# Quantum mechanics Assignment 1 — J L Gouws

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1.

$$
\begin{aligned}
(\langle\psi| + \alpha^* \langle\phi|)(\alpha|\phi\rangle + |\psi\rangle) &= \langle\psi|(\alpha|\phi\rangle + |\psi\rangle) + \alpha^* \langle\phi|(\alpha|\phi\rangle + |\psi\rangle) \\
&= \alpha\langle\psi|\phi\rangle + \langle\psi|\psi\rangle + |\alpha|^2 \langle\phi|\phi\rangle + \alpha^* \langle\phi|\psi\rangle \\
&\geq 0
\end{aligned}
$$

The above statement holds for all $\alpha$, and in particular $\alpha = i\frac{\langle\psi|\phi\rangle}{\langle\phi|\phi\rangle}$. If we substitute this choice of $\alpha$ into the inequality above, we get:

$$
\begin{aligned}
&\Rightarrow i\frac{|\langle\psi|\phi\rangle|^2}{\langle\phi|\phi\rangle} + \langle\psi|\psi\rangle - |\langle\psi|\phi\rangle|^2 \frac{\langle\phi|\phi\rangle}{|\langle\phi|\phi\rangle|^2} - i\frac{|\langle\phi|\psi\rangle|^2}{\langle\phi|\phi\rangle} \geq 0 \\
&\Rightarrow \langle\psi|\psi\rangle - \frac{|\langle\psi|\phi\rangle|^2}{\langle\phi|\phi\rangle} \geq 0 \\
&\Rightarrow \langle\psi|\psi\rangle \langle\phi|\phi\rangle \geq |\langle\psi|\phi\rangle|^2
\end{aligned}
$$

Which is the required result.

2. a). Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a general matrix over $\mathbb{C}$.
Then we need to find $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, such that:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \delta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

For this we have:

$$\begin{aligned} a &= \alpha + \delta \\ b &= \beta + -i\gamma \\ c &= \beta + i\gamma \\ d &= \alpha - \delta \end{aligned}$$

This system of equations is partially decoupled, and so is easily solved.

$$\begin{aligned} \alpha &= \frac{a+d}{2} \\ \beta &= \frac{b+c}{2} \\ \gamma &= i\frac{b-c}{2} \\ \delta &= \frac{a-d}{2} \end{aligned}$$

Since $a, b, c, d$ are in $\mathbb{C}$ and $\mathbb{C}$ is a field, $\alpha, \beta, \gamma, \delta$ are definitely in $\mathbb{C}$ as required.

b). $\operatorname{tr}\left(A^\dagger B\right)$ is an inner product: First, $(A, B) = \operatorname{tr}\left(A^\dagger B\right)$, is indeed a map $\mathbb{C}^{2\times 2} \times \mathbb{C}^{2\times 2} \to \mathbb{C}$.
Let:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$