Physics 711
Electromagnetism Study Guide

These notes are based on the following handbooks:

- D J Griffiths, Introduction to electrodynamics,
- A D Jackson, Classical electrodynamics,
- H C Ohanian, Classical electrodynamics,

and provide a concise summary of several topics discussed in these handbooks. For the purposes of this course, these notes will suffice, but students interested in more detail and the broader context are encouraged to consult these handbooks.

1 Revision

We start with a brief revision of the fundamental formulation of electromagnetism, done in physics 342, after which we move on to more advanced applications.

1.1 Maxwell's equations (see Griffiths section 7.3.3, Third Edition)

The fundamental equations of electromagnetism are Maxwell's equations given by:

$$\begin{aligned}
\vec{\nabla}\cdot\vec{E} &= \frac{1}{\epsilon_0}\rho \quad \text{(Gauss)}\\
\vec{\nabla}\cdot\vec{B} &= 0\\
\vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t} \quad \text{(Faraday)}\\
\vec{\nabla}\times\vec{B} &= \mu_0\vec{J}+\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} \quad \text{(Ampere)}
\end{aligned} \tag{1.1.1}$$

Here, $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields, ρ and $\vec{J}$ are the charge and current densities and ϵ_0 and μ_0 the permittivity and permeability of free space and $\mu_0\epsilon_0 = \frac{1}{c^2}$.

These equations, together with the Lorentz law,

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{1.1.2}$$

contain all the physical information of classical electromagnetism. The continuity equation,

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \tag{1.1.3}$$

reflecting charge conservation, follows from these.

1.2 Potential formulation (see Griffiths section 10.1.1, Third Edition)

The second equation in (1.1.1) allows us to write the magnetic field as the curl of a vector field

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{1.2.1}$$

where $\vec{A}$ is called the vector potential. Upon doing this we automatically satisfy Maxwell's second equation. Substituting this into Faraday's law gives

$$\vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0, \tag{1.2.2}$$

which allows us to write for the quantity in brackets as

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} V, \tag{1.2.3}$$

or

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} V, \tag{1.2.4}$$

where V is called the scalar potential. Upon doing this we automatically satisfy Faraday's law. We have now solved the second and third of Maxwell's equations. Let us consider the other two in potential form. The first equation (Gauss' law) becomes

$$\vec{\nabla}^2 V + \frac{\partial}{\partial t}\left(\vec{\nabla} \cdot \vec{A}\right) = -\frac{1}{\epsilon_0}\rho, \tag{1.2.5}$$

while the last equation (Ampere's law) becomes, upon using the vector identity $\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) - \vec{\nabla}^2 \vec{A}$,

$$\left(\vec{\nabla}^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}\right) - \nabla \left(\nabla \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \vec{J}. \tag{1.2.6}$$

Equations (1.2.5) and (1.2.6) are completely equivalent to Maxwell's equations and encode all the physical information of classical electromagnetism.

1.3 Gauge transformations (see Griffiths section 10.1.2, Third Edition)

The equations (1.2.1) and (1.2.4) do not fix the gauge potentials uniquely for given magnetic and electric fields. This is easily seen by noting that we can add to the vector potential the gradient of any (twice differentiable) scalar functions and the curl will give the same magnetic field, i.e.,

$$\vec{\nabla} \times \left(\vec{A} + \vec{\nabla}\alpha\right) = \vec{\nabla} \times \vec{A} = \vec{B} \tag{1.3.1}$$

Similarly we can add to the scalar potential any function of time only without affecting the electric field

$$-\vec{\nabla}\left(V + f(t)\right) - \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} = \vec{E}. \tag{1.3.2}$$

Thus $\vec{A}' = \vec{A} + \vec{\nabla}\alpha$ and $V' = V + f(t)$ are equally acceptable potentials that give exactly the same physics, i.e. $\vec{E}$ and $\vec{B}$. In this regard it should be noted that the potentials are not physically observable, but only the electric and magnetic fields. It is then also clear that (1.2.5) and (1.2.6) cannot fix the potentials uniquely for a given charge and current density. If we want to determine the potentials uniquely, we need to impose further constraints on them. These constraints are, however, completely arbitrary and we can choose them to our convenience as they only affect the unobservable potentials and not the electric and magnetic fields. We refer to these further conditions as gauge conditions. There are two particularly convenient choices of gauge conditions:

- **Coulomb gauge:** Here the following condition is imposed on the vector potential

$$\vec{\nabla} \cdot \vec{A} = 0. \tag{1.3.3}$$

With this condition the equation for the scalar potential (1.2.5) becomes just Poisson's equation which can easily be solved

$$\vec{\nabla}^2 V = -\frac{1}{\epsilon_0}\rho. \tag{1.3.4}$$

However, the equation for the vector potential remains complicated

$$\left(\vec{\nabla}^2 \vec{A} - \mu_0\epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}\right) = -\mu_0 \vec{J} + \mu_0\epsilon_0 \vec{\nabla}\frac{\partial V}{\partial t}. \tag{1.3.5}$$

- **Lorentz gauge:** Here we choose the following condition

$$\vec{\nabla} \cdot \vec{A} = -\mu_0\epsilon_0 \frac{\partial V}{\partial t} \tag{1.3.6}$$

With this choice (1.2.5) becomes

$$\vec{\nabla}^2 V - \mu_0\epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{1}{\epsilon_0}\rho, \tag{1.3.7}$$

while (1.2.5) reads

$$\vec{\nabla}^2 \vec{A} - \mu_0\epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}. \tag{1.3.8}$$

We see that the same differential operator (d'Alembertian)

$$\Box = \vec{\nabla}^2 - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2} \tag{1.3.9}$$

appears in both equations so that we can write (1.3.7) and (1.3.8) as

$$\Box V = -\frac{1}{\epsilon_0}\rho, \tag{1.3.10}$$

$$\Box \vec{A} = -\mu_0 \vec{J}, \tag{1.3.11}$$

which treats the scalar and vector potential on an equal footing, which is important for the relativistic covariant formulation of electrodynamics.

1.4 Relativity and tensor notation (see Griffiths section 12.1.3, Third edition)

When one transforms from one inertial frame to another, the coordinates in the two frames are related in special relativity by the Lorentz transformations (given here for the case of parallel x-axes, which is actually sufficient since any other case can be obtained from this by a simple rotation)

$$\begin{aligned} x' &= \gamma\left(x - vt\right), \\ y' &= y, \\ z' &= z, \\ t' &= \gamma\left(t - \frac{v}{c^2}x\right). \end{aligned} \tag{1.4.1}$$

and inversely

$$\begin{aligned} x &= \gamma\left(x' + vt'\right), \\ y &= y', \\ z &= z', \\ t &= \gamma\left(t' + \frac{v}{c^2}x'\right). \end{aligned} \tag{1.4.2}$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{1.4.3}$$

These equations can be written more compactly by introducing the following notation for the coordinates

$$x^0 = ct,\ x^1 = x,\ x^2 = y,\ x^3 = z,\ \beta = \frac{v}{c}, \tag{1.4.4}$$

collectively denoted x^μ $(\mu = 0, 1, 2, 3)$. With this notation (1.4.1) and (1.4.2) can be written as

$$x'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu_\nu x^\nu \equiv \Lambda^\mu_\nu x^\nu, \tag{1.4.5}$$

$$x^\mu = \sum_{\nu=0}^{3} \left(\Lambda^{-1}\right)^\mu_\nu x^\nu \equiv \left(\Lambda^{-1}\right)^\mu_\nu x'^\nu, \tag{1.4.6}$$

$$\Lambda^\alpha_\beta \left(\Lambda^{-1}\right)^\beta_\gamma = \delta^\alpha_\gamma, \tag{1.4.7}$$

where we have introduced Einstein's summation convention over repeated indices and the Λ^μ_ν are the matrix elements of the matrix

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.4.8}$$

i.e., $\Lambda^0_0 = \gamma, \Lambda^0_1 = -\gamma\beta \ldots$. We say that 4 quantities a^μ that transform under a change of coordinate system as in (1.4.5) is a contravariant 4-vector, i.e.,

$$a'^\mu = \Lambda^\mu_\nu a^\nu. \tag{1.4.9}$$

The Lorentz transformation was designed such that

$$\Lambda^\alpha_\mu g^{\mu\nu} \Lambda^\beta_\nu = g^{\alpha\beta}, \tag{1.4.10}$$

where $g^{\mu\nu}$ (the Minkowski metric) is the matrix

$$g = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.4.11}$$

i.e. $g^{00} = -1$, $g^{11} = 1 \ldots$. We denote by $g_{\mu\nu}$ the inverse of $g^{\mu\nu}$. Clearly $g_{\mu\nu} = g^{\mu\nu}$ and

$$\Lambda^\mu_\alpha g_{\mu\nu} \Lambda^\nu_\beta = g_{\alpha\beta}. \tag{1.4.12}$$

Let us now consider the quantity

$$s^2 = x^\mu g_{\mu\nu} x^\nu = -\left(x^0\right)^2 + \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2. \tag{1.4.13}$$

If we calculate the same quantity in the transformed reference frame we have

$$s'^2 = x'^\mu g_{\mu\nu} x'^\nu = x^\alpha \Lambda^\mu_\alpha g_{\mu\nu} \Lambda^\nu_\beta x^\beta = x^\alpha g_{\alpha\beta} x^\beta = s^2, \tag{1.4.14}$$

where we have used the fact that α, β are dummy variables. We call this a Lorentz invariant. Let us now introduce the following 4 quantities

$$x_\mu = g_{\mu\nu} x^\nu. \tag{1.4.15}$$

From (1.4.12) we can then write

$$\Lambda^{\mu}_{\alpha} g_{\mu\gamma} = g_{\alpha\beta} \left(\Lambda^{-1}\right)^{\beta}_{\gamma} \tag{1.4.16}$$

by multiplying both sides with $\left(\Lambda^{-1}\right)^{\beta}_{\gamma}$ and using (1.4.7). Under a Lorentz transformation we then have

$$x'_{\mu} = g_{\mu\nu} x'^{\nu} = g_{\mu\nu} \Lambda^{\nu}_{\alpha} x^{\alpha} = g_{\nu\mu} \Lambda^{\nu}_{\alpha} x^{\alpha} = \left(\Lambda^{-1}\right)^{\gamma}_{\mu} g_{\alpha\gamma} x^{\alpha} = \left(\Lambda^{-1}\right)^{\gamma}_{\mu} x_{\gamma}. \tag{1.4.17}$$

These 4 quantities therefore transform according to the inverse of Λ and we say they constitute a covariant 4-vector. In terms of these we can write the Lorentz scalar as

$$s^2 = x^{\mu} x_{\mu}. \tag{1.4.18}$$

We can now also introduce higher rank tensors. A contravariant tensor of rank two is in general 16 quantities $G^{\mu\nu}$ that transform as follow

$$G'^{\mu\nu} = \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} G^{\alpha\beta}, \tag{1.4.19}$$

while a covariant one transforms as

$$G'_{\mu\nu} = \left(\Lambda^{-1}\right)^{\alpha}_{\mu} \left(\Lambda^{-1}\right)^{\beta}_{\nu} G_{\alpha\beta}. \tag{1.4.20}$$

Since the Lorentz transformations are linear we can write them in infinitesimal form as

$$\begin{aligned} dx' &= \gamma\left(dx - v dt\right), \\ dy' &= dy, \\ dz' &= dz, \\ dt' &= \gamma\left(dt - \frac{v}{c^2} dx\right). \end{aligned} \tag{1.4.21}$$

and inversely

$$\begin{aligned} dx &= \gamma\left(dx' + v dt'\right), \\ dy &= dy', \\ dz &= dz', \\ dt &= \gamma\left(dt' + \frac{v}{c^2} dx'\right). \end{aligned} \tag{1.4.22}$$

As we have seen the quantity $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 = -c^2dt'^2 + dx'^2 + dy'^2 + dz'^2$ is a Lorentz invariant. If the particle is in rest in the moving (primed) frame, we have $dx = vdt, dy = dz = 0, dx' = dy' = dz' = 0$. Substituting this in ds^2 we have

$$ds^2 = c^2\left(\frac{v^2}{c^2} - 1\right)dt^2 = -c^2dt'^2 \equiv -c^2d\tau^2. \tag{1.4.23}$$

Since ds^2 and c^2 are invariants, it follows that the time interval

$$d\tau = \sqrt{1 - \frac{v^2}{c^2}}dt, \tag{1.4.24}$$

which is simply the time measured in the rest frame, is an invariant, called the proper time.

A similar argument can be applied to the length interval. If the length of a rod in rest in the moving (primed) frame is measured in the laboratory (unprimed frame), we must read of the positions of the endpoints at the same instant in time in the laboratory frame, i.e., $dt = 0$. From (1.4.22) $dt = 0$ implies $dt' = -\frac{v}{c^2}dx'$. Substituting in ds^2 (with $dy = dz = dy' = dz' = 0$) we have

$$ds^2 = dx^2 = ds'^2 = \left(1 - \frac{v^2}{c^2}\right)dx'^2 \tag{1.4.25}$$

and thus that

$$dx = \sqrt{1 - \frac{v^2}{c^2}}dx'. \tag{1.4.26}$$

Note that dx' is the length of the rod in its rest frame (primed system), while dx is its length in the frame w.r.t. which it is moving. We refer to dx' as the proper length and it is a Lorentz invariant.

1.5 Maxwell's equations in tensor notation (see Griffiths section 12.3.4, Third edition)

We start by considering the first and fourth of Maxwell equation (1.1.1), written in slightly different form:

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0}\rho$$
$$\vec{\nabla} \times \vec{B} - \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t} = \mu_0\vec{J} \tag{1.5.1}$$

If we consider a small volume element dV containing charge Q and moving with velocity $\vec{u}$, the current and charge densities, in the laboratory frame (w.r.t. to which it is moving), are

$$\rho = \frac{Q}{dV}, \quad \vec{J} = \rho\vec{u} \tag{1.5.2}$$

The densities in the rest frame of the charge are

$$\rho_0 = \frac{Q}{dV_0}, \quad \vec{J} = 0 \tag{1.5.3}$$

where from (1.4.26) $dV = \frac{1}{\gamma}dV_0 = \sqrt{1-\frac{u^2}{c^2}}dV_0$ due to the length contraction along the x-axis. Note that dV_0, the volume in the rest frame, is an invariant as was shown in (1.4.26). The density in the laboratory frame can therefore be expressed in terms of the invariant density ρ_0 in the rest frame (proper density) as (it is invariant as dV_0 and the charge Q are invariants)

$$\rho = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\frac{Q}{dV_0} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\rho_0, \quad \vec{J} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\rho_0\vec{u}. \tag{1.5.4}$$

Keeping in mind that the proper time $d\tau = \sqrt{1-\frac{u^2}{c^2}}dt$ (time in the rest system) is an invariant we know that the 4-velocity

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{dt}{d\tau}\frac{dx^\mu}{dt} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}(c,\vec{u}) \tag{1.5.5}$$

transforms as a 4-vector. Comparing (1.5.4) and (1.5.5) we note that

$$J^\mu = (\rho c, \vec{J}) = \rho_0 u^\mu \tag{1.5.6}$$

is a 4-vector, called the current density 4-vector. This means that the quantities $\left(c\epsilon_0\vec{\nabla}\cdot\vec{E}, \frac{1}{\mu_0}\left(\vec{\nabla}\times\vec{B}\right) - \epsilon_0\frac{\partial\vec{E}}{\partial t}\right)$ on the left of (1.5.1) must also be a 4-vector. This can only be the case if the magnetic and electric fields are not vectors, but rather a rank two tensor, $F^{\mu\nu}$, in particular the electric field must correspond to the F^{0i} components so that $\vec{\nabla}\cdot\vec{E}_i = \frac{\partial\vec{E}_i}{\partial x^i} = \frac{\partial F^{0i}}{\partial x^i}$

(summation over repeated index) be the zeroth component of a 4-vector. To do this we introduce the anti-symmetric rank two tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & B_z & -B_y \\ -\frac{E_y}{c} & -B_z & 0 & B_x \\ -\frac{E_z}{c} & B_y & -B_x & 0 \end{pmatrix}, \tag{1.5.7}$$

i.e. $F^{01} = \frac{E_x}{c}$, $F^{02} = \frac{E_y}{c} \ldots$. In terms of this tensor (1.5.1) can be written as

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu \tag{1.5.8}$$

where a summation over the repeated index ν is implied (note that $\frac{\partial}{\partial x^\nu}$ is a covariant 4-vector, while x^ν is a contravariant 4-vector. Similarly, if we introduce the dual tensor

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -\frac{E_z}{c} & \frac{E_y}{c} \\ -B_y & \frac{E_z}{c} & 0 & -\frac{E_x}{c} \\ -B_z & -\frac{E_y}{c} & \frac{E_x}{c} & 0 \end{pmatrix}, \tag{1.5.9}$$

The remaining two homogenous Maxwell equations can be written as

$$\frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0. \tag{1.5.10}$$

Equations (1.5.8) and (1.5.10) are the Maxwell equations in tensor notation. Each of them stands for four equations.

1.6 Potential formulation in tensor notation (see Griffiths section 12.3.5, Third edition)

Let us introduce the four quantities

$$A^\mu = \left(\frac{V}{c}, \vec{A}\right) \tag{1.6.1}$$

where V is the scalar potential and $\vec{A}$ the vector potential. In terms of these we can write

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} \tag{1.6.2}$$

As $F^{\mu\nu}$ is a rank two tensor, A^{μ} must be a 4-vector, called the 4-vector potential. As before, upon introducing the potentials, the homogenous Maxwell equations (1.5.10) are automatically satisfied. The inhomogenous equations (1.5.8) become

$$\frac{\partial}{\partial x^{\nu}}\frac{\partial A^{\nu}}{\partial x_{\mu}} - \frac{\partial}{\partial x^{\nu}}\frac{\partial A^{\mu}}{\partial x_{\nu}} = \frac{\partial}{\partial x_{\mu}}\frac{\partial A^{\nu}}{\partial x^{\nu}} - \frac{\partial}{\partial x^{\nu}}\frac{\partial A^{\mu}}{\partial x_{\nu}} = \mu_0 J^{\mu} \tag{1.6.3}$$

This equation can be simplified if we realize that, as before, the 4-vector potential is not uniquely fixed by (1.5.8), but that we can make gauge transformations. Indeed, replacing A^{μ} by $A'^{\mu} = A^{\mu} + \frac{\partial \lambda}{\partial x_{\mu}}$ where λ is an arbitrary (twice differentiable) function, we see that

$$F'^{\mu\nu} = \frac{\partial A'^{\nu}}{\partial x_{\mu}} - \frac{\partial A'^{\mu}}{\partial x_{\nu}} = \frac{\partial A^{\nu}}{\partial x_{\mu}} - \frac{\partial A^{\mu}}{\partial x_{\nu}} = F^{\mu\nu} \tag{1.6.4}$$

We can therefore always add the gradient of a scalar to the 4-vector potential without changing $F^{\mu\nu}$ and thus the electric and magnetic fields or the physics. To fix the 4- vector potential uniquely we must therefore impose, as before, additional conditions, called gauge conditions. These are arbitrary, but whatever choice is made, it will not affect the physics. Here we take the Lorentz gauge

$$\partial_{\mu} A^{\mu} \equiv \frac{\partial A^{\mu}}{\partial x^{\mu}} = \vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} \tag{1.6.5}$$

With this choice (1.6.3) becomes

$$\Box A^{\mu} = -\mu_0 J^{\mu} \tag{1.6.6}$$

where the d'Alembert operator is

$$\Box \equiv \frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial x_{\mu}} = \vec{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{1.6.7}$$

1.7 Energy of an electromagnetic field (see Griffiths section 8.1, Third edition)

Let us consider the work done by the magnetic and electric fields in a time interval dt on a charge q

$$dW = \vec{F} \cdot d\vec{\ell} = \vec{F} \cdot \vec{v} dt = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \cdot \vec{v} dt = q\vec{E} \cdot \vec{v} dt. \tag{1.7.1}$$

We also have for a volume element d^3x

$$q = \rho d^3x, \quad \rho\vec{v} = \vec{J}, \tag{1.7.2}$$

so that

$$\frac{dW}{dt} = \vec{E} \cdot \vec{J} d^3x. \tag{1.7.3}$$

For a finite volume this gives

$$\frac{dW}{dt} = \int_V \vec{E} \cdot \vec{J} d^3x. \tag{1.7.4}$$

From Ampere's law

$$\vec{J} = \frac{1}{\mu_0}\left(\vec{\nabla} \times \vec{B}\right) - \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{1.7.5}$$

it follows that

$$\vec{E} \cdot \vec{J} = \frac{1}{\mu_0}\vec{E} \cdot \left(\vec{\nabla} \times \vec{B}\right) - \epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}. \tag{1.7.6}$$

But,

$$\vec{\nabla} \cdot \left(\vec{E} \times \vec{B}\right) = \vec{B} \cdot \left(\vec{\nabla} \times \vec{E}\right) - \vec{E} \cdot \left(\vec{\nabla} \times \vec{B}\right) \tag{1.7.7}$$

so that

$$\vec{E} \cdot \left(\vec{\nabla} \times \vec{B}\right) = \vec{B} \cdot \left(\vec{\nabla} \times \vec{E}\right) - \vec{\nabla} \cdot \left(\vec{E} \times \vec{B}\right) \tag{1.7.8}$$

$$= -\vec{B} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \cdot \left(\vec{E} \times \vec{B}\right), \tag{1.7.9}$$

where we have used Faraday's law $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$.

This gives

$$\vec{E} \cdot \vec{J} = \frac{1}{\mu_0}\left[-\vec{B} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \cdot \left(\vec{E} \times \vec{B}\right)\right] - \epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} \tag{1.7.10}$$

$$= -\frac{\partial}{\partial t}\frac{1}{2}\left[\frac{1}{\mu_0}\vec{B}^2 + \epsilon_0 \vec{E}^2\right] - \frac{1}{\mu_0}\vec{\nabla} \cdot \left(\vec{E} \times \vec{B}\right). \tag{1.7.11}$$

It follows that

$$\begin{aligned}
\frac{dW}{dt} &= -\frac{\partial}{\partial t}\int_V d^3x \frac{1}{2}\left[\frac{1}{\mu_0}\vec{B}^2 + \epsilon_0\vec{E}^2\right] - \frac{1}{\mu_0}\int_V d^3x \vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right) d^3x \\
&= -\frac{\partial}{\partial t}\int_V d^3x \frac{1}{2}\left[\frac{1}{\mu_0}\vec{B}^2 + \epsilon_0\vec{E}^2\right] - \frac{1}{\mu_0}\int_S \left(\vec{E}\times\vec{B}\right)\cdot d\vec{a} \\
.
\end{aligned} \tag{1.7.12}$$

Since $\frac{dW}{dt}$ is the rate at which work is done on the charge in the volume V, this implies that $\int_V d^3x \frac{1}{2}\left[\frac{1}{\mu_0}\vec{B}^2 + \epsilon_0\vec{E}^2\right]$ must be the energy stored in V and $\frac{1}{\mu_0}\int_S \left(\vec{E}\times\vec{B}\right)\cdot d\vec{a}$ the rate at which energy flows across the surface bounding V.

We conclude that

$$U = \frac{1}{2}\left[\frac{1}{\mu_0}\vec{B}^2 + \epsilon_0\vec{E}^2\right] \tag{1.7.13}$$

is the energy density. Introducing the Poynting vector

$$\vec{S} = \frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right), \tag{1.7.14}$$

we conclude that $\vec{S}\cdot d\vec{a}$ is the energy flux across the bounding surface, i.e. it is the energy/unit time/unit area.

2 Electromagnetic radiation

In this chapter we investigate electromagnetic radiation from a variety of charge and current distributions. This will prepare the way for our discussion of scattering in the next chapter.

2.1 Retarded potentials (see Griffiths section 10.2.1, Third edition)

In eq. (1.6.6) we have found the equation that governs the vector and scalar potentials to be

$$\Box A^\mu = -\mu_0 J^\mu \tag{2.1.1}$$

with

$$\begin{aligned}
\Box &\equiv \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} = \vec{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}, \\
J^\mu &= \left(\rho c, \vec{J}\right), \\
A^\mu &= \left(\frac{V}{c}, \vec{A}\right). \qquad (2.1.2)
\end{aligned}$$

Thus (2.1.1) explicitly reads (note $\mu_0\epsilon_0 = \frac{1}{c^2}$)

$$\begin{aligned}
&\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} - \vec{\nabla}^2 V = \frac{1}{\epsilon_0}\rho \\
&\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}^2 \vec{A} = \mu_0 \vec{J}. \qquad (2.1.3)
\end{aligned}$$

We now proceed to show that these equations are solved by the so called retarded potentials

$$\begin{aligned}
V\left(\vec{r}, t\right) &= \frac{1}{4\pi\epsilon_0}\int_V d^3\vec{r}^{\prime}\,\frac{\rho\left(\vec{r}^{\prime}, t_r\right)}{\left|\vec{r}-\vec{r}^{\prime}\right|}, \\
\vec{A}\left(\vec{r}, t\right) &= \frac{\mu_0}{4\pi}\int_V d^3\vec{r}^{\prime}\,\frac{\vec{J}\left(\vec{r}^{\prime}, t_r\right)}{\left|\vec{r}-\vec{r}^{\prime}\right|}, \qquad (2.1.4)
\end{aligned}$$

where

$$t_r = t - \frac{\left|\vec{r}-\vec{r}^{\prime}\right|}{c}. \qquad (2.1.5)$$

We now show that (2.1.4) solves (2.1.3), but we do so only for the scalar potential as the calculation for the vector potential is exactly the same. We first compute $\vec{\nabla}^2 V$:

$$\begin{aligned}
\vec{\nabla} V &= \frac{1}{4\pi\epsilon_0}\int_V d^3\vec{r}^{\prime}\,\vec{\nabla}\left(\frac{\rho\left(\vec{r}^{\prime}, t_r\right)}{\left|\vec{r}-\vec{r}^{\prime}\right|}\right) \\
&= \frac{1}{4\pi\epsilon_0}\int_V d^3\vec{r}^{\prime}\left[\frac{1}{\left|\vec{r}-\vec{r}^{\prime}\right|}\frac{\partial\rho\left(\vec{r}^{\prime}, t_r\right)}{\partial t}\vec{\nabla} t_r - \left(\frac{\vec{r}-\vec{r}^{\prime}}{\left|\vec{r}-\vec{r}^{\prime}\right|^3}\right)\rho\left(\vec{r}^{\prime}, t_r\right)\right] \\
&= \frac{1}{4\pi\epsilon_0}\int_V d^3\vec{r}^{\prime}\left[-\frac{\dot{\rho}}{c}\left(\frac{\vec{r}-\vec{r}^{\prime}}{\left|\vec{r}-\vec{r}^{\prime}\right|^2}\right) - \left(\frac{\vec{r}-\vec{r}^{\prime}}{\left|\vec{r}-\vec{r}^{\prime}\right|^3}\right)\rho\right]. \qquad (2.1.6)
\end{aligned}$$

Here we have introduced $\rho\left(\vec{r}',t_r\right)\equiv\rho$, $\frac{\partial\rho}{\partial t}=\dot{\rho}$ and we have used

$$\begin{aligned}\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}'|} &= -\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3},\\ \vec{\nabla}t_r &= -\frac{1}{c}\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}.\end{aligned} \tag{2.1.7}$$

We now have

$$\begin{aligned}&\vec{\nabla}^2V=\vec{\nabla}\cdot\vec{\nabla}V=\\ &\frac{1}{4\pi\epsilon_0}\int_V d^3\vec{r}'\left[-\frac{\vec{\nabla}\dot{\rho}}{c}\cdot\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^2}\right)-\frac{\dot{\rho}}{c}\vec{\nabla}\cdot\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^2}\right)\right.\\ &\left.-\vec{\nabla}\cdot\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right)\rho-\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right)\cdot\vec{\nabla}\rho\right]\end{aligned} \tag{2.1.8}$$

Here we have used

$$\vec{\nabla}\cdot\left(f\vec{A}\right)=\vec{\nabla}f\cdot\vec{A}+f\vec{\nabla}\cdot\vec{A} \tag{2.1.9}$$

We now have to compute a few quantities:

$$\vec{\nabla}\cdot\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^2}\right)=\frac{1}{|\vec{r}-\vec{r}'|^2} \tag{2.1.10}$$

and

$$\vec{\nabla}\dot{\rho}=\ddot{\rho}\vec{\nabla}t_r=-\frac{\ddot{\rho}}{c}\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}, \tag{2.1.11}$$

where we used (2.1.7) (Recall that $\rho=\rho\left(\vec{r}',t_r\right)$ so that $\vec{\nabla}$ only acts on t_r).

Finally

$$\vec{\nabla}\cdot\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right)=0,\quad\text{if}\quad\vec{r}\neq\vec{r}' \tag{2.1.12}$$

Naively one would conclude from (2.1.12) that $\vec{\nabla}\cdot\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right)=0$, but this is incorrect as the expression is ambiguous when $\vec{r}=\vec{r}'$. To see what object we have here, let us apply Gauss' law to a sphere with arbitrary radius ϵ and centered at $\vec{r}$. This gives

$$\int_{S'^3}d^3\vec{r}'\vec{\nabla}\cdot\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right)=\int_{S^3}d^3\vec{r}\,\vec{\nabla}\cdot\frac{\vec{r}}{|\vec{r}|^3}=\int_{\partial S}\frac{\vec{r}}{|\vec{r}|^3}\cdot d\vec{a}=4\pi. \tag{2.1.13}$$

In the first step we shifted the origin and in the second step we applied Gauss' law. Thus the quantity $\vec{\nabla}\cdot\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right)$ vanishes everywhere, except possibly when $\vec{r}=\vec{r}'$ and then it behaves so badly that its integral is finite! There is a function, called the Dirac delta function with exactly this property

$$\begin{aligned}
\delta\left(\vec{r}-\vec{r}'\right) &= 0, \quad \text{if} \quad \vec{r}\neq\vec{r}', \\
\int_\Omega d^3\vec{r}'\,\delta\left(\vec{r}-\vec{r}'\right) &= 1, \quad \text{if} \quad \vec{r}\in\Omega, \\
\int_\Omega d^3\vec{r}'\,\delta\left(\vec{r}-\vec{r}'\right)f(\vec{r}') &= f(\vec{r}), \quad \text{if} \quad \vec{r}\in\Omega,
\end{aligned} \tag{2.1.14}$$

Thus, comparing (2.1.13) and (2.1.14), we conclude

$$\vec{\nabla}\cdot\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right) = 4\pi\delta\left(\vec{r}-\vec{r}'\right). \tag{2.1.15}$$

We can now return to (2.1.8). Using (2.1.10), (2.1.11) and (2.1.15) in (2.1.8), we conclude

$$\begin{aligned}
\vec{\nabla}^2 V &= \frac{1}{4\pi\epsilon_0}\int_V d^3\vec{r}'\left[\frac{1}{c^2}\frac{\ddot{\rho}}{|\vec{r}-\vec{r}'|} - 4\pi\delta\left(\vec{r}-\vec{r}'\right)\rho\left(\vec{r}',t_r\right)\right] \\
&= \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} - \frac{1}{\epsilon_0}\rho\left(\vec{r},t\right),
\end{aligned} \tag{2.1.16}$$

which agrees with (2.1.3).

Equations (2.1.4) give a formal solution of the Maxwell equations for an arbitrary configuration of electric charge and current densities. Obviously the result (2.1.4) is still very difficult to compute, even numerically, and it can only be done for very specific charge and current configurations or in some approximation. We discuss such an approximation in the next section and a specific charge and current configuration for which it can be computed in the section following it.

2.2 Radiation from an arbitrary source (see Griffiths section 11.1.4, Third edition)

The general solution of the Maxwell equations i.t.o. potentials is given by (2.1.4). Let us focus on the scalar potential for the moment

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int_V d^3\vec{r}' \frac{\rho(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|}, \tag{2.2.1}$$

with

$$t_r = t - \frac{|\vec{r} - \vec{r}'|}{c} \tag{2.2.2}$$

$$|\vec{r} - \vec{r}'| = \sqrt{(\vec{r} - \vec{r}') \cdot (\vec{r} - \vec{r}')} = \sqrt{r^2 + r'^2 - 2\vec{r} \cdot \vec{r}'}. \tag{2.2.3}$$

The physical setup we consider is one in which the charge and current densities are non-zero in some finite region V over which we are integrating in (2.2.1). We take the observation point $\vec{r}$ where the potentials are calculated to lie far away from this region. We can thus assume $r >> r'$ for all $\vec{r}'$ in (2.2.1):

$$|\vec{r} - \vec{r}'| = r\sqrt{1 + \frac{r'^2}{r^2} - \frac{2\vec{r} \cdot \vec{r}'}{r^2}} \approx r\left(1 - \frac{\hat{r} \cdot \vec{r}'}{r}\right), \tag{2.2.4}$$

and

$$\frac{1}{|\vec{r} - \vec{r}'|} \approx \frac{1}{r}\left(1 + \frac{\hat{r} \cdot \vec{r}'}{r}\right). \tag{2.2.5}$$

where $\hat{r} \cdot \hat{r} = 1$.

We then have

$$\rho\left(\vec{r}', t - \frac{|\vec{r} - \vec{r}'|}{c}\right) \approx \rho\left(\vec{r}', t - \frac{r}{c} + \frac{\hat{r} \cdot \vec{r}'}{c}\right) \approx \rho(\vec{r}', t_0) + \dot{\rho}(\vec{r}', t_0)\frac{\hat{r} \cdot \vec{r}'}{c}, \tag{2.2.6}$$

where $t_0 = t - \frac{r}{c}$.

This gives for the scalar potential

$$\begin{aligned} V(\vec{r}, t) &= \frac{1}{4\pi\epsilon_0} \int_V d^3\vec{r}' \left[\rho(\vec{r}', t_0) + \dot{\rho}(\vec{r}', t_0)\frac{\hat{r} \cdot \vec{r}'}{c}\right]\left[\frac{1}{r} + \frac{\hat{r} \cdot \vec{r}'}{r^2}\right] \\ &\approx \frac{1}{4\pi\epsilon_0 r}\left[\int_V d^3\vec{r}' \rho(\vec{r}', t_0) + \frac{\hat{r}}{c} \cdot \int_V d^3\vec{r}' \dot{\rho}(\vec{r}', t_0)\,\vec{r}'\right]. \end{aligned} \tag{2.2.7}$$

Defining the electric dipole moment

$$\vec{p}(t) = \int_V d^3\vec{r}'\rho(\vec{r}', t)\,\vec{r}', \tag{2.2.8}$$

we have

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\hat{r}\cdot\dot{\vec{p}}(t_0)}{cr}\right]. \tag{2.2.9}$$

Similarly for the vector potential

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi}\int_V d^3\vec{r}'\frac{\vec{J}(\vec{r}', t_r)}{|\vec{r}-\vec{r}'|} \approx \frac{\mu_0}{4\pi}\int_V d^3\vec{r}'\frac{\vec{J}(\vec{r}', t_0)}{r}. \tag{2.2.10}$$

Next we calculate $\int_V d^3\vec{r}'\vec{J}(\vec{r}', t)$. To do this consider

$$\int_V d^3\vec{r}'\vec{\nabla}'\cdot\left(x_k'\vec{J}(\vec{r}', t)\right) = \int_{\partial V} d^3\vec{r}' x_k'\vec{J}(\vec{r}', t)\cdot d\vec{a}. \tag{2.2.11}$$

If we assume that the currents vanish at infinity (on the boundary) this integral vanishes. We also have

$$\begin{aligned}\vec{\nabla}'\cdot\left(x_k'\vec{J}(\vec{r}', t)\right) &= \left(\vec{\nabla}'x_k'\right)\cdot\vec{J}(\vec{r}', t) + x_k'\vec{\nabla}'\cdot\vec{J}(\vec{r}', t)\\ &= \vec{J}_k(\vec{r}', t) - x_k'\frac{\partial\rho(\vec{r}', t)}{\partial t}\end{aligned} \tag{2.2.12}$$

where we used the continuity equation (1.1.3). We thus conclude

$$\int_V d^3\vec{r}'\left[\vec{J}_k(\vec{r}', t) - x_k'\frac{\partial\rho(\vec{r}', t)}{\partial t}\right] = 0, \tag{2.2.13}$$

from which it follows that

$$\int_V d^3\vec{r}'\vec{J}_k(\vec{r}', t) = \frac{\partial}{\partial t}\int_V d^3\vec{r}' x_k'\rho(\vec{r}', t) = \frac{\partial\vec{p}_k(t)}{\partial t}, \tag{2.2.14}$$

or in vector notation

$$\int_V d^3\vec{r}'\vec{J}(\vec{r}', t) = \frac{\partial p\,\vec{(t)}}{\partial t}. \tag{2.2.15}$$

Finally, we conclude

$$\vec{A}(\vec{r}, t) \approx \frac{\mu_0}{4\pi r}\frac{\partial\vec{p}(t_0)}{\partial t}. \tag{2.2.16}$$

Now that we have the potential, we can compute the electric and magnetic fields at an observation point far away from the finite region V in which the charge and current densities are non-zero. We have for the electric field

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}, \tag{2.2.17}$$

but we only keep terms to order $\frac{1}{r}$. If we apply $\vec{\nabla}$ to $\frac{1}{r}$ it gives terms of order $\frac{1}{r^2}$ and we must disregard these. This means that the term $\frac{Q}{r}$ has to be disregarded when we compute $\vec{\nabla} V$. The only term that we therefore have to keep in computing $\vec{\nabla} V$ is

$$\vec{\nabla} \left(\frac{\hat{r} \cdot \dot{\vec{p}}\,(t_0)}{rc} \right) \approx \frac{\hat{r} \cdot \ddot{\vec{p}}(t_0)}{rc} \vec{\nabla} t_0. \tag{2.2.18}$$

We also have

$$\vec{\nabla} t_0 = \vec{\nabla} \left(t - \frac{r}{c} \right) = -\frac{\hat{r}}{c} \tag{2.2.19}$$

and we conclude

$$\vec{\nabla} V \approx -\frac{1}{4\pi\epsilon_0 c^2} \frac{\hat{r} \cdot \ddot{\vec{p}}(t_0)}{r} \hat{r}. \tag{2.2.20}$$

Also

$$\frac{\partial \vec{A}}{\partial t} \approx \frac{\mu_0}{4\pi r} \ddot{\vec{p}}(t_0) \tag{2.2.21}$$

so that the electric field is given by

$$\vec{E} = \frac{\mu_0}{4\pi r} \left[\left(\hat{r} \cdot \ddot{\vec{p}}(t_0) \right) \hat{r} - \ddot{\vec{p}}(t_0) \right]. \tag{2.2.22}$$

Here we have used $\mu_0 \epsilon_0 = \frac{1}{c^2}$. Using the identity $\vec{A} \times \left(\vec{B} \times \vec{C} \right) = \vec{B} \left(\vec{A} \cdot \vec{C} \right) - \vec{C} \left(\vec{A} \cdot \vec{B} \right)$ we can write this as

$$\vec{E} = \frac{\mu_0}{4\pi r} \left[\hat{r} \times \left(\hat{r} \times \ddot{\vec{p}}(t_0) \right) \right]. \tag{2.2.23}$$

Similarly we have for the magnetic field

$$\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \frac{\mu_0}{4\pi r} \dot{\vec{p}} \approx \frac{\mu_0}{4\pi r} \left(\vec{\nabla} \times \dot{\vec{p}}(t_0) \right) = \frac{\mu_0}{4\pi r} \left(\vec{\nabla} t_0 \times \ddot{\vec{p}}(t_0) \right), \tag{2.2.24}$$

which finally gives

$$\vec{B} = -\frac{\mu_0}{4\pi cr}\left(\hat{r} \times \ddot{\vec{p}}(t_0)\right). \tag{2.2.25}$$

From this it follows that

$$\vec{E} = -c\left(\hat{r} \times \vec{B}\right). \tag{2.2.26}$$

Note that

$$\begin{aligned} \vec{B} \cdot \hat{r} &= 0, \\ \vec{E} \cdot \hat{r} &= 0, \\ \vec{E} \cdot \vec{B} &= 0. \end{aligned} \tag{2.2.27}$$

Finally, using (2.2.27), we can compute the Poynting vector as

$$\vec{S} = \frac{1}{\mu_0}\left(\vec{E} \times \vec{B}\right) = \frac{1}{\mu_0}|\vec{E}||\vec{B}|\hat{r}. \tag{2.2.28}$$

We also have

$$\begin{aligned} |\vec{B}| &= \frac{\mu_0}{4\pi cr}|\hat{r} \times \ddot{\vec{p}}(t_0)\,| = \frac{\mu_0}{4\pi cr}|\ddot{\vec{p}}(t_0)\,|\sin\theta, \\ |\vec{E}| &= \frac{\mu_0}{4\pi r}|\hat{r} \times \ddot{\vec{p}}(t_0)\,| = \frac{\mu_0}{4\pi r}|\ddot{\vec{p}}(t_0)\,|\sin\theta, \end{aligned} \tag{2.2.29}$$

where θ is the angle between $\ddot{\vec{p}}$ and $\hat{r}$. This gives for the Poynting vector

$$\vec{S} = \frac{\mu_0|\ddot{\vec{p}}(t_0)\,|^2}{16\pi^2 c}\frac{\sin^2\theta}{r^2}\hat{r} \tag{2.2.30}$$

The total power radiated is easily computed as

$$P = \int \vec{S} \cdot d\vec{a} = \frac{\mu_0|\ddot{\vec{p}}(t_0)\,|^2}{6\pi c}, \tag{2.2.31}$$

where we have used $d\vec{a} = r^2 \sin\theta d\theta d\phi\, \hat{r}$.

2.3 Radiation from a point particle (see Griffiths section 10.3,Third edition)

In this section we derive the potentials and fields caused by a moving point particle. In spirit the discussion presented here follows that of Griffiths, but in detail it is much different and more concise. We start by deriving the potentials associated with a moving point particle, the so called Lienard-Wiechert potentials.

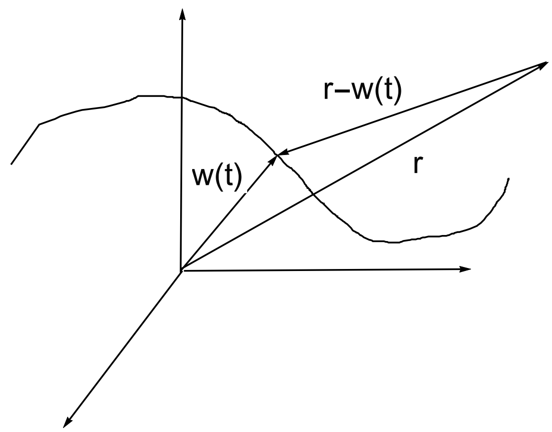

Figure 1: Radiation from a charged point particle

2.3.1 Lienard-Wiechert potentials (see Griffiths section 10.3.1,Third edition)

Consider a particle with charge q that moves along a curve $\vec{w}(t)$ as a function of time (see figure 1).

The charge density is given by

$$\rho\left(\vec{r}, t\right) = q\delta\left(\vec{r} - \vec{w}(t)\right) \tag{2.3.1}$$

and the current density by

$$\vec{J}\left(\vec{r}, t\right) = q\vec{u}(t)\delta\left(\vec{r} - \vec{w}(t)\right) \tag{2.3.2}$$

where $\vec{u} = \dot{\vec{w}}$ is the velocity of the particle. Then we have

$$\begin{aligned} V\left(\vec{r}, t\right) &= \frac{1}{4\pi\epsilon_0}\int_V d^3\vec{r}'\frac{\rho\left(\vec{r}', t_r\right)}{\left|\vec{r} - \vec{r}'\right|} \\ &= \frac{q}{4\pi\epsilon_0}\int_V d^3\vec{r}'\frac{1}{\left|\vec{r} - \vec{r}'\right|}\delta\left(\vec{r}' - \vec{w}\left(t - \frac{\left|\vec{r} - \vec{r}'\right|}{c}\right)\right) \end{aligned} \tag{2.3.3}$$

To do this integral, let us change variables from $\vec{r}'$ to

$$\vec{R} = \vec{r}' - \vec{w}\left(t - \frac{|\vec{r} - \vec{r}'|}{c}\right), \tag{2.3.4}$$

which reads in component form

$$\begin{aligned} X &= x' - w_x\left(t - \frac{|\vec{r} - \vec{r}'|}{c}\right), \\ Y &= y' - w_y\left(t - \frac{|\vec{r} - \vec{r}'|}{c}\right), \\ Z &= z' - w_z\left(t - \frac{|\vec{r} - \vec{r}'|}{c}\right). \end{aligned} \tag{2.3.5}$$

Then

$$dX\,dY\,dZ = |J| dx' dy' dz', \tag{2.3.6}$$

where the Jacobian is given by

$$J = \det\begin{pmatrix} \frac{\partial X}{\partial x'} & \frac{\partial X}{\partial y'} & \frac{\partial X}{\partial z'} \\ \frac{\partial Y}{\partial x'} & \frac{\partial Y}{\partial y'} & \frac{\partial Y}{\partial z'} \\ \frac{\partial Z}{\partial x'} & \frac{\partial Z}{\partial y'} & \frac{\partial Z}{\partial z'} \end{pmatrix}, \tag{2.3.7}$$

These derivatives can easily be calculated. As an example consider

$$\frac{\partial X}{\partial x'} = 1 - \frac{\partial w_x(t_r)}{\partial x'} = 1 - \frac{\partial w_x(t_r)}{\partial t_r}\frac{\partial t_r}{\partial x'} = 1 - u_x(t_r)\frac{x - x'}{c|\vec{r} - \vec{r}'|}. \tag{2.3.8}$$

Calculating all these derivatives, the Jacobian explicitly reads

$$\begin{aligned} J &= \det\begin{pmatrix} 1 - u_x(t_r)\frac{x-x'}{c|\vec{r}-\vec{r}'|} & -u_x(t_r)\frac{y-y'}{c|\vec{r}-\vec{r}'|} & -u_x(t_r)\frac{z-z'}{c|\vec{r}-\vec{r}'|} \\ -u_y(t_r)\frac{x-x'}{c|\vec{r}-\vec{r}'|} & 1 - u_y(t_r)\frac{y-y'}{c|\vec{r}-\vec{r}'|} & -u_y(t_r)\frac{z-z'}{c|\vec{r}-\vec{r}'|} \\ -u_z(t_r)\frac{x-x'}{c|\vec{r}-\vec{r}'|} & -u_z(t_r)\frac{y-y'}{c|\vec{r}-\vec{r}'|} & 1 - u_z(t_r)\frac{z-z'}{c|\vec{r}-\vec{r}'|} \end{pmatrix} \\ &= 1 - \frac{\vec{u}(t_r)\cdot(\vec{r} - \vec{r}')}{c|\vec{r} - \vec{r}'|} \end{aligned} \tag{2.3.9}$$

Now we have

$$\begin{aligned} V(\vec{r}, t) &= \frac{q}{4\pi\epsilon_0}\int_V d^3\vec{r}'\frac{1}{|\vec{r} - \vec{r}'|}\delta\left(\vec{r}' - \vec{w}\left(t - \frac{|\vec{r} - \vec{r}'|}{c}\right)\right) \\ &= \frac{q}{4\pi\epsilon_0}\int_V d^3\vec{R}\frac{|J|^{-1}}{|\vec{r} - \vec{r}'|}\delta\left(\vec{R}\right) \\ &= \frac{q}{4\pi\epsilon_0}\left[\left(1 - \frac{\vec{u}(t_r)\cdot(\vec{r} - \vec{r}')}{c|\vec{r} - \vec{r}'|}\right)^{-1}\frac{1}{|\vec{r} - \vec{r}'|}\right]_{\vec{R}=0}. \end{aligned} \tag{2.3.10}$$

From

$$\vec{R} = \vec{r}' - \vec{w}\left(t - \frac{|\vec{r} - \vec{r}'|}{c}\right) = 0 \tag{2.3.11}$$

we have an equation for $\vec{r}'$ and we denote the solution of this equation by $\vec{r}' = \vec{\xi}\,(\vec{r}, t)$. Thus we can write (2.3.10) also as

$$V\,(\vec{r}, t) = \frac{q}{4\pi\epsilon_0}\left[\left(1 - \frac{\vec{u}\,(t_r)\cdot\left(\vec{r} - \vec{\xi}\,(\vec{r}, t)\right)}{c|\vec{r} - \vec{\xi}\,(\vec{r}, t)\,|}\right)^{-1}\frac{1}{|\vec{r} - \vec{\xi}\,(\vec{r}, t)\,|}\right] \tag{2.3.12}$$

The same calculation for the vector potential A yields

$$\begin{aligned}
\vec{A}\,(\vec{r}, t) &= \frac{\mu_0 q \vec{u}\,(t_r)}{4\pi}\left[\left(1 - \frac{\vec{u}\,(t_r)\cdot\left(\vec{r} - \vec{\xi}\,(\vec{r}, t)\right)}{c|\vec{r} - \vec{\xi}\,(\vec{r}, t)\,|}\right)^{-1}\frac{1}{|\vec{r} - \vec{\xi}\,(\vec{r}, t)\,|}\right] \\
&= \mu_0\epsilon_0\vec{u}\,(t_r)\,V\,(\vec{r}, t) = \frac{\vec{u}\,(t_r)}{c^2}V\,(\vec{r}, t)\,,
\end{aligned} \tag{2.3.13}$$

where $t_r = t - \frac{|\vec{r} - \vec{\xi}(\vec{r}, t)|}{c}$.

Eqs. (2.3.12) and (2.3.13) are called the Lienard-Wiechert potentials.

2.3.2 Radiation of a moving point charge (see Griffiths section 10.3.2 and 11.2.1, Third edition)

Since we have the potentials of a moving point charge, we can now proceed to calculate the electric and magnetic fields, which are the physical observable quantities. However, as already discussed in the preceding sections, when we are interested in radiation, we are only interested in the asymptotic behaviour of the electric and magnetic fields, i.e., the behaviour at large distances. This is what we are going to do in this section; we are going to calculate the electric and magnetic fields caused by a moving point charge, but only to leading order in $\frac{1}{|\vec{r} - \vec{r}'|}$, i.e., the large distance asymptotic behaviour.

Our starting point is the usual expressions for the field in terms of the potentials

$$\begin{aligned}
\vec{E} &= -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}, \\
\vec{B} &= \vec{\nabla}\times\vec{A}.
\end{aligned} \tag{2.3.14}$$

Let us also define

$$\vec{s} = \vec{r} - \vec{\xi}\,(\vec{r}, t)\,, \quad \hat{s} = \frac{\vec{s}}{|\vec{s}|}. \tag{2.3.15}$$

To calculate the fields in (2.3.14) we need the following derivatives $\frac{\partial t_r}{\partial x_i}$, $\frac{\partial \vec{\xi}}{\partial x_i}$, $\frac{\partial t_r}{\partial t}$ and $\frac{\partial \vec{\xi}}{\partial t}$. Recall the equation (2.3.11) that determines $\vec{\xi}\,(\vec{r}, t)$.

$$\vec{\xi} = \vec{w}\left(t - \frac{|\vec{r} - \vec{\xi}|}{c}\right) = \vec{w}\,(t_r) \tag{2.3.16}$$

It then follows that

$$\frac{\partial \vec{\xi}}{\partial x_i} = \frac{\partial \vec{w}\,(t_r)}{\partial x_i} = \frac{\partial \vec{w}\,(t_r)}{\partial t_r}\frac{\partial t_r}{\partial x_i} = \vec{u}\,(t_r)\,\frac{\partial t_r}{\partial x_i} \tag{2.3.17}$$

We also have

$$\begin{aligned}
\frac{\partial t_r}{\partial x_i} &= \frac{\partial}{\partial x_i}\left(t - \frac{|\vec{r} - \vec{\xi}|}{c}\right) = -\frac{1}{c}\frac{\partial}{\partial x_i}\left[\sum_{j=1}^{3}(x_j - \xi_j)^2\right]^{\frac{1}{2}} \\
&= -\frac{1}{c|\vec{r} - \vec{\xi}|}\left[\sum_{j=1}^{3}(x_j - \xi_j)\left(\delta_{ij} - \frac{\partial \xi_j}{\partial x_i}\right)\right] \\
&= -\frac{s_i}{c|\vec{s}|} + \frac{1}{c|\vec{s}|}\vec{s}\cdot\frac{\partial \vec{\xi}}{\partial x_i} \\
&= -\frac{\hat{s}_i}{c} + \frac{1}{c}\hat{s}\cdot\vec{u}\,(t_r)\,\frac{\partial t_r}{\partial x_i}
\end{aligned} \tag{2.3.18}$$

where we have used (2.3.17). We conclude

$$\frac{\partial t_r}{\partial x_i} = -\frac{1}{c}\left[1 - \frac{\vec{u}\,(t_r)\cdot\hat{s}}{c}\right]^{-1}\hat{s}_i. \tag{2.3.19}$$

Or, written in vector notation

$$\vec{\nabla} t_r = -\frac{1}{c}\left[1 - \frac{\vec{u}\,(t_r)\cdot\hat{s}}{c}\right]^{-1}\hat{s}. \tag{2.3.20}$$

We then immediately also have

$$\frac{\partial \vec{\xi}}{\partial x_i} = -\frac{\vec{u}\,(t_r)}{c}\left[1 - \frac{\vec{u}\,(t_r)\cdot\hat{s}}{c}\right]^{-1}\hat{s}_i. \tag{2.3.21}$$

In a similar fashion we have

$$\frac{\partial \vec{\xi}}{\partial t} = \frac{\partial \vec{w}\,(t_r)}{\partial t} = \frac{\partial \vec{w}\,(t_r)}{\partial t_r}\frac{\partial t_r}{\partial t} = \vec{u}\,(t_r)\,\frac{\partial t_r}{\partial t}. \tag{2.3.22}$$

and

$$\begin{aligned}
\frac{\partial t_r}{\partial t} &= \frac{\partial}{\partial t}\left(t - \frac{|\vec{r}-\vec{\xi}|}{c}\right) = 1 - \frac{1}{c}\frac{\partial}{\partial t}\left[\sum_{j=1}^{3}(x_j-\xi_j)^2\right]^{\frac{1}{2}} \\
&= 1 - \frac{1}{c|\vec{r}-\vec{\xi}|}\left[\sum_{j=1}^{3}(x_j-\xi_j)\left(-\frac{\partial \xi_j}{\partial t}\right)\right] \\
&= 1 + \frac{1}{c|\vec{s}|}\vec{s}\cdot\frac{\partial \vec{\xi}}{\partial t} \\
&= 1 + \frac{1}{c}\hat{s}\cdot\vec{u}\,(t_r)\,\frac{\partial t_r}{\partial t}
\end{aligned} \tag{2.3.23}$$

where we used (2.3.22). We conclude

$$\frac{\partial t_r}{\partial t} = \left[1 - \frac{\vec{u}\,(t_r)\cdot\hat{s}}{c}\right]^{-1} \tag{2.3.24}$$

and

$$\frac{\partial \vec{\xi}}{\partial t} = \vec{u}\,(t_r)\left[1 - \frac{\vec{u}\,(t_r)\cdot\hat{s}}{c}\right]^{-1}. \tag{2.3.25}$$

Now we return to the calculation of the fields. From (2.3.12) we have

$$V\,(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\left(1 - \frac{\vec{u}\,(t_r)\cdot\hat{s}}{c}\right)^{-1}\frac{1}{|\vec{s}|} \tag{2.3.26}$$

We first calculate $\vec{\nabla}V$. We have

$$\frac{\partial V}{\partial x_i} = \frac{q}{4\pi\epsilon_0}\left(1 - \frac{\vec{u}\,(t_r)\cdot\hat{s}}{c}\right)^{-2}\frac{1}{|\vec{s}|}\left(\frac{\partial}{\partial x_i}\frac{\vec{u}\,(t_r)\cdot\hat{s}}{c}\right)+\frac{q}{4\pi\epsilon_0}\left(1 - \frac{\vec{u}\,(t_r)\cdot\hat{s}}{c}\right)^{-1}\frac{\partial}{\partial x_i}\frac{1}{|\vec{s}|} \tag{2.3.27}$$

and

$$\begin{aligned}
\frac{\partial}{\partial x_i}\frac{1}{|\vec{s}|} &= \frac{\partial}{\partial x_i}\left[\sum_{j=1}^{3}(x_j-\xi_j)^2\right]^{-\frac{1}{2}} = -\frac{1}{|\vec{s}|^3}\left[(x_i-\xi_i) - \sum_{j=1}^{3}(x_j-\xi_j)\,\frac{\partial \xi_j}{\partial x_i}\right] \\
&= -\frac{s_i}{|\vec{s}|^3} + \frac{\vec{s}}{|\vec{s}|^3}\cdot\frac{\partial \vec{\xi}}{\partial x_i}
\end{aligned} \tag{2.3.28}$$

as well as

$$\frac{\partial}{\partial x_i}\left(\vec{u}\left(t_r\right)\cdot\hat{s}\right)=\left(\vec{a}\left(t_r\right)\cdot\hat{s}\right)\frac{\partial t_r}{\partial x_i}+\vec{u}\left(t_r\right)\cdot\frac{\partial\hat{s}}{\partial x_i} \tag{2.3.29}$$

$$\begin{aligned}\frac{\partial\hat{s}_j}{\partial x_i} &= \frac{\partial}{\partial x_i}\left(\frac{s_j}{|\vec{s}|}\right)=\frac{1}{|\vec{s}|}\frac{\partial s_j}{\partial x_i}+s_j\frac{\partial}{\partial x_i}\left(\frac{1}{|\vec{s}|}\right)\\ &= \frac{1}{|\vec{s}|}\left(\delta_{ij}-\frac{\partial\xi_j}{\partial x_i}\right)+s_j\left(-\frac{s_i}{|\vec{s}|^3}+\frac{\vec{s}}{|\vec{s}|^3}\cdot\frac{\partial\vec{\xi}}{\partial x_i}\right).\end{aligned} \tag{2.3.30}$$

We therefore have

$$\vec{u}\left(t_r\right)\cdot\frac{\partial\hat{s}}{\partial x_i}=\frac{u_i}{|\vec{s}|}-\frac{1}{|\vec{s}|}\vec{u}\cdot\frac{\partial\vec{\xi}}{\partial x_i}+\vec{u}\cdot\vec{s}\left(-\frac{s_i}{|\vec{s}|^3}+\frac{\vec{s}}{|\vec{s}|^3}\cdot\frac{\partial\vec{\xi}}{\partial x_i}\right). \tag{2.3.31}$$

In the asymptotic limit the observation point r is taken to be on a sphere with large radius, $|\vec{r}|\to\infty$ and thus also $|\vec{s}|\to\infty$. We should therefore only keep the leading order terms in $\frac{1}{|\vec{s}|}$. Firstly, note from (2.3.20), (2.3.21), (2.3.24) and (2.3.25) that all the derivatives are of order 1. From this, (2.3.28) and (2.3.30), we easily establish that to leading order in $\frac{1}{|\vec{s}|}$

$$\begin{aligned}\frac{\partial V}{\partial x_i} &= \frac{q}{4\pi\epsilon_0 c}\left(1-\frac{\vec{u}\left(t_r\right)\cdot\hat{s}}{c}\right)^{-2}\frac{\vec{a}\cdot\hat{s}}{|\vec{s}|}\frac{\partial t_r}{\partial x_i}\\ &= -\frac{q}{4\pi\epsilon_0 c^2}\left(1-\frac{\vec{u}\left(t_r\right)\cdot\hat{s}}{c}\right)^{-3}\frac{\vec{a}\cdot\hat{s}}{|\vec{s}|}\hat{s}_i\end{aligned} \tag{2.3.32}$$

from (2.3.20). In vector form this reads

$$\vec{\nabla}V=-\frac{q}{4\pi\epsilon_0 c^2}\left(1-\frac{\vec{u}\left(t_r\right)\cdot\hat{s}}{c}\right)^{-3}\frac{\vec{a}\cdot\hat{s}}{|\vec{s}|}\hat{s} \tag{2.3.33}$$

Next we need $\frac{\partial\vec{A}}{\partial t}$. Since $\vec{A}\left(\vec{r},t\right)=\frac{\vec{u}(t_r)}{c^2}V\left(\vec{r},t\right)$ from (2.3.13), we have

$$\frac{\partial\vec{A}}{\partial t}=\frac{\vec{a}}{c^2}\frac{\partial t_r}{\partial t}V+\frac{\vec{u}\left(t_r\right)}{c^2}\frac{\partial V}{\partial t}, \tag{2.3.34}$$

while from (2.3.26)

$$\frac{\partial V}{\partial t}=\frac{q}{4\pi\epsilon_0 c}\left(1-\frac{\vec{u}\left(t_r\right)\cdot\hat{s}}{c}\right)^{-2}\frac{1}{|\vec{s}|}\left(\frac{\partial}{\partial t}\vec{u}\left(t_r\right)\cdot\hat{s}\right)+\frac{q}{4\pi\epsilon_0 c}\left(1-\frac{\vec{u}\left(t_r\right)\cdot\hat{s}}{c}\right)^{-1}\frac{\partial}{\partial t}\frac{1}{|\vec{s}|}. \tag{2.3.35}$$

Calculating the different terms we have

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{1}{|\vec{s}|} &= \frac{\partial}{\partial t}\frac{1}{|\vec{r}-\vec{\xi}|} = \frac{1}{|\vec{s}|^3}\vec{s}\cdot\frac{\partial\vec{\xi}}{\partial t} \approx \frac{1}{|\vec{s}|^2} \\
\frac{1}{|\vec{s}|}\frac{\partial}{\partial t}\left(\vec{u}\left(t_r\right)\cdot\hat{s}\right) &= \frac{1}{|\vec{s}|}\left[\left(\vec{a}\left(t_r\right)\cdot\hat{s}\right)\frac{\partial t_r}{\partial t}+\vec{u}\left(t_r\right)\cdot\frac{\partial\hat{s}}{\partial t}\right] \approx \frac{\vec{a}\left(t_r\right)\cdot\hat{s}}{|\vec{s}|}\frac{\partial t_r}{\partial t} \\
\frac{\partial\hat{s}_j}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{s_j}{|\vec{s}|}\right) = \frac{1}{|\vec{s}|}\frac{\partial s_j}{\partial t}+s_j\frac{\partial}{\partial t}\frac{1}{|\vec{s}|} = -\frac{1}{|\vec{s}|}\frac{\partial\xi_j}{\partial t}+s_j\frac{\partial}{\partial t}\frac{1}{|\vec{s}|} \approx \frac{1}{|\vec{s}|}
\end{aligned} \tag{2.3.36}$$

To leading order we therefore have from (2.3.35) and (2.3.36)

$$\begin{aligned}
\frac{\partial\vec{A}}{\partial t} &= \frac{\vec{a}}{c^2}V\frac{\partial t_r}{\partial t}+\frac{\vec{u}\left(t_r\right)}{c^2}\left[\frac{q}{4\pi\epsilon_0 c}\left(1-\frac{\vec{u}\left(t_r\right)\cdot\hat{s}}{c}\right)^{-2}\frac{\vec{a}\cdot\hat{s}}{|\vec{s}|}\frac{\partial t_r}{\partial t}\right] \\
&= \frac{q}{4\pi\epsilon_0 c^2}\left(1-\frac{\vec{u}\left(t_r\right)\cdot\hat{s}}{c}\right)^{-2}\frac{1}{|\vec{s}|}\left[\vec{a}+\vec{a}\cdot\vec{s}\left(1-\frac{\vec{u}\left(t_r\right)\cdot\hat{s}}{c}\right)^{-1}\frac{\vec{u}\left(t_r\right)}{c}\right]
\end{aligned} \tag{2.3.37}$$

The electric field is now easily obtained to leading order as

$$\begin{aligned}
\vec{E} &= -\vec{\nabla}V-\frac{\partial\vec{A}}{\partial t} \\
&= \frac{q}{4\pi\epsilon_0 c^2}\left[\left(1-\frac{\vec{u}\cdot\hat{s}}{c}\right)^{-3}\frac{\vec{a}\cdot\hat{s}}{|\vec{s}|}\hat{s}-\left(1-\frac{\vec{u}\cdot\hat{s}}{c}\right)^{-2}\frac{\vec{a}}{|\vec{s}|}-\left(1-\frac{\vec{u}\cdot\hat{s}}{c}\right)^{-3}\frac{\vec{a}\cdot\vec{s}\,\vec{u}}{c|\hat{s}|}\right] \\
&= \frac{q}{4\pi\epsilon_0 c^2}\left[\frac{\hat{s}\times\left[\left(\hat{s}-\frac{\vec{u}}{c}\right)\times\vec{a}\right]}{|\vec{s}|\left(1-\frac{\vec{u}\cdot\hat{s}}{c}\right)^3}\right]
\end{aligned} \tag{2.3.38}$$

Finally we calculate the asymptotic magnetic field

$$\begin{aligned}
\vec{B} &= \vec{\nabla}\times\vec{A} = \vec{\nabla}\times\left(\frac{\vec{u}\left(t_r\right)}{c^2}V\left(\vec{r},t\right)\right) \\
&= \frac{V\left(\vec{r},t\right)}{c^2}\left(\vec{\nabla}\times\vec{u}\left(t_r\right)\right)-\frac{\vec{u}\left(t_r\right)}{c^2}\times\vec{\nabla}V\left(\vec{r},t\right).
\end{aligned} \tag{2.3.39}$$

The second term is known from (2.3.33). Since $\vec{u}$ depends on t_r only, the first term can be calculated as follows

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z} = \hat{x}\frac{\partial t_r}{\partial x}\frac{\partial}{\partial t_r}+\hat{y}\frac{\partial t_r}{\partial y}\frac{\partial}{\partial t_r}+\hat{z}\frac{\partial t_r}{\partial z}\frac{\partial}{\partial t_r} = \vec{\nabla}t_r\frac{\partial}{\partial t_r} \tag{2.3.40}$$

which, when used in (2.3.39), gives

$$\begin{aligned}
\vec{B} &= \frac{V(\vec{r},t)}{c^2}\left(\vec{\nabla}t_r\frac{\partial}{\partial t_r}\times\vec{u}(t_r)\right)-\frac{\vec{u}(t_r)}{c^2}\times\vec{\nabla}V(\vec{r},t)\\
&= \frac{V(\vec{r},t)}{c^2}\left(\vec{\nabla}t_r\times\vec{a}(t_r)\right)-\frac{\vec{u}(t_r)}{c^2}\times\vec{\nabla}V(\vec{r},t)\\
&= -\frac{q}{4\pi\epsilon_0 c^3}\left[\frac{(\hat{s}\times\vec{a}(t_r))\left(1-\frac{\vec{u}(t_r)\cdot\hat{s}}{c}\right)+\frac{1}{c}(\hat{s}\times\vec{u}(t_r))\,\vec{a}(t_r)\cdot\hat{s}}{|\vec{s}|\left(1-\frac{\vec{u}(t_r)\cdot\hat{s}}{c}\right)^3}\right]\\
&= \frac{1}{c}\left(\hat{s}\times\vec{E}\right).
\end{aligned} \tag{2.3.41}$$

Here we have used (2.3.20), (2.3.33) and (2.3.38).

We immediately have

$$\begin{aligned}
\vec{B}\cdot\hat{s} &= 0,\\
\vec{E}\cdot\hat{s} &= 0,\\
\vec{E}\cdot\vec{B} &= 0.
\end{aligned} \tag{2.3.42}$$

2.3.3 Power radiated by a moving point charge (see Griffiths section 10.3.2 and 11.2.1, Third edition)

The compute the power radiated by a particle far away from it, we only need to compute the asymptotic Poynting vector. Since we know the asymptotic electric and magnetic fields from (2.3.38) and (2.3.41) this is easily computed.

$$\vec{S}=\frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right)=\frac{1}{\mu_0}\left(\vec{E}\times\left(\hat{s}\times\vec{E}\right)\right)=\frac{|\vec{E}|^2}{\mu_0 c}\hat{s}. \tag{2.3.43}$$

At non-relativistic speeds where $|\vec{u}| << c$ the expression for the electric field in (2.3.38) simplifies considerably to

$$\vec{E}=\frac{q}{4\pi\epsilon_0 c^2|\vec{s}|}(\hat{s}\times(\hat{s}\times\vec{a}))=\frac{q}{4\pi\epsilon_0 c^2|\vec{s}|}((\hat{s}\cdot\vec{a})\,\hat{s}-\vec{a}), \tag{2.3.44}$$

from which it follows that

$$|\vec{E}|^2=\left(\frac{q}{4\pi\epsilon_0 c^2}\right)^2\frac{1}{|\vec{s}|^2}\left[|\vec{a}|^2-(\hat{s}\cdot\vec{a})^2\right]. \tag{2.3.45}$$

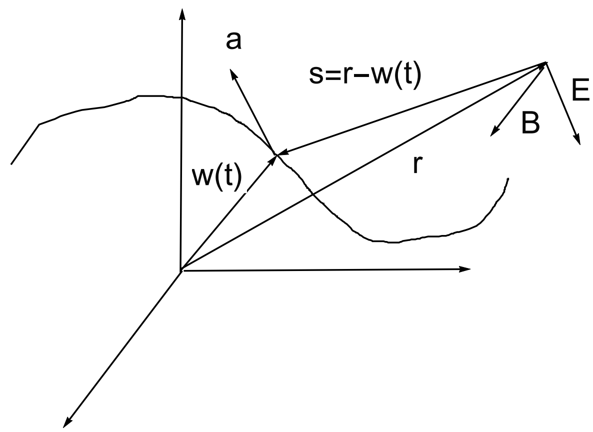

Figure 2: Fields of an accelerating charged point particle

The radiated energy flux emitted by the particle is therefore

$$\begin{aligned}\vec{S} &= \frac{1}{\mu_0 c}\left(\frac{q}{4\pi\epsilon_0 c^2}\right)^2 \frac{1}{|\vec{s}|^2}\left[|\vec{a}|^2 - (\hat{s}\cdot\vec{a})^2\right]\hat{s} \\ &= \frac{1}{4\pi\epsilon_0}\frac{q^2}{4\pi c^3}\frac{|\vec{a}|^2\sin^2\theta}{|\vec{s}|^2}\hat{s} \qquad (2.3.46)\end{aligned}$$

Here θ is the angle between $\hat{s}$ and $\vec{a}$. Note that when the particle is not accelerating, the radiated power is zero. Furthermore, since $|\vec{r}| \to \infty$, the unit vector $\hat{s}$ will point approximately radially outwards. Thus the radiated power is emitted in a radial fashion far away from the accelerating particle.

The total power emitted by the particle is obtained by integrating the energy flux over the area of a sphere at infinity. Thus

$$\begin{aligned}P &= \int_{S^3}\vec{S}\cdot d\vec{a} = \int_0^\pi d\theta\sin\theta\int_0^{2\pi} d\phi r^2\left[\frac{1}{4\pi\epsilon_0}\frac{q^2}{4\pi c^3}\frac{|\vec{a}|^2\sin^2\theta}{r^2}\right] \\ &= \frac{1}{6\pi\epsilon_0}\frac{q^2|\vec{a}|^2}{c^3} = \frac{\mu_0}{6\pi c}q^2|\vec{a}|^2 \qquad (2.3.47)\end{aligned}$$

Here we have used that $\hat{s} \approx \hat{r}$ at large radial distances $|\vec{r}|$, $|\vec{s}| \approx |\vec{r}| = r$ and $d\vec{a} = da\hat{r} = r^2 \sin\theta d\theta d\phi \hat{r}$.

This is known as Lamor's formula and tells us at what rate a non-relativistic accelerating particle will loose energy.

3 Scattering (see Ohanian section 16.1)

In this chapter we apply the results of sections 2.3.3 to investigate scattering of electromagnetic radiation from a free charged particle and a charged particle bound in a potential.

We start by first considering the physical mechanism behind scattering. If electromagnetic radiation falls onto a charged particle, the electric field associated with the radiation will give rise to a resulting force and thus acceleration of the particle. However, as we just learned, an accelerating particle will in turn radiate, thus effectively scattering the incoming radiation. We shall now proceed to compute the differential and total scattering cross sections of such a particle.

3.1 Thomson scattering

From eq. (2.3.46) we have seen that the energy flux radiated by an accelerating particle is given by

$$\vec{S} = \frac{1}{4\pi\epsilon_0} \frac{q^2}{4\pi c^3} \frac{|\vec{a}|^2 \sin^2\alpha}{|\vec{s}|^2} \hat{s} \tag{3.1.1}$$

Where $|\vec{a}|$ is the magnitude of the acceleration and α the angle between $\vec{a}$ and $\vec{s} = \vec{r} - \vec{r}'$, the observation point. If we consider a small solid angle $d\Omega$ in the direction $\vec{s}$ the area spanned by the solid angle is $d\vec{a} = |\vec{s}|^2 d\Omega \hat{s}$ (see figure 3). The total amount of energy passing through this area/unit time, and thus the power, is

$$\vec{S} \cdot d\vec{a} \equiv dP = |\vec{S}||\vec{s}|^2 d\Omega = \frac{1}{4\pi\epsilon_0} \frac{q^2}{4\pi c^3} |\vec{a}|^2 \sin^2\alpha \; d\Omega. \tag{3.1.2}$$

where $\vec{S}$ is the Poynting vector calculated in (2.3.46). We conclude

$$\frac{dP}{d\Omega} = \frac{1}{4\pi\epsilon_0} \frac{q^2}{4\pi c^3} |\vec{a}|^2 \sin^2\alpha. \tag{3.1.3}$$

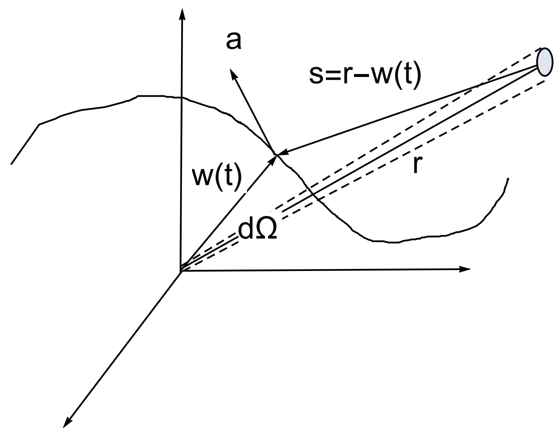

Figure 3: Scattering from a charged point particle

Suppose the incoming radiation is described by a plane wave with direction of propagation $\vec{k}$ and frequency ω. Then

$$\begin{aligned}\vec{E} &= \vec{E}_0 \cos\left(\vec{k}\cdot\vec{x}-\omega t\right), \\ \vec{B} &= \vec{B}_0 \cos\left(\vec{k}\cdot\vec{x}-\omega t\right), \\ \vec{B}_0 &= \frac{1}{c}\left(\hat{k}\times\vec{E}_0\right), \quad \vec{E}_0\cdot\hat{k}=0. \end{aligned} \tag{3.1.4}$$

This incoming radiation will cause an acceleration of the charged particle. In the non-relativistic limit this is simply given by Newton's second law. Furthermore, as the magnitude of the magnetic field in the radiation is $\frac{E_0}{c}$, we can neglect the Lorentz force $\vec{F} = q\vec{v}\times\vec{B}$ in the non-relativistic limit. Thus, assuming no other forces, the acceleration caused by the incoming radiation is simply

$$\vec{a} = \frac{q\vec{E}}{m} = \frac{q\vec{E}_0}{m}\cos\left(\vec{k}\cdot\vec{x}-\omega t\right), \tag{3.1.5}$$

where m is the mass of the particle. Substituting this in (3.1.3) gives

$$\frac{dP}{d\Omega} = \frac{|\vec{E}_0|^2}{4\pi\epsilon_0}\frac{q^4\sin^2\alpha}{4\pi c^3 m^2}\cos^2\left(\vec{k}\cdot\vec{x}-\omega t\right). \tag{3.1.6}$$

Averaging this over the period $\tau = \frac{2\pi}{\omega}$ of the incoming radiation, gives the time averaged energy flux/unit time/solid angle

$$\begin{aligned} \left\langle\frac{dP}{d\Omega}\right\rangle &= \frac{1}{\tau}\int_0^\tau dt\frac{dP}{d\Omega} = \frac{\omega}{2\pi}\frac{|\vec{E}_0|^2}{4\pi\epsilon_0}\frac{q^4\sin^2\alpha}{4\pi c^3 m^2}\int_0^{\frac{2\pi}{\omega}}\cos^2\left(\vec{k}\cdot\vec{x}-\omega t\right) \\ &= \frac{|\vec{E}_0|^2}{8\pi\epsilon_0}\frac{q^4\sin^2\alpha}{4\pi c^3 m^2}. \end{aligned} \tag{3.1.7}$$

Next we define the differential scattering cross section

$$\frac{d\sigma}{d\Omega} = \frac{\text{Energy radiated/unit time/unit solid angle}}{\text{Incident energy/unit time/unit area}} \tag{3.1.8}$$

The numerator is just $\left\langle\frac{dP}{d\Omega}\right\rangle$. To compute the denominator, we note that the Poynting vector for the plane wave is given by

$$\vec{S}_{\text{inc}} = \frac{|\vec{E}_0|^2}{\mu_0 c}\cos^2\left(\vec{k}\cdot\vec{x}-\omega t\right)\hat{k}. \tag{3.1.9}$$

The time averaged is then

$$\begin{aligned} \left\langle\vec{S}_{\text{inc}}\right\rangle &= \frac{1}{\tau}\int_0^\tau dt\vec{S}_{\text{inc}} = \frac{\omega}{2\pi}\frac{|\vec{E}_0|^2}{\mu_0 c}\int_0^{\frac{2\pi}{\omega}}\cos^2\left(\vec{k}\cdot\vec{x}-\omega t\right)\hat{k} \\ &= \frac{|\vec{E}_0|^2}{2\mu_0 c}\hat{k}. \end{aligned} \tag{3.1.10}$$

The incident energy flux/unit time/unit area is therefore simply $|\left\langle\vec{S}_{\text{inc}}\right\rangle| = \frac{|\vec{E}_0|^2}{2\mu_0 c}$. Thus

$$\frac{d\sigma}{d\Omega} = \frac{1}{(4\pi\epsilon_0)^2}\left(\frac{q^2}{mc^2}\right)^2\sin^2\alpha. \tag{3.1.11}$$

Note that this has the units of an area.

This is still an inconvenient form as is it expressed in terms of the angle between the acceleration and point of observation. We would like to express

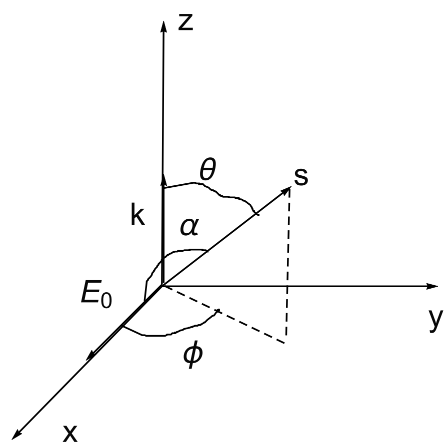

Figure 4: Differential cross section for Thomson scattering

this result in terms of normal spherical angles. We can always choose, without loss of generality, $\hat{z}||\hat{k}$. Let us take the polarization to be in the x-z plane, i.e. $\vec{E}_0||\hat{x}$. As the acceleration is parallel to $\vec{E}_0$, it is in the $\hat{x}$ direction so that α is the angle between $\hat{x}$ and $\vec{s}$ (see figure 4). Now

$$\hat{s}\cdot\hat{x} = \cos\alpha = \sin\theta\cos\phi \qquad (3.1.12)$$

so that

$$\sin^2\alpha = 1 - \sin^2\theta\cos^2\phi. \qquad (3.1.13)$$

This gives for x-polarized incident radiation

$$\frac{d\sigma}{d\Omega} = \frac{1}{(4\pi\epsilon_0)^2}\left(\frac{q^2}{mc^2}\right)^2\left(1 - \sin^2\theta\cos^2\phi\right), \qquad (3.1.14)$$

where, θ, ϕ now specifies the angles of the observation point relative to the direction of the incident radiation. Similarly for y-polarized radiation the result is

$$\frac{d\sigma}{d\Omega} = \frac{1}{(4\pi\epsilon_0)^2}\left(\frac{q^2}{mc^2}\right)^2\left(1 - \sin^2\theta\sin^2\phi\right). \qquad (3.1.15)$$

If the incoming radiation is unpolarized, we have an equal mixture of x and y polarized waves. Then the two scattering cross sections simply add to yield

$$\frac{d\sigma}{d\Omega} = \frac{1}{(4\pi\epsilon_0)^2}\left(\frac{q^2}{mc^2}\right)^2\left(1+\cos^2\theta\right). \tag{3.1.16}$$

This is the Thomson differential scattering cross section. The total scattering cross section is

$$\begin{aligned}\sigma &= \int d\Omega\frac{d\sigma}{d\Omega} = \frac{1}{(4\pi\epsilon_0)^2}\left(\frac{q^2}{mc^2}\right)^2\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\phi\left(1+\cos^2\theta\right)\\ &= \frac{1}{3\pi\epsilon_0^2}\left(\frac{q^2}{mc^2}\right)^2.\end{aligned} \tag{3.1.17}$$

All these results are valid for a free particle since we did not take any possible restoring forces or dissipative forces due to the loss of energy arising from the radiation into account when we calculated the acceleration in (3.1.5). Consider now a particle bound in the minimum of some potential. Assuming small displacements we can make a harmonic approximation. For an isotropic potential we then have

$$V(\vec{x}) = V(\vec{x}_0) + \frac{m\omega_0^2}{2}|\vec{y}|^2, \quad \vec{y} = \vec{x}-\vec{x}_0. \tag{3.1.18}$$

If radiation falls onto the particle, it will be accelerated by the electric field $\vec{E} = \vec{E}_0\cos\left(\vec{k}\cdot\vec{x}-\omega t\right)$, but there will also be a restoring force $\vec{F}_{\rm res} = -\vec{\nabla}V = -m\omega_0^2\vec{y}$. Furthermore we know that the particle looses energy (see the Lamor formula) due to radiation, which should be taken into account through a damping term $\vec{F}_{\rm damp} = -m\Gamma\dot{\vec{y}}$. Thus the more realistic expression for the acceleration is

$$m\ddot{\vec{y}} = q\vec{E} + \vec{F}_{\rm res} + \vec{F}_{\rm damp}, \tag{3.1.19}$$

or

$$m\left(\ddot{\vec{y}} + \Gamma\dot{\vec{y}} + \omega_0^2\vec{y}\right) = q\vec{E}_0\cos\left(\vec{k}\cdot\vec{x}-\omega t\right). \tag{3.1.20}$$

If the displacement $|\vec{y}|$ is small compared to the wave-length λ , we see from $\frac{\omega}{c} = \frac{2\pi}{\lambda}$ that we can neglect the y-dependency on the right of this equation. This is called the dipole approximation. Then we have

$$m\left(\ddot{\vec{y}} + \Gamma\dot{\vec{y}} + \omega_0^2\vec{y}\right) = q\vec{E}_0\cos\omega t. \tag{3.1.21}$$

To solve this equation, take

$$\vec{y} = \vec{A}\cos(\omega t) + \vec{B}\sin(\omega t) \tag{3.1.22}$$

Substituting this in (3.1.21), the coefficients of $\cos(\omega t)$ and $\sin(\omega t)$ have to vanish as they are linear independent functions, which yields

$$\begin{aligned} m\left(\omega_0^2 - \omega^2\right)\vec{A} + m\Gamma\omega\vec{B} &= q\vec{E}_0 \\ m\left(\omega_0^2 - \omega^2\right)\vec{B} - m\Gamma\omega\vec{A} &= 0 \end{aligned} \tag{3.1.23}$$

Solving for $\vec{A}$ and $\vec{B}$ from this, we obtain the steady state solution

$$\vec{y}(t) = \frac{q\left[\left(\omega_0^2 - \omega^2\right)\cos\omega t + \Gamma\omega\sin\omega t\right]}{m\left[\left(\omega_0^2 - \omega^2\right)^2 + \Gamma^2\omega^2\right]}\vec{E}_0. \tag{3.1.24}$$

The acceleration is therefore

$$\vec{a} = \ddot{\vec{y}}(t) = -\frac{\omega^2 q\left[\left(\omega_0^2 - \omega^2\right)\cos\omega t + \Gamma\omega\sin\omega t\right]}{m\left[\left(\omega_0^2 - \omega^2\right)^2 + \Gamma^2\omega^2\right]}\vec{E}_0. \tag{3.1.25}$$

Now

$$\frac{dP}{d\Omega} = \frac{|\vec{E}_0|^2}{4\pi\epsilon_0}\frac{q^2\sin^2\alpha}{4\pi c^3}\frac{\omega^4 q^2\left[\left(\omega_0^2 - \omega^2\right)\cos\omega t + \Gamma\omega\sin\omega t\right]^2}{m^2\left[\left(\omega_0^2 - \omega^2\right)^2 + \Gamma^2\omega^2\right]^2}. \tag{3.1.26}$$

Taking the time average we have

$$\begin{aligned} \left\langle\left(a\cos\omega t + b\sin\omega t\right)^2\right\rangle &= \frac{1}{\tau}\int_0^\tau \left(a\cos\omega t + b\sin\omega t\right)^2 = \frac{1}{2}\left(a^2 + b^2\right) \\ &= \frac{1}{2}\left[\left(\omega_0^2 - \omega^2\right)^2 + \Gamma^2\omega^2\right]. \end{aligned} \tag{3.1.27}$$

Thus

$$\frac{dP}{d\Omega} = \frac{|\vec{E}_0|^2 q^4\sin^2\alpha}{\left(8\pi\epsilon_0\right)\left(4\pi c^3 m^2\right)}\frac{\omega^4}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \Gamma^2\omega^2\right]}. \tag{3.1.28}$$

Comparing with (3.1.7) we note that the only difference is the factor $\frac{\omega^4}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \Gamma^2\omega^2\right]}$. Now we can write down the differential scattering cross section immediately

$$\frac{d\sigma}{d\Omega} = \frac{1}{\left(4\pi\epsilon_0\right)^2}\left(\frac{q^2}{mc^2}\right)^2\frac{\omega^4}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \Gamma^2\omega^2\right]}\sin^2\alpha. \tag{3.1.29}$$

For unpolarized incident radiation this reads in polar angles

$$\frac{d\sigma}{d\Omega} = \frac{1}{(4\pi\epsilon_0)^2}\left(\frac{q^2}{mc^2}\right)^2 \frac{\omega^4}{\left[(\omega_0^2-\omega^2)^2+\Gamma^2\omega^2\right]}\left(1+\cos^2\theta\right). \tag{3.1.30}$$

The total cross section is then also

$$\sigma = \frac{1}{3\pi\epsilon_0^2}\left(\frac{q^2}{mc^2}\right)^2 \frac{\omega^4}{\left[(\omega_0^2-\omega^2)^2+\Gamma^2\omega^2\right]}. \tag{3.1.31}$$

Note that when ω approaches ω_0 , the cross section exhibits a sharp peak, called a resonance. The width of the peak is determined by Γ If $\omega << \omega_0$ (long wave-length approximation)

$$\frac{d\sigma}{d\Omega} = \frac{1}{(4\pi\epsilon_0)^2}\left(\frac{q^2}{mc^2}\right)^2 \left(\frac{\omega}{\omega_0}\right)^4\left(1+\cos^2\theta\right). \tag{3.1.32}$$

The total cross section is then also

$$\sigma = \frac{1}{3\pi\epsilon_0^2}\left(\frac{q^2}{mc^2}\right)^2 \left(\frac{\omega}{\omega_0}\right)^4 \tag{3.1.33}$$

The scattering cross section is thus proportional to ω^4 , or inversely proportional to λ^4 . This is Rayleigh's law.

3.2 Explanation of the blue sky

Rayleigh's law explains the fact that the sky appears blue to us. Consider sunlight falling onto the earth as indicated in the figure 5.

The light we observe is scattered from electrons bound in molecules, but as we saw above the scattering cross section for light of shorter wave-length is much larger than that for long wave length. Thus short wave-length light scattered more and the scattered light is therefore blue shifted, which is what we observe. When the sun is on the horizon at sunset or sunrise, we observe more in the forward direction, i.e., unscattered light. Since the shorter wave-lengths are scattered more effectively, the unscattered light is red shifted, which is what we see at sunrise and sunset.

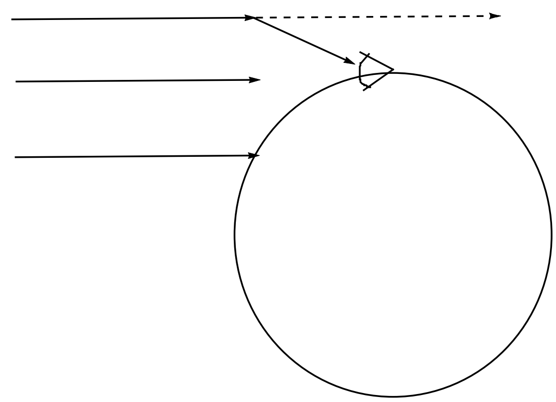

Figure 5: Explanation of blue sky

4 Magneto hydrodynamics and plasmas (see Jackson chapter 10)

Magneto hydrodynamics and plasma physics deal with the behaviour of combined systems of electromagnetic fields and conducting liquids or gasses. The fact that the medium (gas or liquid) is conducting manifests itself in Ohm's law, which relates the current density and electric field as follows:

$$\vec{J} = \sigma \vec{E}, \quad \text{or} \quad \vec{E} = \rho \vec{J} \tag{4.0.1}$$

where σ is the conductivity and ρ the resistivity of the medium.

The difference between magneto hydrodynamics and plasma physics relates to two different regimes. In magneto hydrodynamics the gas, consisting of electrons and ions, moves in such a way that there is no charge separation, i.e., the electron and ions behave as one fluid that can be described with normal hydrodynamical parameters such as density, velocity and pressure. In the plasma regime, however, the electrons and ions tend to separate and act as independent fluids. In this regime a two component fluid description, which treats the electrons and ions as two independent fluids, is more accurate. Magneto hydrodynamics and plasma physics is an essential component of astro-physics, thermo-nuclear reactions and any other processes in which a plasma may be formed.

4.1 Magneto hydrodynamic equations (see Jackson section 10.2)

We consider an electrically neutral conducting fluid moving in an electromagnetic field. The fluid is described by a matter density $\rho(\vec{x},t)$, velocity field $\vec{v}(\vec{x},t)$, pressure $p(\vec{x},t)$ and conductivity σ. Since the fluid is neutral the charge density $\rho_c(\vec{x},t) = 0$.

The basic magneto hydrodynamic equations are

- The continuity equation

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0. \tag{4.1.1}$$

  This equation simply states that matter is conserved, i.e., the rate at which the matter content of a finite volume changes, is compensated by the rate at which matter flows over the bounding surface of the volume.

- The force equation

$$\rho\frac{d\vec{v}}{dt} = -\vec{\nabla}p + \left(\vec{J}\times\vec{B}\right) + \vec{F}_v + \rho\vec{g}. \tag{4.1.2}$$

It is important to note that the total time derivative appears on the left. We now proceed to explain each term appearing on the right.

Consider a small volume element ΔV moving with velocity $\vec{v}(\vec{x},t)$ at position $\vec{x}$ at time t (see figure 6).

The first force on the element arises from the pressure gradient. Consider what happens in the x direction: On the left edge of the cube the pressure is $p(x,y,z,t)$ (without loss of generality we can take the point $\vec{x}(t)$ to be on the left edge). At the right edge the pressure is $p(x+\Delta x,y,z,t)$. The net force in the x-direction is therefore

$$\vec{F}_x = \left[p(x,y,z,t) - p(x+\Delta x,y,z,t)\right]\Delta y\Delta z \tag{4.1.3}$$

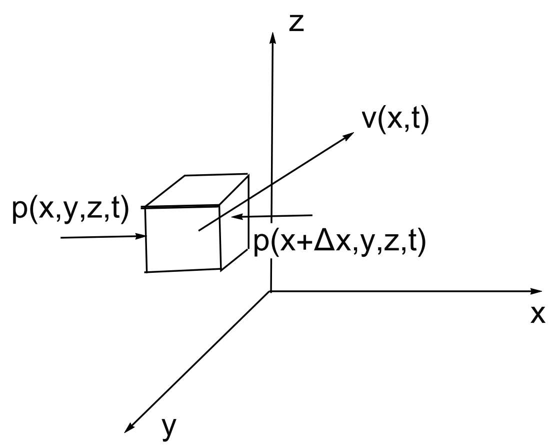

Figure 6: Forces on a small volume element

The acceleration is therefore

$$\Delta m \frac{d\vec{v}_x}{dt} = \rho \Delta V \frac{d\vec{v}_x}{dt} = \left[p\left(x, y, z, t\right) - p\left(x + \Delta x, y, z, t\right) \right] \Delta y \Delta z, \qquad (4.1.4)$$

from which we conclude

$$\rho \frac{d\vec{v}_x}{dt} = -\frac{\partial p}{\partial x} \qquad (4.1.5)$$

The same argument applied in the y and z directions gives

$$\rho \frac{d\vec{v}}{dt} = -\vec{\nabla} p \qquad (4.1.6)$$

This explains the origin of the first term in (4.1.2).

To understand the second term, recall the basic force law for a free particle in an electromagnetic field $\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$ and keep in mind that we assumed charge neutrality, which means that the total charge of the particles in a small volume ads up to zero. Obviously this means that we must have different species of particles with opposing charges, e.g. protons and electrons in the fluid.

Let us first consider the electric force on a small volume element

$$\vec{F}_{\text{Tot}} = \sum_i q_i \vec{E}_i \approx \left(\sum_i q_i \right) \vec{E} = 0. \tag{4.1.7}$$

Here q_i is the charge of the i^{th} particle in the volume element and $\vec{E}_i$ the electric field at its position. We have also made the assumption that the electric field varies slowly over space so that we can approximate the electric field at all points in the small volume element with the same average field. Furthermore we have used the assumption of charge neutrality. We conclude that the net electric force on the volume element is zero.

Let us now consider the total Lorentz force on the volume element. All particles of a given specie has the same charge, q_s, so that the total Lorentz force, denoted $\vec{F}_s$, caused by them on the volume element is

$$\vec{F}_s = \sum_i \left(q_i \vec{v}_i \times \vec{B}_i \right) = q_s \left(\sum_i \vec{v}_i \right) \times \vec{B} = n_s q_s \langle \vec{v} \rangle_s \times \vec{B} \tag{4.1.8}$$

Here $\langle \vec{v}_s \rangle = \frac{1}{n_s} \sum_i \vec{v}_i$ is the average velocity of the particles of a given species in the volume element. We have also again used the assumption that the magnetic field has a slow spatial variation so that we can approximate it by the same average field in the small volume element. Next we note that $n_s = \rho_s \Delta V$ where ρ_s is the particle density of the specie and ΔV the volume of the volume element so that we can write

$$\vec{F}_s = \Delta V \rho_s q_s \langle \vec{v} \rangle_s \times \vec{B} \tag{4.1.9}$$

We now relate this to the current density. Consider particles of specie s moving in the z-direction at an average velocity with z-component $\langle \vec{v}_z \rangle_s$. The total charge that crosses a small area $\Delta x \Delta y$ perpendicular to the z-axis in a time interval Δt is

$$\Delta Q_s = q_s \rho_s \left(\langle \vec{v}_z \rangle_s \Delta t \right) \Delta x \Delta y. \tag{4.1.10}$$

By definition, the current density is the total charge flowing per unit time per unit area

$$\left(\vec{J}_s \right)_z = \frac{\Delta Q_s}{\Delta t \Delta x \Delta y} = q_s \rho_s \langle \vec{v}_z \rangle_s. \tag{4.1.11}$$

The same applies to the x- and y-directions, so that we can write (4.1.9)

$$\vec{F}_s = \Delta V \vec{J}_s \times \vec{B} \tag{4.1.12}$$

The total Lorentz force experienced by the small volume element is the contribution of all species

$$\vec{F}_{\text{Tot}} = \sum_s \vec{F}_s = \Delta V \vec{J} \times \vec{B} \tag{4.1.13}$$

where we have used (4.1.12) and $\vec{J} = \sum_s \vec{J}_s$ is the total current density arising from all species. From Newton's second law, $\vec{F}_{\text{Tot}} = \Delta m \frac{d\vec{v}}{dt}$ and it follows from (4.1.13) that

$$\rho \frac{d\vec{v}}{dt} = \vec{J} \times \vec{B} \tag{4.1.14}$$

with ρ the total mass density. This explains the second term in (4.1.2); it is just the Lorentz force experienced by the current density $\vec{J}$.

The third term $\vec{F}_v$ represents a viscous force, taken to be proportional to the velocity

$$\vec{F}_v = \rho \vec{\nabla}^2 \vec{v}. \tag{4.1.15}$$

The last term is the gravitational force.

The other equations are just Maxwell's equations

$$\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= \frac{1}{\epsilon_0} \rho_c, \\
\vec{\nabla} \cdot \vec{B} &= 0, \\
\vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}, \\
\vec{\nabla} \times \vec{B} &= \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}.
\end{aligned} \tag{4.1.16}$$

From our assumption of charge neutrality of the system, we can set $\rho_c = 0$. Furthermore we make the basic magneto hydrodynamic approximation that we have only low frequency fields, so that we can neglect the time derivative in the fourth equation of (4.1.16). We then have

$$\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= \vec{\nabla} \cdot \vec{B} = 0, \\
\vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}, \\
\vec{\nabla} \times \vec{B} &= \mu_0 \vec{J}.
\end{aligned} \tag{4.1.17}$$

We do not consider the divergence equations $\vec{\nabla}\cdot\vec{E}=\vec{\nabla}\cdot\vec{B}=0$. The reason for this is that the electric field in the rest frame of a small volume element is actually determined by Ohm's law $\vec{E}=\frac{\vec{J}}{\rho}$. Assuming a constant resistivity, this implies $\vec{\nabla}\cdot\vec{E}=\frac{1}{\rho}\vec{\nabla}\cdot\vec{J}=-\frac{1}{\rho}\frac{\partial\rho_c}{\partial t}=0$, where we used the continuity equation for charge conservation. Thus the divergenceless condition is automatically satisfied. From the second equation in (4.1.17) we observe that $\frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{B}\right)=0$. We can therefore always ensure the divergenceless condition of the magnetic field by imposing it as an initial condition. We therefore only consider the last two equations

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t},$$
$$\vec{\nabla}\times\vec{B}=\mu_0\vec{J}, \tag{4.1.18}$$

from here on.

Let us now consider Ohm's law in more detail. We choose a reference frame S' that moves with the volume element ΔV , i.e., the volume element is in rest in this frame. In this frame Ohm's law holds

$$\vec{J}'=\sigma\vec{E}' \tag{4.1.19}$$

The currents in the moving frame and rest frame are simply related by

$$\vec{J}=\vec{J}'+\rho_c\vec{v}=\vec{J}'. \tag{4.1.20}$$

The last term is just the additional contribution to the current coming from the relative motion of the carge. However, for a charge neutral system this vanishes as is indicated by the last equality in (4.1.20). On the other hand we can also determine the relation between $\vec{E}$ and $\vec{E}'$ from the tensorial transformation properties discussed in section 1.5. In particular the field strength $F^{\mu\nu}$ transforms as a second rank tensor so that

$$F'^{\mu\nu}=\Lambda^\mu_\alpha\Lambda^\nu_\beta F^{\alpha\beta} \tag{4.1.21}$$

where for $v << c$ and $\vec{v}||\hat{x}$

$$\Lambda=\begin{pmatrix} 1 & -\frac{v}{c} & 0 & 0 \\ -\frac{v}{c} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.1.22}$$

and

$$F^{\mu\nu} = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & B_z & -B_y \\ -\frac{E_y}{c} & -B_z & 0 & B_x \\ -\frac{E_z}{c} & B_y & -B_z & 0 \end{pmatrix}. \tag{4.1.23}$$

We thus have

$$\begin{aligned} F'^{0i} &= \Lambda^0_\alpha \Lambda^i_\beta F^{\alpha\beta},\ i = 1,2,3 \\ &= \Lambda^i_\beta \left(F^{0\beta} - \frac{v}{c} F^{1\beta} \right) \end{aligned} \tag{4.1.24}$$

For $i = 1$ we thus have

$$F'^{01} = \Lambda^1_\beta \left(F^{0\beta} - \frac{v}{c} F^{1\beta} \right) \approx F^{01} \tag{4.1.25}$$

and thus

$$E'_x = E_x \tag{4.1.26}$$

For $i = 2$ we have

$$F'^{02} = \Lambda^2_\beta \left(F^{0\beta} - \frac{v}{c} F^{1\beta} \right) = F^{02} - \frac{v}{c} F^{12} \tag{4.1.27}$$

and thus

$$E'_y = E_y - vB_z = E_y + \left(\vec{v} \times \vec{B} \right)_y, \quad (\vec{v} || \hat{x}) \tag{4.1.28}$$

For $i = 3$ we have

$$F'^{03} = \Lambda^3_\beta \left(F^{0\beta} - \frac{v}{c} F^{1\beta} \right) = F^{03} - \frac{v}{c} F^{13} \tag{4.1.29}$$

and thus

$$E'_z = E_z + vB_y = E_z + \left(\vec{v} \times \vec{B} \right)_z, \quad (\vec{v} || \hat{x})\,. \tag{4.1.30}$$

In general we can therefore write

$$\vec{E}' = \vec{E} + \vec{v} \times \vec{B}. \tag{4.1.31}$$

Thus from (4.1.20) we have

$$\vec{J} = \vec{J}' = \sigma \left(\vec{E} + \vec{v} \times \vec{B} \right). \tag{4.1.32}$$

The basic magneto hydrodynamic equations are now (4.1.1), (4.1.2), (4.1.18) and (4.1.32)

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0, \\
&\rho \frac{d\vec{v}}{dt} = -\vec{\nabla} p + \left(\vec{J} \times \vec{B}\right) + \vec{F}_v + \rho \vec{g}, \\
&\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \\
&\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}, \\
&\vec{J} = \sigma \left(\vec{E} + \vec{v} \times \vec{B}\right). \qquad (4.1.33)
\end{aligned}
$$

4.2 Magnetic diffusion, viscosity and pressure (see Jackson section 10.3)

Let us take the curl of $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$. This gives

$$
\vec{\nabla} \times \left(\vec{\nabla} \times \vec{B}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{B}\right) - \vec{\nabla}^2 \vec{B} = \mu_0 \left(\vec{\nabla} \times \vec{J}\right) \qquad (4.2.1)
$$

and thus

$$
\begin{aligned}
-\vec{\nabla}^2 \vec{B} &= \mu_0 \sigma \left[\vec{\nabla} \times \left(\vec{E} + \vec{v} \times \vec{B}\right)\right] \\
&= \mu_0 \sigma \left[-\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \left(\vec{v} \times \vec{B}\right)\right], \qquad (4.2.2)
\end{aligned}
$$

(assuming σ spatially constant) from which it follows that

$$
\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{B}\right) + \frac{1}{\mu_0 \sigma} \vec{\nabla}^2 \vec{B}. \qquad (4.2.3)
$$

For a fluid at rest $\vec{v} = 0$ and this becomes a diffusion equation

$$
\frac{\partial \vec{B}}{\partial t} = \frac{1}{\mu_0 \sigma} \vec{\nabla}^2 \vec{B}. \qquad (4.2.4)
$$

The solution is of the form

$$
\vec{B} = \vec{B}_0 e^{\left(i\vec{k}\cdot\vec{x} - \omega t\right)}, \ \omega = \frac{|\vec{k}|^2}{\mu_0 \sigma} > 0. \qquad (4.2.5)
$$

Thus an initial magnetic field configuration decays away on a time scale $\tau = \frac{1}{\omega} = \frac{\mu_0 \sigma}{|\vec{k}|^2}$. We see that slowly varying spatial variations of the magnetic field live long, while fast variation decay quickly.

From $\vec{J} = \sigma \left(\vec{E} + \vec{v} \times \vec{B} \right)$ we see that $\vec{J} \times \vec{B} = \sigma \left[\vec{E} \times \vec{B} + \left(\vec{v} \times \vec{B} \right) \times \vec{B} \right]$. We can always decompose $\vec{v} = \vec{v}_{||} + \vec{v}_{\perp}$ where $\vec{v}_{||}$ is the component of the velocity parallel to the magnetic field and $\vec{v}_{\perp}$ the component perpendicular to the magnetic field, i.e. $\vec{v}_{\perp} \cdot \vec{B} = 0$ and $\vec{v}_{||} \times \vec{B} = 0$. Thus

$$\left(\vec{v} \times \vec{B} \right) \times \vec{B} = \left(\vec{v}_{\perp} \times \vec{B} \right) \times \vec{B} = -|\vec{B}|^2 \vec{v}_{\perp}. \tag{4.2.6}$$

Putting this in the force equation we have

$$\rho \left(\frac{d\vec{v}_{||}}{dt} + \frac{d\vec{v}_{\perp}}{dt} \right) = \vec{F} + \left(\vec{J} \times \vec{B} \right) = \vec{F} + \sigma \left(\vec{E} \times \vec{B} \right) - \sigma |\vec{B}|^2 \vec{v}_{\perp}. \tag{4.2.7}$$

Here $\vec{F} = -\vec{\nabla} p + \vec{F}_v + \rho \vec{g}$ represents all the non-electromagnetic (mechanical) forces. Since $\vec{E} \times \vec{B}$ is also perpendicular to $\vec{B}$, we can combine it with $\vec{v}_{\perp}$. Let us therefore set $\vec{w} = \frac{\vec{E} \times \vec{B}}{|\vec{B}|^2}$ then

$$\rho \left(\frac{d\vec{v}_{||}}{dt} + \frac{d\vec{v}_{\perp}}{dt} \right) = \vec{F} + \left(\vec{J} \times \vec{B} \right) = \vec{F} - \sigma |\vec{B}|^2 \left(\vec{v}_{\perp} - \vec{w} \right). \tag{4.2.8}$$

This equation shows that the velocity, and thus flow, parallel to the magnetic field is determined by the component of the mechanical forces parallel to the magnetic field only. If σ is large, which is the case at high temperatures, we have

$$\rho \frac{d\vec{v}_{\perp}}{dt} = \vec{F}_{\perp} - \sigma |\vec{B}|^2 \left(\vec{v}_{\perp} - \vec{w} \right) \approx -\sigma |\vec{B}|^2 \left(\vec{v}_{\perp} - \vec{w} \right), \tag{4.2.9}$$

i.e. we can ignore the mechanical forces perpendicular to the magnetic field. The solution of this equation is easily found to be

$$\vec{v}_{\perp} = \vec{w} + \vec{v}_{\perp}^0 e^{-\frac{\sigma |\vec{B}|^2 t}{\rho}}. \tag{4.2.10}$$

Thus, $\vec{v}_{\perp}$ decays on a short time scale $\tau = \frac{\rho}{\sigma |\vec{B}|^2}$ from an initial value $\vec{v}_{\perp} = \vec{w} + \vec{v}_{\perp}^0$ to $\vec{v}_{\perp} = \vec{w} = \frac{\vec{E} \times \vec{B}}{|\vec{B}|^2}$. If $\vec{E} = 0$, $\vec{v}_{\perp} = 0$ and flow can only occur along the magnetic field lines. We say that the magnetic field lines are trapped in the plasma as any attempt to move the plasma perpendicular to the field lines will cause the field lines to be dragged along to prevent $\vec{v}_\perp \neq 0$. Note from (4.2.8) that the term proportional $|\vec{B}|^2$ acts as an additional viscous force that prevents flow perpendicular to the magnetic field. We call this the magnetic viscosity.

Next we rewrite the magnetic force term in (4.1.2) as follows

$$\vec{J} \times \vec{B} = -\frac{1}{\mu_0}\left[\vec{B} \times \left(\vec{\nabla} \times \vec{B}\right)\right], \tag{4.2.11}$$

where we have used $\vec{J} = \frac{1}{\mu_0}\left(\vec{\nabla} \times \vec{B}\right)$. Using the identity

$$\vec{B} \times \left(\vec{\nabla} \times \vec{B}\right) = \frac{1}{2}\vec{\nabla}\left(\vec{B} \cdot \vec{B}\right) - \left(\vec{B} \cdot \vec{\nabla}\right)\vec{B} \tag{4.2.12}$$

gives

$$\vec{J} \times \vec{B} = -\frac{1}{\mu_0}\left[\frac{1}{2}\vec{\nabla}|\vec{B}|^2 - \left(\vec{B} \cdot \vec{\nabla}\right)\vec{B}\right]. \tag{4.2.13}$$

The force equation then reads

$$\begin{aligned} \rho\frac{d\vec{v}}{dt} &= -\vec{\nabla}p - \frac{1}{2\mu_0}\vec{\nabla}|\vec{B}|^2 + \frac{1}{\mu_0}\left(\vec{B} \cdot \vec{\nabla}\right)\vec{B} + \vec{F}_v + \rho\vec{g} \\ &= -\vec{\nabla}\left[p + \frac{|\vec{B}|^2}{2\mu_0}\right] + \frac{1}{\mu_0}\left(\vec{B} \cdot \vec{\nabla}\right)\vec{B} + \vec{F}_v + \rho\vec{g} \end{aligned} \tag{4.2.14}$$

We see that the magnetic force is partially equivalent to a pressure

$$p_M = \frac{|\vec{B}|^2}{2\mu_0}, \tag{4.2.15}$$

called the magnetic pressure.

4.3 Debye screening (see Jackson section 10.9)

When charged particles are moving freely in a plasma, e.g., electrons stripped from the ions at high temperature or small ions in a solution (electrolytes) a phenomena known as Debye screening occurs. We discuss this mechanism now. Assume the charged particles are electrons with charge e that are in thermal equilibrium and that they move in an electrostatic potential $V(\vec{x})$

. We know that the probability of finding a particle with energy $E = eV$ is given by the Boltzman factor

$$P(E) = e^{-\beta E} = e^{-\frac{eV}{kT}}. \tag{4.3.1}$$

We can interpret this as the probability of finding the electron at position $\vec{x}$ as the potential is a function of position:

$$P(\vec{x}) = e^{-\beta E} = e^{-\frac{eV(\vec{x})}{kT}}. \tag{4.3.2}$$

The density of electrons at position $\vec{x}$ is of course proportional to $P(\vec{x})$, so that

$$n_e(\vec{x}) = n_0 e^{-\frac{eV(\vec{x})}{kT}}. \tag{4.3.3}$$

The whole plasma is neutral, the positive ions provide the neutralizing background that is uniformly distributed $\rho_{\text{ions}}(\vec{x}) = -en_0$. (Note $n_e(\vec{x}) = n_0 \Rightarrow \rho_e = en_0 \Rightarrow \rho_{\text{ions}} = -en_0$ for charge neutrality) . Now we put a test charge of charge Q at the origin. The charge density is then

$$\begin{aligned} \rho(\vec{x}) &= Q\delta(\vec{x}) + e(n_e(\vec{x}) - n_0) \\ &= Q\delta(\vec{x}) + en_0\left(e^{-\frac{eV(\vec{x})}{kT}} - 1\right). \end{aligned} \tag{4.3.4}$$

Gauss' law then reads

$$\vec{\nabla}^2 V(\vec{x}) = -\frac{1}{\epsilon_0}\left[Q\delta(\vec{x}) + en_0\left(e^{-\frac{eV(\vec{x})}{kT}} - 1\right)\right]. \tag{4.3.5}$$

If we assume $\frac{eV}{kT} << 1$ and expand the exponential we have

$$\vec{\nabla}^2 V(\vec{x}) = -\frac{Q}{\epsilon_0}\delta(\vec{x}) + \frac{e^2 n_0}{\epsilon_0 kT} V(\vec{x}), \tag{4.3.6}$$

or

$$\left(\vec{\nabla}^2 - k_D^2\right) V(\vec{x}) = -\frac{Q}{\epsilon_0}\delta(\vec{x}), \tag{4.3.7}$$

where $k_D^2 = \frac{e^2 n_0}{\epsilon_0 kT}$. It is easy to find the spherical symmetrical solution to this equation which reads

$$V(\vec{x}) = \frac{Qe^{-k_D|\vec{x}|}}{|\vec{x}|}. \tag{4.3.8}$$

This means that the electrostatic potential $V(\vec{x})$ and thus Coulomb force $-\vec{\nabla} V(\vec{x})$ are screened exponentially and decay on a length scale

$$\ell = k_D^{-1} = \sqrt{\frac{\epsilon_0 kT}{e^2 n_0}}. \tag{4.3.9}$$

This is called the Debye screening length. Thus in a plasma electric charges get screened strongly and the Coulomb interaction becomes short ranged.

5 Lagrangian and Hamiltonian of a non-relativistic particle in the presence of electromagnetic fields (see Jackson section 12.1)

A charged particle moving in the presence of electric and magnetic field experiences a force

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{5.0.1}$$

The equation of motion is therefore

$$m\ddot{\vec{x}} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{5.0.2}$$

Now we want to write down the Hamiltonian and Lagrangian that yields (5.0.2).

Recall that for a particle with Lagrangian $L\left(\dot{\vec{x}}, \vec{x}, t\right)$ and action

$$S = \int_{t_i}^{t_f} dt L\left(\dot{\vec{x}}, \vec{x}, t\right) \tag{5.0.3}$$

the equation of motion follows from Hamilton's principle which states that the particle follows a path that makes the action stationery. This path, and thus equation of motion, follows from the Euler Lagrange equations for a stationery action

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0, \ \forall i. \tag{5.0.4}$$

The Lagrangian that yields this equation of motion is

$$L\left(\dot{\vec{x}}, \vec{x}, t\right) = \frac{1}{2}m\dot{\vec{x}}^2 + q\dot{\vec{x}} \cdot \vec{A}(\vec{x}, t) - qV(\vec{x}, t) \tag{5.0.5}$$

where $\vec{A}(\vec{x},t)$ and $V(\vec{x},t)$ are the vector and scalar potentials. Let us verify that this is indeed the case. From the Euler Lagrange equations we have

$$\frac{d}{dt}\left(m\dot{x}_i + qA_i(\vec{x},t)\right) - \left(q\vec{x}\cdot\frac{\partial\vec{A}}{\partial x_i} - q\frac{\partial V}{\partial x_i}\right) = 0 \tag{5.0.6}$$

The important point to realize is that $\frac{d}{dt}$ is the total derivative so that

$$\frac{dA_i(\vec{x},t)}{dt} = \frac{\partial A_i}{\partial t} + \sum_j \frac{\partial A_i}{\partial x_j}\frac{dx_j}{dt} = \frac{\partial A_i}{\partial t} + \sum_j \frac{\partial A_i}{\partial x_j}\dot{x}_j. \tag{5.0.7}$$

We thus have

$$m\ddot{x}_i + q\left[\left(\frac{\partial A_i}{\partial t} + \frac{\partial V}{\partial x_i}\right) + \sum_j \dot{x}_j\left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}\right)\right] = 0, \tag{5.0.8}$$

from which it follows, using (1.2.1) and (1.2.4), that

$$m\ddot{x}_i - q\left[E_i + \left(\dot{\vec{x}}\times\vec{B}\right)_i\right] = 0. \tag{5.0.9}$$

Thus, we indeed have

$$m\ddot{\vec{x}} = q\left[\vec{E} + \left(\vec{v}\times\vec{B}\right)\right]. \tag{5.0.10}$$

Next we can compute the momentum canonically conjugate to x_i:

$$\pi_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + qA_i, \tag{5.0.11}$$

or in vector notation

$$\vec{\pi} = m\dot{\vec{x}} + q\vec{A} = \vec{p} + q\vec{A}. \tag{5.0.12}$$

The Hamiltonian then follows from

$$\begin{aligned} H &= \sum_i \pi_i\dot{x}_i - L = \vec{\pi}\cdot\dot{\vec{x}} - L \\ &= \left(m\dot{\vec{x}} + q\vec{A}\right)\cdot\dot{\vec{x}} - \left(\frac{1}{2}m\dot{\vec{x}}^2 + q\dot{\vec{x}}\cdot\vec{A} - qV\right) \\ &= \frac{1}{2}m\dot{\vec{x}}^2 + qV \\ &= \frac{\left(\vec{\pi} - q\vec{A}\right)^2}{2m} + qV. \end{aligned} \tag{5.0.13}$$

We see that the Hamiltonian has the conventional form of a particle moving in a potential V if we identify the momentum to be $\vec{p} = \vec{\pi} - q\vec{A}$. This is called minimal substitution, which states that the momentum and Hamiltonian of a particle moving in a magnetic field with vector potential $\vec{A}$ is simply obtained by making the substitution

$$\vec{p} \to \vec{\pi} - q\vec{A}. \tag{5.0.14}$$