Topic 1

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September 10, 2023

1. (a) The bead has velocity in the $\hat{\phi}$ direction of $\rho\omega$, leading to a kinetic energy term: $\frac{1}{2}m\omega^2\rho^2$. In the $\hat{\rho}$ direction, the velocity is simply $\dot{\rho}$, leading to a kinetc eneregy term: $\frac{1}{2}m\dot{\rho}^2$. In the z direction, the velocity is $\dot{z}$, but since the bead is constrained to the piece of wire, this velocity can be determined in terms of ρ. Using the constraint equation results in:

$$\begin{aligned}\frac{dz}{dt} &= D_t\left[\alpha\rho^2\right]\\ &= 2\alpha\rho D_t\left[\rho\right]\\ &= 2\alpha\rho\dot{\rho}\\ &\Rightarrow T_z = \frac{1}{2}m4\alpha^2\rho^2\dot{\rho}^2\end{aligned}$$

If we choose $z = 0$ as the point of zero potential, then:

$$V = mgh = mg\alpha\rho^2$$

Combining this gives the Lagrangian of the system:

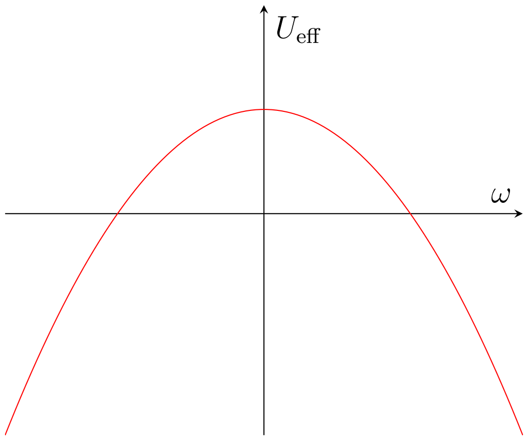
$$\begin{aligned}L &= T - V\\ &= \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\omega^2\rho^2 + \frac{1}{2m}4\alpha^2\rho^2\dot{\rho}^2 - mg\alpha\rho^2\\ &= \frac{1}{2}m\left(\left[1 + 4\alpha^2\rho^2\right]\dot{\rho}^2 + \rho^2\omega^2\right) - mg\alpha\rho^2\end{aligned}$$

This is a simplistic derivation that might run into problems with more complicated systems and coordinates.

(b) In this case the conserved charge is, since there is no explicit z dependence and ω is constant:

$$\begin{aligned}Q_s &= \dot{q}^i p_i - L\\ &= \dot{\rho}m\left(1 + 4\alpha^2\rho^2\right)\dot{\rho} - L\\ &= \frac{1}{2}\dot{\rho}m\left(1 + 4\alpha^2\rho^2\right)\dot{\rho} - \frac{1}{2}m\omega\rho^2\omega + mg\alpha\rho^2\\ &= \frac{1}{2}\dot{\rho}m\left(1 + 4\alpha^2\rho^2\right)\dot{\rho} + \frac{1}{2}m\omega\rho^2\left(2g\alpha - \omega^2\right)\\ &= \frac{1}{2}\dot{\rho}m\left(1 + 4\alpha^2\rho^2\right)\dot{\rho} + \frac{1}{2}m\omega\rho^2\left(\omega_0^2 - \omega^2\right)\\ &= \frac{1}{2}\dot{\rho}m\left(1 + 4\alpha^2\rho^2\right)\dot{\rho} + U_{\text{eff}}\end{aligned}$$

With $\omega_0 = \sqrt{2g\alpha}$.

(c) 

Intuitively:

- when $\omega > \omega_0$ the bead will move upwards
- when $\omega = \omega_0$ the bead will stay still vertically
- when $\omega < \omega_0$ the bead will move down the wire

i. In this case

ii.

iii.

(d) The energy is not conserved here as the motor is giving energy input into the system. Thus we do not expect a conseved energy-like quantity to have the form of the total energy. Mathematically they are not equivalent becuase the kinetic term is not a simple quadratic in the velocities.

2. (a) And the time it takes to traverse the wire is:

$$T[y(x)] = \int_{y_A}^{y_B} (\dot{y})^{-1} dy$$

$\dot{y}$ can be determined from the conservation of energy. Along the bead's path, the conservation of the bead's energy constrains $\dot{y}$, $\dot{x}$ and y so that:

$$E = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy$$

Rerranging this:

$$\begin{aligned}\dot{y}^2 &= 2gy + \dot{x}^2 - \frac{2E}{m} \\ &= 2gy + (x'\dot{y})^2 - \frac{2E}{m} \\ &= \frac{2gy + \frac{2E}{m}}{1 - x'^2}\end{aligned}$$

Substituting this into the time functional makes the y dependence explicit:

$$T[y(x)] = \int_{y_A}^{y_B} \frac{\sqrt{1 - x'^2}}{\sqrt{2gy}} dy$$

(b) In out time actional, there is an equivalent of a Lagragian:

$$L(x, x', y) = \frac{\sqrt{1 - x'^2}}{\sqrt{2gy}}$$

x is cyclical here, which implies that:

$$\frac{\partial L}{\partial x'} = -\frac{2x'}{2\sqrt{2gy(1-x'^2)}}$$

Which gives:

$$0 = \frac{d}{dy}\frac{\partial L}{\partial x'} = -\frac{4x'gy(1-x'^2)}{2\sqrt{2gy(1-x'^2)}}$$

3. (a) Consider a variation of the form:

$$q(t) \rightsquigarrow q(t) + \epsilon\tilde{\delta}_s^f q(t) = q(t) + \epsilon f(t)\tilde{\delta}_s q(t)$$

with $f(t_0) = f(t_1) = 0$. Now the velocity varies as:

$$\dot{q}(t) \rightsquigarrow \dot{q}(t) + \epsilon\tilde{\delta}_s^f \dot{q}(t)$$

With:

$$\dot{\tilde{\delta}}_s^f q(t) = \dot{f}\tilde{\delta}_s q + f(t)\tilde{\delta}_s q$$

The variation in the Lagrangian now is:

$$\begin{aligned}
\tilde{\delta}_s^f L &= \frac{\partial L}{\partial q^i}\tilde{\delta}_s^f q^i + \frac{\partial L}{\partial \dot{q}^i}\tilde{\delta}_s^f \dot{q}^i \\
&= \frac{\partial L}{\partial q^i} f(t)\tilde{\delta}_s q^i + \frac{\partial L}{\partial \dot{q}^i}\dot{f}\tilde{\delta}_s q^i + \frac{\partial L}{\partial \dot{q}^i} f(t)\frac{d}{dt}\delta_s q^i \\
&= \left[\frac{\partial L}{\partial q^i} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i} f(t)\right)\right]\tilde{\delta}_s q^i + \frac{\partial L}{\partial \dot{q}^i}\dot{f}\tilde{\delta}_s q^i
\end{aligned}$$

The Euler-Lagrange equations are evident in the first bracket and $f(t)$ is constant under variations of trajectories; therefore, this term will fall to zero for the physical trajectory. The second term is the only term left in the action's variation:

$$de\tilde{l}ta_s^f S = \int_{t_0}$$