Practical Theoretical Mechanics

Due: Monday September 18th @ 9PM
https://www.dropbox.com/request/4F9OZTfRFJjlPnK2RHIS
(deadline and resubmission policy available in the course outline)

Late deadline: Monday October 2nd @ 9PM
https://www.dropbox.com/request/lzzzIGEJvlC9g0Min9tH

Pass/fail deadline: formal request needed by Monday October 16th

1 Bead on a rotating wire

Consider a bead of mass m placed on a rigid wire of parabolic shape, $z = \alpha\rho^2$. The bead can slide without friction in a homogeneous gravitational field g (pointing in the negative z direction). The wire is powered by a motor and forced to rotate with constant angular velocity ω, see figure 1.

a) Using the cylindrical coordinates (ρ, φ, z), show that the system can be described by the following Lagrangian:

$$L = T - V = \frac{1}{2}m\Big([1 + 4\alpha^2\rho^2]\dot{\rho}^2 + \rho^2\omega^2\Big) - mg\alpha\rho^2\,. \tag{1}$$

b) Argue that there is a conserved quantity, *generalized energy*, that can be written as

$$E = \frac{1}{2}m(1 + 4\alpha^2\rho^2)\dot{\rho}^2 + U_{\text{eff}}\,, \quad U_{\text{eff}} = \frac{1}{2}m\rho^2(\omega_0^2 - \omega^2)\,, \tag{2}$$

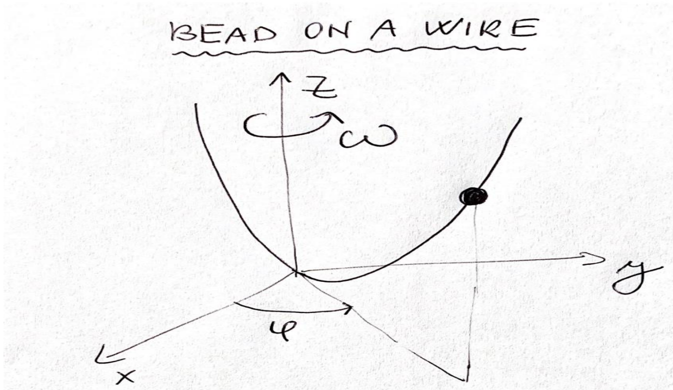

Figure 1: Bead on a wire

where U_{eff} is called the effective potential, and ω_0 is a constant to be determined.

c) Consider now three possibilities: i) $\omega > \omega_0$ ii) $\omega < \omega_0$ and iii) $\omega = \omega_0$. Please sketch U_{eff} and qualitatively describe the behavior of the system in each case.

d) Compare your expression for E with the 'total energy' $E_0 = T + V$. Can you explain why they are not the same? Is E_0 conserved?

2 Brachistochrone

Long before Euler and Lagrange and the invention of variational calculus to rephrase mechanics as the problem of minimizing an action functional subjected to appropriate boundary conditions, people have been interested in other variational problems. A famous one is that of the brachistochrone, that is: What is the shape of a wire between two points A and B so that a bead takes the shortest time to slide down from one to the other in the absence of friction? (See Figure 2.)

a) Use the conservation of energy to show that the time functional of the shape of the curve is given by

$$T[y(x)] = \int_{y_A}^{y_B} \frac{\sqrt{1+x'^2}}{\sqrt{2gy}} dy\,, \quad x' = \frac{dx}{dy}\,. \tag{3}$$

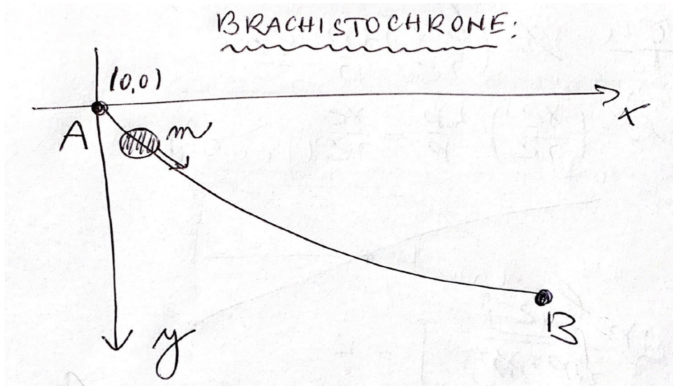

Figure 2: Brachistochrone

b) Find a "conserved quantity" Q associated to this variational problem and use it to recover the following parametric solution:

$$x = \frac{a}{2}(\theta - \sin\theta)\,, \quad y = \frac{a}{2}(1 - \cos\theta)\,, \tag{4}$$

which describes a cycloid of radius a – what is a in terms of the integral of the coordinates of the points A and B? And in terms of the "conserved quantity" Q?

3 A sneaky recipe for the Noether charge

Here we give a "sneaky recipe" for how to extract the Noether charge by turning a *global* infinitesimal symmetry into a *local* infinitesimal variation.[1] We follow here the version of the Noether theorem given in the main text.

Here is the recipe:

a) Let

$$q(t) \mapsto q'(t) = q(t) + \epsilon\widetilde{\delta}_s q(t)$$

be such that, *for ϵ an infinitesimal constant*, it leaves the action functional invariant, $\widetilde{\delta}_s L = \frac{d}{dt}R_s$.

[1] See e.g. Section 7.3 of Weinberg, The Quantum Theory of Fields, Vol. 1, Cambridge University Press (2005).

b) Consider now the same transformation of t and $q(t)$ written at the previous point, but replace the infinitesimal constant ϵ with an arbitrary infinitesimal function $\epsilon f(t)$ *that vanishes at the boundary* of the time interval, i.e.

$$\widetilde{\delta}_s q(t) \rightsquigarrow \delta_s^f q(t) = f(t)\widetilde{\delta}_s q(t) \quad \text{and} \quad f(t_0) = 0 = f(t_1).$$

The variation of the action under the new transformation will not will not be a boundary term anymore. Repeat the steps used to prove Noether's theorem to show that now

$$\delta_s^f S = \int_{t_0}^{t_1} dt\ Q_s \dot{f},$$

with $Q_s = \frac{\partial L}{\partial \dot{q}}\widetilde{\delta}_s q - R_s$ the Noether charge.

c) Show that the above equality implies that $Q_s(t)$ is conserved on-shell of the equations of motion.

d) Use the sneaky recipe to prove the conservation of energy for the Lagrangian $L = \frac{1}{2}m\dot{q}^2 - V(q)$.