Elliptic curves in Nemo

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- hard to invert (exponential time?),

then there is an analogue of the Diffie–Hellman key exchange [2].

$$\begin{tikzcd}[column sep=small]
 & x_0 \arrow[dl, dash, "a"'] \arrow[dr, dash, "b"] & & \\
a \star x_0 \arrow[dr, dash, "b"'] & & b \star x_0 \arrow[dl, dash, "a"] & \text{public} \\
 & (ab) \star x_0 & & \text{shared secret}
\end{tikzcd}$$

The Couveignes–Rostovtsev–Stolbunov scheme

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Use the action of a class group on a set of isogenous elliptic curves.

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Goals

- Explain what this means
- Describe the computations needed
- Discuss our EllipticCurves module for Nemo.

1. Context
2. An example in isogeny-based cryptography
   - Basics
   - Computations
3. The EllipticCurves module for Nemo
   - Contents
   - Further development
   - Some benchmarks
4. Conclusion

Elliptic curves over k

- *Elliptic curves* over a field k are algebraic curves that have an abelian group structure, e.g.

$$\begin{aligned} E_1 &: y^2 + a_1xy = x^3 + a_2x^2 + a_4x + a_6 \\ E_2 &: y^2 = x^3 + ax + b \\ E_3 &: By^2 = x^3 + Ax^2 + x. \end{aligned}$$

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- *Isogenies* are nonzero morphisms. Our isogenies will be defined over k. As rational maps, they have *degrees*.

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If we know this *kernel polynomial*, we can easily find E' using Vélu's formulas.

Action of the class group

Proposition

Let $E/\mathbb{F}_p$ be an ordinary elliptic curve.

- The ring $\mathrm{End}(E)$ is isomorphic to a quadratic order $\mathcal{O}$.
- For each prime number ℓ, there are either 2 (split case), 1 (ramified case) or 0 (inert case) ideals in $\mathcal{O}$ of norm ℓ.
  From now on, ℓ will always be prime, odd and split.
- Ideal of norm ℓ = tuple (ℓ, v), $v \in \mathbb{Z}/\ell\mathbb{Z}$.
- There is an action on the set of elliptic curves with CM by $\mathcal{O}$. Ideals of norm ℓ act as ℓ-isogenies.
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How can we compute this action ?

Main algorithm

Problem

Given $E/\mathbb{F}_p$ and a prime $\ell \neq p$, how can we compute the curves linked to E by an ℓ-isogeny?

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Most general idea

Let $\Phi_\ell(X, Y)$ be the ℓ^{th} classical modular polynomial. The two roots j_1, j_2 of

$$\Phi_\ell(j(E), Y)$$

are the j-invariants of the neighbors of E. To choose the one corresponding to an ideal (ℓ, v):

- compute the kernel $K(x)$ of the isogeny $E \to j_1$
- check if the Frobenius acts on it as scalar mult. by v:

$$(x^p, y^p) \stackrel{?}{=} [v] \cdot (x, y) \mod K(x) \text{ and curve equation.}$$

Bostan–Morain–Salvy–Schost [1]

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Algorithm

- Normalize ϕ (involves evaluating modular polynomials)
- Solve this ODE in power series up to a certain precision with a Newton iteration
- Recover $K(x)$ using the Berlekamp–Massey rational reconstruction algorithm.

Another solution

Problem

Given $E/\mathbb{F}_p$ and a prime $\ell \neq p$, how can we compute the curves linked to E by an ℓ-isogeny?

Finding roots of modular polynomials is costly : $\Phi_\ell(X, Y)$ has degree $\ell + 1$ in both variables.

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More specific idea

Suppose that for some d, K is the only subgroup of order ℓ in E whose points are defined over $\mathbb{F}_{p^d}$.

- Look for ℓ-torsion points over this field to find K, using scalar multiplications
- Compute the curve E/K using Vélu's formulas.

The isogeny $E \to E/K$ has degree ℓ.

Finding adequate curves

This second method is only efficient with small-degree extensions.
Not every curve satisfies the conditions before for small d: we have to look for adequate curves.
In practice, we have to use both algorithms, general and specific.

What we would like Nemo to do

In the general method:

- Define elliptic curves over finite fields and general rings
- Define isogenies, sacalar multiplication and isomorphisms
- Find roots of polynomials over finite fields
- Solve ODEs in power series with Newton iterations
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In the specific method:

- Define points on elliptic curves
- Arithmetic operations on elliptic curves
- Extensions of finite fields.

Types for curves

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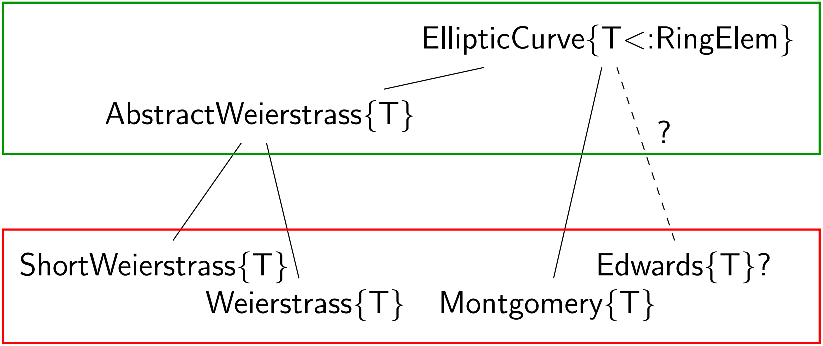

E.g. `j-invariant` is defined for the EllipticCurve type, while `a-invariants` is only defined for AbstractWeierstrass.

Types for maps

Maps can be isomorphisms between different models (evaluate on points), isogenies (compute image and kernels), scalar multiplications (both?).

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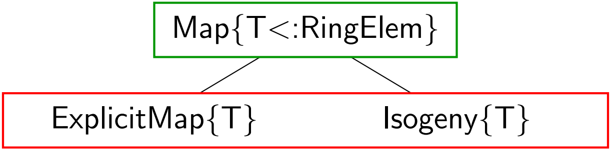


```
immutable ExplicitMap{T} <:  Map{T}
domain::EllipticCurve{T}
image::EllipticCurve{T}
map::Function
end
```

```
immutable Isogeny{T} <:  Map{T}
domain::EllipticCurve{T}
degree::Integer
kernel::PolyElem{T}
image::EllipticCurve{T}
end
```

Types for points

- Should points be attached with a curve?
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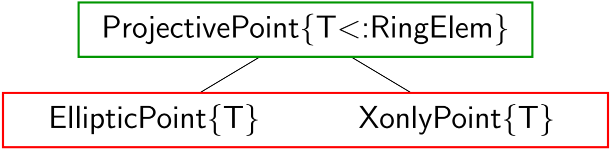



```
type EllipticPoint{T} <:
ProjectivePoint{T}
X::T
Y::T
Z::T
curve::EllipticCurve{T}
end
```

```
type XonlyPoint{T} <:
ProjectivePoint{T}
X::T
Z::T
curve::MontgomeryCurve{T}
end
```

What EllipticCurves can do

- Define curves, points and maps, check for equality and validity
- Basic functions such as invariants
- Compute isomorphisms between different models
- Generic arithmetic on Weierstrass/Montgomery curves
- Efficient x-only arithmetic on Montgomery curves
- Division polynomials for ShortWeierstrass
- Isogeny computations : Vélu's formulas and the BMSS algorithm for short Weierstrass and Montgomery curves
- Modular polynomials (for small ℓ's)
- Over finite fields : random points, torsion points, computation of Frobenius eigenvalues.

Useful things that should be done elsewhere

In finite fields :

- Multiplicative orders
- Random elements
- Square roots
- Roots of polynomials and irreducible polynomials
- Field extensions over *prime* fields

Others:

- Derivatives of multivariate polynomials

Further possible development

- Call (system) PARI to compute the cardinality of curves over finite fields
- Compute modular polynomials/equations on the fly?
- Zeta functions?
- Have p-adic numbers to compute isogenies in small characteristic?
- Go down the arithmetic route for elliptic curves over number fields or local fields?

Three ways to compute roots over $\mathbb{F}_p$

At present, there is no direct way to do this in Nemo.

Sol. 1 (Nemo)

```
function roots(P)
A = parent(P)
X = gen(A)
R = ResidueRing(A, P)
Frob = R(X)^BigInt(p)
Frob = data(Frob)
g = gcd(Frob - X, P)
fact = factor(g)
...  # recover roots
end
```

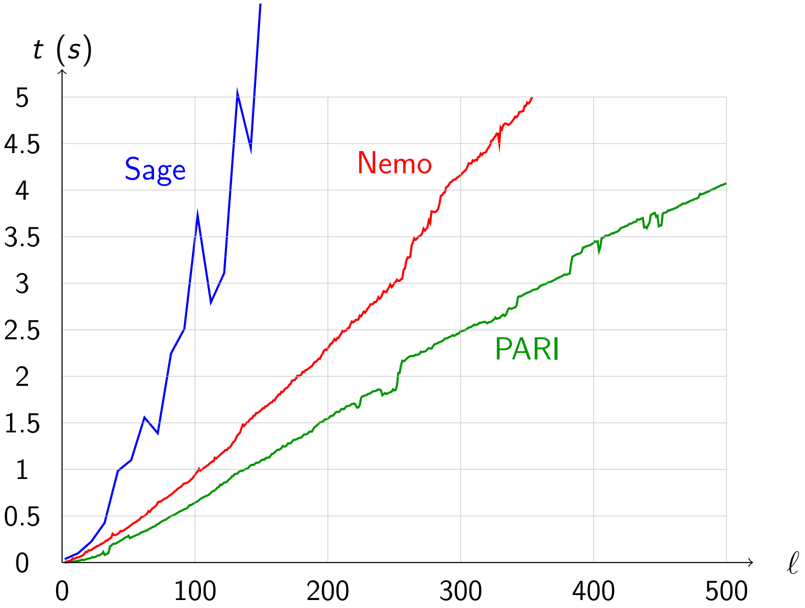
Sol. 2 (Sage/PARI)

```
def roots(P):
Q = pari(P)
rts = Q.polrootsmod(p)
return rts.sage()
```

Sol. 3 (Sage)

```
def roots(P):
A = P.parent()
X = A.gen()
R = A.quotient(P)
Frob = R(X)**p
Frob = Frob.lift()
g = gcd(Frob, P)
return g.roots()
```

Timing results



Three ways to compute scalar multiplications

Sol. 1 (Nemo)

```
E = Weierstrass(...)
Fext, _ = FiniteField(p^d, alpha)
Eext = base_extend(E, Fext)
P = random(Eext)
times(p^d, P)
```

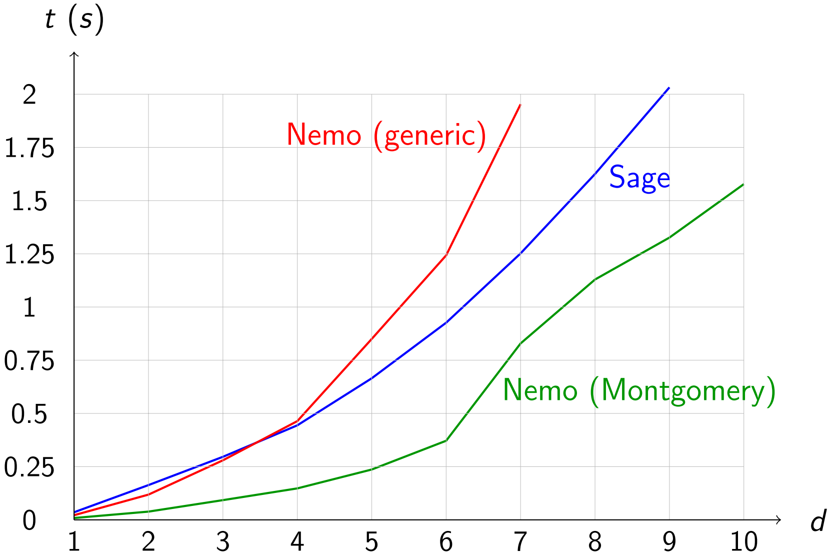
Sol. 2 (Nemo)

```
E = Montgomery(...)
Fext, _ = FiniteField(p, d, alpha)
Eext = base_extend(E, Fext)
P = randomXonly(Eext)
times(p^d, P)
```

Sol. 3 (Sage)

```
E = EllipticCurve(...)
Fext = FiniteField(p**d, "alpha")
Eext = E.base_extend(Fext)
P = Eext.random_element()
C = p**d
C * P
```

Timing results



Questions

- Can we do better to compute roots of polynomials over finite fields?
- Can we do better for non-prime finite fields?

Take home messages

- ...

Thank you!

References

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