Graph Neural Networks: introduction

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Table of Content

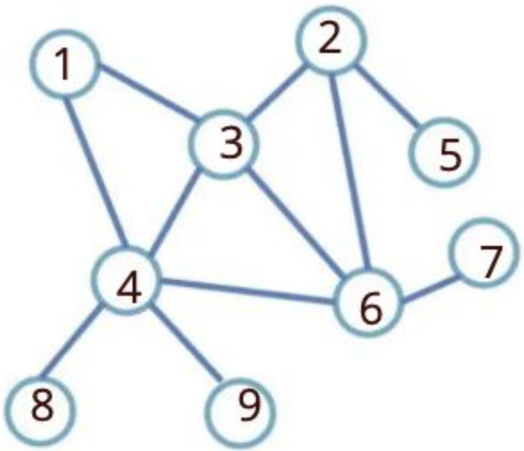
What are graphs?

Definition of a graph

$$G = (V, E)$$

$$V = \{v_i : i \in \{1,2,\ldots,N\}\}$$

$$E \subseteq \{(v_i, v_j) : v_i, v_j \in V\}$$



Many Types of Graphs

Image credit: Medium

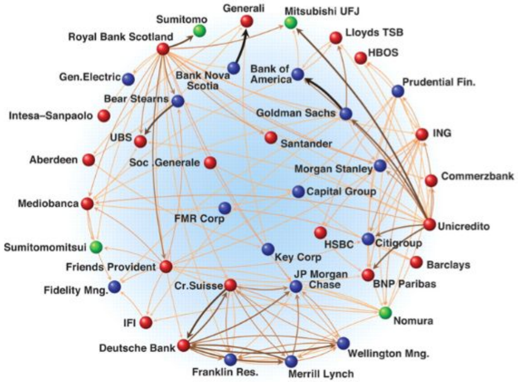


Image credit: Science

Image credit: Lumen Learning

Many Types of Graphs

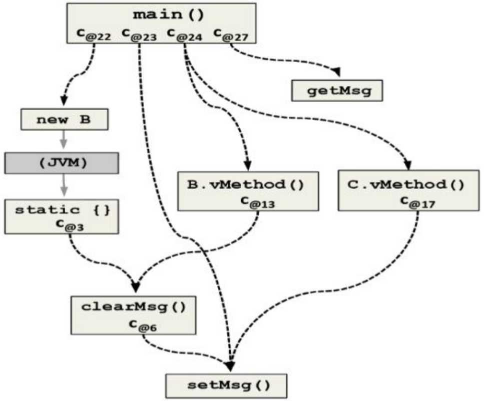


Image credit: ResearchGate

Code Graphs

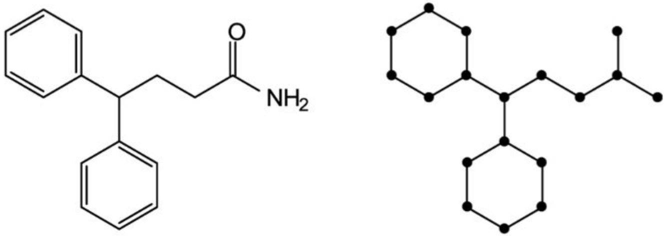


Image credit: MDPI

Molecules

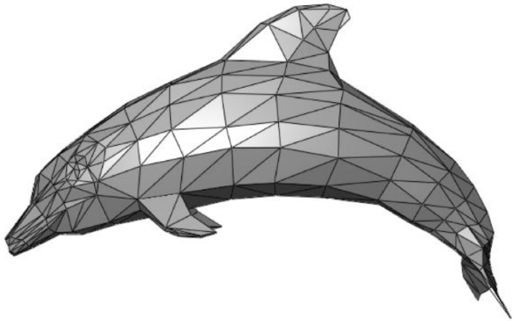
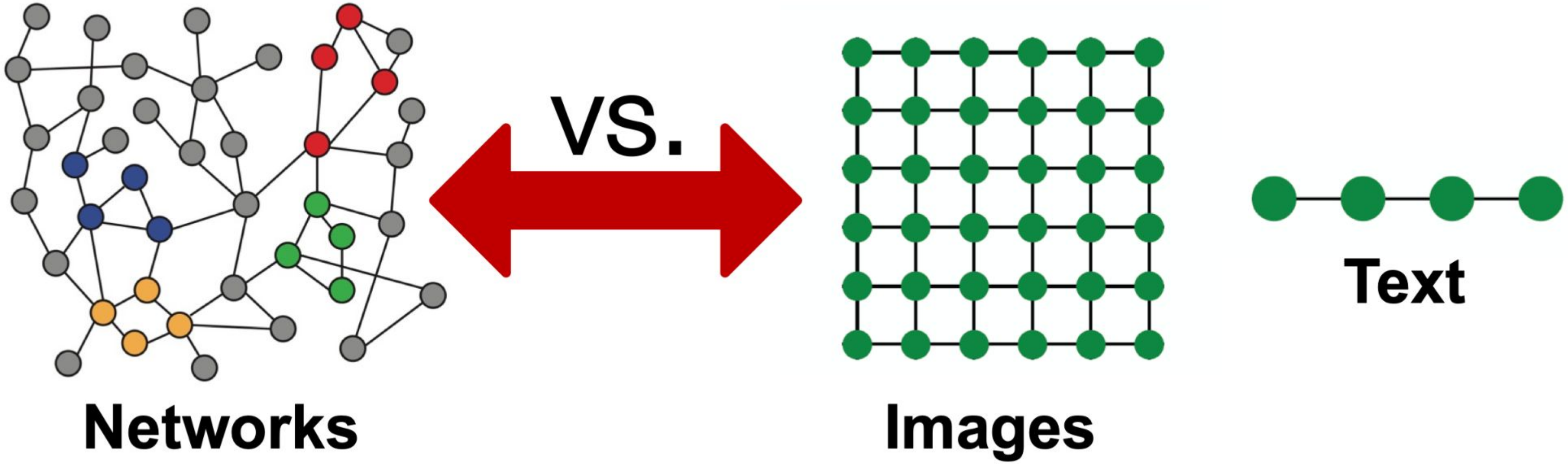


Image credit: Wikipedia

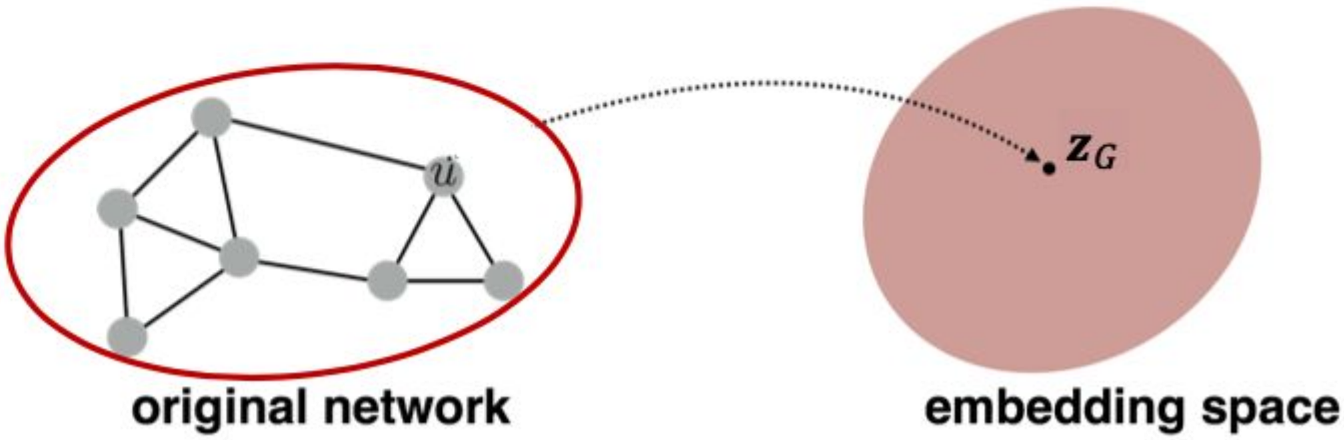
3D Shapes

Graphs are complex!

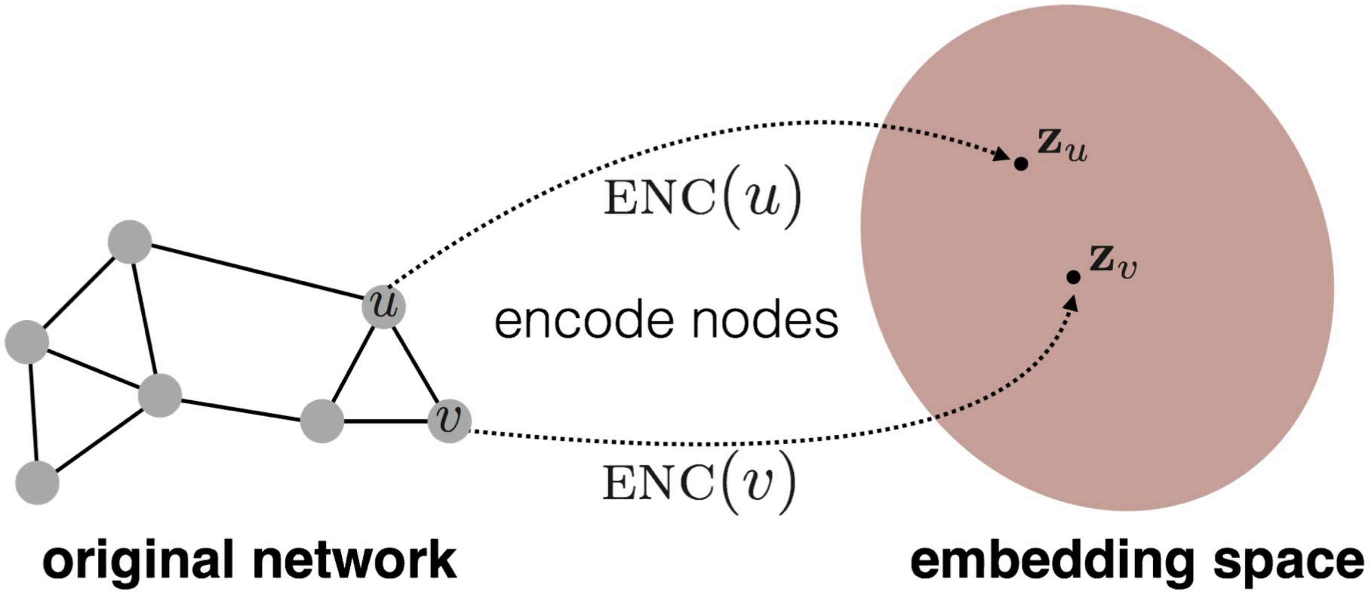


How to embed a graph?

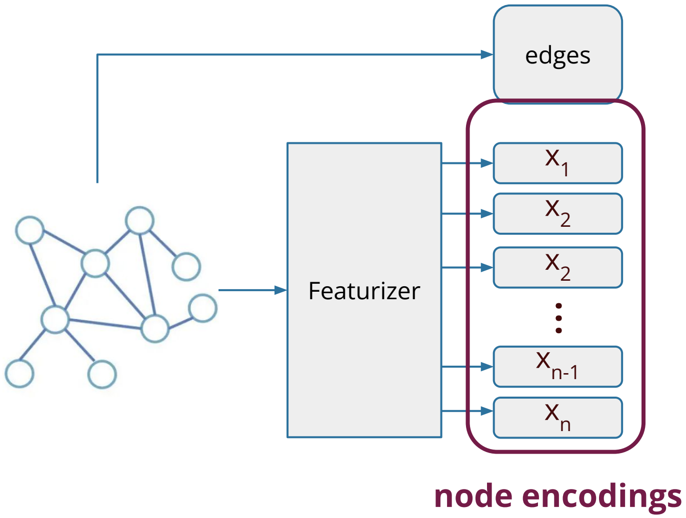
Embedding Space



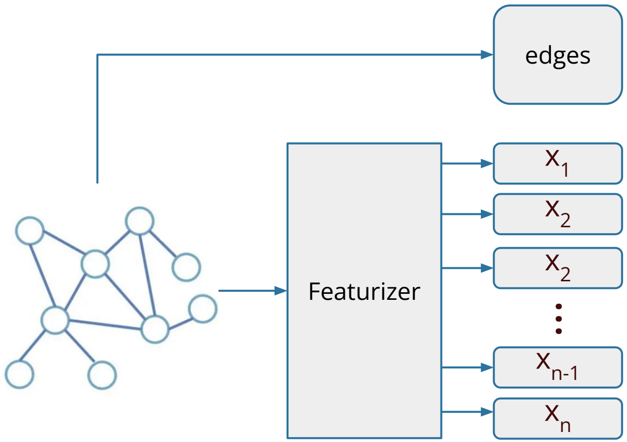
Embedding Space

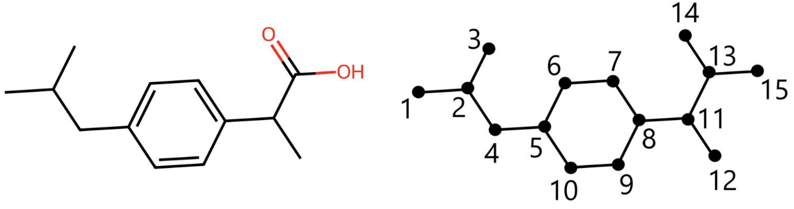


Node Encodings



Node Encodings

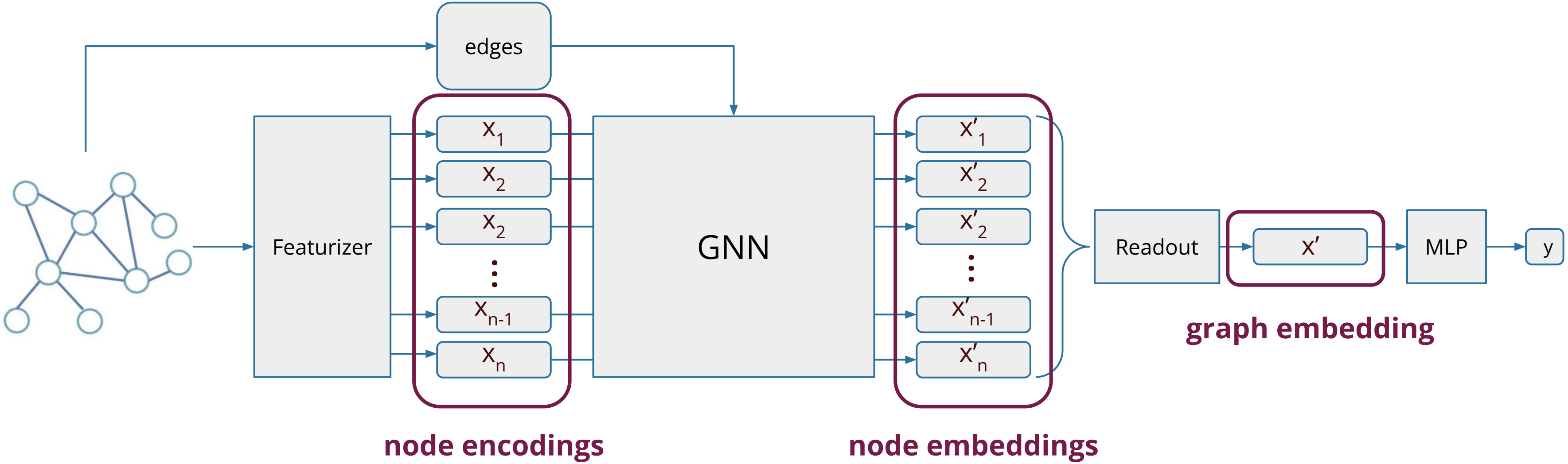




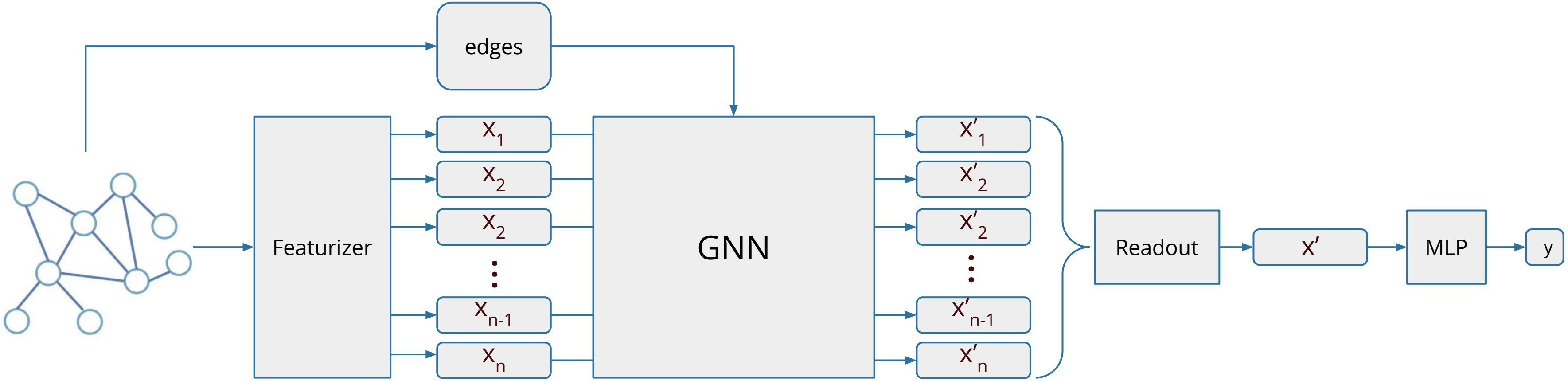
$$X = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \\ 13 \\ 14 \\ 15 \end{array} \begin{array}{c} \begin{array}{cccccc} \mathrm{C} & \mathrm{O} & \mathrm{N} & \#\mathrm{H} & \delta & q^* \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 3 & 1 & 0 \\ 1 & 0 & 0 & 1 & 3 & 0 \\ 1 & 0 & 0 & 3 & 1 & 0 \\ 1 & 0 & 0 & 2 & 2 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \\ 13 \\ 14 \\ 15 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & & 13 & 14 & 15 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Graph Neural Network



How shouldn't GNN look like?



How should GNN look like?

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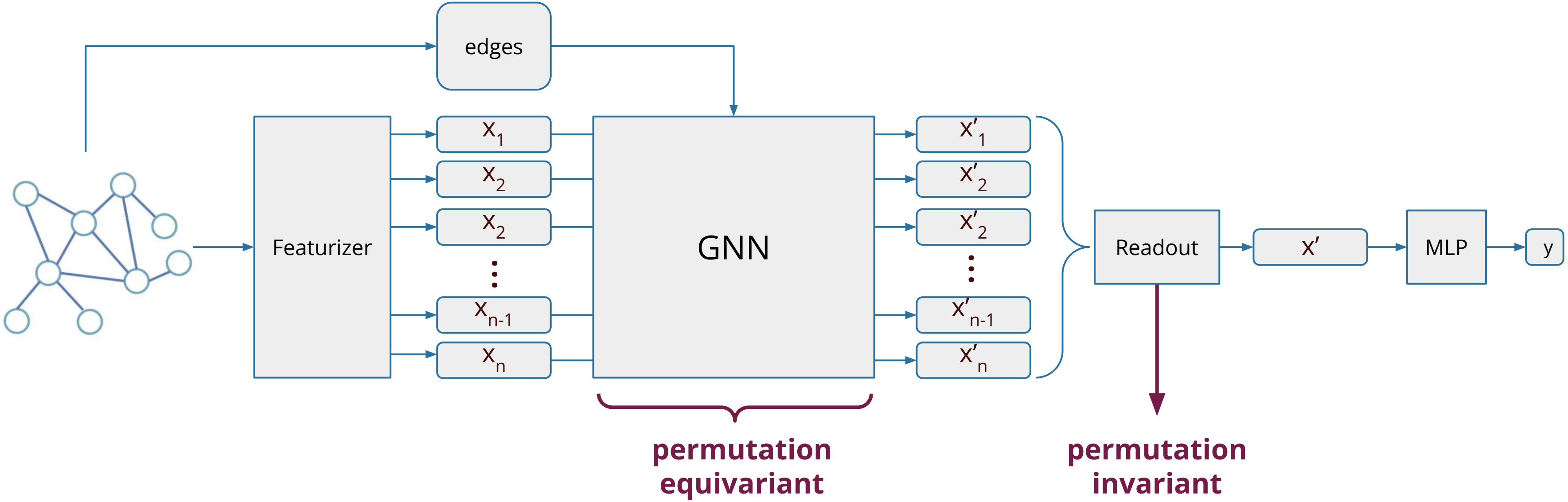
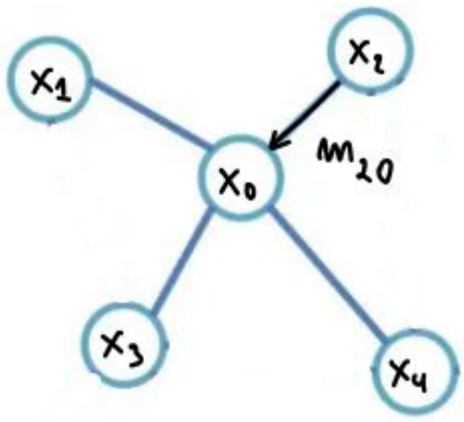


1. GNN should be permutation-equivariant.
2. GNN should deal with the graph structure given by edges.

Message Passing Neural Networks

Message Passing

Message Passing



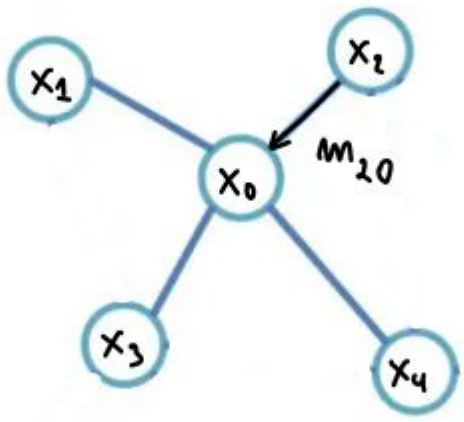
General:

$$m_{ji} = \psi(x_j, x_i)$$

Example:

$$m_{ji} = Wx_j$$

Message Passing



General:

$$m_{ji} = \psi(x_j, x_i)$$

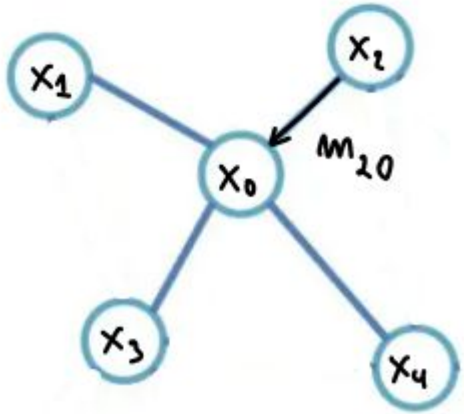
$$m_i = \square_{j \in N(i)} m_{ji}$$

Example:

$$m_{ji} = W x_j$$

$$m_i = \sum_{j \in N(i)} m_{ji}$$

Message Passing



General:

$$m_{ji} = \psi(x_j, x_i)$$

$$m_i = \square_{j \in N(i)} m_{ji}$$

$$x_i' = \rho(x_i, m_i)$$

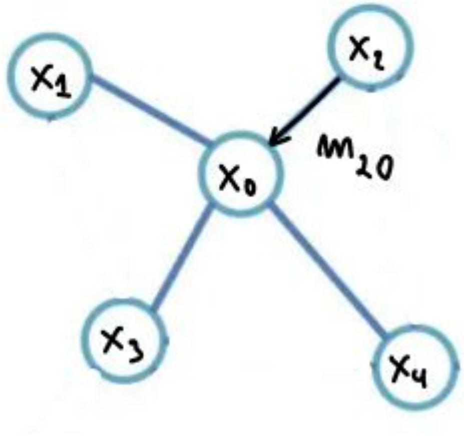
Example:

$$m_{ji} = W x_j$$

$$m_i = \sum_{j \in N(i)} m_{ji}$$

$$x_i' = W_1 x_i + W_2 m_i$$

Message Passing



General:

$$m_{ji} = \psi(x_j, x_i)$$

$$m_i = \square_{j \in N(i)} m_{ji}$$

$$x_i' = \rho(x_i, m_i)$$

$$x_i' = \rho(x_i, \square_{j \in N(i)} \psi(x_j, x_i))$$

Example:

$$m_{ji} = W x_j$$

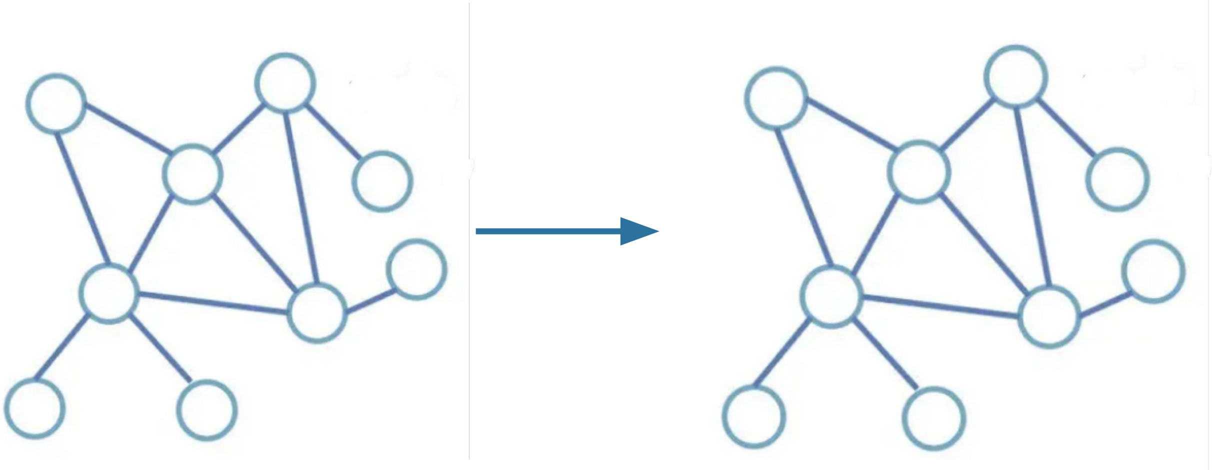
$$m_i = \sum_{j \in N(i)} m_{ji}$$

$$x_i' = W_1 x_i + W_2 m_i$$

$$x_i' = W_1 x_i + W_2 \sum_{j \in N(i)} W_3 x_j$$

Message Passing

Information Propagation

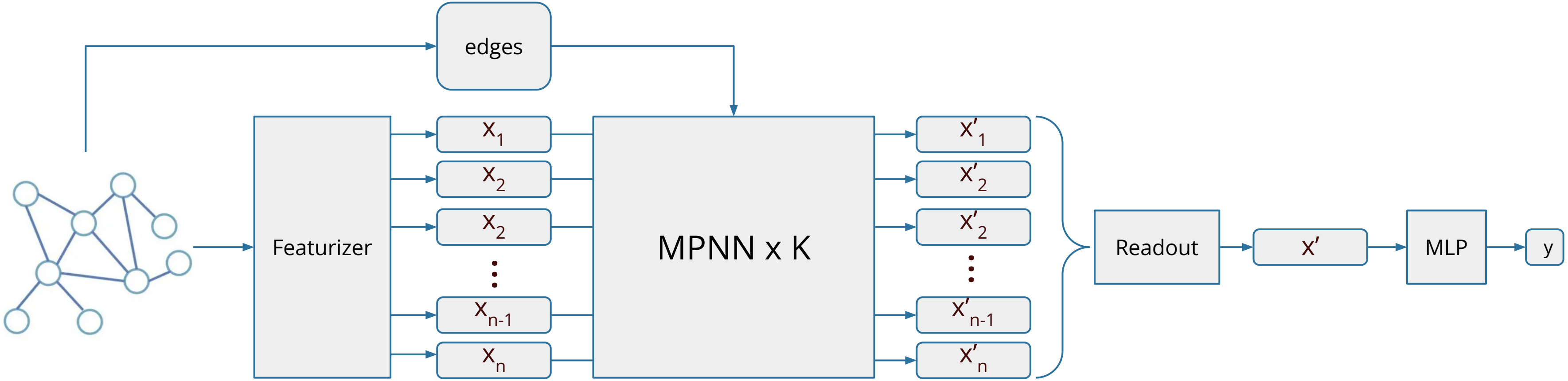


$(x_1^t, x_2^t, \ldots, x_n^t)$ $\quad\rightarrow\quad$ $(x_1^{t+1}, x_2^{t+1}, \ldots, x_n^{t+1})$

$$x_i^{t+1} = \rho(x_i^t, \square_{j \in N(i)} \psi(x_j^t, x_i^t))$$

Message Passing

Is it a proper GNN?



1. MPNN is permutation-equivariant.
2. MPNN deals with the graph structure given by edges.

Message Passing

Examples

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GIN:

$$\mathbf{x}'_i = h_{\mathbf{\Theta}} \left((1 + \epsilon) \cdot \mathbf{x}_i + \sum_{j \in \mathcal{N}(i)} \mathbf{x}_j \right)$$

GCN:

$$\mathbf{x}'_i = \mathbf{\Theta}^{\top} \sum_{j \in \mathcal{N}(i) \cup \{ i \}} \frac{e_{j,i}}{\sqrt{\hat{d}_j \hat{d}_i}} \mathbf{x}_j$$

GraphSAGE:

$$\mathbf{x}'_i = \mathbf{W}_1 \mathbf{x}_i + \mathbf{W}_2 \cdot \mathrm{mean}_{j \in \mathcal{N}(i)} \mathbf{x}_j$$

GAT:

$$\mathbf{x}'_i = \alpha_{i,i}\mathbf{\Theta}_{s}\mathbf{x}_{i} + \sum_{j \in \mathcal{N}(i)} \alpha_{i,j}\mathbf{\Theta}_{t}\mathbf{x}_{j},$$

Message Passing

Issues

What are the issues?

Long-range dependencies

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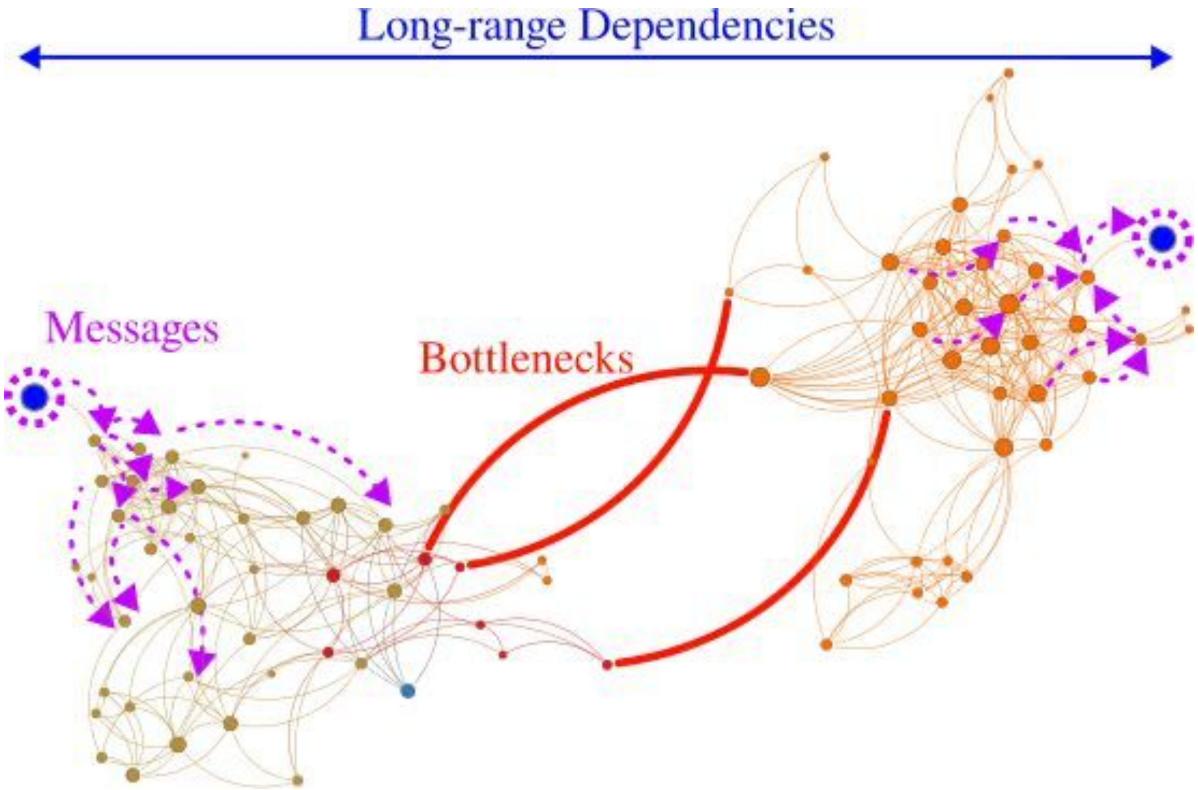
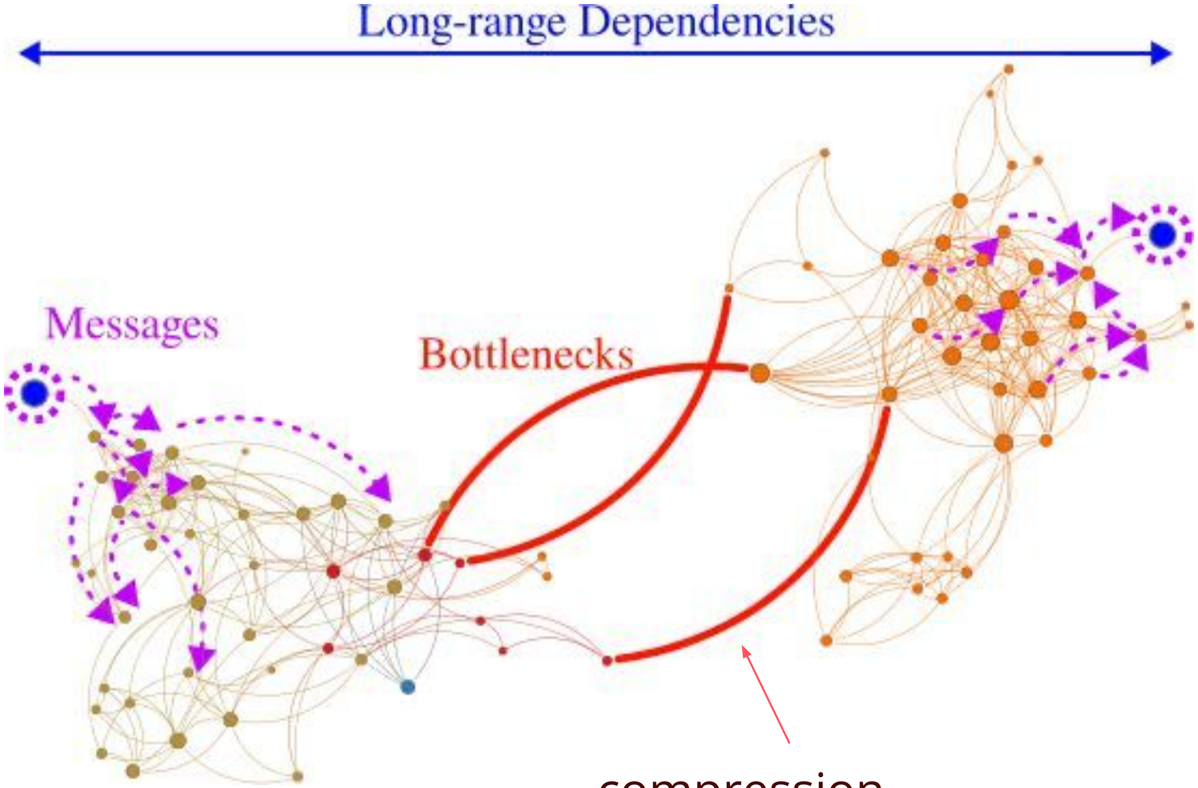
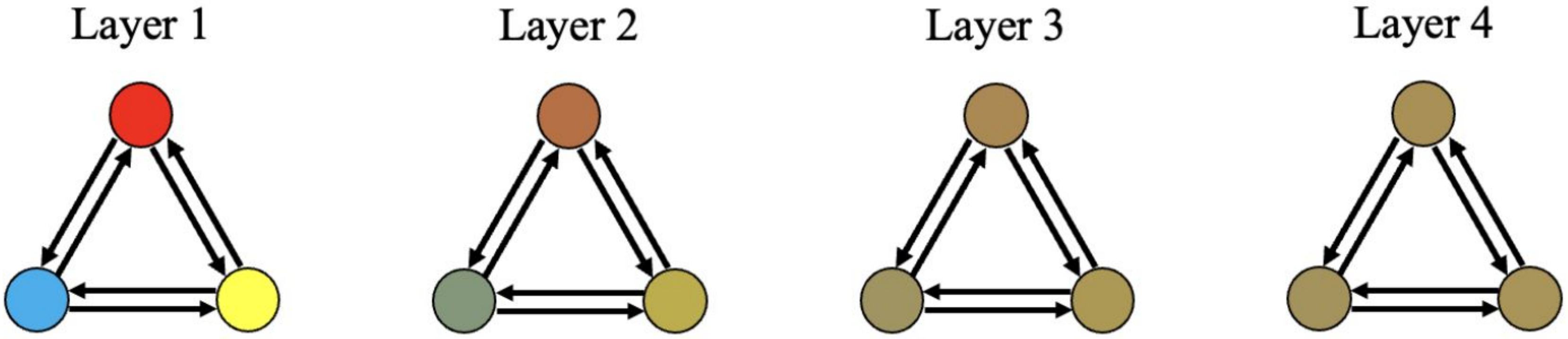


Oversquashing

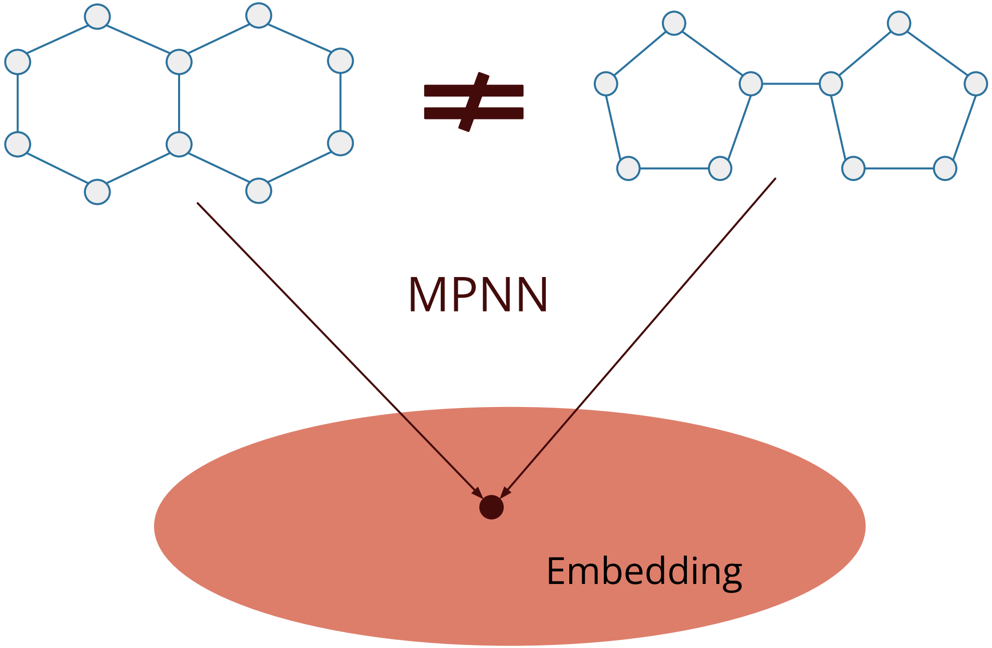


compression

Oversmoothing



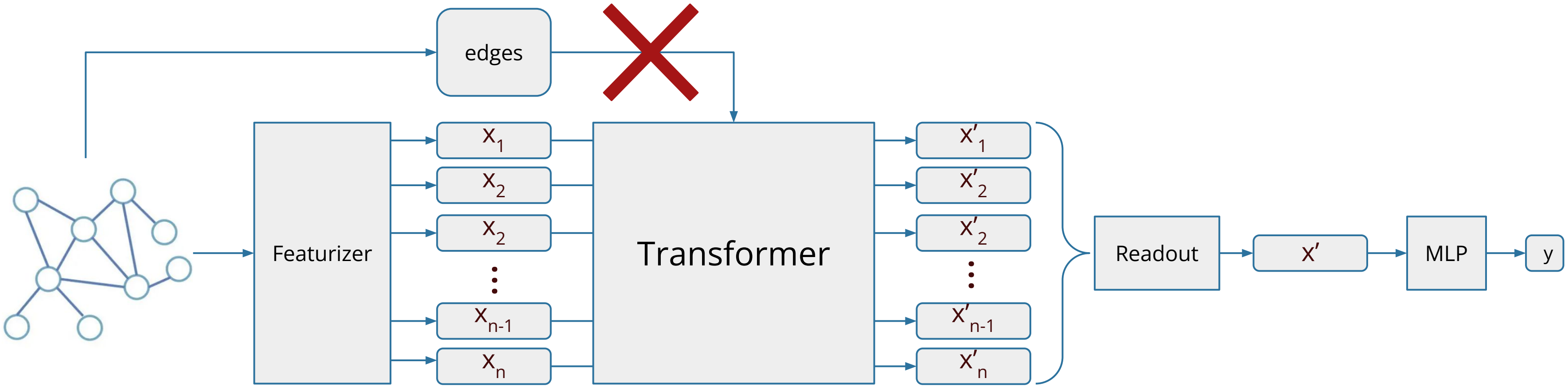
Expressivity



Transformers as GNNs

Transformers

Is it a proper GNN?

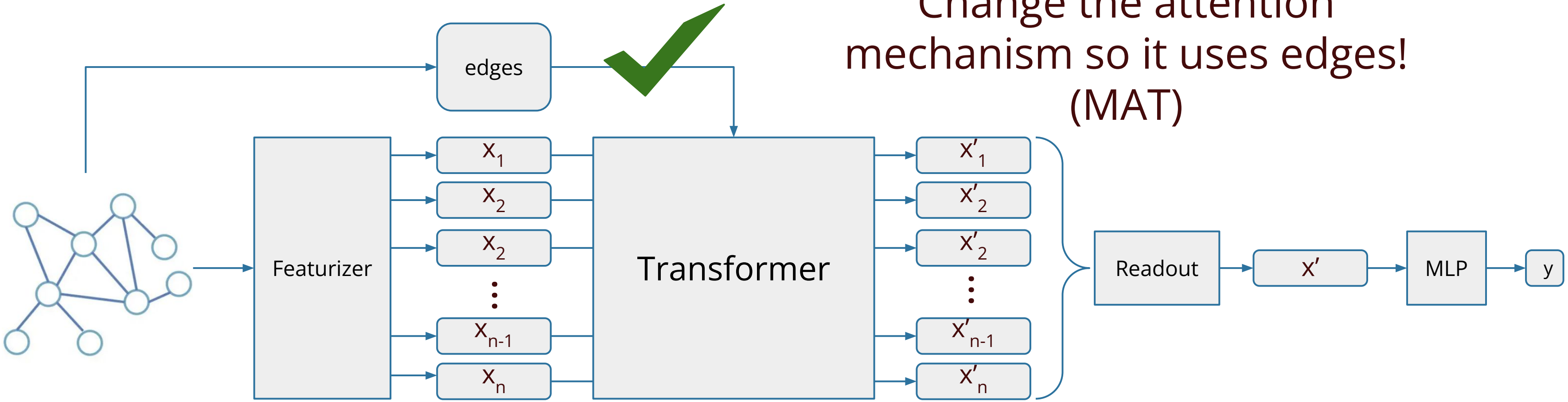


1. Transformer is permutation-equivariant.
2. Transformer cannot deal with the graph structure given by edges by default :<

Transformers

Is it a proper GNN?

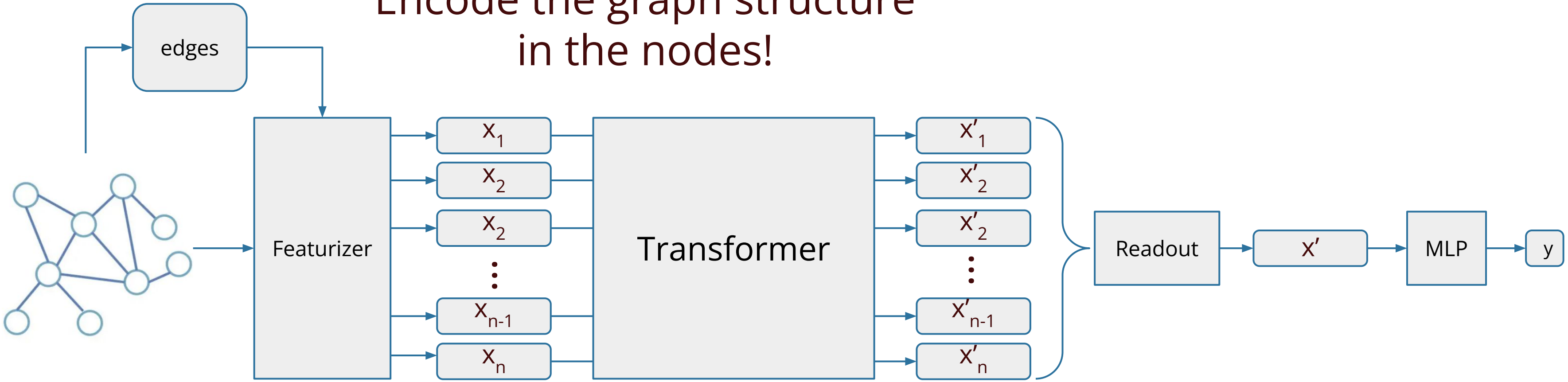
Change the attention mechanism so it uses edges! (MAT)



Transformers

Is it a proper GNN?

Encode the graph structure
in the nodes!



1. It can be done with structural/positional encodings (e.g. Random Walk).
2. Or it can be than with MPNN! (GraphGPS)

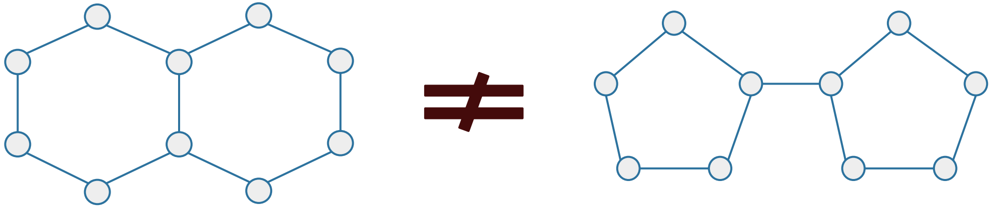
Transformers

Issues

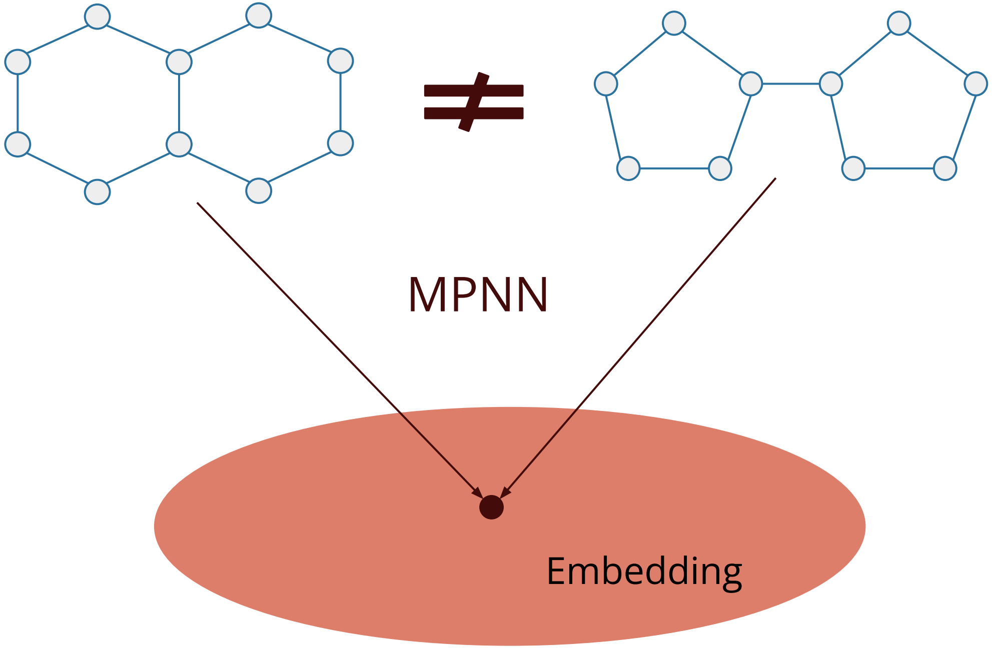
Transformers deal with long-range dependencies and oversquashing.

Expressivity of Message Passing

Can MPNN distinguish all graphs?



MPNN cannot distinguish all graphs!



The most powerful MPNN

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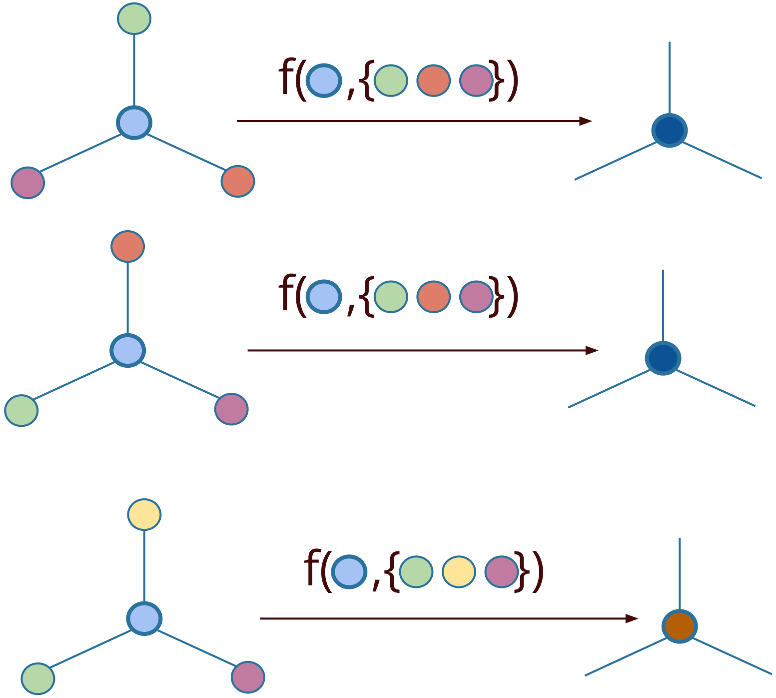
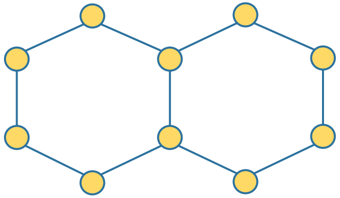
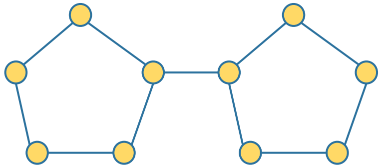
- Let us assume that all nodes in a graph have the same initial encodings.
- Let us denote node embeddings as colors. Different color -> different embedding.
- MPNN can only return different colors for nodes with different neighborhood.
- Our coloring MPNN always returns different color for nodes with different neighbors.



Coloring MPNN





Coloring MPNN

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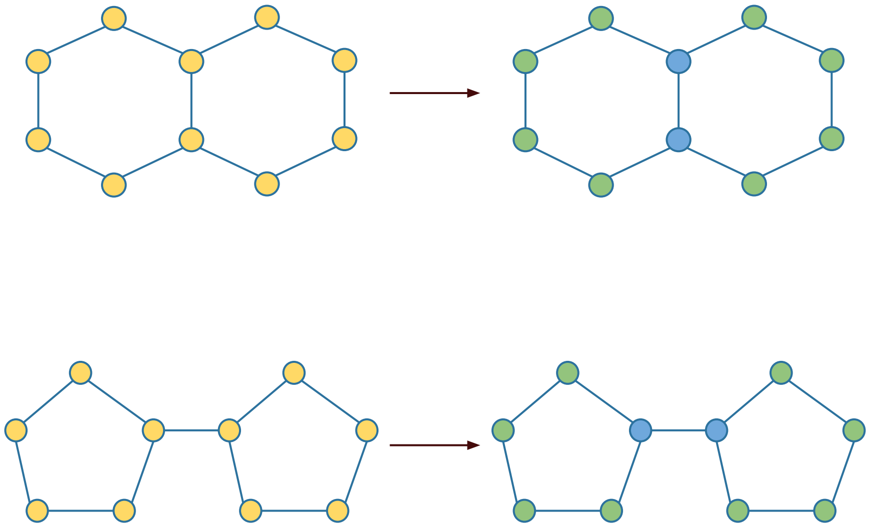
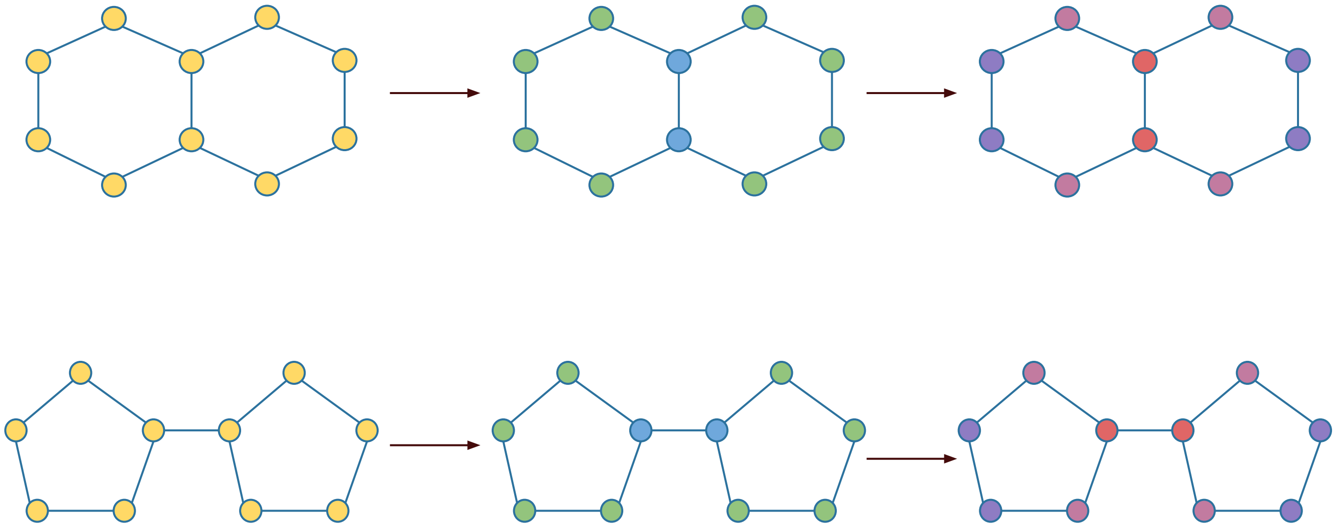
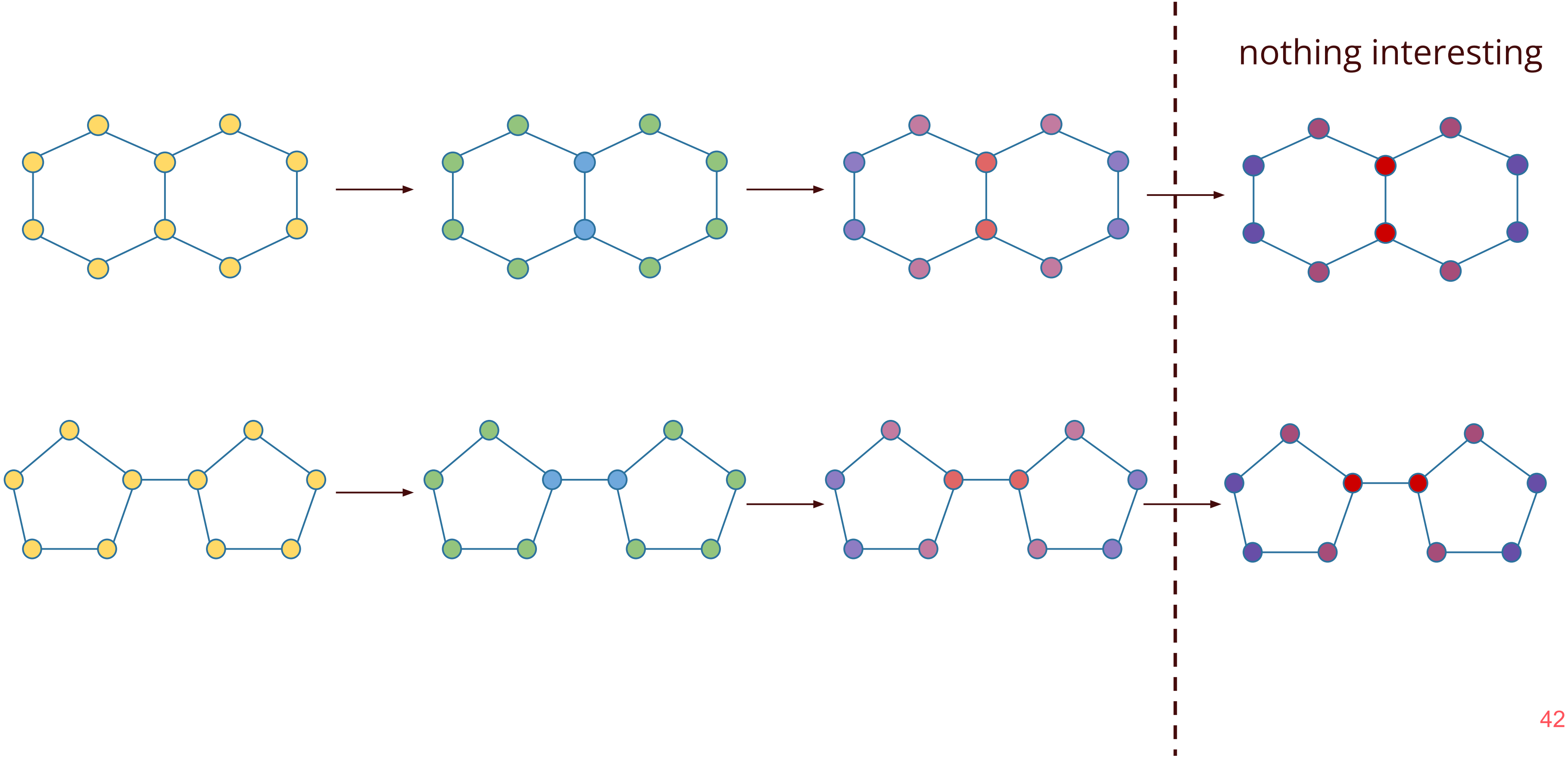


Coloring MPNN

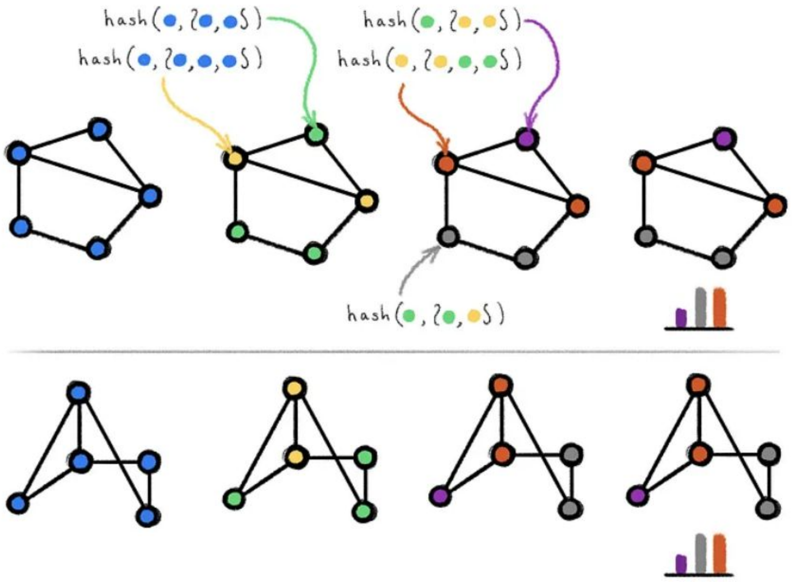


Coloring MPNN



WL test

- Our coloring MPNN works very similar to Weisfeiler-Lehman graph isomorphism test.
- No MPNN is more powerful than WL test.



k-WL test

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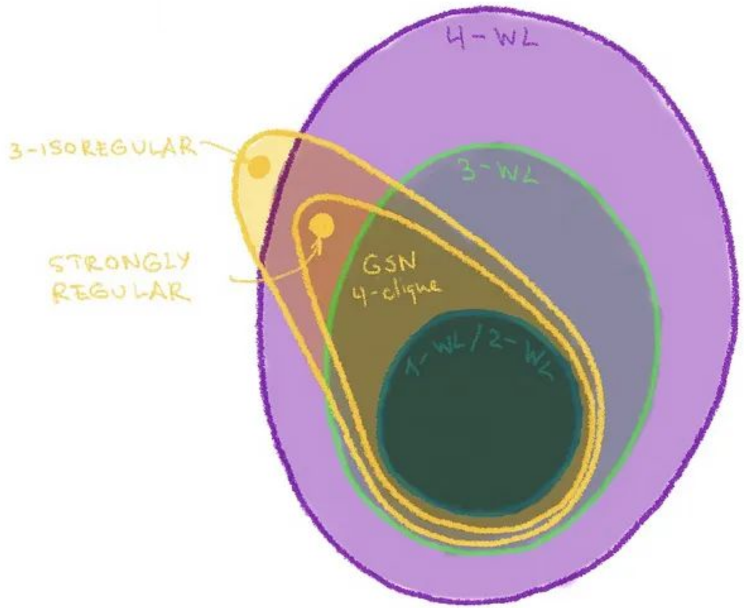
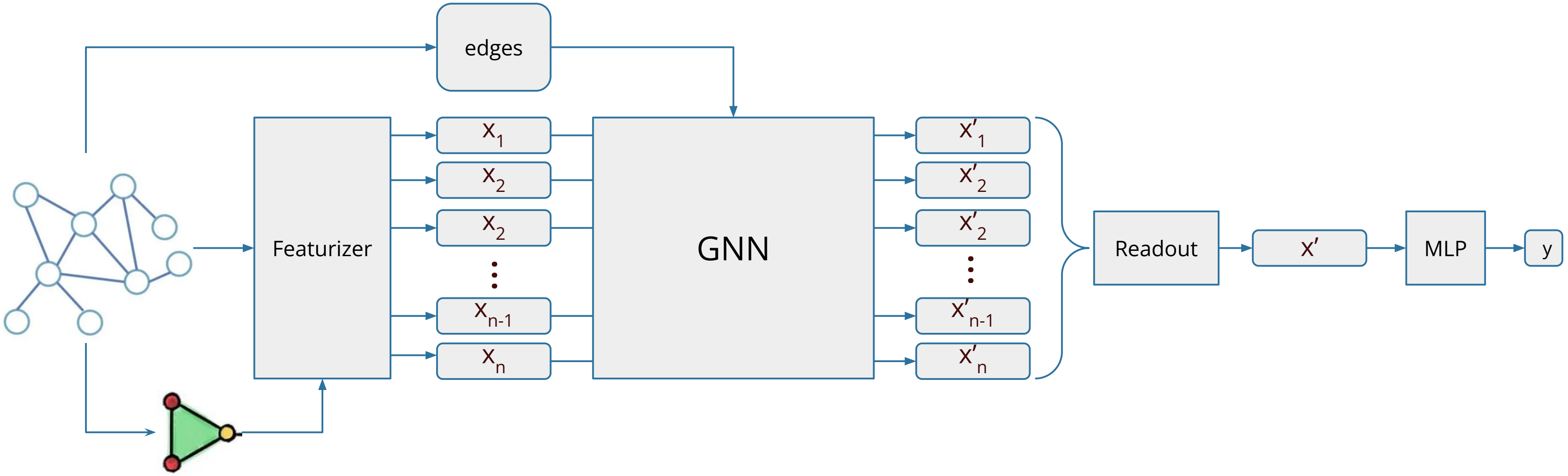
- We can easily generalize the WL test and obtain k-WL test. (k+1)-WL test is strictly more powerful than k-WL test (for k>1).
- There is k-GNN model which mimics the k-WL test and is O(n^k).
- There is an IGN model as powerful as 3-WL and O(n^2).
- But we can escape the k-WL classification...



Beyond k-WL classifiation

Examples of non-isomorphic graphs that cannot be distinguished by 1-WL but can be distinguished by 3-WL due to its capability of counting triangles.

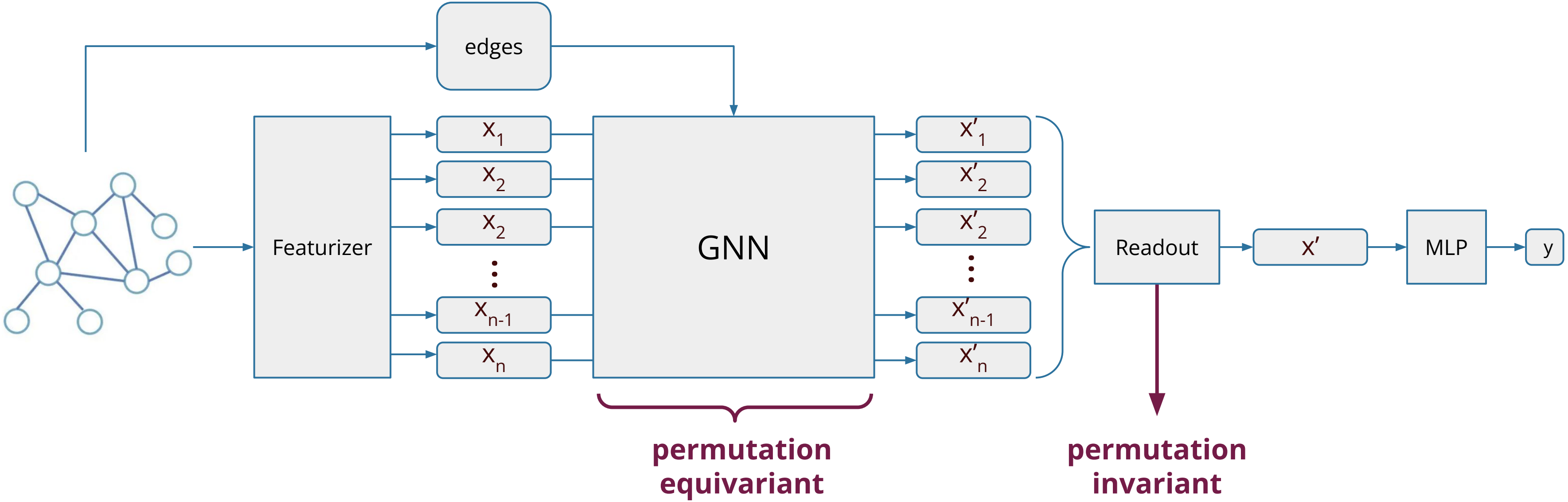
Beyond k-WL classifiation



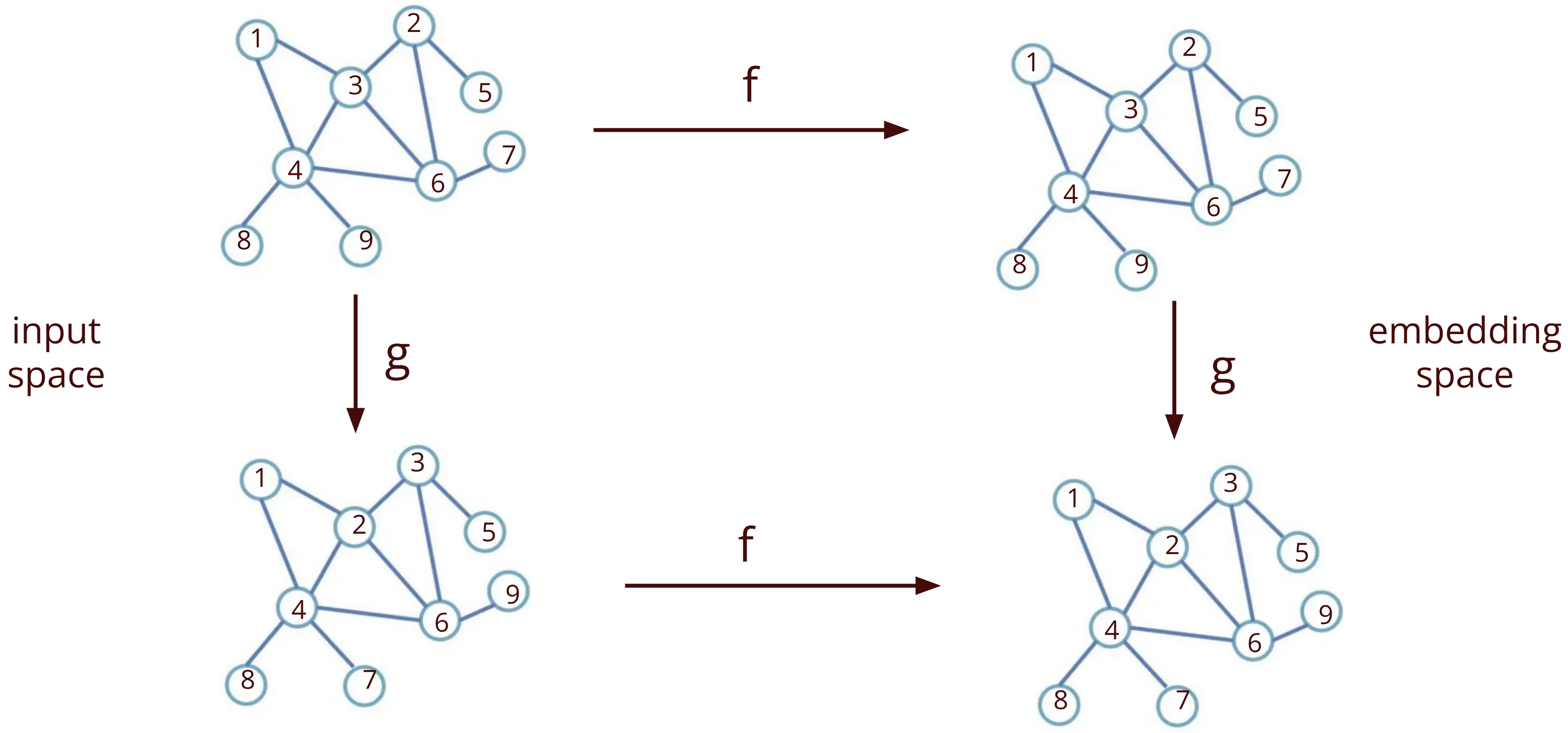
We can simply count triangles and
enrich the node encodings!

Symmetries: Equivariant Deep Learning

Permutation Equivariance

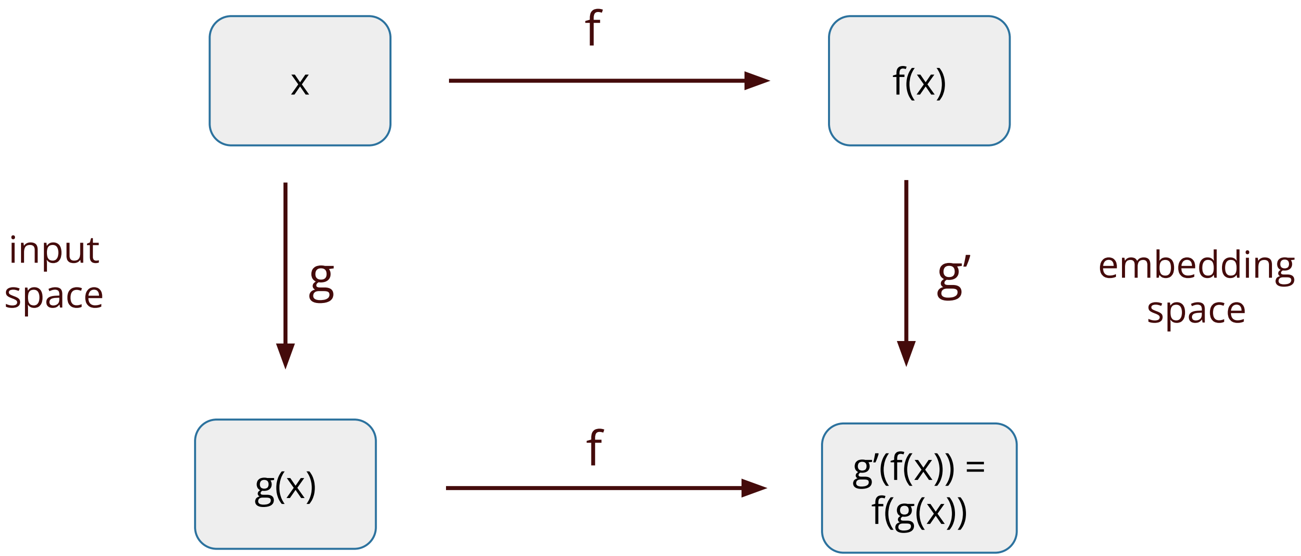


Permutation Equivariance



Equivariance

f - G-equivariant function
g - operation from group G defined in input space
g’ - operation from group G’ defined in embedding space corresponding to G



Rotation Equivariance

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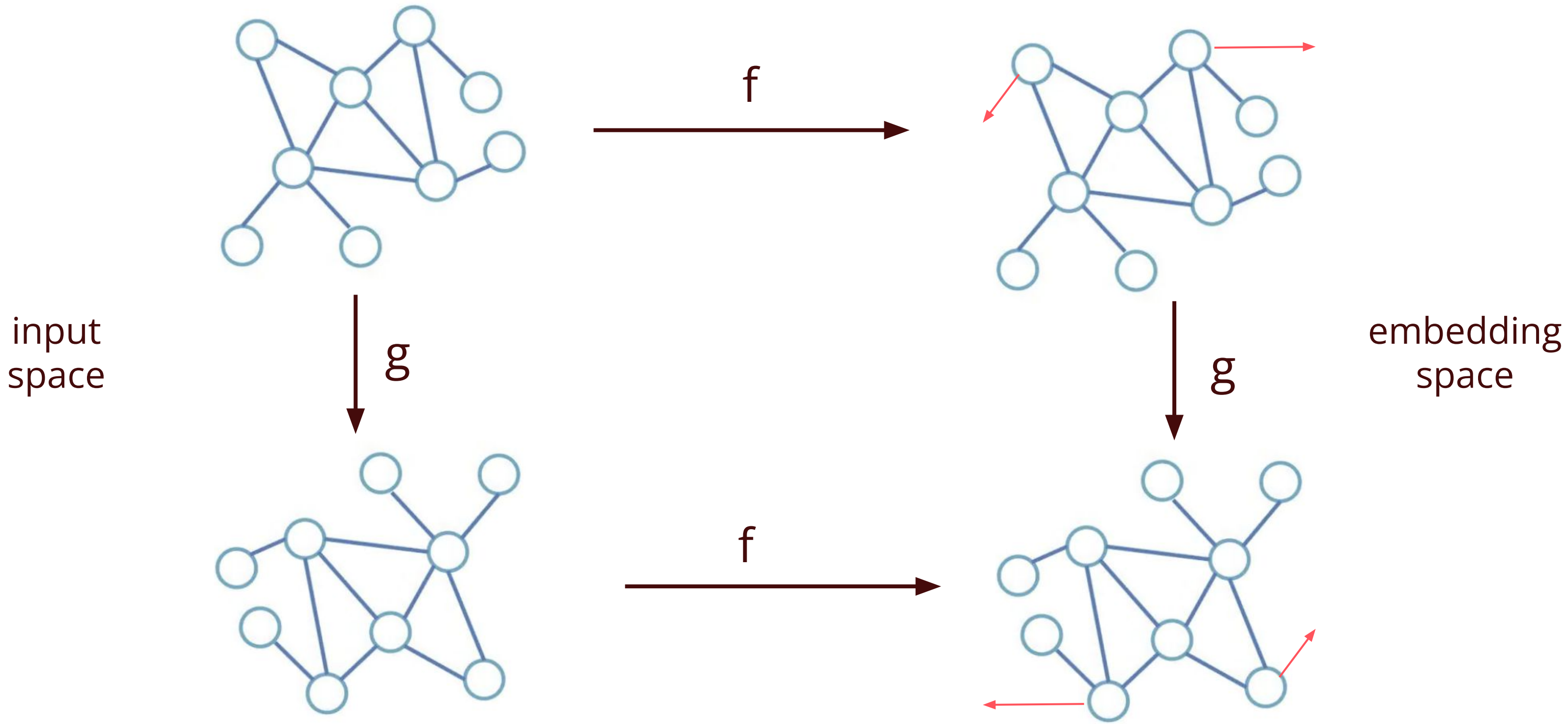

Deep Equivariant Learning

- Some node-level prediction tasks requires equivariance with respect to e.g. rotations.
- Equivariance can be used to incorporate invariance.
- Deep Equivariant Learning is a fast growing field. It definitely requires its own course.

GNNs are great!

Recommended course

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Machine Learning in Drug Discovery

by Sabina Smusz and Tomasz Danel

Inspirations

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LoGG reading group: yt channel

Geometric Deep Learning: The Erlangen Programme of ML: video

Group Equivariant Deep Learning: yt series

Machine Learning in Drug Discovery: UJ lectures and labs

Machine Learning with Graphs course from Stanford: webpage

Worth reading

- MPNNs:
  - GAT: Graph Attention Networks
  - GraphSAGE: Inductive Representation Learning on Large Graphs
  - GIN: How Powerful are Graph Neural Networks?
- Transformers:
  - MAT: Molecule Attention Transformer (GMUM)
  - GraphGPS: Recipe for a General, Powerful, Scalable Graph Transformer
  - SAN: Rethinking Graph Transformers with Spectral Attention
- k-WL GNNs:
  - k-GNN: Weisfeiler and Leman go neural: Higher-order graph neural networks

  - IGN: Convergence of Invariant Graph Networks

Worth reading

- Geometric GNNs:
  - SGCN: Spatial Graph Convolutional Networks (GMUM)
  - Geometric Transformer: Geometric Transformer for End-to-End Molecule Properties Prediction
- Equivariant GNNs:
  - EGNN: E(n) Equivariant Graph Neural Networks
  - SE(3)-Transformers: 3D Roto-Translation Equivariant Attention Networks
- Symmetry-breaking GNNs:
  - ChIRo: Learning 3D Representations of Molecular Chirality with Invariance to Bond Rotations
  - ChiENN: Embracing Molecular Chirality with Graph Neural Networks (GMUM)
- Random:
  - Understanding convolution on graphs via energies

Thanks for your attention!