A (1+1)D MODEL OF FERMION-KINK COUPLING WITH ANALYTIC SOLUTIONS

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1. Introduction

We present a simple system of a fermion coupled to a pseudoscalar field (modelling a topological kink) on the circle. The physical effects of fermions on topological solitons have been investigated both on unbounded $(1+1)d$ spacetimes [1, 3], and on higher-dimensional planes and spheres [2, 5, 6], by both numerical and analytical methods.

In our model, we argue that an internal symmetry shared by the fermion and scalar underlies localisation of the fermion about the soliton. By considering bispinors, we demonstrate that the equations of motion reduce to a simple dynamical system in only bosonic coordinates. This dynamical system can be analytically solved in terms of elliptic functions, yielding qualitative and quantitative insight into physical parameters of the original fermion-kink system.

Further details of this analysis can be found in the author's Ph.D. thesis [4]. This is joint work with Steffen Krusch.

2. The model

The manifold: Spacetime is $\mathbb{R}\times\mathrm{S}^1$, with coördinates $x^\mu=(t,\theta)$ and Minkowski metric $g=\mathrm{d}t^2-\rho^2\mathrm{d}\theta^2$ for a constant length scale ρ. (We take $c=1$.)
The fields: Our model consists of a classical Dirac spinor ψ taking values in $\mathbb{C}^2$, coupled to a periodic (pseudo)scalar field ϕ taking values in $\mathbb{R}$. The periodicity condition on ϕ is specified by its **homotopy class** n_ϕ, where $\phi(t,\theta+2\pi)=\phi(t,\theta)+2n_\phi\pi$. (Preservation of n_ϕ under parity is the reason why ϕ must be pseudoscalar.)
The spin representation: We take a chiral representation for local gamma matrices γ^a,

$$\gamma^0=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \gamma^1=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad \gamma^3=\gamma^0\gamma^1=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\tag{1}$$

and by contracting with a local orthonormal frame we obtain spacetime gamma matrices γ^μ according to $\gamma^t=\gamma^0,\gamma^\theta=-\frac{1}{\rho}\gamma^1$. The Dirac operator is $\partial\!\!\!/=\gamma^\mu\partial_\mu$ and the fermion adjoint is $\overline{\psi}=\psi^\dagger\gamma^0$.
The action functional is

$$S=\int_{\mathbb{R}\times\mathrm{S}^1}\left[\frac{M^2}{2}\partial_\mu\phi\,\partial^\mu\phi+\overline{\psi}\left(\mathrm{i}\hbar\partial\!\!\!/-g\mathrm{e}^{\mathrm{i}\gamma^3\phi}\right)\psi\right]\rho\,\mathrm{d}t\,\mathrm{d}\theta.\tag{2}$$

Here,

- g is the coupling constant with dimension of energy;
- M^2 is a constant with dimension of action, interpreted as the "inertia" of the kink relative to the presence of the fermion.

The axial symmetry: Besides the Poincaré, fermion U(1) phase ("vector"), and discrete parity symmetries, the fields admit a U(1) symmetry under the joint transformations

$$\psi(t,\theta)\mapsto\mathrm{e}^{\mathrm{i}\gamma^3\frac{\alpha}{2}}\psi(t,\theta),\quad \phi(t,\theta)\mapsto\phi(t,\theta)-\alpha.\tag{3}$$

We call this the **axial** symmetry.
Currents and charge densities: The phase symmetry has the usual Noether current $J^\mu=\overline{\psi}\gamma^\mu\psi$. We use its time component to define a "density" X and a normalisation by total charge T:

$$X=\frac{1}{\hbar}J^t=|\psi|^2,\quad T=\int_{\mathrm{S}^1}|\psi|^2(\theta)\,\rho\,\mathrm{d}\theta.\tag{4}$$

The axial symmetry has Noether current

$$J^\mu_{\text{axial}}=M^2\partial^\mu\phi+\frac{\hbar}{2}\overline{\psi}\gamma^\mu\gamma^3\psi.\tag{5}$$

We define the **axial charge** W of the fermion to be

$$W=\overline{\psi}\gamma^3\psi.\tag{6}$$

A measure of parity violation

W is proportional to both J^θ, and to the fermionic contribution to $J^t_{\textbf{axial}}$.
The ansatz: We break relativistic invariance and make the ansatz that **the kink is static** and **the fermion is in a stationary state**:

$$\phi(t,\theta)=\phi(\theta)\text{ only},\quad \psi(t,\theta)=\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_ft}\psi(\theta),\tag{7}$$

where E_f is the energy density of the fermion field. In this ansatz, the equations of motion

The operator on the LHS is the Hamiltonian for the fermion field.

$$\gamma^0\left(\mathrm{i}\frac{\hbar}{\rho}\gamma^1\frac{\partial}{\partial\theta}+g\mathrm{e}^{\mathrm{i}\gamma^3\phi}\right)\psi=E_f\psi,\tag{8}$$

$$\frac{M^2}{\rho^2}\frac{\partial^2\phi}{\partial\theta^2}-\mathrm{i}g\overline{\psi}\gamma^3\mathrm{e}^{\mathrm{i}\gamma^3\phi}\psi=0,\tag{9}$$

From (5), axial current conservation is expressed as

$$\partial_\mu J^\mu_{\text{axial}}=-\frac{M^2}{\rho^2}\frac{\partial^2\phi}{\partial\theta^2}-\frac{\hbar}{2\rho}\frac{\partial|\psi|^2}{\partial\theta}=0.\tag{10}$$

Localisation: $|\psi|^2$ is extremised where $\frac{\partial\phi}{\partial\theta}$ is extremised.

3. A special case: uniform kink without backreaction

Let us further restrict to the case of a *uniform* kink,

$$\phi(\theta)=n_\phi\theta,\tag{11}$$

and ignore backreaction effects of the fermion on the kink. The paired internal axial and spatial translation symmetries are broken to the diagonal subgroup generated by the self-adjoint operator

$$\hat{L}=\hbar\left(-\mathrm{i}\frac{\partial}{\partial\theta}+\frac{n}{2}\gamma^3\right).\tag{12}$$

We call this the **generalised momentum** symmetry. $\hat{L}$ commutes with the fermionic Hamiltonian defined implicitly at (8). A joint eigenstate of energy and generalised momentum satisfying $\hat{L}\psi_l=\hbar l\psi_l$ takes the form

$$\psi_l(\theta)=\begin{pmatrix}\psi_R\\\psi_L\end{pmatrix}=\begin{pmatrix}A\mathrm{e}^{\mathrm{i}(l-\frac{n}{2})\theta}\\B\mathrm{e}^{\mathrm{i}(l+\frac{n}{2})\theta}\end{pmatrix},\tag{13}$$

"$|W|$ is maximal"

for some constants $A,B\in\mathbb{C}$, with $l\pm\frac{n}{2}\in\mathbb{Z}$. Then the eigenvalue problem at (8) is solved algebraically to give

$$E=\frac{n\hbar}{2\rho}\pm\sqrt{\frac{\hbar^2l^2}{\rho^2}+g^2}.\tag{14}$$

Once an energy level is specified, then the coefficients A,B are easily obtained in terms of n, l and g. (Their overall scale is also determined by the total charge T.)
The parity transformation preserves the uniform kink (11) and commutes with $\hat{L}^2$ but not with $\hat{L}$. An alternative joint eigenbasis for energy and parity may be constructed from the generalised momentum eigenstates schematically by taking

$$\psi^\pm_{|l|}=\frac{1}{\sqrt{2}}\left(\psi_l\pm\psi_{-l}\right).\tag{15}$$

Such parity solutions in this special case have non-constant fermionic density $X(\theta)$.

(and $W=0$)

4. Example special case parity eigenstates

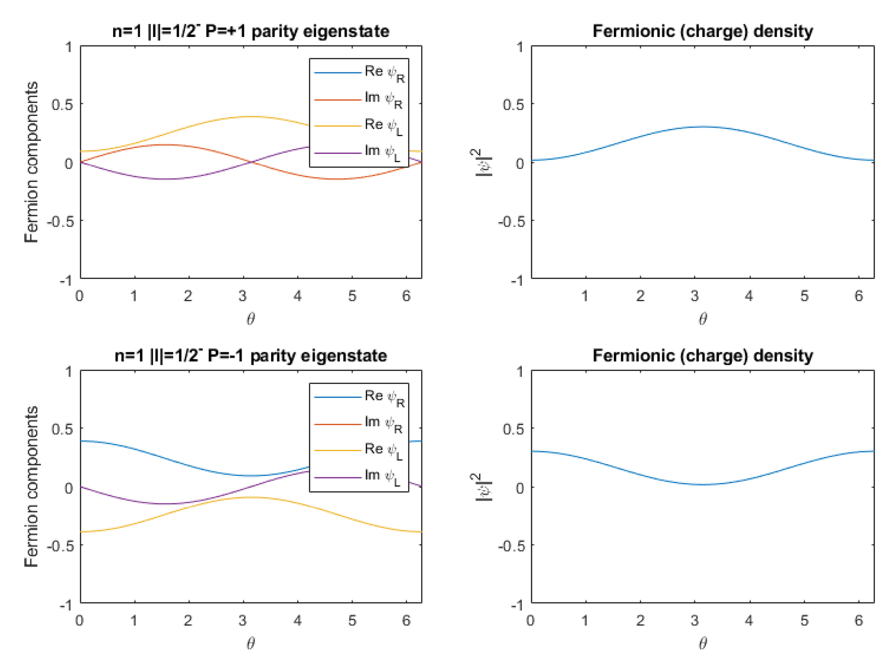

Fig. 1: $n=1$, $|l|=\frac{1}{2}$ parity eigenstates

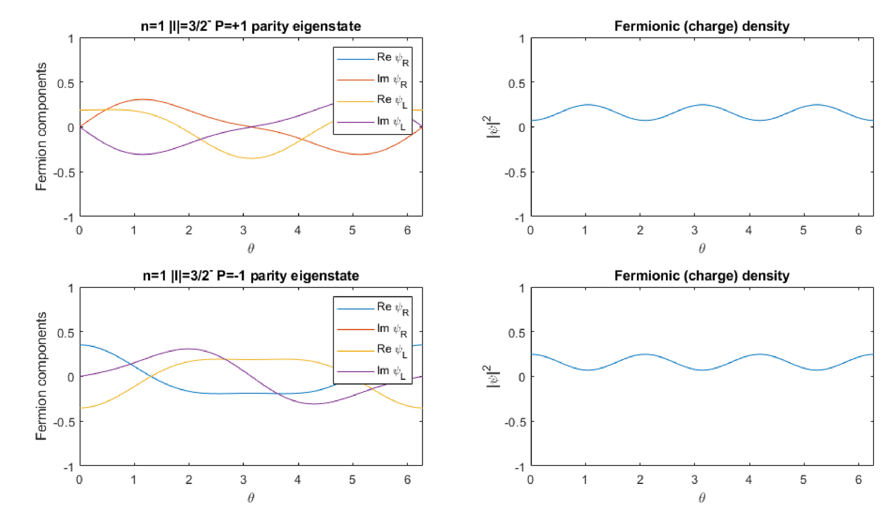

Fig. 2: $n=1$, $|l|=\frac{3}{2}$ parity eigenstates

5. Numerical parity solutions in the general case

Angular momentum eigenstates (13) of the special case are also solutions of the general case, but the parity eigenstates of the special case violate the conservation of axial current (10). Nonetheless, we numerically find solutions with non-uniform kinks where the fermion appears (to visual inspection) to be very close to these invalid parity eigenstates – compare Fig. 3 to the negative parity state in Fig. 2.

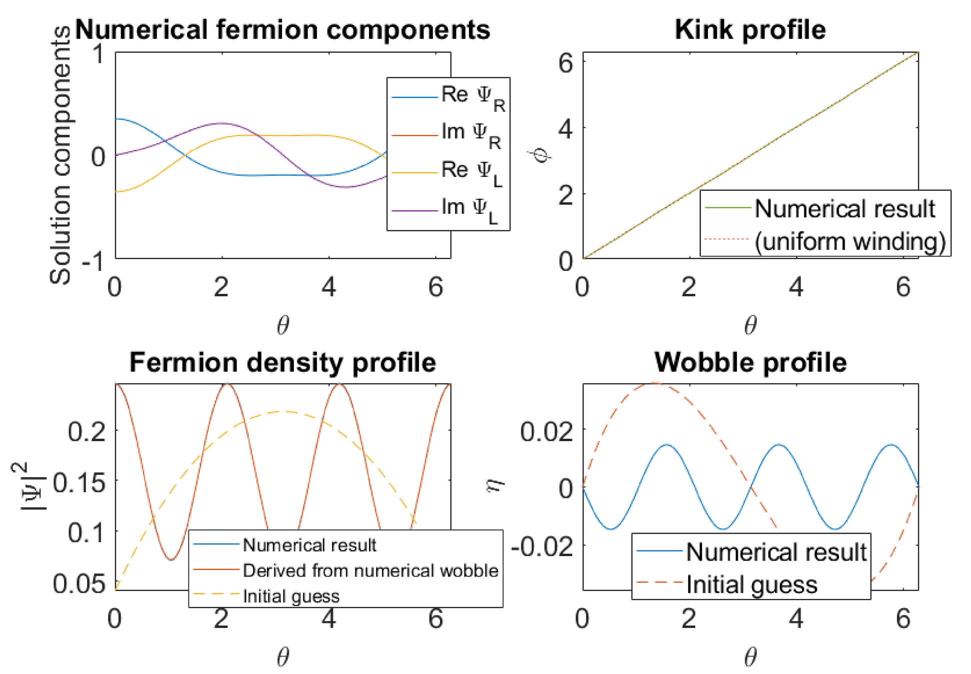

Fig. 3: A numerical solution with apparent negative parity

Moreover, the numerically determined energy densities of such solutions in the general case deviate only slightly from the analytic energy spectra of the special case, cf. Fig 4.

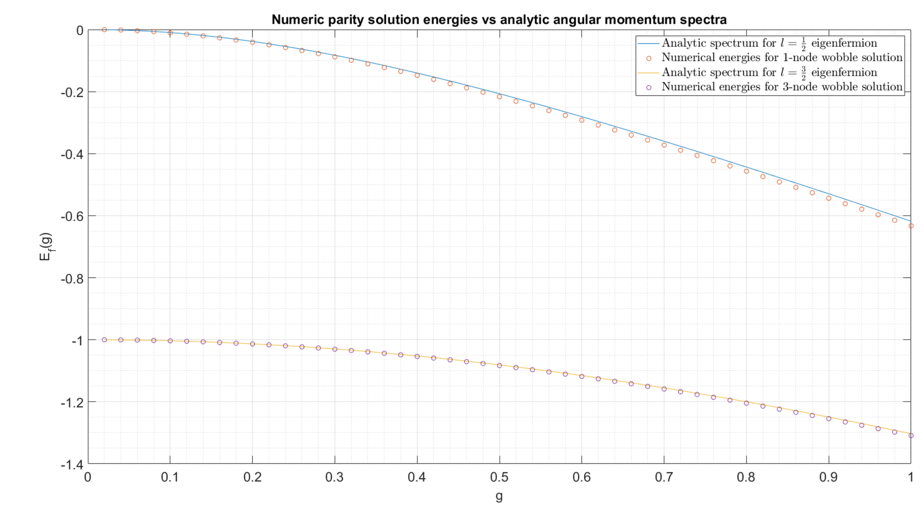

Fig. 4: Numerical and analytic energy density eigenvalues

6. The bispinor picture

In addition to X (4) and W (6), we may define two more real bispinors:

$$Y=\overline{\psi}\mathrm{e}^{\mathrm{i}\gamma^3\phi}\psi,\quad Z=\mathrm{i}\overline{\psi}\gamma^3\mathrm{e}^{\mathrm{i}\gamma^3\phi}\psi.\tag{16}$$

The equations of motion (8-9) together with axial current conservation (10) then imply the system of ODEs:

Equations (18-20) are independent of ϕ. (17) specifies ϕ in terms of X.

$$\frac{\partial\phi}{\partial\theta}=n+\frac{\hbar\rho}{2M^2}\left(\frac{T}{2\pi\rho}-X\right),\tag{17}$$

$$\frac{\hbar}{\rho}\frac{\partial X}{\partial\theta}=-2gZ,\tag{18}$$

$$\frac{\hbar}{\rho}\frac{\partial Y}{\partial\theta}=-\left(2E'+\frac{\hbar^2}{2M^2}X\right)Z,\tag{19}$$

$$\frac{\hbar}{\rho}\frac{\partial Z}{\partial\theta}=\left(2E'+\frac{\hbar^2}{2M^2}X\right)Y-2gX,\tag{20}$$

along with an algebraic identity for the axial charge W as a first integral,

$$W^2=X^2-Y^2-Z^2.\tag{21}$$

(Lorentz invariance!)

The constant E' appearing in (19) and (20) is given by

Fermionic "free energy"

$$E'=E_f-\frac{\hbar}{2\rho}\left(n+\frac{T\hbar}{4\pi M^2}\right).$$

Related to evolution of axial phase.

At the cost of introducing another constant first integral, C_0, the system (18-20) reduces to the first-order non-linear ODE

$$-\left(\frac{\hbar}{\rho}\right)^4\left(\frac{\partial X}{\partial\theta}\right)^2=\frac{\hbar^6}{16\rho^2M^4}X^4+\frac{\hbar^4E'}{\rho^2M^2}X^3+\left(\frac{4\hbar^2}{\rho^2}\left[E'^2-g^2\right]+\frac{\hbar^3C_0}{2\rho M^2}\right)X^2+\frac{4\hbar C_0E'}{\rho}X+\left(C_0^2+\frac{\hbar^2g^2W^2}{\rho^2}\right).\tag{23}$$

An ODE of this form admits analytic solutions in terms of Jacobi elliptic functions. For parity solutions with $W=0$, the RHS of (23) factorises as a pair of quadratics to give

$$-\left(\frac{\partial X}{\partial\theta}\right)^2=\left(\frac{\hbar\rho}{4M^2}X^2+\frac{2\rho}{\hbar}\left[E'-g\right]X+\frac{C_0\rho^2}{\hbar^2}\right)\left(\frac{\hbar\rho}{4M^2}X^2+\frac{2\rho}{\hbar}\left[E'+g\right]X+\frac{C_0\rho^2}{\hbar^2}\right)\tag{24}$$

In principle, E' and C_0 are still unknown constants, but we can choose the rest of the physical parameters.

7. Qualitative analytic outlook

We outline some remarks on the nature of the specific elliptic function solutions for $X(\theta)$ of the ODE (23).

- The unknown constants E' and C_0 are specified by the roots of the quartic polynomial. In the case $W=0$ with the factorisation (24), these relationships are particularly straightforward.
- The reality condition on X constrains these roots in terms of Möbius transformations of the complex plane.
- The two physical constraints of *periodicity* and *normalisation* of the fermion are encoded as a pair of transcendental curves in the space of two independent roots.
- For $W=0$, we are guaranteed that any solution to the bispinor system (18-20) extends to a well-defined solution for the underlying fermion ψ.

Therefore, we hope we can approach the problem of finding the intersections of the transcendental constraint curves numerically, at least for solutions preserved under parity. Finding such an intersection also tells us the values of T, E' and C_0!

8. Conclusions

The axial symmetry of this model makes the scalar field subordinate to the fermion, and endows a rich structure on the space of possible solutions. General solutions are approximated by the case without backreaction. We are interested in advancing the numerical investigation of this model, as well as extending the methods to kink-fermion coupling under larger symmetry groups and analogous higher-dimensional models.

9. References

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