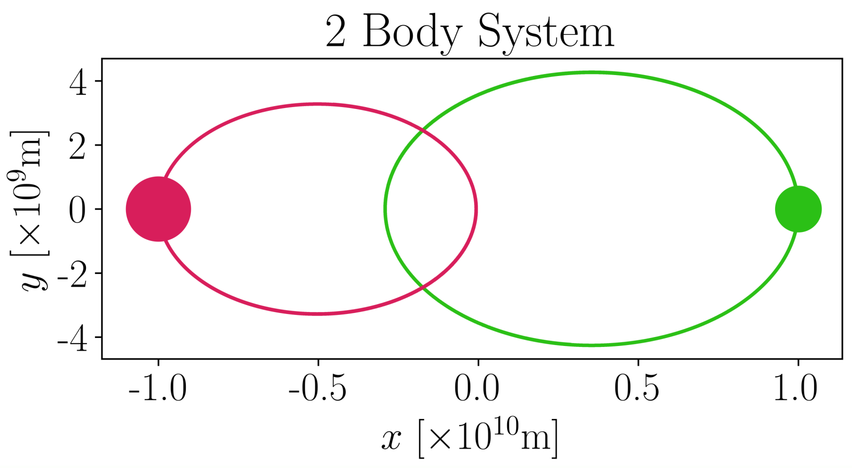
N-body Simulations of Gravitational Systems Using a Leapfrog Integration Method



2-body simulation of similar masses orbiting a common barycentre. (Solvable)

With N>2, there is no analytical solution for the movement of a system acting under gravity. Therefore we must use numerical methods to study system properties. The simplest method involves brute-force calculation of forces between all particles in the system.

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Loop over n timesteps:
   Loop over n particles:
      → Calculate force from
      n other particles
      → Integrate to find
      position and velocity
      → Calculate energy
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The code can be optimised due to equal & opposite force, but still requires O(n^2) time. Optimisation requires complex computation (e.g. the Barnes-Hut[1] method which groups the force from distant particles and runs in O(n log(n)) time).

Currently, simulation can process ~10^5 timesteps/minute for 10 bodies in random conditions.

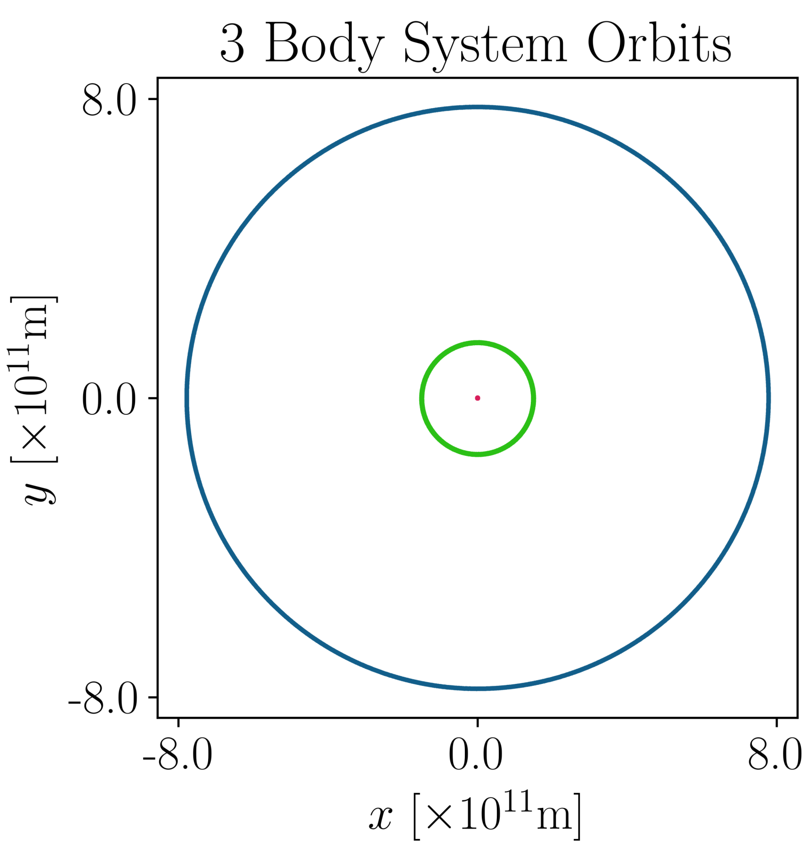
$$\boldsymbol{F}_i = -G\sum_{j\neq i} m_i m_j \frac{\boldsymbol{r}_i - \boldsymbol{r}_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^3}$$

$$\boldsymbol{v}_i^{n+1/2} = \boldsymbol{v}_i^{n-1/2} + \boldsymbol{F}_i dt/m_i$$

$$\boldsymbol{r}_i^{n+1} = \boldsymbol{x}_i^{n} + \boldsymbol{v}_i^{n+1/2} dt$$

Velocity is calculated out of step with position (at $t_0 \pm dt/2$); this is called a Leapfrog integration method. It's a 2nd order method with very few calculations which is well suited to this system.

As $r \to 0$, $F \to \infty$ so particles at close separations will be pushed away at v>>c. We add a softening constant ε to the denominator of F which causes particles to act as spheres of radius ε/2.[2]



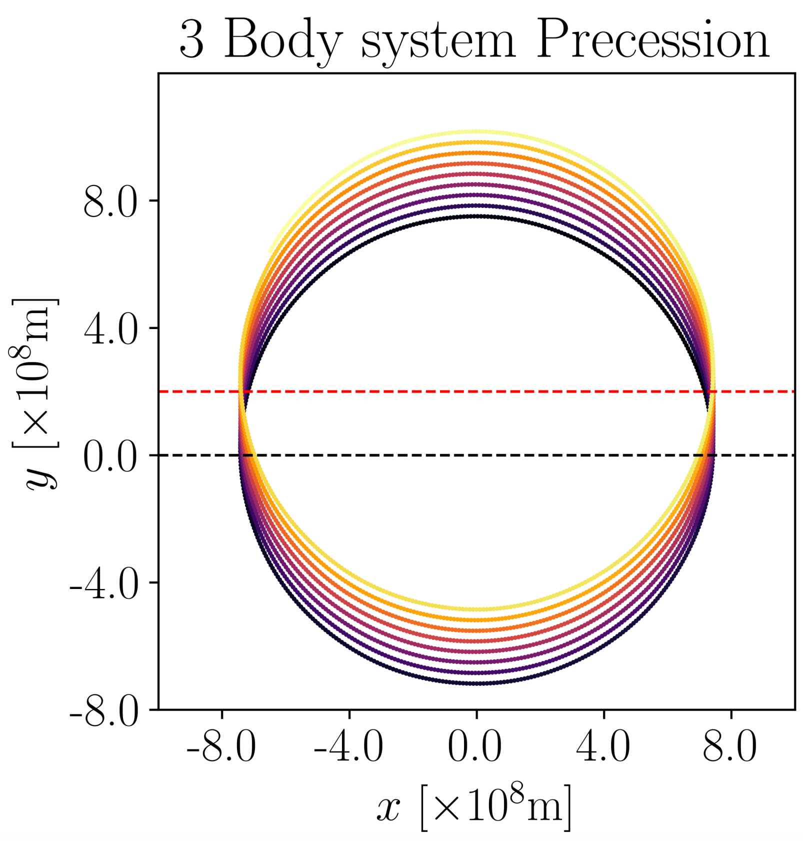
Sun-Earth-Jupiter system orbits over 100yrs. Sun orbit is 1000x smaller than Jupiter's so only visible as a point here.

Sun / Earth / Jupiter

- Simulating the 3-body system of the Earth, Sun, and Jupiter over 100yrs with a timestep of 1 day (L, R)
- Energy loss was slightly under 1% over entire simulation time (A system only acting gravitationally should conserve energy but computationally may lose energy).

Precession of orbits can be observed (R), as system is not stable over longer time periods

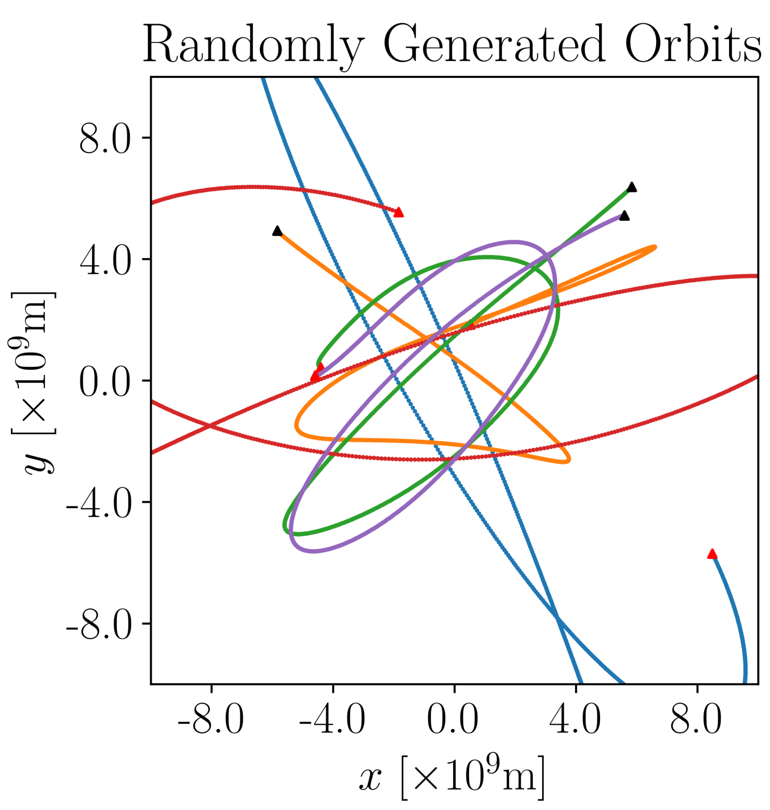
- Initial conditions calculated as if Sun and Jupiter were a 2-body system, then Earth was added by considering the Earth-Sun 2 body system (these can be found analytically)



Precession of Solar orbit (100yrs), from dark to light. Black line shows starting position of orbit midpoint, red line shows end position. Note axis scale is 10^8 m.

This program can be extended to simulate N bodies over an arbitrary 3D region, and can simulate a wide range of situations, including light particles in an existing vector field, multiple clusters of masses travelling towards one another, or collapse of a system over time. This gives many options for the future of this project.

- Galactic collisions (2 clusters of stellar masses travelling towards one another)
- Planetary formation (small particles colliding with a chance of combining into larger particles)
- Testing other simulation methods[1,3]



5 particles generated in 2D plane with random positions and velocities, with equal masses. Simulated for 10^7 s with 10^5 timesteps. Black triangles show start position and red ones show end position.

[1] Barnes, J., Hut, P. A hierarchical O(N log N) force-calculation algorithm. Nature 324, 446–449 (1986) doi:10.1038/324446a0

[2] Dehnen, W. (2001). Towards optimal softening in three-dimensional N-body codes - I. Minimizing the force error. Monthly Notices of the Royal Astronomical Society, 324(2), pp.273-291.

[3] Aarseth, S.J., 2003. Gravitational N-body simulations: tools and algorithms. Cambridge University Press