CM Homework 1

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September 18, 2023

1. (a) The bead has velocity in the $\hat{\phi}$ direction of $\rho\omega$, leading to a kinetic energy term: $\frac{1}{2}m\omega^2\rho^2$. In the $\hat{\rho}$ direction, the velocity is simply $\dot{\rho}$, leading to a kinetc eneregy term: $\frac{1}{2}m\dot{\rho}^2$. In the z direction, the velocity is $\dot{z}$, but since the bead is constrained to the piece of wire, this velocity can be determined in terms of ρ. Using the constraint equation results in:

$$\begin{aligned}\frac{dz}{dt} &= D_t\left[\alpha\rho^2\right] \\ &= 2\alpha\rho D_t\left[\rho\right] \\ &= 2\alpha\rho\dot{\rho} \\ &\Rightarrow T_z = \frac{1}{2}m4\alpha^2\rho^2\dot{\rho}^2\end{aligned}$$

If we choose $z = 0$ as the point of zero potential, then:

$$V = mgh = mg\alpha\rho^2$$

Combining this gives the Lagrangian of the system:

$$\begin{aligned}L &= T - V \\ &= \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\omega^2\rho^2 + \frac{1}{2}m4\alpha^2\rho^2\dot{\rho}^2 - mg\alpha\rho^2 \\ &= \frac{1}{2}m\left(\left[1 + 4\alpha^2\rho^2\right]\dot{\rho}^2 + \rho^2\omega^2\right) - mg\alpha\rho^2\end{aligned}$$

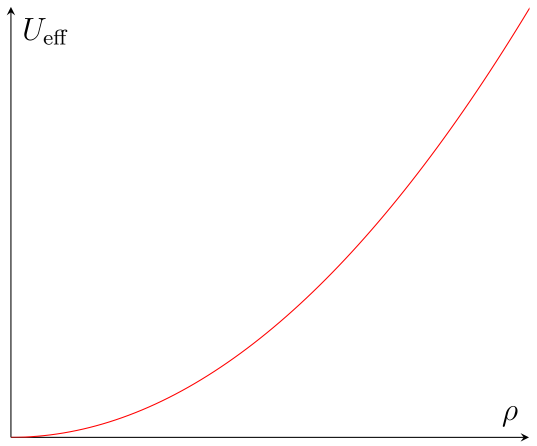
This is a simplistic derivation that might run into problems with more complicated systems and coordinates. A better approach would be to do it interms of rectangular coordiates(the basis vectors are fixed throughout space) and have a constraint equation with a Lagrange multiplier, and doing a change of coordinates.

(b) This Lagrangian has no explicit time dependence, and so there will be a conserved quantity that is "canonically conjugate" to time, which is just the generalized energy:

$$\begin{aligned}Q_s &= \dot{q}^i p_i - L \\ &= \dot{\rho}m\left(1 + 4\alpha^2\rho^2\right)\dot{\rho} - L \\ &= \frac{1}{2}m\left(1 + 4\alpha^2\rho^2\right)\dot{\rho}^2 - \frac{1}{2}m\omega\rho^2\omega + mg\alpha\rho^2 \\ &= \frac{1}{2}m\left(1 + 4\alpha^2\rho^2\right)\dot{\rho}^2 + \frac{1}{2}m\omega\rho^2\left(2g\alpha - \omega^2\right) \\ &= \frac{1}{2}m\left(1 + 4\alpha^2\rho^2\right)\dot{\rho}^2 + \frac{1}{2}m\omega\rho^2\left(\omega_0^2 - \omega^2\right) \\ &= \frac{1}{2}m\left(1 + 4\alpha^2\rho^2\right)\dot{\rho}^2 + U_{\text{eff}}\end{aligned}$$

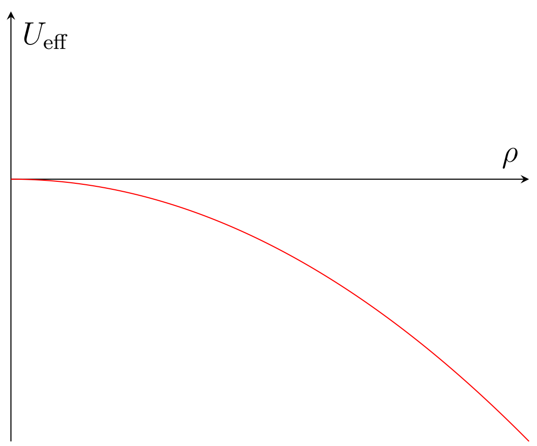
With $\omega_0 = \sqrt{2g\alpha}$.

(c) i.



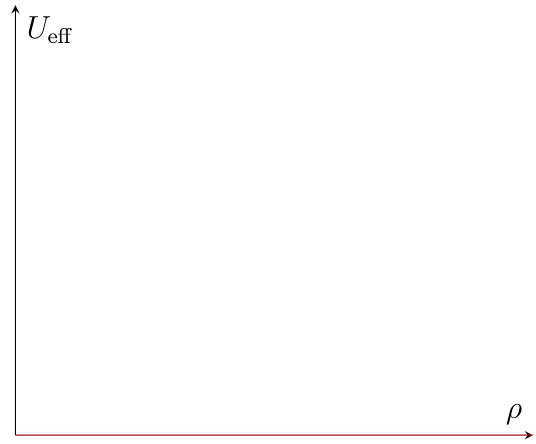
Here, the potential term is a positive parabola. The easiest way to visualize this is imagine that the wire is stationary in the gravitaional field. For some initial conditions there will be a ρ_{max} that ρ cannot increase beyond–for the energy to remain constant $\dot{\rho}$ must decrease when ρ increases, and eventually $\dot{\rho}$ must become zero. When the bead has zero velocity in the ρ direction it will fall down the wire and $\dot{\rho}$ will increase while ρ decreases. With ρ decreasing, it will eventually become zero, and all the particle's energy will be purely kinetic. The bead will move past the origin and swing back to "$-\rho_{\text{max}}$" and the cycle will continue. The bead essentially oscillates on the wire–it can be seen that the bead's generalized energy looks more or less like that of a simple harmonice oscillator with a slight perturbation connecting ρ and $\dot{\rho}$.

ii.



In this case, as the bead falls down the potential, ρ increases, and the bead heads off to infinity. It might be possible(probably but cannot be sure without an explicit check of the system's integrability) to start with $\rho\dot{\rho}$ negative and small enough that the bead will move down and change direction before crossing $\rho = 0$ and then heading off to $\rho \to \pm\infty$. Essentially trying to climb the hill of the potential, failling and rolling back down the hill. If the initial velocity is great enough, the bead might swing past $\rho = 0$ and head off to the infinity on the side of the wire opposite to where it started.

iii.



Here, the potential term is zero. More specifically, the potential and generalized energy are completely indepedent of the gravitaional strength parameter g. This implies that we can imagine the bead on the wire in free space. One option for the bead's trajectory is to stay stationary at a given point ρ for eternity. Another option is for the bead to move in a monotonous direction along the wire and move towards infinity increasingly slowly. This point is a bifurcation in the system.

(d) The energy is not conserved here as the motor is giving energy input into the system. Thus we do not expect a conseved energy-like quantity to have the form of the total energy. Mathematically they are not equivalent because the kinetic term is not a simple quadratic in the velocities.

2. (a) The time it takes to traverse the wire is:

$$T[x(y)] = \int_{y_A}^{y_B} (\dot{y})^{-1} dy$$

$\dot{y}$ can be determined from the conservation of energy. Along the bead's path, the conservation of the bead's energy constrains $\dot{y}$, $\dot{x}$ and y so that:

$$0 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy = \frac{1}{2}m(x'^2 + 1)\dot{y}^2 - mgy$$

Rerranging this:

$$\dot{y}^2 = \frac{2gy}{1 + x'^2}$$
$$\Rightarrow \dot{y} = \frac{\sqrt{2gy}}{\sqrt{1 + x'^2}}$$

Substituting this into the time functional makes the y dependence explicit:

$$T[y(x)] = \int_{y_A}^{y_B} \frac{\sqrt{1 + x'^2}}{\sqrt{2gy}} dy$$

(b) In the time actional, there is an equivalent of a Lagragian:

$$L(x, x', y) = \frac{\sqrt{1 + x'^2}}{\sqrt{2gy}}$$

x is cyclical here, which implies that:

$$\frac{\partial L}{\partial x'} = \frac{x'}{\sqrt{2gy(1 + x'^2)}}$$

is conserved with changing y or:

$$0 = \frac{d}{dy}\frac{\partial L}{\partial x'} \Rightarrow \frac{x'}{\sqrt{2gy(1+x'^2)}} = Q$$

Since the derivative is a total derivative in y and $x(y)$. The following algebra ensues(some of it might be superfluous since there are squaring and square-rooting steps):

$$\begin{aligned}
Q &= -\frac{x'}{\sqrt{2gy(1+x'^2)}} \\
\Rightarrow x'^2 &= 2Q^2 gy(1+x'^2) \\
\Rightarrow x'^2 &= \frac{2Q^2 gy}{1-2Q^2 gy} \\
\Rightarrow \frac{dx}{dy} &= \frac{\sqrt{y}}{\sqrt{1/2Q^2 g - y}}
\end{aligned}$$

This equation is separable so it can be integrated to get:

$$x = \int_{x_A}^{x} d\mathrm{x} + x_A = \int_{y_A}^{y} \frac{\sqrt{\mathrm{y}}}{\sqrt{1/2Q^2 g - \mathrm{y}}} d\mathrm{y} + x_A$$

This integral can be solved with the substitution $y = \frac{1}{2Q^2 g}\sin^2\frac{\theta}{2}$:[1]

$$\begin{aligned}
\int_{y_A}^{y} \frac{\sqrt{\mathrm{y}}}{\sqrt{1/2Q^2 g - \mathrm{y}}} d\mathrm{y} + x_A &= \int_{\theta_A}^{\theta} \frac{\sin(\theta/2)}{\cos(\theta/2)} \frac{1}{2Q^2 g} \sin\frac{\theta}{2}\cos\frac{\theta}{2} d\theta + x_A \\
&= \frac{1}{2Q^2 g}\int_{\theta_A}^{\theta} \sin^2(\theta/2) d\theta + x_A \\
&= \frac{1}{2Q^2 g}\frac{1}{2}(\theta - \sin\theta) - \frac{1}{Q^2 g}\frac{1}{2}(\theta_A - \sin\theta_A) + x_A \\
&= \frac{a}{2}(\theta - \sin\theta)
\end{aligned}$$

Where $a = \frac{1}{2Q^2 g}$, and the dependence on the conserved quantity is clear, and y is:

$$y = \frac{1}{2Q^2 g}\sin^2\frac{\theta}{2} = \frac{a}{2}(1-\cos\theta)$$

Determining a in terms of x_a, x_b, y_a, y_b:

$$x_B - x_A = \frac{a}{2}(\theta_B - \theta_A + \sin\theta_A - \sin\theta_B)$$

And:

$$\theta_* = 2\sin^{-1}\left(\sqrt{\frac{y_*}{a}}\right)$$

So that:

$$\begin{aligned}
x_B - x_A = &\frac{a}{2}\left[2\sin^{-1}\sqrt{\frac{y_B}{a}} - 2\sin^{-1}\sqrt{\frac{y_A}{a}} + \sin\left(2\sin^{-1}\left(\sqrt{\frac{y_A}{a}}\right)\right) + \sin\left(2\sin^{-1}\left(\sqrt{\frac{y_B}{a}}\right)\right)\right] \\
= &a\left[\sin^{-1}\sqrt{\frac{y_B}{a}} - \sin^{-1}\sqrt{\frac{y_A}{a}} + \sqrt{\frac{y_A}{a}}\cos\left(\sin^{-1}\left(\sqrt{\frac{y_A}{a}}\right)\right)\right. \\
&\left. - \sqrt{\frac{y_B}{a}}\cos\left(\sin^{-1}\left(\sqrt{\frac{y_B}{a}}\right)\right)\right] \\
= &a\left[\sin^{-1}\sqrt{\frac{y_B}{a}} - \sin^{-1}\sqrt{\frac{y_A}{a}} + \sqrt{\frac{y_A}{a}}\sqrt{1-\frac{y_A}{a}} - \sqrt{\frac{y_B}{a}}\sqrt{1-\frac{y_B}{a}}\right]
\end{aligned}$$

And this is an implicit relation for a in terms of the coordinates.

[1] I got stuck here and cheated by looking at this link

3. (a) Noted.

(b) Consider a variation of the form:

$$q(t) \rightsquigarrow q(t) + \epsilon\tilde{\delta}_s^f q(t) = q(t) + \epsilon f(t)\tilde{\delta}_s q(t)$$

with $f(t_0) = f(t_1) = 0$. Now the velocity varies as:

$$\dot{q}(t) \rightsquigarrow \dot{q}(t) + \epsilon\tilde{\delta}_s^f \dot{q}(t)$$

With:

$$\tilde{\delta}_s^f \dot{q}(t) = \dot{f}\tilde{\delta}_s q + f(t)\tilde{\delta}_s \dot{q}$$

The variation in the Lagrangian now is:

$$\begin{aligned}
\tilde{\delta}_s^f L &= \frac{\partial L}{\partial q^i}\tilde{\delta}_s^f q^i + \frac{\partial L}{\partial \dot{q}^i}\tilde{\delta}_s^f \dot{q}^i \\
&= \frac{\partial L}{\partial q^i} f(t)\tilde{\delta}_s q^i + \frac{\partial L}{\partial \dot{q}^i}\dot{f}\tilde{\delta}_s q^i + \frac{\partial L}{\partial \dot{q}^i} f(t)\delta_s \dot{q}^i \\
&= f(t)\left(\frac{\partial L}{\partial q^i}\tilde{\delta}_s q^i + \frac{\partial L}{\partial \dot{q}^i}\delta_s \dot{q}^i\right) + \frac{\partial L}{\partial \dot{q}^i}\dot{f}\tilde{\delta}_s q^i \\
&= f(t)\left(\tilde{\delta}_s L\right) + \frac{\partial L}{\partial \dot{q}^i}\dot{f}\tilde{\delta}_s q^i \\
&= f(t)\frac{d}{dt}R_s + \frac{\partial L}{\partial \dot{q}^i}\dot{f}\tilde{\delta}_s q^i \\
&= \frac{d}{dt}\left(f(t)R_s\right) - R_s\dot{f} + \frac{\partial L}{\partial \dot{q}^i}\dot{f}\tilde{\delta}_s q^i \\
&= \frac{d}{dt}\left(f(t)R_s\right) + \left(\frac{\partial L}{\partial \dot{q}^i}\tilde{\delta}_s q^i - R_s\right)\dot{f} \\
&= \frac{d}{dt}\left(f(t)R_s\right) + Q_s\dot{f}
\end{aligned}$$

Now the variation in the action is:

$$\begin{aligned}
\tilde{\delta}_s^f S &= \int_{t_0}^{t_1} \frac{d}{dt}\left(f(t)R_s\right) + Q_s\dot{f}dt \\
&= f(t)R_s|_{t_0}^{t_1} + \int_{t_0}^{t_1} Q_s\dot{f}dt \\
&= \int_{t_0}^{t_1} dtQ_s\dot{f}
\end{aligned}$$

Where the first integral vanishes owing to f's boundary conditions.

(c) On shell, the action's variation goes to zero:

$$\begin{aligned}
0 &= \int_{t_0}^{t_1} dtQ_s\dot{f} \\
&= \int_{f_0}^{f_1} dfQ_s \\
&= fQ_s|_{t_0}^{t_1} + \int_{t_0}^{t_1} f\frac{dQ_s}{dt}dt \\
&= \int_{t_0}^{t_1} f\frac{dQ_s}{dt}dt
\end{aligned}$$

Since f is an arbitrary function, integral being identically zero implies:

$$\frac{dQ_s}{dt} = 0$$

(d) The time translation symmetry is $\tilde{\delta}_s^f q = f\dot{q}$:

$$\begin{aligned}
\tilde{\delta}_s^f S = \int_{t_0}^{t_1} dt \tilde{\delta}_s^f L &= \int_{t_0}^{t_1} dt \left[-\frac{\partial V}{\partial q} f(t)\dot{q}^i + m\dot{q}\dot{f}\dot{q} + m\dot{q}f(t)\ddot{q}^i \right] \\
&= \int_{t_0}^{t_1} dt \left[-\frac{dV}{dt} f(t) + m\dot{q}^2 \dot{f} + m\dot{q}f(t)\ddot{q}^i \right] \\
&= \int_{t_0}^{t_1} dt \left[V\dot{f} + \frac{1}{2}m\dot{q}^2\dot{f} + \frac{d}{dt}\left(\frac{1}{2}m\dot{q}^2 f(t)\right) - \frac{d}{dt}(Vf) \right] \\
&= \int_{t_0}^{t_1} dt \left(V + \frac{1}{2}m\dot{q}^2 \right) \dot{f}
\end{aligned}$$

The total time derivative terms go to zero upon integration because $f(t_1) = f(t_0) = 0$, and what is left is the conserved energy times $\dot{f}$. Note that this only works because the potential has no explicit time dependence. This derivation is a nice way to see that time dependent potentials do not conserve energy.