32B CHALLENGE PROBLEM REPORT 1

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ABSTRACT. In this challenge problem report, we strive to further develop our understandings of triple integrals by applying the principles of calculating volumes and masses in real life circumstances. We accomplish the modelling of a region inside the sphere but outside the central cylinder's mass throguh solving a series of practical sub-problems.

1. INTRODUCTION

Multi-variable Integral Calculus

Multi-variable calculus can be categorised as having 6 broad generalizations for the 1-variable integral of scalar-valued functions. Fluency in multi-variable integral calculus can give one the ability to integrate functions over 1D curves, 2D planar surfaces, as well as 3D volumes and much beyond. The 6 categorisations include: the line integral of scalar-valued function, line integral of vector fields, surface integral of scalar-valued functions, surface integral of vector fields, double integrals, and triple integrals. I imagine we will likely encounter all, if not even more, of the forms throughout our study of the 32B Multi-variable Calculus course. [1]

Today, we are particularly concerned with the practical applications of triple integrals in the context of volume and masses.

2. PROBLEM 1

We begin straight away by considering the region outside the central cylinder but inside of the sphere. First, we imagine there to exist a sphere of radius R centered at origin; then, for $R > r_0 > 0$, we imagine a central cylinder defined by the equation $x^2 + y^2 = r_0^2$.

The two regions are defined as follows:

$$\begin{cases} x^2 + y^2 + z^2 = R^2 \\ x^2 + y^2 = r_0^2 \end{cases} \tag{2.1}$$

It is important to immediately sketch the region to gain an intuition of the problem before we proceed further:

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The author would like to thank Professor Wong for an opportunity to learn LaTeX and investigate/formulate a mathematical problem in this format.

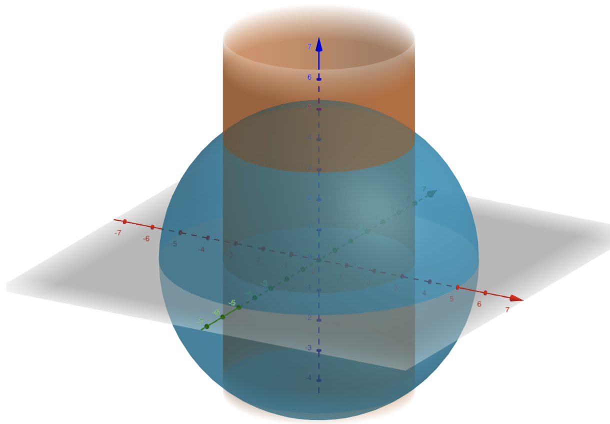

FIGURE 1. The 3D region: sphere with a central cylinder drilled through the inside. [2]

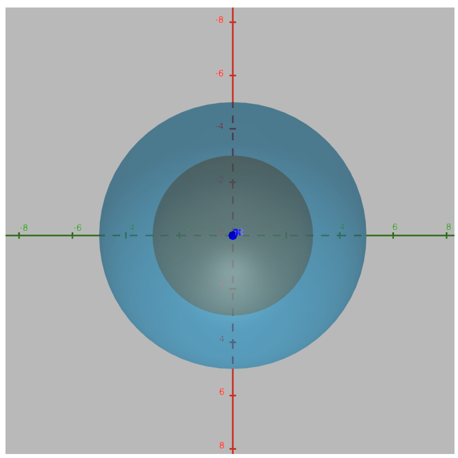

FIGURE 2. The above region in the X-Y Plane. We omit the X-Z and Y-Z cases as they are the same and basically identical to the previous figure[2]

Hence, through observation of the bounds, we can describe the region W with which we integrate over. Before that, it is perhaps necessary to introduce the concept of triple integrals and how they enable us to calculate volumes of 3D regions.

Theorem 2.1. ***(Definitions of Triple Integrals)***

The triple integrals of functions $f(x, y, z)$ are simply generalization of double integrals. For a 3D region, we break it down into infinitesimally small boxes. For example, if the region has the following bounds:

$$\mathfrak{B} = [a, b] \times [c, d] \times [p, q]$$

which is equivalent to:

$$a \leq x \leq b$$

$$c \leq y \leq d$$
$$p \leq z \leq q$$

We can then subdivide the box into subboxes:

$$\mathfrak{B}_{\mathfrak{ijk}} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

Where each x_i, y_j, z_k are bounded in between the respective letters $[a,b], [c,d], [p,q]$, and the difference between two adjacent values is defined as $\Delta x_i, \Delta y_i, \Delta z_k$ respectively.

We can then form the Riemann sum with each sample point P_{ijk}:

$$S_{N,M,L} = \sum_{i=1}^{N} \sum_{j=1}^{M} \sum_{k=1}^{L} f(P_{ijk}) \Delta x_i \Delta y_j \Delta z_k$$

*If the sum $S_{N,M,L}$ approaches a limit as $|\mathfrak{P}| \to 0$ (for arbitrary sample point choices), then f is **integrable** over $\mathfrak{B}$, where:*

$$\iiint_{\mathfrak{B}} f(x,y,z)\, dV = \lim_{|\mathfrak{P}| \to 0} S_{N,M,L}$$

More important and relevant to our results, the volume V of a region $\mathfrak{W}$ is defined as the triple integral of the constant function $f(x,y,z) = 1$:

$$V = \iiint_{\mathfrak{W}} 1\, dV \tag{2.2}$$

This is very similar to how the area of a region is given by the double integral $A = \iint_{\mathfrak{R}} 1\, dA$.

It is also important to know how to calculate the integral when x, y or z are bounded by functions.

Theorem 2.2. ***(Iterated Integral)***

The triple integral of continuous f over the region

$$\mathfrak{W} : (x,y) \subseteq \mathfrak{D}$$
$$z_1(x,y) \leq z \leq z_2(x,y)$$

is equivalent to the iterated integral

$$\iiint_{\mathfrak{W}} f(x,y,z)\, dV = \iint_{\mathfrak{D}} (\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\, dz)\, dA \tag{2.3}$$

Knowing this, we may begin partaking on the journey of identifying the bounds for the drilled sphere.

For this question, it is easier to utilise the spherical coordinates since the region is centrally simple. This then requires us to introduce polar and spherical coordinates.

Theorem 2.3. ***(Definition of Polar Coordinates)***

The polar coordinates are typically more useful in situations when the function is better modelled circular than rectangular. For each point,

$$P = \begin{cases} (x, y)\ (rectangular) \\ (r, \theta)\ (polar) \end{cases} \tag{2.4}$$

To convert from Polar to Rectangular:

$$x = r\ cos\,\theta$$
$$y = r\ sin\,\theta$$

To convert from Rectangular to Polar:

$$r = \sqrt{x^2 + y^2}$$
$$tan\,\theta = \frac{y}{x}\ (x \neq 0)$$

Similar to rectangular coordinates, we can also integrate in polar coordinates. This is usually simpler when the domain is an angular sector, or, as the textbook preferred, as ***polar rectangle****:*

$$\mathfrak{R} = \begin{cases} \theta_1 \leq \theta \leq \theta_2 \\ r_1 \leq r \leq r_2 \end{cases}$$

We assume $r_1/ge0$ and all coordinates are non-negative. Using the Change of Variables Formula for a polar rectangle $\mathfrak{R}$ is:

$$\iint_{\mathfrak{R}} f(x, y)\, dA = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(rcos\,\theta, rsin\,\theta) r\ dr\ d\theta \tag{2.5}$$

Theorem 2.4. ***(Definition of Cylindrical Coordinates)***

Cylindrical Coordinates are more or less polar coordinates in 3D.

$$P = \begin{cases} (x, y, z)\ (rectangular) \\ (r, \theta, z)\ (cylindrical) \end{cases}$$

To generalise, for continuous function f on the region:

$$\mathfrak{W} = \begin{cases} \theta_1 \leq \theta \leq \theta_2 \\ r_1(\theta) \leq r \leq r_2(\theta) \\ z_1(r,\theta) \leq z \leq z_2(r,\theta) \end{cases}$$

The integral:

$$\iiint_{\mathfrak{W}} f(x,y,z)\, dV = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2}\int_{z_1(r\ \theta)}^{z_2(r,\ \theta)} f(rcos\,\theta, rsin\,\theta, z) r\, dz\, dr\, d\theta \tag{2.6}$$

With the theorem and definitions being laid out as groundwork for our future endeavours, we can start by describing the region W which we will integrate over to find the volume of the region.

A reminder for (2.1)/bounds as given in the question:

$$\begin{cases} x^2 + y^2 + z^2 = R^2 \\ x^2 + y^2 = r_0^2 \end{cases}$$

According to the two earlier pictures and (2.1), we can discern the θ covers the entirety of the sphere so it will be from 0 to 2π. r is given in the question to be between r_0 and R. We then need to do a bit of work for z. Since the cylinder is drilled through the sphere, the limiting z coordinates will be from the sphere itself. From (2.1), we can find z in the following way:

$$\begin{aligned} &z^2 = R^2 - x^2 - y^2 \\ \Rightarrow &z = \sqrt{R^2 - (x^2 + y^2)} \\ \Rightarrow &\sqrt{R^2 - (x^2 + y^2)} \leq z \leq \sqrt{R^2 - (x^2 + y^2)} \\ \Rightarrow &\sqrt{R^2 - r^2} \leq z \leq \sqrt{R^2 - r^2} \end{aligned}$$

The bounds are then clearly defined as:

$$\begin{cases} 0 \leq \theta \leq 2\pi \\ r_0 \leq r \leq R \\ -\sqrt{R^2 - r^2} \leq z \leq \sqrt{R^2 - r^2} \end{cases}$$

2.1. **Result 1(A)**

With the bounds being clearly defined, and combining how the volume is represented with spherical coordinates, we can present the integral as:

$$\iiint_{\mathfrak{W}} 1\, dV = \boxed{\int_0^{2\pi}\int_{r_0}^{R}\int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} r\, dz\, dr\, d\theta} \tag{2.7}$$

After defining the integral, we are left to calculate it.

$$
\begin{aligned}
\int_0^{2\pi}\int_{r_0}^{R}\int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} r\,dz\,dr\,d\theta &= \int_0^{2\pi}\int_{r_0}^{R} r\sqrt{R^2-r^2}-(-\sqrt{R^2-r^2})\,dr\,d\theta \\
&= 2\int_0^{2\pi}\int_{r_0}^{R} r(R^2-r^2)^{\frac{1}{2}}\,dr\,d\theta \\
&= 2\int_0^{2\pi} \frac{-1}{2}\frac{2}{3}(R^2-r^2)^{\frac{3}{2}}\Big|_{r_0}^{R}\,d\theta \\
&= \frac{-2}{3}\int_0^{2\pi} (R^2-r^2)^{\frac{3}{2}}\Big|_{r_0}^{R}\,d\theta \\
&= \frac{-2}{3}\int_0^{2\pi} (R^2-R^2)^{\frac{3}{2}}-(R^2-r_0^2)^{\frac{3}{2}}\,d\theta \\
&= \frac{2}{3}(R^2-r_0^2)^{\frac{3}{2}}\int_0^{2\pi} 1 d\theta \\
&= \frac{4\pi}{3}(R^2-r_0^2)^{\frac{3}{2}}
\end{aligned}
$$

2.2. **Result 1(B)**

The volume of the solid region $\mathfrak{W}$ is thus

$$\boxed{\frac{4\pi}{3}(R^2-r_0^2)^{\frac{3}{2}}} \tag{2.8}$$

3. Problem 2

For the second problem we will using the expression of the integral in P1 for investigating real world problems.

First, we should find an expression for the height of the solid region $\mathfrak{W}$.

This is a simple geometric problem. We first sketch the graph in 2D Y-Z Plane, at $x = 0$.

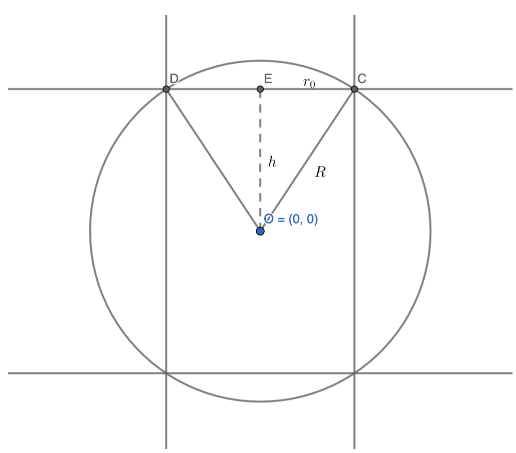

Figure 3. The geometric problem in Y-Z Plane. [2]

As illustrated here, we want to find the value of $2h$ which represents the height of the solid region.
By Pythagoras, since h is in a right triangle:

$$\begin{aligned} h_{\mathfrak{W}} &= 2h \\ &= \boxed{2\sqrt{R^2 - r_0^2}} \end{aligned}$$

3.1. **Result 2(A)**

We can then re-write the previous volume expression in temrs of h.

$$\begin{aligned} V_{\mathfrak{W}} &= \frac{4\pi}{3}(R^2 - r_0^2)^{\frac{3}{2}} \\ &= \frac{\pi}{6} \times 8(R^2 - r_0^2)^{\frac{3}{2}} \\ &= \frac{\pi}{6} \times (2(R^2 - r_0^2)^{\frac{1}{2}})^3 \\ &= \boxed{\frac{\pi}{6}h^3} \end{aligned}$$

After acquiring the volume expression in terms of h only, we can start investigating real life issues. For a sphere of radius 2 meters and a central cylinder of radius 1 meter, we can easily find the remaining region $\mathfrak{B}$ using the integrals we already have.

3.2. **Result 2(B)**

We have $R = 2, r_0 = 1$, therefore,

$$\begin{aligned} h &= 2\sqrt{2^2 - 1^1} = 2\sqrt{3} \\ \Rightarrow V_{\mathfrak{W}} &= \frac{\pi}{6} \times (2\sqrt{3})^3 \\ &= \boxed{4\sqrt{3}\pi} \end{aligned}$$

Finally, we turn our attention to a UCLA-specific question. If we imagine the entire campus is contained inside a sphere with $R = 1km$, and we also cut a cylinder out, it would be very interesting to find out the volume of the remaining region. Instead of the inner radius r_0, we are given height of the region $\mathfrak{C}$ and asked to find r_0.

Similar to before, this can be achieved in a fairly straightforward manner.

3.3. **Result 2(C)**

$$
\begin{aligned}
h &= 2\sqrt{R^2 - r_0^1} = 2\sqrt{3} \\
\Rightarrow R^2 - r_0^2 &= 3 \\
\Rightarrow r_0 &= \sqrt{R^3 - 3} \\
&= \sqrt{1000^3 - 3} \\
&= \boxed{999.9985m}
\end{aligned}
$$

4. Problem 3

Next, we move on to investigating regions of different size and radius but with the same density function. For the regions from 2(B) and 2(C), we are investigating whether they have the same mass.

Again, our lives will be made a great deal more expedient if we utilise the explanatory power of pictures to our aid.

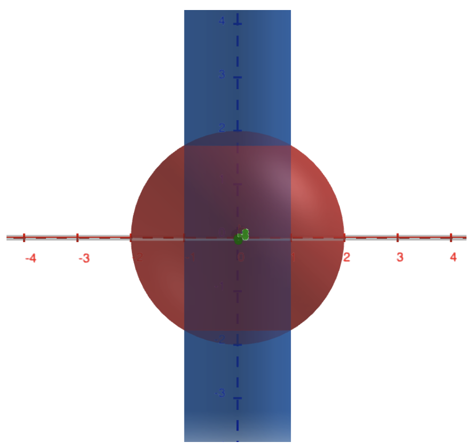

Figure 4. B: The region from (2b). $R = 4$, $r_0 = 1$m, $h = 2\sqrt{3}$m. [2]

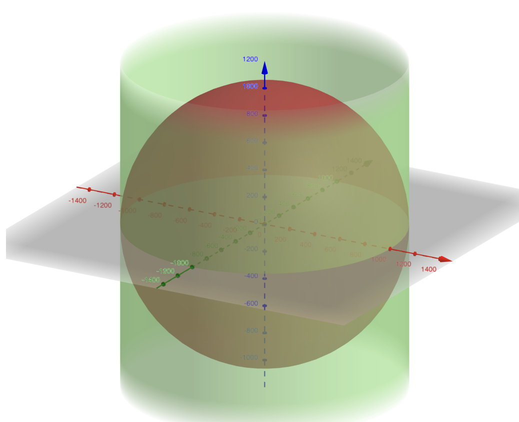

Figure 5. C: The region from (2c). $R = 1000$, $r_0 = 999.9985$m, $h = 2\sqrt{3}$m. [2]

As illustrated by the scale, despite the region $\mathfrak{W}$ sharing the same height $h = 2\sqrt{3}$m in both graphs, the region themselves are vastly different. If they have the same density $\delta(r, \theta, z) = \frac{1}{r}$, we can still investigate whether the masses are the same.

So far, we have only dealt with volumes before, so it is worthwhile to first examine the definition of mass in the context of 3D regions.

Theorem 4.1. ***(Applications of Multiple Integrals - Mass)***

Unlike volume, how mass is distributed depends on the density function. The total amount of a quantity in $\mathfrak{R}^3$ is defined as the triple integral:

$$mass = total\ amount = \iiint_{\mathfrak{W}} \delta(x, y, z)\, dV \tag{4.1}$$

where δ represents the density function per unit volume.

The only difference between volume and mass is the replacement of the density function as the integral rather than 1. We will first integrate region B, with $\delta(r, \theta, z) = \frac{1}{r}$.

$$\begin{aligned} B_{\mathfrak{M}} &= \int_0^{2\pi}\int_1^2\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} \frac{1}{r} r\, dV \\ &= \int_0^{2\pi}\int_1^2\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} 1\, dz\, dr\, d\theta \\ &= \int_0^{2\pi}\int_1^2 2(4-r^2)^{\frac{1}{2}}\, dr\, d\theta \end{aligned}$$

Here we can use a trig substitution, where $r = 2cos(\phi)$, $\phi = cos^{-1}(\frac{r}{2})$, and $dr = -2sin(\phi)$. For brevity, we will also examine the inner integral

$$\int_1^2 2(4-r^2)^{\frac{1}{2}}\, dr$$

first, before returning to the actual mass calculation with the remaining double integral.

We first have

$$\begin{aligned} \sqrt{4-r^2} &= \sqrt{4-4cos^2\phi} \\ &= 2\sqrt{1-cos^2\phi} \\ &= 2sin\phi \end{aligned}$$

which is much easier to integrate.

$$
\begin{aligned}
\int_1^2 2(4-r^2)^{\frac{1}{2}}\,dr &= \int_{\arccos\frac{1}{2}}^{\arccos 1} 4\sin\phi\,dr \\
&= \int_{\arccos\frac{1}{2}}^{\arccos 1} 4\sin\phi(-2\sin\phi)\,d\phi \\
&= -8\int_{\arccos\frac{1}{2}}^{\arccos 1} \sin^2\phi\,d\phi \\
&= -8\int_{\arccos\frac{1}{2}}^{\arccos 1} \frac{1}{2}(1-\cos 2)\phi\,d\phi \\
&= -4(\phi-\frac{1}{2}\sin 2\phi\ \Big|_{\arccos\frac{1}{2}}^{\arccos 1}) \\
&= -4(\arccos 1-\frac{1}{2}\sin(2\arccos 1)-\arccos\frac{1}{2}+\frac{1}{2}\sin(2\arccos\frac{1}{2}) \\
&= -4(0-\frac{1}{2}\sin 0-\frac{\pi}{3}+\frac{1}{2}\sin\frac{2\pi}{3}) \\
&= -4(\frac{-\pi}{3}+\frac{\sqrt{3}}{4}) \\
&= \frac{4}{3}\pi-\sqrt{3}
\end{aligned}
$$

Then, combining with the original double integral:

$$
\begin{aligned}
\int_0^{2\pi}\frac{4}{3}\pi-\sqrt{3}\,d\theta &= (\frac{4}{3}\pi-\sqrt{3})2\pi \\
&\approx \boxed{15.436}
\end{aligned}
$$

Using the same approach, we arrive at a similar integral for region C.

$$
\begin{aligned}
C_{\mathfrak{M}} &= \int_0^{2\pi}\int_{\sqrt{999997}}^{1000}\int_{-\sqrt{1000^2-r^2}}^{\sqrt{1000^2-r^2}} 1r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_{\sqrt{999997}}^{1000} 2\sqrt{1000^2-r^2}\,dr\,d\theta \\
&= -1000^2\int_0^{2\pi}(\phi-\frac{1}{2}\sin 2\phi\ \Big|_{\arccos\frac{\sqrt{999997}}{1000}}^{\arccos 1})\,d\theta \\
&\approx 0.003464\times 2\pi \\
&\approx \boxed{0.02177}
\end{aligned}
$$

4.1. **Result 3(A)**

It is clear that $\boxed{15.436 \neq 0.02177}$, hence the regions do not have the same mass.

Finally, and perhaps most interestingly, we are asked to find a non-constant density function such that the masses are indeed the same. This time, both regions are centered at the origin. P drawing them together would provide us with insights on how to approach this challenge.

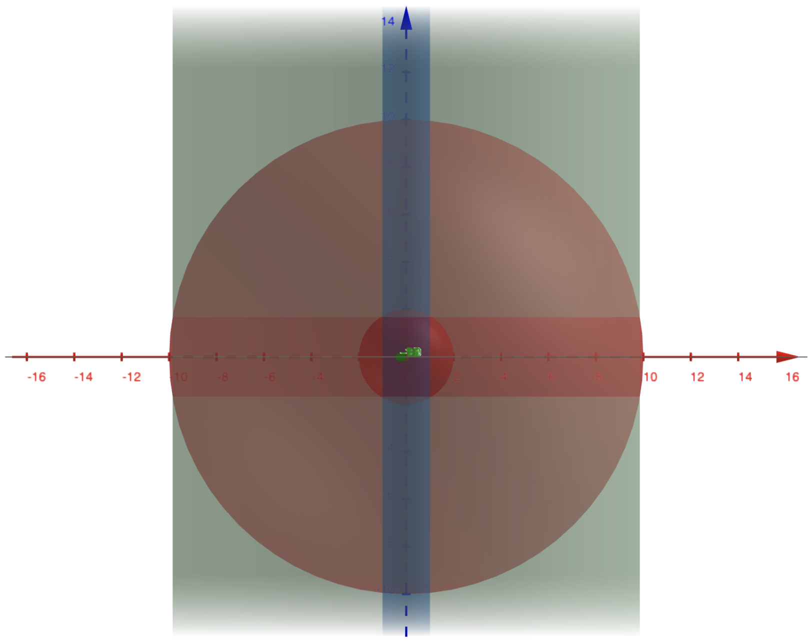

FIGURE 6. Both regions together: note, not up to scale, for otherwise shape B would not be discernible. [2]

One important detail here is the horizontal red shade across the center - this represents the height of the region between both spheres and their respective central cylinders. As can be seen the height is the same for both.

Demonstrated by the previous problem, with a density function of $\delta(r, \theta, z) = \frac{1}{r}$, shape B has a much greater mass than C. This is intuitive, for the density function suggests as $r \to \infty$, $\delta \to 0$. while as $r \to 0$, $\delta \to \infty$.

Hence being closer to the center as a clear bias of having a greater mass. Consequentially, we should adjust our density function to favour being further away from the center to make the masses equivalent.

For starters, we can experiment with $\delta(r, \theta, z) = 1$, or $\delta(r, \theta, z) = r$ instead of $\frac{1}{r}$. This means the further away the region is from center, the greater its mass. Then, since it is an exploratory problem, we can experiment with computational calculators such as Wolfram Alpha to see which density functions actually do the job. We can also omit the last integral with respect to θ as it is unlikely the density function depends on θ. It should only depend on r, and the last integral does nothing but multiplying the expression with 2π.

It turns out our first guess happens to be perfectly right:

$$\int_1^2 2r(4 - r^2)^{\frac{1}{2}}\, dr = \int_{\sqrt{999997}}^{1000000} 2r(1000^2 - r^2)^{\frac{1}{2}}\, dr = 2\sqrt{3}$$

This also means any other density function such as $\delta(r, \theta, z) = r$ would favour the perimeter too much and the mass of region C would instead supersede region B.

4.2. **Result 3(B)**

The density function we want is therefore

$$\boxed{\delta(r, \theta, z) = 1} \tag{4.2}$$

References

[1] Nykamp, D. (n.d.). The integrals of multivariable calculus. Math Insight.https://mathinsight.org/integrals$_{m}ultivariable_{c}alculus_{s}ummary$
[2] GeoGebra 3D Graphing Software, https://www.geogebra.org/3d

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