An infinite limit of Conway's Soldiers

The fifth row but no further

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SMSAS PGR seminar

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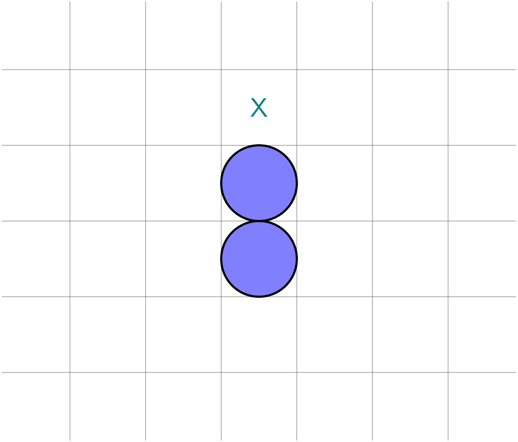
Conway's Soldiers – review

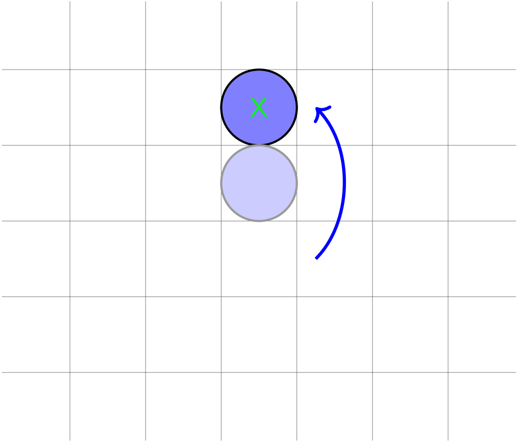
- Conway's Soldiers is a game similar to English draughts or chequers, played on an infinite chequerboard grid. Mathematically, we can think of it taking place on the lattice $\mathbb{Z}^2$.
- Pieces are placed on squares of the board, or equivalently on points $(x, y) \in \mathbb{Z}^2$.
- Two pieces are neighbours if they are one square apart vertically or horizontally (but not diagonally), i.e. if exactly one of their coordinates differs by exactly 1.
- A piece can move by jumping vertically or horizontally over a neighbouring piece. This removes the piece that is jumped over from the game.

Note: most of the slides in this talk will just be diagrams, with me talking over them!

Conway's Soldiers: example move



Conway's soldiers: example move



Conway's Soldiers: the problem

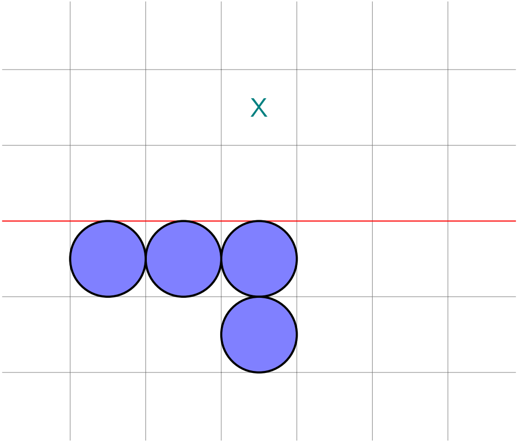
Suppose we restrict the starting state of the game so that all initially existent pieces are in the lower half plane (or have y-coordinate below some fixed value).

Question

How high up is it possible to get a piece using only legal moves from such a starting configuration?

Some examples of optimal starting configurations follow for getting two, three and four rows up respectively:

Two rows up (four pieces)



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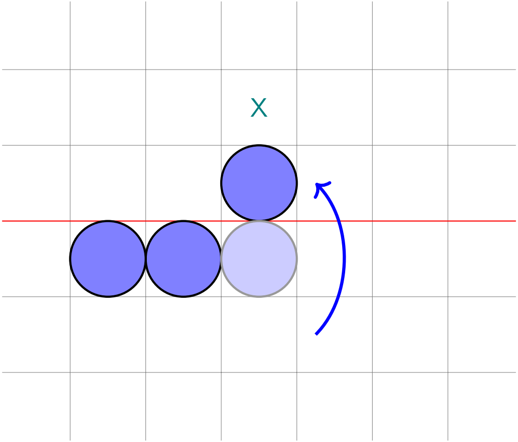

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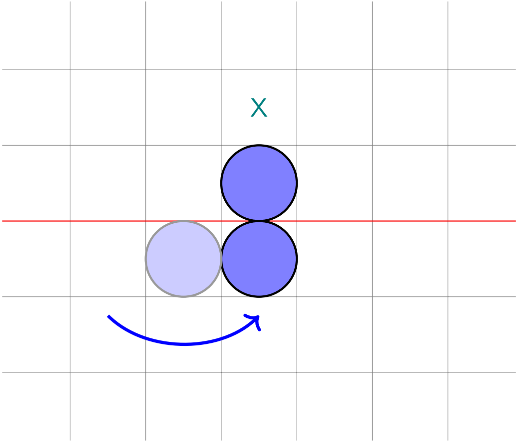

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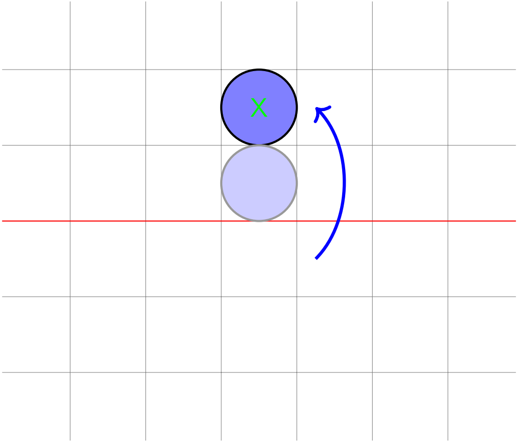
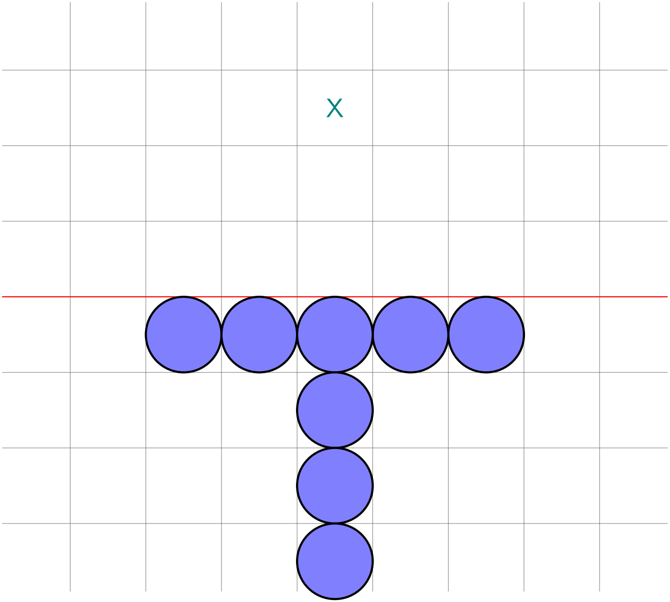
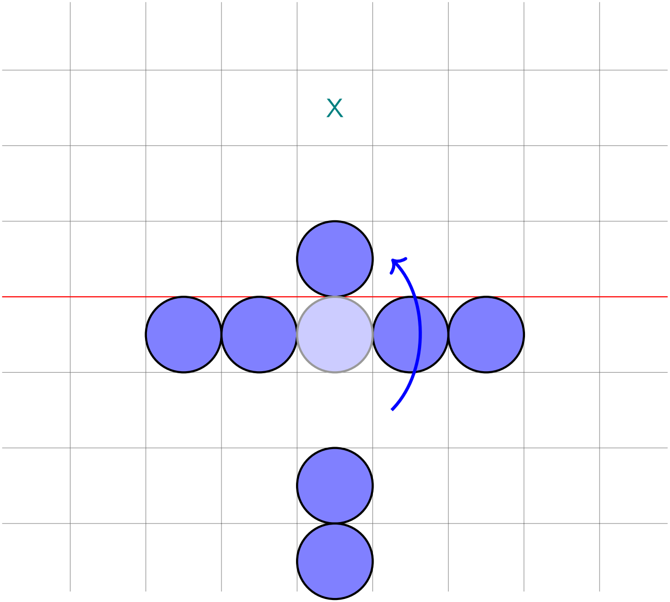


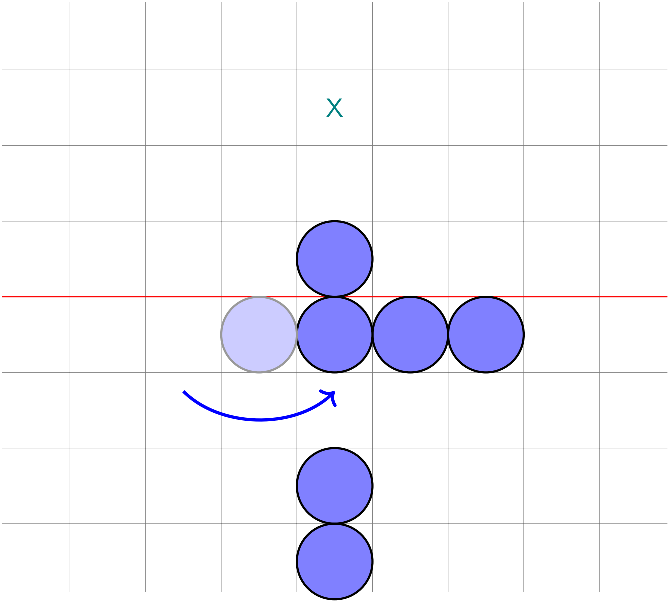
Three rows up (eight pieces)



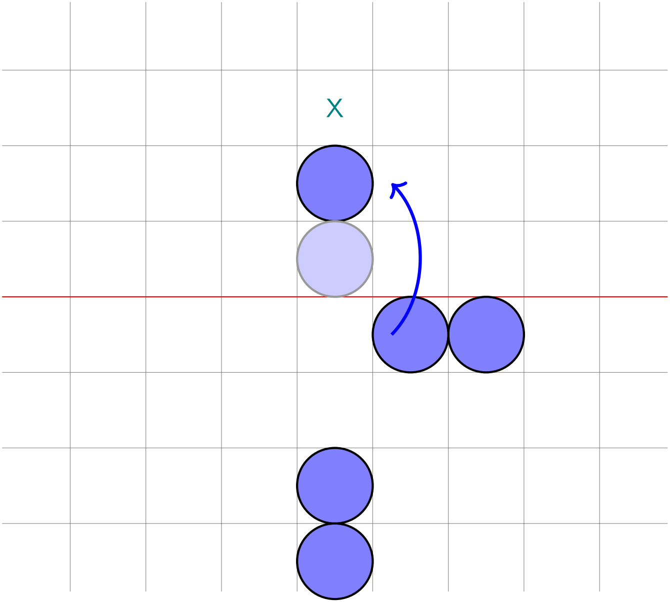
Three rows up (eight pieces)



Three rows up (eight pieces)



Three rows up (eight pieces)



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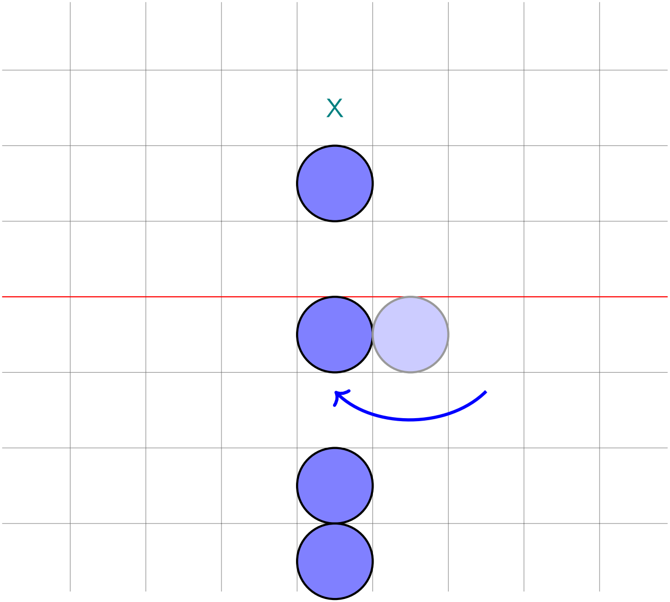

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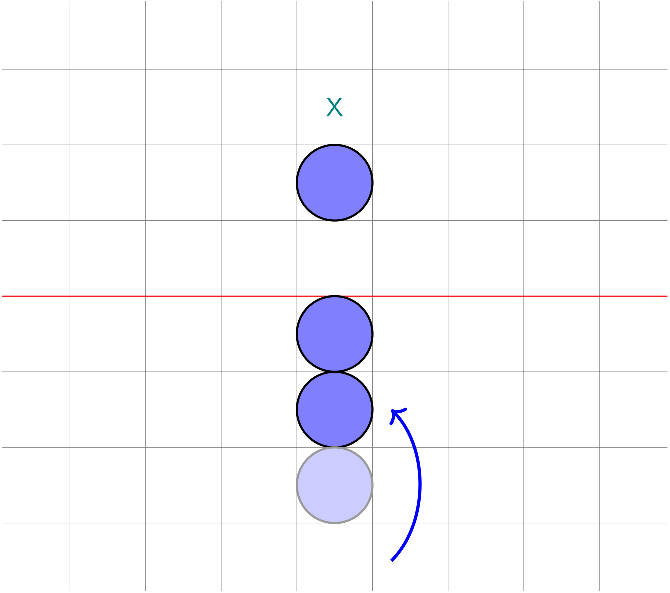

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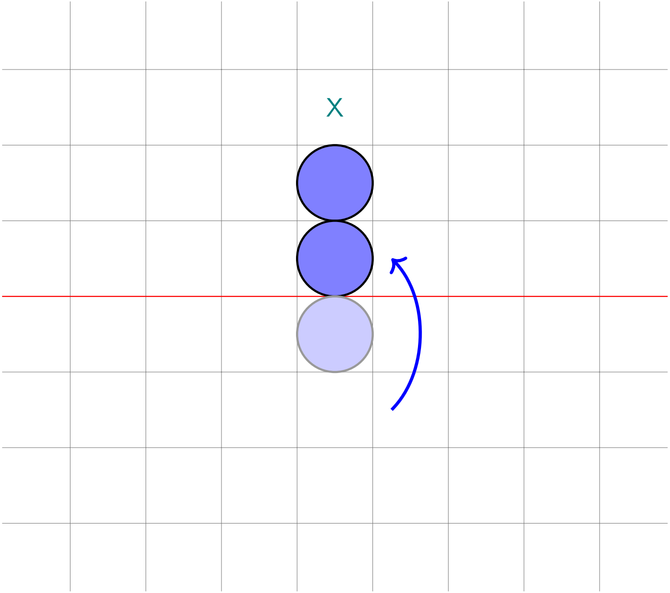

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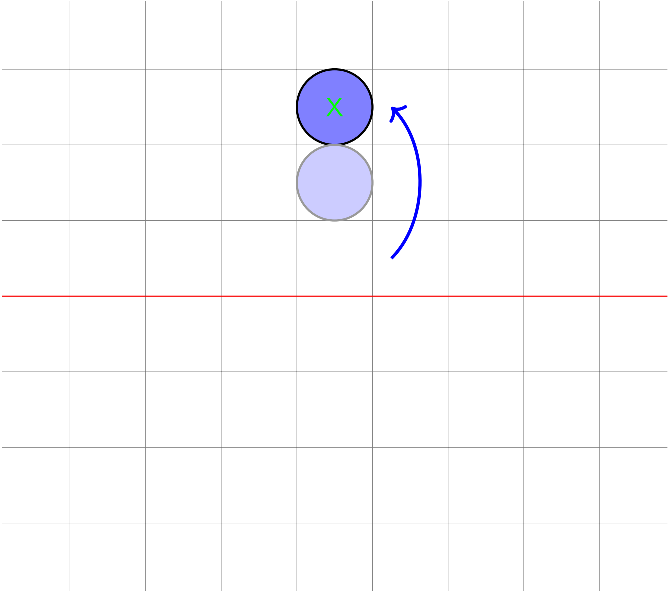
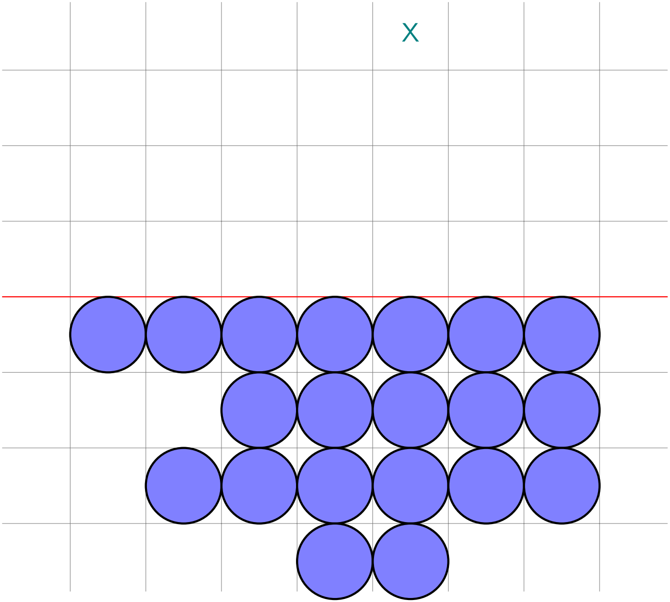


Four rows up (20 pieces)



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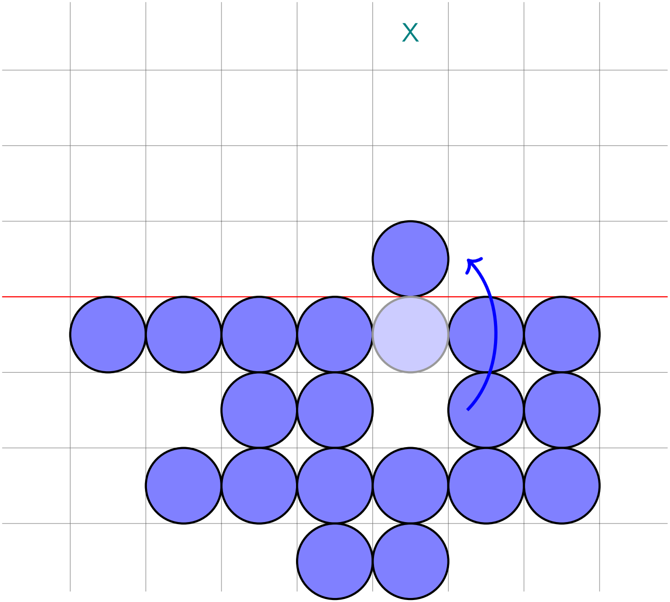

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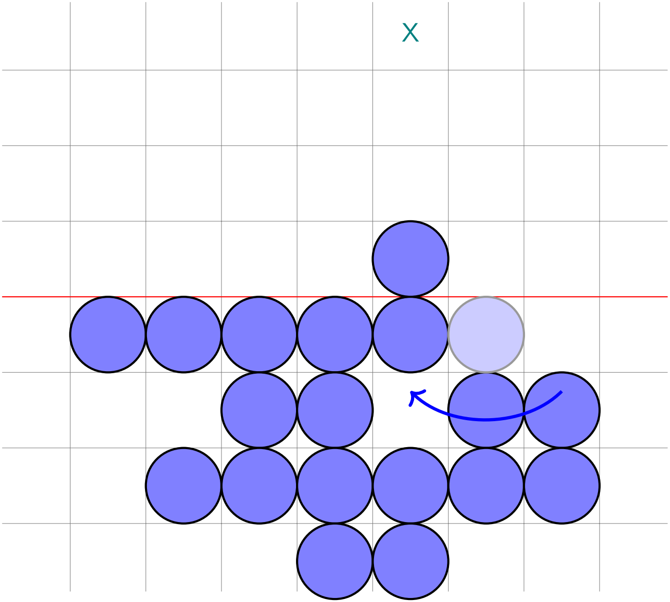

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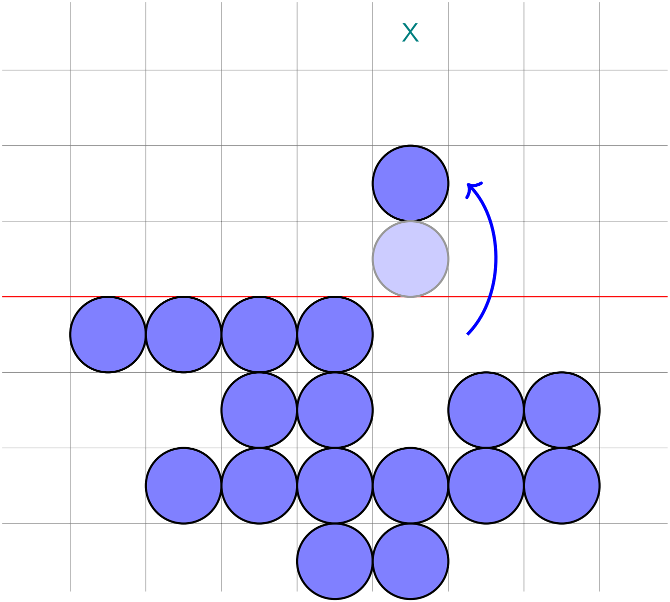

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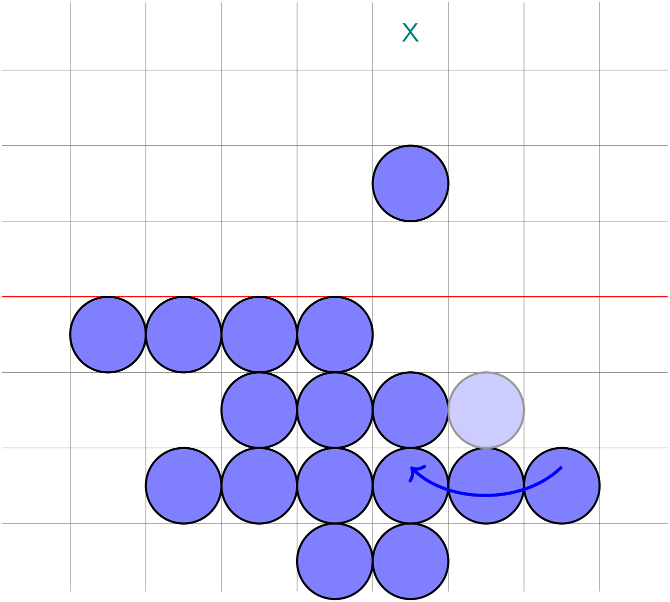

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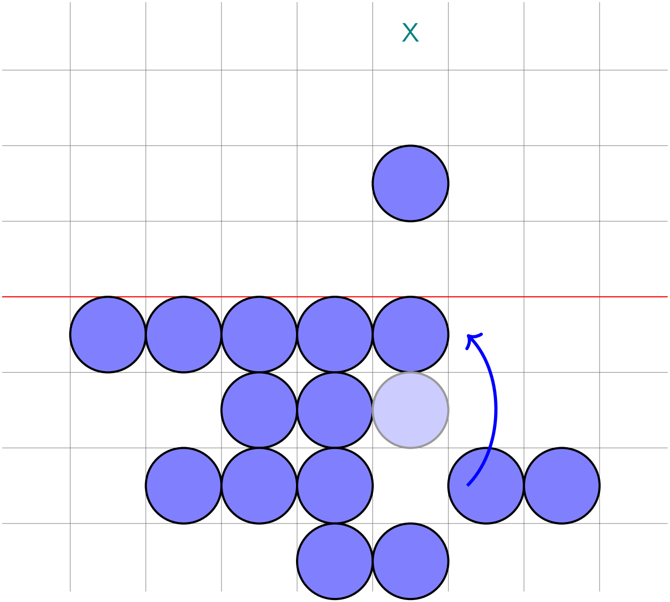

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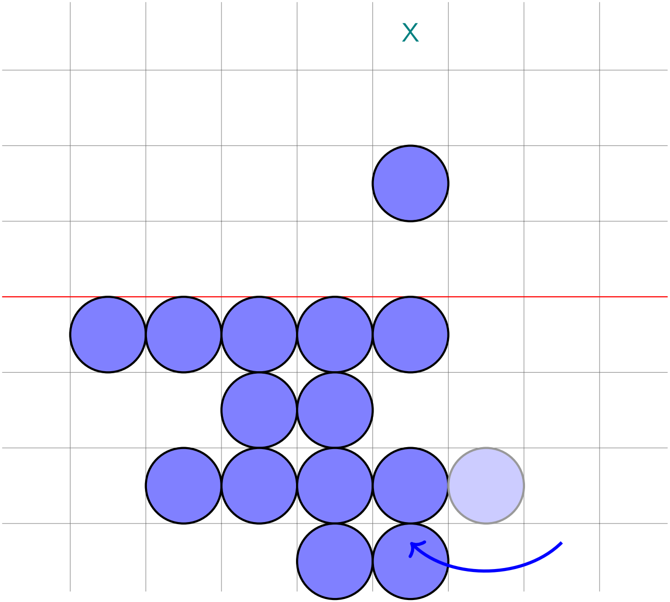

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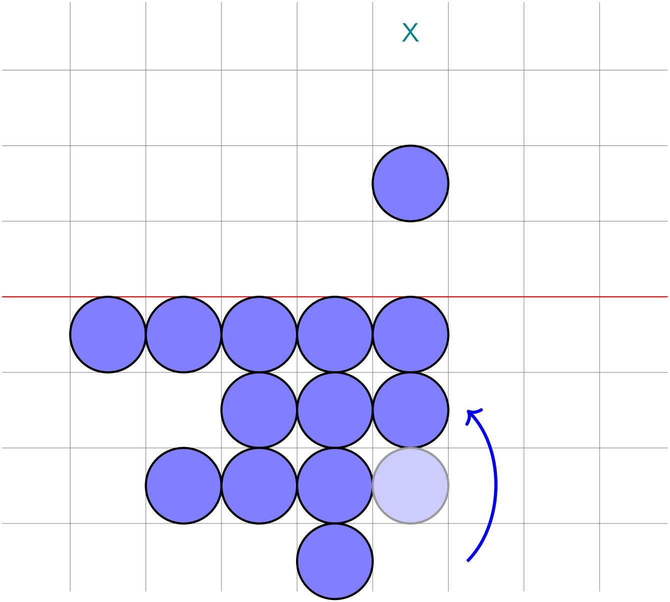

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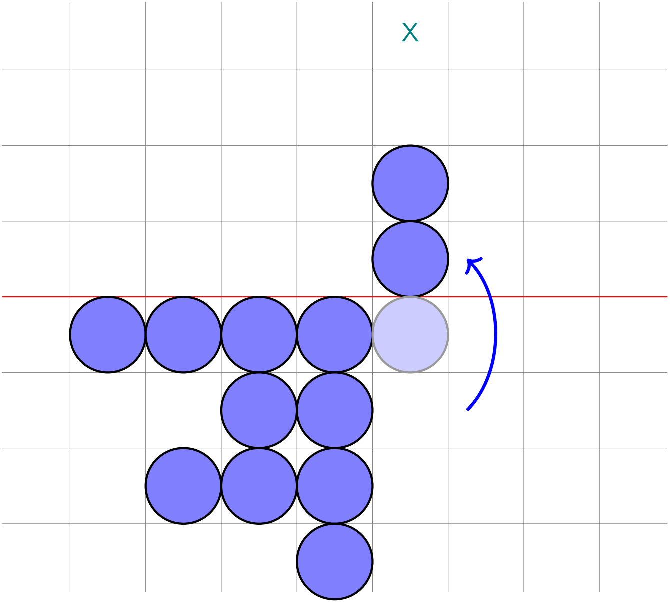

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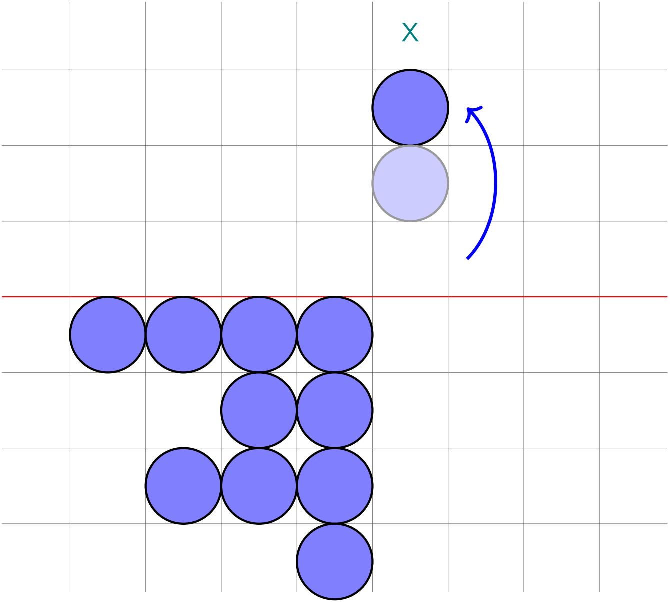

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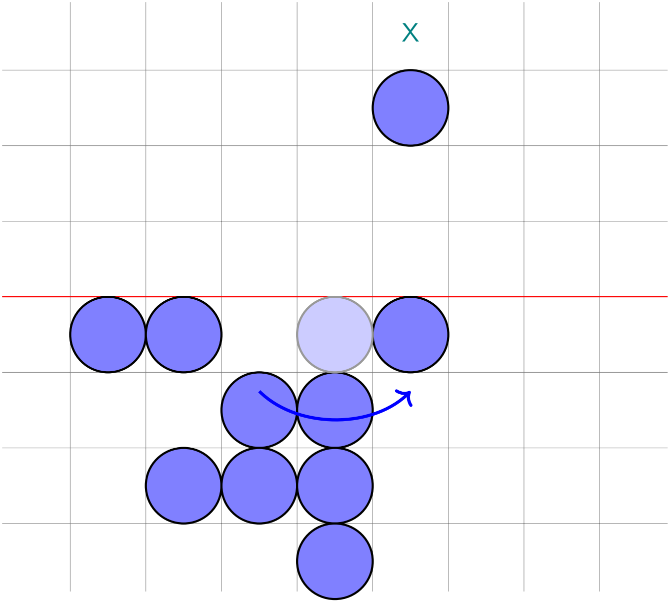

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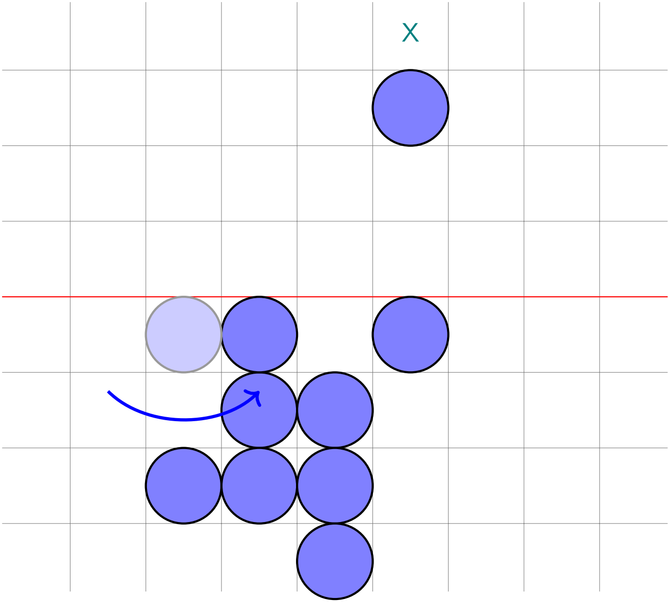

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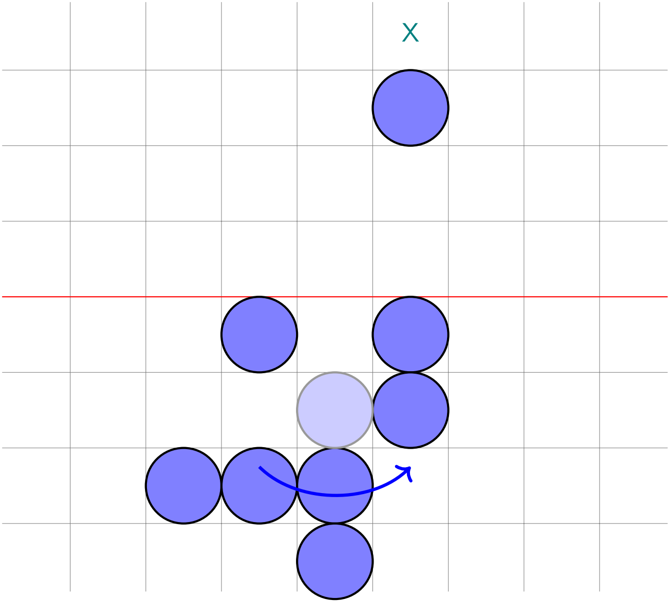

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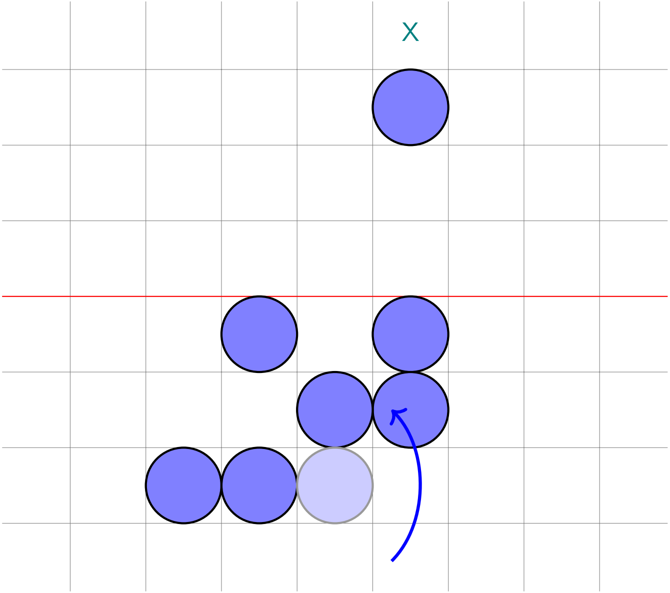

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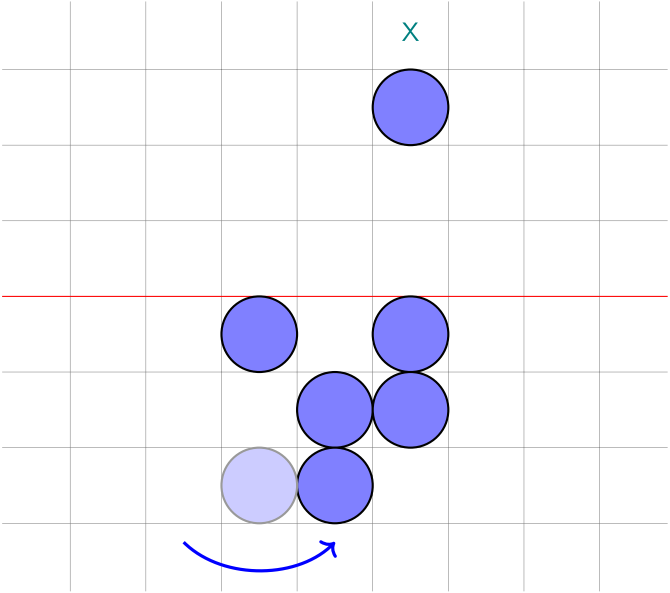

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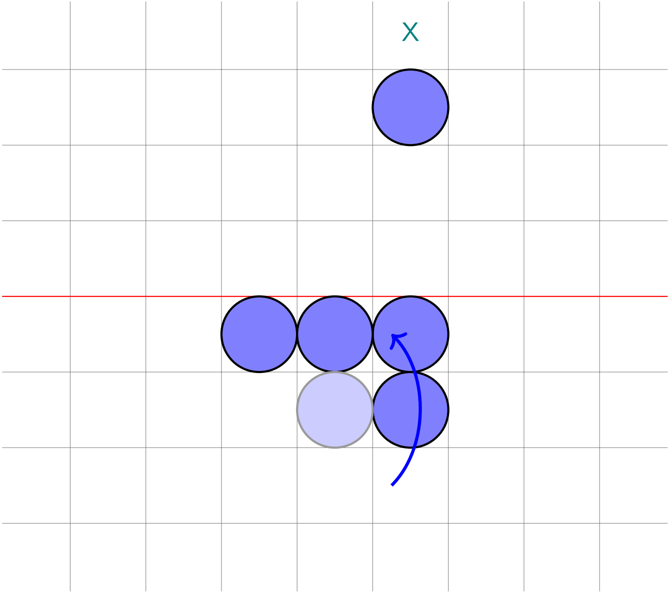

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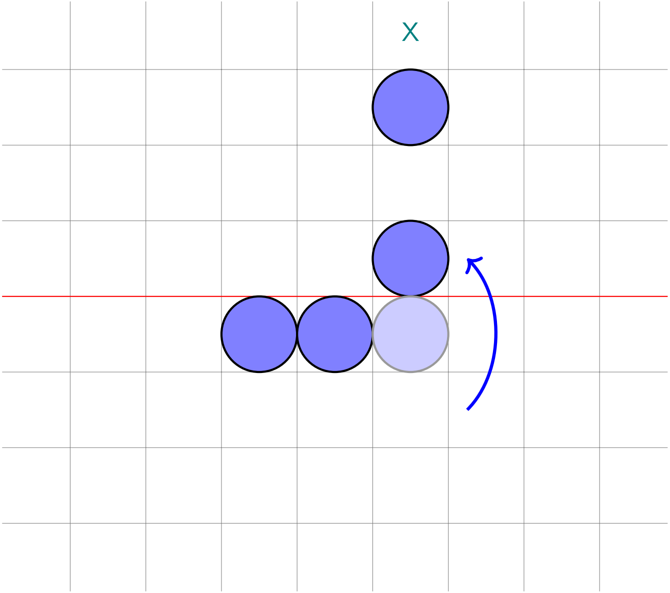

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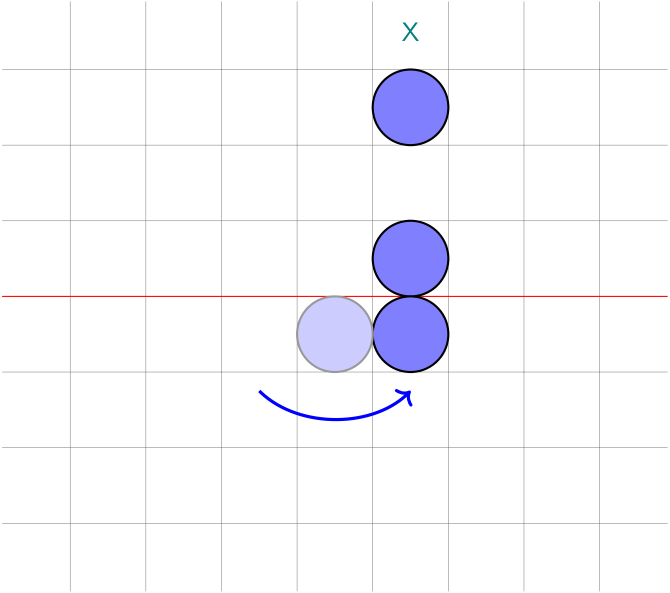

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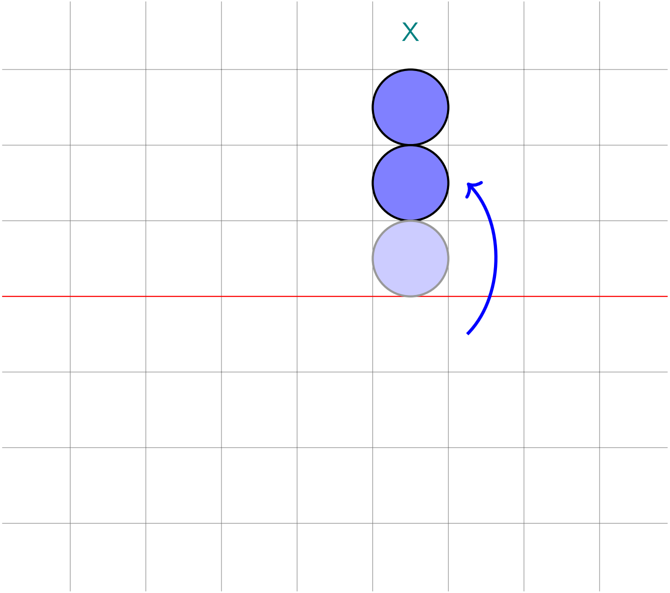

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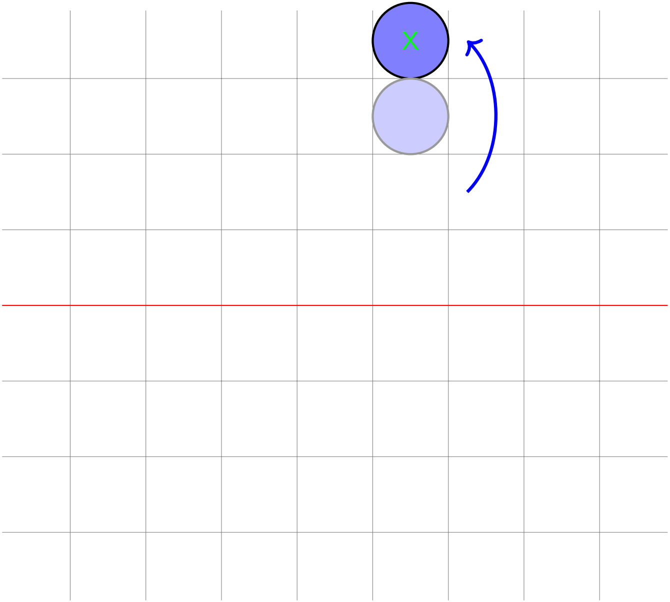
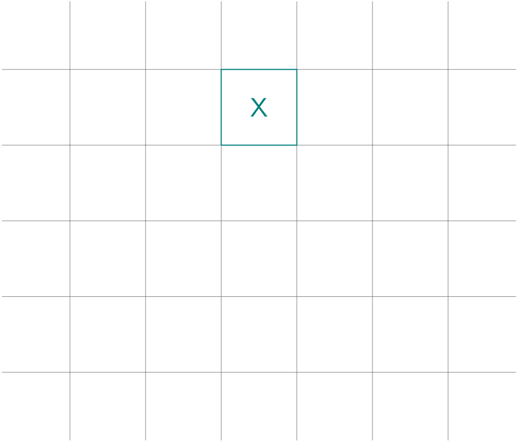


Five rows up?

- It's not possible to reach the fifth row with a finite number of starting pieces.
- Even if the lower half plane is completely filled with pieces, it's not possible to reach the fifth row in a finite number of legal moves.

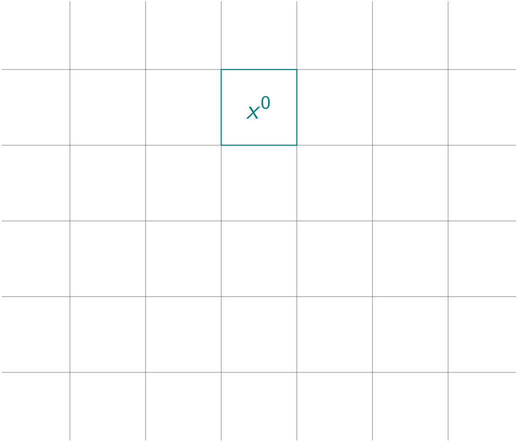
To prove this, we seek a method of assigning a value called a *weight* to each square cell. We will give configurations of pieces a *score* according to the total weight of occupied cells.

Assigning a weight to the cells



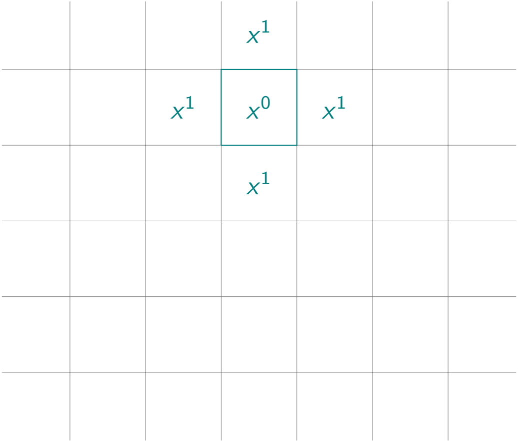
We will assign weights by specifying a target cell. (Our aim is to occupy this target cell after a number of moves.)

Assigning a weight to the cells



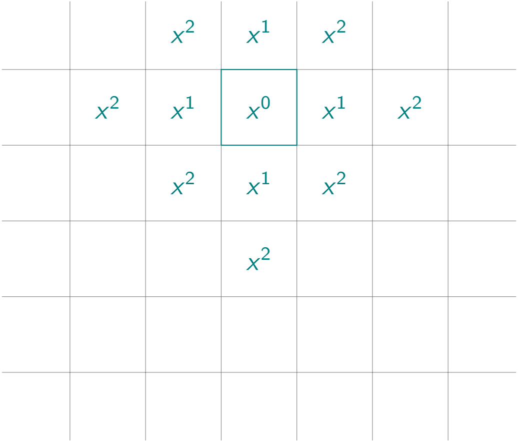
The target cell is assigned weight 1, which may be expressed x^0 for some as yet unknown value x.

Assigning a weight to the cells



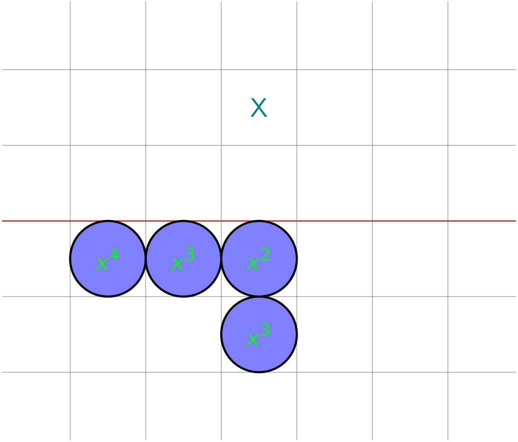
The target cell's neighbours are assigned a weight of x^1 for the same unspecified x.

Assigning a weight to the cells



All other cells are assigned a weight in powers of the value x according to their distance from the target measured through neighbours.

The score of a configuration

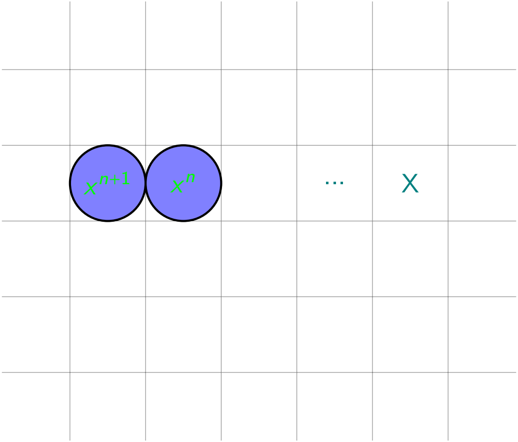


This configuration has the score $x^2 + 2x^3 + x^4$.

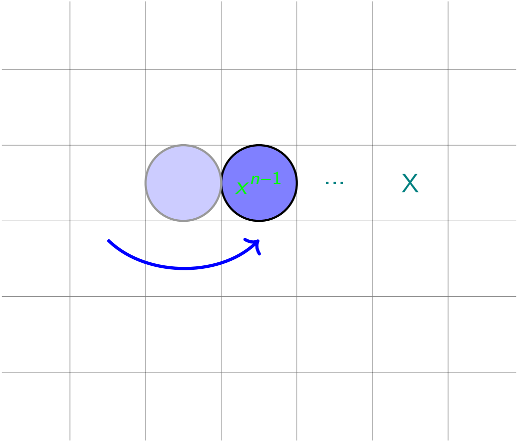
What value for x?

Our goal is to fix a specific value of the weight x which will allow us to learn something about the problem of reaching the fifth row up. To work out what value will be useful, we consider how the score changes when we perform a legal move.
We see three types of legal move in terms of how the score changes: moves towards the target, moves away from the target, and neutral moves where the moving piece is the same distance from the target before and after the move.

Moving towards the target

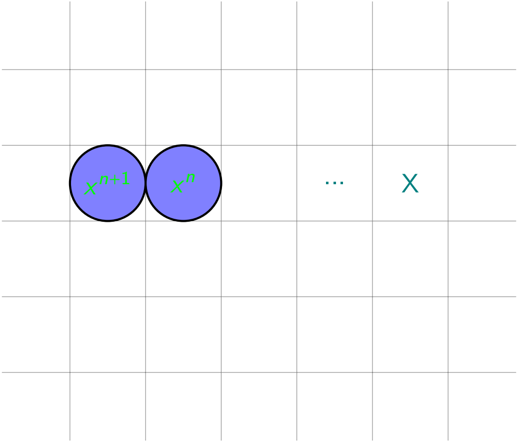


Moving towards the target

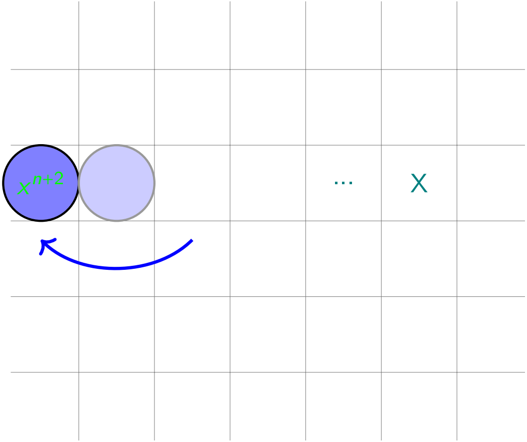


A move towards the target changes the score by
$x^{n-1} - x^n - x^{n+1} = x^{n-1}(1 - x - x^2)$.

Moving away from the target

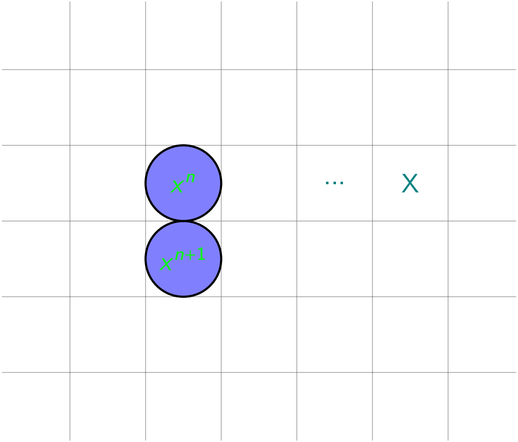


Moving away from the target

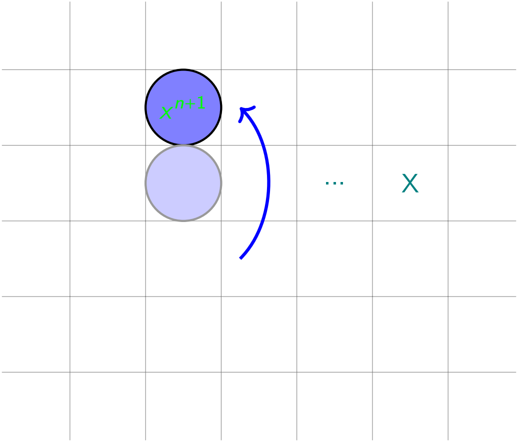


A move away from the target changes the score by $x^{n+2} - x^{n+1} - x^n = x^n(-1 - x + x^2)$.

Neutral moves



Neutral moves



A neutral move changes the score by $x^{n+1} - x^n - x^{n+1} = -x^n$.

The choice of weight x

We observe that, if $x \geq 0$, the score of any configuration must be non-negative. Thus it may be useful if *legal moves can only decrease the score*.
From the three types of moves, we see that for this to hold, x must satisfy the inequalities

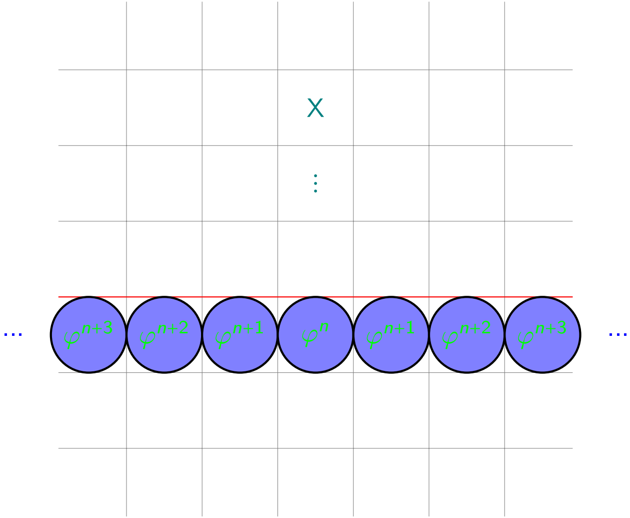
1. $1 - x - x^2 \leq 0$
2. $x \geq 0$
3. $x^2 - x - 1 \leq 0$

Thankfully, (2.) is consistent with the condition on non-negative scores. Solving the remaining quadratic equations for their positive roots gives

$$\frac{\sqrt{5}-1}{2} \leq x \leq \frac{\sqrt{5}+1}{2}$$

We're getting the idea that we want scores to be as small as possible. So let's pick $x = \varphi := \frac{\sqrt{5}-1}{2} \approx 0.61803\ldots$. Note in particular that $\varphi^2 = 1 - \varphi$.

The score of a full row



The score of a full infinite row

A full row with the target cell n rows above has the score

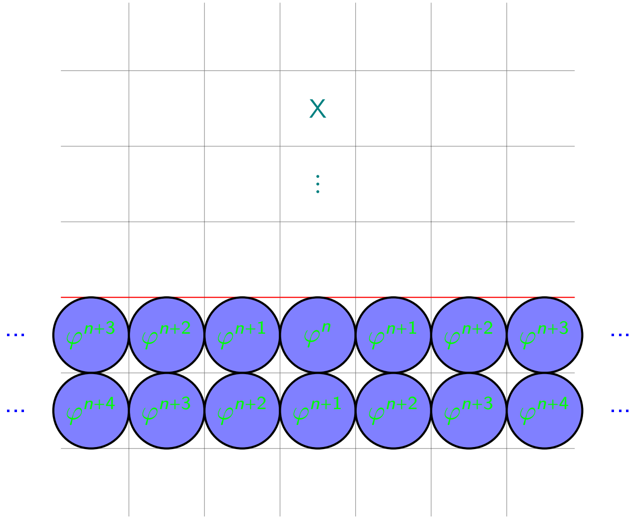
$$\begin{aligned}\varphi^n + 2\varphi^{n+1} + 2\varphi^{n+2} + \dots &= \varphi^n \left(1 + 2\varphi + 2\varphi^2 + \dots\right) \\ &= \varphi^n \left(1 + 2\sum_{i=1}^{\infty} \varphi^i\right)\end{aligned}$$

Either evaluate the geometric series, or notice that

$$\begin{aligned}\varphi^2 &= 1 - \varphi \\ \varphi^3 &= \varphi - \varphi^2 \\ \varphi^4 &= \varphi^2 - \varphi^3 \dots\end{aligned}$$

So $\varphi^2 + \varphi^3 + \varphi^4 + \cdots = 1$, and the infinite row has the score $\varphi^n (3 + 2\varphi)$.

Adding a second full infinite row



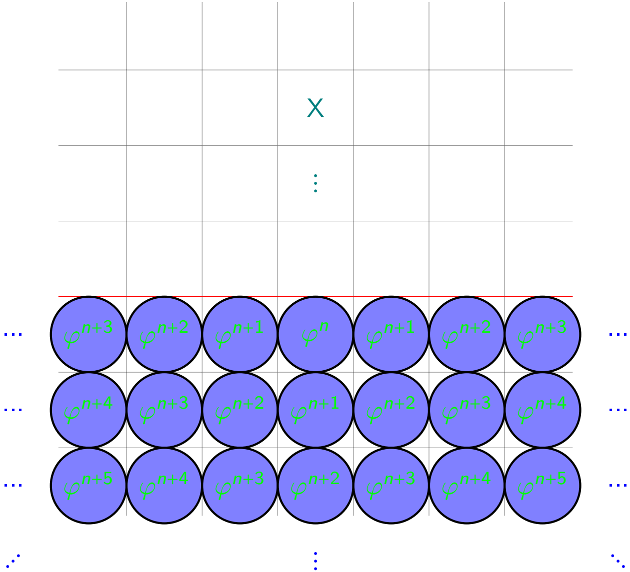
Adding a second full infinite row

Adding a second infinite row below the first multiplies the score by $1 + \varphi$. With two infinite rows, the score is

$$\begin{aligned} \varphi^n(3+2\varphi)(1+\varphi) &= \varphi^n(3+5\varphi+2\varphi^2) \\ &= \varphi^n(5+3\varphi) \end{aligned}$$

Note that we can eliminate all powers of φ greater than 1 by successively replacing φ^2 with $1 - \varphi$.

Filling the lower half plane

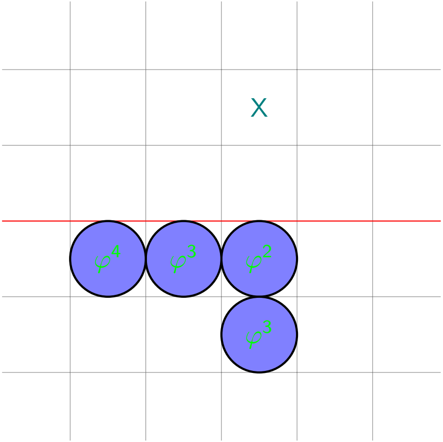


Filling the lower half plane

Suppose instead we fill every cell in the lower half plane. Then the first row's score gets multiplied by $1 + \varphi + \varphi^2 + \varphi^3 + \cdots = 2 + \varphi$.
So the full half-plane has score

$$\begin{aligned} \varphi^n(3 + 2\varphi)(2 + \varphi) &= \varphi^n(6 + 7\varphi + 2\varphi^2) \\ &= \varphi^n(8 + 5\varphi). \end{aligned}$$

An example revisited



Explicitly, the score of this configuration is

$$\begin{aligned}\varphi^2 + 2\varphi^3 + \varphi^4 &= \varphi^2 + 2\varphi^3 + \varphi^2(1-\varphi) \\ &= 2\varphi^2 + \varphi^3 \\ &= 2\varphi^2 + \varphi(1-\varphi) \\ &= \varphi + \varphi^2 \\ &= 1\end{aligned}$$

This is the sense in which this configuration is *optimal* for getting two rows up.

Maximum possible scores

Now we can calculate the maximum possible scores for the filled infinite half-plane depending on the height n of the target square. Let S_n be the score of the filled half-plane when the target is at height n. Then:

$$
\begin{aligned}
S_1 &= \varphi(8 + 5\varphi) = 5 + 3\varphi \\
S_2 &= \varphi(5 + 3\varphi) = 3 + 2\varphi \\
S_3 &= \varphi(3 + 2\varphi) = 2 + \varphi \\
S_4 &= \varphi(2 + \varphi) = 1 + \varphi \\
S_5 &= \varphi(1 + \varphi) = 1
\end{aligned}
$$

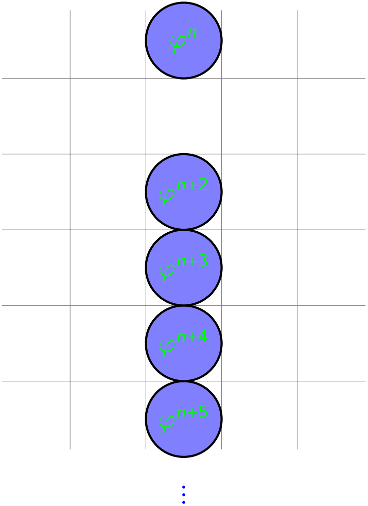
$S_5 = 1$ confirms the impossibility of getting to the fifth row with a finite number of pieces, i.e. in a finite number of moves.

A suggestive limit

It's interesting to consider how we can relax the rules to allow infinite moves in a way that makes sense. The convergent series

$$\varphi^2 + \varphi^3 + \cdots = 1$$

suggests that this may be possible in some sense: an infinite row of pieces starting at position φ^{n+2} is "worth as much" as a single piece with score φ^n.



Relaxing the rules

We follow Tatham and Taylor's[1][2] treatment of infinite moves.

Starting question

What sort of "infinite move" should we allow?

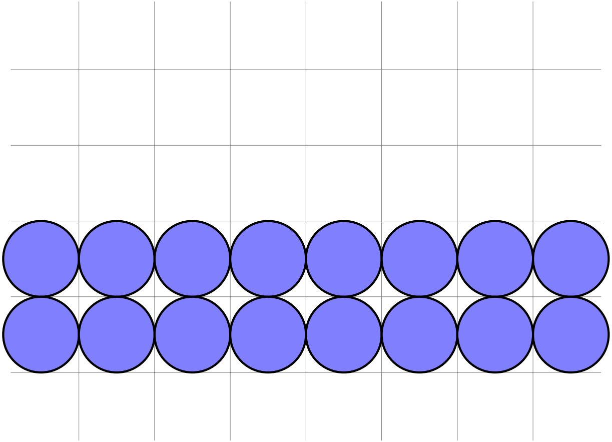
Two possibilities come to mind:

1. Moving pieces in parallel,
2. Moving pieces in sequence.

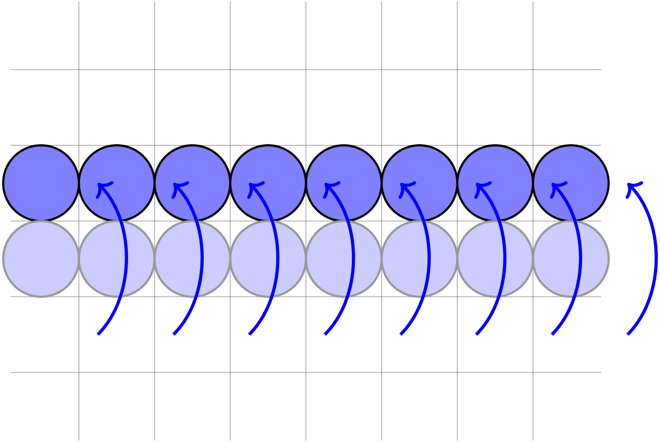
[1] S. Tatham, *Reaching row 5 in Solitaire Army*, `https://www.chiark.greenend.org.uk/~sgtatham/solarmy/`

[2] S. Tatham & G. Taylor, *Reaching row five in solitaire army*. `https://tartarus.org/gareth/maths/stuff/solarmy.pdf`

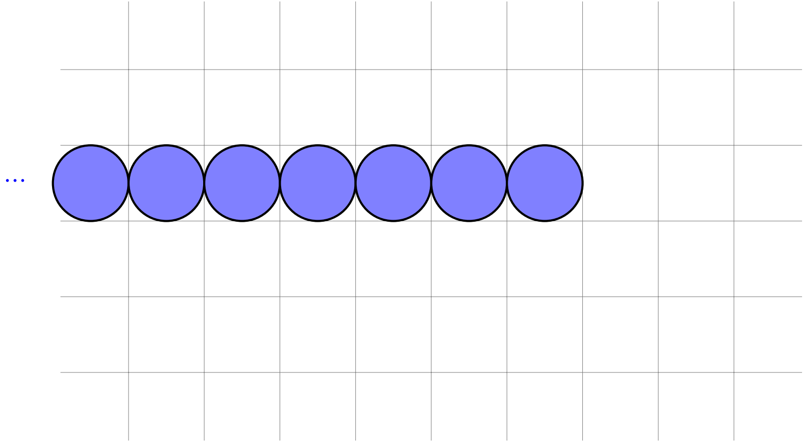
Moving in parallel



Moving in parallel



Moving in sequence



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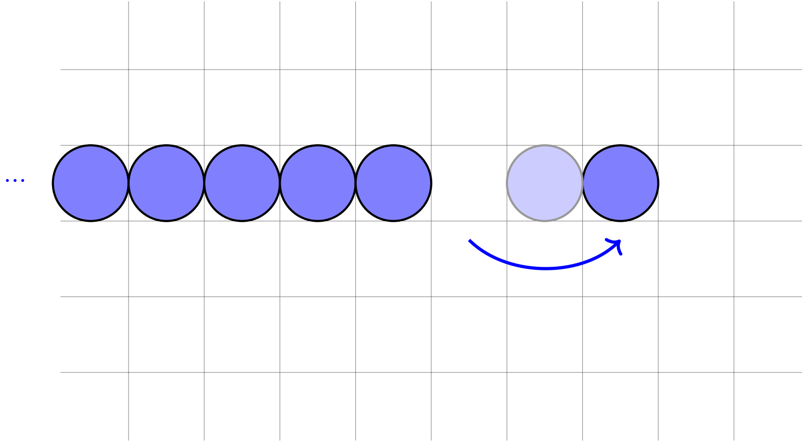
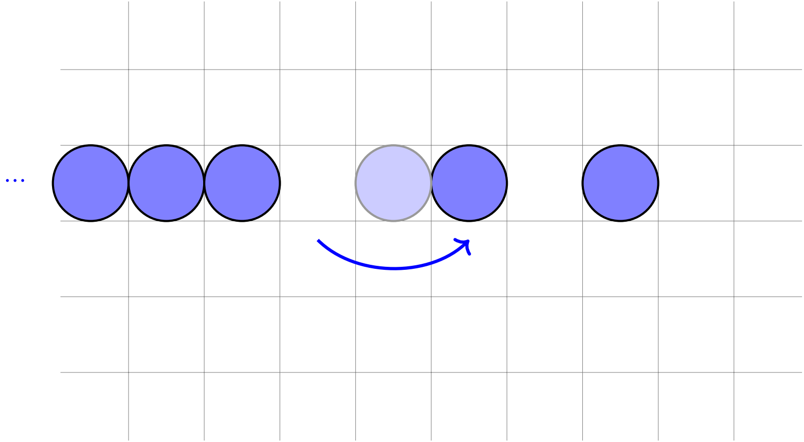
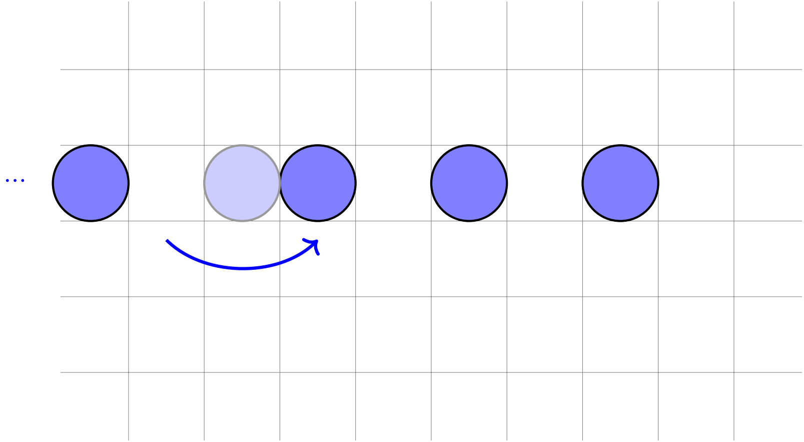


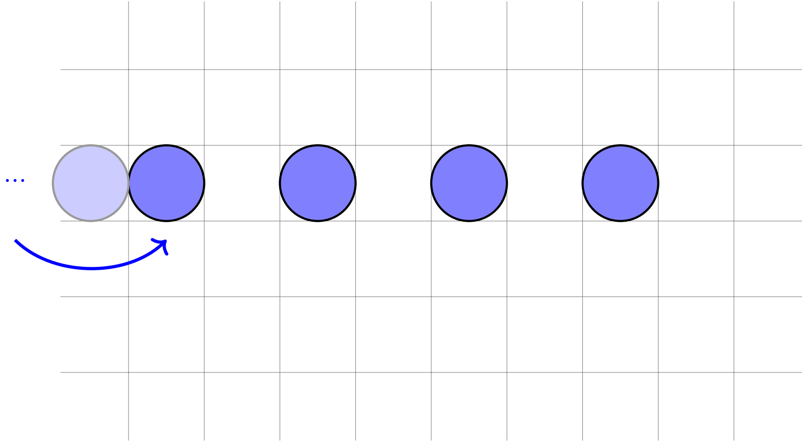
Moving in sequence



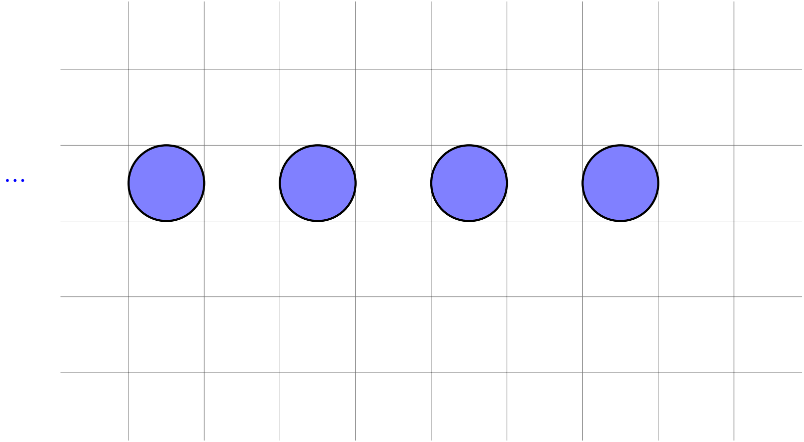
Moving in sequence



Moving in sequence



Moving in sequence



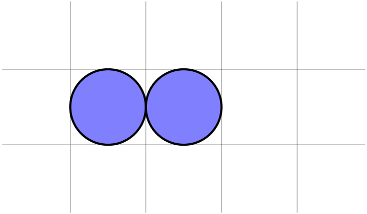
Moving in sequence

We will allow such an infinite sequence of moves to terminate in finite time, all taking place between times $t = 0$ and $t = 1$. For example, we might say that each move "accelerates":

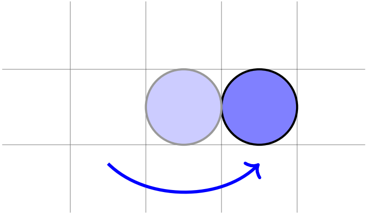
- the first jump takes place at $t = \frac{1}{2}$,
- the second jump takes place at $t = \frac{3}{4}$,
- the third jump takes place at $t = \frac{7}{8}$ ...

At time $t = 1$ the configuration sequence has completed and we can consider making other legal moves at times $t \geq 1$.
This is the only new move we add to the game.

Viewing the game in reverse



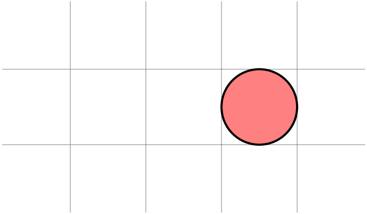
$t = t_0$



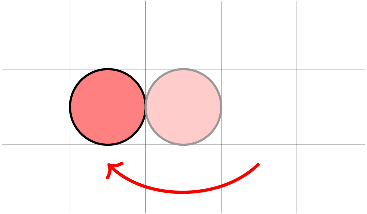
$t = t_0 + 1$

The game rules tell us what a legal move is by comparing the state of the board at two time steps, say $t = t_0$ and $t = t_0 + 1$.
One interpretation is that, as time *increases*, pieces are *removed* from the board.
But we could equally interpret it as, as time *decreases*, pieces are *added* to the board.

Viewing the game in reverse



$t = t_0'$

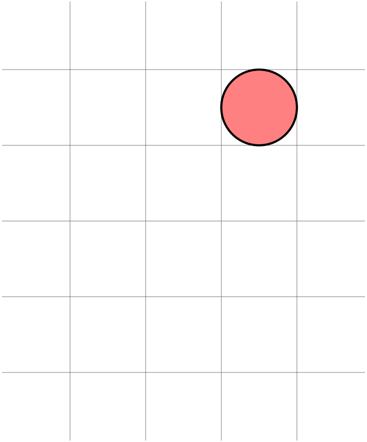


$t = t_0' - 1$

The game rules tell us what a legal move is by comparing the state of the board at two time steps, say $t = t_0$ and $t = t_0 + 1$.
One interpretation is that, as time *increases*, pieces are *removed* from the board.
But we could equally interpret it as, as time *decreases*, pieces are *added* to the board.

Viewing the game in reverse

Note that this is not adding anything new to the game! Every legal move removing a piece in the "forwards" game is equivalent to a legal move adding a piece in the "reversed" game, and vice versa.



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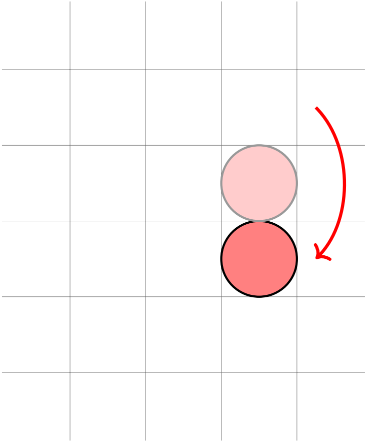

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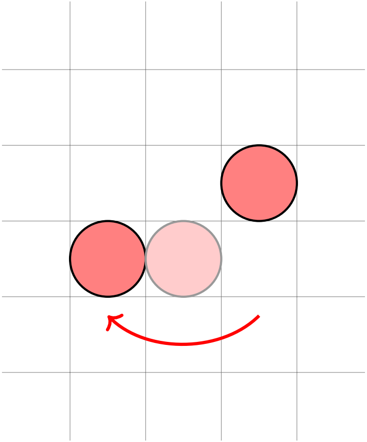

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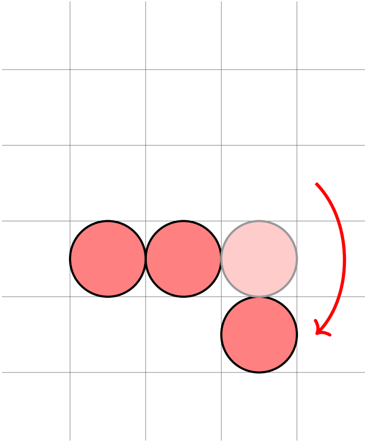

Viewing the game in reverse

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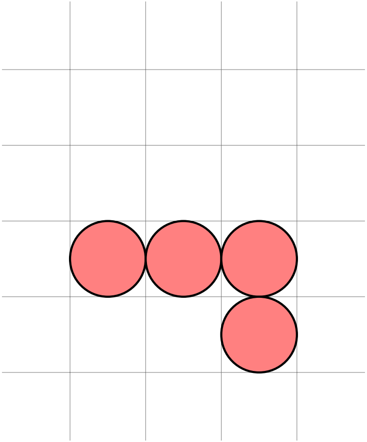
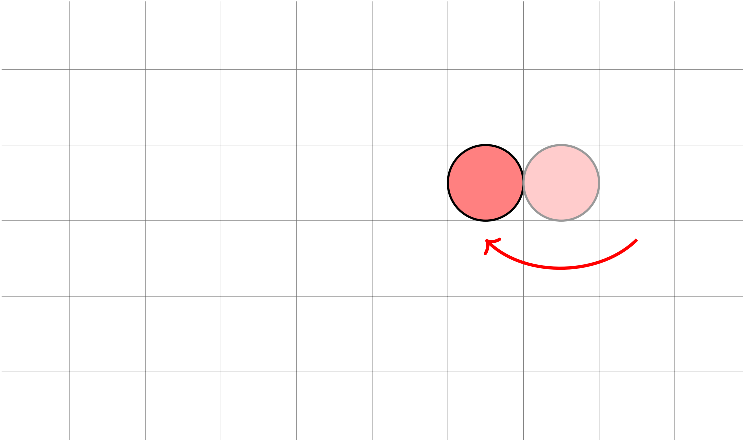


An infinite converging sequence in the reversed game

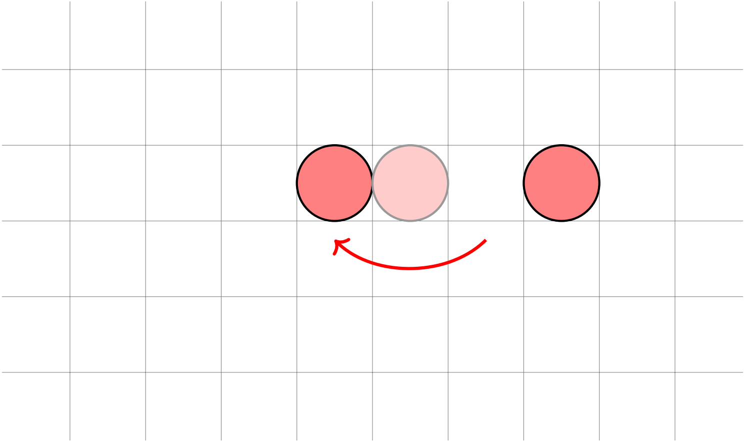
$t > t_0$

An infinite converging sequence in the reversed game



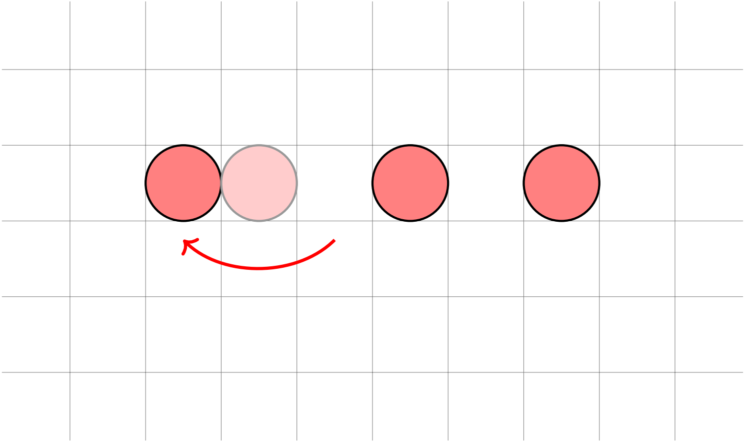
$t = t_0$

An infinite converging sequence in the reversed game



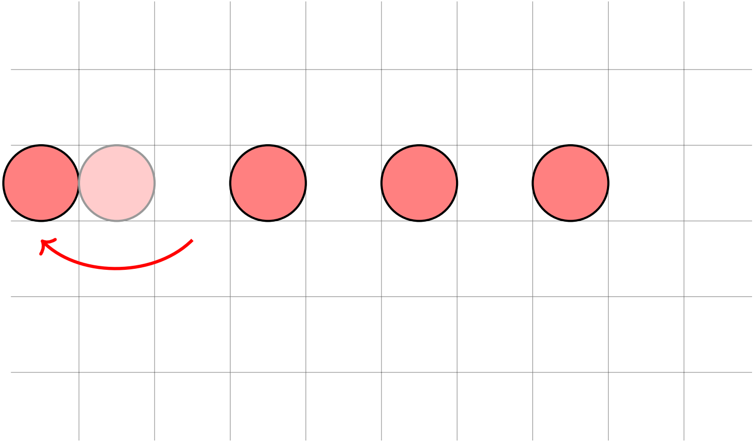
$t = t_0 - \frac{1}{2}$

An infinite converging sequence in the reversed game



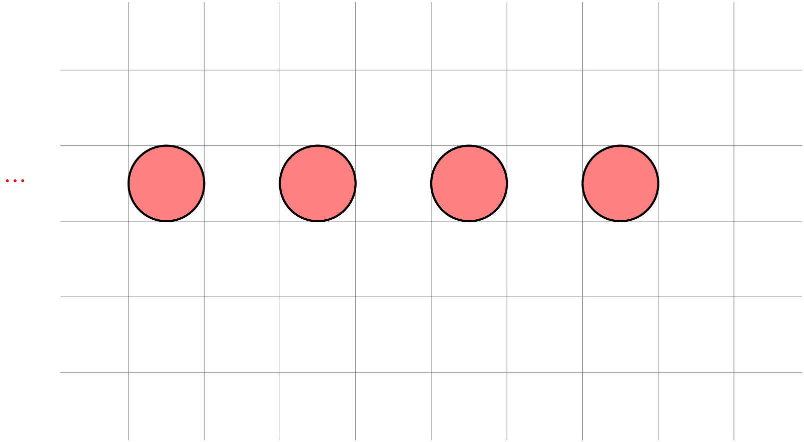
$t = t_0 - \frac{3}{4}$

An infinite converging sequence in the reversed game



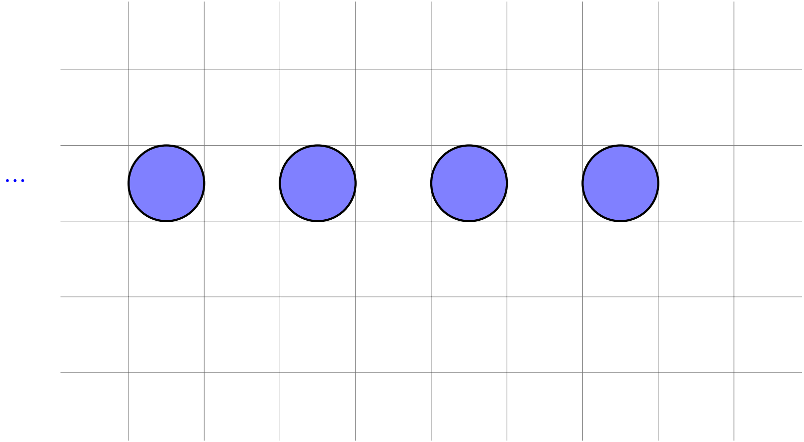
$t = t_0 - \frac{7}{8}$

An infinite converging sequence in the reversed game



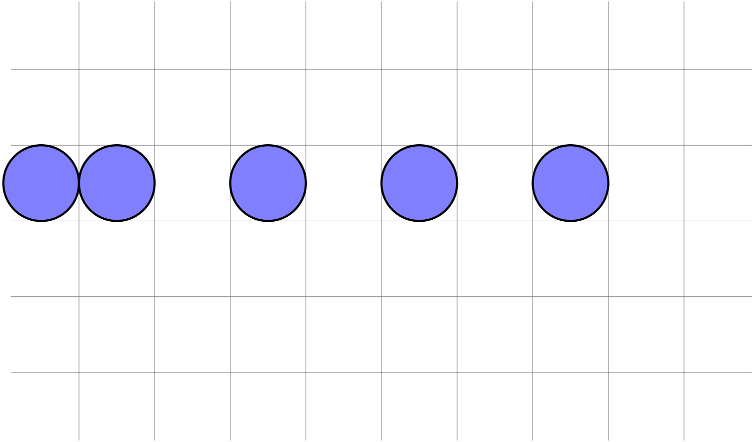
$t = t_0 - 1$

The forwards equivalent



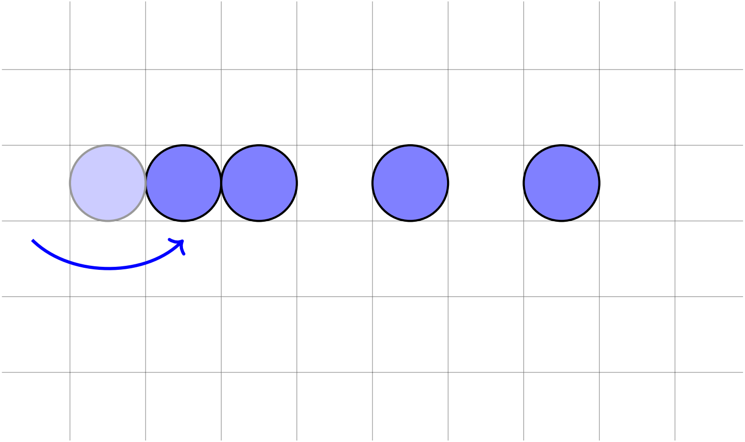
$t = t_0'$

The forwards equivalent



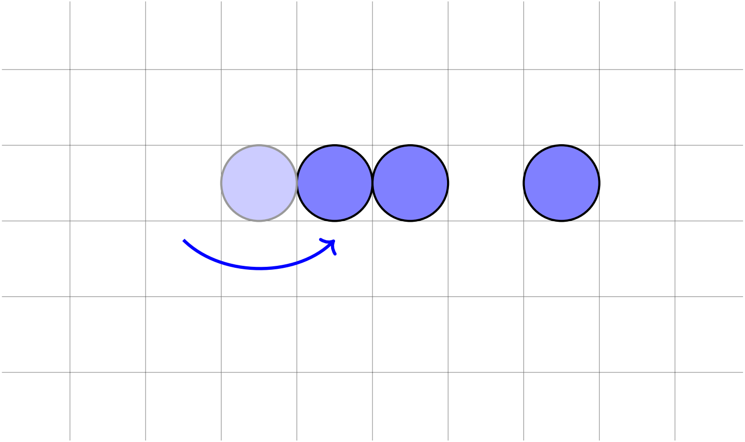
$$t = t_0' + \frac{1}{16}$$

The forwards equivalent



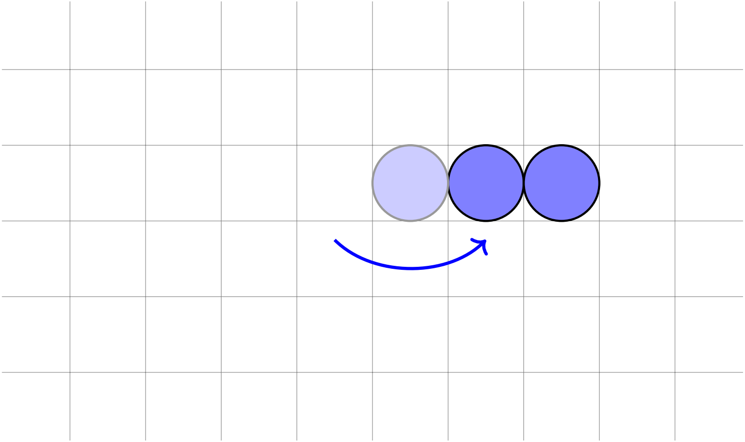
$$t = t_0' + \frac{1}{8}$$

The forwards equivalent



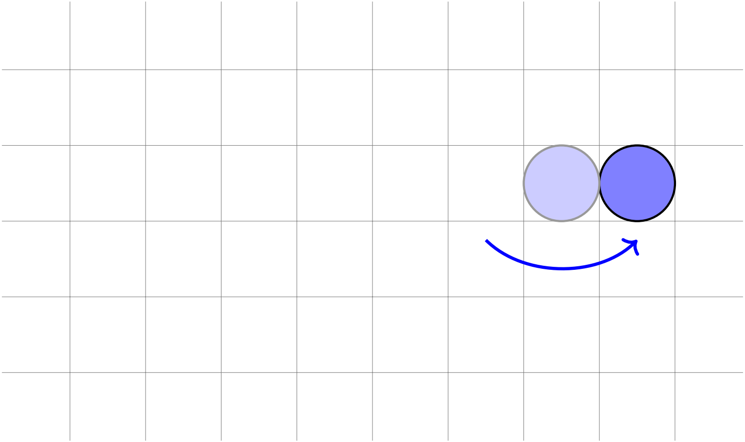
$$t = t_0' + \frac{1}{4}$$

The forwards equivalent



$$t = t_0' + \frac{1}{2}$$

The forwards equivalent

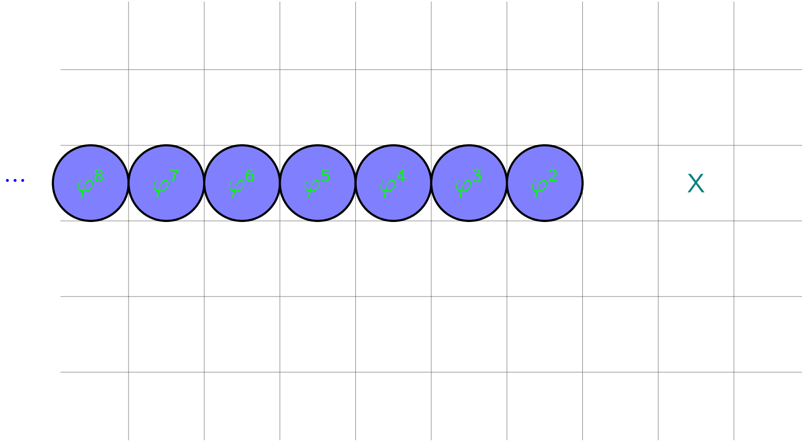


$$t = t_0' + 1$$

The forwards equivalent

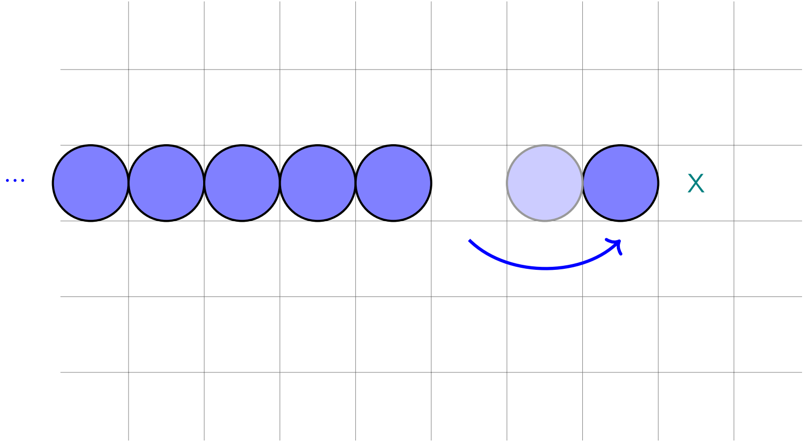
$t > t_0' + 1$

The *whoosh*



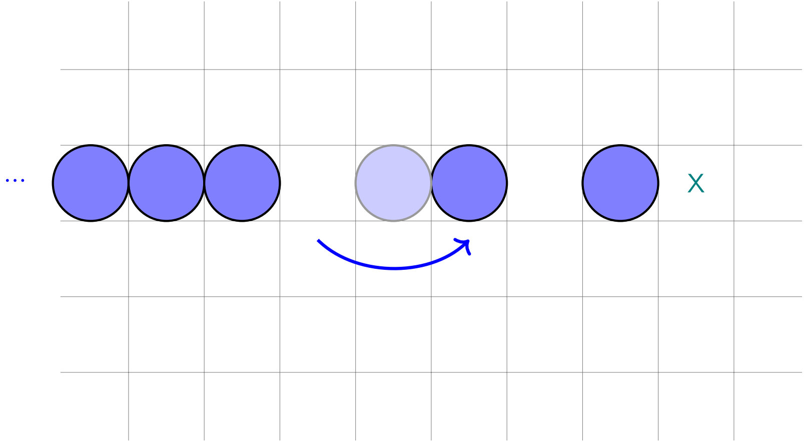
$t = t_0$

The *whoosh*



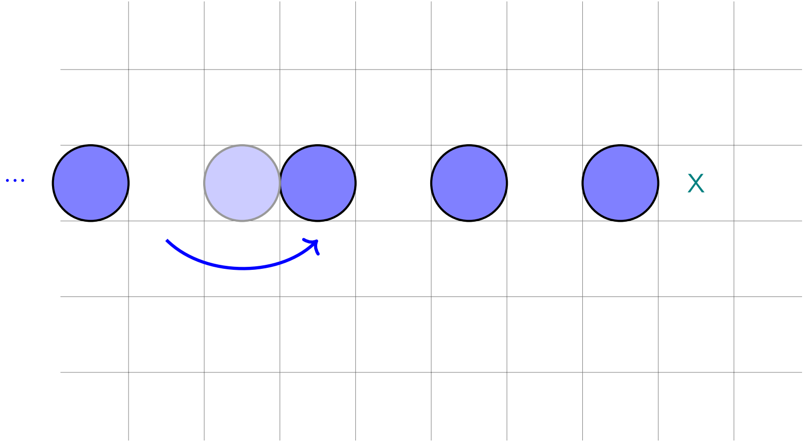
$$t = t_0 + \frac{1}{2}$$

The *whoosh*



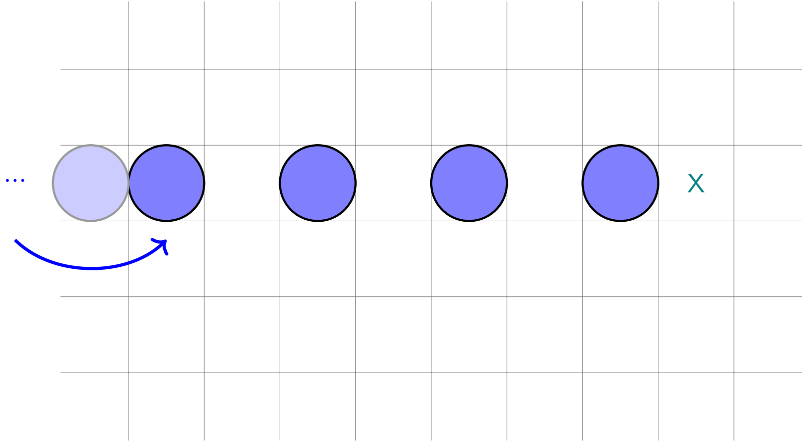
$t = t_0 + \frac{3}{4}$

The *whoosh*



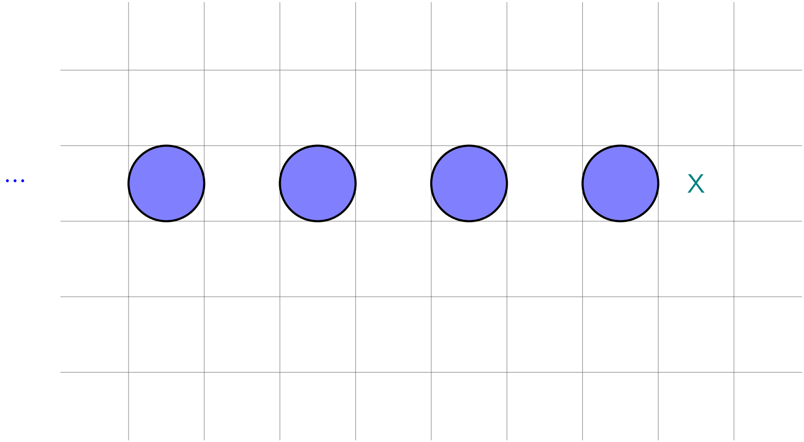
$$t = t_0 + \frac{7}{8}$$

The *whoosh*



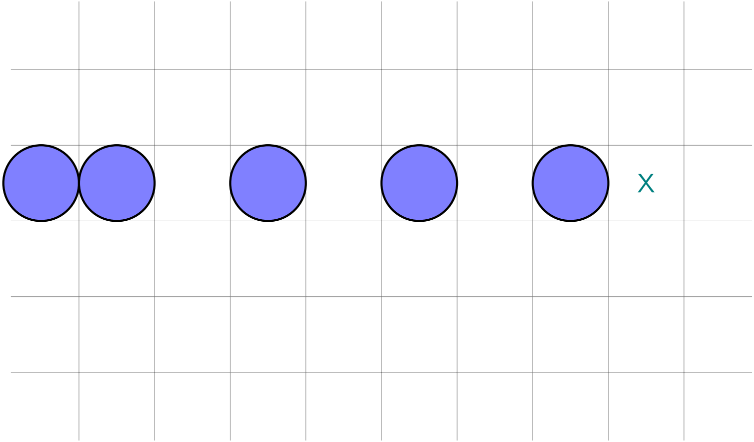
$$t = t_0 + \frac{15}{16}$$

The *whoosh*



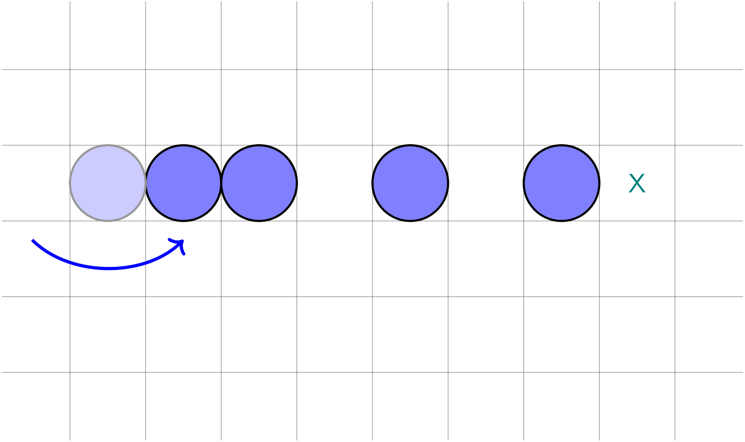
$t = t_0 + 1$

The *whoosh*



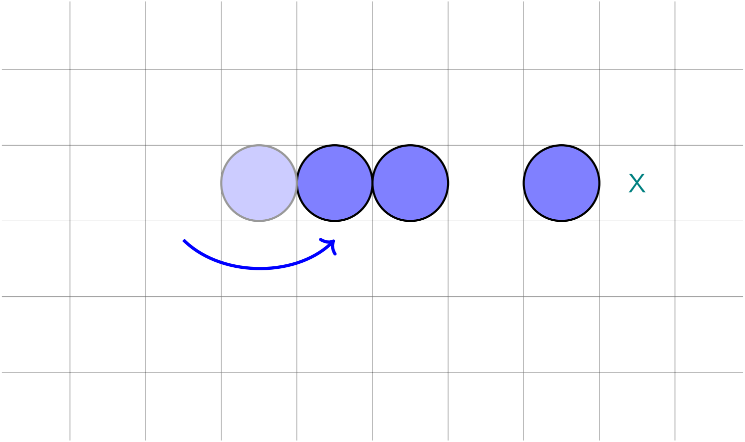
$$t = t_0 + \frac{17}{16}$$

The *whoosh*



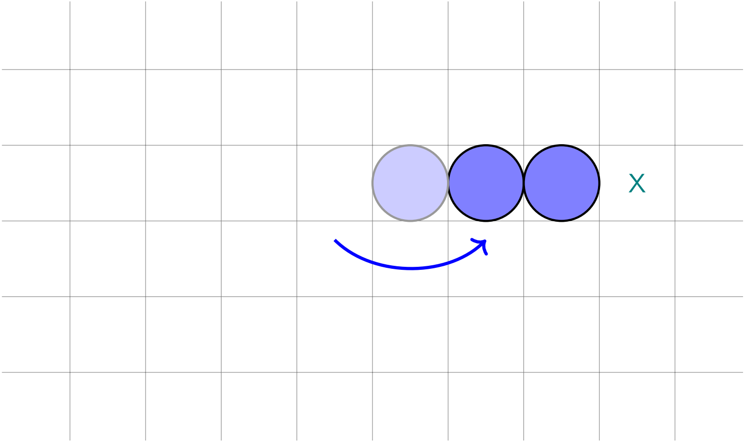
$$t = t_0 + \frac{9}{8}$$

The *whoosh*



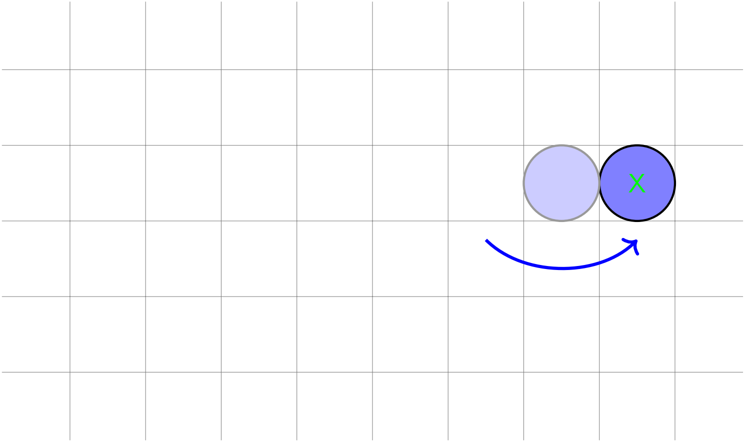
$$t = t_0 + \frac{5}{4}$$

The *whoosh*



$$t = t_0 + \frac{3}{2}$$

The *whoosh*

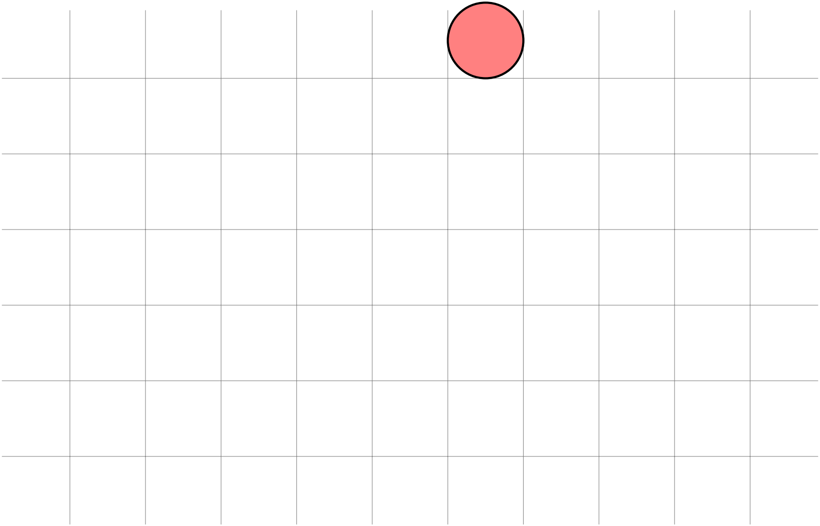


$t = t_0 + 2$

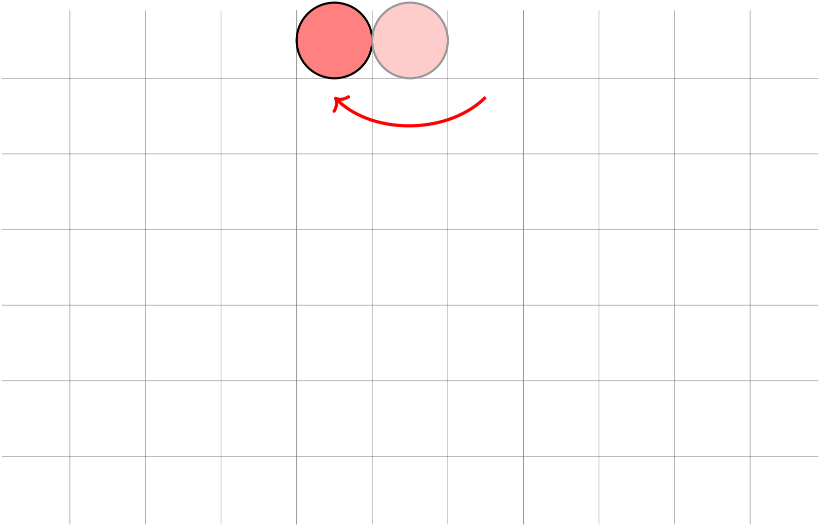
The forwards equivalent

$t > t_0 + 2$

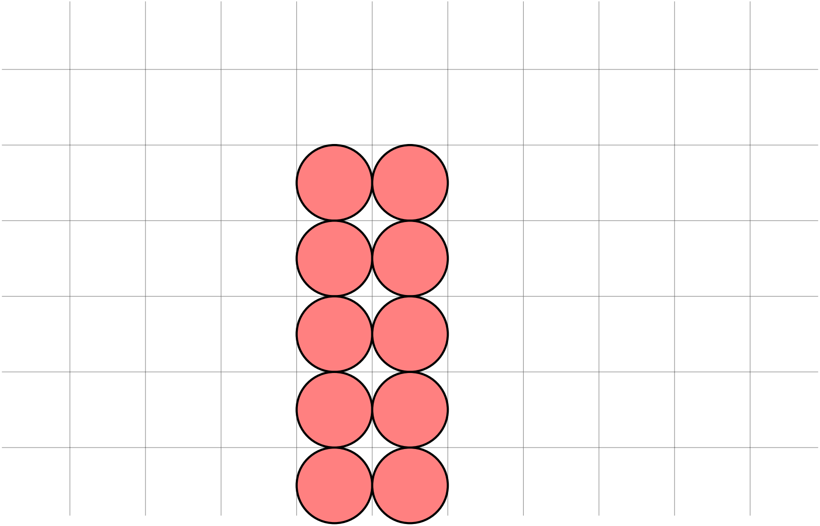
Defining the *megawhoosh* in reverse



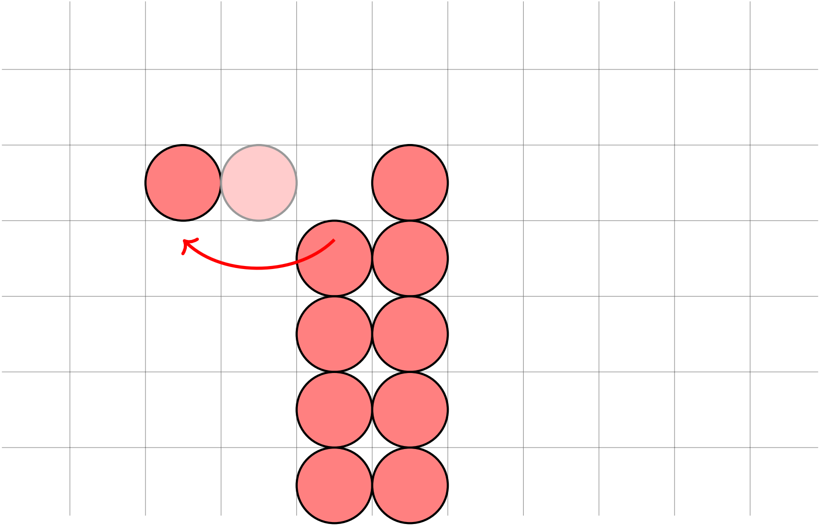
Defining the *megawhoosh* in reverse



Defining the *megawhoosh* in reverse



Defining the *megawhoosh* in reverse



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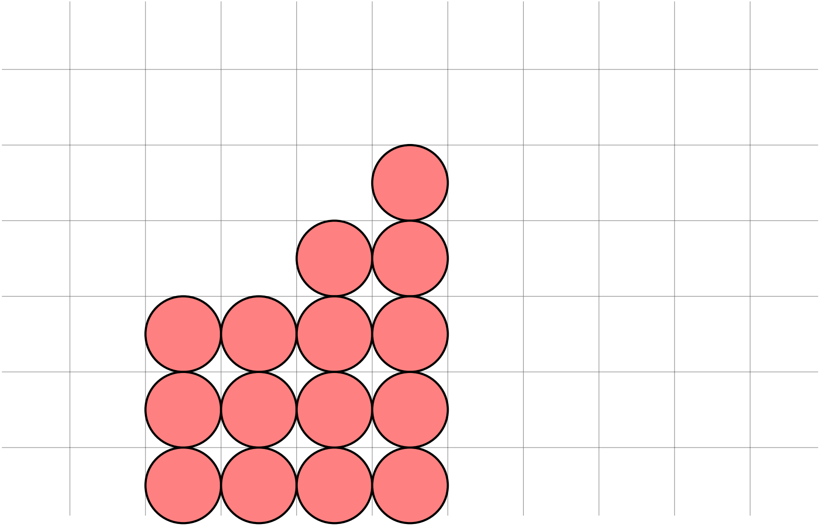

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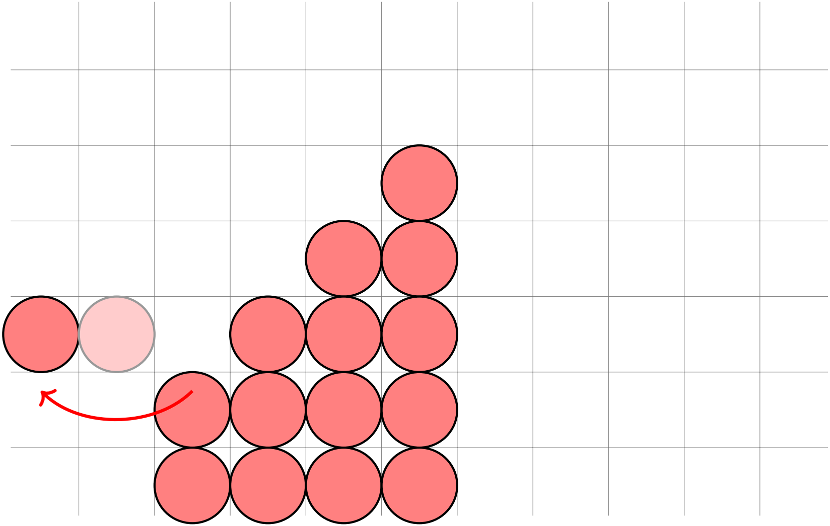
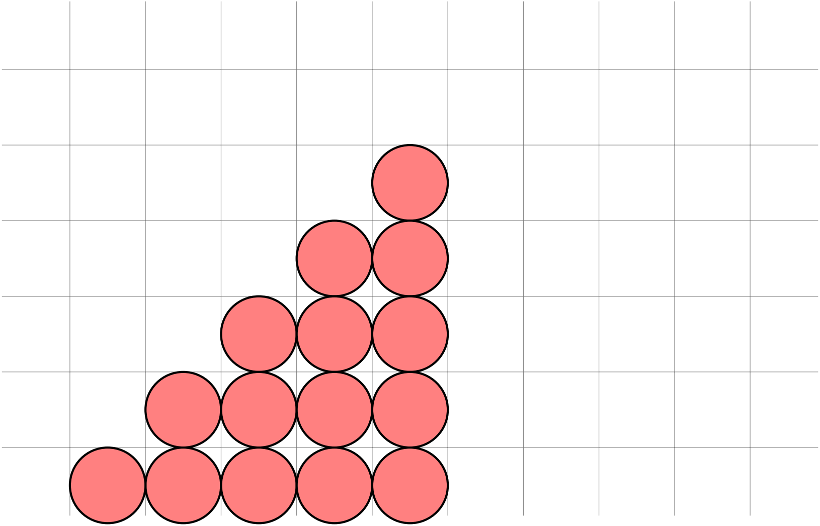


Defining the *megawhoosh* in reverse



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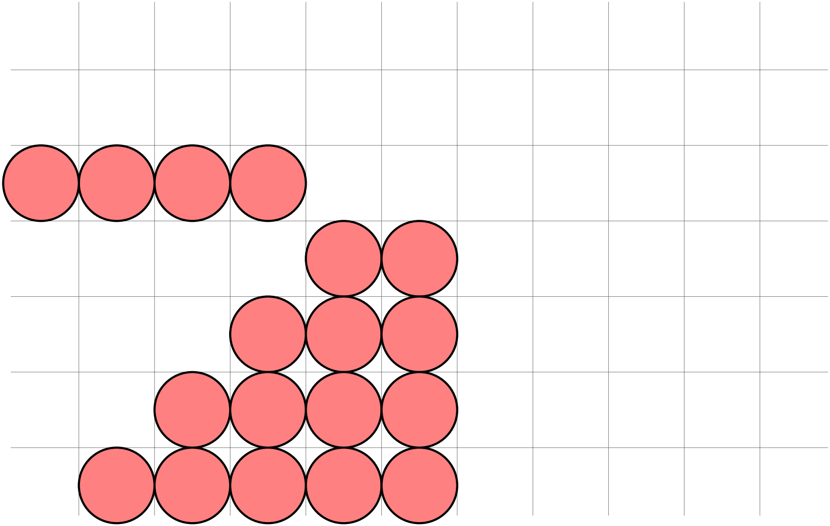
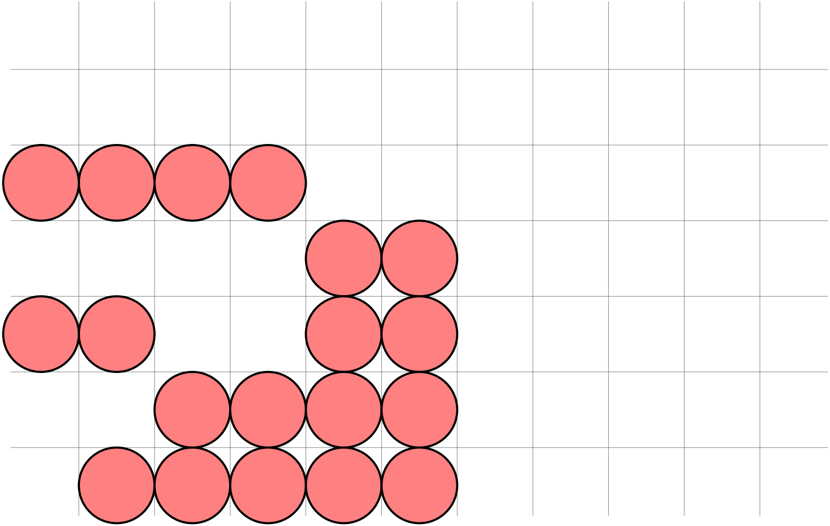
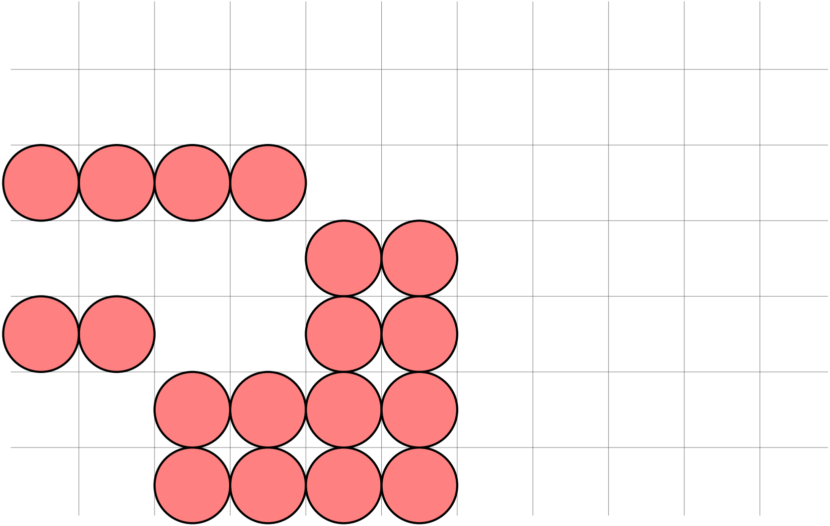


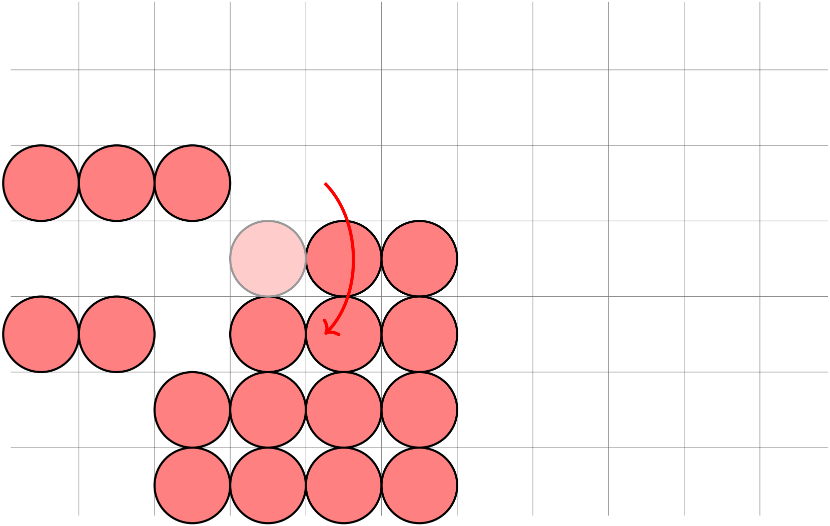
Defining the *megawhoosh* in reverse



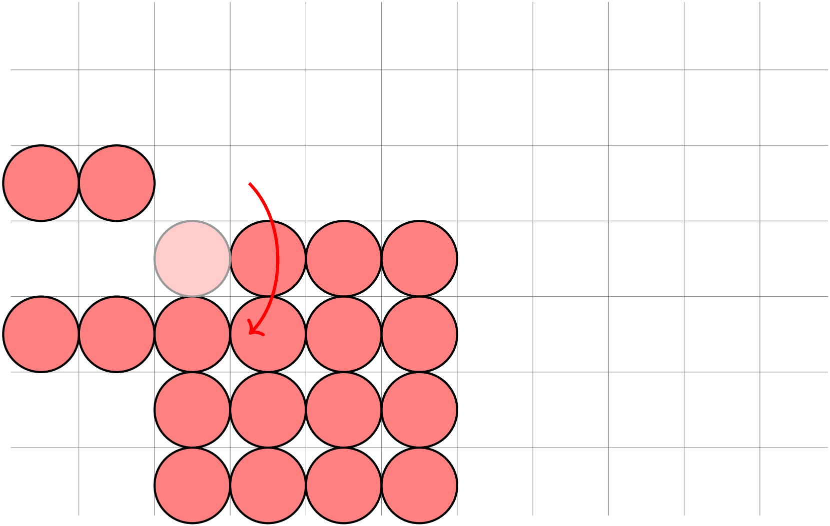
Defining the *megawhoosh* in reverse



Defining the *megawhoosh* in reverse



Defining the *megawhoosh* in reverse



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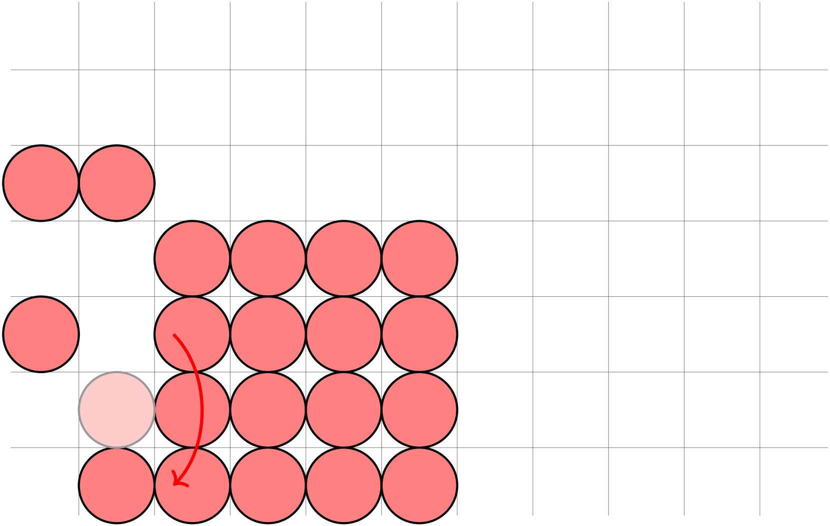
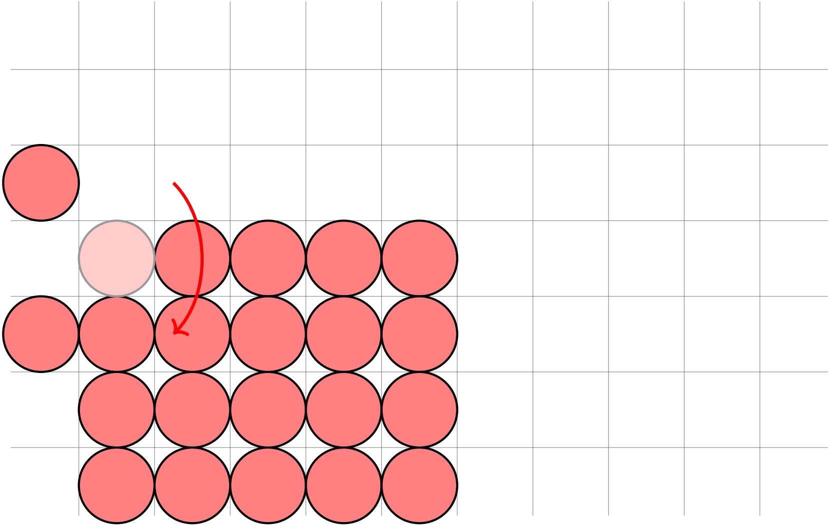


Defining the *megawhoosh* in reverse



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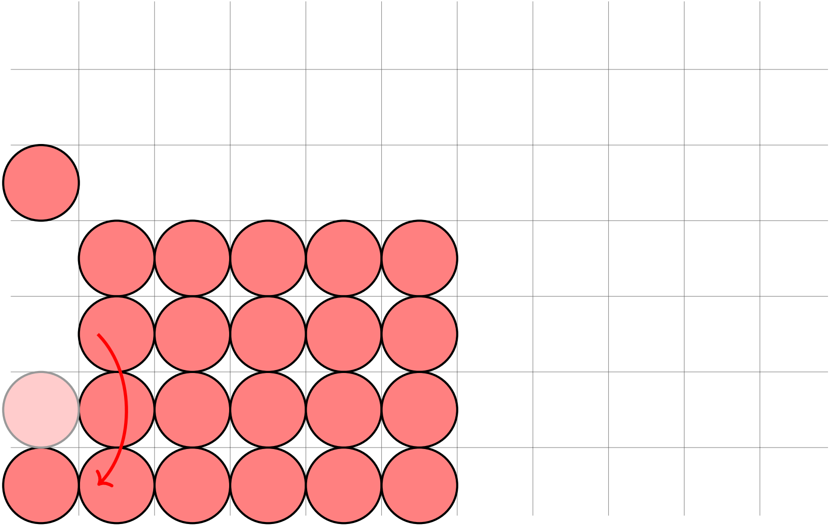

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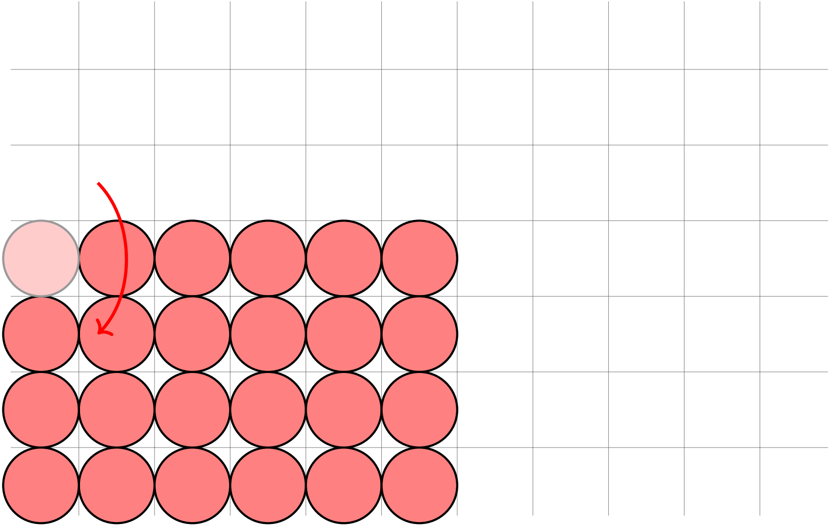

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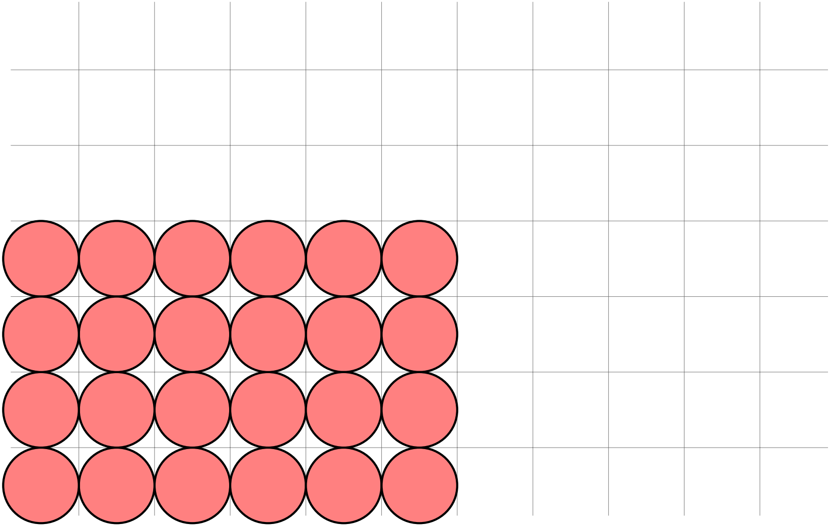
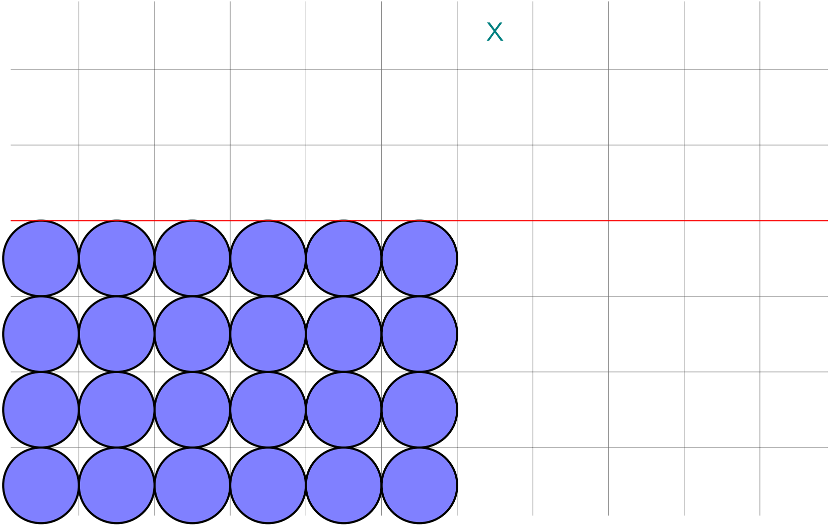


The forwards *megawhoosh*



All together now

Finally, the combination of a whoosh up the central column and a pair of megawhooshes in the quarter-planes to either side allows us to end up with one piece five rows above the starting half-plane.
As I have reached the limit of my ability to animate these moves, let's just watch Simon Tatham's original animation of the full process from his website:
`https://www.chiark.greenend.org.uk/~sgtatham/solarmy/solution.gif`

The rigorous construction

To make the analysis rigorous, we define a state function $f(x, y, t)$ of time and the board coordinates, such that

- $f(x, y, t) = 0$ if position (x, y) is unoccupied at time t.
- $f(x, y, t) = 1$ if position (x, y) is occupied at time t.
- $f(x, y, t) = \frac{1}{2}$ if t is precisely the time that the value of f changes, i.e. there are time neighbourhoods befpore and after t where the state is constant, but the state before is not the state after.

We want to constrain the function so that

1. Positions only change state during a move.
2. Only one move takes place at a time.
3. No position changes state infinitely often.
4. For any fixed time we only allow the basic legal move.

The rigorous construction

Rewriting the constraints more mathematically,

1. **Positions only change during moves:** for any fixed x, y and any continuous interval of time during which $f(x, y, t)$ is never equal to $\frac{1}{2}$, $f(x, y, t)$ is constant on that interval.
2. **One move at a time:** at any instant t, there are either exactly three values of x, y for which $f(x, y, t) = \frac{1}{2}$, or there are none.
3. **No position changes infinitely often:** For any fixed x, y, there are only finitely many values of t for which $f(x, y, t) = \frac{1}{2}$.
4. **Defining the basic legal move:** For any t which represents a move (i.e. there are three x, y with $f(x, y, t) = \frac{1}{2}$), the three coordinates either occupy three adjacent x-coordinates for a fixed y or vice versa. Also, the centre space of the three must change its value of f from 1 to 0, while one of the outer two does the same and the other changes from 0 to 1.

It can be shown that the state function we define by allowing whooshes and megawhooshes satisfies these criteria.

The constraint theorem

The constraints imply the following theorem:

1. for every time instant t
2. for every (x, y)
3. there exists some $\epsilon > 0$ such that
4. within the time interval $(t - \epsilon, t + \epsilon)$
5. the position (x, y) is not involved in any move which is not at time t.

Swapping lines 2 and 3 would disallow convergent infinite sequences. Allowing the infinite convergent sequence is equivalent to breaking this theorem from applying *absolutely* to only *pointwise*. In this sense, this may be the "best" or "weakest" way to break the finite game and allow infinite moves.

Black lives matter

If you want to support anti-racist and police accountability causes in the wake of the murder of George Floyd, consider donating.

In the U.S.

Act Blue can help you split a donation between bail funds, mutual aid groups and activist organisations: visit `https://secure.actblue.com/donate/bail_funds_george_floyd`

In the U.K.

StopWatch support research and action for fair and accountable policing: visit `http://www.stop-watch.org`

Black Lives Matter UK oppose imperialism, capitalism, white supremacy and patriarchy in the U.K. and around the world: visit `https://www.gofundme.com/f/ukblm-fund`