Physics 721: Tutorial
2022.03.04

This assignment is due on Friday, 11 March 2022.

1. Non-interacting systems — revision from 3rd year. Calculate the partition functions and average magnetisations for the following *non-interacting* magnetic systems:

   (a) The spin–$\frac{1}{2}$ Ising model with N spins, external field h.

   (b) The spin–1 Ising model with N spins, external field h.

   Note: Why are we still calculating for non-interacting systems? The calculations for non-interacting systems can be "leveraged" to help us deal with approximations to the interacting systems. It is always good to know the results for some systems where the calculations are possible.

2. Transfer matrix: Go through the calculation of the partition function and the magnetisation for the spin-1/2, 1D, periodic, nearest-neighbour Ising model using the transfer matrix method. Calculate the partition function and the magnetisation per spin as functions of β, h, and J. (*Recall:* For the magnetisation you can use the idea of the generating function — think derivatives with respect to βh.)

3. Do not use the transfer matrix method in this question. Calculate the partition function for the 1D, nearest-neighbour, spin–$\frac{1}{2}$ Ising system with the following Hamiltonian. The systems is *not* periodic. (*Hint:* Simply start summing over possible configurations starting with the last spin, and inspect the result.)

$$H = -\sum_{\langle i,j\rangle} J_{ij}\, s_i s_j.$$

   Also calculate the correlation function $\langle s_i\, s_{i+m}\rangle$.

4. A three-state model for magnet has the following spin states for each $\vec{s}_i$ where $i \in \{1, \ldots, N\}$

$$\vec{s}_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ or } \begin{pmatrix} -\frac{1}{2} \\ \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \text{ or } \begin{pmatrix} -\frac{1}{2} \\ \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \tag{1}$$

   with the total energy given by the following expression:

$$H = -J \sum_{\langle ij\rangle} \vec{s}_i \cdot \vec{s}_j, \tag{2}$$

   where usual notation for nearest-neighbour summation applies above. In the steps below you shall compute the mean-field solution using the following variational Hamiltonian

$$H_{\mathrm{v}} = -\sum_{i=1}^{N} \begin{pmatrix} h \\ 0 \end{pmatrix} \cdot \vec{s}_i. \tag{3}$$

   (a) Why do you think the choice of one component of the external variational field is sufficient? *Hint:* Draw the three vectors in eq. (1) to help you answer this question.

   (b) Compute the Helmholtz free energy and the average $\langle \vec{s}_i \rangle_{\mathrm{v}}$ for the variational Hamiltonian H_{v} of eq. (3).

   (c) Show that the mean-field theory results in the following expression for h

$$\frac{h}{Jz} = \frac{e^{3\beta h/2} - 1}{e^{3\beta h/2} + 2} \tag{4}$$

   (don't solve this explicitly) and the variational free energy

$$\frac{\widetilde{F}}{N} = -k_{\mathrm{B}} T \ln\left[e^{\beta h} + 2e^{-\beta h/2}\right] + \frac{h^2}{2Jz}. \tag{5}$$

   Here z is the number of nearest neighbours for any site, as per convention.