1. We have:

$$z\frac{\partial \ln \mathfrak{Z}}{\partial z} = z\frac{\partial \mathfrak{Z}}{\partial z}\Big/ \mathfrak{Z} = \frac{z}{\mathfrak{Z}}\sum_N Nz^{N-1}Z_N = \sum_N N\frac{z^N Z_N}{\mathfrak{Z}} = \sum_N N\frac{e^{\beta \mu N}\sum_r e^{-\beta E_r}}{\mathfrak{Z}} = \sum_N NP_N = \langle N \rangle$$

2. The cannonical partition function for an ideal gas is:

$$Z_N = \frac{1}{h^3 N!}\int d^{3N}p\, d^{3N}q\, e^{-\beta H}$$

$$= \frac{1}{\lambda^{3N}N!}\int d^{3N}q\, e^{-\beta \sum_{i<j}u_{ij}}$$

$$= \frac{1}{\lambda^{3N}N!}\int d^{3N}q\, e^{-\beta \times 0}$$

$$= \frac{1}{\lambda^{3N}N!}\int d^{3N}q$$

$$= \frac{V^N}{\lambda^{3N}N!}$$

Since there is no interaction between ideal gas molecules. We can use this to evaluate the grand partion function for an ensemble of ideal gas samples:

$$\mathfrak{Z} = \sum_{N=0}^{\infty} z^N \frac{V^N}{\lambda^{3N}N!}$$

$$= \sum_{N=0}^{\infty} \frac{(zV/\lambda^3)^N}{N!}$$

$$= e^{zV/\lambda^3}$$

We also have that:

$$P = \frac{k_B T}{V}\ln \mathfrak{Z} = z\frac{k_B T}{\lambda^3} \Rightarrow \frac{P}{k_B T} = \frac{z}{\lambda^3}$$

and:

$$\langle N \rangle = z\frac{\partial}{\partial z}\left[\frac{zV}{\lambda^3}\right] = \frac{zV}{\lambda^3} \Rightarrow \frac{\langle N \rangle}{V} = \frac{z}{\lambda^3}$$

From which we deduce:

$$\frac{P}{k_B T} = \frac{\langle N \rangle}{V} \Rightarrow PV = \langle N \rangle k_B T$$

3. We have:

$$\frac{1}{\lambda^3} \sum_{l=1}^{\infty} b_l z^l = \frac{P}{k_B T} = \frac{1}{\lambda^3} \sum_{l=1}^{\infty} a_l \times \left(\lambda^3 \frac{\langle N \rangle}{V}\right)^l$$

$$\Rightarrow \sum_{l=1}^{\infty} b_l z^l = \sum_{l=1}^{\infty} a_l \times \left(\sum_{k=1}^{\infty} k b_k z^k\right)^l$$

Now we match coefficients of $z$. The only way to get a $z$ term on the right handside is if $l = k = 1$:

$$b_1 z = a_1 \times b_1 z \Rightarrow a_1 = 1$$

Also note that:

$$b_1 = \frac{1}{V} \int_V d^3 r = 1 \Rightarrow a_1 = b_1 = 1$$

For the $z^2$ terms, the only way to get $z^2$ on the right is if $l = 1$ and $k = 2$ or if $l = 2$ and $k = 1$

$$b_2 z^2 = a_1 \times (2b_2 z^2) + a_2 \times (b_1 z^1)^2$$
$$\Rightarrow b_2 = 2b_2 + a_2$$
$$\Rightarrow a_2 = -b_2$$

Where I used the fact that $a_1 = b_1 = 1$. For the $z^3$ terms, the only way to get $z^3$ on the right is if $l = l$ and $k = 3$ or if $l = 2$ and $k = 1, 2$ (cross multiplied terms) and $l = 3$. This is because 3 is a prime number.

$$b_3 z^3 = a_1 \times (3b_3 z^3) + a_2 \times (b_1 z \times 2b_2 z^2 + 2b_2 z^2 \times b_1 z) + a_3 \times (b_1 z)^3$$
$$\Rightarrow b_3 = 3b_3 + (-b_2)(4b_2) + a_3$$
$$\Rightarrow a_3 = 4b_2^2 - 2b_3$$

4.

$$a_2 = -\frac{2\pi}{\lambda^3} \int_0^{\infty} (e^{-V(r)/kT} - 1) r^2 dr$$

$$= -\frac{2\pi}{\lambda^3} \int_0^{\alpha} (0 - 1) r^2 dr - \frac{2\pi}{\lambda^3} \int_{\alpha}^{\infty} (1 - 1) r^2 dr$$

$$= \frac{2\pi}{\lambda^3} \int_0^{\alpha} r^2 dr$$

$$= \frac{2\pi \alpha^3}{3\lambda^3}$$

5.

$$a_2 = -\frac{2\pi}{\lambda^3} \int_0^{\infty} (e^{-V(r)/kT} - 1) r^2 dr$$

$$= -\frac{2\pi}{\lambda^3} \left[ \int_0^{\alpha} (0 - 1) r^2 dr + \int_{\alpha}^{2\alpha} (e^{v_0} + 1) r^2 dr + \int_{2\alpha}^{\infty} (1 - 1) r^2 dr \right]$$

$$= -\frac{2\pi}{\lambda^3} \left[ \frac{\alpha^3}{3} + (e^{v_0} + 1) \left( \frac{8\alpha^3}{3} - \frac{\alpha^3}{3} \right) \right]$$

$$= -\frac{2\pi \alpha^3 (7 e^{v_0} + 8)}{3\lambda^3}$$