Statistical Physics Tutorial 1 — J L Gouws

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1. We have:

$$z\frac{\partial \ln \mathcal{Z}}{\partial z} = z\,\frac{\partial \mathcal{Z}}{\partial z}\Big/\,\mathcal{Z} = \frac{z}{\mathcal{Z}}\sum_N N z^{N-1} Z_N = \sum_N N\frac{z^N Z_N}{\mathcal{Z}} = \sum_N N\frac{e^{\beta\mu N}\sum_r e^{-\beta E_r}}{\mathcal{Z}} = \sum_N N P_N = \langle N\rangle$$

2. a). Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a general matrix over $\mathbb{C}$.

   Then we need to find $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, such that:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \gamma\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \delta\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

   For this we have:

$$\begin{aligned} a &= \alpha + \delta \\ b &= \beta - i\gamma \\ c &= \beta + i\gamma \\ d &= \alpha - \delta \end{aligned}$$

   This system of equations is partially decoupled, and so is easily solved.

$$\begin{aligned} \alpha &= \frac{a+d}{2} \\ \beta &= \frac{b+c}{2} \\ \gamma &= i\frac{b-c}{2} \\ \delta &= \frac{a-d}{2} \end{aligned}$$

   Since a, b, c, d are in $\mathbb{C}$ and $\mathbb{C}$ is a field, $\alpha, \beta, \gamma, \delta$ are definitely in $\mathbb{C}$ as required.

b). $\mathrm{tr}\left(A^{\dagger}B\right)$ is an inner product: First, $(A, B) = \mathrm{tr}\left(A^{\dagger}B\right)$, is indeed a map $\mathbb{C}^{2\times 2} \times \mathbb{C}^{2\times 2} \to \mathbb{C}$. Let:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \qquad C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

Note:

$$\begin{aligned} (A, B) &= \mathrm{tr}\left\{\begin{pmatrix} a_{11}^* & a_{21}^* \\ a_{12}^* & a_{22}^* \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right\} \\ &= \mathrm{tr}\left\{\begin{pmatrix} a_{11}^*b_{11} + a_{21}^*b_{21} & a_{11}^*b_{12} + a_{21}^*b_{22} \\ a_{12}^*b_{11} + a_{22}^*b_{21} & a_{12}^*b_{12} + a_{22}^*b_{22} \end{pmatrix}\right\} \\ &= a_{11}^*b_{11} + a_{21}^*b_{21} + a_{12}^*b_{12} + a_{22}^*b_{22} \end{aligned}$$

Similarly

$$(B, A) = b_{11}^*a_{11} + b_{21}^*a_{21} + b_{12}^*a_{12} + b_{22}^*a_{22}$$

It is now clear that:

$$(B, A)^* = \left(b_{11}^*a_{11} + b_{21}^*a_{21} + b_{12}^*a_{12} + b_{22}^*a_{22}\right)^* = a_{11}^*b_{11} + a_{21}^*b_{21} + a_{12}^*b_{12} + a_{22}^*b_{22} = (A, B)$$

As required, now moving onto linearity in the second argument:

$$\begin{aligned} (A, cB + dC) &= \mathrm{tr}\left\{A^{\dagger}\left(cB + dC\right)\right\} \\ &= \mathrm{tr}\left\{\begin{pmatrix} a_{11}^* & a_{21}^* \\ a_{12}^* & a_{22}^* \end{pmatrix}\begin{pmatrix} cb_{11} + dc_{11} & cb_{12} + dc_{12} \\ cb_{21} + dc_{21} & cb_{22} + dc_{22} \end{pmatrix}\right\} \\ &= a_{11}^*(cb_{11} + dc_{11}) + a_{21}^*(cb_{21} + dc_{21}) + a_{12}^*(cb_{12} + dc_{12}) + a_{22}^*(cb_{22} + dc_{22}) \\ &= c\left(a_{11}^*b_{11} + a_{21}^*b_{21} + a_{12}^*b_{12} + a_{22}^*b_{22}\right) + d\left(a_{11}^*c_{11} + a_{21}^*c_{21} + a_{12}^*c_{12} + a_{22}^*c_{22}\right) \\ &= c\,\mathrm{tr}(A^{\dagger}B) + d\,\mathrm{tr}(A^{\dagger}C) \\ &= c\,(A, B) + d\,(A, C) \end{aligned}$$

Positive definiteness is shown by:

$$\begin{aligned} (A, A) &= a_{11}^*a_{11} + a_{12}^*a_{12} + a_{21}^*a_{21} + a_{22}^*a_{22} \\ &= |a_{11}|^2 + |a_{12}|^2 + |a_{21}|^2 + |a_{22}|^2 \\ &\geq 0 \end{aligned}$$

and since $|c|^2 = 0 \Leftrightarrow c = 0$ for any $c \in \mathbb{C}$. We have form the above:

$$(A, A) = 0 \Leftrightarrow A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This is actually obvious if one realizes that there is an isomorphism:

$$\begin{aligned} I &: \mathbb{C}^{2\times 2} \to \mathbb{C}^4 \\ &: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \end{aligned}$$

Then $\mathrm{tr}\left\{A^{\dagger}B\right\}$ is the standard inner product on $\mathbb{C}^4$.

The product of a 2×2 diagonal matrix and an anti-diagonal matrix zero:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow (I, \sigma_x) = (I, \sigma_y) = (\sigma_z, \sigma_x) = (\sigma_z, \sigma_y) = 0$$

It now remains to check the product of the two diagonal and anti-diagonal matrices:

$$(I, \sigma_z) = 1 + 0 + 0 - 1 = 0$$
$$(\sigma_x, \sigma_y) = 0 + 1^*(-i) + 1^*(i) + 0 = 0$$

This shows that the four matrices are mutually orthogonal.

All matrices have two elements, and each element has modulus of 1. This makes the norm of each matrix $\sqrt{2}$, hence:

$$\left\{\frac{I}{\sqrt{2}}, \frac{\sigma_x}{\sqrt{2}}, \frac{\sigma_y}{\sqrt{2}}, \frac{\sigma_z}{\sqrt{2}}\right\}$$

Forms an orthonormal basis for this space.

3. Let the basis of our ket space be given by $\{|a\rangle\}$, that is the set of eigenkets of A. Now:

$$A^n = A^n \sum_a |a\rangle \langle a| = \sum_a A^n |a\rangle \langle a| = \sum_a a^n |a\rangle \langle a|$$

And similarly:

$$A^m = \sum_a a^m |a\rangle \langle a|$$

Thus

$$\begin{aligned}
A^n A^m &= \left(\sum_a a^n |a\rangle \langle a|\right)\left(\sum_{a'} a'^m |a'\rangle \langle a'|\right) \\
&= \sum_a \sum_{a'} a^n a'^m |a\rangle \langle a|a'\rangle \langle a'| \\
&= \sum_a \sum_{a'} a^n a'^m |a\rangle \delta^{a'}{}_a \langle a'| \\
&= \sum_a a^n a^m |a\rangle \langle a| \\
&= \sum_a a^{n+m} |a\rangle \langle a| \\
&\equiv A^{n+m}
\end{aligned}$$

4. a). By definition:

$$\operatorname{tr} A = \sum_i \langle i|A|i\rangle$$

Now:

$$\begin{aligned}
\operatorname{tr}(AB) &= \sum_i \langle i|AB|i\rangle \\
&= \sum_i \sum_j \langle i|A|j\rangle \langle j|B|i\rangle \\
&= \sum_i \sum_j \langle j|B|i\rangle \langle i|A|j\rangle \\
&= \sum_j \langle j|BA|j\rangle \\
&= \operatorname{tr}(BA)
\end{aligned}$$

b). We have:

$$\begin{aligned}
AB &= \sum_i AB\,|i\rangle \langle i| \\
&= \sum_{i,j} A\,|j\rangle \langle j|\, B\,|i\rangle \langle i| \\
&= \sum_{i,j,k} |k\rangle \langle k|\, A\,|j\rangle \langle j|\, B\,|i\rangle \langle i| \\
&= \sum_{i,j,k} (\langle k|\, A\,|j\rangle \langle j|\, B\,|i\rangle)\, |k\rangle \langle i|
\end{aligned}$$

Now last expression above is just the sum of a number, $\langle k|A|j\rangle \langle j|B|i\rangle$, times by an operator and so:

$$\begin{aligned}
(AB)^\dagger &= \sum_{i,j,k} [(\langle k|\, A\,|j\rangle \langle j|\, B\,|i\rangle)\, |k\rangle \langle i|]^\dagger \\
&= \sum_{i,j,k} (\langle k|\, A\,|j\rangle \langle j|\, B\,|i\rangle)^*\, |i\rangle \langle k| \\
&= \sum_{i,j,k} \left(\langle j|\, A^\dagger\,|k\rangle \langle i|\, B^\dagger\,|j\rangle\right) |i\rangle \langle k| \\
&= \sum_{i,j,k} |i\rangle \langle i|\, B^\dagger\,|j\rangle \langle j|\, A^\dagger\,|k\rangle \langle k| \\
&= \sum_i |i\rangle \langle i|\, B^\dagger A^\dagger \\
&= B^\dagger A^\dagger
\end{aligned}$$

As required.

c).

$$\begin{aligned}
A &= |\alpha\rangle \langle\beta| \\
&= \sum_n |n\rangle \langle n|\alpha\rangle \langle\beta| \\
&= \sum_n \sum_m |n\rangle \langle n|\alpha\rangle \langle\beta|m\rangle \langle m|
\end{aligned}$$

This is a matrix with the *nm*-th element having the value: $\langle n|\alpha\rangle \langle\beta|m\rangle$.

d).

$$\begin{aligned}
A^\dagger &= \left(\sum_n \sum_m |n\rangle \langle n|\alpha\rangle \langle \beta|m\rangle \langle m|\right)^\dagger \\
&= \sum_n \sum_m \left(\langle n|\alpha\rangle \langle \beta|m\rangle\right)^* |m\rangle \langle n| \\
&= \sum_n \sum_m \left(\langle m|\alpha\rangle \langle \beta|n\rangle\right)^* |n\rangle \langle m|
\end{aligned}$$

This shows that the *nm*-th element of $A^\dagger$ is the complex conjugate of the *mn*-th element of A. This is the definition of the Hermitian conjugate of a matrix.