Chapter 2

Boolean Arithmetic

These slides support chapter 2 of the book

The Elements of Computing Systems

By Noam Nisan and Shimon Schocken

MIT Press

Chapter 2: Boolean arithmetic

- Binary numbers
- Binary addition
- Negative numbers
- Arithmetic Logic Unit
- Project 2 overview

`0` `1`

`00` `01` `10` `11`

3 bits – 8 possibilities

N bits – 2^N possibilities

Representing numbers

Binary	Decimal
0	0
1	1
10	2
11	3
100	4
101	5
...	

Representing numbers

10^2 10^1 10^0

100s 10s 1s

$7\ 8\ 9_{\text{ten}}$

Representing numbers

10^2 10^1 10^0

100s 10s 1s

789_{ten}

$$= 7 \cdot 10^2 + 8 \cdot 10^1 + 9 \cdot 10^0 = 789_{ten}$$

Binary → Decimal

2^2 $\quad$ 2^1 $\quad$ 2^0

4s $\quad$ 2s $\quad$ 1s

$$1\;0\;1_{\text{two}}$$

$$= 1 \cdot 10^2 + 0 \cdot 10^1 + 1 \cdot 10^0 = 5_{\text{ ten}}$$

Binary → Decimal

$$b_n \ b_{n\text{-}1} \ b_{n\text{-}2} \ \ldots \ b_1 \ b_0$$

$$= \sum_{i=0}^{n} b_i \cdot 2^i$$

Maximum value represented by *k* bits:

$$1 + 2 + 4 + \ldots + 2^{k-1} = 2^k - 1$$

Fixed word size

We will use a fixed number of bits.
Say 8 bits.

$$\left.\begin{array}{l}
\texttt{0000 0000} \\
\texttt{0000 0001} \\
\texttt{0000 0010} \\
\texttt{0000 0011} \\
\ldots \\
\texttt{0111 1111} \\
\texttt{1000 0000} \\
\texttt{1000 0001} \\
\ldots \\
\texttt{1111 1110} \\
\texttt{1111 1111}
\end{array}\right\} 2^8 = 256 \text{ values}$$

Representing signed numbers

We will use a fixed number of bits.
Say 8 bits.

```
0000 0000
0000 0001
0000 0010
0000 0011
…
0111 1111
1000 0000
1000 0001
…
1111 1110
1111 1111
```

positive values

negative values

- That's one possible representation
- We'll use a better one, later

Decimal → Binary

$87_{ten} = ? ? ? ? ? ? ? ?_{\ two}$

Decimal → Binary

$87_{ten} = 64 + 16 + 4 + 2 + 1$

$= ?\ ?\ ?\ ?\ \ ?\ ?\ ?\ ?_{\,two}$

Decimal → Binary

$$87_{ten} = 64 + 16 + 4 + 2 + 1$$

$$= 0\ 1\ 0\ 1\ \ 0\ 1\ 1\ 1_{two}$$

32 8

Chapter 2: Boolean arithmetic

- ✓ Binary numbers
- ➡ Binary addition
- Negative numbers
- Arithmetic Logic Unit
- Project 2 overview

Boolean arithmetic

- Addition
- Subtraction
- Comparison (<,>,=)
- Multiplication
- Division

implement

get for free

get for free

postpone
to software

postpone
to software

Addition

```
  0 0 0 1 0 1 0 1
+
  0 1 0 1 1 1 0 0
  _______________
  ? ? ? ? ? ? ? ?
```

Addition

```
  0 0 0 1 0 1 0 1        21
+                      +
  0 1 0 1 1 1 0 0        92
  ---------------       ---
  0 1 1 1 0 0 0 1       113
```

Addition

$$
\begin{array}{r}
5\;7\;8\;3 \\
+\quad 2\;4\;5\;6 \\
\hline
?\;?\;?\;?
\end{array}
$$

Addition

$$\begin{array}{r} 5\ 7\ 8\ 3 \\ +\ \ 2\ 4\ 5\ 6 \\ \hline \end{array}$$

Addition

```
    1 1
    5 7 8 3
+
    2 4 5 6
    -------
    8 2 3 9
```

Addition

```
  0 0 0 1 0 1 0 1
+
  0 1 0 1 1 1 0 0
  ---------------
  ? ? ? ? ? ? ? ?
```

Addition

```
      1 1 1
  0 0 0 1 0 1 0 1
+
  0 1 0 1 1 1 0 0
  ---------------
  0 1 1 1 0 0 0 1
```

Overflow

```
  1 0 0 1 0 1 0 1
+
  1 1 0 1 1 1 0 0
  _______________
  ? ? ? ? ? ? ? ?
```

Overflow

```
1 0 0 0 1 1 1 0 0
   1 0 0 1 0 1 0 1
+
   1 1 0 1 1 1 0 0
  ----------------
1 0 1 1 1 0 0 0 1
```

Building an Adder

```
    0 0 1 0 1 1 0
    1 0 0 1 0 1 1 1
 +
    0 1 0 1 0 0 1 0
   ----------------
    1 1 1 0 1 0 0 1
```

- Half adder: adds two bits
- Full adder: adds three bits
- Adder: adds two integers

Half adder

a	b	sum	carry
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

carry bit

```
    0 0 1 0 1 1 0
    1 0 0 1 0 1 1 1
+   0 1 0 1 0 0 1 0
   ----------------
    1 1 1 0 1 0 0 1
```

sum bit

Half adder

a	b	sum	carry
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

HalfAdder.hdl

```
/** Computes the sum of two bits. */

CHIP HalfAdder {
    IN a, b;
    OUT sum, carry;

    PARTS:
    // Put your code here:
}
```

Full adder

```
    0 0 1 0 1 1 0
    1 0 0 1 0 1 1 1
+
    0 1 0 1 0 0 1 0
   ----------------
    1 1 1 0 1 0 0 1
```

Full adder

a	b	c	sum	carry
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

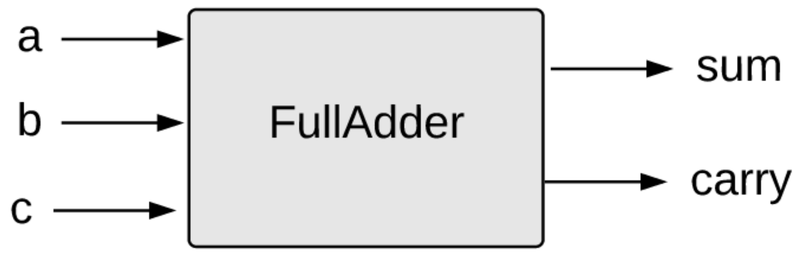
carry bit

```
    0 0 1 0 1 1 0
    1 0 0 1 0 1 1 1
+   0 1 0 1 0 0 1 0
    ---------------
    1 1 1 0 1 0 0 1
```

sum bit

Full adder

a	b	c	sum	carry
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1



FullAdder.hdl

```
/** Computes the sum of three bits. */

CHIP HalfAdder {
    IN a, b, c;
    OUT sum, carry;

    PARTS:
    // Put your code here:
}
```

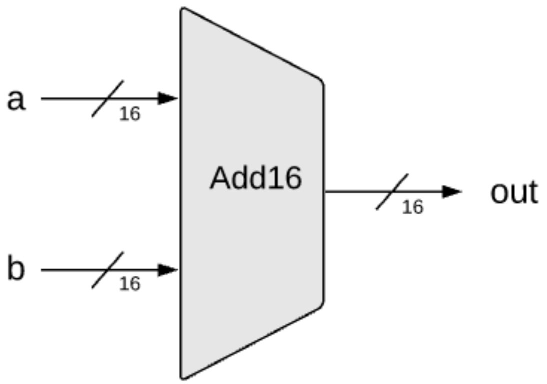
Multi-bit Adder

```
    0 0 1 0 1 1 0
    1 0 0 1 0 1 1 1
 +
    0 1 0 1 0 0 1 0
   ----------------
    1 1 1 0 1 0 0 1
```

Multi-bit Adder

```
    0 0 1 0 1 1 0
    1 0 0 1 0 1 1 1
+
    0 1 0 1 0 0 1 0
   ----------------
    1 1 1 0 1 0 0 1
```

Multi-bit Adder



	0	0	1	0	1	1	0	
	1	0	0	1	0	1	1	1
+	0	1	0	1	0	0	1	0
	1	1	1	0	1	0	0	1

Add16.hdl

```
/* Adds two 16-bit, two's-complement values.
 * The most-significant carry bit is ignored. */

CHIP Add16 {
    IN a[16], b[16];
    OUT out[16];

    PARTS:
   // Put you code here:
}
```

Chapter 2: Boolean arithmetic

- ✓ Binary numbers
- ✓ Binary addition
- ➡ Negative numbers
- Arithmetic Logic Unit
- Project 2 overview

Representing numbers (using 4 bits)

0000	0
0001	1
0010	2
0011	3
0100	4
0101	5
0110	6
0111	7
1000	8
1001	9
1010	10
1011	11
1100	12
1101	13
1110	14
1111	15

Using n bits, we can represent the positive integers in the range $0 \ldots 2^n-1$

Representing negative numbers

```
0000
0001
0010
0011
0100
0101
0110
0111
1000
1001
1010
1011
1100
1101
1110
1111
```

Possible solution: use a sign bit

0000	0
0001	1
0010	2
0011	3
0100	4
0101	5
0110	6
0111	7
1000	-0
1001	-1
1010	-2
1011	-3
1100	-4
1101	-5
1110	-6
1111	-7

Use the left-most bit to represent the sign, -/+

Use the remaining bits to represent a positive number

Complications:

- -0?
- $x + (-x)$ is not 0
- more complications

Two's Complement

0000	0	
0001	1	
0010	2	
0011	3	
0100	4	
0101	5	
0110	6	
0111	7	
1000	-8	(16 - 8)
1001	-7	(16 - 9)
1010	-6	(16 - 10)
1011	-5	(16 - 11)
1100	-4	(16 - 12)
1101	-3	(16 - 13)
1110	-2	(16 - 14)
1111	-1	(16 - 15)

Represent the negative number $-x$ using the positive number $2^n - x$

Two's Complement

```
0000   0
0001   1
0010   2
0011   3
0100   4
0101   5
0110   6
0111   7
1000  -8  (16 - 8)
1001  -7  (16 - 9)
1010  -6  (16 - 10)
1011  -5  (16 - 11)
1100  -4  (16 - 12)
1101  -3  (16 - 13)
1110  -2  (16 - 14)
1111  -1  (16 - 15)
```

positive numbers range:
$0 \ldots 2^{n-1} - 1$

negative numbers range:
$-1 \ldots -2^{n} - 1$

Addition using two's complement

```
   -2          14        1110
+          +          +
   -3          13        1101
  ----        ----      -----
   -5          11       11011
```


Two's complement rationale:

- representation is modulu 2^n
- addition is modulu 2^n

Computing *-x*

Input: x

Output: $-x$ (in two's complement)

Insight: if we solve this we'll know how to subtract:

$$y - x = y + (-x)$$

Computing -*x*

Input: x

Output: $-x$ (in two's complement)

Idea: $2^n - x = 1 + (2^n - 1) - x$

11111111_{two}

```
  11111111
- 10101100   (some x example)
  --------
  01010011   (flip all the bits)
```

Now add 1 to the result

Computing -x (example)

Input: 4

Output: should be 12 (representing -4 in two's compelemt)

Input:	`0100`
Flip the bits:	`1011`
Add one:	+ `1`
Output:	`1100`
	= 12_{ten}

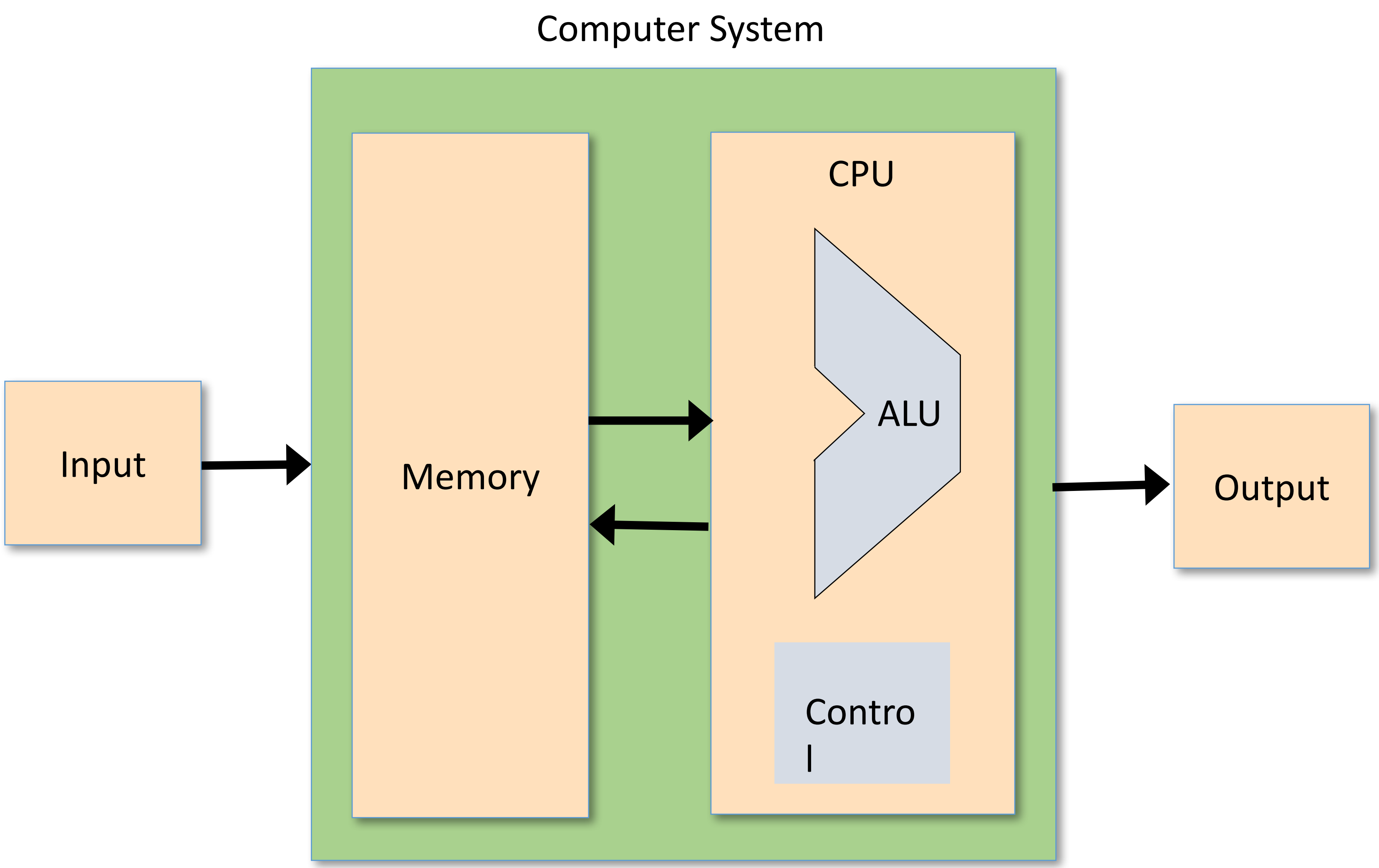
<u>To add 1:</u>

Flip all the bits from right to left, stop when the first 0 flips to 1

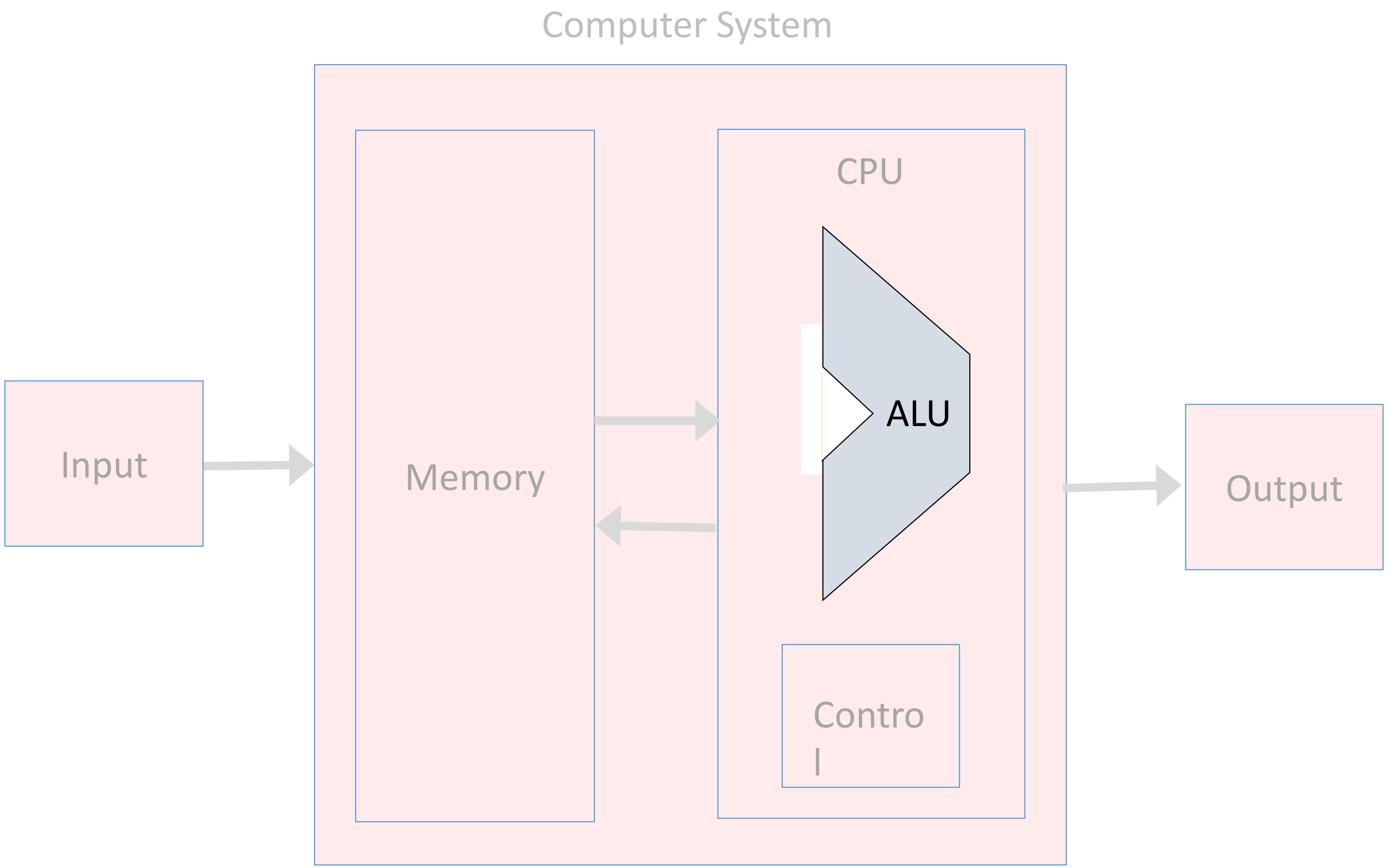
Chapter 2: Boolean arithmetic

- ✓ Binary numbers
- ✓ Binary addition
- ✓ Negative numbers
- ➡ Arithmetic Logic Unit
- Project 2 overview

Von Neumann Architecture

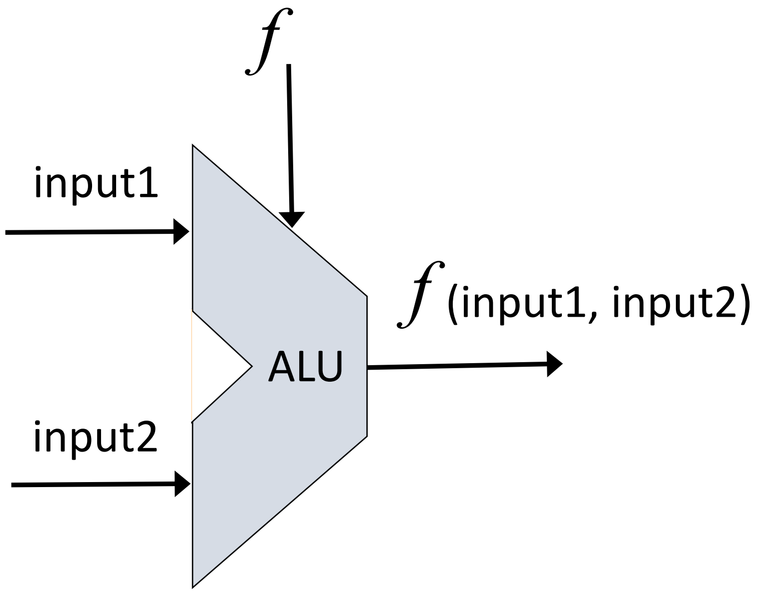


The Arithmetic Logical Unit



The Arithmetic Logical Unit

The ALU computes a function on the two inputs, and outputs the result



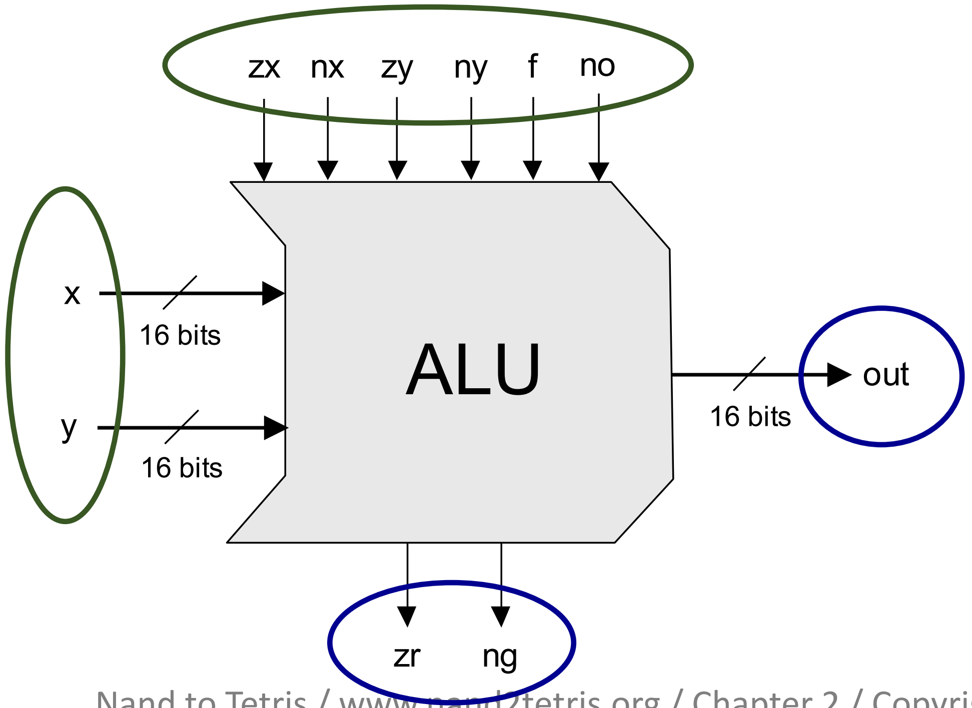
f : one out of a family of pre-defined arithmetic and logical functions

- Arithmetic functions: integer addition, multiplication, division, ...
- logical functions: And, Or, Xor, …

Which functions should the ALU perform?
A hardware / software tradeoff.

The Hack ALU

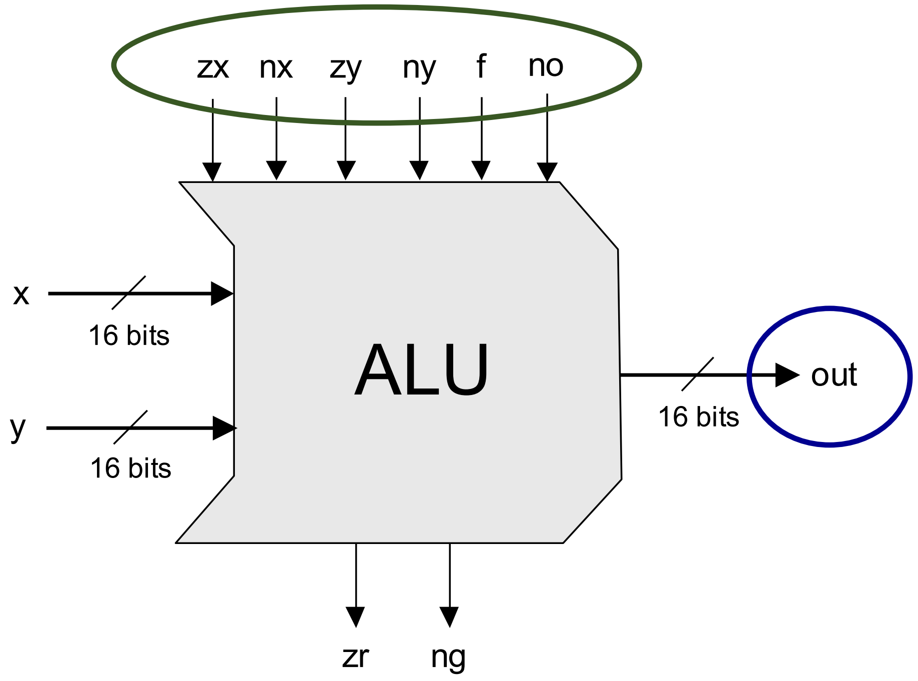
- Operates on two 16-bit, two's complement values
- Outputs a 16-bit, two's complement value
- Which function to compute is set by six 1-bit inputs
- Computes one out of a family of 18 functions
- Also outputs two 1-bit values (to be discussed later).



out
0
1
-1
x
y
!x
!y
-x
-y
x+1
y+1
x-1
y-1
x+y
x-y
y-x
x&y
x|y

The Hack ALU

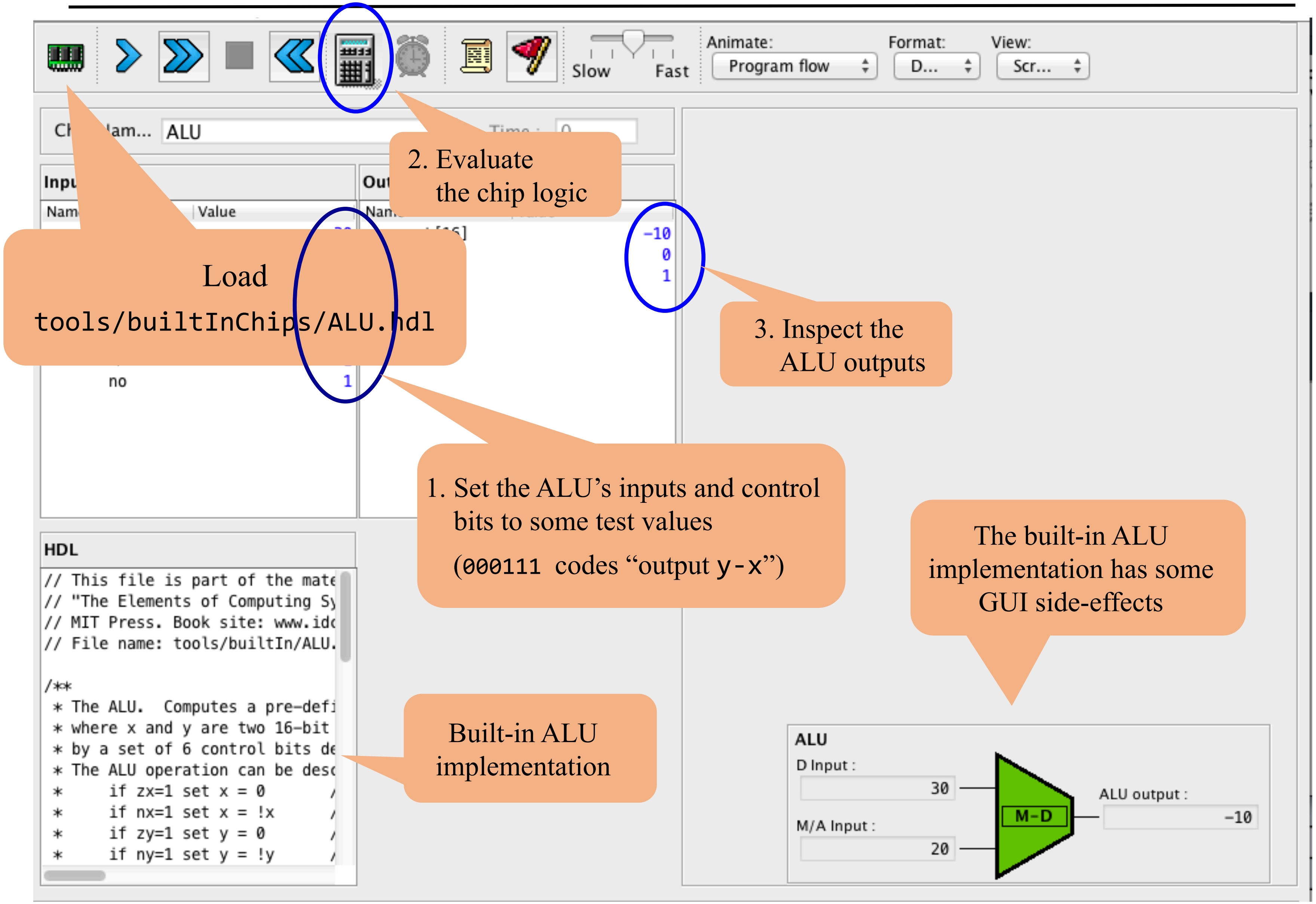
To cause the ALU to compute a function, set the control bits to one of the binary combinations listed in the table.



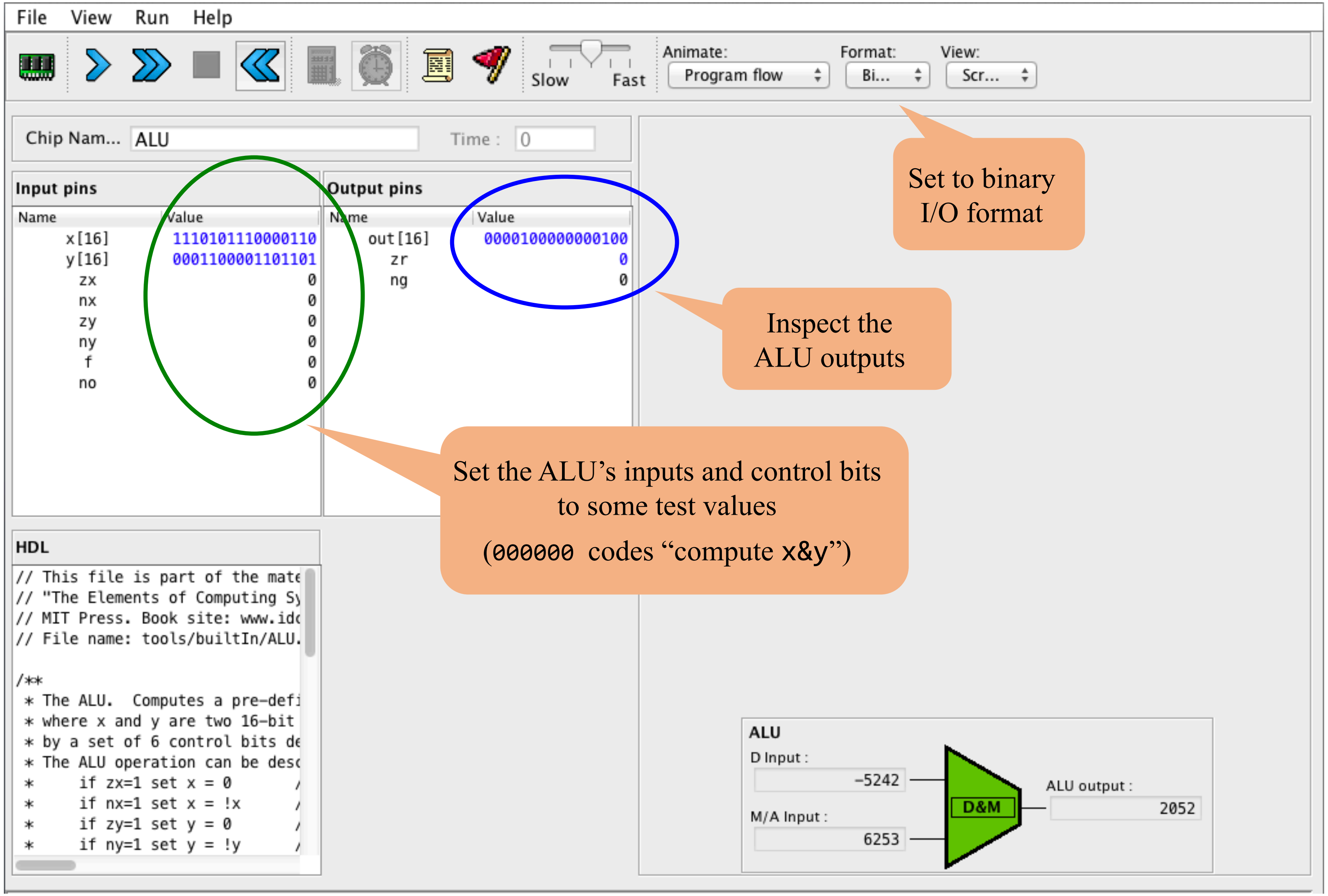
control bits

zx	nx	zy	ny	f	no	out
1	0	1	0	1	0	0
1	1	1	1	1	1	1
1	1	1	0	1	0	-1
0	0	1	1	0	0	x
1	1	0	0	0	0	y
0	0	1	1	0	1	!x
1	1	0	0	0	1	!y
0	0	1	1	1	1	-x
1	1	0	0	1	1	-y
0	1	1	1	1	1	x+1
1	1	0	1	1	1	y+1
0	0	1	1	1	0	x-1
1	1	0	0	1	0	y-1
0	0	0	0	1	0	x+y
0	1	0	0	1	1	x-y
0	0	0	1	1	1	y-x
0	0	0	0	0	0	x&y
0	1	0	1	0	1	x|y

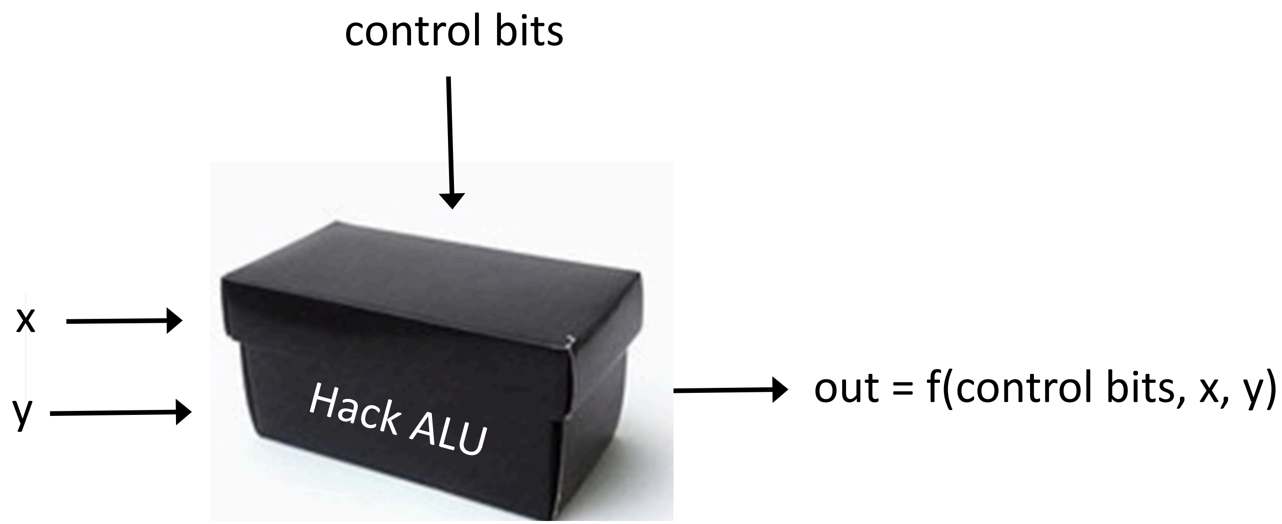
The Hack ALU in action: compute `y-x`



The Hack ALU in action: compute `x & y`

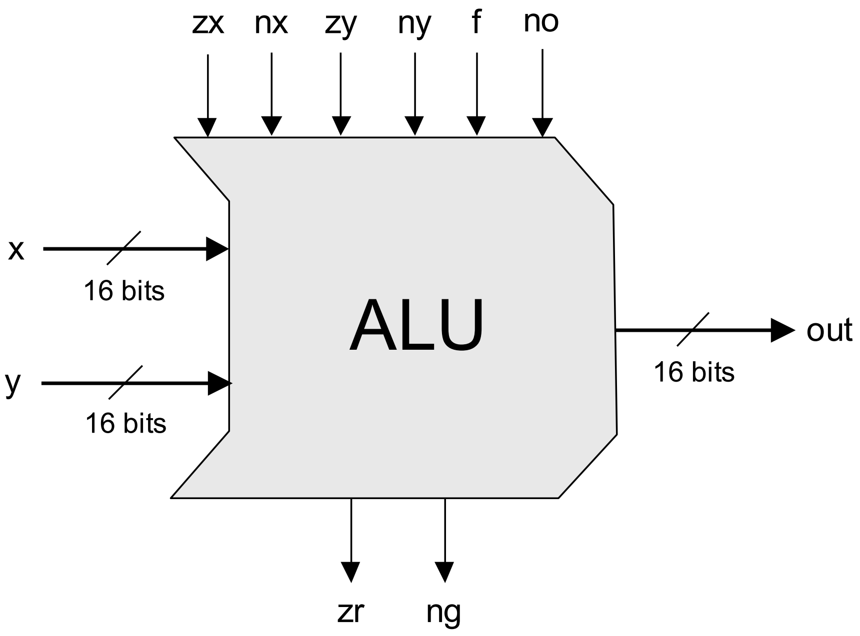


Opening up the Hack ALU black box



The Hack ALU operation

pre-setting the x input		pre-setting the y input		selecting between computing + or &	post-setting the output	Resulting ALU output
zx	nx	zy	ny	f	no	out
if zx then x=0	if nx then x=!x	if zy then y=0	if ny then y=!y	if f then out=x+y else out=x&y	if no then out=!out	out(x,y)=



The Hack ALU operation

pre-setting the x input		pre-setting the y input		selecting between computing + or &	post-setting the output	Resulting ALU output
zx	nx	zy	ny	f	no	out
if zx then x=0	if nx then x=!x	if zy then y=0	if ny then y=!y	if f then out=x+y else out=x&y	if no then out=!out	out(x,y)=
1	0	1	0	1	0	0
1	1	1	1	1	1	1
1	1	1	0	1	0	-1
0	0	1	1	0	0	x
1	1	0	0	0	0	y
0	0	1	1	0	1	!x
1	1	0	0	0	1	!y
0	0	1	1	1	1	-x
1	1	0	0	1	1	-y
0	1	1	1	1	1	x+1
1	1	0	1	1	1	y+1
0	0	1	1	1	0	x-1
1	1	0	0	1	0	y-1
0	0	0	0	1	0	x+y
0	1	0	0	1	1	x-y
0	0	0	1	1	1	y-x
0	0	0	0	0	0	x&y
0	1	0	1	0	1	x|y

ALU operation example: compute `!x`

pre-setting the x input		pre-setting the y input		selecting between computing + or &	post-setting the output	Resulting ALU output
zx	nx	zy	ny	f	no	out
if zx then x=0	if nx then x=!x	if zy then y=0	if ny then y=!y	if f then out=x+y else out=x&y	if no then out=!out	out(x,y)=
1	0	1	0	1	0	0
1	1	1	1	1	1	1
1	1	1	0	1	0	-1
0	0	1	1	0	0	x
1	1	0	0	0	0	y
0	0	1	1	0	1	!x
1	1	0				!y
0	0	1				-x
1	1	0				-y
0	1	1				x+1
1	1	0				y+1
0	0	1				x-1
1	1	0				y-1
0	0	0				x+y
0	1	0				x-y
0	0	0				y-x
0	0	0				x&y
0	1	0				x|y

Example: compute `!x`

```
x:         1 1 0 0
y:         1 0 1 1
```

Following pre-setting:

```
x:         1 1 0 0
y:         1 1 1 1
```

Computation and post-setting:

```
x&y:       1 1 0 0
!(x&y):    0 0 1 1   (!x)
```

ALU operation example: compute y-x

pre-setting the x input		pre-setting the y input		selecting between computing + or &	post-setting the output	Resulting ALU output
zx	nx	zy	ny	f	no	out
if zx then x=0	if nx then x=!x	if zy then y=0	if ny then y=!y	if f then out=x+y else out=x&y	if no then out=!out	out(x,y)=
1	0	1	0	1	0	0
				1	1	1
				1	0	-1
				0	0	x
				0	0	y
				0	1	!x
				0	1	!y
				1	1	-x
				1	1	-y
				1	1	x+1
				1	1	y+1
				1	0	x-1
				1	0	y-1
				1	0	x+y
				1	1	x-y
0	0	0	1	1	1	y-x
0	0	0	0	0	0	x&y
0	1	0	1	0	1	x|y

Example: compute y-x

```
x:       0 0 1 0   (2)
y:       0 1 1 1   (7)
```

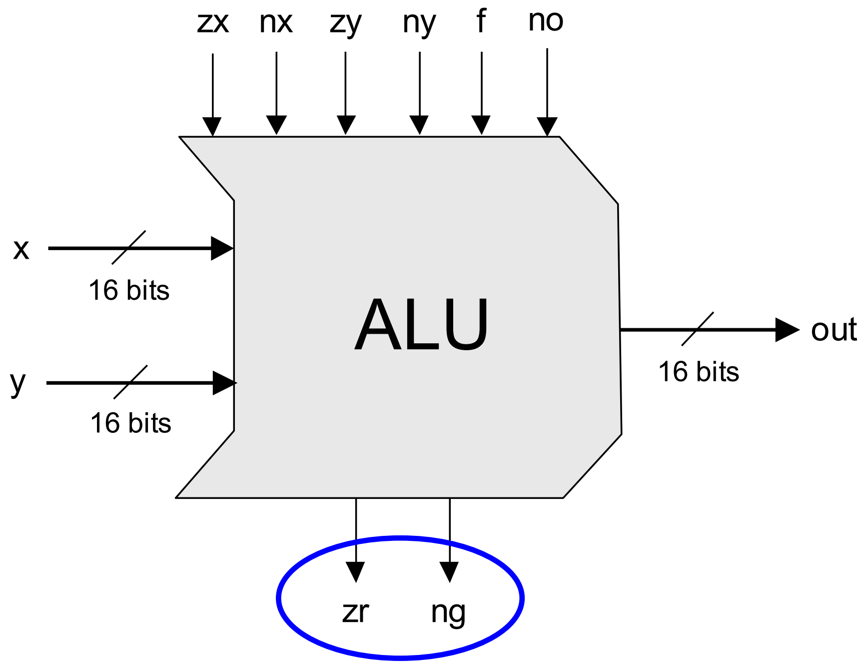
Following pre-setting:

```
x:       0 0 1 0
y:       1 0 0 0
```

Computation and post-setting:

```
x+y:     1 0 1 0
!(x+y):  0 1 0 1   (5)
```

The Hack ALU output control bits




```
if (out == 0) then zr = 1, else zr = 0

if (out < 0)  then ng = 1, else ng = 0
```

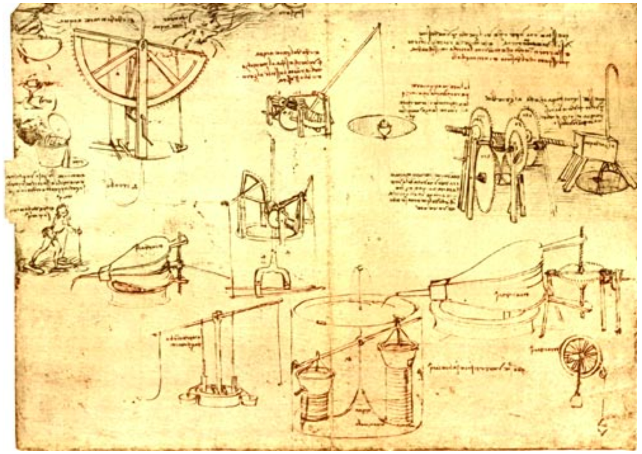
These two control bits will come into play when we build the complete computer's architecture.

Perspective

<u>The Hack ALU is:</u>

- Simple
- Elegant
- To implement it this ALU,
  you only need to know how to:
  - Set a 16-bit value to `0000000000000000`
  - Set a 16-bit value to `1111111111111111`
  - Negate a 16-bit value (bit-wise)
  - Compute `plus` or `And` on two 16-bit values

  That's it!



"Simplicity is the ultimate sophistication."
— Leonardo da Vinci

Chapter 2: Boolean arithmetic

- ✓ Binary numbers
- ✓ Binary addition
- ✓ Negative numbers
- ✓ Arithmetic Logic Unit
- ➡ Project 2 overview

Project 2

Given: All the chips built in Project 1

Goal: Build the following chips:

- `HalfAdder`
- `FullAdder`
- `Add16`
- `Inc16`
- `ALU`

A family of *combinational* chips, from simple adders to an Arithmetic Logic Unit.

Half Adder

a	b	sum	carry
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

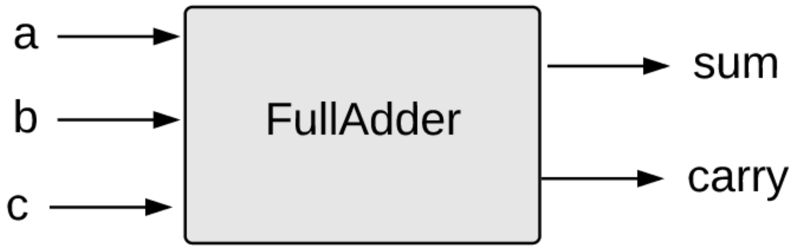
HalfAdder.hdl

```
/** Computes the sum of two bits. */

CHIP HalfAdder {
    IN a, b;
    OUT sum, carry;

    PARTS:
    // Put your code here:
}
```

Implementation tip

Can be built using two very elementary gates.

Full Adder



a	b	c	sum	carry
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

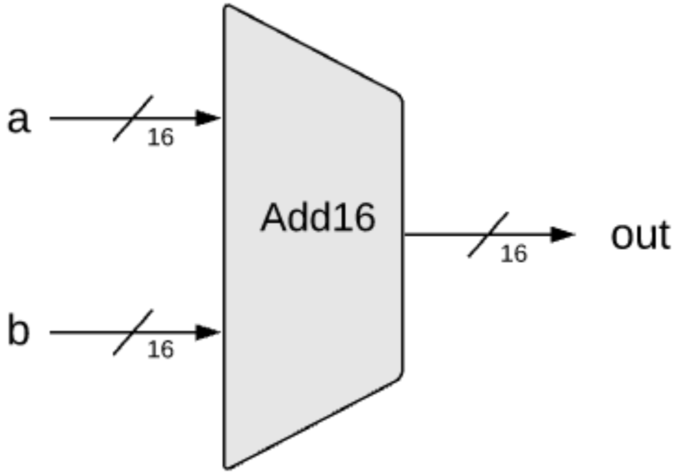
`FullAdder.hdl`

```
/** Computes the sum of three bits. */

CHIP HalfAdder {
    IN a, b, c;
    OUT sum, carry;

    PARTS:
    // Put your code here:
}
```

<u>Implementation tips</u>

Can be built using two half-adders.

16-bit adder



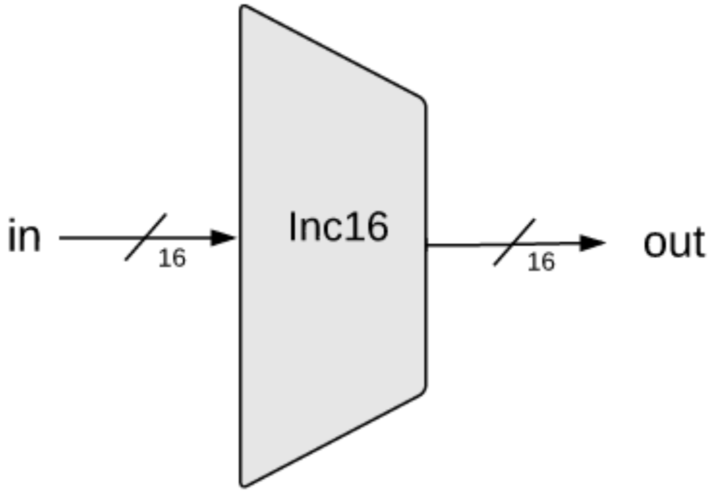
Implementation tips

- An *n*-bit adder can be built from *n* full-adder chips
- The carry bit is "piped" from right to left
- The MSB carry bit is ignored.

Add16.hdl

```
/*
 * Adds two 16-bit, two's-complement values.
 * The most-significant carry bit is ignored.
 */

CHIP Add16 {
    IN a[16], b[16];
    OUT out[16];

    PARTS:
   // Put you code here:
}
```

16-bit incrementor



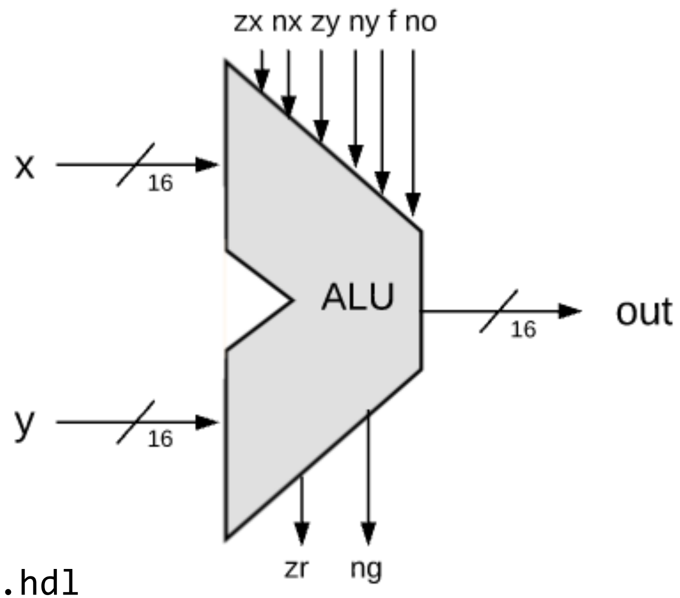
Inc16.hdl

```
/*
 * Outputs in + 1.
 * The most-significant carry bit is ignored.
 */

CHIP Inc16 {
    IN in[16];
    OUT out[16];

    PARTS:
    // Put you code here:
}
```

Implementation tip

The single-bit `0` and `1` values are represented in HDL as `false` and `true`.

ALU



Implementation tips

- Building blocks: `Add16`, and some gates built in project 1
- Can be built with ~20 lines of HDL code

ALU.hdl

```
/** The ALU. */
// Manipulates the x and y inputs as follows:
// if (zx  == 1) sets x = 0        // 16-bit true constant
// if (nx  == 1) sets x = !x       // bitwise Not
// if (zy  == 1) sets y = 0        // 16-bit true constant
// if (ny  == 1) sets y = !y       // bitwise Not
// if (f   == 1) sets out = x + y  // int. 2's-complement addition
// if (f   == 0) sets out = x & y  // bitwise And
// if (no  == 1) sets out = !out   // bitwise Not
// if (out == 0) sets zr = 1       // 1-bit true constant
// if (out < 0)  sets ng = 1       // 1-bit true constant
...
```

Project 2 Resources

From NAND to Tetris
Building a Modern Computer From First Principles
www.nand2tetris.org

- Home
- Prerequisites
- Syllabus
- Course
- Book
- Software
- Terms
- Papers
- Talks
- Cool Stuff
- About
- Team
- Q&A

Project 2: Combinational Chips

Background

The centerpiece of the computer's architecture is the *CPU*, or *Central Processing Unit*, and the centerpiece of the CPU is the *ALU*, or *Arithmetic-Logic Unit*. In this project you will gradually build a set of chips, culminating in the construction of the *ALU* chip of the *Hack* computer. All the chips built in this project are standard, except for the ALU itself, which differs from one computer architecture to another.

Objective

Build all the chips described in Chapter 2 (see list below), leading up to an *Arithmetic Logic Unit* - the Hack computer's ALU. The only building blocks that you can use are the chips described in chapter 1 and the chips that you will gradually build in this project.

Chips

Chip (HDL)	Description	Test script	Compare file
HalfAdder	Half Adder	HalfAdder.tst	HalfAdder.cmp
FullAdder	Full Adder	FullAdder.tst	FullAdder.cmp
Add16	16-bit Adder	Add16.tst	Add16.cmp
Inc16	16-bit incrementer	Inc16.tst	Inc16.cmp
ALU	Arithmetic Logic Unit	ALU.tst	ALU.cmp

All the necessary project 2 files are available in:
nand2tetris / projects / 02

More resources

- HDL Survival Guide
- Hardware Simulator Tutorial
- nand2tetris Q&A forum

All available in: `www.nand2tetris.org`

Best practice advice

- Try to implement the chips in the given order
- If you don’t implement some of the chips required in project 2, you can still use them as chip-parts in other chips. Just rename the given stub-files; this will cause the simulator to use the built-in versions of these chips
- You can invent new, “helper chips”; however, this is not required: you can build any chip using previously-built chips only
- Strive to use as few chip-parts as possible
- You will have to use chips implemented in Project 1
- For efficiency and consistency’s sake, use their built-in versions rather than your own implementation.

Chapter 2: Boolean arithmetic

- ✓ Binary numbers
- ✓ Binary addition
- ✓ Negative numbers
- ✓ Arithmetic Logic Unit
- ✓ Project 2 overview

Chapter 2

Boolean Arithmetic

These slides support chapter 2 of the book

The Elements of Computing Systems

By Noam Nisan and Shimon Schocken

MIT Press