Simplex Method

1. LP Model in Equation Form

- LP Model in Equation Form
  - Two requirements

1. All the constraints (with the exception of the nonnegativity of the variables) are equations with nonnegative right-hand side.
2. All the variables are nonnegative.

- Converting Inequalities into Equations with Nonnegative RHS
  - Slack Variable $6x_1 + 4x_2 \le 24$ $\quad$ $6x_1 + 4x_2 + s_1 = 24, s_1 \ge 0$
  - Surplus Variable $x_1 + x_2 \ge 800$ $\quad$ $x_1 + x_2 - S_1 = 800, S_1 \ge 0$
  - Nonnegative RHS $-x_1 + x_2 \le -3$ $\quad$ $-x_1 + x_2 + s_1 = -3, s_1 \ge 0$

$$x_1 - x_2 - s_1 = 3$$

LP Model in Equation Form

- Unrestricted Variable

$$y_{i+1} = y_{i+1}^- - y_{i+1}^+, \text{ where } y_{i+1}^- \geq 0 \text{ and } y_{i+1}^+ \geq 0$$

- Example

$$\text{Maximize } z = 2x_1 + 3x_2$$

$$\begin{aligned} 2x_1 + x_2 &\leq 4 \\ x_1 + 2x_2 &\leq 5 \\ x_1, x_2 &\geq 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 2x_1 + x_2 + s_1 &= 4 \\ x_1 + 2x_2 + s_2 &= 5 \\ x_1, x_2, s_1, s_2 &\geq 0 \end{aligned}$$

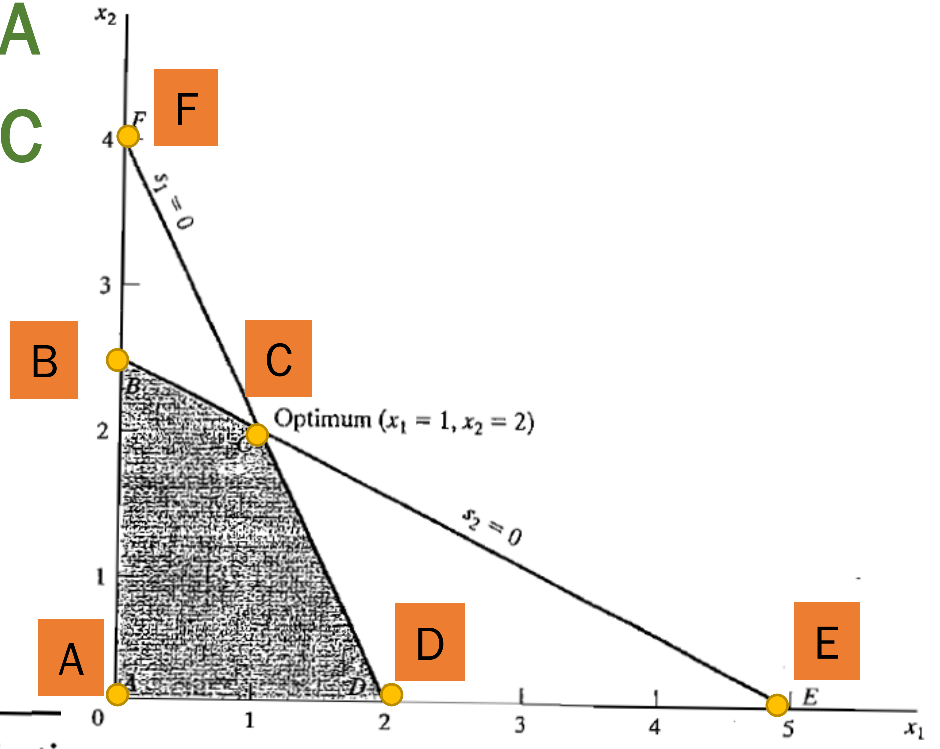
  - m = 2 equations and n = 4 variables
  - Corner points can be found by putting n-m = 2 variables zero.

2. Transition from Graphical to Algebraic Solution

- Put $x_1 = 0$, $x_2 = 0$, and s1=4, s2 = 5 **Point A**
- Put s1=0, s2=0, and $x_1 = 1$, $x_2 = 2$ **Point C**

- Basic variables = m,
- Nonbasic variables = n-m

$$\boxed{\begin{aligned} 2x_1 + x_2 + s_1 \quad &= 4 \\ x_1 + 2x_2 \quad + s_2 &= 5 \end{aligned}}$$



Nonbasic (zero) variables	Basic variables	Basic solution	Associated corner point	Feasible?	Objective value, z
(x_1, x_2)	(s_1, s_2)	(4, 5)	A	Yes	0

Maximum number of corner points

$$C_m^n = \frac{n!}{m!(n-m)!}$$

If m = 10, n = 20, then 184,756 corner points

Simplex Method

- Selectively investigate few corner points and locate the optimum solution
- Reddy Mikks Model

$$\text{Maximize } z = 5x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$$

  - Rewrite objective function $\boxed{z - 5x_1 - 4x_2 = 0}$

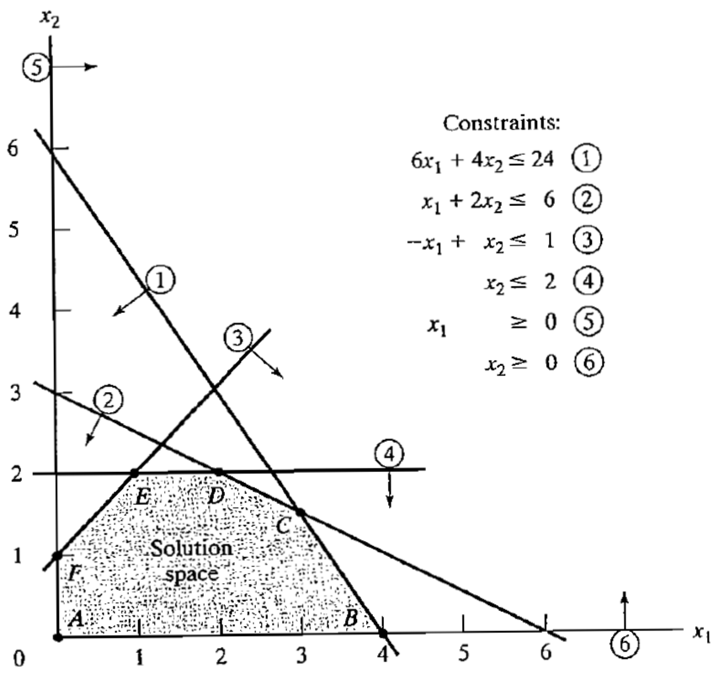
$$\text{Maximize } z = 5x_1 + 4x_2$$

$$\begin{aligned} 6x_1 + 4x_2 &\le 24 \\ x_1 + 2x_2 &\le 6 \\ -x_1 + x_2 &\le 1 \\ x_2 &\le 2 \\ x_1, x_2 &\ge 0 \end{aligned}$$

$$\begin{aligned}
6x_1 + 4x_2 + s_1 &= 24 \quad \text{(Raw material } M1\text{)} \\
x_1 + 2x_2 + s_2 &= 6 \quad \text{(Raw material } M2\text{)} \\
-x_1 + x_2 + s_3 &= 1 \quad \text{(Market limit)} \\
x_2 + s_4 &= 2 \quad \text{(Demand limit)} \\
x_1, x_2, s_1, s_2, s_3, s_4 &\ge 0
\end{aligned}$$

Transition from Graphical to Algebraic Solution

Basic	z	x1	x2	s1	s2	s3	s4	Solution	Ratio
z	1	-5	-4	0	0	0	0	0	
s1	0	6	4	1	0	0	0	24	
s2	0	1	2	0	1	0	0	6	
s3	0	-1	1	0	0	1	0	1	
s4	0	0	1	0	0	0	1	2	



Nonbasic (zero) variables: (x_1, x_2)

Basic variables: (s_1, s_2, s_3, s_4)

nonbasic variables $(x_1, x_2) = (0, 0)$

$$z = 0$$
$$s_1 = 24$$
$$s_2 = 6$$
$$s_3 = 1$$
$$s_4 = 2$$

3. Simplex Tableau

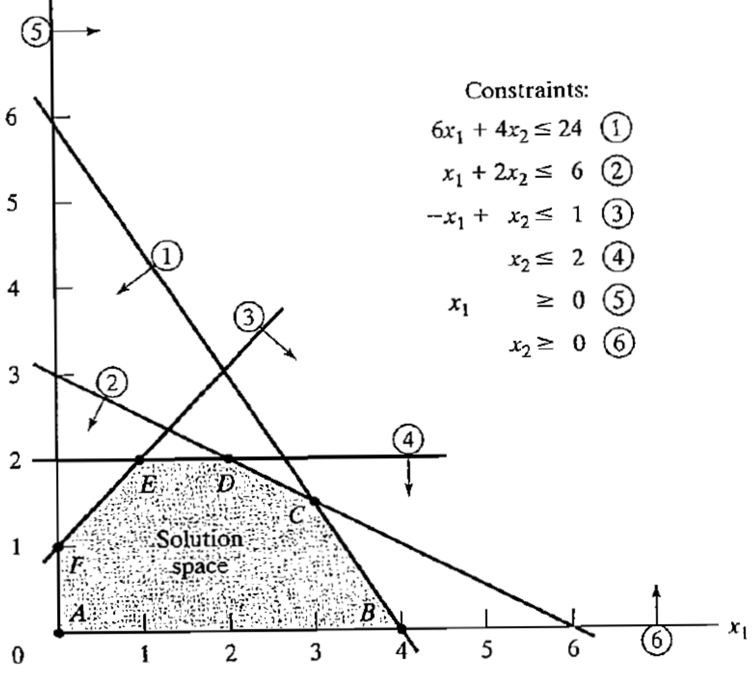
- Entering variable
  - Which nonbasic variable (x_1 or x_2) should enter such that the objective function should improve maximally?
  - Most negative coefficient of the **maximization objective function**
    - Optimality condition

Basic	z	x1	x2	s1	s2	s3	s4	Solution	Ratio
z	1	-5	-4	0	0	0	0	0	
s1	0	6	4	1	0	0	0	24	
s2	0	1	2	0	1	0	0	6	
s3	0	-1	1	0	0	1	0	1	
s4	0	0	1	0	0	0	1	2	

Entering variable x_1

Simplex Tableau

- Leaving variable
  - Minimum nonnegative ratio of RHS of the equation to the corresponding constraint coefficient under the entering variable
    - Feasible condition



Basic	z	x1	x2	s1	s2	s3	s4	Solution	Ratio
z	1	-5	-4	0	0	0	0	0	
s1	0	6	4	1	0	0	0	24	24/6=4
s2	0	1	2	0	1	0	0	6	6/1=6
s3	0	-1	1	0	0	1	0	1	1/-1=-1
s4	0	0	1	0	0	0	1	2	2/0

Nonbasic (zero) variables at B: (s_1, x_2)

Basic variables at B: (x_1, s_2, s_3, s_4)

Leaving Variables s_1

Simplex Tableau

Pivot element

Entering Variable

Basic	z	x1	x2	s1	s2	s3	s4	Solution	Ratio
z	1	-5	-4	0	0	0	0	0	
s1	0	6	4	1	0	0	0	24	
s2	0	1	2	0	1	0	0	6	
s3	0	-1	1	0	0	1	0	1	
s4	0	0	1	0	0	0	1	2	

Leaving variable

Pivot row

Pivot column

Gauss-Jordon Row Operation

1. *Pivot row*

a. Replace the leaving variable in the *Basic* column with the entering variable.

b. New pivot row = Current pivot row ÷ Pivot element

Basic	z	x1	x2	s1	s2	s3	s4	Solution	Ratio
z	1	-5	-4	0	0	0	0	0	
x1	0	1	2/3	1/6	0	0	0	4	
s2	0	1	2	0	1	0	0	6	
s3	0	-1	1	0	0	1	0	1	
s4	0	0	1	0	0	0	1	2	

Pivot row

- Pivot element = 6

Gauss-Jordon Row Operation

2. *All other rows, including z*

New Row = (Current row) − (Its pivot column coefficient) × (New pivot row)

- For row z: current row coefficient (1, -5, -4, 0, 0, 0, 0, 0); pivot column coefficient= -5; new pivot row coefficient (0, 1, 2/3, 1/6, 0, 0, 0, 4)

Basic	z	x1	x2	s1	s2	s3	s4	Solution	Ratio
z	1	-5	-4	0	0	0	0	0	
x1	0	1	2/3	1/6	0	0	0	4	
s2	0	1	2	0	1	0	0	6	
s3	0	-1	1	0	0	1	0	1	
s4	0	0	1	0	0	0	1	2	

- New z-row is (1, 0, -2/3, 5/6, 0, 0, 0, 20)

Gauss-Jordon Row Operation

2. *All other rows, including z*

New Row = (Current row) − (Its pivot column coefficient) × (New pivot row)

- For row s2: current row coefficient (0, 1, 2, 0, 1, 0, 0, 6);
  pivot column coefficient= 1;
  new pivot row coefficient (0, 1, 2/3, 1/6, 0, 0, 0, 4)

Basic	z	x1	x2	s1	s2	s3	s4	Solution	Ratio
z	1	0	-2/3	5/6	0	0	0	20	
x1	0	1	2/3	1/6	0	0	0	4	
s2	0	1	2	0	1	0	0	6	
s3	0	-1	1	0	0	1	0	1	
s4	0	0	1	0	0	0	1	2	

- For row s2: new row (0, 0, 4/3, -1/6, 1, 0, 0, 2)

Gauss-Jordon Row Operation

2. *All other rows, including z*

New Row = (Current row) − (Its pivot column coefficient) × (New pivot row)

- For row s3: current row coefficient (0, -1, 1, 0, 0, 1, 0, 1);
  pivot column coefficient= -1;
  new pivot row coefficient (0, 0, 1, 2/3, 1/6, 0, 0, 0, 4)

Basic	z	x1	x2	s1	s2	s3	s4	Solution	Ratio
z	1	0	-2/3	5/6	0	0	0	20	
x1	0	1	2/3	1/6	0	0	0	4	
s2	0	0	4/3	-1/6	1	0	0	2	
s3	0	-1	1	0	0	1	0	1	
s4	0	0	1	0	0	0	1	2	

- For row s3: new row (0, 0, 5/3, 1/6, 0, 1, 0, 5)

Gauss-Jordon Row Operation

2. *All other rows, including z*

New Row = (Current row) − (Its pivot column coefficient) × (New pivot row)

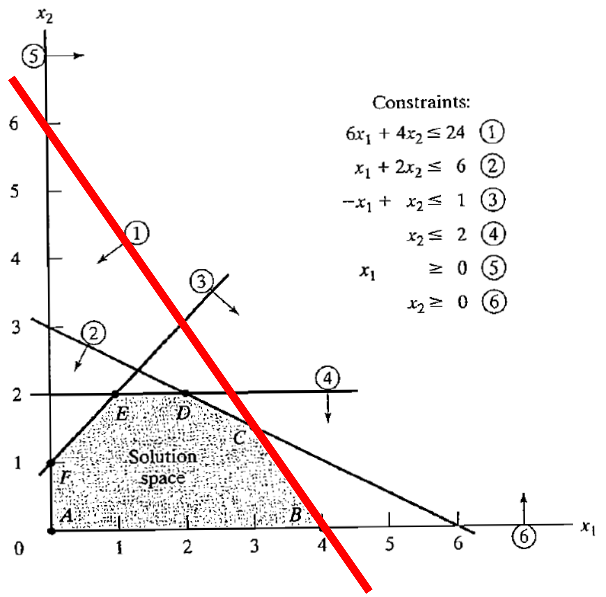
- For row s4: current row coefficient (0, 0, 1, 0, 0, 0, 1, 2); pivot column coefficient= 0; new pivot row coefficient (0, 0, 1, 2/3, 1/6, 0, 0, 0, 4)

Basic	z	x1	x2	s1	s2	s3	s4	Solution	Ratio
z	1	0	-2/3	5/6	0	0	0	20	
x1	0	1	2/3	1/6	0	0	0	4	
s2	0	0	4/3	-1/6	1	0	0	2	
s3	0	0	5/3	1/6	0	1	0	5	
s4	0	0	1	0	0	0	1	2	

- For row s3: new row (0, 0, 1, 0, 0, 0, 1, 2)

Simplex Tableau

- 1st iteration is over



Entering variable: Most negative coefficient for maximization problem

Leaving variable: minimum nonnegative ratio

Basic	z	x1	x2	s1	s2	s3	s4	Solution	Ratio
z	1	0	-2/3	5/6	0	0	0	20	
x1	0	1	2/3	1/6	0	0	0	4	6
s2	0	0	4/3	-1/6	1	0	0	2	3/2
s3	0	0	5/3	1/6	0	1	0	5	3
s4	0	0	1	0	0	0	1	2	2

Pivot Row and Column

- Gauss-Jordon Row operations

1. New pivot x_2-row = Current s_2-row $\div \frac{4}{3}$
2. New z-row = Current z-row $- \left(-\frac{2}{3}\right) \times$ New x_2-row
3. New x_1-row = Current x_1-row $- \left(\frac{2}{3}\right) \times$ New x_2-row
4. New s_3-row = Current s_3-row $- \left(\frac{5}{3}\right) \times$ New x_2-row
5. New s_4-row = Current s_4-row $- (1) \times$ New x_2-row

Simplex Tableau

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	0	0	$\frac{3}{4}$	$\frac{1}{2}$	0	0	21
x_1	0	1	0	$\frac{1}{4}$	$-\frac{1}{2}$	0	0	3
x_2	0	0	1	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\frac{3}{2}$
s_3	0	0	0	$\frac{3}{8}$	$-\frac{5}{4}$	1	0	$\frac{5}{2}$
s_4	0	0	0	$\frac{1}{8}$	$-\frac{3}{4}$	0	1	$\frac{1}{2}$

Any entering Variable?

Decision variable	Optimum value	Recommendation
x_1	3	Produce 3 tons of exterior paint daily
x_2	$\frac{3}{2}$	Produce 1.5 tons of interior paint daily
z	21	Daily profit is $21,000

Constraints

Resource	Slack value	Status
Raw material, $M1$	$s_1 = 0$	Scarce
Raw material, $M2$	$s_2 = 0$	Scarce
Market limit	$s_3 = \frac{5}{2}$	Abundant
Demand limit	$s_4 = \frac{1}{2}$	Abundant

Steps of Simplex Method

Optimality condition	
Maximization problem	**Minimization problem**
Most negative coefficient of nonbasic variable	Most positive coefficient of nonbasic variable
Feasibility condition	
Smallest nonnegative ratio	Smallest nonnegative ratio

Gauss-Jordan row operations.

1. Pivot row

a. Replace the leaving variable in the *Basic* column with the entering variable.

b. New pivot row = Current pivot row ÷ Pivot element

2. All other rows, including z

New row = (Current row) − (pivot column coefficient) × (New pivot row)

Steps of Simplex Method

Step 1. Determine a starting basic feasible solution.

4. Artificial Starting Solution

- Constraints are ($\leq$) with nonnegative right hand sides offers a convenient all-slack starting basic feasible solution.
- Models with $\geq$ or = constraints do not.
- **Artificial Variable**: Starting “ill-behaved” LPs with $\geq$ or = constraints is to use artificial variable that play the role of slacks at the first iteration, and then dispose them legitimately at a later iteration.
- Two methods
  - M-method
  - Two phase method

M-Method

- Use x_3 surplus with constraint 2 and slack variable x_4 with constraint 3

$$\text{Minimize } z = 4x_1 + x_2$$

$$\begin{aligned} 3x_1 + x_2 &= 3 \\ 4x_1 + 3x_2 &\geq 6 \\ x_1 + 2x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$$\Rightarrow$$

$$\text{Minimize } z = 4x_1 + x_2$$

$$\begin{aligned} 3x_1 + x_2 \qquad\qquad &= 3 \\ 4x_1 + 3x_2 - x_3 \qquad &= 6 \\ x_1 + 2x_2 \qquad + x_4 &= 4 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

- Constraint 1 and constraint 2 do not have slack variable
- Add artificial variable R_1 and R_2 and penalize them in the objective function

M-Method

$$\text{Minimize } z = 4x_1 + x_2 + MR_1 + MR_2$$

$Z - 4x1 - x2 - MR1 - MR2 = 0$

$$3x_1 + x_2 + R_1 = 3$$

$$4x_1 + 3x_2 - x_3 + R_2 = 6$$

$$x_1 + 2x_2 + x_4 = 4$$

$$x_1, x_2, x_3, x_4, R_1, R_2 \geq 0$$

Minimization problem:
Add MR_i

- Basic variables: (R_1, R_2, x_4)
- What should be the value of M?
  - It should be large enough relative to the original objective coefficient
  - For the given problem, M = 100

M-Method

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
Z	-4	-1	0	-100	-100	0	0
R_1	3	1	0	1	0	0	3
R_2	4	3	-1	0	1	0	6
x_4	1	2	0	0	0	1	4

Inconsistency: Non zero coefficient of R_1 and R_2

- Substitution such that coefficient of R_1 and R_2 becomes zero
  - For the given problem:

$$\text{New } z\text{-row} = \text{Old } z\text{-row} + (100 \times R_1\text{-row} + 100 \times R_2\text{-row})$$

M-Method

Pivot column

Minimization problem

Pivot row

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution	Ratio
Z	696	399	-100	0	0	0	900	
R_1	3	1	0	1	0	0	3	1
R_2	4	3	-1	0	1	0	6	3/2
x_4	1	2	0	0	0	1	4	4

- Apply simplex method steps
  - **Entering variable:**
    - x_1 (most positive coefficient in z for minimization objective function)
  - **Leaving variable:**
    - R_1 (Minimum nonnegative ratio)

M-Method

- Apply Gauss-Jordon row operations

Pivot column

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution	Ratio
Z	0	167	-100	-232	0	0	204	
x_1	1	1/3	0	1/3	0	0	1	3
R_2	0	5/3	-1	-4/3	1	0	2	6/5
x_4	0	5/3	0	-1/3	0	1	3	9/5

Pivot row

- **Entering variable:** x_2
- **Leaving variable:** R_2

M-Method

- Apply Gauss-Jordon row operations

Pivot column

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution	Ratio
Z	0	0	1/5	-492/5	-501/5	0	18/5	
x_1	1	0	1/5	3/5	-1/5	0	3/5	3
x_2	0	1	-3/5	-4/5	3/5	0	6/5	-2
Pivot row x_4	0	0	1	1	-1	1	1	1

- **Entering variable:** x_3
- **Leaving variable:** x_4

M-Method

- Apply Gauss-Jordon row operations

Any entering Variable?

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
Z	0	0	0	-493/5	-100	-1/5	17/5
x_1	1	0	0	2/5	0	-1/5	2/5
x_2	0	1	0	-1/5	0	3/5	9/5
x_3	0	0	1	1	-1	1	1

- $x_1 = 2/5$, $x_2 = 9/5$ and $z = 17/5$

Two Phase Method

- M-method uses penalty M
  - Possibility of round-off error that may impair the accuracy of simplex calculations
- Two phase method
  - Phase I attempts to find starting basic feasible solution
  - Phase II is invoked to solve the original problem
- Problem solved in the last section

$$\text{Minimize } z = 4x_1 + x_2$$

$$3x_1 + x_2 = 3$$

$$4x_1 + 3x_2 \geq 6$$

$$x_1 + 2x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Phase-I of Two Phase Method

$$\text{Minimize } r = R_1 + R_2$$

$$\begin{aligned} 3x_1 + x_2 + R_1 &= 3 \\ 4x_1 + 3x_2 - x_3 + R_2 &= 6 \\ x_1 + 2x_2 + x_4 &= 4 \\ x_1, x_2, x_3, x_4, R_1, R_2 &\geq 0 \end{aligned}$$

- Simplex tableau

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
r	0	0	0	−1	−1	0	0
R_1	3	1	0	1	0	0	3
R_2	4	3	−1	0	1	0	6
x_4	1	2	0	0	0	1	4

Inconsistence

Phase-I of Two Phase Method

- Substitution $\quad$ New r-row $=$ Old r-row $+ (1 \times R_1\text{-row} + 1 \times R_2\text{-row})$

- Apply simplex steps and Gauss-Jordon row operation
- The optimum solution of Phase I is

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
r	0	0	0	-1	-1	0	0
x_1	1	0	$\frac{1}{5}$	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{3}{5}$
x_2	0	1	$-\frac{3}{5}$	$-\frac{4}{5}$	$\frac{3}{5}$	0	$\frac{6}{5}$
x_4	0	0	1	1	-1	1	1

- r=0, basic feasible solution $x_1 = 3/5$, $x_2 = 6/5$, $x_4 = 1$
- Eliminate columns of artificial variables for Phase II

Phase-II of Two Phase Method

- Phase II problem

$$\text{Minimize } z = 4x_1 + x_2$$

$$\begin{aligned} x_1 \quad + \tfrac{1}{5}x_3 \quad &= \tfrac{3}{5} \\ x_2 - \tfrac{3}{5}x_3 \quad &= \tfrac{6}{5} \\ x_3 + x_4 &= 1 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

- Simplex tableau

Basic	x_1	x_2	x_3	x_4	Solution
z	-4	-1	0	0	0
x_1	1	0	$\frac{1}{5}$	0	$\frac{3}{5}$
x_2	0	1	$-\frac{3}{5}$	0	$\frac{6}{5}$
x_4	0	0	1	1	1

Phase-II of Two Phase Method

- Substitution to make coefficient of basic variables x_1 and x_2 zero

$$\text{New } z\text{-row} = \text{Old } z\text{-row} + (4 \times x_1\text{-row} + 1 \times x_2\text{-row})$$

- Simplex tableau

Basic	x_1	x_2	x_3	x_4	Solution
z	0	0	$\frac{1}{5}$	0	$\frac{18}{5}$
x_1	1	0	$\frac{1}{5}$	0	$\frac{3}{5}$
x_2	0	1	$-\frac{3}{5}$	0	$\frac{6}{5}$
x_4	0	0	1	1	1

- Apply simplex method steps and Gauss-Jordon row operation to include x_3
- Widely used method.

Phase-II of Two Phase Method

The removal of the artificial variables and their columns at the end of Phase I can take place only when they are all *nonbasic* (as Example 3.4-2 illustrates). If one or more artificial variables are *basic* (at *zero* level) at the end of Phase I, then the following additional steps must be undertaken to remove them prior to the start of Phase II.

Step 1. Select a zero artificial variable to leave the basic solution and designate its row as the *pivot row*. The entering variable can be *any* nonbasic (nonartificial) variable with a *nonzero* (positive or negative) coefficient in the pivot row. Perform the associated simplex iteration.

Step 2. Remove the column of the (just-leaving) artificial variable from the tableau. If all the zero artificial variables have been removed, go to Phase II. Otherwise, go back to Step 1.

The logic behind Step 1 is that the feasibility of the remaining basic variables will not be affected when a zero artificial variable is made nonbasic regardless of whether the pivot element is positive or negative. Problems 5 and 6, Set 3.4b illustrate this situation. Problem 7 provides an additional detail about Phase I calculations.

5. Special Cases in Simplex Method

- Degeneracy
- Alternative Optima
- Unbounded Solution
- Nonexistence (or infeasible) solution

Degeneracy

- A tie at minimum ratio (leaving variable)
  - Choose arbitrarily
  - One basic variable become zero in the next iteration (Degeneracy)
- One constraint is redundant
- Example

$$\text{Maximize } z = 3x_1 + 9x_2$$

$$x_1 + 4x_2 \leq 8$$
$$x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Equation form

$z - 3x_1 - 9x_2 = 0$

$x_1 + 4x_2 + x_3 = 8$
$x_1 + 2x_2 + x_4 = 4$
$x_1, x_2 \geq 0$

- Use x_3 and x_4 as slack variables

Degeneracy

Iteration	Basic	x_1	x_2	x_3	x_4	Solution	Ratio
0	z	-3	-9	0	0	0	
	x_3	1	4	1	0	8	8/4=2
	x_4	1	2	0	1	4	4/2=2

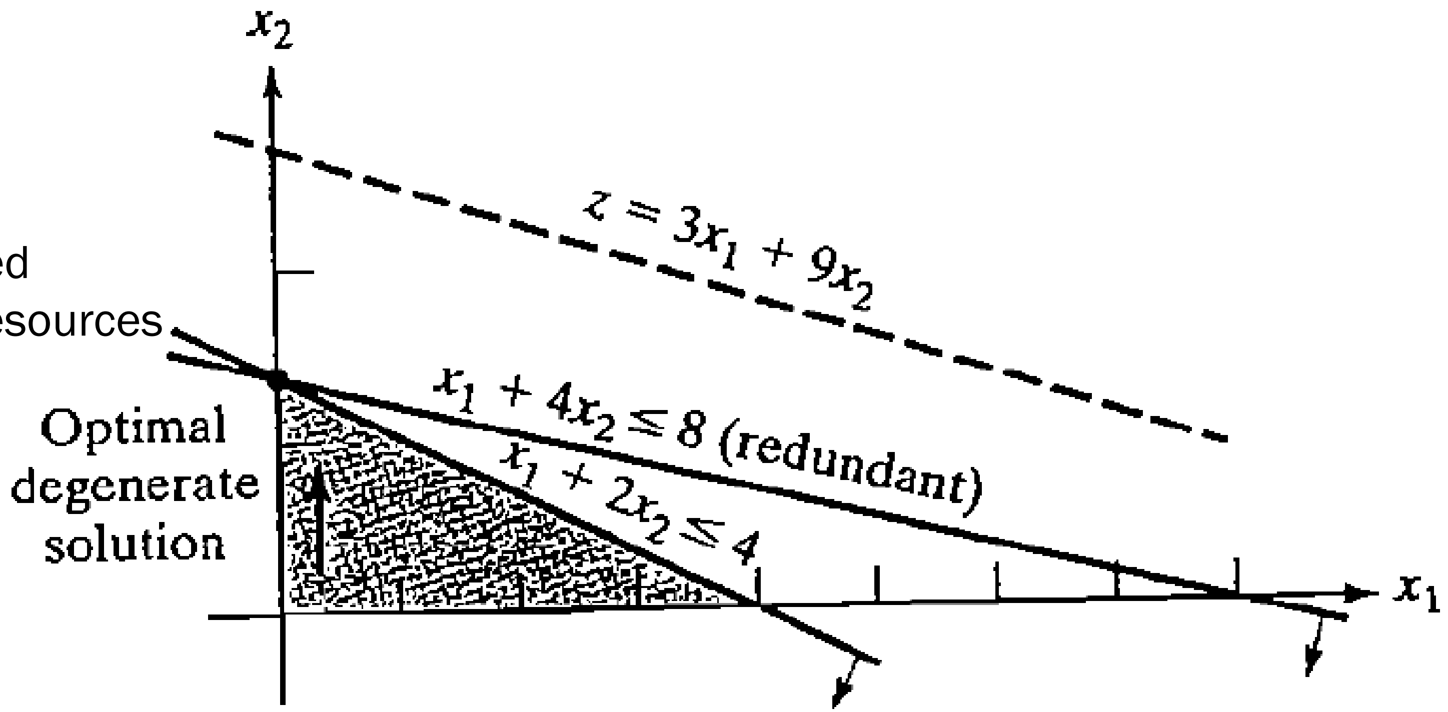
Tie

Cycling
Or
Circling

- Can we stop at iteration 1? No.
  - Temporally degenerate

Degeneracy

- Overdetermined
- Superfluous resources



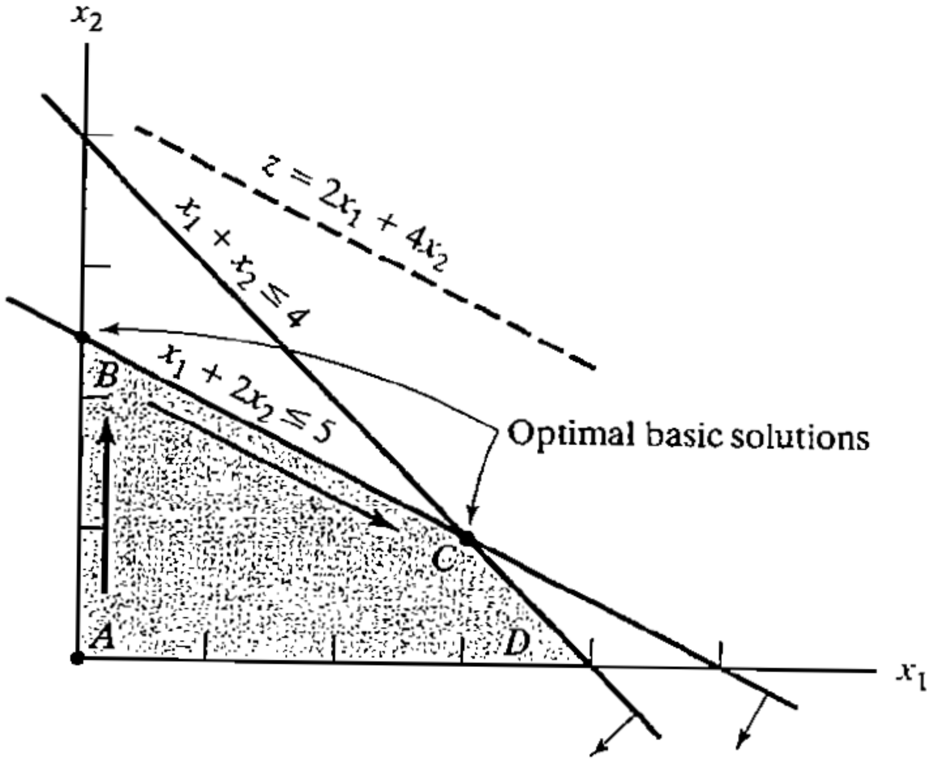
Alternate Optima

- Objective function is parallel to nonredundant binding constraint
- Binding constraint: A constraint that is satisfied as an equation at the optimal solution.
- | Maximize $z = 2x_1 + 4x_2$

$$x_1 + 2x_2 \leq 5$$

$$x_1 + x_2 \leq 4$$

$$x_1, x_2 \geq 0$$



Alternate Optima

Iteration	Basic	x_1	x_2	x_3	x_4	Solution
0	z	-2	-4	0	0	0
x_2 enters	x_3	1	2	1	0	5
x_3 leaves	x_4	1	1	0	1	4

- Already get the optima
- Point B in graph

$= 0, x_2 = \frac{5}{2}$, and $z = 10$,

- Optima
- Point C in graph
- Nonzero x_1

$x_1 = 3, x_2 = 1, z = 10$

- All solutions along line BC are optimal.

Unbounded Solution

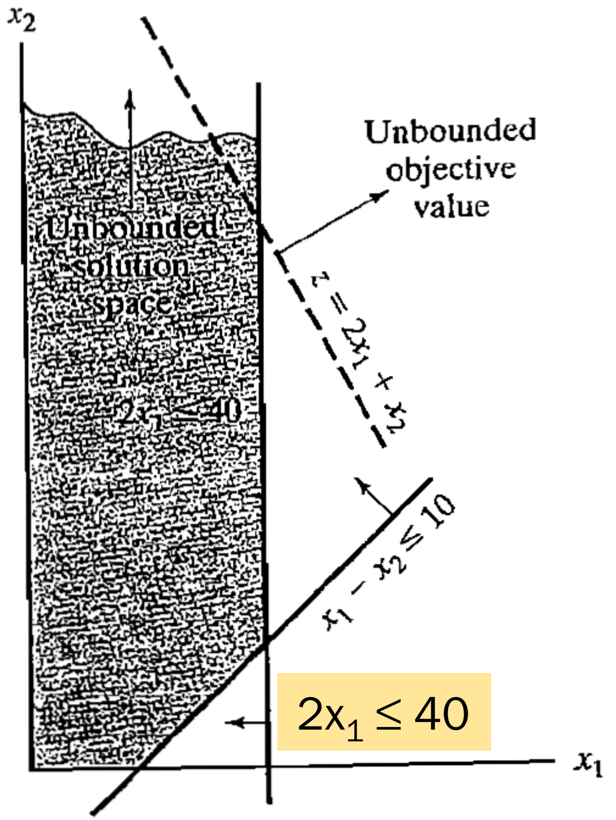
- Objective function value keeps on improving infinitely without violating any constraint
  - At least one variable is unbounded
  - Leads to the conclusion that the model is poorly constructed
- Example

$$\text{Maximize } z = 2x_1 + x_2$$

$$x_1 - x_2 \leq 10$$

$$2x_1 \leq 40$$

$$x_1, x_2 \geq 0$$



Unbounded Solution

Iteration	Basic	X1	X2	X3	X4	Solution	Ratio
0 X1 enters X3 leaves	Z	-2	-1	0	0	0	
	X3	1	-1	1	0	10	10
	X4	2	0	0	1	40	20

- All constraint coefficients under x_3 are either 0 or negative
  - Means no leaving variable and that x_3 can be increased infinitely without violating any constraints.
  - Unbounded problem.

Infeasible Solution

- LP model with inconsistence constraints has no feasible solution.
- This situation will never occur for ≤ type constraints because we can start with slack variables as our basic feasible solutions.
- For other type of constraints, we use artificial variables
  - These artificial variables are forced to become zero at the optima if the model has feasible solution.
  - Otherwise at least one artificial variable will be positive in the optimum iteration

Infeasible Solution

- Example

$$\text{Maximize } z = 3x_1 + 2x_2$$

- Using M-method with M = 100

$$2x_1 + x_2 \leq 2$$

$$3x_1 + 4x_2 \geq 12$$

$$x_1, x_2 \geq 0$$

Iteration	Basic	x_1	x_2	x_4	x_3	R	Solution
0	z	−303	−402	100	0	0	−1200
x_2 enters	x_3	2	1	0	1	0	2
x_3 leaves	R	3	4	−1	0	1	12

Infeasible Solution

- By allowing R to be positive, the simplex method in essence, has reversed the direction of the inequality from

$$3x_1 + 4x_2 \geq 12 \text{ to } 3x_1 + 4x_2 \leq 12$$

- The result is pseudo-optimal solution.