

Quantum mechanics Assignment 1

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1.

$$\begin{aligned}(\langle\psi| + \alpha^* \langle\phi|)(\alpha|\phi\rangle + |\psi\rangle) &= \langle\psi|(\alpha|\phi\rangle + |\psi\rangle) + \alpha^* \langle\phi|(\alpha|\phi\rangle + |\psi\rangle) \\&= \alpha\langle\psi|\phi\rangle + \langle\psi|\psi\rangle + |\alpha|^2 \langle\phi|\phi\rangle + \alpha^* \langle\phi|\psi\rangle \\&\geq 0\end{aligned}$$

The above statement holds for all α , and in particular $\alpha = -\frac{\langle\phi|\psi\rangle}{\langle\phi|\phi\rangle}$. If we substitute this choice of α into the inequality above, we get:

$$\begin{aligned}-\frac{|\langle\psi|\phi\rangle|^2}{\langle\phi|\phi\rangle} + \langle\psi|\psi\rangle + |\langle\psi|\phi\rangle|^2 \frac{\langle\phi|\phi\rangle}{|\langle\phi|\phi\rangle|^2} - \frac{|\langle\psi|\phi\rangle|^2}{\langle\phi|\phi\rangle} &\geq 0 \\ \Rightarrow \langle\psi|\psi\rangle - \frac{|\langle\psi|\phi\rangle|^2}{\langle\phi|\phi\rangle} &\geq 0 \\ \Rightarrow \langle\psi|\psi\rangle \langle\phi|\phi\rangle &\geq |\langle\psi|\phi\rangle|^2\end{aligned}$$

Which is the required result.

2. a). Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a general matrix over \mathbb{C} .

Then we need to find $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, such that:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \delta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

For this we have:

$$\begin{aligned}a &= \alpha + \delta \\ b &= \beta - i\gamma \\ c &= \beta + i\gamma \\ d &= \alpha - \delta\end{aligned}$$

This system of equations is partially decoupled, and so is easily solved.

$$\begin{aligned}\alpha &= \frac{a+d}{2} \\ \beta &= \frac{b+c}{2} \\ \gamma &= i\frac{b-c}{2} \\ \delta &= \frac{a-d}{2}\end{aligned}$$

Since a, b, c, d are in \mathbb{C} and \mathbb{C} is a field, $\alpha, \beta, \gamma, \delta$ are definitely in \mathbb{C} as required.

b). $\text{tr}(A^\dagger B)$ is an inner product: First, $(A, B) = \text{tr}(A^\dagger B)$, is indeed a map $\mathbb{C}^{2 \times 2} \times \mathbb{C}^{2 \times 2} \rightarrow \mathbb{C}$.

Let:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \quad C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

Note:

$$\begin{aligned} (A, B) &= \text{tr} \left\{ \begin{pmatrix} a_{11}^* & a_{21}^* \\ a_{12}^* & a_{22}^* \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right\} \\ &= \text{tr} \left\{ \begin{pmatrix} a_{11}^* b_{11} + a_{21}^* b_{21} & a_{11}^* b_{12} + a_{21}^* b_{22} \\ a_{12}^* b_{11} + a_{22}^* b_{21} & a_{12}^* b_{12} + a_{22}^* b_{22} \end{pmatrix} \right\} \\ &= a_{11}^* b_{11} + a_{21}^* b_{21} + a_{12}^* b_{12} + a_{22}^* b_{22} \end{aligned}$$

Similarly

$$(B, A) = b_{11}^* a_{11} + b_{21}^* a_{21} + b_{12}^* a_{12} + b_{22}^* a_{22}$$

It is now clear that:

$$(B, A)^* = (b_{11}^* a_{11} + b_{21}^* a_{21} + b_{12}^* a_{12} + b_{22}^* a_{22})^* = a_{11}^* b_{11} + a_{21}^* b_{21} + a_{12}^* b_{12} + a_{22}^* b_{22} = (A, B)$$

As required, now moving onto linearity in the second argument:

$$\begin{aligned} (A, cB + dC) &= \text{tr} \{ A^\dagger (cB + dC) \} \\ &= \text{tr} \left\{ \begin{pmatrix} a_{11}^* & a_{21}^* \\ a_{12}^* & a_{22}^* \end{pmatrix} \begin{pmatrix} cb_{11} + dc_{11} & cb_{12} + dc_{12} \\ cb_{21} + dc_{21} & cb_{22} + dc_{22} \end{pmatrix} \right\} \\ &= a_{11}^* (cb_{11} + dc_{11}) + a_{21}^* (cb_{21} + dc_{21}) + a_{12}^* (cb_{12} + dc_{12}) + a_{22}^* (cb_{22} + dc_{22}) \\ &= c(a_{11}^* b_{11} + a_{21}^* b_{21} + a_{12}^* b_{12} + a_{22}^* b_{22}) + d(a_{11}^* c_{11} + a_{21}^* c_{21} + a_{12}^* c_{12} + a_{22}^* c_{22}) \\ &= c \text{tr}(A^\dagger B) + d \text{tr}(A^\dagger C) \\ &= c(A, B) + d(A, C) \end{aligned}$$

Positive definiteness is shown by:

$$\begin{aligned} (A, A) &= a_{11}^* a_{11} + a_{12}^* a_{12} + a_{21}^* a_{21} + a_{22}^* a_{22} \\ &= |a_{11}|^2 + |a_{12}|^2 + |a_{21}|^2 + |a_{22}|^2 \\ &\geq 0 \end{aligned}$$

and since $|c|^2 = 0 \Leftrightarrow c = 0$ for any $c \in \mathbb{C}$. We have from the above:

$$(A, A) = 0 \Leftrightarrow A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This is actually obvious if one realizes that there is an isomorphism:

$$\begin{aligned} I : \mathbb{C}^{2 \times 2} &\rightarrow \mathbb{C}^4 \\ &: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \end{aligned}$$

Then $\text{tr}\{A^\dagger B\}$ is the standard inner product on \mathbb{C}^4 .

The product of a 2×2 diagonal matrix and an anti-diagonal matrix zero:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow (I, \sigma_x) = (I, \sigma_y) = (\sigma_z, \sigma_x) = (\sigma_z, \sigma_y) = 0$$

It now remains to check the product of the two diagonal and anti-diagonal matrices:

$$\begin{aligned} (I, \sigma_z) &= 1 + 0 + 0 - 1 = 0 \\ (\sigma_x, \sigma_y) &= 0 + 1^*(-i) + 1^*(i) + 0 = 0 \end{aligned}$$

This shows that the four matrices are mutually orthogonal.

All matrices have two elements, and each element has modulus of 1. This makes the norm of each matrix $\sqrt{2}$, hence:

$$\left\{ \frac{I}{\sqrt{2}}, \frac{\sigma_x}{\sqrt{2}}, \frac{\sigma_y}{\sqrt{2}}, \frac{\sigma_z}{\sqrt{2}} \right\}$$

Forms an orthonormal basis for this space.

3. Let the basis of our ket space be given by $\{|a\rangle\}$, that is the set of eigenkets of A . Now:

$$A^n = A^n \sum_a |a\rangle \langle a| = \sum_a A^n |a\rangle \langle a| = \sum_a a^n |a\rangle \langle a|$$

And similarly:

$$A^m = \sum_a a^m |a\rangle \langle a|$$

Thus

$$\begin{aligned} A^n A^m &= \left(\sum_a a^n |a\rangle \langle a| \right) \left(\sum_{a'} a'^m |a'\rangle \langle a'| \right) \\ &= \sum_a \sum_{a'} a^n a'^m |a\rangle \langle a|a'\rangle \langle a'| \\ &= \sum_a \sum_{a'} a^n a'^m |a\rangle \delta_{a'}^a \langle a'| \\ &= \sum_a a^n a^m |a\rangle \langle a| \\ &= \sum_a a^{n+m} |a\rangle \langle a| \\ &\equiv A^{n+m} \end{aligned}$$

4. a). By definition:

$$\text{tr } A = \sum_i \langle i|A|i\rangle$$

Now:

$$\begin{aligned}\text{tr}(AB) &= \sum_i \langle i|AB|i\rangle \\ &= \sum_i \sum_j \langle i|A|j\rangle \langle j|B|i\rangle \\ &= \sum_i \sum_j \langle j|B|i\rangle \langle i|A|j\rangle \\ &= \sum_j \langle j|BA|j\rangle \\ &= \text{tr}(BA)\end{aligned}$$

b). We have:

$$\begin{aligned}AB &= \sum_i AB|i\rangle \langle i| \\ &= \sum_{i,j} A|j\rangle \langle j|B|i\rangle \langle i| \\ &= \sum_{i,j,k} |k\rangle \langle k|A|j\rangle \langle j|B|i\rangle \langle i| \\ &= \sum_{i,j,k} (\langle k|A|j\rangle \langle j|B|i\rangle) |k\rangle \langle i|\end{aligned}$$

Now last expression above is just the sum of a number, $\langle k|A|j\rangle \langle j|B|i\rangle$, times by an operator and so:

$$\begin{aligned}(AB)^\dagger &= \sum_{i,j,k} [(\langle k|A|j\rangle \langle j|B|i\rangle) |k\rangle \langle i|]^\dagger \\ &= \sum_{i,j,k} (\langle k|A|j\rangle \langle j|B|i\rangle)^* |i\rangle \langle k| \\ &= \sum_{i,j,k} (\langle j|A^\dagger|k\rangle \langle i|B^\dagger|j\rangle) |i\rangle \langle k| \\ &= \sum_{i,j,k} |i\rangle \langle i|B^\dagger|j\rangle \langle j|A^\dagger|k\rangle \langle k| \\ &= \sum_i |i\rangle \langle i|B^\dagger A^\dagger \\ &= B^\dagger A^\dagger\end{aligned}$$

As required.

c).

$$\begin{aligned}A &= |\alpha\rangle \langle \beta| \\ &= \sum_n |n\rangle \langle n|\alpha\rangle \langle \beta| \\ &= \sum_n \sum_m |n\rangle \langle n|\alpha\rangle \langle \beta|m\rangle \langle m|\end{aligned}$$

This is a matrix with the nm -th element having the value: $\langle n|\alpha\rangle \langle \beta|m\rangle$.

d).

$$\begin{aligned}
 A^\dagger &= \left(\sum_n \sum_m |n\rangle \langle n|\alpha\rangle \langle \beta|m\rangle \langle m| \right)^\dagger \\
 &= \sum_n \sum_m (\langle n|\alpha\rangle \langle \beta|m\rangle)^* |m\rangle \langle n| \\
 &= \sum_n \sum_m (\langle m|\alpha\rangle \langle \beta|n\rangle)^* |n\rangle \langle m|
 \end{aligned}$$

This shows that the nm -th element of A^\dagger is the complex conjugate of the mn -th element of A . This is the definition of the Hermitian conjugate of a matrix.