A model of fermion-kink coupling on $\mathbb{R} \times \mathrm{S}^1$

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Abstract

We present a (1+1)-dimensional model of a (classical) Dirac fermion coupled to a scalar field as an idealised model of a fermion in the presence of a topological kink. Our spacetime manifold is $\mathbb{R} \times S^1$, with an unbounded time direction and a periodic space direction. Although the model is Lorentz-invariant, we break the relativistic symmetry to work in a privileged frame, and consider static solutions for the kink and steady state solutions for the fermion in this non-relativistic perspective, seeking to solve the energy eigenvalue problem for the fermion. We first consider the case where the kink is fixed with an idealised uniform winding in the background, so there is no back-reaction from the fermion on the kink. In this case, the problem is linear, and a symmetry of the coupling term allows us to define an appropriate "generalised angular momentum" operator, the square of whose eigenvalue is a quantum number and permits the organisation of the fermion solutions by energy levels. The energy levels are typically degenerate of dimension 2, and there are two useful bases for such energy levels: a (signed) angular momentum basis, and a parity basis. There is a conserved Noether charge W, the axial charge of the fermion, which can be understood as a measure of how badly the discrete parity symmetry is broken for any fermion solution: it vanishes for parity eigensolutions, but is maximised, in an exact sense, for angular momentum eigensolutions.

Next, we take the kink "out of the background" by introducing a Dirichlet term for the kink into the action, weighted by a parameter interpreted as the kink's "inertial mass", which resists the back-reaction from the fermion on the kink. In the limit of infinite inertial mass, the kink is restored to the background and the problem reduces to the previous special case. In general, the problem with the fully coupled kink is non-linear. Moreover, the symmetries of the system are altered in such a way that the generalised angular momentum of the previous case decomposes into two separate symmetries: spacetime translation of the fields, and a new continuous internal symmetry combining a target space rotation of the kink with an axial spin transformation of the fermion. There is no longer an immediately clear quantum number corresponding to the previous role of angular momentum. The angular momentum eigensolutions of the fermion, with a uniform kink, do remain as solutions of the general model. However, the parity eigensolutions of the special case do not remain solutions of the general case, as they violate the new internal symmetry. There is strong numerical evidence, nonetheless, that there is a new class of solutions wherein the fermion is in a definite-parity state and the kink solution is non-trivial. By considering the Lorentz-invariant bispinors of the model, we observe that under the ansatz of a static kink and a stationary-state fermion, the behaviour of the kink is fully determined by the density profile $|\psi|^2$ of the fermion field ψ, which itself obeys a non-linear differential equation that admits solutions in terms of elliptic functions. We employ the theory of the Jacobi elliptic functions to describe a specific class of analytical solutions, in particular exhibiting fermionic states of definite parity, and compare these to our numerical results. We can again observe the role of the axial Noether charge W in "interpolating" between definite-parity solutions and the preserved "angular momentum" solutions. This framework further allows us to numerically compute the energy spectrum of the fermion for this class of solutions.

Finally, we observe a connection between our model and another model of soliton-fermion coupling motivated by the so-called "chiral bag" model, where the fields also carry isospin and the coupling term is an interaction of their isospin representations. Analogous couplings have been considered in $(1+1)$-, $(2+1)$- and $(3+1)$-dimensional models of fermions and topological solitons. We find that in our case of the $(1+1)$ scalar field on spatial S^1, all of the solutions of our simpler model appear as special cases of the extended isospin model, and that the extended model improves the interpretation of some slightly unphysical features of the simple model such as the behaviour of zero-crossing modes.

Contents

1 Introduction

The dynamics of fermions coupled to topological solitons have been examined in various low-dimensional models where the coupling term is motivate by analogy to the "chiral bag" model of nucleon and fermion fields, where the fermion also carries isospin and couples via a bispinor term to the soliton in isovector representation. Spacetime is typically chosen to be either flat Minkowski space, or such that a canonical spatial slice is taken to be an n-sphere (perhaps by compactification of the original flat spacetime). Often, it is easiest to consider the soliton to be in the background of the fermion, so there is no back-reaction and the fermion can be (numerically or analytically) solved for a choice of known soliton solution. Occasionally the soliton is instead fully coupled to the fermion and solutions of the full the system are obtained. When the fermions can be solved analytically, the symmetries of the model often aid classification of solutions, and solutions may sometimes be expressed in terms of hypergeometric functions. Scattering data can be calculated for some models. A frequent feature of such models is the existence of fermionic zero modes and spectral flow.

In one space dimension, the soliton is a scalar or pseudoscalar field in a model that exhibits kink solutions interpolating between topologically distinct vacua, such as the ϕ^4 model or the sine-Gordon model on the real line. Fermions on a background ϕ^4 kink solution modelled with a very simple chiral coupling term have been solved numerically [9, 11], giving insight into scattering coefficients as well as the role of the topological winding number and fermionic zero modes to the Casimir energy of the system. Numerical solutions have also been found when the kink is fully coupled to the fermion [8]. The conventional chiral bag coupling terms models a fermion on a background sine-Gordon kink by interpreting the kink as an isovector and promoting the fermion to an isospinor representation [10]. Fermion scattering states have wavefunctions that can be expressed in terms of Heun functions. The dependence of the transmission and reflection coefficients on physical parameters such as the fermion mass is investigated numerically, and it is observed that fermionic zero modes polarise the vacuum.

In two spatial dimensions, fermions on a background baby Skyrmion in flat Minkowski space exhibit localisation by the soliton. [5] In the absence of backreaction, there is a combined rotation-isorotation symmetry of the fermions corresponding to a generalised angular momentum generator, and the conserved eigenvalue aids classification of the fermion solutions. When back-reaction is considered, it is numerically observed that a strongly coupled fermion deforms the soliton solution. One zero-crossing fermion mode is observed under spectral flow. Similarly, fermions have been considered coupled to the so-called "magnetic Skyrmion", where the role of the soliton is played by an isovector field with a Dzyaloshinskii-Moriya interaction. [4]. The backreaction of localised fermions has a significant effect on the solitons, leading to bound multisoliton solutions which did not exist in their absence.

In three spatial dimensions, the Skyrme model on a spatial 3-sphere exhibits BPS solutions. The $B = 1$ solution is particularly straightforward and has been used as a background for spin-isospin fermions. [6, 7] Similarly to the baby Skyrmion model in 2+1 dimensions, there is a generalised angular momentum of the fermion around the background Skyrmion providing a quantum number. A full description of fermion solutions is possible with some analytical tools by observing that part of the energy eigenvalue problem is expressed as a Fuchsian differential equation. A family of solutions can be expressed in terms of a basis of spinorial eigenfunctions of the Dirac operator, similar to the role of so-called "spinor harmonics" or "monopole harmonics" on the 2-sphere [1, 2].

In this paper, we examine a model of fermion-kink coupling on $\mathbb{R} \times S^1$. By analogy with the simplicity of the $B = 1$ BPS Skyrmion on S^3, we begin with a fixed scalar field taking values in S^1 in the background of the kink, prescribed to wind uniformly n times over a period of the base space. A suitable definition of generalised angular momentum permits a full decomposition of fermion solutions over an angular momentum eigenbasis, with the energy spectrum determined by this angular momentum quantum number and a discrete choice of energy branch. We demonstrate that the discrete parity symmetry can also be taken to define an eigenbasis which is just an orthogonal transformation of the angular momentum basis. A conserved Noether charge measures the "mixing" of these bases in an exact sense.

We extend the model to fully incorporate the scalar field dynamically. The angular momentum solutions are preserved, but the parity solutions of the previous special case are not solutions of the extended model. Nonetheless, there is numerical evidence that there is a new class of definite-parity fermion solutions with "non-trivial", i.e. not just uniformly winding, kink solutions. By restricting to an ansatz of static kink and stationary state fermions, and changing variables via considering Lorentz-invariant bispinors, we demonstrate that solutions to this ansatz may be expressed in terms of elliptic functions, and describe some features of these analytical solutions. Finally, we show that all solutions of our primary model appear as a subset of solutions of a similar model where the fields also carry isospin, in a manner which clarifies some of the behaviour of zero-crossing fermionic modes.

In a future paper we will extend a similar treatment to a model of baby Skyrmions coupled to kinks on S^2.

1.1 Conventions and representations

Throughout, we specifically retain the reduced Planck's constant $\hbar$, and the length scale ρ giving the radius of S^1, so that we can examine limiting behaviour of these parameters in the future. We set $c = 1$, and take the spacetime metric

$$g = \mathrm{d}t \otimes \mathrm{d}t - \rho^2 \,\mathrm{d}\theta \otimes \mathrm{d}\theta. \tag{1}$$

We take a local orthonormal frame to be ${e_\alpha}^\mu = \mathrm{diag}(1, -\frac{1}{\rho})$. The local tangent vector basis is given by

$$\hat{e}_0 = \frac{\partial}{\partial t}, \hat{e}_1 = -\frac{1}{\rho}\frac{\partial}{\partial \theta} \tag{2}$$

and the local dual basis is

$$\hat{\vartheta}^0 = \mathrm{d}t, \hat{\vartheta}^1 = -\rho \,\mathrm{d}\theta. \tag{3}$$

With respect to this frame, $g(\hat{e}_\alpha, \hat{e}_\beta) = \eta_{\alpha\beta} = \mathrm{diag}(1, -1)$.

For the spin representation, we take the chiral representation of matrices γ^α satisfying $\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}$:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \Rightarrow \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

It is useful to introduce an overall energy scale,

$$a = \frac{\hbar}{\rho}. \tag{5}$$

Adopting this notation, the Dirac operator can be expressed quite simply in terms of the familiar local gamma matrices γ^0, γ^1 as

$$\mathrm{i}\hbar\partial\!\!\!/ = \mathrm{i}\hbar\gamma^0\frac{\partial}{\partial t} - \mathrm{i}a\gamma^1\frac{\partial}{\partial\theta}. \tag{6}$$

(Note that in two dimensions, there is no spin connection.)

We conventionally view all transformations as active: under a (say, Lorentz) transformation Λ of spacetime coordinate,

$$x \mapsto \Lambda x, \tag{7}$$

a generic field $\phi(x)$ on spacetime transforms as

$$\phi'(x) = R_\Lambda \phi(\Lambda^{-1}x), \tag{8}$$

where R_Λ is the appropriate representation of the transformation Λ on the field ϕ.

In the local basis, define the Lorentz invariant and alternating tensor $\hat{\varepsilon}_{\alpha\beta}$ according to

$$\hat{\varepsilon}_{01} = 1 = -\hat{\varepsilon}_{10}, \tag{9}$$

the indices of which we can raise with the local Minkowski metric $\eta^{\alpha\beta}$. Then we have the following useful identities in the local Clifford algebra:

(i) $\gamma^3 = \frac{1}{2}\hat{\varepsilon}_{\alpha\beta}\gamma^\alpha\gamma^\beta$,

(ii) $\gamma^\alpha\gamma^\beta = \eta^{\alpha\beta} - \hat{\varepsilon}^{\alpha\beta}\gamma^3$,

(iii) $\gamma^\alpha\gamma^3 = \hat{\varepsilon}^\alpha{}_\beta\gamma^\beta$.

Contracting with the frame converts this to a Lorentz invariant, alternating tensor in the spacetime basis, but note that e.g. $\hat{\varepsilon}_{\mu\nu} = \hat{\varepsilon}_{\alpha\beta}e^\alpha{}_\mu e^\beta{}_\nu$ is *not* the usual Levi-Civita symbol, as it is not isotropic.

2 The prescribed kink

2.1 Lagrangian and symmetries

Consider a model for a dynamical Dirac fermion $\psi(t,\theta)$ with Lagrangian density

$$\mathcal{L}_f = \overline{\psi}\left(\mathrm{i}\hbar\partial\!\!\!/ - g\mathrm{e}^{\mathrm{i}\gamma^3 n\theta}\right)\psi, \tag{10}$$

where g is the coupling constant and $n \in \mathbb{Z}$. This models a fermion coupled to a kink field $\phi(t,\theta) : S^1 \to S^1$ where the kink is prescribed to be uniform and static with winding number n. The Dirac equation(s) are

$$\left(\mathrm{i}\hbar\partial\!\!\!/ - g\mathrm{e}^{\mathrm{i}\gamma^3 n\theta}\right)\psi = 0 \tag{11}$$

and the spinorial conjugate equation.

The Lagrangian is manifestly time-invariant, leading to a conserved Noether charge corresponding to the total energy,

$$\mathrm{E}_{\mathrm{tot}} = \int_0^{2\pi} \psi^\dagger\gamma^0\left(\mathrm{i}a\gamma^1\frac{\partial}{\partial\theta} + g\mathrm{e}^{\mathrm{i}\gamma^3 n\theta}\right)\psi\,\rho\,\mathrm{d}\theta = \int_0^{2\pi}\psi^\dagger\hat{H}\psi\,\rho\,\mathrm{d}\theta, \tag{12}$$

with $\frac{\partial \mathrm{E}_{\mathrm{tot}}}{\partial t} = 0$. We use the above expression to define the self-adjoint Hamiltonian operator $\hat{H}$. The equation of motion (11) can be written in as a Schrödinger-type equation,

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi. \tag{13}$$

We will seek in particular *steady state* solutions, of the form

$$\psi(t,\theta) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_f t}\psi(\theta), \tag{14}$$

where the fermionic energy density E_f must therefore be a (real, constant) eigenvalue of $\hat{H}$. If we normalise the fermion so that

$$\int_0^{2\pi}\psi^\dagger\psi\,\rho\,\mathrm{d}\theta = 1, \tag{15}$$

there is no need to distinguish between the fermionic energy density eigenvalue E_f, and the total energy $\mathrm{E}_{\mathrm{tot}}$; we will exclusively refer to E_f and call it the energy of the fermion.

When we have prescribed the kink in the background like so, and are only free to transform the fermion field, we observe a continuous symmetry under which

$$\psi(t,\theta) \mapsto \mathrm{e}^{-\mathrm{i}\alpha\frac{n}{2}\gamma^3}\psi(t,\theta-\alpha), \tag{16}$$

with conserved charge (by Noether's theorem)

$$Q_L = \int_{S^1} \left(-\frac{\partial \mathcal{L}_f}{\partial(\partial_t \psi)} \left(\mathrm{i}\gamma^3 \frac{n}{2} \right) + \vartheta^t{}_\theta \right) \mathrm{d}\theta \tag{17}$$

(where $\vartheta^\mu{}_\nu$ is the energy-momentum tensor)

$$= \int_{S^1} \psi^\dagger \left(-i\hbar \frac{\partial}{\partial\theta} + \hbar \frac{n}{2}\gamma^3 \right) \psi \, \mathrm{d}\theta = \int_{S^1} \psi^\dagger \hat{L} \psi \, \mathrm{d}\theta \tag{18}$$

We define the self-adjoint *generalised angular momentum* operator $\hat{L}$ according to the above:

$$\hat{L} = \hbar \left(-\mathrm{i}\frac{\partial}{\partial\theta} + \frac{n}{2}\gamma^3 \right). \tag{19}$$

It is easily checked that

$$[\hat{L}, \hat{H}] = 0, \tag{20}$$

so we can seek a joint eigenbasis of static solutions for both energy and this angular momentum. There is also a discrete parity symmetry,

$$\psi(t,\theta) \mapsto \gamma^0 \psi(t, 2\pi - \theta). \tag{21}$$

2.2 Solutions in angular momentum and parity bases

Our aim is to solve the eigenvalue equation for a static fermion,

$$\hat{H}\psi(\theta) = E\psi(\theta). \tag{22}$$

We first derive the eigenspinors $\psi_l(\theta)$ of the operator $\hat{L}$ defined above at (19), satisfying

$$\hat{L}\psi_l(\theta) = \hbar l \psi_l(\theta). \tag{23}$$

In the chiral basis, we write the fermion as

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \tag{24}$$

where the subscripts R and L refer respectively to the right- and left-handed chiral (Weyl) components. In this basis, equation (23) is expressed

$$\hbar \begin{pmatrix} -\mathrm{i}\frac{\partial}{\partial\theta} + \frac{n}{2} & 0 \\ 0 & -\mathrm{i}\frac{\partial}{\partial\theta} - \frac{n}{2} \end{pmatrix} \begin{pmatrix} \psi_l^R \\ \psi_l^L \end{pmatrix} = \hbar l \begin{pmatrix} \psi_l^R \\ \psi_l^L \end{pmatrix}. \tag{25}$$

Thus the angular momentum eigenspinors are

$$\psi_l(\theta) = \begin{pmatrix} A\mathrm{e}^{\mathrm{i}(l-\frac{n}{2})\theta} \\ B\mathrm{e}^{\mathrm{i}(l+\frac{n}{2})\theta} \end{pmatrix}, \tag{26}$$

where A, B are complex coefficients and $l \pm \frac{n}{2}$ must be an integer.

Susbstituting the above expression for an angular momentum eigenspinor into the energy eigenvalue equation (22), we observe that he energy eigenvalue problem reduces to the small finite-dimensional eigenvalue problem

$$\det(H_{\mathrm{alg}} - E\mathbb{I}) = 0, \tag{27}$$

where we define H_{alg} to be the *algebraic Hamiltonian* for angular momentum eigenvalue l:

$$H_{\mathrm{alg}} = \begin{pmatrix} -a(l - \frac{n}{2}) & g \\ g & a(l + \frac{n}{2}) \end{pmatrix}, \tag{28}$$

The algebraic eigenvalue problem (27) gives a quadratic equation in the energy E,

$$\left(E - a\left(l + \frac{n}{2}\right)\right)\left(E + a\left(l - \frac{n}{2}\right)\right) - g^2 = 0. \tag{29}$$

The solutions give the energy spectrum in terms of n, l and g:

$$E = \frac{an}{2} \pm \sqrt{a^2l^2 + g^2}. \tag{30}$$

Thinking of the spectrum as dependent on the coupling constant g, we see that there are two branches for each choice of n and l. On the upper branch, taking $E = E^+ = \frac{an}{2} + \sqrt{a^2l^2 + g^2}$, the eigenvector condition on the coefficients A, B becomes

$$\frac{B}{A} = \frac{al + \sqrt{a^2l^2 + g^2}}{g}. \tag{31}$$

Thus an explicit (not normalised) solution on the upper energy branch is

$$\psi_{l,E^+}(t,\theta) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\left(\frac{an}{2} + \sqrt{a^2l^2+g^2}\right)t} \begin{pmatrix} g\mathrm{e}^{\mathrm{i}(l-\frac{n}{2})\theta} \\ (al + \sqrt{a^2l^2+g^2})\mathrm{e}^{\mathrm{i}(l+\frac{n}{2})\theta} \end{pmatrix}. \tag{32}$$

On the lower energy branch, taking $E = E^- = \frac{an}{2} - \sqrt{a^2l^2 + g^2}$, the condition on the coefficients is

$$\frac{B}{A} = \frac{al - \sqrt{a^2l^2 + g^2}}{g}, \tag{33}$$

and an explicit solution on the lower branch is

$$\psi_{l,E^-}(t,\theta) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\left(\frac{an}{2} - \sqrt{a^2l^2+g^2}\right)t} \begin{pmatrix} g\mathrm{e}^{\mathrm{i}(l-\frac{n}{2})\theta} \\ (al - \sqrt{a^2l^2+g^2})\mathrm{e}^{\mathrm{i}(l+\frac{n}{2})\theta} \end{pmatrix}. \tag{34}$$

Consider the behaviour of these solutions as g tends to 0. Evaluating the limit carefully – noting, for example, that

$$\frac{al - \sqrt{a^2l^2 + g^2}}{g} = \frac{-g}{al + \sqrt{a^2l^2 + g^2}} \tag{35}$$

– we see that the solution on the upper branch becomes a left-handed Weyl spinor, while the solution on the lower branch becomes a right-handed Weyl spinor:

$$\lim_{g\to 0} \psi_{l,E^+}(\theta) = \begin{pmatrix} 0 \\ 2al\mathrm{e}^{\mathrm{i}(l+\frac{n}{2})\theta} \end{pmatrix} \tag{36}$$

$$\lim_{g\to 0} \psi_{l,E^-}(\theta) = \begin{pmatrix} 2al\mathrm{e}^{\mathrm{i}(l-\frac{n}{2})\theta} \\ 0 \end{pmatrix}. \tag{37}$$

The parity operation $\hat{P}\psi(t,\theta) = \gamma^0\psi(t, 2\pi - \theta)$ commutes with the Hamiltonian, but not with the angular momentum operator $\hat{L}$; parity exchanges an eigenstate of angular momentum l with the corresponding eigenstate of angular momentum $-l$. We define an energy eigenstate $\psi^P_{|l|\pm} = \psi^P_{|l|,E^\pm}$ of definite parity $P = \pm 1$ to be a state satisfying

$$\gamma^0\psi^P_{|l|,E^\pm}(t, 2\pi - \theta) = P\psi^P_{|l|,E^\pm}(t,\theta). \tag{38}$$

Over the angular momentum basis, we see that (appropriately normalised) states of definite parity are (WLOG $l \geq 0$):

$$\psi^{+1}_{|l|} = \frac{1}{\sqrt{2}}\left(\psi_l + \psi_{-l}\right), \tag{39}$$

$$\psi^{-1}_{|l|} = \frac{1}{\sqrt{2}}\left(\psi_l - \psi_{-l}\right). \tag{40}$$

Explicit co-ordinate expressions can be given as the following:

$$\psi^{+}_{|l|\pm}(\theta) = \frac{1}{4\sqrt{\pi(a^2l^2+g^2)}}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\frac{n}{2}\theta}\left[\left(\sqrt{g^2+(-al\pm\sqrt{a^2l^2+g^2})^2}+\sqrt{g^2+(al\pm\sqrt{a^2l^2+g^2})^2}\right)\cos(l\theta)\right. \\ \left. +\,\mathrm{i}\left(\sqrt{g^2+(-al\pm\sqrt{a^2l^2+g^2})^2}-\sqrt{g^2+(al\pm\sqrt{a^2l^2+g^2})^2}\right)\sin(l\theta)\right] \\ \mathrm{e}^{\mathrm{i}\frac{n}{2}\theta}\left[\left(\sqrt{g^2+(al\pm\sqrt{a^2l^2+g^2})^2}+\sqrt{g^2+(-al\pm\sqrt{a^2l^2+g^2})^2}\right)\cos(l\theta)\right. \\ \left. +\,\mathrm{i}\left(\sqrt{g^2+(al\pm\sqrt{a^2l^2+g^2})^2}-\sqrt{g^2+(-al\pm\sqrt{a^2l^2+g^2})^2}\right)\sin(l\theta)\right] \end{pmatrix} \tag{41}$$

$$\psi^{-}_{|l|\pm}(\theta) = \frac{1}{4\sqrt{\pi(a^2l^2+g^2)}}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\frac{n}{2}\theta}\left[\left(\sqrt{g^2+(-al\pm\sqrt{a^2l^2+g^2})^2}-\sqrt{g^2+(al\pm\sqrt{a^2l^2+g^2})^2}\right)\cos(l\theta)\right. \\ \left. +\,\mathrm{i}\left(\sqrt{g^2+(-al\pm\sqrt{a^2l^2+g^2})^2}+\sqrt{g^2+(al\pm\sqrt{a^2l^2+g^2})^2}\right)\sin(l\theta)\right] \\ \mathrm{e}^{\mathrm{i}\frac{n}{2}\theta}\left[\left(\sqrt{g^2+(al\pm\sqrt{a^2l^2+g^2})^2}-\sqrt{g^2+(-al\pm\sqrt{a^2l^2+g^2})^2}\right)\cos(l\theta)\right. \\ \left. +\,\mathrm{i}\left(\sqrt{g^2+(al\pm\sqrt{a^2l^2+g^2})^2}+\sqrt{g^2+(-al\pm\sqrt{a^2l^2+g^2})^2}\right)\sin(l\theta)\right] \end{pmatrix} \tag{42}$$

We present some plots at Figure 1 of the energy spectra $E(g)$ as functions of g for some small non-negative values of n and l. In all cases we have set the energy scale at $a = 1$.

Note that there is a U(1) symmetry of the fermion corresponding to a global phase change:

$$\psi \mapsto \mathrm{e}^{\mathrm{i}\alpha}\psi, \tag{43}$$

for real constant α. The Noether current is the usual spinor U(1) current, $\frac{\hbar}{2}\overline{\psi}\gamma^\mu\psi$. Observe that the time component is proportional to the local density $\psi^\dagger\psi$ we have used to normalise the fermion, invariant under purely spatial transformations. Since this is time-independent for stationary states, the space component $-\frac{a}{2}\psi^\dagger\gamma^3\psi$ is independent of θ and proportional to $|\psi_R|^2 - |\psi_L|^2$. It is convenient to define $W = \psi^\dagger\gamma^3\psi$ and call this the axial charge of the fermion. Note that for parity solutions, $W = 0$. For angular momentum solutions $\psi_{l,E^\pm}$, we find

$$W = \mp\frac{al}{\sqrt{a^2l^2+g^2}}\frac{T}{2\pi}. \tag{44}$$

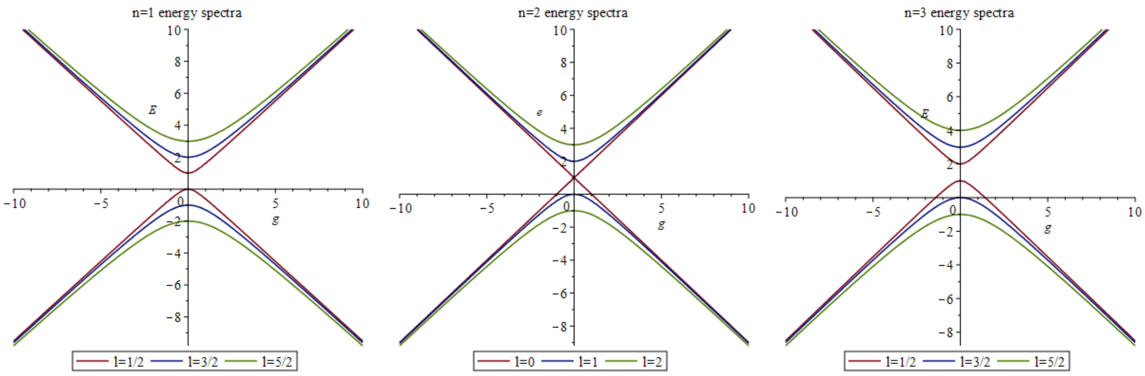

Figure 1: Plots of the fermion energy spectra $E(g)$ in the pure chiral model for small non-negative values of n and l. ($a = 1$)

We will later show that this strictly maximises the possible value of W^2 within the given energy level.

Having determined the general dependence of the energy spectrum on n and l, we can comment on the existence of zero-crossing modes. We solve the energy spectrum (30) for $E = 0$, obtaining

$$g_*^2 = \frac{a^2n^2}{4} - a^2l^2, \tag{45}$$

where g_* is the value of the coupling constant which admits a zero-energy solution for the given n and l. Since we require the coupling constant to be real, we see that the existence of zero-crossing modes is controlled by the inequality

$$|2l| \le |n|. \tag{46}$$

When this inequality is strictly satisfied, the fermion solution exhibits zero energy at non-trivial values of the coupling constant, whereas when it is saturated, the fermion goes to zero energy when the coupling constant goes to zero. We observe that zero-crossing modes can only exist of the angular momentum of the fermion is sufficiently low in comparison to the winding number of the kink.

2.3 A numerical perspective

It will be useful to numerically explore soliton-fermion systems where the determination of the spectrum is not so analytically straightforward. To this end, we can also demonstrate numerical solutions to the stationary-state equation (13) of this model. We use the MATLAB package `bvp4c` which can handle ordinary differential equation boundary value problems with unknown parameters (such as, in our case, the energy eigenvalue $E = E_f$). In order to implement this package, it is necessary to re-write the system in terms of real first order ODEs. We express ψ as

$$\psi = \begin{pmatrix} y_1 + \mathrm{i}y_2 \\ y_3 + \mathrm{i}y_4 \end{pmatrix}, \tag{47}$$

with equations of motion:

$$y_1' = \frac{E}{a}y_2 - \frac{g}{a}y_4\cos(n\theta) + \frac{g}{a}y_3\sin(n\theta) \tag{48}$$

$$y_2' = -\frac{E}{a}y_1 + \frac{g}{a}y_3\cos(n\theta) + \frac{g}{a}y_4\sin(n\theta) \tag{49}$$

$$y_3' = -\frac{E}{a}y_4 + \frac{g}{a}y_2\cos(n\theta) + \frac{g}{a}y_1\sin(n\theta) \tag{50}$$

$$y_4' = \frac{E}{a}y_3 - \frac{g}{a}y_1\cos(n\theta) + \frac{g}{a}y_2\sin(n\theta) \tag{51}$$

E is treated as a further unknown which `bvp4c` will also attempt to determine, so in total there are five unknown functions. We will need five appropriate boundary conditions on the vector $\mathbf{y}(\theta)$, as well as an initial guess of the solution.

This linear system of ODEs is linear, and thenormalisation of the fermion is unimportant. However, we will see in future models that the choice of normalisation can be physically relevant, and mathematically relevant to solving the equations of motion. It is useful therefore to also numerically implement the integral condition that, for some chosen real constant T,

$$\int_0^{2\pi} \psi^\dagger \psi \, \rho \, \mathrm{d}\theta = T. \tag{52}$$

We achieve this by introducing one more a priori unknown function, y_5, representing the cumulative global fermionic density (formally, a U(1) charge),

$$y_5(\theta) = \int_0^{\theta} |\psi(\theta')|^2 \, \rho \, \mathrm{d}\theta'. \tag{53}$$

Thus to the above equations, we add

$$y_5' = y_1^2 + y_2^2 + y_3^2 + y_4^2, \tag{54}$$

with the boundary conditions $y_5(0) = 0$ and $y_5(2\pi) = T$. Even though this introduces a further unknown, it provides two natural boundary conditions on the system, so we only require four further boundary conditions; it is natural to impose periodicity of each real component of the fermion in order to provide the correct number. For our initial guess to an eigensolution for chosen n and $l^{\pm}$, we use our knowledge of the analytic solutions to choose an approximate energy, and give a 'solution' which has the correct Fourier modes of the fermion components active, but scaled uniformly,

$$y_1 = \frac{T}{4\pi}\cos\left(\left(l - \frac{n}{2}\right)\theta\right) \tag{55}$$

$$y_2 = \frac{T}{4\pi}\sin\left(\left(l - \frac{n}{2}\right)\theta\right) \tag{56}$$

$$y_3 = \frac{T}{4\pi}\cos\left(\left(l + \frac{n}{2}\right)\theta\right) \tag{57}$$

$$y_4 = \frac{T}{4\pi}\sin\left(\left(l + \frac{n}{2}\right)\theta\right), \tag{58}$$

and uniform charge density for y_5,

$$y_5 = \frac{T\theta}{2\pi}. \tag{59}$$

As long as the initial guess of the energy eigenvalue is not too far from the true value, `bvp4c` will typically converge fairly quickly to the correct eigensolution. We display some numerically obtained angular momentum solutions at Figures 2 and 3; typically, the calculated energy eigenvalue is accurate to approximately one part in 10^9. In Figures 4 and 5, we plot some numerically determined parity eigenstates along with their local densities, which are nonconstant for parity solutions, unlike angular momentum solutions.

As a numerical tool to explore the structure of the energy spectrum for varying g (which, again, will be useful in more complicated models), we can iterate this process, by using a numerical solution computed at one value of g as the initial guess for the code run at a slightly different value of g. Once again, provided the initial guess of the energy eigenvalue is good, we can iteratively reproduce the analytic spectra to good accuracy; some results are plotted at Figure 6.

3 The fully coupled model

3.1 Adding kink dynamics

To introduce kink dynamiocs to the model, we introduce a Dirichlet energy term for the kink into the Lagrangian density,

$$\mathcal{L}_{\text{kink}} = \frac{M^2}{2}\partial_\mu\phi\,\partial^\mu\phi. \tag{60}$$

M^2 is a parameter with dimensions of mass squared, which must be inserted on dimensional grounds. It is interpreted as an inertial mass for the kink with respect to the effect of the fermion: when $M^2 \gg g$, the behaviour of the kink does not depend on the fermion.

Observe that a dynamical system with a Lagrangian density consisting of just the above term on the circle is a real Klein-Gordon system on the circle, admitting the static idealised solutions of the form $\phi(\theta) = n\theta$. Such solution are the lowest-energy solution in the homotopy class of solutions with winding number n. This justifies introducing this term into the Lagrangian for the kink-fermion system to study the back-reaction of the fermion on the kink. We now take the Lagrangian density,

$$\mathcal{L} = \frac{M^2}{2}\partial_\mu\phi\,\partial^\mu\phi + \overline{\psi}\left(\mathrm{i}\hbar\partial\!\!\!/ - g\mathrm{e}^{\mathrm{i}\gamma^3\phi}\right)\psi. \tag{61}$$

The Euler-Lagrange equations for (61) are the Dirac equation(s) for the fermion,

$$\left(\mathrm{i}\hbar\partial\!\!\!/ - g\mathrm{e}^{\mathrm{i}\gamma^3\phi}\right)\psi = 0 \tag{62}$$

and its fermionic conjugate, as previously, in addition to a new equation of motion for the kink,

$$M^2\partial_\mu\partial^\mu\phi + \mathrm{i}g\overline{\psi}\gamma^3\mathrm{e}^{\mathrm{i}\gamma^3\phi}\psi = 0. \tag{63}$$

By taking the Hermitian conjugate and using the anticommutation relations of the γ matrices, we see that this equation is real, as we require for the real kink. Note that, as claimed, if $M \gg g$ then the kink dynamics decouple from the fermion.

As the Lagrangian density is explicitly independent of θ, the previous generalised angular momentum decouples into two distinct continuous symmetries: translation of the fields on the base space is now a symmetry in its own right, while there is also the internal symmetry of the fields

$$\psi(t,\theta) \mapsto \mathrm{e}^{\mathrm{i}\gamma^3\frac{\alpha}{2}}\psi(t,\theta), \quad \phi(t,\theta) \mapsto \phi(t,\theta) - \alpha. \tag{64}$$

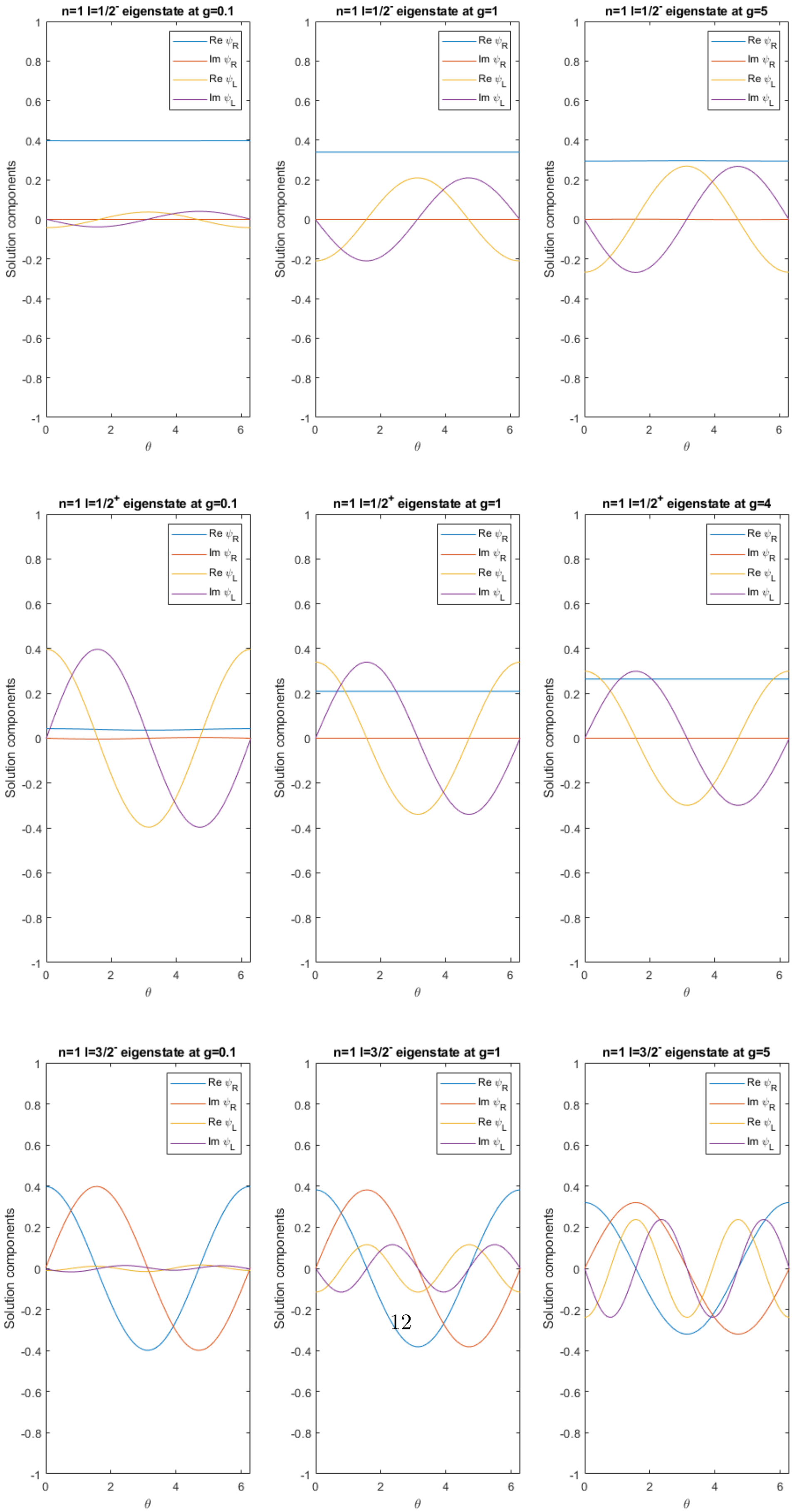

Figure 2: Numerical solutions for some eigenfermions of the uniform kink across values of g ($T = 1$).

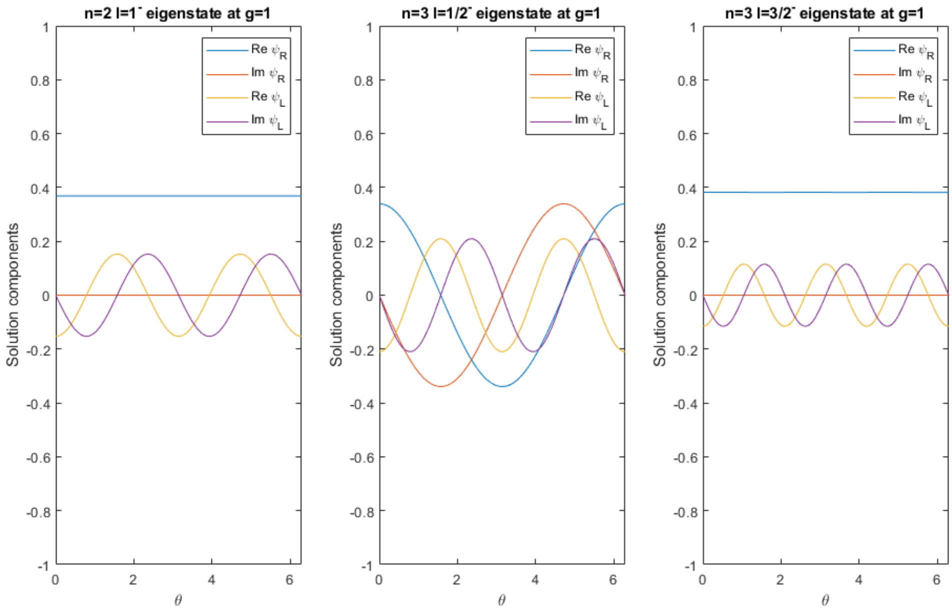

Figure 3: Some other eigensolutions on negative energy branches at $g = 1$.

We refer to this as the axial transformation. The corresponding axial current is

$$J^\mu_{\text{axial}} = \frac{M^2}{\rho}\partial^\mu \phi + \frac{\hbar}{2\rho}\overline{\psi}\gamma^\mu\gamma^3\psi \tag{65}$$

with conservation equation $\partial_\mu J^\mu_{\text{axial}} = 0$, and time-independent total axial charge

$$Q^0_{\text{axial}} = \frac{1}{\rho}\int_0^{2\pi}\left(M^2\frac{\partial\phi}{\partial t} + \frac{\hbar}{2}\psi^\dagger\gamma^3\psi\right)\rho\,\mathrm{d}\theta. \tag{66}$$

The Lagrangian also exhibits the following discrete parity symmetry:

$$\psi(t,\theta) \mapsto \gamma^0\psi(t, 2\pi - \theta) \tag{67}$$

$$\phi(t,\theta) \mapsto 2n\pi - \phi(t, 2\pi - \theta) \tag{68}$$

The effect of the transformation on the kink is specifically chosen to preserve its overall homotopy class as n, rather than switching it to $-n$. This somewhat arbitrarily enforces that ϕ is a Lorentz pseudoscalar rather than scalar; we will comment further on this matter later. Note that the prescribed uniform kink $\phi(t,\theta) = n\theta$ is invariant under parity, and the parity transform on the system as a whole squares to the identity.

We take the ansatz that the kink is static, i.e. time-independent, and that the fermion is a stationary state as before. Then the axial charge density W is once again constant. Moreover, the axial current conservation in this static case gives a relationship between the fermionic charge density and the non-trivial behaviour of the kink:

$$\partial_\mu J^\mu_{\text{axial}} = -\frac{M^2}{\rho^2}\frac{\partial^2\phi}{\partial\theta^2} - \frac{\hbar}{2\rho}\frac{\partial|\psi|^2}{\partial\theta} = 0. \tag{69}$$

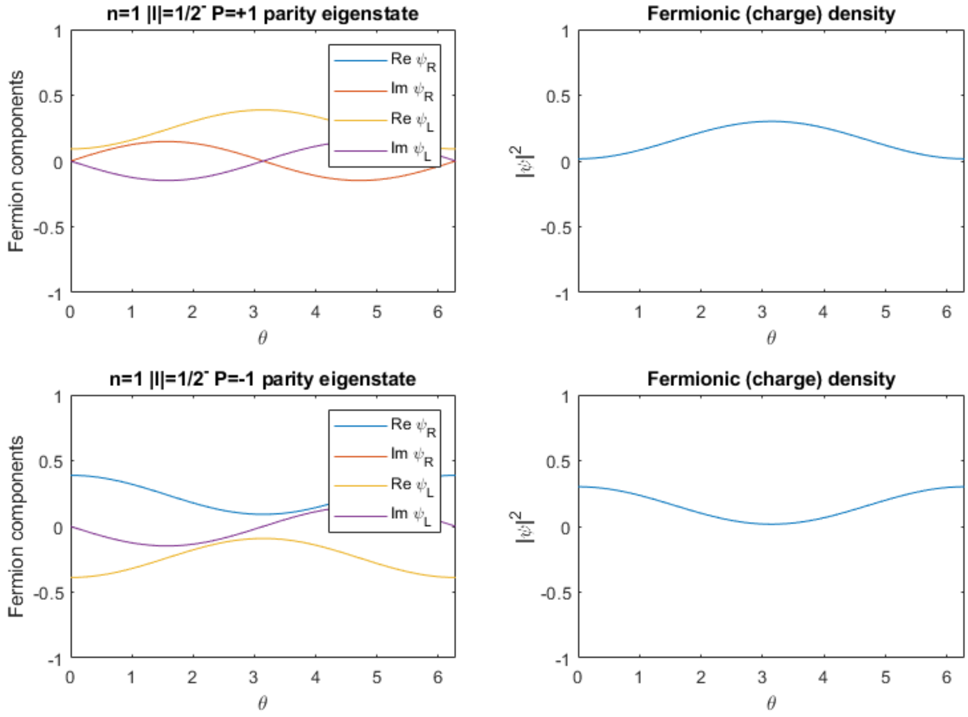

Figure 4: $|l|^{\pm} = \frac{1}{2}^{-}$ parity eigenstates

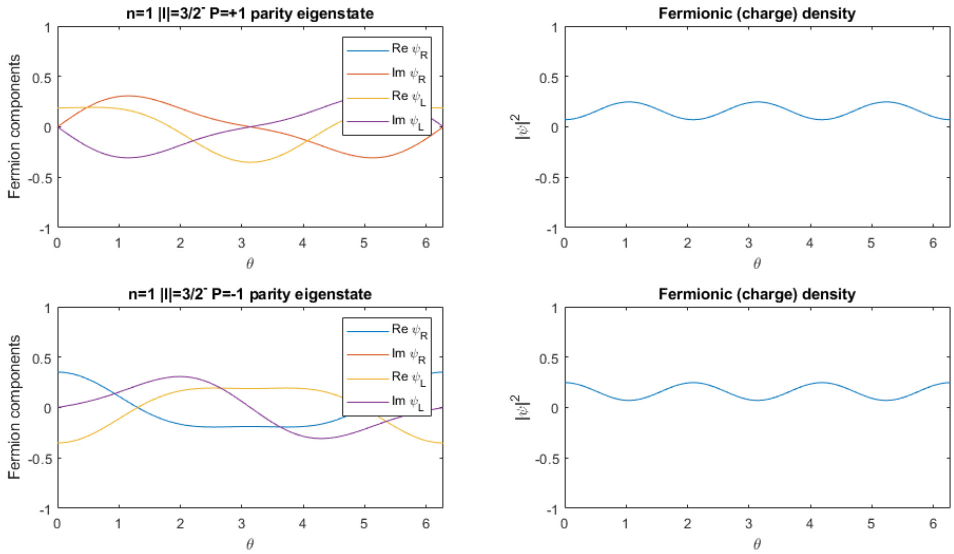

Figure 5: $|l|^{\pm} = \frac{3}{2}^{-}$ parity eigenstates

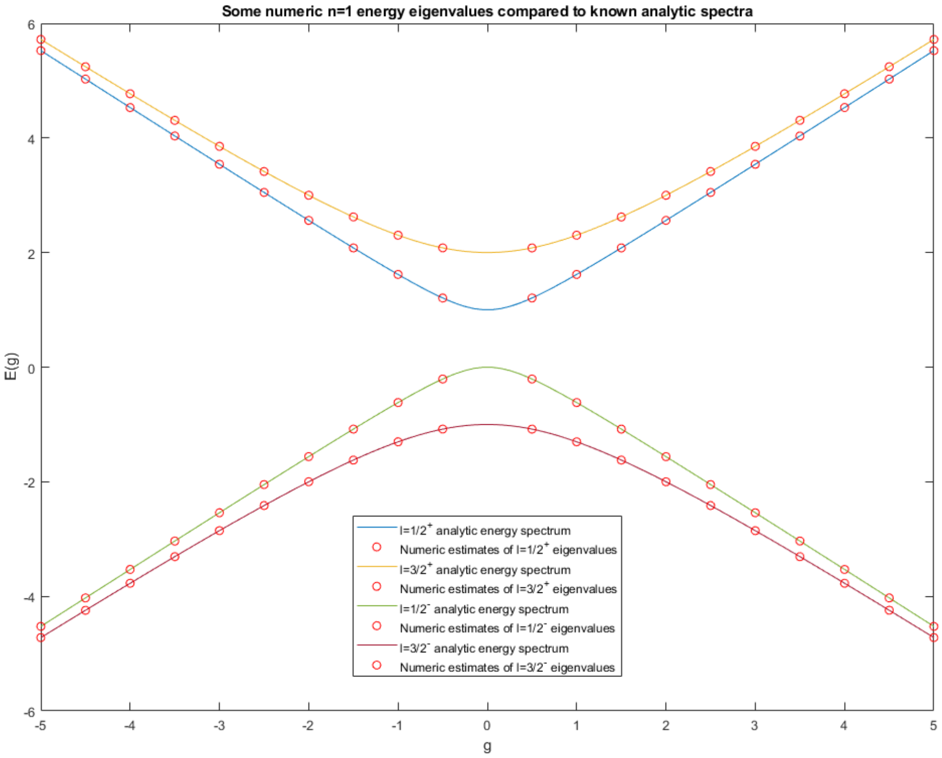

Figure 6: Some iteratively computed energy eigenvalues for low angular momentum n=1 solutions.

Integrating twice, we fix the constants of integration by knowing the overall normalisation of the fermion and the homotopy class of the kink, to obtain:

$$|\psi|^2 = \frac{T}{2\pi} + \frac{2M^2}{\hbar\rho}\left(n - \frac{\partial\phi}{\partial\theta}\right) \tag{70}$$

Writing the kink as

$$\phi(\theta) = n\theta + \eta(\theta), \tag{71}$$

the above becomes

$$|\psi|^2 = \frac{T}{2\pi} - \frac{2M^2}{\hbar\rho}\frac{\partial\eta}{\partial\theta}. \tag{72}$$

Thus the fermion density only deviates from the mean by the derivative of the non-trivial behaviour of the kink, where by "trivial" behaviour we mean the uniform winding for its homotopy class. We will frequently refer to the function $\eta(\theta)$ defined here as the kink's *wobble*.

It is straightforward to check that any pair of fields $(\phi(t,\theta) = n\theta, \psi(t,\theta) = \psi_{l,E^\pm}(t,\theta))$ solves all the equations of motion for the fully coupled model. However, the fermionic parity solutions $\psi^P_{|l|,E^\pm}$ have non-constant local density, so they do not remain solutions when paired with a uniformly winding kink $\psi(t,\theta) = n\theta$ because this violates the axial current conservation.

3.2 Initial numerical treatment of the fully coupled model

To search for new static solutions to the system with non-zero wobble, we introduce the kink and its derivative as further unknown functions to our `bvp4c` code. Writing y_5 for ϕ and y_6 for $\frac{\partial\phi}{\partial\theta}$, and changing the label of the cumulative fermionic charge to y_7, we have a system of first order ODEs as follows:

$$y_1' = \frac{E}{a}y_2 - \frac{g}{a}y_4\cos(y_5) + \frac{g}{a}y_3\sin(y_5) \tag{73}$$

$$y_2' = -\frac{E}{a}y_1 + \frac{g}{a}y_3\cos(y_5) + \frac{g}{a}y_4\sin(y_5) \tag{74}$$

$$y_3' = -\frac{E}{a}y_4 + \frac{g}{a}y_2\cos(y_5) + \frac{g}{a}y_1\sin(y_5) \tag{75}$$

$$y_4' = \frac{E}{a}y_3 - \frac{g}{a}y_1\cos(y_5) + \frac{g}{a}y_2\sin(y_5) \tag{76}$$

$$y_5' = y_6 \tag{77}$$

$$y_6' = \frac{2g\rho^2}{M}\left[(y_1y_4 - y_2y_3)\cos(y_5) - (y_1y_3 + y_2y_4)\sin(y_5)\right] \tag{78}$$

$$y_7' = y_1^2 + y_2^2 + y_3^2 + y_4^2 \tag{79}$$

Including E, we have eight unknowns. We take the following boundary conditions:

$$y_1(0) = y_1(2\pi),\ y_2(0) = 0,\ y_3(0) = y_3(2\pi),\ y_4(0) = y_4(2\pi),$$
$$y_5(0) = 0,\ y_5(2\pi) = 2n\pi,\ y_7(0) = 0,\ y_7(2\pi) = T.$$

Observe that we can use the overall phase symmetry to make the right-handed fermion component (say) real at the origin, and to fix the initial value of the kink to 0 there also; we make the other fermion modes periodic, give the kink the correct homotopy class, and implement y_7 as the cumulative fermion charge density.

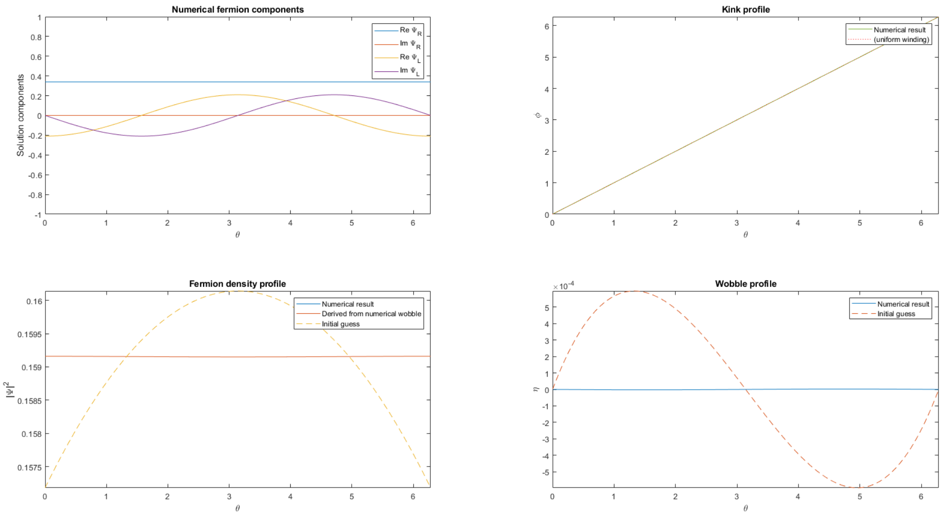

Figure 7: The $n = 1$ $l = 1/2^-$ eigensolution in the static system

Similarly to the case of the uniform kink, we use the Fourier components of the previously known solutions as a template for our initial guess. Recalling that the current conservation law tells us that a non-zero wobble on the kink requires a non-uniform density profile for the fermion, we tweak these Fourier modes by multiplying them by a non-constant envelope function, for which we use a cubic polynomial in θ with roots at $0, \pi$ and 2π. The overall amplitude of this cubic is a parameter that we can vary for different initial guesses. We also add a small relative axial phase between the chiral components of the fermion, the magnitude of which is also a variable parameter.

It is straightforward to check that our previously known angular momentum eigensolutions paired with a uniform kink remain solutions to our more general static system. These solutions are easily recreated by the broader numerical process by taking the perturbation parameters to be extremely small or zero; see examples at Figure 7 and Figure 8.

We also found new solutions, with non-zero wobbles. To broad inspection, the wobbles appear to be simple trigonometric functions, and the numerical results demonstrate very little relative axial phase between the chiral components of the fermion in the most part. The number of nodes on the wobble persist, while their amplitude varies, with varying g. Examples follow in Figures 9-11. We note that these new fermionic solutions appear to exhibit definite parity by comparison with our earlier plots of definite-parity solutions in the special case.

The energy eigenvalue of a $2l$-node wobble appears to track closely to the known spectrum of an angular momentum l eigenstate on the uniform kink, with greater divergence at greater $|g|$: see Figures 12 and 13.

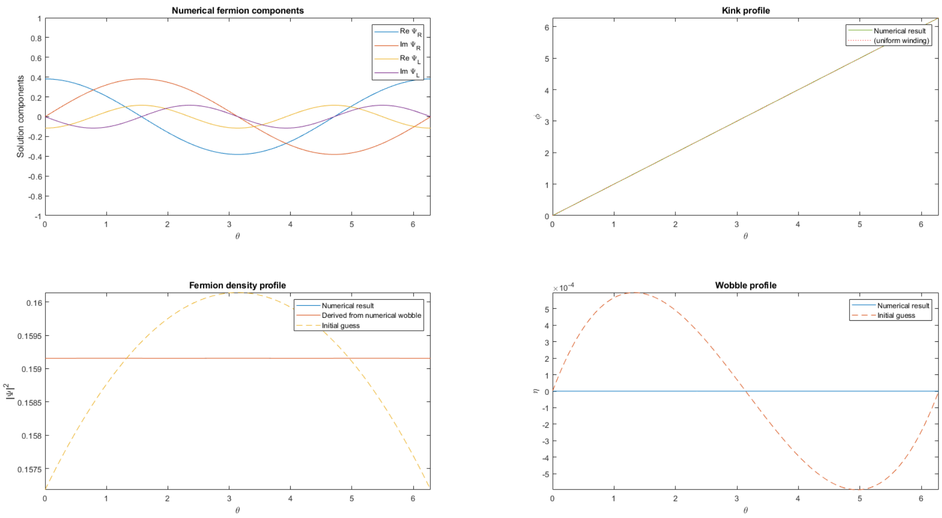

Figure 8: The $n = 1\, l = 3/2^-$ eigensolution in the static system

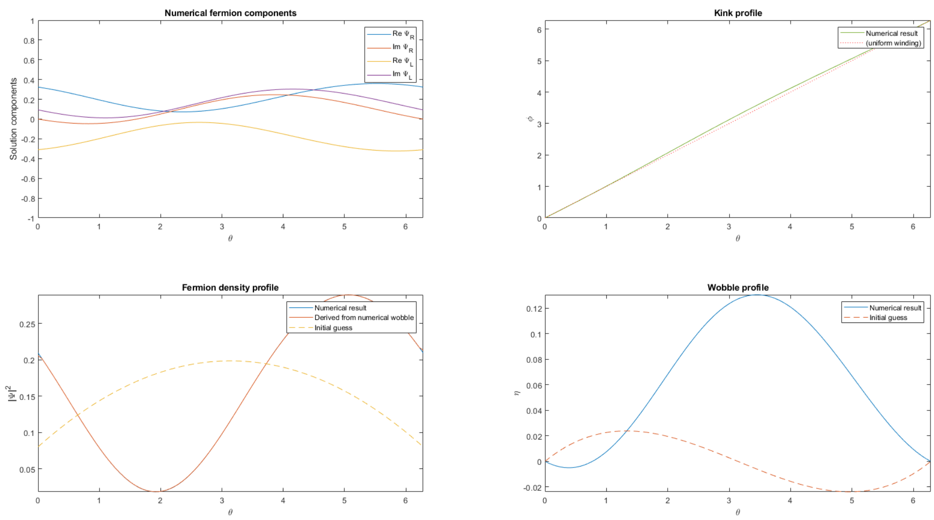

Figure 9: A non-trivial static solution with one-node wobble, at $g = 1$.

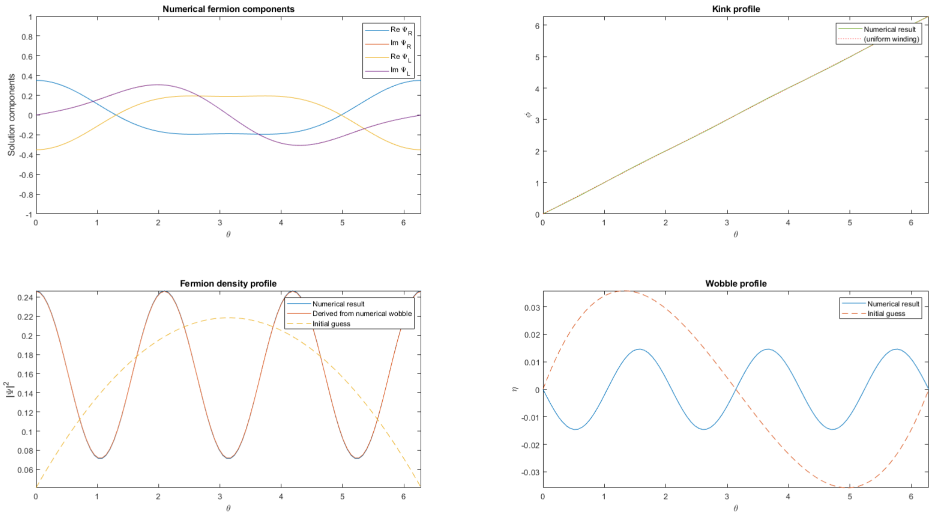

Figure 10: A non-trivial static solution with three-node wobble, at $g = 1$.

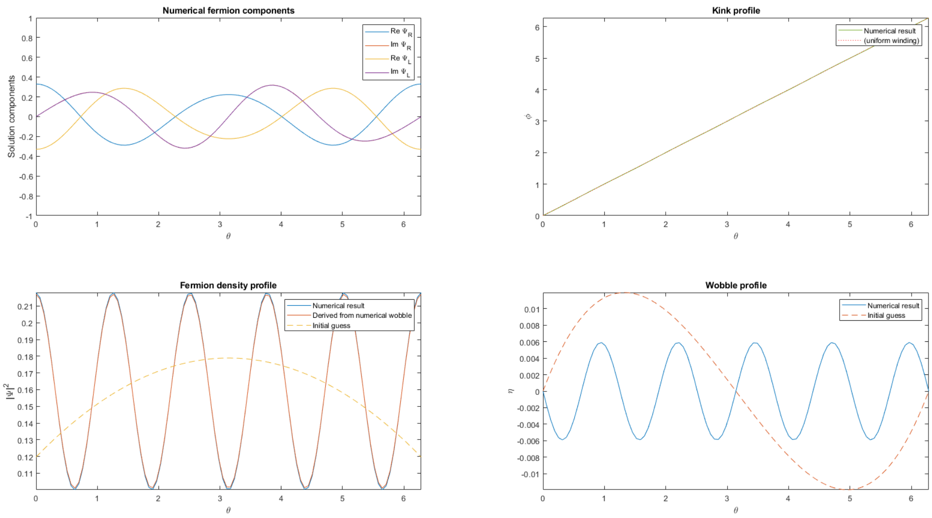

Figure 11: A non-trivial static solution with five-node wobble, at $g = 1$.

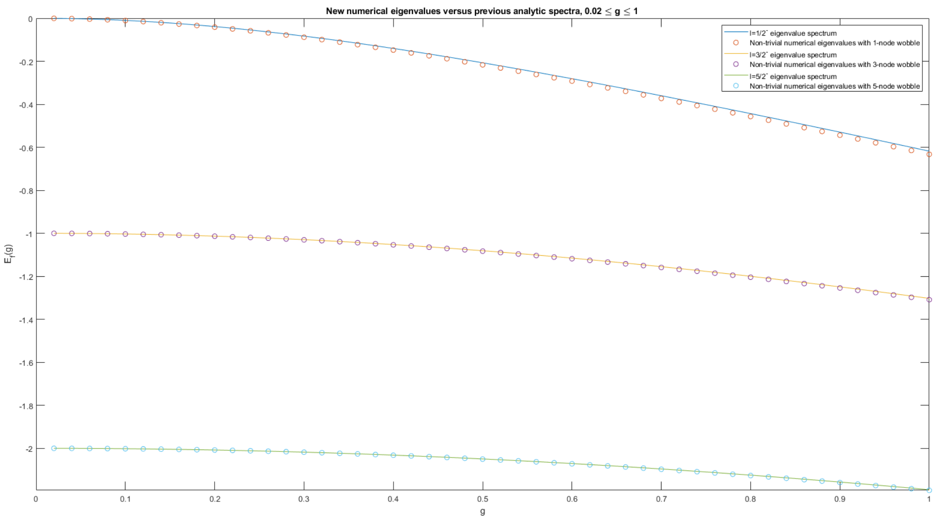

Figure 12: Energy eigenvalues of nontrivial solutions, $0 \leq g \leq 1$

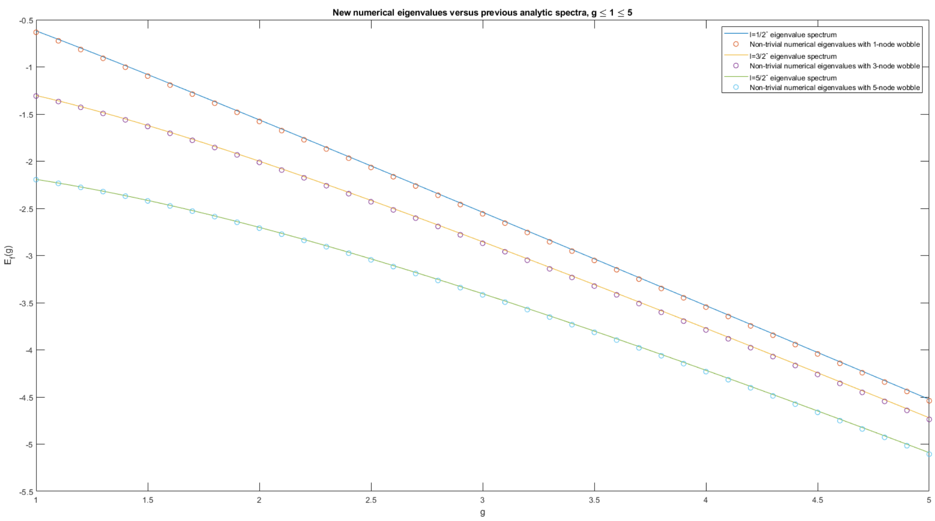

Figure 13: Energy eigenvalues of nontrivial solutions, $1 \leq g \leq 5$

4 Bispinor co-ordinates and static solutions

4.1 The $XYZW$ picture

We observe that the role of the kink in the coupling term $-g\overline{\psi}\mathrm{e}^{\mathrm{i}\gamma^3\phi}\psi$ appears to act something like a local change of basis for the inner product of the fermion with itself. However, it is not a gauge connection: note that since (in the chiral representation) the matrix $\mathrm{e}^{\mathrm{i}\gamma^3\phi}$ is diagonal, it cannot be written as $\gamma^\mu A_\mu$ for some gauge connection A_μ. Nonetheless, we can learn something by absorbing this feature of the kink into the fermion, as follows.

Let us change variables to a new fermion field $\xi(x^\mu)$, given by

$$\xi(x^\mu) = \mathrm{e}^{\mathrm{i}\gamma^3 K(x^\mu)}\psi(x^\mu) \tag{80}$$

for some as yet unspecified function of spacetime $K(x^\mu)$. In the new co-ordinates, the fermionic part of the Lagrangian is

$$\mathcal{L}_f = \overline{\xi}\left(\mathrm{i}\hbar\partial\!\!\!/ - \hbar(\partial\!\!\!/ K)\gamma^3 - g\mathrm{e}^{\mathrm{i}\gamma^3(\phi+2K)}\right)\xi. \tag{81}$$

If we now take, for example, $K = -\frac{\phi}{2}$, the explicit appearance of the kink is removed from the Lagrangian, and we are left with

$$\mathcal{L} = \frac{M^2}{2}\partial_\mu\phi\partial^\mu\phi + \overline{\xi}\left(\mathrm{i}\hbar\partial\!\!\!/ + \frac{\hbar}{2}(\partial\!\!\!/\phi)\gamma^3 - g\right)\xi. \tag{82}$$

In these co-ordinates, the coupling constant g now appears as a mass term for the new fermion field ξ, and the kink itself does not explicitly appear, but we have introduced a coupling between the fermion and the derivative of the kink field.

Strictly speaking, the change of variables as written gives the new fermion ξ either periodic or antiperiodic boundary conditions, according to $\xi(t, \theta + 2\pi) = (-1)^n\xi(t, \theta)$. However, we could instead choose to define e.g. $K = -\frac{\phi}{2} + n\pi$ to restore truly periodic boundary conditions to ξ while preserving the new form of the Lagrangian. It is straightforward to check that ξ still transforms as a Dirac spinor under parity in either case: $\hat{P}\xi(t, \theta) = \gamma^0\xi(t, -\theta)$.

In a sense, the kink derivative now even more resembles a gauge connection: we could write the first two fermionic terms as $\overline{\xi}\mathrm{i}\hbar\gamma^\mu(\partial_\mu + A_\mu)\xi$ where $A_\mu = -\frac{\mathrm{i}}{2}\gamma^3\partial_\mu\phi = -\frac{\mathrm{i}}{2}\hat{\varepsilon}^\nu{}_\mu\partial_\nu\phi$. However, recall that parity symmetry requires that ϕ is a pseudoscalar rather than scalar field, so this “gauge potential” A_μ would be a Lorentz pseudovector rather than vector. Moreover, even if the original coupling were not chiral (so γ^3 would not appear, and ϕ could be a true scalar field), the kinetic term for the kink would be a mass term for the gauge connection, $\frac{M^2}{2}A_\mu A^\mu$, rather than the field energy $\sim F_{\mu\nu}F^{\mu\nu}$. As written, this would be a system with a zero-energy gauge field and where the gauge symmetry is explicitly broken by the kink (inertial) mass.

The U(1) phase action is unchanged from the previous co-ordinates, and the expression for the phase current retains the same form:

$$J^\mu = \hbar\overline{\psi}\gamma^\mu\psi = \hbar\overline{\xi}\gamma^\mu\xi. \tag{83}$$

We see that the new fermion ξ is in fact invariant under the axial transformation; in the new co-ordinates, then, that axial transformation is simply the shunt or internal translation of the kink. Concordantly, the axial current conservation equation is simply the Euler-Lagrange equation for ϕ derived from the new Lagrangian:

$$0 = \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu\phi)}\right) = \partial_\mu J^\mu_{\text{axial}} = \partial_\mu\left(M^2\partial^\mu\phi + \frac{\hbar}{2}\overline{\xi}\gamma^\mu\gamma^3\xi\right) = \partial_\mu\left(M^2\partial^\mu\phi + \frac{1}{2}\hat{\varepsilon}^\mu{}_\nu J^\nu\right). \tag{84}$$

Now consider the Lorentz covariant bispinors we can form using ξ. We have already seen that the phase current is such a Lorentz vector, $J^\mu = \hbar\bar{\xi}\gamma^\mu\xi$. Define variables X and W from its components according to $J^\mu = (\hbar X, -aW)$, so

$$X = \bar{\xi}\gamma^t\xi = \xi^\dagger\xi = \psi^\dagger\psi, \tag{85}$$

$$W = -\rho\bar{\xi}\gamma^\theta\xi = \xi^\dagger\gamma^3\xi = \psi^\dagger\gamma^3\psi \tag{86}$$

Note that X and W are both real; moreover X is non-negative. Furthermore, X is the fermionic (phase charge) density, while W is the fermionic contribution to the axial charge density. X is invariant under parity, but W picks up a minus sign under a parity transformation: explicitly, $\hat{P}W(t,\theta) = -W(t,-\theta)$. There are two more real bispinors Y and Z we can form, defined as

$$Y = \bar{\xi}\xi = \xi^\dagger\gamma^0\xi = \psi^\dagger\gamma^0 e^{i\gamma^3\phi}\psi, \tag{87}$$

$$Z = i\bar{\xi}\gamma^3\xi = i\xi^\dagger\gamma^1\xi = i\psi^\dagger\gamma^1 e^{i\gamma^3\phi}\psi. \tag{88}$$

Again, Y is invariant under parity, while Z picks up a minus sign. (Here, Y is a true Lorentz scalar, while Z is pseudoscalar.)

There are no more bispinors that cannot be expressed in terms of these four real co-ordinates: note, for example, that if we construct the rank two Lorentz tensor $V^{\mu\nu} = \bar{\xi}\gamma^\mu\gamma^\nu\xi$, we find

$$V^{\mu\nu} = \bar{\xi}\gamma^\mu\gamma^\nu\xi = \bar{\xi}\left(g^{\mu\nu} - \hat{\varepsilon}^{\mu\nu}\gamma^3\right)\xi = g^{\mu\nu}Y + i\hat{\varepsilon}^{\mu\nu}Z. \tag{89}$$

It may be worth noting that since $g^{\mu\nu}$ and $\hat{\varepsilon}^{\mu\nu}$ are themselves invariant under Lorentz transforms, this tensor $V^{\mu\nu}$ is actually a complex Lorentz-invariant grade 2 multivector, sent to its complex conjugate under the parity transformation.

Many of the dynamical features of the kink-fermion system can be captured using these co-ordinates X, Y, Z and W, without reference to the "bare" fermion ξ (or ψ). A key observation is that all possible evolutions of the system (not just classical solutions) take place on a three-dimensional cone in this $XYZW$ picture:

$$X^2 - W^2 - Y^2 - Z^2 = 0. \tag{90}$$

This is most easily demonstrated by passing to polar coordinates for the original complex components of ψ: writing

$$\psi(t,\theta) = \begin{pmatrix} R(t,\theta)e^{i\mu(t,\theta)} \\ L(t,\theta)e^{i\nu(t,\theta)} \end{pmatrix}, \tag{91}$$

with μ, ν periodic real functions and R, L non-negative real functions, it is easily seen that

$$X = R^2 + L^2, \tag{92}$$

$$W = R^2 - L^2, \tag{93}$$

$$Y = 2RL\cos(\phi + \mu - \nu), \tag{94}$$

$$Z = -2RL\sin(\phi + \mu - \nu), \tag{95}$$

from which (90) follows. This can be expressed in a manifestly Lorentz-invariant form: note that

$$J^\mu J_\mu = \hbar^2(X^2 - W^2), \tag{96}$$

while, defining $V^{\mu\nu}$ as above at (89),

$$V^{\mu\nu}V_{\mu\nu} = 2(Y^2 + Z^2). \tag{97}$$

Thus the equation of the cone (90) can be expressed

$$\frac{J^\mu J_\mu}{\hbar^2} = \frac{1}{2} V^{\mu\nu} V_{\mu\nu}. \tag{98}$$

We stress that this does not require imposing the equations of motion: it is a consequence of Lorentz covariance alone.

Expressing the conservation laws for classical solutions in the $XYZW$ picture, we see

$$\partial_\mu J^\mu = 0 \quad \equiv \quad \hbar \left(\frac{\partial X}{\partial t} - \frac{1}{\rho} \frac{\partial W}{\partial \theta} \right) = 0; \tag{99}$$

$$\partial_\mu J^\mu_{\text{axial}} = 0 \quad \equiv \quad \hbar \left(\frac{\partial W}{\partial t} - \frac{1}{\rho} \frac{\partial X}{\partial \theta} \right) = \partial_\mu \varepsilon^\mu{}_\nu J^\nu = -2M^2 \partial_\mu \partial^\mu \phi. \tag{100}$$

The Dirac equations for ξ and $\overline{\xi}$ are

$$\mathrm{i}\hbar \partial\!\!\!/ \xi + \frac{\hbar}{2} (\partial\!\!\!/ \phi) \gamma^3 \xi - g\xi = 0, \tag{101}$$

$$\mathrm{i}\hbar \overline{\xi} \overleftarrow{\partial\!\!\!/} + \frac{\hbar}{2} \overline{\xi} \gamma^3 (\partial\!\!\!/ \phi) + g\overline{\xi} = 0. \tag{102}$$

These provide two further Lorentz-covariant equations of motion by contracting spinors and summing appropriately: one for each combination of derivatives of X and W seen above in the current conservation equations. The $\left(\frac{\partial X}{\partial t}, \frac{\partial W}{\partial \theta}\right)$ equation is equivalent to the phase current conservation equation $\partial_\mu J^\mu = 0$. For $\left(\frac{\partial W}{\partial t}, \frac{\partial X}{\partial \theta}\right)$ we obtain

$$\partial_\mu \varepsilon^\mu{}_\nu J^\nu = \hbar \left(\frac{\partial W}{\partial t} - \frac{1}{\rho} \frac{\partial X}{\partial \theta} \right) = 2gZ \tag{103}$$

which by comparison with axial current conservation tells us

$$-M^2 \partial^\mu \partial_\mu \phi = gZ. \tag{104}$$

Derivatives of Y and Z do not arise in such a neat (fermion-independent) manner, because for example the contraction needed to isolate $\frac{\partial Y}{\partial t}$ introduces the combination $\frac{\partial \overline{\xi}}{\partial \theta} \gamma^3 \xi - \overline{\xi} \gamma^3 \frac{\partial \xi}{\partial \theta}$, which is not directly expressible in terms of Z only. In total, the remaining four equations we can derive from the Dirac equations are:

$$\hbar \frac{\partial Y}{\partial t} = \hbar \frac{\partial \phi}{\partial t} Z - a \left[\frac{\partial \overline{\xi}}{\partial \theta} \gamma^3 \xi - \overline{\xi} \gamma^3 \frac{\partial \xi}{\partial \theta} \right] \tag{105}$$

$$a \frac{\partial Y}{\partial \theta} = a \frac{\partial \phi}{\partial \theta} Z - \hbar \left[\frac{\partial \overline{\xi}}{\partial t} \gamma^3 \xi - \overline{\xi} \gamma^3 \frac{\partial \xi}{\partial t} \right] \tag{106}$$

$$\hbar \frac{\partial Z}{\partial t} = -\hbar \frac{\partial \phi}{\partial t} Y - 2gW - \mathrm{i}a \left[\frac{\partial \overline{\xi}}{\partial \theta} \xi - \overline{\xi} \frac{\partial \xi}{\partial \theta} \right] \tag{107}$$

$$a \frac{\partial Z}{\partial \theta} = -a \frac{\partial \phi}{\partial \theta} Y - 2gX - \mathrm{i}\hbar \left[\frac{\partial \overline{\xi}}{\partial t} \xi - \overline{\xi} \frac{\partial \xi}{\partial t} \right]. \tag{108}$$

4.2 Static kinks and stationary state fermions

Let us now elide any discussion of the general dynamics in terms of the $XYZW$ picture, and immediately impose the same ansätze as we did before in the special case:

$$\phi(t,\theta) = \phi(\theta) \text{ only}, \quad \psi(t,\theta) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_f t}\psi(\theta). \tag{109}$$

The system will vastly simplify. To abbreviate this "Static kink and Stationary State fermion" ansatz, we will refer to it as the *3S* ansatz. Immediately, $\frac{\partial X}{\partial t}$ and $\frac{\partial W}{\partial t}$ vanish. Thus by (99), W is uniformly constant in both t and θ. Z is now directly proportional to $\frac{\partial X}{\partial \theta}$, as well as acting as a source for ϕ. Moreover, time-dependence for Y and Z can only possibly arise from the kink, so $\frac{\partial Y}{\partial t}$ and $\frac{\partial Z}{\partial t}$ also vanish. We are left with the much simpler system of ODEs:

$$-a\frac{\partial X}{\partial \theta} = \frac{2M^2}{\rho^2}\frac{\partial^2\phi}{\partial\theta^2} = 2gZ, \tag{110}$$

$$a\frac{\partial Y}{\partial \theta} = \left(a\frac{\partial\phi}{\partial\theta} - 2E_f\right)Z, \tag{111}$$

$$a\frac{\partial Z}{\partial \theta} = \left(-a\frac{\partial\phi}{\partial\theta} + 2E_f\right)Y - 2gX. \tag{112}$$

Recall the definition of the total fermionic (phase charge) density (per unit radius of the spatial circle), T, as

$$T = \int_0^{2\pi} X(\theta)\,\rho\,\mathrm{d}\theta. \tag{113}$$

Then, we can integrate the first equation at (110) and use the known winding number n of the kink to write

$$\frac{\partial\phi}{\partial\theta} = n + \frac{\hbar\rho}{2M^2}\left(\frac{T}{2\pi} - X(\theta)\right). \tag{114}$$

It is useful now to define two new constants:

$$\lambda = \frac{\hbar\rho}{2M^2}; \quad E' = E_f - \frac{a}{2}\left(n + \frac{T\lambda}{2\pi}\right). \tag{115}$$

Substituting the expression for $\frac{\partial\phi}{\partial\theta}$ into the equations for $\frac{\partial Y}{\partial\theta}$ and $\frac{\partial Z}{\partial\theta}$, we observe that the problem is reduced to the system of four first-order ODEs:

$$\frac{\partial\phi}{\partial\theta} = n + \lambda\left(\frac{T}{2\pi} - X\right), \tag{116}$$

$$a\frac{\partial X}{\partial\theta} = -2gZ, \tag{117}$$

$$a\frac{\partial Y}{\partial\theta} = -\left(2E' + a\lambda X\right)Z, \tag{118}$$

$$a\frac{\partial Z}{\partial\theta} = \left(2E' + a\lambda X\right)Y - 2gX. \tag{119}$$

In particular, the equations for the bispinors X, Y and Z are not explicitly dependent on the kink ϕ. If we can solve for the bispinors first, we can then deduce the matching kink solution.

It is immediately clear that $X^2 - Y^2 - Z^2$ is a first integral of the system; of course, it equals W^2. Another first integral is apparent by substituting (117) into (118):

$$2ag\frac{\partial Y}{\partial \theta} = (2E' + a\lambda X)(-2gZ) \tag{120}$$

$$= (2E' + a\lambda X)\left(a\frac{\partial X}{\partial \theta}\right) \tag{121}$$

$$= a\frac{\partial}{\partial \theta}\left[2E'X + \frac{a\lambda}{2}X^2\right] \tag{122}$$

Hence there is a constant C_0 given by

$$C_0 = 2agY - 2aE'X - \frac{a^2\lambda}{2}X^2. \tag{123}$$

This equation also determines Y in terms of X and C_0; so, finally, we can plug it and (117) into (119) to reduce the XYZ system to a single non-linear differential equation in X. It is convenient to express it in the following manner:

$$(2ag)a\frac{\partial Z}{\partial \theta} = (2E' + a\lambda X)(2agY) - 4ag^2X \tag{124}$$

$$a^2\left(2g\frac{\partial Z}{\partial \theta}\right) = (2E' + a\lambda X)\left(C_0 + 2aE'X + \frac{a^2\lambda}{2}X^2\right) - 4ag^2X \tag{125}$$

$$a^2\left(-a\frac{\partial^2 X}{\partial \theta^2}\right) = \frac{a^3\lambda^2}{2}X^3 + 3a^2\lambda E'X^2 + \left(4a\left[E'^2 - g^2\right] + a\lambda C_0\right)X + 2E'C_0 \tag{126}$$

At the cost of introducing one more constant of integration, which we will call C_1, we can multiply by $2\frac{\partial X}{\partial \theta}$ and integrate once more, yielding

$$-a^3\left(\frac{\partial X}{\partial \theta}\right)^2 = \frac{a^3\lambda^2}{4}X^4 + 2a^2\lambda E'X^3 + \left(4a\left[E'^2 - g^2\right] + a\lambda C_0\right)X^2 + 4E'C_0X + C_1. \tag{127}$$

In general, a non-linear differential equation of this form can be solved by an elliptic function (possibly reducing to a trigonometric or hyperbolic function in certain degenerate cases) [3]. However, the specific solution depends on the coefficients of the polynomial appearing on the RHS above, and at present the parameters E', C_0 and C_1 are all unknown. We can eliminate one unknown by comparing the equation (127) to the condition $X^2 - W^2 - Y^2 - Z^2 = 0$:

$$-a^2(2gZ)^2 = (2agY)^2 - 4a^2g^2X^2 + 4a^2g^2W^2 \tag{128}$$

$$-a^2\left(-a\frac{\partial X}{\partial \theta}\right)^2 = \left(C_0 + 2aE'X + \frac{a^2\lambda}{2}X^2\right)^2 - 4a^2g^2X^2 + 4a^2g^2W^2 \tag{129}$$

$$-a^4\left(\frac{\partial X}{\partial \theta}\right)^2 = \frac{a^4\lambda^2}{4}X^4 + 2a^3\lambda E'X^3 + \left(4a^2\left[E'^2 - g^2\right] + a^2\lambda C_0\right)X^2 + 4aE'C_0X + C_0^2 + 4a^2g^2W^2 \tag{130}$$

Hence $aC_1 = C_0^2 + 4a^2g^2W^2$. If we specifically consider definite-parity solutions where $W = 0$, then $aC_1 = C_0^2$, and in fact the quartic in X factorises into two real quadratics, giving

$$-a^4\left(\frac{\partial X}{\partial \theta}\right)^2 = \left(\frac{a^2\lambda}{2}X^2 + 2a\left[E' - g\right]X + C_0\right)\left(\frac{a^2\lambda}{2}X^2 + 2a\left[E' + g\right]X + C_0\right). \tag{131}$$

It is convenient now to perform some rescaling. First let us divide powers of the energy scale a from the parameters E', g and C_0 according to

$$\tilde{E}' = \frac{E'}{a}; \quad \tilde{g} = \frac{g}{a}; \quad \tilde{C}_0 = \frac{C_0}{a^2}. \tag{132}$$

This yields

$$-\left(\frac{\partial X}{\partial \theta}\right)^2 = \frac{\lambda^2}{4} X^4 + 2\lambda \tilde{E}' X^3 + \left(4\left[\tilde{E}'^2 - \tilde{g}^2\right] + \lambda \tilde{C}_0\right) X^2 + 4\tilde{E}'\tilde{C}_0 X + \tilde{C}_0^2 + 4\tilde{g}^2 W^2 \tag{133}$$

Finally, when λ is non-zero, we can also rescale X to remove the (remaining) leading coefficients of $\frac{\lambda}{2}$ on the quadratics by setting

$$\tilde{X}(\theta) = \frac{\lambda}{2} X(\theta), \tag{134}$$

leaving

$$-\left(\frac{\partial \tilde{X}}{\partial \theta}\right)^2 = \tilde{X}^4 + 4\tilde{E}'\tilde{X}^3 + \left(4\left[\tilde{E}'^2 - \tilde{g}^2\right] + \lambda \tilde{C}_0\right) \tilde{X}^2 + 2\lambda \tilde{E}'\tilde{C}_0 \tilde{X} + \frac{\lambda^2 \tilde{C}_0^2}{4} + \lambda^2 \tilde{g}^2 W^2. \tag{135}$$

5 Elliptic functions and the general solution to the 3S ansatz

5.1 The Jacobi sn function

The *Jacobi sn function*, $\mathrm{sn}(u,k)$, where u is a complex variable and k is a parameter called the elliptic modulus, may be defined in terms of its inverse function $\mathrm{sn}^{-1}(u,k)$, where, for $0 < |k|^2 < 1$,

$$\mathrm{sn}^{-1}(u,k) = \int_0^u \frac{\mathrm{d}z}{\sqrt{1-z^2}\sqrt{1-k^2z^2}}. \tag{136}$$

This integral converges for $0 < u < 1$, but by taking a branch cut ending on the point $u = 1$, $\mathrm{sn}(u,k)$ can be analytically continued to the rest of the complex plane. It can similarly be analytically continued for values of $|k| > 1$. (Note that in the limiting cases $k = 0$ then the integral reduces to $\sin^{-1}(u)$, while for $k = 1$ it becomes $\tanh^{-1}(u)$.)

Using the above integral expression, it may be checked that the general solution to the non-linear DE

$$\left(\frac{\partial \tilde{X}}{\partial \theta}\right) = -\sqrt{(\tilde{X}(\theta)-t_1)(\tilde{X}(\theta)-t_2)(\tilde{X}(\theta)-t_3)(\tilde{X}(\theta)-t_4)} \tag{137}$$

for four distinct (but not necessarily real) constants $t_i \in \mathbb{C}, i = 1, \ldots 4$, may be expressed in terms of the inverse function $\theta(\tilde{X})$ as

$$\theta(\tilde{X}) - \theta_c = p^{-1}\mathrm{sn}\left(\left[\tilde{X}, t_4; t_3, t_2\right]^{\frac{1}{2}}, [t_1, t_4; t_2, t_3]^{\frac{1}{2}}\right), \tag{138}$$

where θ_c is the constant of integration,

$$p = \frac{\sqrt{(t_1-t_3)(t_2-t_4)}}{2} \tag{139}$$

and the notation $[a,b;c,d]$ denotes the cross-ratio,

$$[a,b;c,d] = \frac{(c-a)(d-b)}{(c-b)(d-a)}. \tag{140}$$

(The theory of the Jacobi elliptic functions ensures that this expression is well-defined: it is invariant under arbitrary reordering of the four roots t_i, and will be correct even when $k = [t_1, t_4; t_2, t_3]^{\frac{1}{2}}$ does not satisfy $0 \le |k|^2 \le 1$. See Whittaker and Watson [3] for full details.)

Inverting 138 for $\tilde{X}(\theta)$ (and omitting θ_c), we have

$$\tilde{X}(\theta) = \frac{t_3(t_4-t_2) - t_2(t_4-t_3)\mathrm{sn}^2(p\theta,k)}{(t_4-t_2)-(t_4-t_3)\mathrm{sn}^2(p\theta,k)}. \tag{141}$$

We observe that, supposing there is a θ_0 such that $\mathrm{sn}(p\theta_0,k)$ vanishes, then $\tilde{X}(\theta_0) = t_3$; similarly, supposing that there is a regular singular point $\theta = \theta_\infty$ of $\mathrm{sn}(p\theta,k)$, then $\tilde{X}(\theta_\infty) = t_2$. Now, $\mathrm{sn}(u,k)$ does have both zeroes and regular singular points. It is a doubly-periodic complex function, the periodic properties being expressed in terms of two values called the *quarter-periods* depending on k and denoted K, K' respectively. When $0 < |k|^2 < 1$, and by suitable analytic continuation otherwise, the quarter-periods are defined in terms of the *complete elliptic integrals of the first kind*:

$$K(k) = \mathrm{sn}^{-1}(1,k) = \int_0^1 \frac{\mathrm{d}z}{\sqrt{1-z^2}\sqrt{1-k^2z^2}}, \tag{142}$$

$$K'(k) = \mathrm{sn}^{-1}(1,\sqrt{1-k^2}) = \int_0^1 \frac{\mathrm{d}z}{\sqrt{1-z^2}\sqrt{1-(1-k^2)z^2}}. \tag{143}$$

Generally, the doubly periodic behaviour of $\mathrm{sn}(u,k)$ is captured by the identity

$$\mathrm{sn}(u + 2\alpha K(k) + 2\mathrm{i}\beta K'(k), k) = (-1)^\alpha \mathrm{sn}(u,k); \tag{144}$$

then, points $u \in \mathbb{C}$ congruent to 0 are zeroes of $\mathrm{sn}(u,k)$, while points congruent to $\mathrm{i}K'$ are regular singular points. Thus, supposing we can order the t_i such that p is pure imaginary and further that sn^2 is always real, we see that t_2 and t_3 will be the extrema of $\tilde{X}(\theta)$. Now, for our purposes, we will require that $\tilde{X}(\theta)$ has the real period 2π. Then it is easily seen that we must have

$$2\pi p = 2\mathrm{i}JK' \tag{145}$$

for some $J \in \mathbb{Z}$.

[TBD: Normalisation of $\int_0^{2\pi} \tilde{X}\,\mathrm{d}\theta$ using the complete elliptic integral of the third kind, $\Pi(\nu,k)$. Principal values over fundamental periods.]

5.2 Parity solutions and parameter-matching

[TBD: In the case $W = 0$ of the 3S ansatz, the factorisation of the quartic in 131 allows us a lot of control over the relationship between the roots of the quartic and the physical parameters of the model. In particular we can look for cases where the quartic has four real roots, two of which are physical, and p is purely imaginary as desired above. This involves finding intersections in the (t_2, t_3) plane of two curves enforcing the periodicity and normalisation conditions, defined in terms of complete elliptic integrals – a transcendental problem in general, but numerically tractable. Then it is straightforward to obtain numberical values of the roots from our numerical calculations of E', C_0

and choices of g, T, &c, and give an expression of the form 141 with numerically calculated t_i for our numerical solutions. Actually the analysis and process of matching parameters is not much more difficult if we drop the condition $W = 0$, but our numerical BVP method seems to like to converge on parity solutions. We can then generally run an iterative process in g to obtain numerical energy spectra curves $E(g)$ to good accuracy in the fully coupled model.

Observe that solving for $|\psi|^2$, by axial current conservation, is enough to determine ϕ. Thus, we can write explicit expressions for kink solutions (at least in terms of (exponents of arctans of) elliptic functions with numerical parameters), and obtain (direct) numerical solutions with good accuracy.]

5.3 Angular momentum solutions and the role of the axial charge W

Let us consider how, from the $XYZW$ picture, solutions of the special case with background kink arise. Moving the kink to the background corresponds to taking the limit $M^2 \to \infty$, which in the $XYZW$ picture corresponds to $\lambda \to 0$. In this limit we cannot perform the final rescaling (134), but (133) reduces to

$$-\left(\frac{\partial X}{\partial \theta}\right)^2 = 4\left(\tilde{E}'^2 - \tilde{g}^2\right)X^2 + 4\tilde{E}'\tilde{C}_0 X + \tilde{C}_0^2 + 4\tilde{g}^2 W^2. \tag{146}$$

With a little algebra, this can be transformed to a standard integral:

$$\int 2\sqrt{\tilde{E}'^2 - \tilde{g}^2}\,\mathrm{d}\theta = \int \frac{\pm\,\mathrm{d}X}{\sqrt{1 - \left[\frac{2\left(\tilde{E}'^2 - \tilde{g}^2\right)}{\tilde{g}\sqrt{\tilde{C}_0^2 + 4W^2\left(\tilde{E}'^2 - \tilde{g}^2\right)}}X + \frac{\tilde{E}'\tilde{C}_0^2}{\tilde{g}\sqrt{\tilde{C}_0^2 + 4W^2\left(\tilde{E}'^2 - \tilde{g}^2\right)}}\right]^2}} \tag{147}$$

This gives two general solutions: one on each of two (positive and negative) energy branches, according to

$$X(\theta) = \frac{2}{k^2}\left[\tilde{g}\sqrt{\tilde{C}_0^2 - k^2W^2}\sin k\left(\theta - \theta_0\right) \mp \tilde{C}_0\sqrt{\tilde{g}^2 + \frac{k^2}{4}}\right], \; \tilde{E}' = \pm\sqrt{\tilde{g}^2 + \frac{k^2}{4}}, \; k \in \mathbb{Z}, \tag{148}$$

where θ_0 is the constant of integration. We immediately see, for real solutions, the bound on W^2:

$$0 \le W^2 \le \frac{\tilde{C}_0^2}{k^2}. \tag{149}$$

We will show that we obtain the known parity eigenstates when $W^2 = 0$, and the known angular momentum eigenstates when $W^2 = \frac{\tilde{C}_0^2}{k^2}$; in each case, the interpretation of the mode k is that $k = 2l$ where l is the angular momentum eigenvalue.

Let us consider some effects of setting $\lambda = 0$ on the kink and physical invariants. From the known eigensolutions (either parity or angular momentum), we can exactly determine X, Y, Z and W, whence C_0, directly in terms of l, T, &c. For the angular momentum solutions,

$$X = \frac{T}{2\pi}, \quad Y = \pm\frac{g}{\sqrt{a^2l^2 + g^2}}\frac{T}{2\pi}, \quad Z = 0, \quad W = \mp\frac{al}{\sqrt{a^2l^2 + g^2}}\frac{T}{2\pi}, \tag{150}$$

with the choice of $\pm$ sign always corresponding to the energy branch. For the parity solutions, let

$P = \pm 1$ be the parity. Then

$$X = \left[1 + P \frac{alg}{a^2 l^2 + g^2} \cos 2l \left(\theta - \theta_0 \right) \right] \frac{T}{2\pi}, \tag{151}$$

$$Y = \left[\frac{g}{\sqrt{a^2 l^2 + g^2}} + P \cos 2l \left(\theta - \theta_0 \right) \right] \frac{T}{2\pi}, \tag{152}$$

$$Z = \left[P \frac{a^2 l^2}{a^2 l^2 + g^2} \sin 2l \left(\theta - \theta_0 \right) \right] \frac{T}{2\pi}, \tag{153}$$

and by construction $W = 0$.

Consider again the quadratic on the RHS of 146. Its roots are

$$X = \frac{-\tilde{E}' \tilde{C}_0 \pm \tilde{g} \sqrt{\tilde{C}_0^2 - 4 \left(\tilde{E}'^2 - \tilde{g}^2 \right) W^2}}{2 \left(\tilde{E}'^2 - \tilde{g}^2 \right)} \tag{154}$$

For physical periodicity, we require that both roots are non-negative (since $\left(\tilde{E}'^2 - \tilde{g}^2 \right) > 0$ for reality of 148). This gives the same physical bound on W^2, and in particular demonstrates the double root when the upper bound is saturated. But on such a physical solution where X is constant, by axial current conservation the kink is uniform – so we immediately reduce to the special case as a subset of the general case, implying that the solution is an angular momentum solution.

[TBD: Some straightforward, if a little tedious, linear algebra, shows that we obtain exactly the above expressions for X, Y, Z, and W in the relevant saturated bounds: this will go in an appendix.]

How do we interpret W in the special case of the prescribed kink? Fix an energy level $E^{\pm}$, determined by $|l| \neq 0$ and the choice of branch, and identify $k = 2l$. All such energy levels are two-dimensional; we have exhibited the angular momentum and parity bases. Individiual elements of the parity basis $\{\psi^+, \psi^-\}$ have $W = 0$, while the individual elements of the angular momentum basis are proportional to linear combinations of the parity basis solutions according to

$$\psi_{l^{\pm}} \sim (\psi^+ + \psi^-), \tag{155}$$

$$\psi_{-l^{\pm}} \sim (\psi^+ - \psi^-). \tag{156}$$

Now, a general solution (*of the special case*) in the chosen energy level may be expressed as a linear combination of the parity basis solutions,

$$\psi = u\psi^+ + v\psi^-, \quad u, v \in \mathbb{C}. \tag{157}$$

Calculating e.g. W, C_0, X, Y in terms of the known values for the parity basis, we find:

$$W = \mp \left(8al\sqrt{a^2 l^2 + g^2} \right) (u^* v + v^* u) \tag{158}$$

$$C_0 = 16a^3 l^2 \sqrt{a^2 l^2 + g^2} \left(|u|^2 + |v|^2 \right). \tag{159}$$

Then the bound $C_0^2 - (2alW)^2 \geq 0$ simply corresponds to the fact that $|u|^2 + |v|^2 \geq u^* v + v^* u$. In this sense, axial charge arises from the cross-terms in $|\psi|^2$ when expressed in the parity basis: the projection, or inner product, of the positive and negative parity components.

[TBD: Without moving the kink to the background, we can use W to shift the quartic 131 down in the (X, X') plane until the extremal roots (t_2, t_3) coalesce into a double root, so that X is fixed: observe again that this must be, and is, an angular momentum solution.]

5.4 The interpretation of the invariants E' and C_0

[TBD: E' admits an interpretation as a "free energy", or (total – interaction) energy of fermions: the "budget" after paying the "ante" to obey axial current conservation.

C_0 is related by an affine transformation to the Noether pressure term, i.e. the element $p = \vartheta^{\theta}{}_{\theta}$ of the energy-momentum tensor, as follows:

$$p = \frac{\partial \mathcal{L}}{\partial(\partial_\theta \phi)} \frac{\partial \phi}{\partial \theta} + \frac{\partial \mathcal{L}}{\partial(\partial_\theta \psi)} \frac{\partial \psi}{\partial \theta} - \mathcal{L} \tag{160}$$

$$= -\frac{a}{4\lambda} \left(n + \frac{T\lambda}{2\pi} \right)^2 - E'X - \frac{a\lambda}{4} X^2 + gY \tag{161}$$

in the 3S ansatz. Thus, by comparison with (123),

$$C_0 = 2ap + \frac{a^2}{2\lambda} \left(n + \frac{T\lambda}{2\pi} \right)^2 . \tag{162}$$

p is not well defined in the limit $M^2 \to \infty$, but C_0 is: it has the limit

$$C_0 \to 2a \left(gY - \left[E_f - \frac{an}{2} \right] X \right) = 2a \left(gY - E'_{(\lambda=0)} X \right) . \tag{163}$$

Note that $gY = \vartheta^{\mu}{}_{\mu}$ looks like the usual mass-squared content of the system appearing in a Noether pressure; we might therefore interpret $p' = \frac{C_0}{2a}$ as a sort of "regularised pressure" for the fermion.]

6 The isospin model

6.1 Lagrangian, symmetries, prescribed kink special case and solution

We introduce isospin to the model by promoting the fermion to also live in a spinor representation of isospin. It is typical to also promote the scalar field ϕ to be an isovector $\boldsymbol{\phi} = (\phi_1, \phi_2)$; let us consider this to begin with, though we will show we can still reduce this to a single isoscalar field.

We denote the new four-component fermion by Ψ, and label its components $\Psi_{\alpha a}$ by a pair of indices denoting the components in tensor product of the Lorentz spinor and isospinor, ordered so that

$$\Psi = \begin{pmatrix} \Psi_{11} \\ \Psi_{12} \\ \Psi_{21} \\ \Psi_{22} . \end{pmatrix} \tag{164}$$

To distinguish between the Lorentz spin and Euclidean isospin representations, let us denote the "new" γ matrices in terms of the previous ones by $\hat{\gamma}^\alpha = \gamma^\alpha \otimes \mathbb{I}$, and the isospin matrices by $\tau_i = \mathbb{I} \otimes \sigma_i$ for the standard Pauli matrices σ_i. There is a (co)vector of isospin matrices in the Clifford algebra $\boldsymbol{\tau} = (\tau_1, \tau_2)$ and the chiral isospin matrix τ_3. We now couple the fermion and isovector field with the term

$$g\overline{\Psi}\boldsymbol{\phi} \cdot \boldsymbol{\tau} \Psi. \tag{165}$$

(Note that, owing to the difference in signature, a *Euclidean* bispinor such as $\overline{\Psi}\boldsymbol{\tau}\Psi$ is in the *dual* isovector representation: that is, the combination $\overline{\Psi}\boldsymbol{\phi} \cdot \boldsymbol{\tau}\Psi$ is an isoscalar just when $\boldsymbol{\phi}$ is also a *dual* isovector, so transforms *backwards* under isorotations.)

Observe that, if we can define $\chi = \arccos\phi_1 = \arcsin\phi_2$, then $\boldsymbol{\phi}\cdot\boldsymbol{\tau} = \tau_1 \mathrm{e}^{\mathrm{i}\chi\tau_3}$. Thus, if we require that the isovector obeys $\boldsymbol{\phi}\cdot\boldsymbol{\phi} = 1$, as we should, we may simply write $\boldsymbol{\phi} = (\cos\tilde{\phi}, \sin\tilde{\phi})$, and the coupling term as

$$g\overline{\Psi}\tau_1 \mathrm{e}^{\mathrm{i}\tilde{\phi}\tau_3}\Psi. \tag{166}$$

Moreover, a Dirichlet term for the isovector $\boldsymbol{\phi}$ is equivalent to a Dirichlet term for this angular isoscalar $\tilde{\phi}$: by direct calculation,

$$\frac{1}{2}\partial_\mu\boldsymbol{\phi}\cdot\partial^\mu\boldsymbol{\phi} = \frac{1}{2}\partial_\mu\tilde{\phi}\,\partial^\mu\tilde{\phi}. \tag{167}$$

Thus, we will eliminate any explicit appearance of an isovector, and just use ϕ to denote a scalar kink field as before. We will now work in a general fully-coupled model with the Lagrangian density, which may include a fermionic mass m_f,

$$\mathcal{L} = \frac{M^2}{2}\partial_\mu\phi\,\partial^\mu\phi + \overline{\Psi}\left(\mathrm{i}\hbar\partial\!\!\!/ - g\mathrm{e}^{\mathrm{i}\tau_3\phi} - m_f\right)\Psi, \tag{168}$$

and the special (linear) case of a prescribed static uniform kink $\phi(t,\theta) = n\theta$ given by the Lagrangian density

$$\mathcal{L}_f = \overline{\Psi}\left(\mathrm{i}\hbar\partial\!\!\!/ - g\mathrm{e}^{\mathrm{i}\tau_3 n\theta} - m_f\right)\Psi. \tag{169}$$

Much of the analysis of the special case either follows immediately from, or is directly analagous to, the model without isospin. Fermionic energy is conserved and solutions obey a Schrödinger-type equation,

$$\mathrm{i}\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi, \tag{170}$$

where now the self-adjoint Hamiltonian is

$$\hat{H} = \mathrm{i}a\hat{\gamma}^3\frac{\partial}{\partial\theta} + g\hat{\gamma}^0\tau_1\mathrm{e}^{\mathrm{i}n\theta\tau_3} + m_f\gamma^0; \tag{171}$$

stationary state solutions obey $\hat{H}\Psi = E_f\Psi$. There is again a continuous symmetry combining translation on the base space with isorotation,

$$\Psi(t,\theta) \to \mathrm{e}^{\mathrm{i}\frac{\alpha n}{2}\tau_3}\Psi(t,\theta-\alpha), \tag{172}$$

with corresponding self-adjoint generalised angular momentum generator

$$\hat{L} = -\mathrm{i}\frac{\partial}{\partial\theta} + \frac{n}{2}\tau_3 \tag{173}$$

which commutes with $\hat{H}$. In the case $m_f = 0$, there is another continuous internal symmetry that we call the *biaxial* symmetry,

$$\Psi \to \mathrm{e}^{\mathrm{i}\alpha\gamma^3\tau_3}\Psi, \tag{174}$$

with self-adjoint generator $\gamma^3\tau_3$ which commutes with both $\hat{H}$ and $\hat{L}$. Since $\left(\gamma^3\tau_3\right)^2 = 1$, this operator has eigenvalues ± 1. For eigenstates, this eigenvalue is the product of the Lorentzian and Euclidean chiralities of its spinor representations, and we refer to it as the state's *orthochirality.* In a slight abuse of existing nomenclature, we refer to states in the $+1$ eigenspace as *homochiral*, as their individual chiralities are equal, while states in the -1 eigenspace are called *heterochiral* as the individual chiralities are opposites.

An eigenstate of generalised angular momentum, satisfying $\hat{L}\Psi = l\Psi$, is quickly seen to be of the form

$$\Psi_k(\theta) = \begin{pmatrix} A\mathrm{e}^{\mathrm{i}\left(l-\frac{n}{2}\right)\theta} \\ B\mathrm{e}^{\mathrm{i}\left(l+\frac{n}{2}\right)\theta} \\ C\mathrm{e}^{\mathrm{i}\left(l-\frac{n}{2}\right)\theta} \\ D\mathrm{e}^{\mathrm{i}\left(l+\frac{n}{2}\right)\theta} \end{pmatrix} \tag{175}$$

for some constant complex coefficients A, B, C, D. Then homochiral energy eigenstates have $E_f = \frac{an}{2} \pm \sqrt{a^2l^2 + g^2}$ and are respectively proportional to

$$\Psi_{l\pm,\mathrm{hom}}(\theta) = \begin{pmatrix} g\mathrm{e}^{\mathrm{i}\left(l-\frac{n}{2}\right)\theta} \\ 0 \\ 0 \\ \left(al \pm \sqrt{a^2l^2+g^2}\right)\mathrm{e}^{\mathrm{i}\left(l-\frac{n}{2}\right)\theta} \end{pmatrix}, \tag{176}$$

while heterochiral energy eigenstates have $E_f = -\frac{an}{2} \pm \sqrt{a^2l^2 + g^2}$ and are respectively proportional to

$$\Psi_{l\pm,\mathrm{het}} = \begin{pmatrix} 0 \\ g\mathrm{e}^{\mathrm{i}\left(l+\frac{n}{2}\right)\theta} \\ \left(al \pm \sqrt{a^2l^2+g^2}\right)\mathrm{e}^{\mathrm{i}\left(l-\frac{n}{2}\right)\theta} \\ 0 \end{pmatrix}. \tag{177}$$

We observe now, without loss of generality, that zero-crossing only occurs when a homochiral lower branch curve $E_f(g)$ intersects a heterochiral upper branch. We will later use this to provide a physical interpretation of what occurs at these zero-crossing points.

6.2 Fully coupled bispinor co-ordinates and reduction

[TBD: As before, in the fully coupled model the generalised angular momentum splits into translation on the base space and the internal isorotation. The massless fully coupled model has the following equations of motion:

$$\text{Dirac:} \qquad \left(\mathrm{i}\hbar\partial - g\tau_1\mathrm{e}^{\mathrm{i}\phi\tau_3}\right)\Psi = 0 \tag{178}$$

(and spinorial conjugate) (179)

$$\text{Kink:} \qquad M^2\partial_\mu\partial^\mu\phi - g\overline{\Psi}\tau_2\mathrm{e}^{\mathrm{i}\phi\tau_3}\Psi = 0 \tag{180}$$

$$\text{Phase current conservation:} \qquad \partial_\mu\left[\overline{\Psi}\hat{\gamma}^\mu\Psi\right] = 0 \tag{181}$$

$$\text{Isocurrent conservation:} \qquad M^2\partial_\mu\partial^\mu\phi + \frac{\hbar}{2}\partial_\mu\left[\overline{\Psi}\hat{\gamma}^\mu\tau_3\Psi\right] = 0 \tag{182}$$

$$\text{(implying via the kink equation)} \qquad \frac{\hbar}{2}\partial_\mu\left[\overline{\Psi}\hat{\gamma}^\mu\tau_3\Psi\right] = -g\overline{\Psi}\tau_2\mathrm{e}^{\mathrm{i}\phi\tau_3}\Psi \tag{183}$$

$$\text{Biaxial current conservation:} \qquad \partial_\mu\hat{\varepsilon}^\mu{}_\nu\left[\overline{\Psi}\hat{\gamma}^\nu\tau_3\Psi\right] = 0. \tag{184}$$

In the 3S ansatz, all the equations of motion are captured by the collection of bispinors (either Lorentz vectors or isovectors):

$$(\hat{X}, \hat{W}) = \mathcal{J}^\alpha = \overline{\Psi}\hat{\gamma}^\alpha\Psi, \tag{185}$$

$$(\mathcal{U}, \mathcal{V}) = \mathcal{K}^\alpha = \overline{\Psi}\hat{\gamma}^\alpha\tau_3\Psi, \tag{186}$$

$$(G, F) = \overline{\Psi}\tau_i\mathrm{e}^{\mathrm{i}\phi\tau_3}\Psi. \tag{187}$$

The equations of motion are compactly expressed

$$\hbar\frac{\partial\hat{X}}{\partial t} - a\frac{\partial\hat{W}}{\partial\theta} = 0, \tag{188}$$

$$M^2\partial_\mu\partial^\mu\phi - gF = 0, \tag{189}$$

$$M^2\partial_\mu\partial^\mu\phi + \frac{\hbar}{2}\frac{\partial\mathcal{U}}{\partial t} - \frac{a}{2}\frac{\partial\mathcal{V}}{\partial\theta} = 0, \tag{190}$$

$$\text{(the previous two implying)}\quad \hbar\left(\frac{\partial\mathcal{U}}{\partial t} - \frac{1}{\rho}\frac{\partial\mathcal{V}}{\partial\theta}\right) = -2gF, \tag{191}$$

$$\frac{\hbar}{2}\frac{\partial\mathcal{V}}{\partial t} - \frac{a}{2}\frac{\partial\mathcal{U}}{\partial\theta} = 0. \tag{192}$$

We show that all the definite orthochirality angular momentum solutions remain solutions of the fully coupled model. But we can go further. By direct computation, we see that

$$\hat{X} = |\Psi_{11}|^2 + |\Psi_{12}|^2 + |\Psi_{21}|^2 + |\Psi_{22}|^2, \tag{193}$$

$$\hat{W} = |\Psi_{11}|^2 + |\Psi_{12}|^2 - |\Psi_{21}|^2 - |\Psi_{22}|^2, \tag{194}$$

$$\mathcal{U} = |\Psi_{11}|^2 - |\Psi_{12}|^2 + |\Psi_{21}|^2 - |\Psi_{22}|^2, \tag{195}$$

$$\mathcal{V} = |\Psi_{11}|^2 - |\Psi_{12}|^2 - |\Psi_{21}|^2 + |\Psi_{22}|^2. \tag{196}$$

Therefore, comparison of the bispinor equations of motion of the two models show that all the 3S solutions of the model *without* isospin appear as *definite orthochirality* solutions of the model with isospin: either set the homochiral components to zero and identify $X = \hat{X} = \mathcal{U}$ and $W = \hat{W} = \mathcal{V}$, or set the heterochiral components to zero and identify $X = \hat{X} = -\mathcal{V}$ and $W = \hat{W} = -\mathcal{U}$. Any new solutions of the isospin model must explicitly mix orthochiralities. This remains to be explored.]

6.3 Discrete symmetries and zero-crossing modes

[TBD: The parity symmetry still exists, and is improved: owing to the better symmetry of the energy spectra in the isospin model, we are allowed to exchange upper and lower energy branches, and thus no longer need to awkwardly insist that ϕ be a pseudoscalar. A parity transformation identifies e.g. a homochiral state on a kink with winding number n with a heterochiral state on a kink with winding number $-n$. A C symmetry now exists directly swapping upper and lower energy branches, which gives us a better interpretation of "antiparticles" / the Dirac sea. So perhaps we should strictly consider the previous model to exist only as a subset of the model with isospin. In that case, it appears that spectral flow across the critical coupling value of a zero mode might be interpreted heuristically as swapping all particles for their antiparticles.]

7 Conclusion and outlook

[TBD: Obvious next questions to ask:

1. What happens as we send $\rho \to \infty$? (This should probably be part of this paper, not just mentioned in the conclusion.)
2. What about adding a potential $U(\phi)$ for the kink such as the sine-Gordon potential, or (if we can make sense of it with careful modification) the ϕ^4 potential, which actually admit proper (topological soliton) kinks in the absence of the fermion?

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[TBD: Fix bizarro BibTex formatting!]