Dempster
Rubin
Laird

x: datos completos
y: datos observados

$$f(x,y\mid\phi) = f(x\mid\phi)$$

$$= f(x\mid y,\phi)\, f(y\mid\phi)$$

$$\Rightarrow f(x\mid\phi) = f(x\mid y,\phi)\, f(y\mid\phi)$$

$$\log f(x\mid\phi) = \log f(x\mid y,\phi) + \log f(y\mid\phi)$$

Supongamos que $X \sim f(x\mid\phi^{(0)})$

Paso E

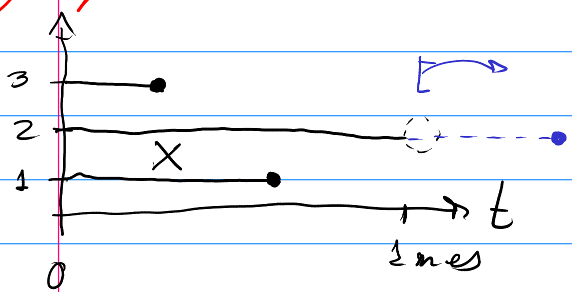
$$\mathbb{E}_{X\sim f(x\mid\phi^{(0)})} \log f(x\mid\phi) = \mathbb{E}_{X\sim f(x\mid\phi^{(0)})} \log f(x\mid y,\phi) + \log g(y\mid\phi)$$

$$= Q(\phi\mid\phi^{(0)})$$

Paso M

$$\phi^{(1)} = \arg\max_{\phi} Q(\phi\mid\phi^{(0)})$$

Ejemplo (Normal con censura por la derecha)



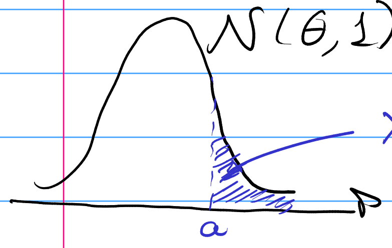
$X_1,\ldots,X_n \overset{iid}{\sim} \mathcal{N}(\theta,1)$, pero si $X>a$, se observa a

Supongamos que $x_1, \ldots, x_m$ están observados y $x_{m+1} > a, \ldots, x_n > a$ están censurados.

* Datos Observados

$$y_1 = x_1,\ y_2 = x_2, \ldots, y_m = x_m,$$

$$y_{m+1} = \{x_{m+1} > a\}, \ldots, y_n = \{x_n > a\}$$



$X|X>a \sim$ Normal truncada$(\theta, 1, a)$

$$X \sim N(\theta, 1)$$

$$f(x|\theta) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(x-\theta)^2\right\}$$

$$\propto \exp\left\{-\frac{1}{2}(x-\theta)^2\right\}$$

$$\log f(x|\theta) = cte. - \frac{1}{2}(x-\theta)^2$$

$$\log f(\vec{x}|\theta) = cte - \frac{1}{2}\sum_{i=1}^{n}(x_i-\theta)^2$$

$$= cte - \frac{1}{2}\sum_{i=1}^{m}\underbrace{(x_i-\theta)^2}_{\text{observados}} - \frac{1}{2}\sum_{i=m+1}^{n}\underbrace{(x_i-\theta)^2}_{\text{censurados}}$$

$$= cte - \frac{1}{2}\sum_{i=1}^{m} x_i^2 - \frac{1}{2}\sum_{i=1}^{m}\theta^2 + \theta\sum_{i=1}^{m} x_i$$

$$- \frac{1}{2}\sum_{i=m+1}^{n} x_i^2 - \frac{1}{2}\sum_{i=m+1}^{n}\theta^2 + \theta\sum_{i=m+1}^{n} x_i$$

$$= \text{cte} - \frac{1}{2}\sum_{i=1}^{n} x_i^2 - \frac{1}{2} n\theta^2 + \theta\left(\sum_{i=1}^{m} x_i + \sum_{i=m+1}^{n} x_i\right)$$

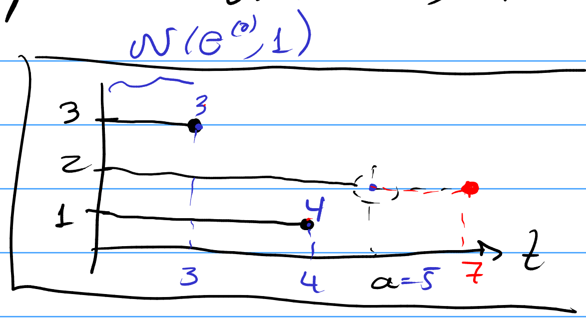
- Paso E: $\mathbb{E}_{X\sim f(x|\theta^{(0)})}$

$$Q(\theta|\theta^{(0)}) = \text{cte} - \text{cte} - \frac{1}{2} n\theta^2 + \theta\, \mathbb{E}_{X\sim f(x|\theta^{(0)})}\left[\sum_{i=1}^{m} X_i + \sum_{i=m+1}^{n} X_i\right]$$

+ $\mathbb{E}(X_i|Y_i) = \mathbb{E}(X_i|X_i) = X_i \quad , \; i = 1, \dots, m$

+ $\mathbb{E}(X_j|Y_j) = \mathbb{E}(Z)$ donde Z es una normal truncada en (a,∞) con parámetros $(\theta, 1)$:
  $j = m+1, \dots, n$

$$\theta + \frac{\phi(a-\theta)}{1-\Phi(a-\theta)} = \omega_\theta$$



ϕ: densidad de $\mathcal{N}(0,1)$

Φ: función de distribución de $\mathcal{N}(0,1)$

$Q(\theta|\theta^{(0)})$

$$= \text{cte} - \frac{1}{2} n\theta^2 + \theta\, \mathbb{E}_{X\sim f(x|\theta^{(0)})}\left[\sum_{i=1}^{m} X_i + \sum_{i=m+1}^{n} X_i\right]$$

$$= \text{cte} - \frac{1}{2} n\theta^2 + \theta\left(\sum_{i=1}^{m} x_i + (n-m)\,\omega_{\theta^{(0)}}\right)$$

Paso m

$$\frac{dQ(\theta|\theta^{(0)})}{d\theta} = -n\theta + \sum_{i=1}^{m} x_i + (n-m)\,\omega_{\theta^{(0)}}$$

$$= -n\theta + m\bar{x}_{obs} + (n-m)\,\omega_{\theta^{(0)}}$$

donde $\bar{x}_{obs} = \frac{1}{m}\sum_{i=1}^{m} x_i$

$$\Rightarrow \quad \theta^{(1)} = \frac{m\bar{x}_{obs} + (n-m)\,\omega_{\theta^{(0)}}}{n}$$

$$= \frac{m}{n}\bar{x}_{obs} + \frac{n-m}{n}\left(\theta^{(0)} + \frac{\phi(a-\theta^{(0)})}{1-\Phi(a-\theta^{(0)})}\right)$$

$$= \frac{m}{n}\bar{x}_{obs} + \frac{n-m}{n}\theta^{(0)} + \frac{n-m}{n}\,\frac{\phi(a-\theta^{(0)})}{1-\Phi(a-\theta^{(0)})}$$

* Paso EM

$$\theta^{(j+1)} = \frac{m}{n}\bar{x}_{obs} + \frac{n-m}{n}\theta^{(j)} + \frac{n-m}{n}\,\frac{\phi(a-\theta^{(j)})}{1-\Phi(a-\theta^{(j)})}$$

También pudimos haber maximizado la logverosimilitud observada, i.e.

$$\log f(y|\phi) \equiv \log \mathcal{L}(\phi|y)$$

$$\mathcal{L}(\theta|y) \propto \left[\prod_{i=1}^{m} \phi(x_i \mid \theta, 1)\right] \left[\prod_{i=m+1}^{n} (1-\Phi(a-\theta))\right]$$

$$\phi(x|\theta, 1) = \text{densidad } \mathcal{N}(\theta, 1)$$

$$= \left(\prod_{i=1}^{m} \phi(x_i \mid \theta, 1)\right)(1-\Phi(a-\theta))^{n-m}$$

$$\ell(\theta|y) = \sum_{i=1}^{m} \log \phi(x_i|\theta, 1) + (n-m)(1-\Phi(a-\theta))$$

$$\left.\frac{d\ell}{d\theta}\right|_{\theta^*} = 0$$

$$\Rightarrow \theta^* = \bar{x}_{obs} + \frac{n-m}{m} \frac{\phi(a-\theta^*)}{1-\Phi(a-\theta^*)}$$

Propuesta

$$\theta^{(i+1)} = \bar{x}_{obs} + \frac{n-m}{m} \frac{\phi(a-\theta^{(i)})}{1-\Phi(a-\theta^{(i)})}$$

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from scipy.stats import norm
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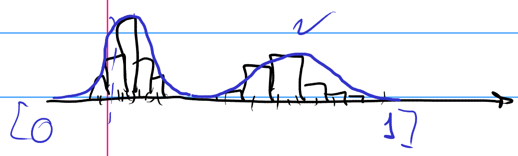
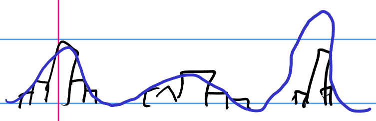
$$\begin{cases} \text{norm.rvs(size=m, loc=}\theta\text{, scale=1)} \\ [a]*(n-m) \end{cases}$$



$k=2$



$k=3$