Quantum mechanics Assignment 1 J L Gouws
February 18, 2022 **26634554**

1.

$$\begin{aligned}\left(\langle\psi| + \alpha^* \langle\phi|\right)\left(\alpha|\phi\rangle + |\psi\rangle\right) &= \langle\psi|\left(\alpha|\phi\rangle + |\psi\rangle\right) + \alpha^* \langle\phi|\left(\alpha|\phi\rangle + |\psi\rangle\right) \\ &= \alpha\langle\psi|\phi\rangle + \langle\psi|\psi\rangle + |\alpha|^2 \langle\phi|\phi\rangle + \alpha^* \langle\phi|\psi\rangle \\ &\geq 0\end{aligned}$$

The above statement holds for all α, and in particular $\alpha = i\frac{\langle\psi|\phi\rangle}{\langle\phi|\phi\rangle}$. If we substitute this choice of α into the inequality above, we get:

$$\begin{aligned}&\Longrightarrow i\frac{|\langle\psi|\phi\rangle|^2}{\langle\phi|\phi\rangle} + \langle\psi|\psi\rangle - |\langle\psi|\phi\rangle|^2 \frac{\langle\phi|\phi\rangle}{|\langle\phi|\phi\rangle|^2} - i\frac{|\langle\phi|\psi\rangle|^2}{\langle\phi|\phi\rangle} \geq 0 \\ &\Longrightarrow \langle\psi|\psi\rangle - \frac{|\langle\psi|\phi\rangle|^2}{\langle\phi|\phi\rangle} \geq 0 \\ &\Longrightarrow \langle\psi|\psi\rangle\langle\phi|\phi\rangle \geq |\langle\psi|\phi\rangle|^2\end{aligned}$$

Which is the required result.

2. a). Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a general matrix over $\mathbb{C}$.

Then we need to find $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, such that:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \gamma\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \delta\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

For this we have:

$$\begin{aligned} a &= \alpha + \delta \\ b &= \beta + -i\gamma \\ c &= \beta + i\gamma \\ d &= \alpha - \delta \end{aligned}$$

This system of equations is partially decoupled, and so is easily solved.

$$\begin{aligned} \alpha &= \frac{a+d}{2} \\ \beta &= \frac{b+c}{2} \\ \gamma &= i\frac{b-c}{2} \\ \delta &= \frac{a-d}{2} \end{aligned}$$

Since a, b, c, d are in $\mathbb{C}$ and $\mathbb{C}$ is a field, $\alpha, \beta, \gamma, \delta$ are definitely in $\mathbb{C}$ as required.

b). $\operatorname{tr}\left(A^\dagger B\right)$ is an inner product: First, $(A,B) = \operatorname{tr}\left(A^\dagger B\right)$, is indeed a map $\mathbb{C}^{2\times 2} \times \mathbb{C}^{2\times 2} \longrightarrow \mathbb{C}$.
Let:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \qquad C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

Note:

$$\begin{aligned}(A,B) &= \operatorname{tr}\left\{\begin{pmatrix} a_{11}^* & a_{21}^* \\ a_{12}^* & a_{22}^* \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right\} \\ &= \operatorname{tr}\left\{\begin{pmatrix} a_{11}^*b_{11} + a_{21}^*b_{21} & a_{11}^*b_{12} + a_{21}^*b_{22} \\ a_{12}^*b_{11} + a_{22}^*b_{21} & a_{12}^*b_{12} + a_{22}^*b_{22} \end{pmatrix}\right\} \\ &= a_{11}^*b_{11} + a_{21}^*b_{21} + a_{12}^*b_{12} + a_{22}^*b_{22}\end{aligned}$$

Similarly

$$(B,A) = b_{11}^*a_{11} + b_{21}^*a_{21} + b_{12}^*a_{12} + b_{22}^*a_{22}$$

It is now clear that:

$$(B,A)^* = \left(b_{11}^*a_{11} + b_{21}^*a_{21} + b_{12}^*a_{12} + b_{22}^*a_{22}\right)^* = a_{11}^*b_{11} + a_{21}^*b_{21} + a_{12}^*b_{12} + a_{22}^*b_{22} = (A,B)$$

As required, now moving onto linearity in the second argument:

$$\begin{aligned}(A, cB + dC) &= \operatorname{tr}\left\{A^\dagger\left(cB + dC\right)\right\} \\ &= \operatorname{tr}\left\{\begin{pmatrix} a_{11}^* & a_{21}^* \\ a_{12}^* & a_{22}^* \end{pmatrix}\begin{pmatrix} cb_{11} + dc_{11} & cb_{12} + dc_{12} \\ cb_{21} + dc_{21} & cb_{22} + dc_{22} \end{pmatrix}\right\} \\ &= a_{11}^*(cb_{11} + dc_{11}) + a_{21}^*(cb_{21} + dc_{21}) + a_{12}^*(cb_{12} + dc_{12}) + a_{22}^*(cb_{22} + dc_{22}) \\ &= c\left(a_{11}^*b_{11} + a_{21}^*b_{21} + a_{12}^*b_{12} + a_{22}^*b_{22}\right) + d\left(a_{11}^*c_{11} + a_{21}^*c_{21} + a_{12}^*c_{12} + a_{22}^*c_{22}\right) \\ &= c\operatorname{tr}(A^\dagger B) + d\operatorname{tr}(A^\dagger C) \\ &= c(A,B) + d(A,C)\end{aligned}$$

Positive definiteness is shown by:

$$\begin{aligned}(A,A) &= a_{11}^*a_{11} + a_{12}^*a_{12} + a_{21}^*a_{21} + a_{22}^*a_{22} \\ &= |a_{11}|^2 + |a_{12}|^2 + |a_{21}|^2 + |a_{22}|^2 \\ &\geq 0\end{aligned}$$

and since $|c| = 0 \iff c = 0$ for any $c \in \mathbb{C}$. We have form the above:

$$(A,A) = 0 \iff A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

3. Let the basis of our ket space be given by $\{|a\rangle\}$, that is the set of eigenkets of A. Now:

$$A^n = A^n \sum_a |a\rangle \langle a| = \sum_a A^n |a\rangle \langle a| = \sum_a a^n |a\rangle \langle a|$$

And similarly:

$$A^m = \sum_a a^m |a\rangle \langle a|$$

Thus

$$\begin{aligned}
A^n A^m &= \left(\sum_a a^n |a\rangle \langle a|\right)\left(\sum_{a'} a'^m |a'\rangle \langle a'|\right) \\
&= \sum_a \sum_{a'} a^n a'^m |a\rangle \langle a|' a\rangle \langle a| \\
&= \sum_a \sum_{a'} a^n a'^m |a\rangle \delta^a{}_{a'} \langle a| \\
&= \sum_a a^n a^m |a\rangle \langle a| \\
&= \sum_a a^{n+m} |a\rangle \langle a| \\
&\equiv A^{n+m}
\end{aligned}$$

4. a). By definition:

$$\operatorname{tr} A = \sum_i \langle i|A|i\rangle$$

Now:

$$\begin{aligned}
\operatorname{tr}(AB) &= \sum_i \langle i|AB|i\rangle \\
&= \sum_i \sum_j \langle i|A|j\rangle \langle j|B|i\rangle \\
&= \sum_i \sum_j \langle j|B|i\rangle \langle i|A|j\rangle \\
&= \sum_j \langle j|BA|j\rangle \\
&= \operatorname{tr}(BA)
\end{aligned}$$

b). We have:

$$\begin{aligned}
AB &= \sum_i AB\,|i\rangle \langle i| \\
&= \sum_i \sum_j A\,|j\rangle \langle j|\,B\,|i\rangle \langle i| \\
&= \sum_i \sum_j \sum_k |k\rangle \langle k|\,A\,|j\rangle \langle j|\,B\,|i\rangle \langle i| \\
&= \sum_i \sum_j \sum_k \langle k|\,A\,|j\rangle \langle j|\,B\,|i\rangle\,|k\rangle \langle i|
\end{aligned}$$

Now last expression above is just the sum of a number, $\langle k|A|j\rangle \langle j|B|i\rangle$, times by an operator and so:

$$\begin{aligned}
(AB)^\dagger &= \sum_i \sum_j \sum_k \big(\langle k|\,A\,|j\rangle \langle j|\,B\,|i\rangle\,|k\rangle \langle i|\big)^\dagger \\
&= \sum_i \sum_j \sum_k \big(\langle k|\,A\,|j\rangle \langle j|\,B\,|i\rangle\big)^*\,|i\rangle \langle k| \qquad = \sum_i \sum_j \sum_k \langle j|\,A^\dagger\,|k\rangle \langle k|\,B^\dagger\,|j\rangle\,|i\rangle \langle k|
\end{aligned}$$