Electromagnetism Assignment 1 — J L Gouws

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1. An observation of some source is being made at a position $\vec{r}$. Electromagnetic waves take time to travel through space, they travel at a speed of c in free space for all reference frames. Since we are observing the source from some point $\vec{r}$, we are actually seeing what was happening at the source when the EM waves were emmitted. Since EM waves travel at speed c in free space, the time it takes for the waves to travel from source point $\vec{r}'$ to an observation point at $\vec{r}$ is $\frac{|r-r'|}{c}$. Hence, the waves that are observed at time t originiated at time:

$$t_r = t - \frac{|\vec{r}-\vec{r}'|}{c}$$

Thus to find the potential that we are observing we need to integrate over the source at space-time point $(t_r, \vec{r}')$.

2. a). We have:

$$\frac{\partial}{\partial x_i} f(|\vec{r}-\vec{r}'|) = f'(|\vec{r}-\vec{r}'|)\frac{\partial|\vec{r}-\vec{r}'|}{\partial x_i} = f'(|\vec{r}-\vec{r}'|)\frac{\partial}{\partial x_i}\left(\sum_i (x_i - x_i')^2\right)^{1/2} = f'(|\vec{r}-\vec{r}'|)\frac{2(x_i - x_i')}{2|r-r'|}$$

Similarly:

$$\frac{\partial}{\partial x_i'} f(|\vec{r}-\vec{r}'|) = f'(|\vec{r}-\vec{r}'|)\frac{\partial|\vec{r}-\vec{r}'|}{\partial x_i'} = f'(|\vec{r}-\vec{r}'|)\frac{\partial}{\partial x_i'}\left(\sum_i (x_i - x_i')^2\right)^{1/2} = f'(|\vec{r}-\vec{r}'|)\frac{2(x_i - x_i')\times(-1)}{2|r-r'|}$$

From which it is clear that:

$$\frac{\partial}{\partial x_i} f(|\vec{r}-\vec{r}'|) = f'(|\vec{r}-\vec{r}'|)\frac{(x_i - x_i')}{|r-r'|} = -\frac{\partial}{\partial x_i'} f(|\vec{r}-\vec{r}'|)$$

b) First calculating $\frac{\partial V}{\partial t}$:

$$\begin{aligned}
\frac{\partial V}{\partial t} &= \frac{1}{4\pi\epsilon_0}\frac{\partial}{\partial t}\int_V d^3\vec{r}'\frac{\rho(\vec{r}', t_r)}{|r-r'|} \\
&= \frac{1}{4\pi\epsilon_0}\int_V d^3\vec{r}'\frac{1}{|r-r'|}\frac{\partial\rho(\vec{r}', t_r)}{\partial t} \\
&= \frac{1}{4\pi\epsilon_0}\int_V d^3\vec{r}'\frac{1}{|r-r'|}\frac{\partial\rho(\vec{r}', t_r)}{\partial t_r}\frac{\partial t_r}{\partial t} \\
&= \frac{1}{4\pi\epsilon_0}\int_V d^3\vec{r}'\frac{1}{|r-r'|}\frac{\partial\rho(\vec{r}', t_r)}{\partial t_r}
\end{aligned}$$

2. And now, calculating $\nabla \cdot \vec{A}$:

$$\begin{aligned}
\nabla \cdot \vec{A} &= \frac{\mu_0}{4\pi} \nabla \cdot \int_V d^3\vec{r}' \frac{\vec{J}(\vec{r}', t_r)}{|r - r'|} \\
&= \frac{\mu_0}{4\pi} \int_V d^3\vec{r}' \nabla \cdot \left(\frac{\vec{J}(\vec{r}', t_r)}{|r - r'|} \right) \\
&= \frac{\mu_0}{4\pi} \int_V d^3\vec{r}' \left(\frac{1}{|r - r'|} \nabla \cdot \vec{J}(\vec{r}', t_r) + J(\vec{r}', t_r) \cdot \nabla \frac{1}{|r - r'|} \right) \\
&= \frac{\mu_0}{4\pi} \int_V d^3\vec{r}' \left(\frac{1}{|r - r'|} \nabla \cdot \vec{J}(\vec{r}', t_r) - J(\vec{r}', t_r) \cdot \nabla' \frac{1}{|r - r'|} \right)
\end{aligned}$$

Note that:

$$\nabla' \cdot \left(\frac{\vec{J}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} \right) = \frac{1}{|\vec{r} - \vec{r}'|} \nabla' \cdot \vec{J}(\vec{r}', t_r) + \vec{J}(\vec{r}', t_r) \cdot \nabla' \left(\frac{1}{|\vec{r} - \vec{r}'|} \right)$$

And so:

$$\nabla \cdot \vec{A} = \frac{\mu_0}{4\pi} \int_V d^3\vec{r}' \left(\frac{1}{|r - r'|} \nabla \cdot \vec{J}(\vec{r}', t_r) + \frac{1}{|\vec{r} - \vec{r}'|} \nabla' \cdot \vec{J}(\vec{r}', t_r) - \nabla' \cdot \left(\frac{\vec{J}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} \right) \right) \tag{1}$$

This last integral is:

$$\begin{aligned}
\frac{\mu_0}{4\pi} \int_V d^3\vec{r}' \nabla' \cdot \left(\frac{\vec{J}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} \right) &= \frac{\mu_0}{4\pi} \int_S \frac{\vec{J}(\vec{r}', t_r) \cdot d\vec{a}}{|\vec{r} - \vec{r}'|} \\
&= 0
\end{aligned}$$

Since there is no current density at the edges of the surface. Also note that, where $J(\vec{r}')$ is the current density with the time held constant:

$$\nabla' \cdot \vec{J}(\vec{r}', t_r) = \nabla' \left(\vec{J}(\vec{r}') \right) + \frac{\partial \vec{J}}{\partial t_r} \nabla' t_r(|\vec{r} - \vec{r}'|) = -\frac{\partial \rho}{\partial t_r} - \frac{\partial \vec{J}}{\partial t_r} \nabla t_r(|\vec{r} - \vec{r}|)$$

Where I used the continuity condition and the result from 2.a). Also:

$$\nabla \cdot \vec{J}(r', t_r) = \frac{\partial \vec{J}}{\partial t_r} \nabla t_r$$

substituting this back into Eq. 1 we see:

$$\begin{aligned}
\nabla \cdot \vec{A} &= \frac{\mu_0}{4\pi} \int_V d^3\vec{r}' \left[\frac{1}{|r - r'|} \frac{\partial \vec{J}}{\partial t_r} \nabla t_r + \frac{1}{|\vec{r} - \vec{r}'|} \left(-\frac{\partial \rho}{\partial t_r} - \frac{\partial \vec{J}}{\partial t_r} \nabla t_r(|\vec{r} - \vec{r}|) \right) \right] \\
&= -\frac{\mu_0}{4\pi} \int_V d^3\vec{r}' \frac{1}{|\vec{r} - \vec{r}'|} \frac{\partial \rho}{\partial t_r} \\
&= -\mu_0 \epsilon_0 \left(\frac{1}{4\pi\epsilon_0} \int_V d^3\vec{r}' \frac{1}{|\vec{r} - \vec{r}'|} \frac{\partial \rho}{\partial t_r} \right) \\
&= -\mu_0 \epsilon_0 \frac{\partial V}{\partial t} \\
&\Rightarrow \nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial V}{\partial t}
\end{aligned}$$

3. a).

$$\vec{J}(\vec{r},t) = \frac{Q}{\Delta A \Delta t}\hat{n} = \frac{Q\Delta l}{\Delta A \Delta l \Delta t}\hat{n} = \frac{Q}{\Delta A \Delta l}\frac{\Delta l}{\Delta t}\hat{n} = \frac{Q}{\Delta V}v\hat{n} = \rho(\vec{r},t)\vec{v}$$

b).

$$\begin{aligned}
V &= \frac{1}{4\pi\epsilon_0}\int_V d^3\vec{r}'\,\frac{\rho(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|} \\
&= \frac{1}{4\pi\epsilon_0}\int_V r'dr'd\theta'dz'\,\frac{\lambda\delta(r'-b)\delta(z')}{|\vec{r}-\vec{r}'|} \\
&= \frac{\lambda}{4\pi\epsilon_0}\int_V r'dr'd\theta'dz'\,\frac{\delta(r'-b)\delta(z')}{\sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')+(z-z')^2}} \\
&= \frac{\lambda}{4\pi\epsilon_0}\int_{\mathbb{R}^2} r'dr'd\theta'\,\frac{\delta(r'-b)}{\sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')+z^2)}} \\
&= \frac{b\lambda}{4\pi\epsilon_0}\int_0^{2\pi} d\theta'\,\frac{1}{\sqrt{r^2+b^2+z^2-2rb\cos(\theta-\theta')}}
\end{aligned}$$

$$\begin{aligned}
\vec{A} &= \frac{\mu_0}{4\pi}\int_V d^3\vec{r}'\,\frac{\vec{J}(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|} \\
&= \frac{\mu_0}{4\pi}\int_V d^3\vec{r}'\,\frac{\vec{v}\rho(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|} \\
&= \frac{\mu_0}{4\pi}\int_V r'dr'd\theta'dz'\,\frac{\omega b\lambda\delta(r'-b)\delta(z')}{|\vec{r}-\vec{r}'|}
\end{aligned}$$