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Abstract

1 Problem

$$AX - XB = C; A, B, C, X \in \mathbb{R}^{nxn} \tag{1}$$

Check correctness with:

$$||AX - XB - C||_{\mathbb{F}} = 0 \tag{2}$$

Steps:
You can decompose B as follows:

$$B = QTQ^T; QQ^T = I, T \in \text{upper triang} \tag{3}$$

Then:

$$AX - XB = C = AX - XQTQ^T \tag{4}$$
$$(AX - XQTQ^T)Q = CQ \tag{5}$$
$$= AXQ - XQTQ^TQ = CQ \tag{6}$$
$$= AXQ - XQT = CQ \tag{7}$$
$$\tag{8}$$

If we consider $XQ = Y$ and $CQ = D$, we get:

$$AY - YT = D \tag{9}$$

At this point, there are two options:

1. T is upper triangular.
2. T is in Schur form.

1.1 Option 1: T is upper triangular

In the first case:

$$AY_{\cdot,1} - Y_{\cdot,1}t_{1,1} = D_1 \tag{10}$$
$$(A - t_{1,1}I)Y_{\cdot,1} = D_1 \tag{11}$$
$$\tag{12}$$

Where $(A - t_{1,1}I)$ is a known matrix, and then it becomes:

$$Zx = b \tag{13}$$

Where $Z = (A - t_{1,1}I)$, $x = Y_{\cdot,1}$ and $b = D_1$.
After solving it, we can continue as follows:

$$AY_{\cdot,2} - t_{1,2}Y_{\cdot,1} - t_{2,2}Y_{\cdot,2} = D_2 \tag{14}$$
$$(A - t_{2,2}I)Y_{\cdot,2} = D_2 + t_{1,2}Y_{\cdot,1} \tag{15}$$
$$\tag{16}$$

Where $(A - t_{2,2}I)$ is a known matrix, and then it becomes:
TO DO

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} \begin{pmatrix} y_{1,1} & y_{1,2} & y_{1,3} & y_{1,4} \\ y_{2,1} & y_{2,2} & y_{2,3} & y_{2,4} \\ y_{3,1} & y_{3,2} & y_{3,3} & y_{3,4} \\ y_{4,1} & y_{4,2} & y_{4,3} & y_{4,4} \end{pmatrix} - \begin{pmatrix} y_{1,1} & y_{1,2} & y_{1,3} & y_{1,4} \\ y_{2,1} & y_{2,2} & y_{2,3} & y_{2,4} \\ y_{3,1} & y_{3,2} & y_{3,3} & y_{3,4} \\ y_{4,1} & y_{4,2} & y_{4,3} & y_{4,4} \end{pmatrix} \begin{pmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} \\ 0 & t_{2,2} & t_{2,3} & t_{2,4} \\ 0 & 0 & t_{3,3} & t_{3,4} \\ 0 & 0 & 0 & t_{4,4} \end{pmatrix} = \begin{pmatrix} d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} \\ d_{2,1} & d_{2,2} & d_{2,3} & d_{2,4} \\ d_{3,1} & d_{3,2} & d_{3,3} & d_{3,4} \\ d_{4,1} & d_{4,2} & d_{4,3} & d_{4,4} \end{pmatrix} \tag{17}$$

$$AY_{\cdot,1} - Y_{\cdot,1} \begin{pmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} \\ 0 & t_{2,2} & t_{2,3} & t_{2,4} \\ 0 & t_{3,2} & t_{3,3} & t_{3,4} \\ 0 & 0 & 0 & t_{4,4} \end{pmatrix} = D_{\cdot,1} \tag{18}$$

$$AY_{\cdot,1} - Y_{\cdot,1} t_{1,1} = D_{\cdot,1} \tag{19}$$

$$\begin{pmatrix} y_{1,1} & y_{1,2} & y_{1,3} & y_{1,4} \\ y_{2,1} & y_{2,2} & y_{2,3} & y_{2,4} \\ y_{3,1} & y_{3,2} & y_{3,3} & y_{3,4} \\ y_{4,1} & y_{4,2} & y_{4,3} & y_{4,4} \end{pmatrix} \begin{pmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} \\ 0 & t_{2,2} & t_{2,3} & t_{2,4} \\ 0 & t_{3,2} & t_{3,3} & t_{3,4} \\ 0 & 0 & 0 & t_{4,4} \end{pmatrix} = \begin{pmatrix} y_{1,1}t_{1,1} & y_{1,1}t_{1,2} + y_{1,2}t_{2,2} + y_{1,3}t_{3,2} & t_{1,3} & t_{1,4} \\ 0 & y_{2,1}t_{1,2} + y_{2,2}t_{2,2} + y_{2,3}t_{3,2} & t_{2,3} & t_{2,4} \\ 0 & y_{3,1}t_{1,2} + y_{3,2}t_{2,2} + y_{3,3}t_{3,2} & t_{3,3} & t_{3,4} \\ 0 & y_{4,1}t_{1,2} + y_{4,2}t_{2,2} + y_{4,3}t_{3,2} & 0 & t_{4,4} \end{pmatrix} \tag{20}$$

$$AY_{\cdot,2} - Y_{\cdot,1} \begin{pmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} \\ 0 & t_{2,2} & t_{2,3} & t_{2,4} \\ 0 & t_{3,2} & t_{3,3} & t_{3,4} \\ 0 & 0 & 0 & t_{4,4} \end{pmatrix} = D_{\cdot,1} \tag{21}$$

$$\begin{pmatrix} y_{1,1} & y_{1,2} & y_{1,3} & y_{1,4} \\ y_{2,1} & y_{2,2} & y_{2,3} & y_{2,4} \\ y_{3,1} & y_{3,2} & y_{3,3} & y_{3,4} \\ y_{4,1} & y_{4,2} & y_{4,3} & y_{4,4} \end{pmatrix} \begin{pmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} \\ 0 & t_{2,2} & t_{2,3} & t_{2,4} \\ 0 & t_{3,2} & t_{3,3} & t_{3,4} \\ 0 & 0 & 0 & t_{4,4} \end{pmatrix} = \begin{pmatrix} y_{1,1}t_{1,1} & \cdots & y_{1,1}t_{1,3} + y_{2,2}t_{2,3} + y_{3,3}t_{3,3} & t_{1,4} \\ 0 & \cdots & y_{2,1}t_{1,3} + y_{2,2}t_{2,3} + y_{3,3}t_{3,3} & t_{2,4} \\ 0 & \cdots & y_{3,1}t_{1,3} + y_{2,2}t_{2,3} + y_{3,3}t_{3,3} & t_{3,4} \\ 0 & \cdots & y_{4,1}t_{1,3} + y_{2,2}t_{2,3} + y_{3,3}t_{3,3} & t_{4,4} \end{pmatrix} \tag{22}$$

$$Y_{\cdot,1}t_{1,2} + Y_{\cdot,2}t_{2,2} + Y_{\cdot,3}t_{3,2} = \begin{pmatrix} y_{1,1}t_{1,1} & y_{1,1}t_{1,2} + y_{1,2}t_{2,2} + y_{1,3}t_{3,2} & t_{1,3} & t_{1,4} \\ 0 & y_{2,1}t_{1,2} + y_{2,2}t_{2,2} + y_{2,3}t_{3,2} & t_{2,3} & t_{2,4} \\ 0 & y_{3,1}t_{1,2} + y_{3,2}t_{2,2} + y_{3,3}t_{3,2} & t_{3,3} & t_{3,4} \\ 0 & y_{4,1}t_{1,2} + y_{4,2}t_{2,2} + y_{4,3}t_{3,2} & 0 & t_{4,4} \end{pmatrix} \to D_2 \tag{23}$$

$$Y_{\cdot,1}t_{1,3} + Y_{\cdot,2}t_{2,3} + Y_{\cdot,3}t_{3,3} \to \begin{pmatrix} y_{1,1}t_{1,1} & \cdots & y_{1,1}t_{1,3} + y_{1,2}t_{2,3} + y_{1,3}t_{3,3} & t_{1,4} \\ 0 & \cdots & y_{2,1}t_{1,3} + y_{2,2}t_{2,3} + y_{3,3}t_{3,3} & t_{2,4} \\ 0 & \cdots & y_{3,1}t_{1,3} + y_{2,2}t_{2,3} + y_{3,3}t_{3,3} & t_{3,4} \\ 0 & \cdots & y_{4,1}t_{1,3} + y_{2,2}t_{2,3} + y_{3,3}t_{3,3} & t_{4,4} \end{pmatrix} \to D_3 \tag{24}$$

$$AY_{\cdot,2} - Y_{\cdot,1}t_{1,2} - Y_{\cdot,2}t_{2,2} - Y_{\cdot,3}t_{3,2} = D_2 \tag{25}$$

$$AY_{\cdot,3} - Y_{\cdot,1}t_{1,3} - Y_{\cdot,2}t_{2,3} - Y_{\cdot,3}t_{3,3} = D_3 \tag{26}$$

$$(A - t_{2,2}I)Y_{\cdot,2} - Y_{\cdot,1}t_{1,2} - Y_{\cdot,3}t_{3,2} = D_2 \tag{27}$$

$$AY_{\cdot,3} - Y_{\cdot,1}t_{1,3} - Y_{\cdot,2}t_{2,3} - Y_{\cdot,3}t_{3,3} = D_3 \tag{28}$$

$$(A - t_{2,2}I)Y_{\cdot,2} - Y_{\cdot,1}t_{1,2} - D_2 = Y_{\cdot,3}t_{3,2} \tag{29}$$
$$AY_{\cdot,3} - Y_{\cdot,1}t_{1,3} - Y_{\cdot,2}t_{2,3} - Y_{\cdot,3}t_{3,3} = D_3 \tag{30}$$

$$Y_{\cdot,3} = \frac{(A - t_{2,2}I)Y_{\cdot,2} - Y_{\cdot,1}t_{1,2} - D_2}{t_{3,2}} \tag{31}$$
$$AY_{\cdot,3} - Y_{\cdot,1}t_{1,3} - Y_{\cdot,2}t_{2,3} - Y_{\cdot,3}t_{3,3} = D_3 \tag{32}$$

$$Y_{\cdot,3} = \frac{(A - t_{2,2}I)Y_{\cdot,2}}{t_{3,2}} - \frac{Y_{\cdot,1}t_{1,2}}{t_{3,2}} - \frac{D_2}{t_{3,2}} \tag{33}$$

$$Y_{\cdot,3} = KY_{\cdot,2} - S \tag{34}$$
$$K = \frac{(A - t_{2,2}I)}{t_{3,2}} \tag{35}$$
$$S = \frac{Y_{\cdot,1}t_{1,2}}{t_{3,2}} - \frac{D_2}{t_{3,2}} \tag{36}$$
$$\tag{37}$$

$$AY_{\cdot,3} - Y_{\cdot,1}t_{1,3} - Y_{\cdot,2}t_{2,3} - Y_{\cdot,3}t_{3,3} = D_3 \tag{38}$$
$$A(KY_{\cdot,2} - S) - Y_{\cdot,1}t_{1,3} - Y_{\cdot,2}t_{2,3} - (KY_{\cdot,2} - S)t_{3,3} = D_3 \tag{39}$$
$$AKY_{\cdot,2} - AS - Y_{\cdot,1}t_{1,3} - Y_{\cdot,2}t_{2,3} - KY_{\cdot,2}t_{3,3} - St_{3,3} = D_3 \tag{40}$$
$$(AK - Kt_{3,3} - t_{1,3}I)Y_{\cdot,2} - AS - Y_{\cdot,1}t_{1,3} - St_{3,3} = D_3 \tag{41}$$
$$(AK - Kt_{3,3} - t_{1,3}I)Y_{\cdot,2} = D_3 + AS + Y_{\cdot,1}t_{1,3} + St_{3,3} \tag{42}$$
$$\tag{43}$$