1. a)

$$H_N\{s_i\} = -h \sum_{i=1}^{N} s_i$$

Thus:

$$Z_N = \sum_{s_1 = \pm 1} \sum_{s_2 = \pm 1} \cdots \sum_{s_N = \pm 1} \exp\left\{-\beta h \sum_{i=1}^{N} s_i\right\}$$

$$= \sum_{s_1 = \pm 1} \sum_{s_2 = \pm 1} \cdots \sum_{s_{N-1} = \pm 1} \exp\left\{-\beta h \sum_{i=1}^{N-1} s_i\right\}\left(\sum_{s_N = \pm 1} \exp\{-\beta h s_N\}\right)$$

$$= \sum_{s_1 = \pm 1} \sum_{s_2 = \pm 1} \cdots \sum_{s_{N-1} = \pm 1} \exp\left\{-\beta h \sum_{i=1}^{N-1} s_i\right\}\left(e^{\beta h} + e^{-\beta h}\right)$$

$$= \sum_{s_1 = \pm 1} \sum_{s_2 = \pm 1} \cdots \sum_{s_{N-1} = \pm 1} \exp\left\{-\beta h \sum_{i=1}^{N-1} s_i\right\} 2\cosh(\beta h)$$

$$= 2\cosh(\beta h) \sum_{s_1 = \pm 1} \sum_{s_2 = \pm 1} \cdots \sum_{s_{N-1} = \pm 1} \exp\left\{-\beta h \sum_{i=1}^{N-1} s_i\right\}$$

$$= 2\cosh(\beta h) Z_{N-1}$$

And $Z_1 = e^{\beta h} + e^{-\beta h} = 2\cosh(\beta h)$. It follows that:

$$Z_N = [2\cosh(\beta h)]^N$$

Now for the average magnetization $\langle s_i \rangle$:

$$\langle s_i \rangle = \frac{1}{Z_N} \sum_{s_1 = \pm 1} s_1 \sum_{s_2 = \pm 1} s_2 \cdots \sum_{s_N = \pm 1} s_N \exp\left\{-\beta h \sum_{i=1}^{N} s_i\right\}$$

$$= \frac{1}{Z_N} \sum_{s_1 = \pm 1} s_1 \sum_{s_2 = \pm 1} s_2 \cdots \sum_{s_{N-1} = \pm 1} \exp\left\{-\beta h \sum_{i=1}^{N-1} s_i\right\} \sum_{s_N = \pm 1} s_N \exp\{-\beta h s_N\}$$

$$= \frac{1}{Z_N} \sum_{s_1 = \pm 1} s_1 \sum_{s_2 = \pm 1} s_2 \cdots \sum_{s_{N-1} = \pm 1} \exp\left\{-\beta h \sum_{i=1}^{N-1} s_i\right\} [-2\sinh(\beta h)]$$

$$\vdots$$

$$= \frac{[-2\sinh(\beta h)]^N}{Z_N}$$

$$= [-\tanh(\beta h)]^N$$

2. The cannonical partition function for an ideal gas is:

$$Z_N = \frac{1}{h^3 N!} \int d^{3N}p \, d^{3N}q \, e^{-\beta H}$$

$$= \frac{1}{\lambda^{3N} N!} \int d^{3N}q \exp\left\{-\beta \sum_{i<j} u_{ij}\right\}$$

$$= \frac{1}{\lambda^{3N} N!} \int d^{3N}q \, e^{-\beta \times 0}$$

$$= \frac{1}{\lambda^{3N} N!} \int d^{3N}q$$

$$= \frac{V^N}{\lambda^{3N} N!}$$

Since there is no interaction between ideal gas molecules. We can use this to evaluate the grand partion function for an ensemble of ideal gas samples:

$$\mathfrak{Z} = \sum_{N=0}^{\infty} z^N \frac{V^N}{\lambda^{3N} N!}$$

$$= \sum_{N=0}^{\infty} \frac{(zV/\lambda^3)^N}{N!}$$

$$= e^{zV/\lambda^3}$$

We also have that:

$$P = \frac{k_B T}{V} \ln \mathfrak{Z} = z\frac{k_B T}{\lambda^3} \implies \frac{P}{k_B T} = \frac{z}{\lambda^3}$$

and:

$$\langle N \rangle = z\frac{\partial}{\partial z}\left[\frac{zV}{\lambda^3}\right] = \frac{zV}{\lambda^3} \implies \frac{\langle N \rangle}{V} = \frac{z}{\lambda^3}$$

From which we deduce:

$$\frac{P}{k_B T} = \frac{\langle N \rangle}{V} \implies PV = \langle N \rangle k_B T$$

3. We have:

$$\frac{1}{\lambda^3} \sum_{l=1}^{\infty} b_l z^l = \frac{P}{k_B T} = \frac{1}{\lambda^3} \sum_{l=1}^{\infty} a_l \times \left(\lambda^3 \frac{\langle N \rangle}{V}\right)^l$$

$$\Rightarrow \sum_{l=1}^{\infty} b_l z^l = \sum_{l=1}^{\infty} a_l \times \left(\sum_{k=1}^{\infty} k b_k z^k\right)^l$$

Now we match coefficients of z. The only way to get a z term on the right handside is if $l = k = 1$:

$$b_1 z = a_1 \times b_1 z \Rightarrow a_1 = 1$$

Also note that:

$$b_1 = \frac{1}{V} \int_V d^3 r = 1 \Rightarrow a_1 = b_1 = 1$$

For the z^2 terms, the only way to get z^2 on the right is if $l = 1$ and $k = 2$ or if $l = 2$ and $k = 1$

$$b_2 z^2 = a_1 \times (2b_2 z^2) + a_2 \times (b_1 z^1)^2$$
$$\Rightarrow b_2 = 2b_2 + a_2$$
$$\Rightarrow a_2 = -b_2$$

Where I used the fact that $a_1 = b_1 = 1$. For the z^3 terms, the only way to get z^3 on the right is if $l = l$ and $k = 3$ or if $l = 2$ and $k = 1, 2$ (cross multiplied terms) and $l = 3$. This is because 3 is a prime number.

$$b_3 z^3 = a_1 \times (3b_3 z^3) + a_2 \times (b_1 z \times 2b_2 z^2 + 2b_2 z^2 \times b_1 z) + a_3 \times (b_1 z)^3$$
$$\Rightarrow b_3 = 3b_3 + (-b_2)(4b_2) + a_3$$
$$\Rightarrow a_3 = 4b_2^2 - 2b_3$$

4. We wish to find $a_2 = -b_2$, to this end note that:

$$b_2 = \frac{\lambda^3}{2!V\lambda^{3\times 2}} \int d^3q_1 \int d^3q_2 f_{12}$$

$$= \frac{1}{2V\lambda^3} \int d^3q_1 \int d^3(q_2 - q_1) \left[e^{-\beta V(|q_2 - q_1|)} - 1 \right]$$

$$= \frac{1}{2V\lambda^3} \int d^3q_1 \int d^3x \left[e^{-\beta V(|x|)} - 1 \right]$$

Where $x = q_2 - q_1$. Provided that the interaction range of the potential is short the integrals can be treated as approximately independent. This follows from the fact that if q_1 starts differing largely from q_2, the integrand dependent on x becomes negligible ($V \to 0$). Thus the integral over x cannot vary much with changing q_1. Hence:

$$b_2 \approx \frac{1}{2V\lambda^3} \left(\int d^3q_1 \right)\left(\int d^3x \left[e^{-\beta V(|x|)} - 1 \right] \right)$$

$$= \frac{1}{2\lambda^3} \int d^3x \left[e^{-\beta V(|x|)} - 1 \right]$$

$$= \frac{1}{2\lambda^3} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{\infty} r^2 dr \left[e^{-\beta V(|r|)} - 1 \right]$$

$$= \frac{2\pi}{\lambda^3} \int_0^{\infty} r^2 dr \left[e^{-\beta V(|r|)} - 1 \right]$$

Now we can use this to evaluate a_2.

$$a_2 = -b_2 = -\frac{2\pi}{\lambda^3} \int_0^{\infty} (e^{-\beta V(r)} - 1) r^2 dr$$

$$= -\frac{2\pi}{\lambda^3} \int_0^{\alpha} (0 - 1) r^2 dr - \frac{2\pi}{\lambda^3} \int_{\alpha}^{\infty} (1 - 1) r^2 dr$$

$$= \frac{2\pi}{\lambda^3} \int_0^{\alpha} r^2 dr$$

$$= \frac{2\pi\alpha^3}{3\lambda^3}$$

5. Doing a similar calculation to the above:

$$a_2 = -\frac{2\pi}{\lambda^3} \int_0^{\infty} (e^{-\beta V(r)} - 1) r^2 dr$$

$$= -\frac{2\pi}{\lambda^3} \left[\int_0^{\alpha} (0 - 1) r^2 dr + \int_{\alpha}^{2\alpha} (e^{\beta v_0/kT} - 1) r^2 dr - \int_{2\alpha}^{\infty} (1 - 1) r^2 dr \right]$$

$$= -\frac{2\pi}{\lambda^3} \left[-\frac{\alpha^3}{3} + (e^{\beta v_0} - 1)\left(\frac{8\alpha^3}{3} - \frac{\alpha^3}{3} \right) \right]$$

$$= -\frac{2\pi\alpha^3(7e^{v_0/kT} - 8)}{3\lambda^3}$$

It can been seen from this that a_2 is negative for low T, and is positive for high T when the exponential is large and small respevtively. Thus a_2 does change sign with temperature.