Physics 714 (Quantum Mechanics B)
Handed out: 16/02/22
Due: 28/02/22
Total: 27

Problem 1 [3]

Proof the Schwarz inequality directly from the defining properties of the inner-product. Use bra-ket notation.

Hint: Note that the inner-product of $\alpha|\phi\rangle + |\psi\rangle$ with itself is positive for all α, and in particular for the α and α^* that minimises this quantity.

Problem 2 [9]

a) Show that any 2×2 matrix can be expressed as a linear combination of the identity and Pauli matrices, which are defined as follows:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ & -1 \end{pmatrix}$$

[2]

b) If A and B are matrices, show that $\mathrm{tr}\left(A^\dagger B\right)$ is an inner-product, i.e. verify the properties of an inner-product. Show that the matrices in (a) are orthogonal and normalise them so that they form an orthonormal basis.

[7]

Hint: Use the fact that C^D is an inner product space with the standard inner product.

Problem 3 [4]

Use the definition of a function of an operator $f(A) = \sum_n f(a_n)|n\rangle\langle n|$, where $A|n\rangle = a_n|n\rangle$ and $\langle n|m\rangle = \delta_{n,m}$ to show that $A^n A^m = A^{n+m}$.

Problem 4 [11]

Use bra-ket notation to proof or calculate the following:

a) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$

[3]

b) $(AB)^\dagger = B^\dagger A^\dagger$

[5]

c) For two kets $|\alpha\rangle$ and $|\beta\rangle$ the numbers $\langle n|\alpha\rangle$ and $|\beta\rangle$ are known in some complete orthonormal basis. Find the matrix representation of the operator $A = |\alpha\rangle\langle\beta|$ in this basis.

[1]

d) Show that the matrix representation of the adjoint, $A^\dagger$, of an operator A in some orthonormal basis $|n\rangle$ is the hermitian conjugate of the matrix representation of A in the same basis.

[2]