1.

$$\left(\langle\psi| + \alpha^*\langle\phi|\right)\left(\alpha|\phi\rangle + |\psi\rangle\right) = \langle\psi|\left(\alpha|\phi\rangle + |\psi\rangle\right) + \alpha^*\langle\phi|\left(\alpha|\phi\rangle + |\psi\rangle\right)$$
$$= \alpha\langle\psi|\phi\rangle + \langle\psi|\psi\rangle + |\alpha|^2\langle\phi|\phi\rangle + \alpha^*\langle\phi|\psi\rangle$$
$$\geq 0$$

The above statement holds for all α, and in particular $\alpha = i\frac{\langle\psi|\phi\rangle}{\langle\phi|\phi\rangle}$. If we substitute this choice of α into the inequality above, we get:

$$\Longrightarrow i\frac{|\langle\psi|\phi\rangle|^2}{\langle\phi|\phi\rangle} + \langle\psi|\psi\rangle - |\langle\psi|\phi\rangle|^2\frac{\langle\phi|\phi\rangle}{|\langle\phi|\phi\rangle|^2} - i\frac{|\langle\phi|\psi\rangle|^2}{\langle\phi|\phi\rangle} \geq 0$$

$$\Longrightarrow \langle\psi|\psi\rangle - \frac{|\langle\psi|\phi\rangle|^2}{\langle\phi|\phi\rangle} \geq 0$$

$$\Longrightarrow \langle\psi|\psi\rangle\langle\phi|\phi\rangle \geq |\langle\psi|\phi\rangle|^2$$

Which is the required result.

2. a). Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a general matrix over $\mathbb{C}$.

Then we need to find $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, such that:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \gamma\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \delta\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

For this we have:

$$a = \alpha + \delta$$
$$b = \beta + -i\gamma$$
$$c = \beta + i\gamma$$
$$d = \alpha - \delta$$

This system of equations is partially decoupled, and so is easily solved.

$$\alpha = \frac{a + d}{2}$$
$$\beta = \frac{b + c}{2}$$
$$\gamma = i\frac{b - c}{2}$$
$$\delta = \frac{a - d}{2}$$

Since a, b, c, d are in $\mathbb{C}$ and $\mathbb{C}$ is a field, $\alpha, \beta, \gamma, \delta$ are definitely in $\mathbb{C}$ as required.

b). $\text{tr}\left(A^\dagger B\right)$ is an inner product: First, $(A, B) = \text{tr}\left(A^\dagger B\right)$, is indeed a map $\mathbb{C}^{2\times 2} \times \mathbb{C}^{2\times 2} \to \mathbb{C}$.
Let:
$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \qquad C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

Note:
$$(A, B) = \text{tr}\left\{ \begin{pmatrix} a_{11}^* & a_{21}^* \\ a_{12}^* & a_{22}^* \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right\}$$
$$= \text{tr}\left\{ \begin{pmatrix} a_{11}^* b_{11} + a_{21}^* b_{21} & a_{11}^* b_{12} + a_{21}^* b_{22} \\ a_{12}^* b_{11} + a_{22}^* b_{21} & a_{12}^* b_{12} + a_{22}^* b_{22} \end{pmatrix} \right\}$$
$$= a_{11}^* b_{11} + a_{21}^* b_{21} + a_{12}^* b_{12} + a_{22}^* b_{22}$$

Similarly

$$(B, A) = b_{11}^* a_{11} + b_{21}^* a_{21} + b_{12}^* a_{12} + b_{22}^* a_{22}$$

It is now clear that:

$$(B, A)^* = \left(b_{11}^* a_{11} + b_{21}^* a_{21} + b_{12}^* a_{12} + b_{22}^* a_{22} \right)^* = a_{11}^* b_{11} + a_{21}^* b_{21} + a_{12}^* b_{12} + a_{22}^* b_{22} = (A, B)$$

As required, now moving onto linearity in the second argument:

$$(A, cB + dC) = \text{tr}\left\{ A^\dagger \left(cB + dC \right) \right\}$$
$$= \text{tr}\left\{ \begin{pmatrix} a_{11}^* & a_{21}^* \\ a_{12}^* & a_{22}^* \end{pmatrix} \begin{pmatrix} cb_{11} + dc_{11} & cb_{12} + dc_{12} \\ cb_{21} + dc_{21} & cb_{22} + dc_{22} \end{pmatrix} \right\}$$
$$= a_{11}^*(cb_{11} + dc_{11}) + a_{21}^*(cb_{21} + dc_{21}) + a_{12}^*(cb_{12} + dc_{12}) + a_{22}^*(cb_{22} + dc_{22})$$
$$= c\left(a_{11}^* b_{11} + a_{21}^* b_{21} + a_{12}^* b_{12} + a_{22}^* b_{22} \right) + d\left(a_{11}^* c_{11} + a_{21}^* c_{21} + a_{12}^* c_{12} + a_{22}^* c_{22} \right)$$
$$= c\,\text{tr}(A^\dagger B) + d\,\text{tr}(A^\dagger C)$$
$$= c\,(A, B) + d\,(A, C)$$

Positive definiteness is shown by:

$$(A, A) = a_{11}^* a_{11} + a_{12}^* a_{12} + a_{21}^* a_{21} + a_{22}^* a_{22}$$
$$= |a_{11}|^2 + |a_{12}|^2 + |a_{21}|^2 + |a_{22}|^2$$
$$\geq 0$$

and since $|c|^2 = 0 \iff c = 0$ for any $c \in \mathbb{C}$. We have form the above:

$$(A, A) = 0 \iff A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This is actually obvious if one realizes that there is an isomorphism:

$$I : \mathbb{C}^{2\times 2} \to \mathbb{C}^4$$

$$: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

Then $\text{tr}\{A^\dagger B\}$ is the standard inner product on $\mathbb{C}^4$.

The product of a 2×2 diagonal matrix and an anti-diagonal matrix zero:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \implies (I, \sigma_x) = (I, \sigma_y) = (\sigma_z, \sigma_x) = (\sigma_z, \sigma_y) = 0$$

It now remains to check the product of the two diagonal and anti-diagonal matrices:

$$(I, \sigma_z) = 1 + 0 + 0 - 1 = 0$$
$$(\sigma_x, \sigma_y) = 0 + 1^*(-i) + 1^*(i) + 0 = 0$$

This shows that the four matrices are mutually orthogonal.

All matrices have two elements, and each element has modulus of 1. This makes the norm of each matrix $\sqrt{2}$, hence:

$$\left\{ \frac{I}{\sqrt{2}}, \frac{\sigma_x}{\sqrt{2}}, \frac{\sigma_y}{\sqrt{2}}, \frac{\sigma_z}{\sqrt{2}} \right\}$$

Forms an orthonormal basis for this space.

3. Let the basis of our ket space be given by $\{|a\rangle\}$, that is the set of eigenkets of A. Now:

$$A^n = A^n \sum_a |a\rangle \langle a| = \sum_a A^n |a\rangle \langle a| = \sum_a a^n |a\rangle \langle a|$$

And similarly:

$$A^m = \sum_a a^m |a\rangle \langle a|$$

Thus

$$\begin{aligned} A^n A^m &= \left(\sum_a a^n |a\rangle \langle a| \right) \left(\sum_{a'} a'^m |a'\rangle \langle a'| \right) \\ &= \sum_a \sum_{a'} a^n a'^m |a\rangle \langle a|a'\rangle \langle a'| \\ &= \sum_a \sum_{a'} a^n a'^m |a\rangle \delta^{a'}{}_a \langle a'| \\ &= \sum_a a^n a^m |a\rangle \langle a| \\ &= \sum_a a^{n+m} |a\rangle \langle a| \\ &\equiv A^{n+m} \end{aligned}$$

4. a). By definition:
$$\text{tr}\, A = \sum_i \langle i|A|i \rangle$$

Now:

$$\begin{aligned}
\text{tr}(AB) &= \sum_i \langle i|AB|i \rangle \\
&= \sum_i \sum_j \langle i|A|j \rangle \langle j|B|i \rangle \\
&= \sum_i \sum_j \langle j|B|i \rangle \langle i|A|j \rangle \\
&= \sum_j \langle j|BA|j \rangle \\
&= \text{tr}(BA)
\end{aligned}$$

b). We have:

$$\begin{aligned}
AB &= \sum_i AB |i \rangle \langle i| \\
&= \sum_i \sum_j A |j \rangle \langle j| B |i \rangle \langle i| \\
&= \sum_i \sum_j \sum_k |k \rangle \langle k| A |j \rangle \langle j| B |i \rangle \langle i| \\
&= \sum_i \sum_j \sum_k \langle k| A |j \rangle \langle j| B |i \rangle |k \rangle \langle i|
\end{aligned}$$

Now last expression above is just the sum of a number, $\langle k|A|j \rangle \langle j|B|i \rangle$, times by an operator and so:

$$\begin{aligned}
(AB)^\dagger &= \sum_i \sum_j \sum_k \left(\langle k| A |j \rangle \langle j| B |i \rangle |k \rangle \langle i| \right)^\dagger \\
&= \sum_i \sum_j \sum_k \left(\langle k| A |j \rangle \langle j| B |i \rangle \right)^* |i \rangle \langle k| \\
&= \sum_i \sum_j \sum_k \langle j| A^\dagger |k \rangle \langle i| B^\dagger |j \rangle |i \rangle \langle k| \\
&= \sum_i \sum_j \sum_k |i \rangle \langle i| B^\dagger |j \rangle \langle j| A^\dagger |k \rangle \langle k| \\
&= \sum_i \sum_j |i \rangle \langle i| B^\dagger A^\dagger |k \rangle \langle k| \\
&= B^\dagger A^\dagger
\end{aligned}$$

As required.

c).

$$\begin{aligned}
A &= |\alpha \rangle \langle \beta| \\
&= \sum_n |n \rangle \langle n|\alpha \rangle \langle \beta| \\
&= \sum_n \sum_m |n \rangle \langle n|\alpha \rangle \langle \beta|m \rangle \langle m| \\
&= \sum_n \sum_m \left(\langle n|\alpha \rangle \langle \beta|m \rangle \right) |n \rangle \langle m|
\end{aligned}$$

d).

$$A^\dagger = \left(\sum_n \sum_m |n\rangle \langle n|\alpha\rangle \langle\beta|m\rangle \langle m| \right)^\dagger$$

$$= \sum_n \sum_m \left(\langle n|\alpha\rangle \langle\beta|m\rangle \right)^* |m\rangle \langle n|$$

$$= \sum_n \sum_m \left(\langle m|\alpha\rangle \langle\beta|n\rangle \right)^* |n\rangle \langle m|$$