Introduction, Modeling to LP

Course Resources

- **Texts:**

  1. S. L. Narasimhan, D. W. McLeavey, and P. J. Billington, *Production, Planning and Inventory Control*, Prentice Hall, 1997.

  2. J. L. Riggs, *Production Systems: Planning, Analysis and Control*, 3rd Ed., Wiley, 1981.

- **References:**

  1. A. Muhlemann, J. Oakland and K. Lockyer, *Productions and Operations Management*, Macmillan, 1992.

  2. H. A. Taha, *Operations Research - An Introduction*, Prentice Hall of India, 1997.

  3. J. K. Sharma, *Operations Research*, Macmillan, 1997.

1. Operations Research Models

Imagine that you have a 5-week business commitment between Fayetteville (FYV) and Denver (DEN). You fly out of Fayetteville on Mondays and return on Wednesdays. A regular round-trip ticket costs $400, but a 20% discount is granted if the dates of the ticket span a weekend. A one-way ticket in either direction costs 75% of the regular price. How should you buy the tickets for the 5-week period?

1. What are the decision **alternatives?**
2. Under what **restrictions** is the decision made?
3. What is an appropriate **objective criterion** for evaluating the alternatives?

Operations Research Models

1. Buy five regular FYV-DEN-FYV for departure on Monday and return on Wednesday of the same week.
2. Buy one FYV-DEN, four DEN-FYV-DEN that span weekends, and one DEN-FYV.
3. Buy one FYV-DEN-FYV to cover Monday of the first week and Wednesday of the last week and four DEN-FYV-DEN to cover the remaining legs. All tickets in this alternative span at least one weekend.

$$\text{Alternative 1 cost} = 5 \times 400 = \$2000$$

$$\text{Alternative 2 cost} = .75 \times 400 + 4 \times (.8 \times 400) + .75 \times 400 = \$1880$$

$$\text{Alternative 3 cost} = 5 \times (.8 \times 400) = \mathbf{\$1600}$$

Operations Research Models

- Wire of L inches.
- Make a rectangle from the wire such that area of rectangle should be maximum.

- Alternates
  - Width = w
  - Height = h
- Restriction
  - $2(w+h) = L$
  - $w,h \geq 0$
- Objective function: Maximize $z = wh$

Search and Optimization

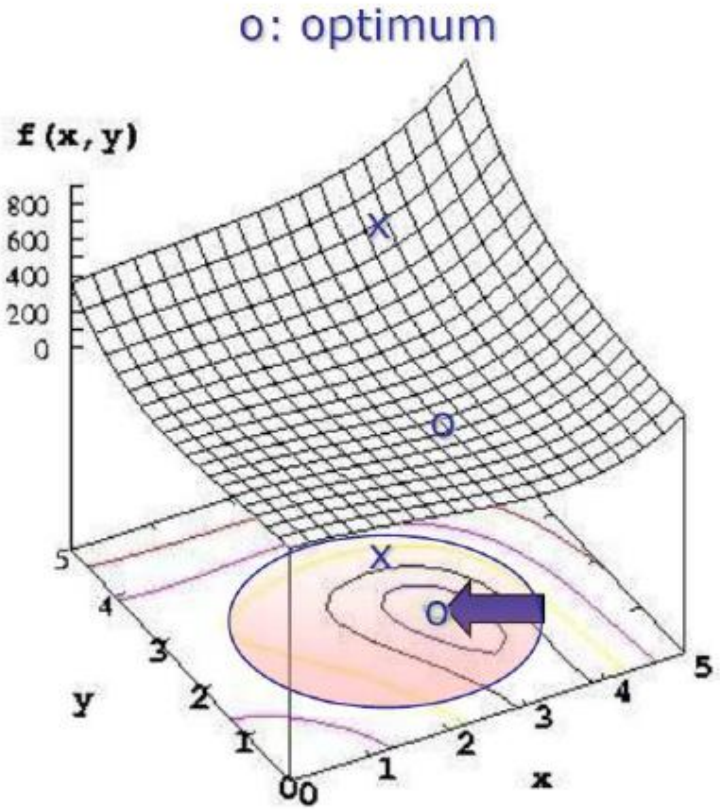
A task of searching for a set of decision variables which would minimize or maximize objective function(s) subject to satisfying constraints and bounds on decision variables.

- Decision variables: (x,y)
- Objective function: f(x,y)
- Constraints: g(x,y) or h(x,y)

- Optimization Modeling

Minimize f(x,y)

$$g(x,y) \geq 0$$
$$h(x,y) = 0$$
$$x^l \leq x \leq x^u$$
$$y^l \leq y \leq y^u$$



Solutions

- Feasible solution
  - A solution satisfies all the constraints.
- Optimal solution
  - A feasible solution which yields the best value of objective function in the entire feasible search space
  - Sub-optimal or local optimal solution
    - The feasible solution which is optimal in its vicinity.
- Search space
  - The space defined by the constraints and limits on the variables.
- Feasible search space
  - The space in which any point in it is always feasible.

Steps in an Optimization Task

- Need for optimization
- Problem formulation or modeling
  - Identify problem parameters
  - Choose design variables from parameters
  - Formulate constraints
  - Formulate objective function
  - Set up variable bounds
  - Requires 50-60% of the effort
- Choose an optimization algorithm
- Obtain solution
- Reformulation and rerun, if desired

Design Variables and their Bounds

- List any and every parameter related to the problem
- Identify parameters sensitive for the given design or problem
- Specify the type of each parameter (binary, discrete, real)
- Choose few of them as design variables
- **First thumb rule:** Use as few variables as possible
  - Usually from the experience of the user
  - From minimum variability consideration
  - From sensitivity analysis etc.
- Bounds or limit on decision variables

$$x^l \leq x \leq x^u$$

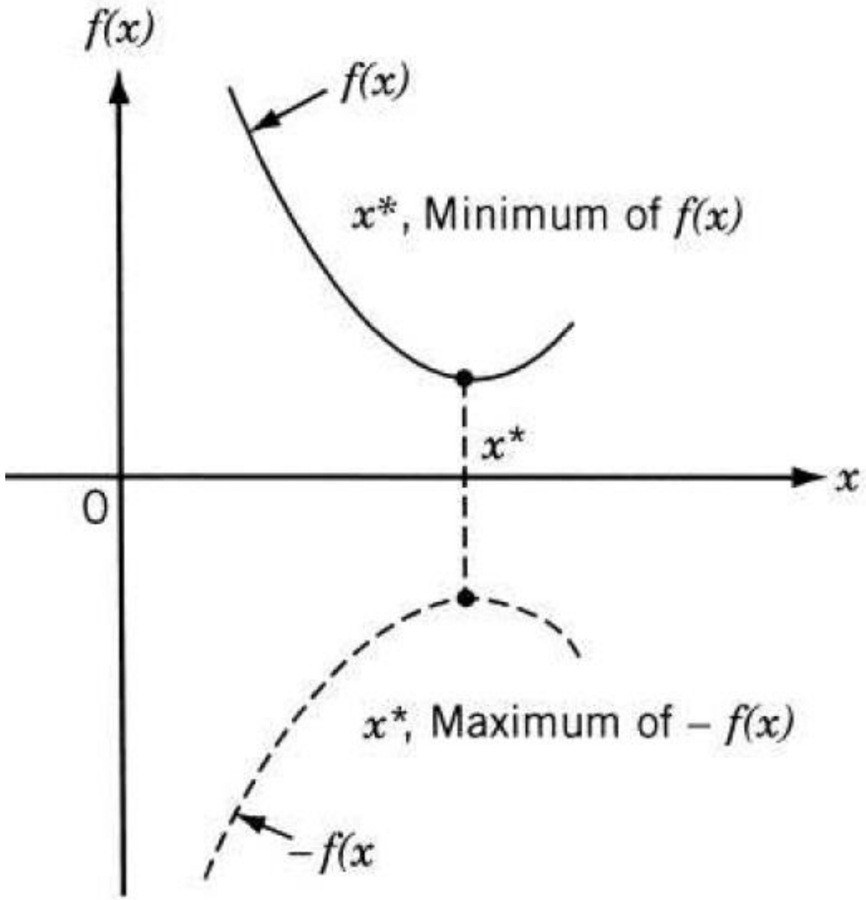
$$y^l \leq y \leq y^u$$

Constraints

- Represent limit on resource or on certain physical phenomenon, for example satisfy stress and deflection limitation
- **Inequality constraint:** $g(x) \geq 0$ or $g(x) \leq 0$
  - mostly encounter in engineering design problems
- **Equality constraint:** $h(x) = 0$,
  - In linear programming (LP), or satisfying demand etc.

Objective Function

- Minimize or maximize
- Optimization methods are generally developed for minimization
- Use duality principle
  - Min f(x) = - Max f(x)



Solving OR Models

- Linear Programming
- Integer Programming
- Dynamic Programming
- Network Programming
- Nonlinear Programming

- OR Techniques do not find solutions in closed form
- Algorithm
  - Fixed computational rules that are applied repetitively to the problem
  - Repetition is called iteration
  - Each iteration a solution is getting closer to the optimum.
  - Tedious and Voluminous
  - Executed on the computer

Thank you.

2. Modeling with Linear Programming (LP)

- Two-variable LP Model

Example 2.1-1 (The Reddy Mikks Company)

Reddy Mikks produces both interior and exterior paints from two raw materials, *M1* and *M2*. The following table provides the basic data of the problem:

	Tons of raw material per ton of		Maximum daily availability (tons)
	Exterior paint	*Interior paint*	
Raw material, *M1*	6	4	24
Raw material, *M2*	1	2	6
Profit per ton ($1000)	5	4	

Alternatives or Decision variables?

Constraints

A market survey indicates that the daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton. Also, the maximum daily demand for interior paint is 2 tons.

Reddy Mikks wants to determine the optimum (best) product mix of interior and exterior paints that maximizes the total daily profit.

Objective function

Modeling: Reddy Mikks Problem

- Decision variables

x_1 = Tons produced daily of exterior paint

x_2 = Tons produced daily of interior paint

Total profit from exterior paint $= 5x_1$ (thousand) dollars

Total profit from interior paint $= 4x_2$ (thousand) dollars

- Objective function

$$\text{Maximize } z = 5x_1 + 4x_2$$

- Constraints

$$\begin{pmatrix}\text{Usage of a raw material}\\ \text{by both paints}\end{pmatrix} \leq \begin{pmatrix}\text{Maximum raw material}\\ \text{availability}\end{pmatrix}$$

	Tons of raw material per ton of		Maximum daily availability (tons)
	Exterior paint	*Interior paint*	
Raw material, $M1$	6	4	24
Raw material, $M2$	1	2	6
Profit per ton (\$1000)	5	4	

Usage of raw material $M1$ by exterior paint $= 6x_1$ tons/day

Usage of raw material $M1$ by interior paint $= 4x_2$ tons/day

Modeling: Reddy Mikks Problem

$$\text{Usage of raw material } M1 \text{ by both paints} = 6x_1 + 4x_2 \text{ tons/day}$$

$$\text{Usage of raw material } M2 \text{ by both paints} = 1x_1 + 2x_2 \text{ tons/day}$$

$$6x_1 + 4x_2 \leq 24 \quad \text{(Raw material } M1\text{)}$$

$$x_1 + 2x_2 \leq 6 \quad \text{(Raw material } M2\text{)}$$

$$x_2 - x_1 \leq 1 \quad \text{(Market limit)}$$

$$x_2 \leq 2 \quad \text{(Demand limit)}$$

A market survey indicates that the daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton. Also, the maximum daily demand for interior paint is 2 tons.

Modeling: Reddy Mikks Problem

$$\text{Maximize } z = 5x_1 + 4x_2$$

- Subject to

$$\begin{aligned} 6x_1 + 4x_2 &\le 24 \\ x_1 + 2x_2 &\le 6 \\ -x_1 + x_2 &\le 1 \\ x_2 &\le 2 \\ x_1, x_2 &\ge 0 \end{aligned}$$

- Feasible solution: A solution (x_1, x_2) satisfying all constraints
- Infeasible solution: Otherwise
- Check
  - Solution (3,1)
  - Another solution (4,1)

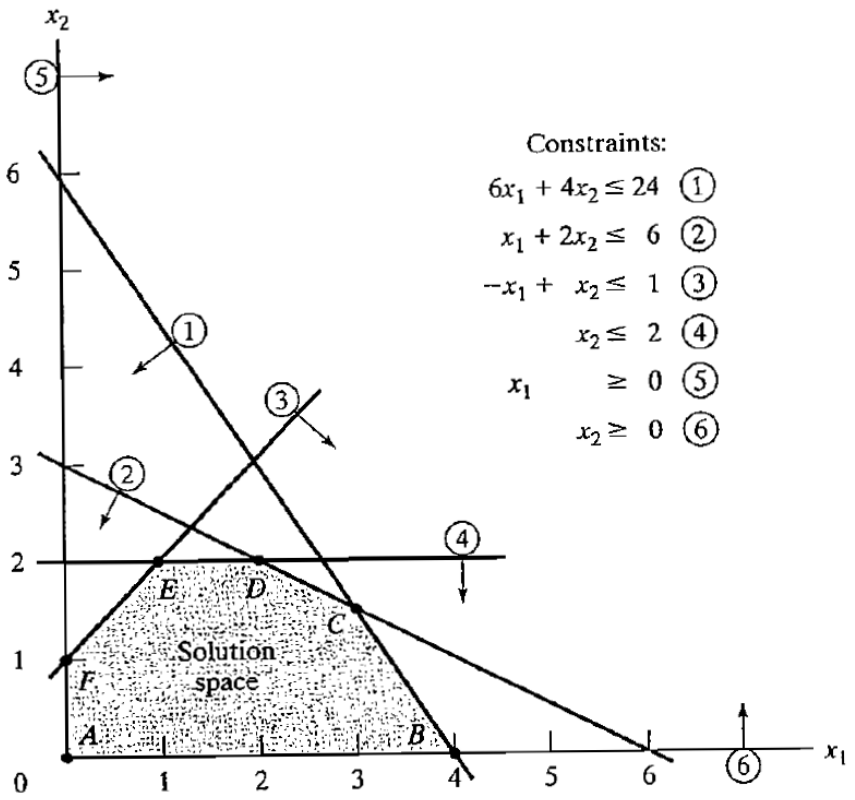
Properties of LP Model

- Objective function and constraints are linear. Linearity implies that LP must satisfy three basic properties
  - Proportionality
    - Contribution of each decision variable in both the objective function and constraints is directly proportional to the value of variable.
  - Additivity
    - Total contribution of all decision variables in the objective function and in the constraints to be the direct sum of the individual contribution of each variable.
  - Certainty
    - Coefficients of the objective function and the constraints are deterministic.

Graphical Solution to Reddy Mikks Model

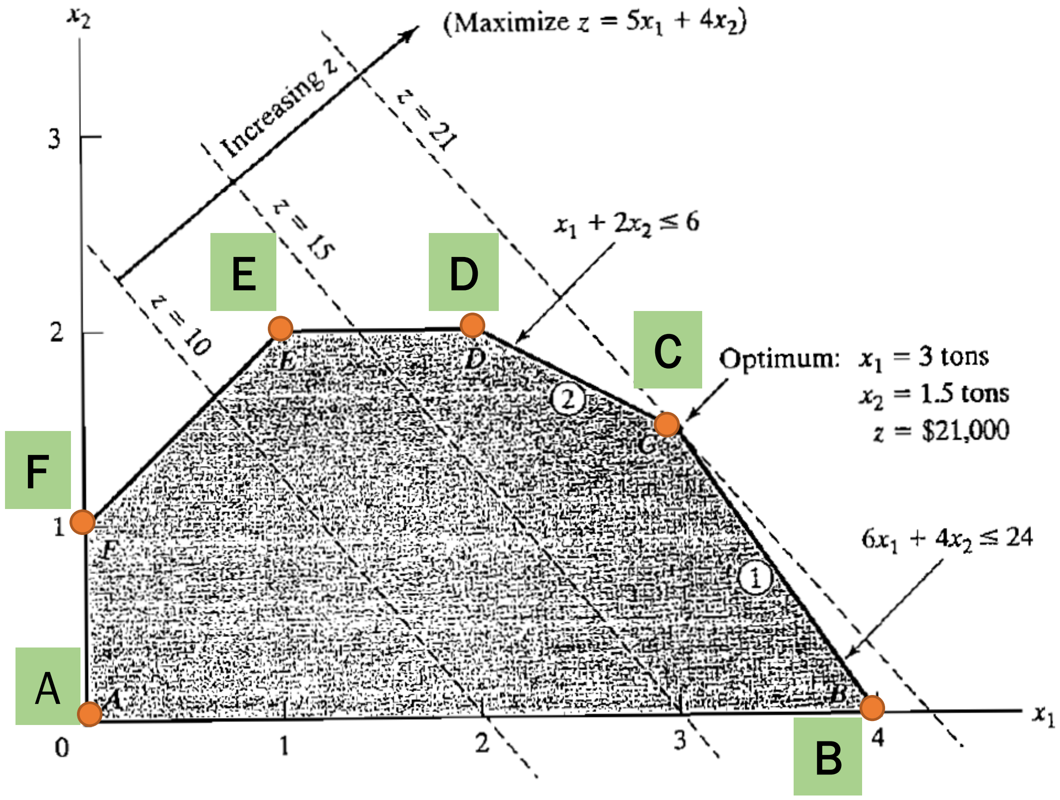
- Determination of the feasible solution space
- Determination of the optimum solution among all the feasible points in the solution space.

Reddy Mikks Model: Determine the feasible space



Graphical Solution to Reddy Mikks Model

- Determine optimal solution



Graphical Solution to Reddy Mikks Model

- Corner Solution
  - Find the optimum solution from the corner solutions only.

Corner point	(x_1, x_2)	z
A	(0, 0)	0
B	(4, 0)	20
C	**(3, 1.5)**	**21 (OPTIMUM)**
D	(2, 2)	18
E	(1, 2)	13
F	(0, 1)	4

Diet Model

Example 2.2-2 (Diet Problem)

Ozark Farms uses at least 800 lb of special feed daily. The special feed is a mixture of corn and soybean meal with the following compositions:

	lb per lb of feedstuff		
Feedstuff	*Protein*	*Fiber*	Cost (\$/lb)
Corn	.09	.02	.30
Soybean meal	.60	.06	.90

The dietary requirements of the special feed are at least 30% protein and at most 5% fiber. Ozark Farms wishes to determine the daily minimum-cost feed mix.

- Decision Variables

x_1 = lb of corn in the daily mix

x_2 = lb of soybean meal in the daily mix

Diet Model

- Objective function:

$$\text{Minimize } z = .3x_1 + .9x_2$$

- Constraints

$$x_1 + x_2 \geq 800$$

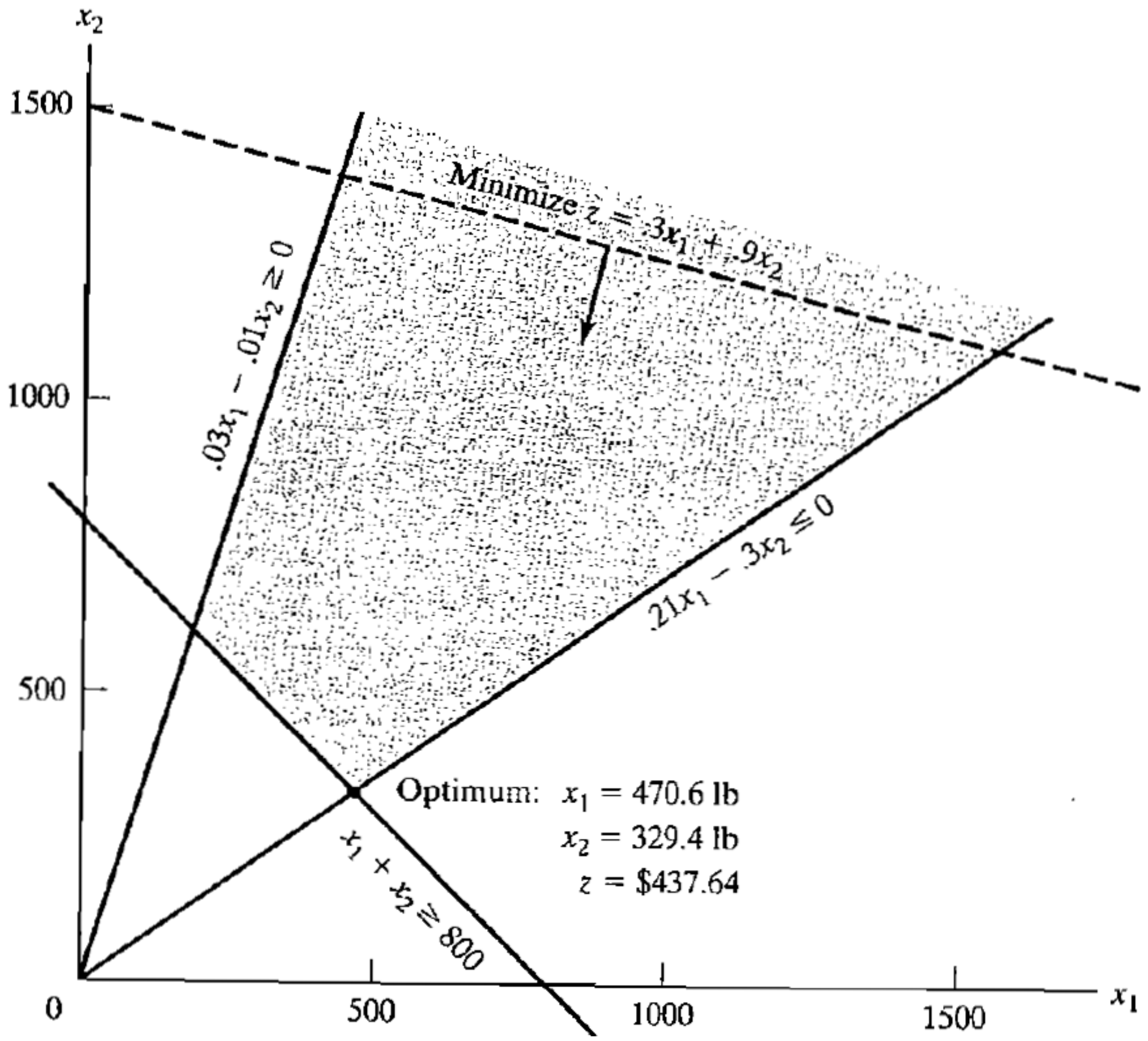
$$.09x_1 + .6x_2 \geq .3(x_1 + x_2)$$

$$.02x_1 + .06x_2 \leq .05(x_1 + x_2)$$

- Variable bound

$$x_1, x_2 \geq 0$$

Graphical Solution to Diet Model



LP Applications

- Many examples are given in the book by TAHA (Self reading)
- Selected LP applications
  - Production Planning and Inventory Control
  - Bus Scheduling
  - Trim Loss

Production Planning and Inventory Control Model

In preparation for the winter season, a clothing company is manufacturing parka and goose overcoats, insulated pants, and gloves. All products are manufactured in four different departments: cutting, insulating, sewing, and packaging. The company has received firm orders for its products. The contract stipulates a penalty for undelivered items. The following table provides the pertinent data of the situation.

	Time per units (hr)				
Department	*Parka*	*Goose*	*Pants*	*Gloves*	Capacity (hr)
Cutting	.30	.30	.25	.15	1000

Devise an optimal production plan for the company.

Production Planning and Inventory Control Model

- Decision Variables

x_1 = number of parka jackets

x_2 = number of goose jackets

x_3 = number of pairs of pants

x_4 = number of pairs of gloves

- Objective function

Net receipts = Total profit − Total penalty

s_j = Number of shortage units of product j, $j = 1, 2, 3, 4$

Demand

$$x_1 + s_1 = 800,\ x_2 + s_2 = 750,\ x_3 + s_3 = 600,\ x_4 + s_4 = 500$$

$$x_j \geq 0,\ s_j \geq 0,\ j = 1, 2, 3, 4$$

Nonnegative condition

$$\text{Maximize } z = 30x_1 + 40x_2 + 20x_3 + 10x_4 - (15s_1 + 20s_2 + 10s_3 + 8s_4)$$

Production Planning and Inventory Control Model

- Constraints

$$.30x_1 + .30x_2 + .25x_3 + .15x_4 \quad 1000 \quad \text{(Cutting)}$$
$$.25x_1 + .35x_2 + .30x_3 + .10x_4 \quad 1000 \quad \text{(Insulating)}$$
$$.45x_1 + .50x_2 + .40x_3 + .22x_4 \quad 1000 \quad \text{(Sewing)}$$
$$.15x_1 + .15x_2 + .10x_3 + .05x_4 \quad 1000 \quad \text{(Packaging)}$$

	Time per units (hr)				
Department	*Parka*	*Goose*	*Pants*	*Gloves*	Capacity (hr)
Cutting	.30	.30	.25	.15	1000
Insulating	.25	.35	.30	.10	1000
Sewing	.45	.50	.40	.22	1000
Packaging	.15	.15	.1	.05	1000

Production Planning and Inventory Control Model

$$\text{Maximize } z = 30x_1 + 40x_2 + 20x_3 + 10x_4 - (15s_1 + 20s_2 + 10s_3 + 8s_4)$$

$$.30x_1 + .30x_2 + .25x_3 + .15x_4 \leq 1000 \quad \text{(Cutting)}$$

$$.25x_1 + .35x_2 + .30x_3 + .10x_4 \leq 1000 \quad \text{(Insulating)}$$

$$.45x_1 + .50x_2 + .40x_3 + .22x_4 \leq 1000 \quad \text{(Sewing)}$$

$$.15x_1 + .15x_2 + .10x_3 + .05x_4 \leq 1000 \quad \text{(Packaging)}$$

$$x_1 + s_1 = 800,\ x_2 + s_2 = 750,\ x_3 + s_3 = 600,\ x_4 + s_4 = 500$$

$$x_j \geq 0,\ s_j \geq 0,\ j = 1, 2, 3, 4$$

Bus Scheduling Model

Progress City is studying the feasibility of introducing a mass-transit bus system that will alleviate the smog problem by reducing in-city driving. The study seeks the minimum number of buses that can handle the transportation needs. After gathering necessary information, the city engineer noticed that the minimum number of buses needed fluctuated with the time of the day and that the required number of buses could be approximated by constant values over successive 4-hour intervals. Figure 2.8 summarizes the engineer's findings. To carry out the required daily maintenance, each bus can operate 8 successive hours a day only.

x_1 = number of buses starting at 12: 01 A.M.

x_2 = number of buses starting at 4:01 A.M.

x_3 = number of buses starting at 8:01 A.M.

x_4 = number of buses starting at 12:01 P.M.

x_5 = number of buses starting at 4:01 P.M.

x_6 = number of buses starting at 8:01 P.M.

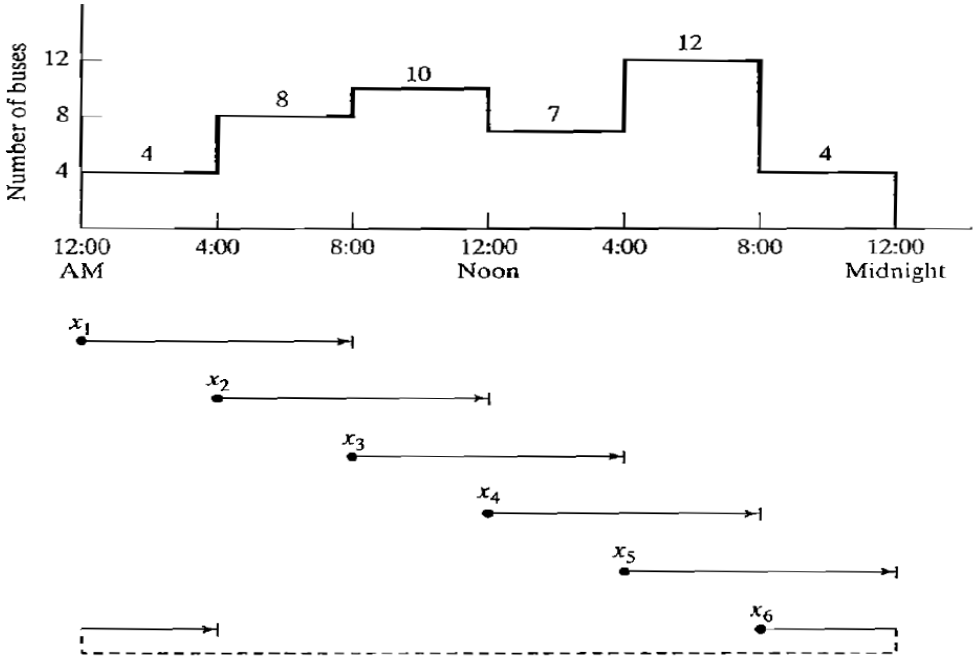


FIGURE 2.8
Number of buses as a function of the time of the day

Bus Scheduling Model

Time period	Number of buses in operation
12:01 A.M. – 4:00 A.M.	$x_1 + x_6$
4:01 A.M. – 8:00 A.M.	$x_1 + x_2$
8:01 A.M. – 12:00 noon	$x_2 + x_3$
12:01 P.M. – 4:00 P.M.	$x_3 + x_4$
4:01 P.M. – 8:00 P.M.	$x_4 + x_5$
8:01 A.M. – 12:00 A.M.	$x_5 + x_6$

$$\text{Minimize } z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

subject to

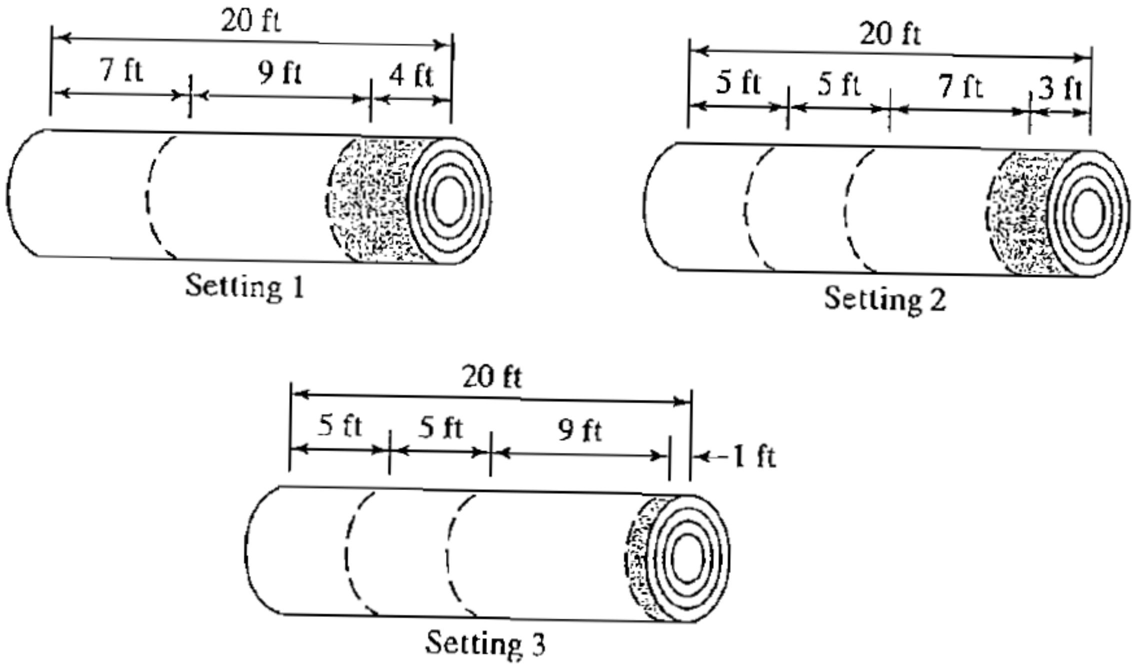
$$\begin{array}{lll}
x_1 + x_6 & \geq 4 & \text{(12:01 A.M.-4:00 A.M.)} \\
x_1 + x_2 & \geq 8 & \text{(4:01 A.M.-8:00 A.M.)} \\
x_2 + x_3 & \geq 10 & \text{(8:01 A.M.-12:00 noon)} \\
x_3 + x_4 & \geq 7 & \text{(12:01 P.M.-4:00 P.M.)} \\
x_4 + x_5 & \geq 12 & \text{(4:01 P.M.-8:00 P.M.)} \\
x_5 + x_6 & \geq 4 & \text{(8:01 P.M.-12:00 P.M.)}
\end{array}$$

$$x_j \geq 0,\ j = 1, 2, \ldots, 6$$

Trim Loss or Stock Slitting Model

The Pacific Paper Company produces paper rolls with a standard width of 20 feet each. Special customer orders with different widths are produced by slitting the standard rolls. Typical orders (which may vary daily) are summarized in the following table:

Order	Desired width (ft)	Desired number of rolls
1	5	150
2	7	200
3	9	300



- Setting knife to the desired width
- A case of 3 settings as shown in Fig.
- Example of feasible combination
  - Slit 300 rolls using setting 1 and 75 rolls using setting 2
  - Slit 200 rolls using setting 1 and 100 rolls using setting 3
  - Which combination is better?

Trim Loss or Stock Slitting Model

- Trim loss as evaluation of goodness of solution
- Assuming the standard roll is of length L
- Trim loss
  - Combination 1: 300 (4XL) + 75 (3XL) = 1425 ft^2
  - Combination 2: 200 (4XL) + 100 (1XL) = 900 ft^2
- Surplus of 5-, 7-, and 9-ft rolls must be considered in the computation of trim loss
  - Combination 1: Surplus of 300-200 = 100 extra 7-ft rolls in setting 1 and 75 extra 7–ft rolls in setting 2. Total waste = 175 (7XL) = 1225 ft^2.
  - Combination 2: Setting 3 produces 200-150 = 50 extra 5-ft rolls and total waste is 50 (5XL) = 250L ft^2.
- Total trim loss area is
  - Combination 1: 1425L + 1225L = 2650L ft^2.
  - Combination 2: 900L + 250L = 1150L ft^2. **(BETTER)**

Mathematica Model: Trim Loss

- Minimize trim-loss area by satisfying the demand.
- Promising setting cannot yield a trim-loss roll of width 5 feet or larger.

Required width (ft)	Knife setting 1	2	3	4	5	6	Minimum number of rolls
5	0	2	2	4	1	0	150
7	1	1	0	0	2	0	200
9	1	0	1	0	0	2	300
Trim loss per foot of length	4	3	1	0	1	2	

x_j = number of standard rolls to be slit according to setting j, $j = 1, 2, \ldots, 6$

$$\begin{aligned}
\text{Number of 5-ft rolls produced} &= 2x_2 + 2x_3 + 4x_4 + x_5 \geq 150 \\
\text{Number of 7-ft rolls produced} &= x_1 + x_2 + 2x_5 \geq 200 \\
\text{Number of 9-ft rolls produced} &= x_1 \quad x_2 + x_3 + 2x_5 + 2x_6 \geq 300
\end{aligned}$$

Mathematica Model: Trim Loss

$$\text{Total area of standard rolls} = 20L(x_1 + x_2 + x_3 + x_4 + x_5 + x_6)$$

$$\text{Total area of orders} = L(150 \times 5 + 200 \times 7 + 300 \times 9) = 4850L$$

$$\text{Minimize } z = 20L(x_1 + x_2 + x_3 + x_4 + x_5 + x_6) - 4850L$$

$$\text{Minimize } z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

subject to

$$\begin{aligned}
2x_2 + 2x_3 + 4x_4 + x_5 &\geq 150 \text{ (5-ft rolls)} \\
x_1 + x_2 + 2x_5 &\geq 200 \text{ (7-ft rolls)} \\
x_1 + x_3 + 2x_6 &\geq 300 \text{ (9-ft rolls)} \\
x_j \geq 0, j = 1, 2, \ldots, 6 &
\end{aligned}$$