Physics 714 (Quantum Mechanics B)
Handed out: 28/02/22
Due: 7/03/22
Total:22

Problem 1 [12]

Consider an arbitrary state $|\psi\rangle$ in the Hilbert space of some quantum system. Let A and B be two hermitian observables (operators) on the Hilbert space and denote the expectation values of these observables in the state $|\psi\rangle$ by $\langle A\rangle = \langle\psi|A|\psi\rangle$ and $\langle B\rangle = \langle\psi|B|\psi\rangle$. Set $U = A - \langle A\rangle$, $V = B - \langle B\rangle$ and define the state $|\phi\rangle = U|\psi\rangle + i\lambda V|\rangle\psi$, with λ real.

1) Show that

$$\langle\phi|\phi\rangle = (\Delta A)^2 + \lambda^2 (\Delta B)^2 + i\lambda\langle[A,B]\rangle$$

where $(\Delta A)^2 = \langle\psi|A^2|\psi\rangle - \langle A\rangle^2$, $(\Delta B)^2 = \langle\psi|B^2|\psi\rangle - \langle B\rangle^2$ and $\langle[A,B]\rangle = \langle\psi|\,[A,B]\,|\psi\rangle$.

[6]

2) Find the value of λ that minimizes $\langle\phi|\phi\rangle$. Using $\langle\phi|\phi\rangle \geq 0\ \forall\lambda$, derive the generalized uncertainty relation

$$(\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4}\langle i\,[A,B]\rangle^2$$

[3]

3) Discuss the implications of the generalized uncertainty relations in terms of the simultaneous measurement of two non-commuting observables (operators). Use position and momentum as an example to demonstrate your conclusion.

[3]

Problem 2 [10]

Proof the following identities:

1) $[A,BC] = [A,B]\,C + B\,[A,C]$

[1]

2) Use (1) to show that if $[A,B] = cI$, then $[A,B^n] = ncB^{n-1}$. Here c is a complex number and I the identity operator.

[3]

3) Use (2) to show that if $[A,B] = cI$, then $[A, f(B)] = cf'(B)$. Here c is a complex number and f' is the derivative of the function f

[2]

4) $e^{-A}Be^{A} = B + [B,A] + \frac{1}{2}\,[[B,A],A] + \frac{1}{3!}\,[[[B,A],A],A] + \ldots$. Use the hint provided on p.22 under point 6 of the notes.

[4]