Some recreational mathematics of the late John H. Conway

The Doomsday algorithm and Conway's Soldiers

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SMSAS PGR seminar

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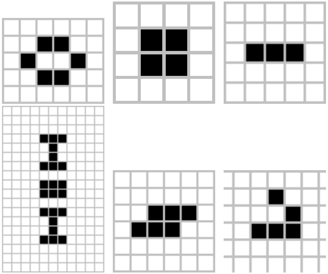
John Horton Conway

- Born 26th December 1937, Liverpool, U.K.
- Died 11th April 2020, New Jersey, U.S.A.
- Educated at Cambridge, he was a fellow and professor there from 1964 to 1986.
- Took up the John von Neumann Chair of Mathematics at Princeton in '86.
- Awards and honours included the Berwick Prize ('71), FRS ('81), inaugural Pólya prize ('87).
- Primary interests included geometry, geometric topology, group theory and, especially, combinatorial game theory.

Conway's Game of Life

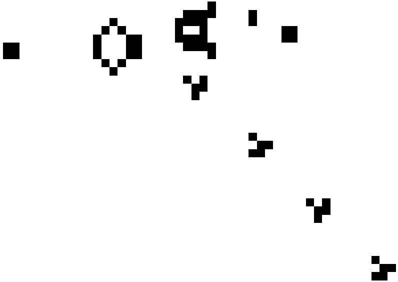
Conway's most popular creation outside academia was undoubtedly the cellular automaton (or "zero-player game") known as the Game of Life. This is an infinite square grid where the square cells are either "alive" or "dead" at each discrete time step according to some simple rules:

1. A dead cell becomes alive if it has exactly three live neighbours (vertically, horizontally and/or diagonally).
2. A live stell stays alive if it has exactly two or three live neighbours.
3. All other live cells die and all other dead cells stay dead.



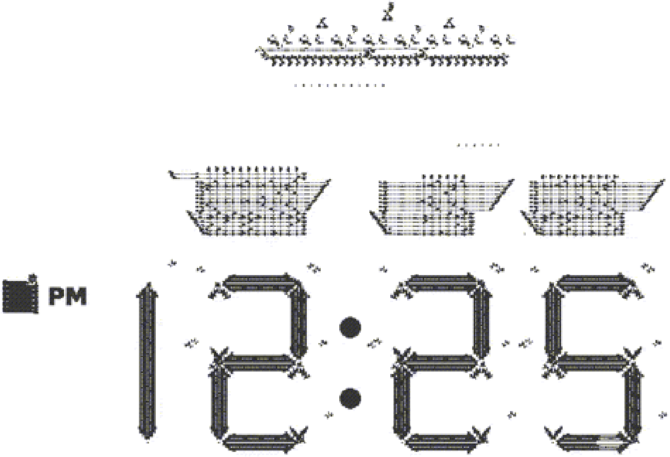
Reproduction in Life

Structures in Life can grow infinitely by producing gliders or similar spaceships. Here are examples of linear and quadratic growth.



A "programme" in Life

The Game of Life is Turing-complete, and fans have created many complex programmes within it.



The ??? sequence

What comes next in the following sequence?

$$1, 11, 21, 1211, 111221, \ldots$$

The look-and-say sequence

What comes next in the following sequence?

$$1, 11, 21, 1211, 111221, 312211, 13112221, 1112213211, \ldots$$

Read off the digits of the current term to get the next term:
"one 1, two 1s, one 2 two 1s, one 1 one 2 two 1s, ..."
Conway proved that, for all such sequences (except in the case of the degenerate sequence $22, 22, 22, \ldots$)

- the sequence eventually "decays" into finite subsequences which do not interact, and
- letting L_n denote the number of digits in the n^{th} term,

$$\lim_{n\to\infty} \frac{L_{n+1}}{L_n} = \lambda = 1.303577269034\ldots$$

The look-and-say sequence

This constant λ, called Conway's constant, is the same for all such look-and-say sequences. He further proved that it is algebraic of degree 71: it is the unique positive real root of the polynomial

$$
\begin{aligned}
&-6+3x-6x^2+12x^3-4x^4+7x^5-7x^6+x^7+5x^9-2x^10-4x^{11}\\
&-12x^{12}+2x^{13}+7x^{14}+12x^{15}-7x^{16}-10x^{17}-4x^{18}+3x^{19}+9x^{20}-7x^{21}\\
&-8x^{23}+14x^{24}-3x^{25}+9x^{26}+2x^{27}-3x^{28}-10x^{29}-2x^{30}-6x^{31}+x^{32}\\
&+10x^{33}-3x^{34}+x^{35}+7x^{36}-7x^{37}+7x^{38}-12x^{39}-5x^{40}+8x^{41}+6x^{42}\\
&+10x^{43}-8x^{44}-8x^{45}-7x^{46}-3x^{47}+9x^{48}+x^{49}+6x^{50}+6x^{51}-2x^{52}\\
&-3x^{53}-10x^{54}-2x^{55}+3x^{56}+5x^{57}+2x^{58}-x^{59}-x^{60}-x^{61}-x^{62}-x^{63}\\
&+x^{64}+2x^{65}+2x^{66}-x^{67}-2x^{68}-x^{69}+x^{71}
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$$

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I will now attempt to demonstrate this party trick.

The Doomsday algorithm – how it works

- The starting point is to notice that certain dates within a year always fall on the same day of the week. In particular, 4/4, 6/6, 8/8, 10/10 and 12/12 are always the same weekday in a given year. This day is called the year's *Doomsday*.

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- In odd months from May onwards, the Doomsday falls on 9/5, 11/7, 5/9 and 7/11. Conway's mnemonic was "I work **nine-to-five** at the **7-11**."

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- The correspondence in January and February depends on whether the year is a leap year – but the last day of February is always a Doomsday.

Mnemonics for the months

If you know what a year's Doomsday is, you can find where it appears in the month.

Month	Doomsday	Conway's mnemonic
January	3/1 (4/1)	"the 3rd three years and the 4th the fourth"
February	28/2 (29/2)	last day of February
March	"0/3"	calculate from the last day of February
April	4/4	"even for even"
May	9/5	"nine-to-five at the 7-11"
June	6/6	"even for even"
July	11/7	"nine-to-five at the 7-11"
August	8/8	"even for even"
September	5/9	"nine-to-five at the 7-11"
October	10/10	"even for even"
November	7/11	"nine-to-five at the 7-11"
December	12/12	"even for even"

Table: The Doomsday in each month (in leap years)

Counting days of the week

Arithmetic modulo 7 gets you from the Doomsday to the date that you want. To help this, Conway recommended giving days of the week a new, numerical naming scheme and fixing the *index value* of the days according to this scheme.

Weekday	Index (mod 7)	Conway's name
Sunday	0	None-day
Monday	1	One-day
Tuesday	2	Two-sday
Wednesday	3	Treble-sday
Thursday	4	Four-sday
Friday	5	Five-day
Saturday	6	Six-a-day

Table: Conway's weekdays

This works quite well in English – but may be confusing for our Greek and Portuguese speakers!

Calculating the Doomsday for a given year

This is the most elaborate part of the trick. Take the last two digits of the year. (E.g. for 1991, we start with 91.)

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3. Add the sum of these values to the century's *anchor day* (mod 7) to obtain the year's Doomsday. (E.g. for 1991, we need the anchor day of 1900, which is Wednesday.)

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Thus, Sunday is the Doomsday for 1991.

Anchor days

Anchor days simply repeat every four centuries (in the Gregorian calendar!).

Century	Anchor day	Conway's mnemonic
1800	Friday	
1900	Wednesday	"We-in-this"-day
2000	Tuesday	"Y-**Two**-K is Two-sday"
2100	Sunday	"Sunday for Twenty-One-days"

Table: Anchor days by century

Putting it all together

So the full Doomsday algorithm for calculating the weekday of a date is:

1. Work out the century's anchor day.
2. Calculate the year's Doomsday from the century's anchor day.
3. Go to a Doomsday in the right month, and count from there to the date you want.

Conway could do this in his head in about two seconds. Allegedly, he set his computer to test him when he tried to log in.

Why it works – the mathematics

A non-leap year has 365 days, which is 52 full 7-day weeks and one day. A leap year simply adds one extra day. So the relative shift in the day of the week after y years is

$$365y + \left\lfloor \frac{y}{4} \right\rfloor \mod 7 = y + \left\lfloor \frac{y}{4} \right\rfloor \mod 7$$

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The values a, b and c in Conway's algorithm are exactly tracking the one-day shifts which come from 12-year cycles, the distance into the current 12-year cycle, and the leap year contributions so far in the current 12-year cycle.

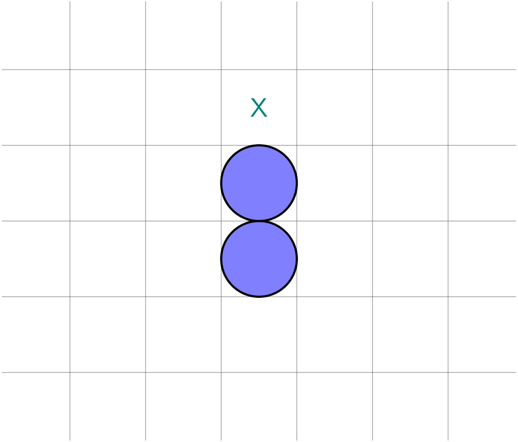
A modern shortcut

A confession: I'm actually using a slightly easier method to get the Doomsday from the anchor day, called the "odd + 11" method developed by Fong and Walters in 2010:

1. Take the final two digits of the year, as before.
2. If this is odd, add 11.
3. Divide by 2.
4. If this is odd, add 11.
5. Take the remainder mod 7, and *subtract* this from the anchor day to get the year's Doomsday.

Fong and Walters proved in 2011 [arXiv:1010.0765] that in modular arithmetic this is equivalent to Conway's method, but it's certainly mentally quicker!

Conway's Soldiers: Moving



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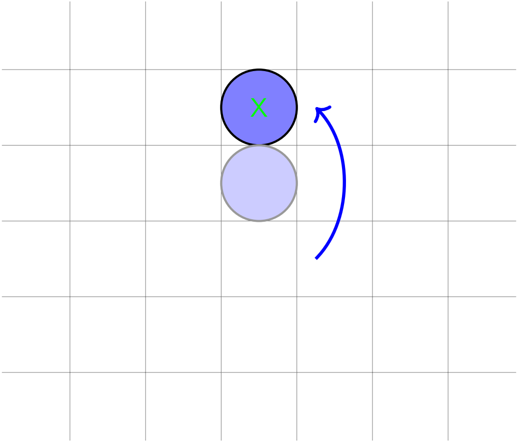

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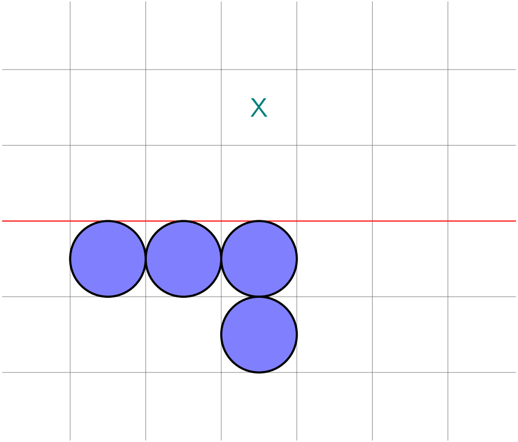

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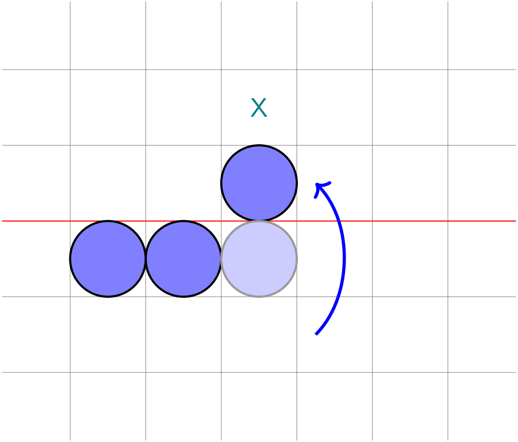

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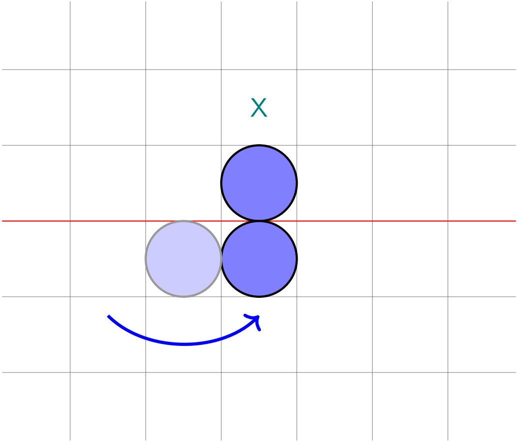

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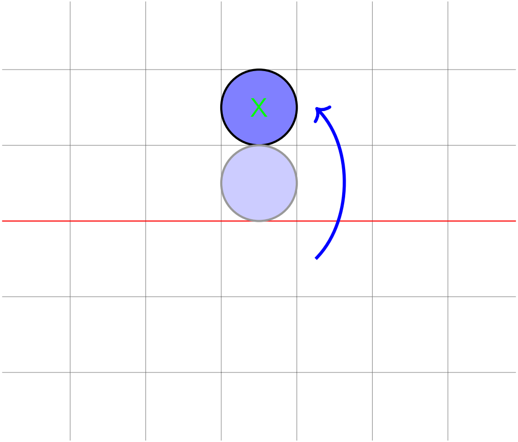

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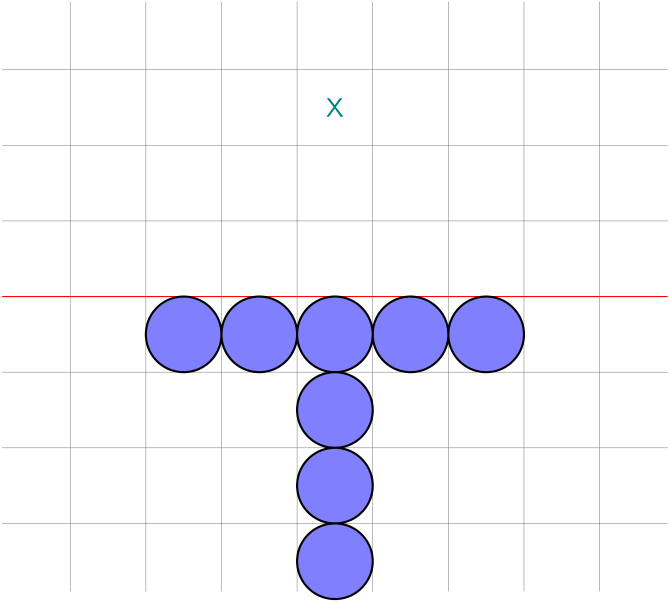

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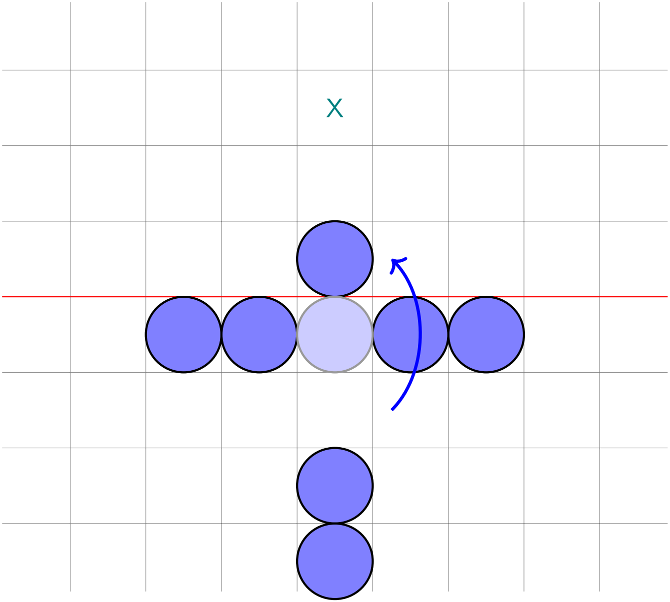

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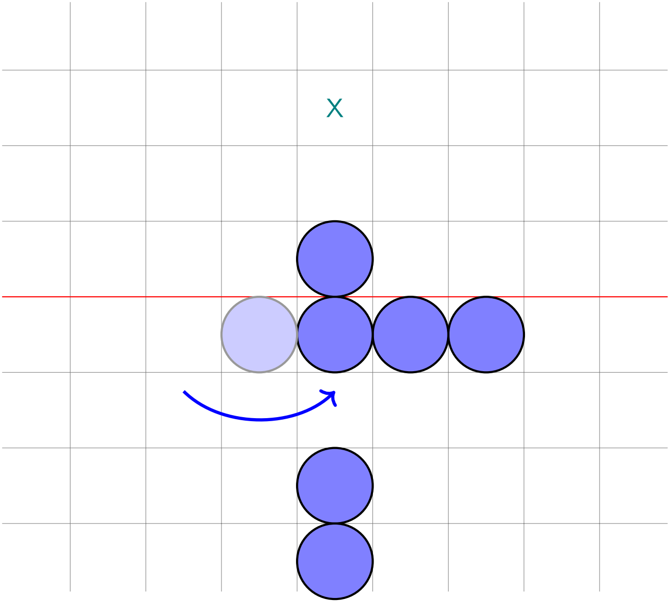

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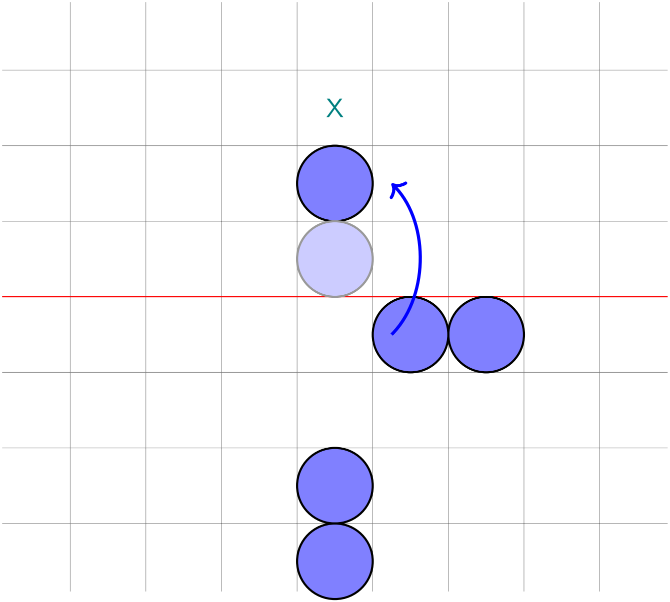

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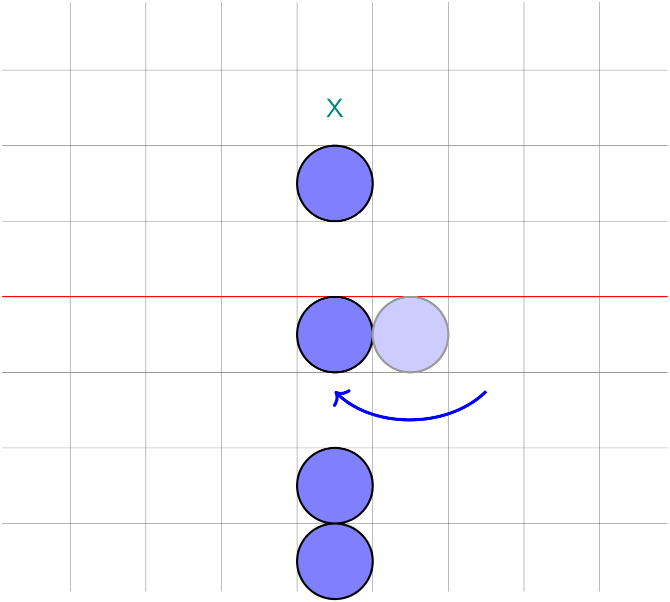

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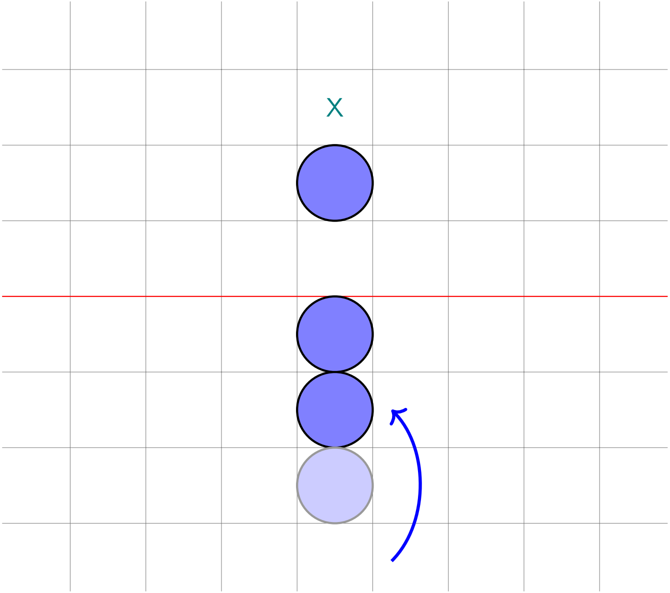

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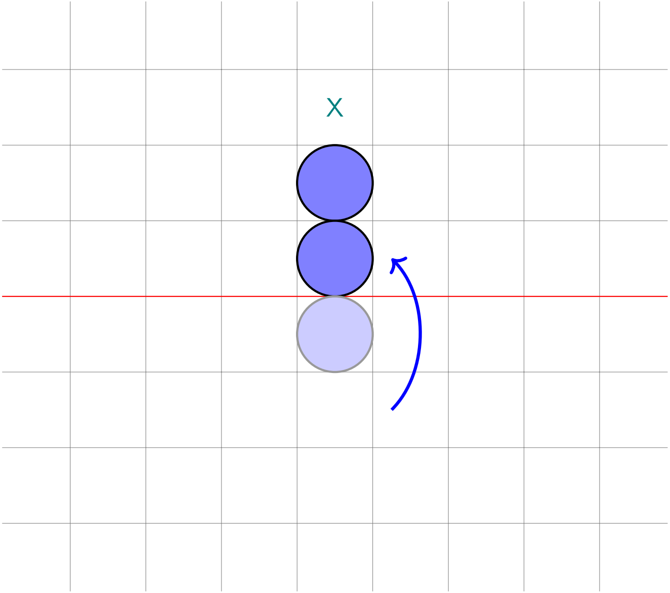

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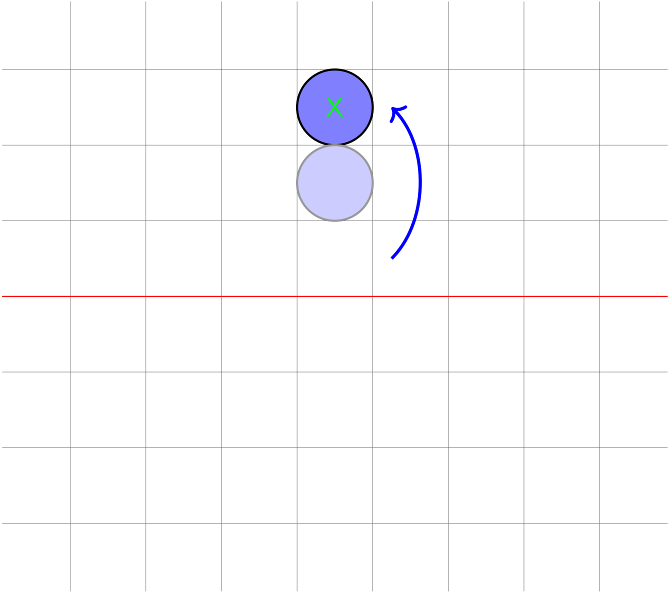
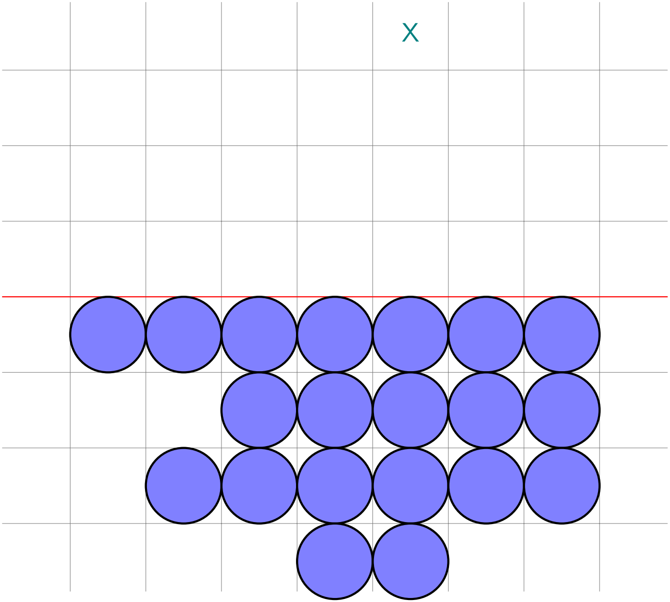


Four Up: a potential starting configuration



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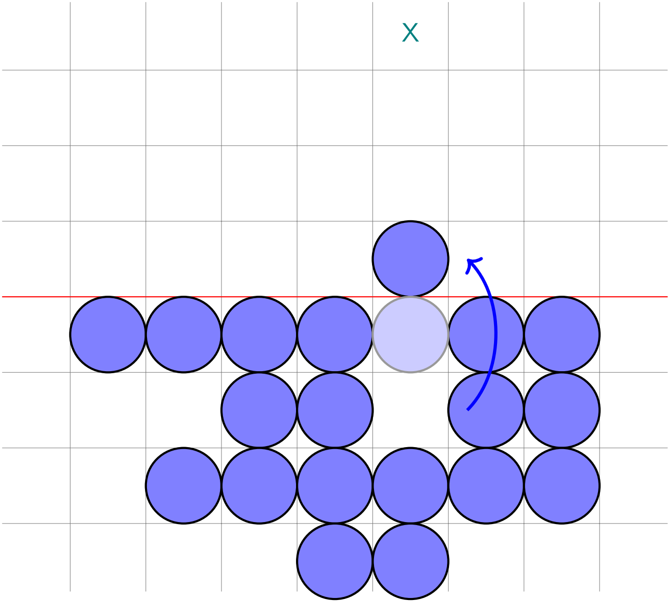

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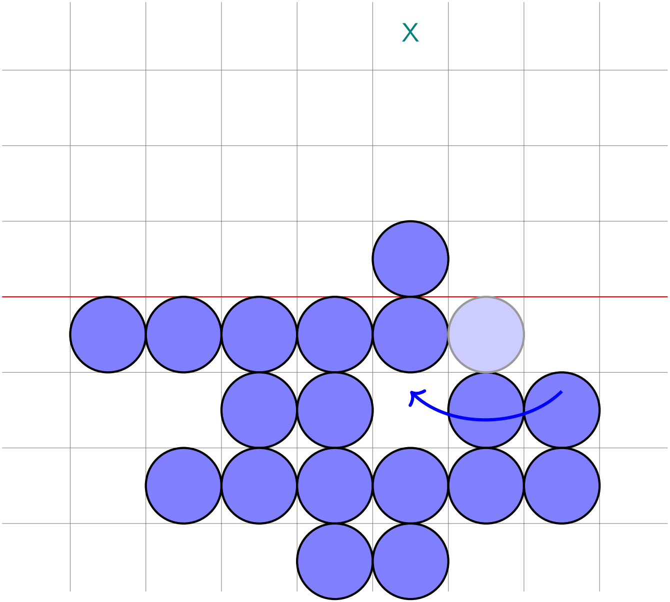

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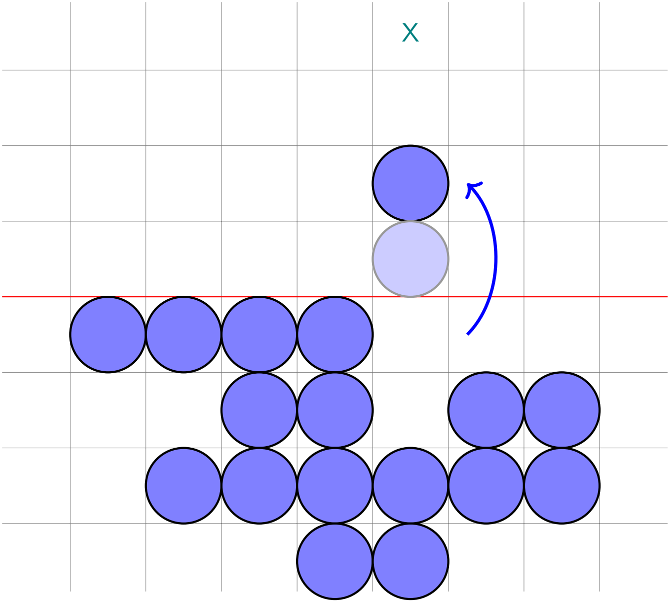

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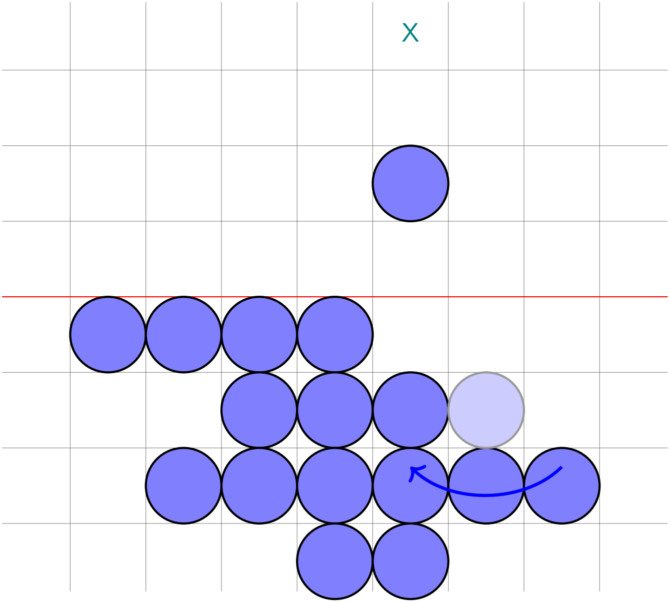

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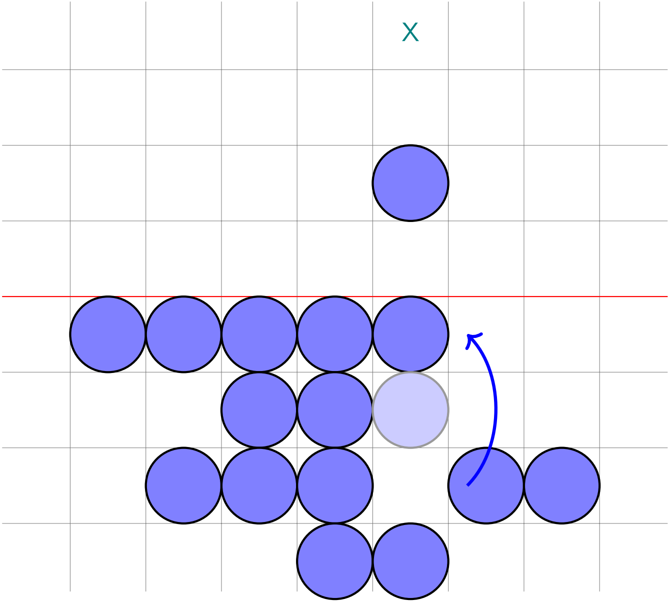

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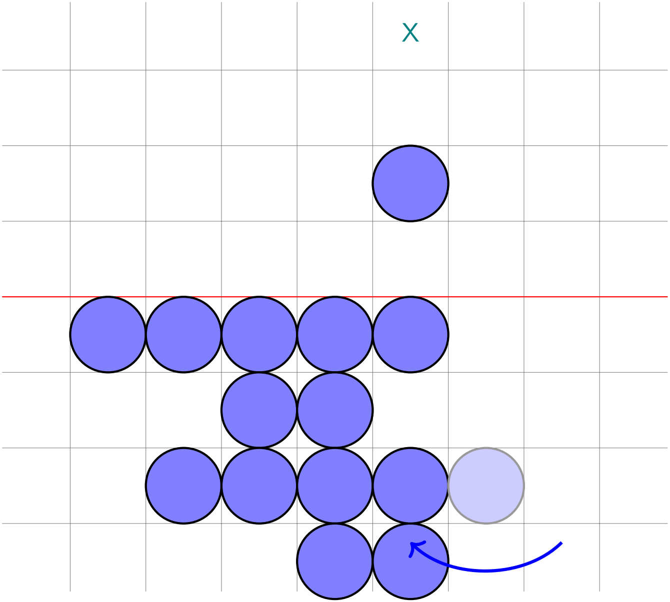

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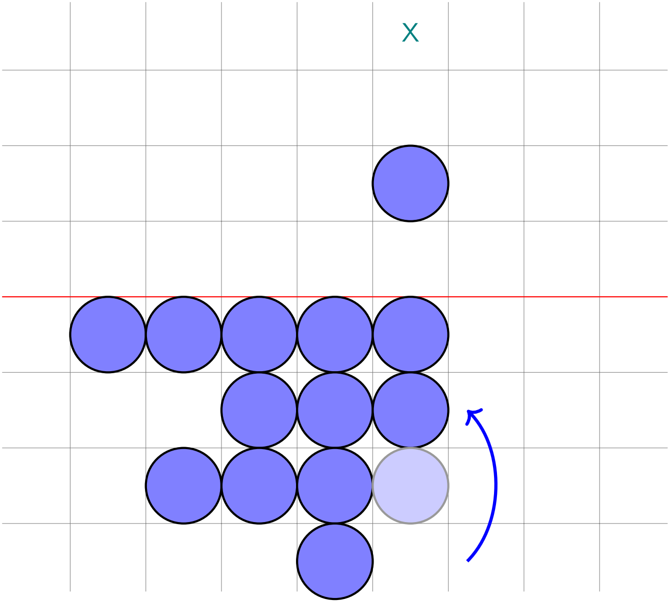

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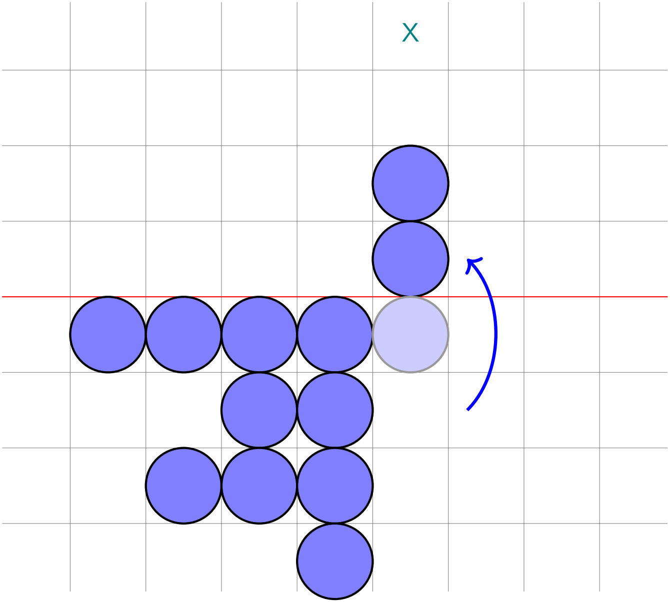

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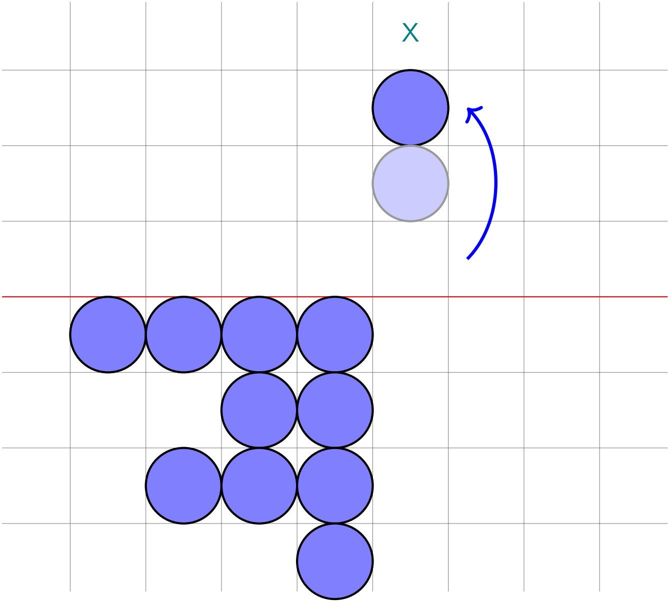

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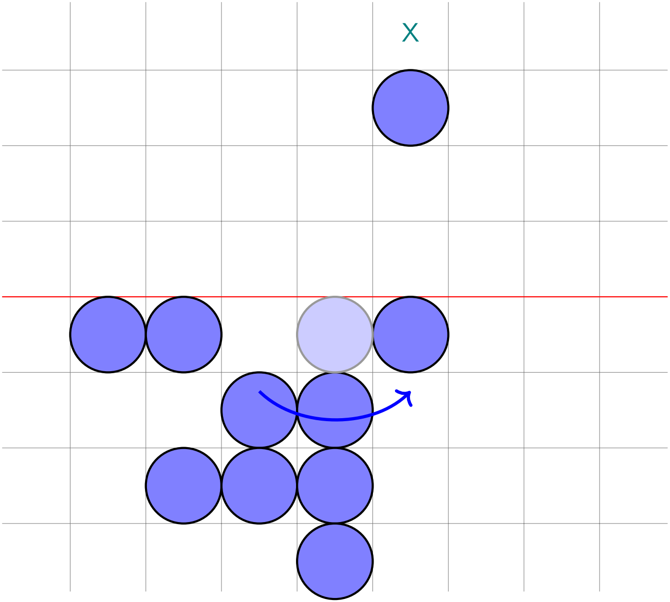

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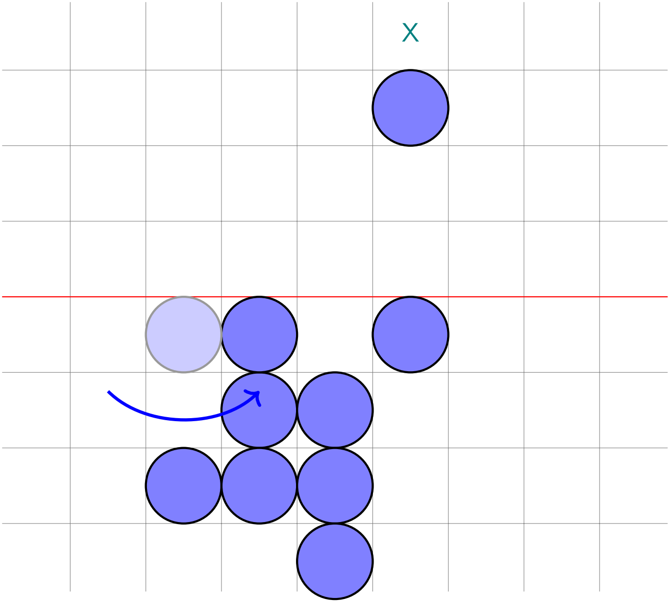

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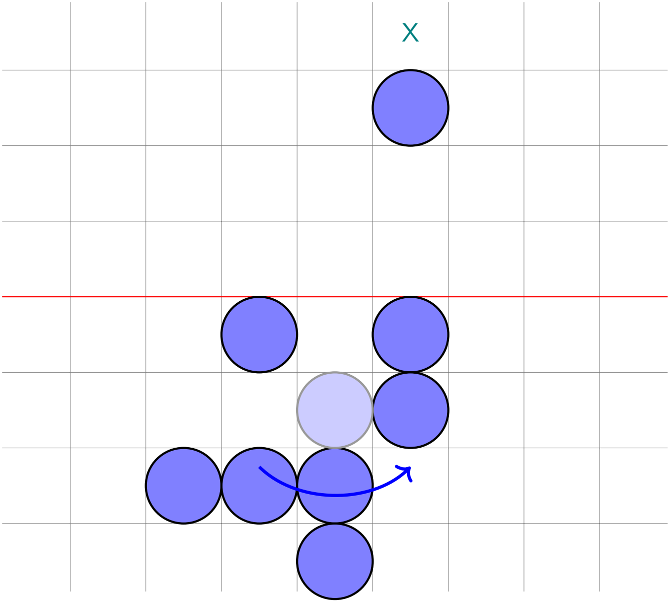

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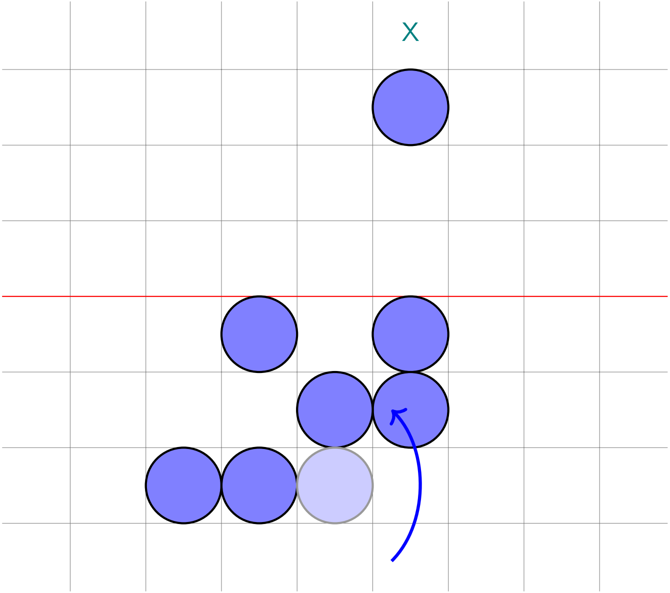

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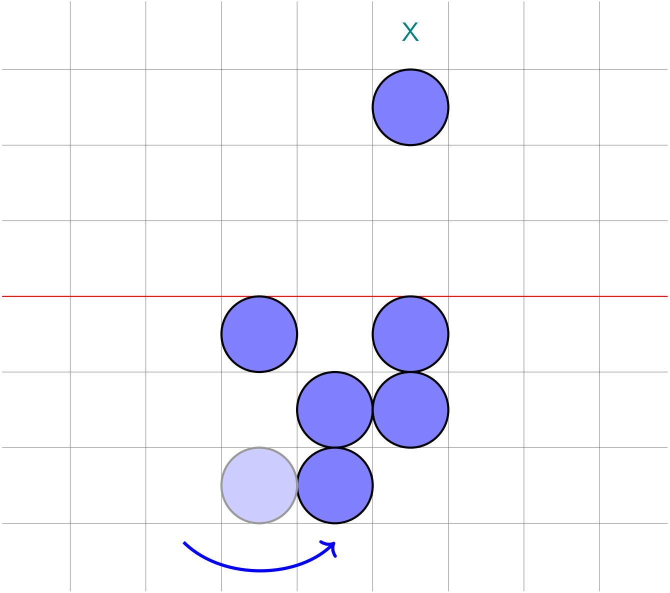

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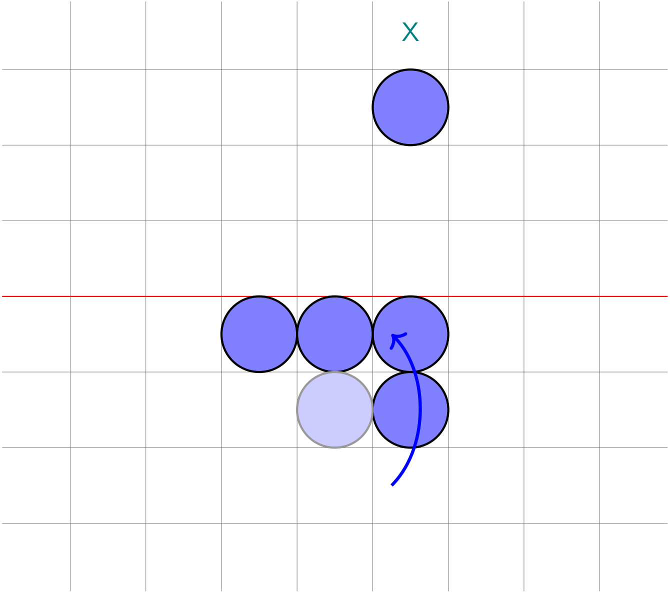

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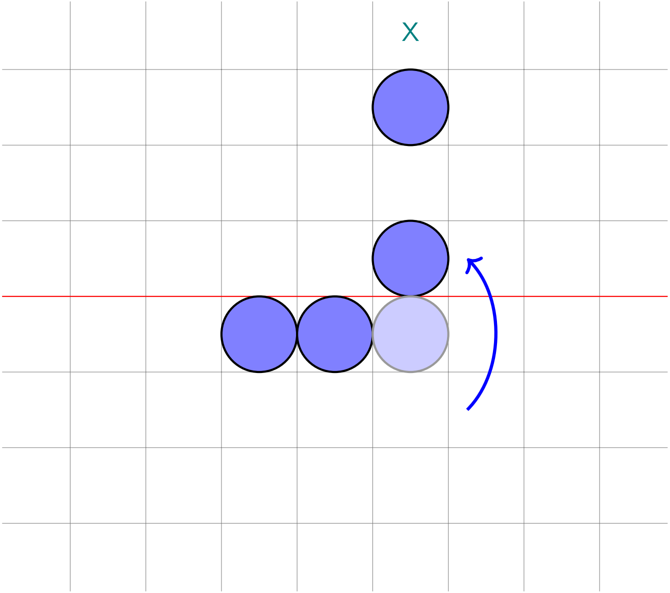

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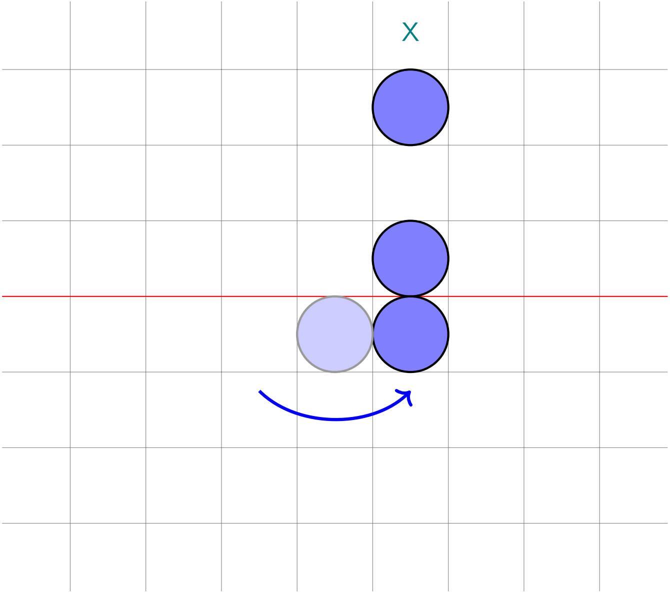

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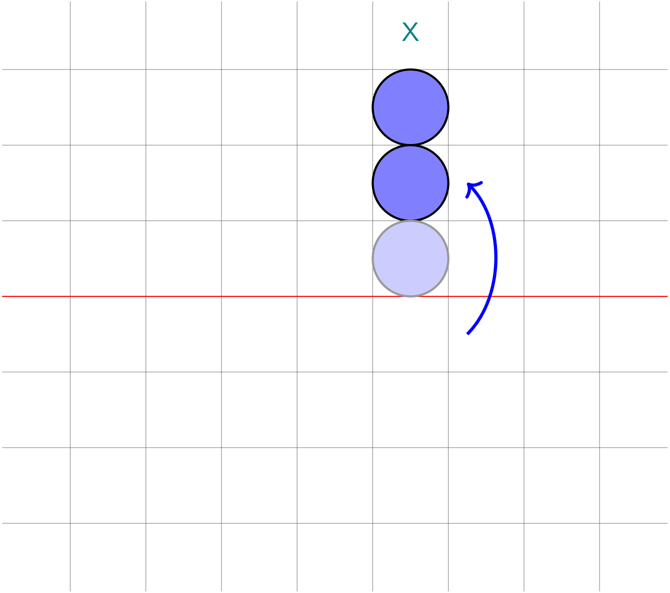

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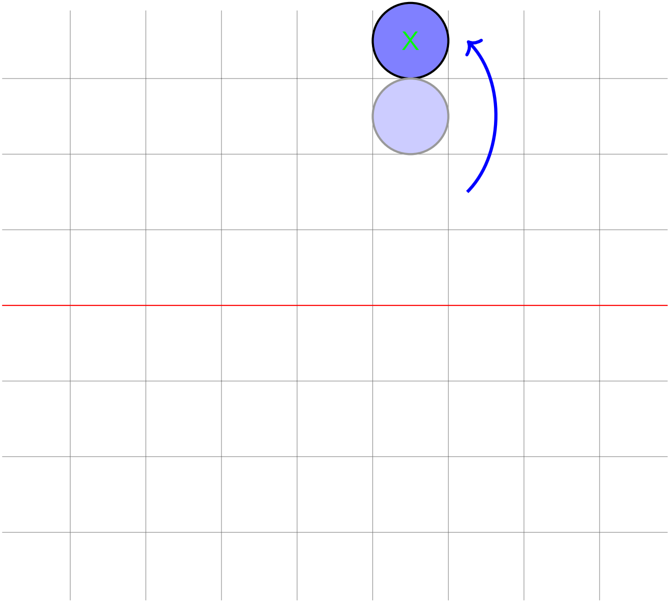
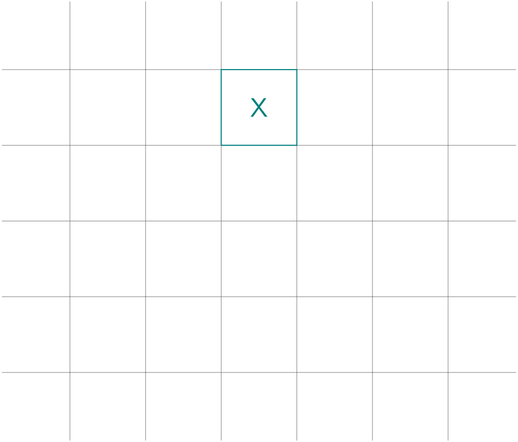


Assigning a weight to the cells



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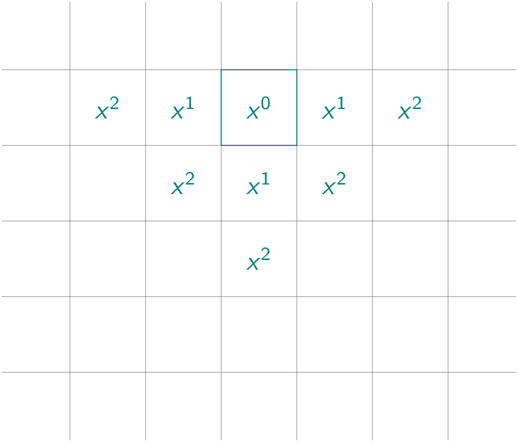
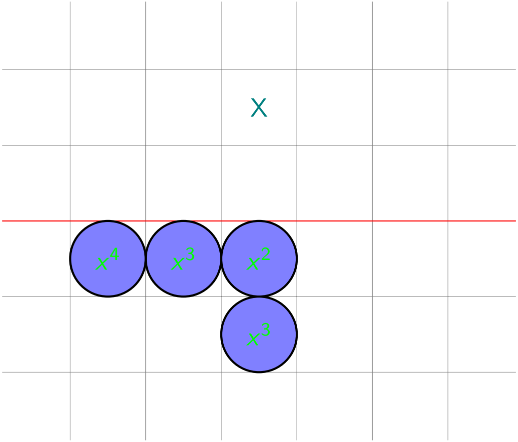


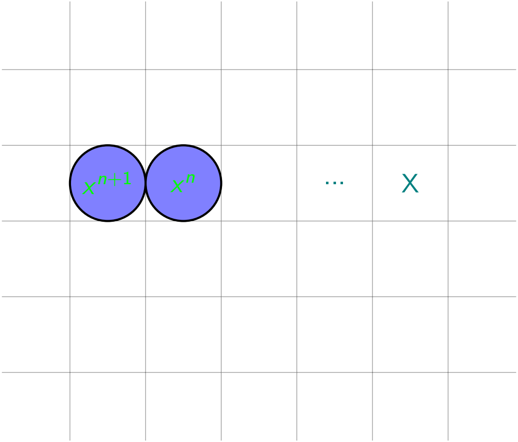
All cells are given a *weight* in powers of some value x according to their distance from the target. Occupied cells contribute their weight to the *score*.

The score of a configuration



This configuration has the score $x^2 + 2x^3 + x^4$.

Moving changes the score



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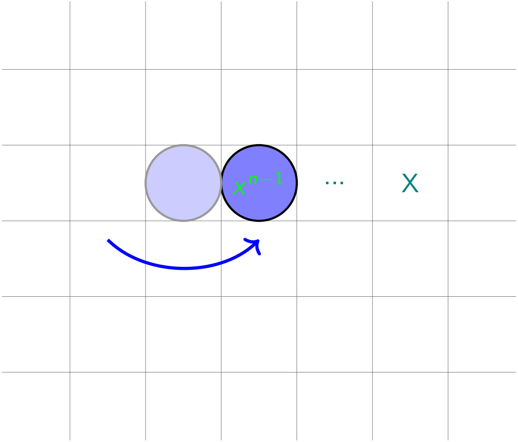
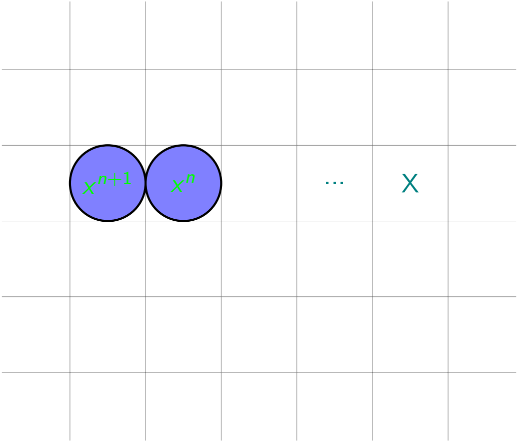
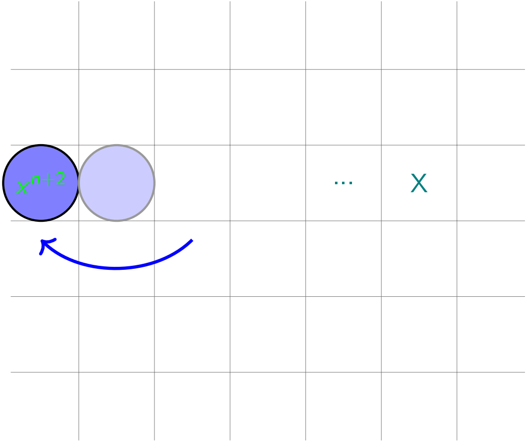


A move towards the target changes the score by $x^{n-1} - x^n - x^{n+1} = x^{n-1}(1 - x - x^2)$.

Moving changes the score

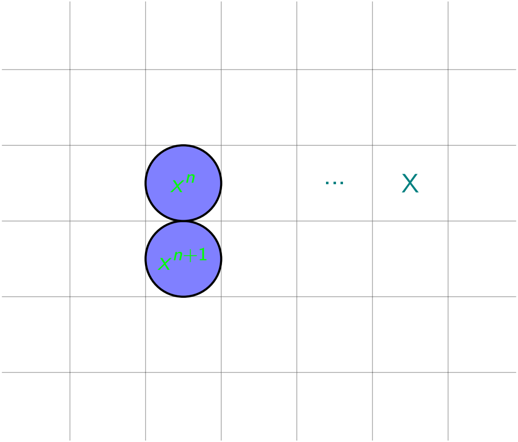


Moving changes the score

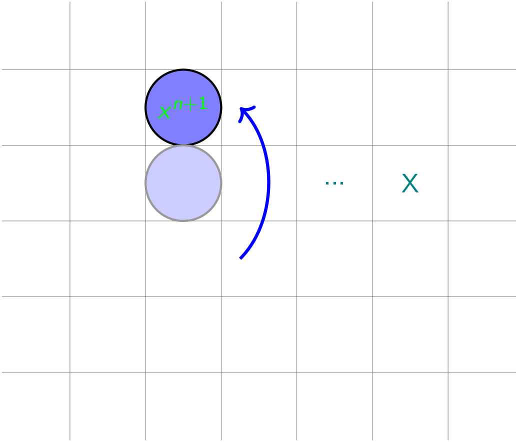


A move away from the target changes the score by $x^{n+2} - x^{n+1} - x^n = x^n(-1 - x + x^2)$.

Moving changes the score



Moving changes the score



A neutral move changes the score by $x^{n+1} - x^n - x^{n+1} = -x^n$.

The score must decrease

Suppose we only want to allow the score to decrease or stay the same, but not increase. From the three types of moves, we then get the inequalities

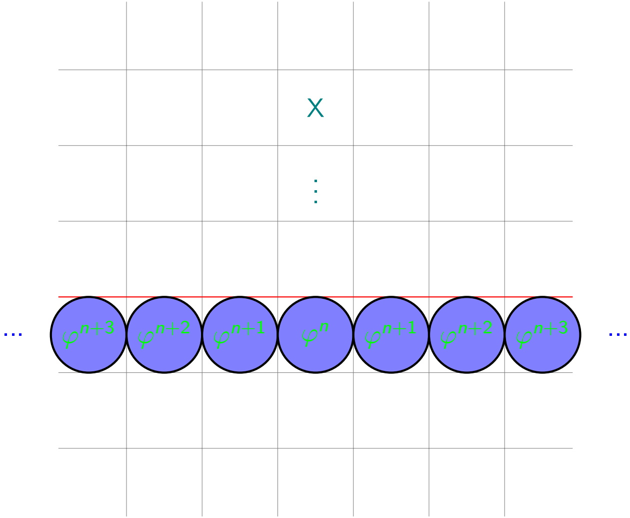
1. $1 - x - x^2 \leq 0$
2. $x \geq 0$
3. $x^2 - x - 1 \leq 0$

Solving the quadratic equations for their positive roots gives

$$\frac{\sqrt{5}-1}{2} \leq x \leq \frac{\sqrt{5}+1}{2}$$

Since we want scores to be as small as possible, we pick $x = \varphi := \frac{\sqrt{5}-1}{2} \approx 0.61803\ldots$. Note in particular that $\varphi^2 = 1 - \varphi$.

The score of a full row



The score of a full row

A full row like the previous has a score

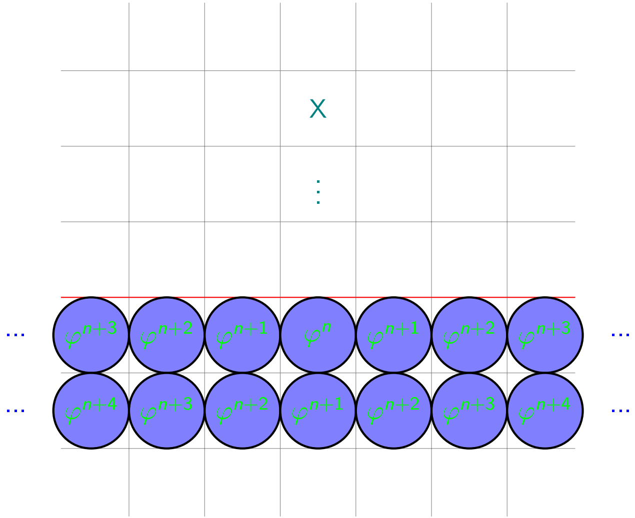
$$\begin{aligned}\varphi^n + 2\varphi^{n+1} + 2\varphi^{n+2} + \ldots &= \varphi^n \left(1 + 2\varphi + 2\varphi^2 + \ldots\right) \\ &= \varphi^n \left(1 + 2\sum_{i=1}^{\infty} \varphi^i\right)\end{aligned}$$

Either evaluate the geometric series, or notice that

$$\begin{aligned}\varphi^2 &= 1 - \varphi \\ \varphi^3 &= \varphi - \varphi^2 \\ \varphi^4 &= \varphi^2 - \varphi^3 \ldots\end{aligned}$$

So $\varphi^2 + \varphi^3 + \varphi^4 + \cdots = 1$, and the infinite row has the score $\varphi^n\,(3 + 2\varphi)$.

Adding more infinite rows



Adding more infinite rows

Adding a second infinite row below the first multiplies the score by $1 + \varphi$. With two infinite rows, the score is

$$\begin{aligned}\varphi^n(3 + 2\varphi)(1 + \varphi) &= \varphi^n(3 + 5\varphi + 2\varphi^2) \\ &= \varphi^n(5 + 3\varphi)\end{aligned}$$

Suppose instead we fill every cell in the allowed half-plane below the red line. Then the first row's score gets multiplied by
$1 + \varphi + \varphi^2 + \varphi^3 + \cdots = 2 + \varphi$.
So the full half-plane has score

$$\begin{aligned}\varphi^n(3 + 2\varphi)(2 + \varphi) &= \varphi^n(6 + 7\varphi + 2\varphi^2) \\ &= \varphi^n(8 + 5\varphi).\end{aligned}$$

Maximum possible scores

Now we can calculate the maximum possible scores for the filled infinite half-plane depending on the height n of the target square. Let S_n be the score of the filled half-plane when the target is at height n. Then:

$$
\begin{aligned}
S_1 &= \varphi(8+5\varphi) = 5+3\varphi \\
S_2 &= \varphi(5+3\varphi) = 3+2\varphi \\
S_3 &= \varphi(5+2\varphi) = 2+\varphi \\
S_4 &= \varphi(2+\varphi) = 1+\varphi \\
S_5 &= \varphi(1+\varphi) = 1
\end{aligned}
$$

Thus the score of a finite number of pieces (relative to the fifth row) is less than 1, and the score cannot increase – so no finite amount of pieces can reach the fifth row.