}{3}-\frac{5}{4}+\frac{2}{5}\right)=\frac{q^2l^5}{320EI}=3.125\times10^{-3}\,\frac{q^2l^5}{EI}$$

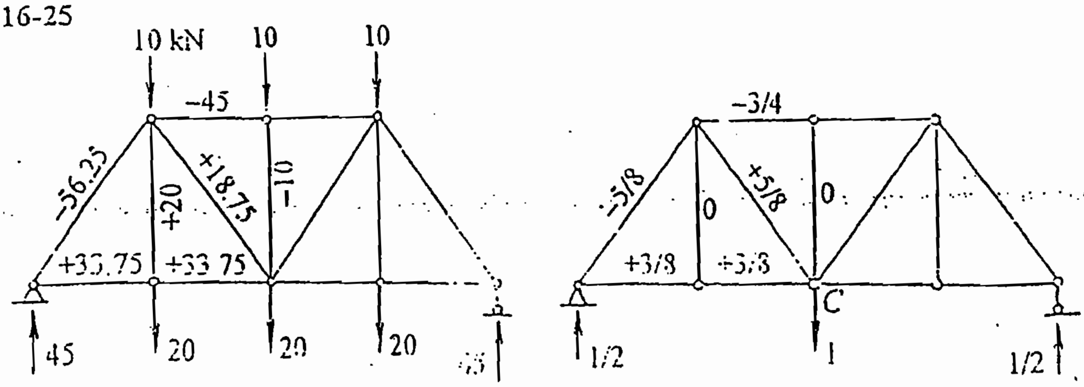
$$\int_0^l my^2\,dx=\frac{mq^2}{(48EI)^2}\int_0^l(3l^2x^2-5lx^3+2x^4)^2\,dx$$

$$=\frac{mq^2}{(48EI)^2}\int_0^l(9l^4x^4-25l^2x^6+4x^8-30l^3x^5-20lx^7+12l^2x^6)\,dx$$

$$=\frac{mq^2l^9}{(48EI)^2}\left(\frac{9}{5}+\frac{25}{7}+\frac{4}{9}-\frac{30}{6}-\frac{20}{8}+\frac{12}{7}\right)=\frac{mq^2l^9}{(48EI)^2}\cdot\frac{19}{630}=1.309\times10^{-5}\,\frac{mq^2l^9}{(EI)^2}$$

$$\omega^2=\frac{\int_0^l qy\,dx}{\int_0^l my^2\,dx}=238.7\,\frac{EI}{ml^4}\qquad \omega=\frac{15.45}{l^2}\sqrt{\frac{EIg}{q}}$$

*16-25



$$\Delta_{CP}=\frac{2}{EA}\left(56.25\times\frac{5}{8}\times5+45\times\frac{3}{4}\times3+33.75\times\frac{3}{8}\times6+18.75\times\frac{5}{8}\times5\right)$$

$$=\frac{823.125}{EA}=\frac{823.125}{210\times10^9\times10^{-3}\times2\times10^{-3}}=1.96\times10^{-3}\ \text{m}$$

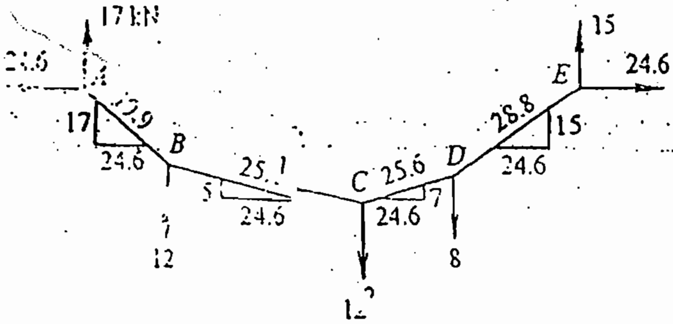
$$\omega=1.13\sqrt{\frac{g}{\Delta_{CP}}}=1.13\sqrt{\frac{9.81}{1.96\times10^{-3}}}=79.9\ 1/\text{s}$$

*第十七章 悬索计算

17-1

$$V_A = V_A^0 = 17\ \text{kN}$$
$$V_E = V_E^0 = 15\ \text{kN}$$
$$H = \frac{M_C^0}{f} = \frac{17\times 35 - 12\times 25}{12}$$
$$= 24.6\ \text{kN}$$

各索段内力见图。



17-2

$$\Sigma M_A = 0 \qquad H = 45\ \text{kN}$$

(a) 悬链线

$$\beta = \frac{0.135\times 20}{2\times 45} = 0.03$$

$$\alpha = \text{sh}^{-1}\left(\frac{0.03\times\frac{6}{20}}{\text{sh}\,0.03}\right) + 0.03 = 0.325630$$

$$y = \frac{45}{0.135}\left[\text{ch}\,\alpha - \text{ch}\left(\frac{2\beta}{l}x - \alpha\right)\right] = 351.1625 - \frac{45}{0.135}\text{ch}(0.003x - \alpha)$$

$$y' = -\text{sh}(0.003x - \alpha)$$

$$x = 0,\quad y' = y'_{\max} = -\text{sh}(-\alpha) = 0.331415$$

$$T_{\max} = H\sqrt{1+(y'_{\max})^2} = 45\sqrt{1+(0.331415)^2} = 47.407\ \text{kN}$$

(b) 抛物线

$$q_y = 0.135\frac{\sqrt{20^2+6^2}}{20} = 0.141\ \text{kN/m}$$

$$f = \frac{0.141\times 20^2}{8\times 45} = 0.1566\ \text{m}$$

$$y'_{\max} = \frac{c+4f}{l} = \frac{6+4\times 0.1566}{20} = 0.331321$$

$$T_{\max} = H\sqrt{1+(y'_{\max})^2} = 45\sqrt{1+(0.331321)^2} = 47.406\ \text{kN}$$