A model of fermion-baby Skyrmion coupling on $\mathbb{R} \times \mathrm{S}^2$

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Abstract

We present a $(2+1)$-dimensional model of a (classical) Dirac fermion coupled to an isovector field as an idealised model of a fermion in the presence of a topological soliton. Our spacetime manifold is $\mathbb{R} \times S^2$, with an unbounded time direction and a periodic space direction. Although the model is Lorentz-invariant, we break the relativistic symmetry to work in a privileged frame, and consider static solutions for the soliton and steady state solutions for the fermion in this non-relativistic perspective, seeking to solve the energy eigenvalue problem for the fermion. We first consider the case where the soliton is fixed with an idealised uniform winding in the background, so there is no back-reaction from the fermion on the soliton. In this case, the problem is linear, and we hope to find appropriate quantum numbers in order to determine a complete basis of eigensolutions. A symmetry of the coupling term allows us to define an appropriate "generalised angular momentum" operator, the square of whose eigenvalue is a quantum number. Separating variables, we are left with the problem of determining the latitude profile function in terms of the polar angle. The differential equation for this function is Fuchsian when the fermion is massless. Analysis of the singular points of the differential operator tells us to expect polynomial solutions up to an overall factor. We exhibit some explicit solutions and demonstrate how their degree determines their energy spectra. We conjecture that all physical solutions will arise as similar polynomial solutions in the latitude function, and that the polynomial degree provides the final quantum number necessary to understand the energy spectra. We also explore numerically the case where the fermion mass is non-zero and the differential equation for the latitude function is no longer Fuchsian.

Contents

1 Introduction

The dynamics of fermions coupled to topological solitons have been examined in various low-dimensional models where the coupling term is motivate by analogy to the "chiral bag" model of nucleon and fermion fields, where the fermion also carries isospin and couples via a bispinor term to the soliton in isovector representation. Spacetime is typically chosen to be either flat Minkowski space, or such that a canonical spatial slice is taken to be an n-sphere (perhaps by compactification of the original flat spacetime). Often, it is easiest to consider the soliton to be in the background of the fermion, so there is no back-reaction and the fermion can be (numerically or analytically) solved for a choice of known soliton solution. Occasionally the soliton is instead fully coupled to the fermion and solutions of the full the system are obtained. When the fermions can be solved analytically, the symmetries of the model often aid classification of solutions, and solutions may sometimes be expressed in terms of hypergeometric functions. Scattering data can be calculated for some models. A frequent feature of such models is the existence of fermionic zero modes and spectral flow.

In one space dimension, the soliton is a scalar or pseudoscalar field in a model that exhibits kink solutions interpolating between topologically distinct vacua, such as the ϕ^4 model or the sine-Gordon model on the real line. Fermions on a background ϕ^4 kink solution modelled with a very simple chiral coupling term have been solved numerically [8, 10], giving insight into scattering coefficients as well as the role of the topological winding number and fermionic zero modes to the Casimir energy of the system. Numerical solutions have also been found when the kink is fully coupled to the fermion [7]. The conventional chiral bag coupling terms models a fermion on a background sine-Gordon kink by interpreting the kink as an isovector and promoting the fermion to an isospinor representation [9]. Fermion scattering states have wavefunctions that can be expressed in terms of Heun functions. The dependence of the transmission and reflection coefficients on physical parameters such as the fermion mass is investigated numerically, and it is observed that fermionic zero modes polarise the vacuum.

In two spatial dimensions, fermions on a background baby Skyrmion in flat Minkowski space exhibit localisation by the soliton. [4] In the absence of backreaction, there is a combined rotation-isorotation symmetry of the fermions corresponding to a generalised angular momentum generator, and the conserved eigenvalue aids classification of the fermion solutions. When back-reaction is considered, it is numerically observed that a strongly coupled fermion deforms the soliton solution. One zero-crossing fermion mode is observed under spectral flow. Similarly, fermions have been considered coupled to the so-called "magnetic Skyrmion", where the role of the soliton is played by an isovector field with a Dzyaloshinskii-Moriya interaction. [3]. The backreaction of localised fermions has a significant effect on the solitons, leading to bound multisoliton solutions which did not exist in their absence.

In three spatial dimensions, the Skyrme model on a spatial 3-sphere exhibits BPS solutions. The $B = 1$ solution is particularly straightforward and has been used as a background for spin-isospin fermions. [5, 6] Similarly to the baby Skyrmion model in 2+1 dimensions, there is a generalised angular momentum of the fermion around the background Skyrmion providing a quantum number. A full description of fermion solutions is possible with some analytical tools by observing that part of the energy eigenvalue problem is expressed as a Fuchsian differential equation. A family of solutions can be expressed in terms of a basis of spinorial eigenfunctions of the Dirac operator, similar to the role of so-called “spinor harmonics” or “monopole harmonics” on the 2-sphere [1, 2].

In this paper, we examine a model of fermion-baby Skyrmion coupling on $\mathbb{R} \times S^2$. By analogy with the simplicity of the $B = 1$ BPS Skyrmion on S^3, we begin with a fixed scalar field taking values in S^2 in the background of the fermion, prescribed to wind uniformly azimuthally n times over a period of the base space. This simple azimuthal winding permits a straightforward definition of generalised angular momentum about the polar axis. Separating variables, we reduce the eigenvalue problem for solving for the latitude profile as a function of the polar angle. We observe that if at least one of the fermion mass and the coupling constant vanishes, the differential equation for the latitude profile is Fuchsian when considered to extend to the entire complex plane, and therefore amenable to solution by Frobenius series. We check the Fuchs relations and observe that, up to the critical exponent factor, normalisable solutions are polynomials. We present some explicit such polynomial solutions and can understand how they fit into energy levels for given azimuthal angular momentum. We conjecture how the degree of such polynomial solutions should generally be the second and final quantum number needed to fully understand the energy spectra, and to describe the solutions in terms of spinor spherical harmonics on s^2. We outline numerical treatments to go beyond this special case, such as permitting both fermion mass and coupling to be non-zero, and moving the baby Skyrmion out of the background to fully couple with the fermion.

2 Constructing the model

2.1 Conventions and representations

Consider the local Clifford algebra for $\mathbb{R} \times S^2$ with Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1)$. The Lie algebra $\mathfrak{so}(1, 2)$ has three generators: J corresponding to a positive spatial rotation, and K^1, K^2 corresponding to positive boosts in the x^1, x^2 directions respectively. In the vector representation they are

$$J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad K^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{1}$$

Let us rewrite them as M^μ, where $(M^0, M^1, M^2) = (J, K^2, -K^1)$. Then M^μ rotates or boosts the $(x^{\mu+1}, x^{\mu+2})$ plane (addition modulo 3) and respects this orientation. Define the totally alternating tensor $\varepsilon^{\lambda\mu\nu}$ by $\varepsilon^{012} = 1$ with indices raised and lowered by η. It's then easily seen that

$$(M^\mu)^\lambda{}_\rho = \varepsilon^{\mu\lambda}{}_\rho \tag{2}$$

and

$$[M^\mu, M^\nu] = -\varepsilon^{\mu\nu}{}_\kappa M^\kappa. \tag{3}$$

A general Lorentz transformation is given in terms of parameters a_0, a_1, a_2 by

$$\Lambda^\mu{}_\nu = \left(e^{a_\lambda M^\lambda}\right)^\mu{}_\nu. \tag{4}$$

The *chiral* representation of the Pauli matrices is

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

They have composition $\sigma_i\sigma_j = \delta_{ij} + \mathrm{i}\varepsilon_{ijk}\sigma_k$. The Clifford algebra is generated by the gamma matrices γ^μ obeying $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}$. In two space and one time dimensions, we may take

$$\gamma^0 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = -\mathrm{i}\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^2 = -\mathrm{i}\sigma_3 = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}. \tag{6}$$

They act on the space of Dirac spinors $\mathbb{C}^2$ by standard multiplication. The product $\gamma^0\gamma^1\gamma^2$ is proportional to the identity, so there is no further decomposition of the Dirac spinor into Weyl spinors. (Note that in this convention, γ^0 is symmetric while γ^1 and γ^2 are anti-symmetric; this is equivalent to the usual physical convention that γ^0 is anti-Hermitian while γ^1 and γ^2 are Hermitian.)

Recall that the Lorentz action on Dirac spinors is usually given by constructing a set of spin generators $S^{\mu\nu}$ proportional to the commutators $[\gamma^\mu, \gamma^\nu]$ and showing that these form a representation of the Lorentz algebra. In $(2+1)$ dimensions, there is no need to have two indices on the spin generators. Let us simply define

$$S^\mu = -\frac{\mathrm{i}}{2}\gamma^\mu. \tag{7}$$

It's easily checked that

$$[S^\mu, S^\nu] = -\varepsilon^{\mu\nu}{}_\kappa S^\kappa, \tag{8}$$

so these S^μ do form a representation of the M^μ. Thus, for any Lorentz action Λ which in the vector representation has $\Lambda = \mathrm{e}^{a_\lambda M^\lambda}$, we can define its action on spinors $S[\Lambda]$ via

$$(S[\Lambda])^\alpha{}_\beta = \left(\mathrm{e}^{a_\lambda S^\lambda}\right)^\alpha{}_\beta. \tag{9}$$

Let $\Psi \in \mathbb{C}^2$ be a Dirac spinor. The spinor conjugate $\overline{\Psi}$ is defined as $\overline{\Psi} = \Psi^\dagger\gamma^0$. It remains to show that the spinor bilinear $\overline{\Psi}\gamma^\mu\Psi$ belongs to the vector representation, i.e.

$$\overline{(S[\Lambda]\Psi)}\gamma^\mu\,(S[\Lambda]\Psi) = \Lambda^\mu{}_\nu(\overline{\Psi}\gamma^\nu\Psi) = \overline{\Psi}\Lambda^\mu{}_\nu\gamma^\nu\Psi. \tag{10}$$

Recall that $\overline{(S[\Lambda]\Psi)} = \Psi^\dagger S[\Lambda]^\dagger\gamma^0 = \overline{\Psi}S[\Lambda]^{-1}$, so it is sufficient to check that $S[\Lambda]^{-1}\gamma^\mu S[\Lambda] = \Lambda^\mu{}_\nu\gamma^\nu$. Expanding the transformations as exponentials, that is (suppressing spinor indices)

$$\mathrm{e}^{-a_\lambda S^\lambda}\gamma^\mu\mathrm{e}^{a_\kappa S^\kappa} = \left(\mathrm{e}^{a_\rho M^\rho}\right)^\mu{}_\nu\gamma^\nu \tag{11}$$

$$\Leftrightarrow (1 - a_\lambda S^\lambda)\gamma^\mu(1 + a_\kappa S^\kappa) = (1 + a_\rho M^\rho)^\mu{}_\nu\gamma^\nu + O(a^2) \tag{12}$$

Hence it is sufficient to show that $[\gamma^\mu, S^\lambda] = \left(M^\lambda\right)^\mu{}_\nu\gamma^\nu$. By comparison with (8) above, we see that

$$[\gamma^\mu, S^\lambda] = -\varepsilon^{\mu\lambda}{}_\nu\gamma^\nu = \varepsilon^{\lambda\mu}{}_\nu\gamma^\nu = \left(M^\lambda\right)^\mu{}_\nu\gamma^\nu \tag{13}$$

as required. Thus indeed the spinor bilinear $\overline{\Psi}\gamma^\mu\Psi$ is a Lorentz vector.

We proceed similarly for the isospinor, which is constructed as the "Dirac" spinor for $\mathfrak{so}(0,3) = \mathfrak{so}(3)$. The Euclidean metric is just the Kronecker delta δ_{ij}. As the metric is of definite signature, there is no need to distinguish between covariant and contravariant indices, so we will write all indices downstairs. $\mathfrak{so}(3)$ is generated by the three elements J_i, each of which generates a positive rotation in the (x_{i+1}, x_{i+2}) plane (addition modulo 3). In the vector representation they have elements

$$(J_i)_{jk} = -\varepsilon_{ijk}. \tag{14}$$

(In particular, J_1 is the same as the J given above for $\mathfrak{so}(1,2)$.) They obey the familiar commutation relation

$$[J_i, J_j] = \varepsilon_{ijk} J_k. \tag{15}$$

A rotation of $\mathbb{R}^3$ parametrised by the components of a vector $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$ is given by

$$R_{ij} = \left(\mathrm{e}^{\boldsymbol{\theta}\cdot\mathbf{J}}\right)_{ij} = \left(\mathrm{e}^{\theta_k J_k}\right)_{ij}. \tag{16}$$

Here the Pauli matrices satisfy to generate the Clifford algebra, as they certainly obey $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{I}$. They act by standard multiplication on an *isospinor* $\Xi \in \mathbb{C}^2$. Again, to find "spin generators" proportional to commutators $[\sigma_i, \sigma_j]$ it is sufficient to simply rescale the σ_i. Let

$$\Sigma_i = -\frac{\mathrm{i}}{2}\sigma_i \tag{17}$$

It is immediate that these form a representation of the J_i as

$$[\Sigma_i, \Sigma_j] = \varepsilon_{ijk}\Sigma_k. \tag{18}$$

Thus a rotation $R = \mathrm{e}^{\boldsymbol{\theta}\cdot\mathbf{J}}$ induces an action on an isospinor according to

$$(S[R])_{ab} = \left(\mathrm{e}^{\boldsymbol{\theta}\cdot\boldsymbol{\Sigma}}\right)_{ab}. \tag{19}$$

However, in contrast to the spinors in indefinite signature, we see that for the isospinors the corresponding identity to (13) is

$$[\sigma_i, \Sigma_j] = -\frac{\mathrm{i}}{2}[\sigma_i, \sigma_j] = \varepsilon_{ijk}\sigma_k = -(J_i)_{jk}\sigma_k. \tag{20}$$

The relative minus sign here implies that the isospinor bilinear, $\Xi^\dagger\sigma_i\Xi$, in fact belongs to the conjugate of the vector representation: it rotates *backwards* under the action of rotations, i.e.

$$(S[R]\Xi)^\dagger\sigma_i(S[R]\Xi) = \Xi^\dagger(R^{-1})_{ij}\sigma_j\Xi. \tag{21}$$

Thus, to create a rotation-invariant scalar of the form $\Xi^\dagger\boldsymbol{\phi}\cdot\boldsymbol{\sigma}\Xi$, the object $\boldsymbol{\phi}$ should also be in the dual (iso)vector representation, and transform backwards under (iso)rotations.

Note that the constant of proportionality of the spin generators to the basis elements of the Clifford algebras, which was $-\frac{\mathrm{i}}{2}$ in both (7) and (17) above, was fixed by the desire to equate the commutator in each case. In particular, our choice of orientation fixed the sign, and hence our conclusion that the isospinor bilinear transformed backwards relative to the spinor bilinear. Simply setting $\Sigma_i = +\frac{\mathrm{i}}{2}\sigma_i$ instead at (17), for example, would not have made the isospinor bilinear transform forwards, because the commutator would break the orientation of the algebra.

2.2 Geometry and the Lagrangian

We embed $\mathbb{R}\times S^2$ in $\mathbb{R}\times\mathbb{R}^3$ with coordinates $x^\mu = (t, x, y, z)$. At each fixed time t, we perform the usual stereographic projections from S^2 to $\mathbb{R}^2$ in order to provide a smooth atlas of charts. Explicitly, at each time t, and not explicitly labelling the dependence on t, we define the north pole N to be $(t, 0, 0, 1)$ and the south pole S to be $(t, 0, 0, -1)$. We take charts

$$U_N = \{t\}\times S^2 \setminus \{S\} \tag{22}$$

$$U_S = \{t\}\times S^2 \setminus \{N\} \tag{23}$$

The stereographic projection operators φ_N, φ_S map U_N, U_S respectively to $\mathbb{R}\times\mathbb{R}^2$ according to

$$\varphi_N(t,x,y,z) = \left(t, \frac{x}{1+z}, \frac{y}{1+z}\right) \tag{24}$$

$$\varphi_S(t,x,y,z) = \left(t, \frac{x}{1-z}, \frac{y}{1-z}\right) \tag{25}$$

Let us work in $\operatorname{im}\varphi_S$ for now. We will refer to the local coordinates as $X^\mu = (t, X, Y)$, and define $R^2 = X^2 + Y^2$. The pullback of the standard Minkowski metric $\eta_{\mu\nu} = \operatorname{diag}(1,-1,-1,-1)$ via φ_S becomes

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}t^2 - \frac{4}{(1+R^2)^2}\mathrm{d}X^2 - \frac{4}{(1+R^2)^2}\mathrm{d}Y^2 \tag{26}$$

We take a dreibein $\{e_\alpha{}^\mu\}$ with local tangent vector basis

$$\hat{e}_\alpha = e_\alpha{}^\mu \frac{\partial}{\partial X^\mu} \text{ such that } g(\hat{e}_\alpha, \hat{e}_\beta) = \eta_{\alpha\beta} = \operatorname{diag}(1,-1,-1) \tag{27}$$

$$\text{i.e. } g_{\mu\nu} = e^\alpha{}_\mu e^\beta{}_\nu \eta_{\alpha\beta}, \tag{28}$$

where the $\{e^\alpha{}_\mu\}$ are the matrix inverses of the dreibein. A suitable dreibein has components

$$e_0{}^0 = 1, \quad e_1{}^1 = -\frac{(1+R^2)}{2} = e_2{}^2, \quad \text{others } 0. \tag{29}$$

The dual frame basis is $\hat{\theta}^\alpha = e^\alpha{}_\mu \mathrm{d}X^\mu$, so

$$\hat{\theta}^0 = \mathrm{d}t, \quad \hat{\theta}^1 = -\frac{2}{1+R^2}\mathrm{d}X, \quad \hat{\theta}^2 = -\frac{2}{1+R^2}\mathrm{d}Y. \tag{30}$$

To derive the connection 1-form $\omega^\alpha{}_\beta \in \mathfrak{so}(1,2)\otimes\Omega^1(\mathbb{R}\times\mathbb{R}^2)$, it is easiest to use the torsion-free (antisymmetry) condition

$$\omega_{\alpha\beta} = -\omega_{\beta\alpha} \tag{31}$$

with the first of Cartan's structure equations,

$$\mathrm{d}\hat{\theta}^\alpha + \omega^\alpha{}_\beta \wedge \hat{\theta}^\beta = 0 \quad (= T^\alpha)\,. \tag{32}$$

Recall that the torsion 2-form $T^\alpha \in T(\mathbb{R}\times\mathbb{R}^2)\otimes\Omega^2(\mathbb{R}\times\mathbb{R}^2)$ is obtained as follows. Given a connection ∇, the torsion tensor in the physicist's convention[1] is the type-(1,2) tensor T defined by

$$T(X,Y,\omega) = \omega(\nabla_X Y - \nabla_Y X - [X,Y]). \tag{33}$$

Its components are $T^\alpha{}_{\beta\gamma} = T(\hat{e}_\beta, \hat{e}_\gamma, \hat{\theta}^\alpha)$, and then the torsion 2-form is obtained by contracting

$$T^\alpha = \frac{1}{2}T^\alpha{}_{\beta\gamma}\,\hat{\theta}^\beta \wedge \hat{\theta}^\gamma. \tag{34}$$

In terms of the local frame, the connection components are defined via

$$\nabla_{\hat{e}_\alpha}(\hat{e}_\beta) = \Gamma^\gamma{}_{\alpha\beta}\hat{e}_\gamma, \tag{35}$$

[1] The mathematician's convention carries an overall relative minus sign.

and the connection 1-form is obtained by the contraction

$$\omega^{\alpha}{}_{\beta} = \Gamma^{\alpha}{}_{\gamma\beta}\hat{\theta}^{\gamma}. \tag{36}$$

By evaluating components in the local frame it is easy to check that the condition $T^{\alpha} = 0$ implies the antisymmetry condition on the connection 1-form expressed at (31).

We will evaluate the components that appear in the structure equation (32) for $\alpha = 0, 1, 2$, writing

$$\omega^{\alpha}{}_{\beta} = \omega_t{}^{\alpha}{}_{\beta}\mathrm{d}t + \omega_X{}^{\alpha}{}_{\beta}\mathrm{d}X + \omega_Y{}^{\alpha}{}_{\beta}\mathrm{d}Y. \tag{37}$$

The 9 *a priori* independent components we wish to solve for are WLOG the three components each of $\omega^0{}_1, \omega^0{}_2$ and $\omega^1{}_2$.

- $\alpha = 0$

$$\begin{aligned} 0 &= \mathrm{d}\hat{\theta}^0 + \omega^0{}_1 \wedge \hat{\theta}^1 + \omega^0{}_2 \wedge \hat{\theta}^2 \\ &= \left(-\frac{2}{1+R^2}\right)\left(\omega_t{}^0{}_1\mathrm{d}t \wedge \mathrm{d}X + \omega_Y{}^0{}_1\mathrm{d}Y \wedge \mathrm{d}X + \omega_t{}^0{}_2\mathrm{d}t \wedge \mathrm{d}Y + \omega_X{}^0{}_2\mathrm{d}X \wedge \mathrm{d}Y\right) \end{aligned}$$

This implies $\omega_t{}^0{}_1 = 0 = \omega_t{}^0{}_2$ and $\omega_Y{}^0{}_1 = \omega_X{}^0{}_2$.

- $\alpha = 1$

$$\begin{aligned} 0 &= \mathrm{d}\hat{\theta}^1 + \omega^1{}_0 \wedge \hat{\theta}^0 + \omega^1{}_2 \wedge \hat{\theta}^2 \\ &= \mathrm{d}\left(-\frac{2}{1+X^2+Y^2}\mathrm{d}X\right) - \omega_X{}^0{}_1\mathrm{d}X \wedge \mathrm{d}t + \left(\omega_t{}^1{}_2\mathrm{d}t + \omega_X{}^1{}_2\mathrm{d}X\right) \wedge \left(-\frac{2}{1+X^2+Y^2}\mathrm{d}Y\right) \\ &= \frac{4Y}{(1+R^2)^2}\mathrm{d}Y \wedge \mathrm{d}X - \omega_X{}^0{}_1\mathrm{d}X \wedge \mathrm{d}t - \frac{2\omega_t{}^1{}_2}{1+R^2}\mathrm{d}t \wedge \mathrm{d}Y - \frac{2\omega_X{}^1{}_2}{1+R^2}\mathrm{d}X \wedge \mathrm{d}Y \end{aligned}$$

This implies $\omega_X{}^0{}_1 = 0 = \omega_t{}^1{}_2$ and $\omega_X{}^1{}_2 = -\frac{2Y}{1+R^2}$.

- $\alpha = 2$

$$\begin{aligned} 0 &= \mathrm{d}\hat{\theta}^2 + \omega^2{}_0 \wedge \hat{\theta}^0 + \omega^2{}_1 \wedge \hat{\theta}^1 \\ &= \mathrm{d}\left(-\frac{2}{1+X^2+Y^2}\mathrm{d}Y\right) - \omega_X{}^0{}_2\mathrm{d}X \wedge \mathrm{d}t - \omega_Y{}^0{}_2\mathrm{d}Y \wedge \mathrm{d}t - \omega_Y{}^1{}_2\mathrm{d}Y \wedge \left(-\frac{2}{1+X^2+Y^2}\mathrm{d}X\right) \\ &= \frac{4X}{(1+R^2)^2}\mathrm{d}X \wedge \mathrm{d}Y - \omega_X{}^0{}_2\mathrm{d}X \wedge \mathrm{d}t - \omega_Y{}^0{}_2\mathrm{d}Y \wedge \mathrm{d}t + \frac{2\omega_Y{}^1{}_2}{1+R^2}\mathrm{d}Y \wedge \mathrm{d}X \end{aligned}$$

We finally obtain $\omega_X{}^0{}_2 (= \omega_Y{}^0{}_1) = 0 = \omega_Y{}^0{}_2$ and $\omega_Y{}^1{}_2 = \frac{2X}{1+R^2}$.

Hence the spin connection has as its only non-zero contribution

$$\omega^1{}_2 = \frac{2}{1+R^2}\left(X\mathrm{d}Y - Y\mathrm{d}X\right) \tag{38}$$

or equivalently

$$\omega^{12} = -\frac{2}{1+R^2}\left(X\mathrm{d}Y - Y\mathrm{d}X\right). \tag{39}$$

The spin connection satisfies

$$\Omega_\mu = -\frac{\mathrm{i}}{2}{\omega_\mu}^{\alpha\beta}\Sigma_{\alpha\beta} \tag{40}$$

Hence

$$\Omega_0 = 0 \tag{41}$$

$$\begin{aligned}\Omega_1 &= -\frac{\mathrm{i}}{2}\left({\omega_X}^{12}\Sigma_{12} + {\omega_X}^{21}\Sigma_{21}\right)\\ &= -\mathrm{i}({\omega_X}^{12}\Sigma_{12})\\ &= -\mathrm{i}\left(\frac{2Y}{1+R^2}\right)\frac{1}{2}\gamma^0\\ &= -\frac{\mathrm{i}Y}{1+R^2}\gamma^0 \end{aligned}\tag{42}$$

and similarly

$$\begin{aligned}\Omega_2 &= -\frac{\mathrm{i}}{2}({\omega_Y}^{12}\Sigma_{12} + {\omega_Y}^{21}\Sigma_{21})\\ &= -\mathrm{i}({\omega_Y}^{12}\Sigma_{12})\\ &= -\mathrm{i}\left(-\frac{2X}{1+R^2}\right)\frac{1}{2}\gamma^0\\ &= \frac{\mathrm{i}X}{1+R^2}\gamma^0. \end{aligned}\tag{43}$$

For fermions on this curved spacetime, the Dirac operator is

$$\begin{aligned}\mathrm{i}\not{D} &= \mathrm{i}(\not\partial + \not\Omega) = \mathrm{i}\gamma^\mu(\partial_\mu + \Omega_\mu)\\ &= \mathrm{i}\gamma^\alpha {e_\alpha}^\mu(\partial_\mu + \Omega_\mu)\\ &= \mathrm{i}\gamma^0 1\cdot 1(\partial_0) + \mathrm{i}\gamma^1\left(-\frac{1+R^2}{2}\right)\left(\partial_1 - \frac{\mathrm{i}Y}{1+R^2}\gamma^0\right) + \mathrm{i}\gamma^2\left(-\frac{1+R^2}{2}\right)\left(\partial_2 + \frac{\mathrm{i}X}{1+R^2}\gamma^0\right)\\ &= \mathrm{i}\gamma^0\partial_0 - \mathrm{i}\gamma^i\left(\frac{(1+R^2)}{2}\partial_i - \frac{X_i}{2}\right). \end{aligned}\tag{44}$$

Compare this with the expression derived at (11) of Goatham & Krusch: note that in the contribution for the spin connection, there is a relative factor of $\frac{1}{2}$. This is because on the 2-sphere, contributions to the spatial component of the spin connection only come from the single other spatial component, whereas there are two such contributions from the two other spatial components on the 3-sphere.

We next wish to introduce the (baby) Skyrme field via the real triplet isospinor $\boldsymbol{\phi}$, so the fermion field should also be in a spinor representation for isospin. We construct our representation as the tensor product of representations (Spin) $\otimes$ (Isospin): we promote the spin matrices (i.e. Dirac matrices) to

$$\hat{\gamma}^\mu = \gamma^\mu \otimes \mathbb{I}, \tag{45}$$

and take standard (flat) isospin matrices

$$\boldsymbol{\tau} = \mathbb{I}\otimes\boldsymbol{\sigma}, \tag{46}$$

where $\boldsymbol{\sigma}$ is the usual vector of Pauli matrices.

The fermionic Lagrangian (allowing now for non-zero fermion mass) will be

$$\mathcal{L}_f = \overline{\Psi}(\mathrm{i}\hat{\not{D}} - g\boldsymbol{\tau}\cdot\boldsymbol{\phi} - m)\Psi, \tag{47}$$

where now $\hat{\not{D}} = \hat{\gamma}^\mu D_\mu = \hat{\gamma}^\mu(\partial_\mu + \Omega_\mu)$, g is the spin-isospin coupling constant, m is the fermion mass, $\Psi(t, X_i)$ is the fermion field taking values in $\mathbb{C}^2\otimes$ (Isospin) $= \mathbb{C}^2 \otimes \mathbb{C}^2$, and $\overline{\Psi} = \Psi^\dagger\hat{\gamma}^0$ is the spinor conjugate. That is,

$$\mathcal{L}_f = \Psi^\dagger\hat{\gamma}^0\left[\mathrm{i}\hat{\gamma}^0\partial_0 - \mathrm{i}\hat{\gamma}^i\left(\frac{1+R^2}{2}\partial_i - \frac{X_i}{2}\right) - g\boldsymbol{\tau}\cdot\boldsymbol{\phi} - m\mathbb{I}_2\otimes\mathbb{I}_2\right]\Psi. \tag{48}$$

We prescribe the soliton field to simply be the identity map as a map $\boldsymbol{\phi} : S^2 \to S^2$.

3 Stationary state dynamics

3.1 The Hamiltonian and the latitude profile function

It will be convenient to change coordinates so that on each patch of S^2 we use the usual spherical polar coordinates (θ, ϕ). As an intermediate step, we can change from the stereographic coordinates (X, Y) to holomorphic (stereographic) complex coordinates $(Z, \overline{Z})$, where $Z = X + \mathrm{i}Y$. By varying the Lagrangian with respect to $\overline{\Psi}$, we easily read off the equations of motion. There is an obvious representation of Ψ as a 4-dimensional complex vector, but conceptually we can label the components of the spinor-isospinor field Ψ as a 2×2 complex matrix as follows:

$$\Psi = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{C}) \tag{49}$$

Each row of this 2×2 matrix comprises a spin component, and each column comprises an isospin component. In this convention, the Dirac (spin) matrices $\hat{\gamma}_\mu$ act on the Ψ field via multiplication by the 2×2 γ_μ matrices from the left, and the (isospin) matrices τ_i act via multiplication by $-\sigma_2\sigma_i\sigma_2$ from the right.

In the homogeneous coordinates, the equations of motion are

$$\begin{aligned}
0 &= \mathrm{i}\frac{\partial\Psi_{11}}{\partial t} - (1+Z\overline{Z})\frac{\partial\Psi_{21}}{\partial Z} + \frac{\overline{Z}}{2}\Psi_{21} + g\frac{1-Z\overline{Z}}{1+Z\overline{Z}}\Psi_{11} - 2g\frac{\overline{Z}}{1+Z\overline{Z}}\Psi_{12} - m\Psi_{11},\\
0 &= \mathrm{i}\frac{\partial\Psi_{12}}{\partial t} - (1+Z\overline{Z})\frac{\partial\Psi_{22}}{\partial Z} + \frac{\overline{Z}}{2}\Psi_{22} - g\frac{1-Z\overline{Z}}{1+Z\overline{Z}}\Psi_{12} - 2g\frac{Z}{1+Z\overline{Z}}\Psi_{11} - m\Psi_{12},\\
0 &= \mathrm{i}\frac{\partial\Psi_{21}}{\partial t} + (1+Z\overline{Z})\frac{\partial\Psi_{11}}{\partial \overline{Z}} - \frac{Z}{2}\Psi_{11} - g\frac{1-Z\overline{Z}}{1+Z\overline{Z}}\Psi_{21} + 2g\frac{\overline{Z}}{1+Z\overline{Z}}\Psi_{22} + m\Psi_{21},\\
0 &= \mathrm{i}\frac{\partial\Psi_{22}}{\partial t} + (1+Z\overline{Z})\frac{\partial\Psi_{12}}{\partial \overline{Z}} - \frac{Z}{2}\Psi_{12} + g\frac{1-Z\overline{Z}}{1+Z\overline{Z}}\Psi_{22} + 2g\frac{Z}{1+Z\overline{Z}}\Psi_{21} + m\Psi_{22}.
\end{aligned} \tag{50}$$

We quickly obtain an expression for the Hamiltonian operator $\hat{H}$ from the equations of motion by recognising its role in first-quantised theory as the generator of time translations, so satisfying the expression

$$\mathrm{i}\frac{\partial\Psi}{\partial t} = \hat{H}\Psi.$$

It is most clearly expressed as a 4×4 matrix acting on the Ψ field, where now Ψ is represented as a 4-dimensional complex vector. In comparison to the previous matrix representation, its components are

$$\Psi = \begin{pmatrix} \Psi_{11} \\ \Psi_{12} \\ \Psi_{21} \\ \Psi_{22} \end{pmatrix}. \tag{51}$$

In this representation, the Hamiltonian operator is

$$\hat{H} = \begin{pmatrix} -g\left(\frac{1-Z\overline{Z}}{1+Z\overline{Z}}\right)+m & 2g\frac{\overline{Z}}{1+Z\overline{Z}} & (1+Z\overline{Z})\frac{\partial}{\partial Z}-\frac{\overline{Z}}{2} & 0 \\ 2g\frac{Z}{1+Z\overline{Z}} & g\left(\frac{1-Z\overline{Z}}{1+Z\overline{Z}}\right)+m & 0 & (1+Z\overline{Z})\frac{\partial}{\partial Z}-\frac{\overline{Z}}{2} \\ -(1+Z\overline{Z})\frac{\partial}{\partial \overline{Z}}+\frac{Z}{2} & 0 & g\left(\frac{1-Z\overline{Z}}{1+Z\overline{Z}}\right)-m & -2g\frac{\overline{Z}}{1+Z\overline{Z}} \\ 0 & -(1+Z\overline{Z})\frac{\partial}{\partial \overline{Z}}+\frac{Z}{2} & -2g\frac{Z}{1+Z\overline{Z}} & -g\left(\frac{1-Z\overline{Z}}{1+Z\overline{Z}}\right)-m \end{pmatrix}. \tag{52}$$

(We note that this operator is self-adjoint with respect to the inner product on the space of spinor-isospinor fields $\langle\Psi,\Phi\rangle \sim \int \overline{\Psi}\Phi$ with suitable boundary conditions; more details in Appendix A.)

It is straightforward to change to the usual polar coordinates (θ, ϕ) with the identifications

$$Z = \frac{\mathrm{e}^{\mathrm{i}\phi}\sin\theta}{1-\cos\theta}, \quad \overline{Z} = \frac{\mathrm{e}^{-\mathrm{i}\phi}\sin\theta}{1-\cos\theta}. \tag{53}$$

In spherical polars, the Hamiltonian becomes

$$\hat{H} = \begin{pmatrix} g\cos\theta+m & g\mathrm{e}^{-\mathrm{i}\phi}\sin\theta & \hat{H}_a & 0 \\ g\mathrm{e}^{\mathrm{i}\phi}\sin\theta & -g\cos\theta+m & 0 & \hat{H}_a \\ \hat{H}_b & 0 & -g\cos\theta-m & -g\mathrm{e}^{-\mathrm{i}\phi}\sin\theta \\ 0 & \hat{H}_b & -g\mathrm{e}^{\mathrm{i}\phi}\sin\theta & g\cos\theta-m \end{pmatrix}, \tag{54}$$

where, simply for brevity and ease of typesetting, we have named the expressions

$$\hat{H}_a = -\mathrm{e}^{-\mathrm{i}\phi}\left[\frac{\partial}{\partial\theta} + \frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\phi} + \frac{1}{2}\cot\frac{\theta}{2}\right], \tag{55}$$

$$\hat{H}_b = \mathrm{e}^{\mathrm{i}\phi}\left[\frac{\partial}{\partial\theta} - \frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\phi} + \frac{1}{2}\cot\frac{\theta}{2}\right]. \tag{56}$$

Note that $\hat{H}_a = -\overline{\hat{H}_b}$; this is consistent with the requirement that the overall operator H is self-adjoint with respect to an integral scalar product. (See Appendix A for full details on adjointness of operators of this nature.)

There is a continuous symmetry combining a Lorentz rotation with an isorotation under which the fermion transforms as

$$\Psi(t,\phi,\theta) \to \mathrm{e}^{\mathrm{i}\frac{\alpha}{2}\left(-\hat{\gamma}^0+\tau_3\right)}\Psi(t,\phi-\alpha,\theta). \tag{57}$$

This gives us a generator of generalised angular momentum,

$$\hat{K} = -\mathrm{i}\frac{\partial}{\partial\phi} + \frac{1}{2}\gamma^0 + \frac{1}{2}\tau_3. \tag{58}$$

$\hat{K}$ is self-adjoint and commutes with $\hat{H}$, so we may seek a joint eigenbasis of energy and generalised angular momentum. We find that such states must be of the form

$$\Psi(t,\phi,\theta) = \begin{pmatrix} \Psi_{11}(t,\phi,\theta) \\ \Psi_{12}(t,\phi,\theta) \\ \Psi_{21}(t,\phi,\theta) \\ \Psi_{22}(t,\phi,\theta) \end{pmatrix} = \mathrm{e}^{\mathrm{i}E_f t} \begin{pmatrix} \mathrm{e}^{\mathrm{i}(k-1)\phi}\Theta_{11}(\theta) \\ \mathrm{e}^{\mathrm{i}k\phi}\Theta_{12}(\theta) \\ \mathrm{e}^{\mathrm{i}k\phi}\Theta_{21}(\theta) \\ \mathrm{e}^{\mathrm{i}(k+1)\phi}\Theta_{22}(\theta) \end{pmatrix}. \tag{59}$$

The problem reduces to a system of four first-order ODEs for the latitude profile functions Θ_{ij}. Defining

$$u = \cos\theta, \tag{60}$$

we obtain

$$(E_f - gu - m_f)\Theta_{11} = g\sqrt{1-u^2}\Theta_{12} + \left(\sqrt{1-u^2}\frac{\mathrm{d}}{\mathrm{d}u} + \frac{k}{\sqrt{1-u^2}} - \frac{1}{2}\sqrt{\frac{1+u}{1-u}}\right)\Theta_{21}, \tag{61}$$

$$(E_f + gu - m_f)\Theta_{12} = g\sqrt{1-u^2}\Theta_{11} + \left(\sqrt{1-u^2}\frac{\mathrm{d}}{\mathrm{d}u} + \frac{k+1}{\sqrt{1-u^2}} - \frac{1}{2}\sqrt{\frac{1+u}{1-u}}\right)\Theta_{22}, \tag{62}$$

$$(E_f + gu + m_f)\Theta_{21} = -g\sqrt{1-u^2}\Theta_{22} + \left(-\sqrt{1-u^2}\frac{\mathrm{d}}{\mathrm{d}u} + \frac{k-1}{\sqrt{1-u^2}} + \frac{1}{2}\sqrt{\frac{1+u}{1-u}}\right)\Theta_{11}, \tag{63}$$

$$(E_f - gu + m_f)\Theta_{22} = -g\sqrt{1-u^2}\Theta_{21} + \left(-\sqrt{1-u^2}\frac{\mathrm{d}}{\mathrm{d}u} + \frac{k}{\sqrt{1-u^2}} + \frac{1}{2}\sqrt{\frac{1+u}{1-u}}\right)\Theta_{12}. \tag{64}$$

3.2 The master fourth order differential equation

[TBD: To employ Fuchsian analysis on the two subcases – the massless limit $m_f = 0$, and the decoupled limit $g = 0$ – we need to eliminate three of the $\Theta_{\alpha a}$ components to leave a single fourth order differential equation. It is straightforward to derive with the assistance of computer algebra, but it deserves to be walked through in some more detail in order to make the details easier to confirm.

The clearest way to proceed is to take it in stages. First, we eliminate two of the four components. We choose to eliminate Θ_{12} and Θ_{21}. This leaves two second order equations relating the remaining components, generally of the form

$$A_1(u)\frac{\mathrm{d}^2\Theta_{11}}{\mathrm{d}u^2} + B_1(u)\frac{\mathrm{d}\Theta_{11}}{\mathrm{d}u} + C_1(u)\Theta_{11} = D_1(u)\frac{\mathrm{d}\Theta_{22}}{\mathrm{d}u} + F_1(u)\Theta_{22}, \tag{65}$$

$$A_2(u)\frac{\mathrm{d}^2\Theta_{22}}{\mathrm{d}u^2} + B_2(u)\frac{\mathrm{d}\Theta_{22}}{\mathrm{d}u} + C_2(u)\Theta_{22} = D_2(u)\frac{\mathrm{d}\Theta_{11}}{\mathrm{d}u} + F_2(u)\Theta_{11}, \tag{66}$$

where the coefficient functions are most generally the following:

$$A_1(u) = A_2(u) = (u-1)^2(1+u)^2(gu+E+m_f)(gu+E-m_f) \tag{67}$$

$$B_1(u) = 2\,(gu+E-m_f)\left(1/2\,gu^2 + (E+m_f)\,u + g/2\right)(u-1)(1+u) \tag{68}$$

$$B_2(u) = 2\,(u-1)(1+u)(gu+E+m_f)\left(1/2\,gu^2 + (E-m_f)\,u + g/2\right) \tag{69}$$

$C_1(u)$ and $C_2(u)$ are still horrendous expressions which still need to be converted to nicely rendering LaTeX, but here is an attempt

$$
\begin{aligned}
C_1(u) = &-g^2\left(-g^2 + 1/4 - 2\,Em + E^2 + m^2\right)u^4\\
&- 2\,g\left((-E-m)\,g^2 + (k-1/2)\,g + E^3 - E^2m - Em^2 + m^3 - m/4\right)u^3\\
&- \left(g^4 + \left(-2\,E^2 - 2\,m^2 + (k-1/2)^2\right)g^2 + 3\,(k-1/2)\,(E-m/3)\,g + (E-m)\,(E+m)\left(E^2 - m^2 - 1/\right.\right.\\
&- \left((2\,E+2\,m)\,g^3 + (1/2-k)\,g^2 + \left(-2\,E^3 + 2\,E^2m + \left(2\,k^2 + 2\,m^2 - 2\,k + 1\right)E - 2\,m^3 + m/2\right)g + (E\right.\\
&(E+m)^2\,g^2 + (E-m)\,(k-1/2)\,g + (E-m)\,(E+m)\left(E^2 - k^2 - m^2 + k - 3/4\right)
\end{aligned}
\tag{70}
$$

$$
\begin{aligned}
C_2(u) = &-g^2\left(E^2 + 2\,Em - g^2 + m^2 + 1/4\right)u^4\\
&- 2\,g\left((-E+m)\,g^2 + (-k-1/2)\,g + E^3 + E^2m - Em^2 - m^3 + m/4\right)u^3\\
&- \left(g^4 + \left(-2\,E^2 - 2\,m^2 + (1/2+k)^2\right)g^2 - 3\,(E+m/3)\,(1/2+k)\,g + (E-m)\,(E+m)\left(E^2 - m^2 - 1/\right.\right.\\
&- \left((2\,E-2\,m)\,g^3 + (1/2+k)\,g^2 + \left(-2\,E^3 - 2\,E^2m + \left(2\,k^2 + 2\,m^2 + 2\,k + 1\right)E + 2\,m^3 - m/2\right)g - (E\right.\\
&- (E-m)^2\,g^2 - (1/2+k)\,(E+m)\,g + \left(E^2 - k^2 - m^2 - k - 3/4\right)(E-m)\,(E+m)
\end{aligned}
\tag{71}
$$

$$D_1(u) = 2\,(u-1)^2\,(1+u)^2\,gm_f\,(gu + E + m_f) \tag{72}$$

$$D_2(u) = -2\,(u-1)^2\,(1+u)^2\,gm_f\,(gu + E - m_f) \tag{73}$$

$$
\begin{aligned}
F_1(u) = &-(u-1)\,(1+u)\,g\left((2\,k-2\,u)\,{m_f}^2 + (((1+2\,k)\,g - E)\,u - g + (1+2\,k)\,E)\,m_f\right.\\
&\left.+ (1+u)\,(E+g)\,(gu+E)\right)
\end{aligned}
\tag{74}
$$

$$
\begin{aligned}
F_2(u) = &-(u-1)\,(1+u)\,g\left((-2\,k-2\,u)\,{m_f}^2 + (((2\,k-1)\,g + E)\,u + g + (2\,k-1)\,E)\,m_f\right.\\
&\left.+ (1+u)\,(E+g)\,(gu+E)\right)
\end{aligned}
\tag{75}
$$

Obviously these are still not particularly tractable expressions. However, we do see hints of how it will simplify when we eliminate either m_f or g. Observe:

1. The dependence of B_1 on m_f is the same as the dependence of B_2 on $-m_f$, so $B_1 = B_2$ when $m_f = 0$.

2. D_1 and D_2 have explicit overall factors of both m_f and g, so these vanish in either special case.

3. The dependence of F_1 on m_f is the same as the dependence of $-F_2$ on $-m_f$, so $F_1 = -F_2$ when $m_f = 0$.

Now, we can perform some more manipulation on equations 65 and 66 to obtain a single master fourth order differential equation. However, observing that in either special case when $gm_f = 0$, the vanishing of D_1 and D_2 makes the substitution much simpler. So it is generally convenient to go to one of the special cases before making the final reduction. However, we will also derive the full generality fourth order DE with non-zero parameters for both m_f and g, in order to explicitly demonstrate that it is *not* Fuchsian outside those special cases.

In either special case, we expect to be able to use polynomial solutions from the Fuchsian analysis to give a classification of the stationary state solutions in terms of suitable spinor(-isospinor) spherical harmonics.]

4 The massless fermion

4.1 Fuchsian analysis

Let us now set $m_f = 0$. After some algebra which is significant in amount but straightforward in principle, we may reduce the above system to a single, fourth order ODE in one of the components, say Θ_{11}. Explicitly, it is the rather unwieldy expression:

$$\begin{aligned}
0 =&(u+1)^4(u-1)^4\frac{\mathrm{d}^4\Theta_{11}}{\mathrm{d}u^4}\\
&+6(u+1)^3(u-1)^3\left(u+\frac{1}{3}\right)\frac{\mathrm{d}^3\Theta_{11}}{\mathrm{d}u^3}\\
&-2(u+1)^2(u-1)^2\left(\left[E^2-g^2-\frac{13}{4}\right]u^2-\frac{7}{2}u-E^2+g^2+k^2+\frac{11}{4}\right)\frac{\mathrm{d}^2\Theta_{11}}{\mathrm{d}u^2}\\
&-2(u+1)(u-1)^2\left(\left[E^2-g^2-\frac{1}{4}\right]u^2+\left[2E^2-2g^2-2k-\frac{1}{2}\right]u+E^2-g^2-k^2+2k-\frac{13}{4}\right)\frac{\mathrm{d}\Theta_{11}}{\mathrm{d}u}\\
&+\left\{\left(-g^2-g+\left[E-\frac{1}{2}\right]^2\right)\left(-g^2+g+\left[E+\frac{1}{2}\right]^2\right)u^4+\left(\frac{3}{4}-2k+E^2-g^2\right)u^3\right.\\
&+\left(-2g^4+\left[4E^2-2k^2+4\right]g^2+4Eg+2k^2E^2-\frac{15}{2}k^2+10k-\frac{37}{8}-2E^4\right)u^2\\
&+\left(7k^2-14k+\frac{27}{4}-E^2+g^2\right)u\\
&\left.+g^4+\left(2k^2-2E^2-\frac{5}{2}\right)g^2-2Eg+E^4+\left(-2k^2+\frac{1}{2}\right)E^2-\frac{7}{2}k^2+6k+k^4-\frac{47}{16}\right\}\Theta_{11}.
\end{aligned} \tag{76}$$

It is, at least, readily apparent that this differential equation has regular singular points at $u = 1$ and $u = -1$. We can also show that it has a regular singular point at (complex) infinity, by changing co-ordinate to $v = u^{-1}$. Upon doing so, we obtain:

$$\begin{aligned}
0 =& v^4(v+1)^4(v-1)^4\frac{\mathrm{d}^4\Theta_{11}}{\mathrm{d}v^4}\\
&+2v^3(v+1)^3(v-1)^3(6v^2+v-3)\frac{\mathrm{d}^3\Theta_{11}}{\mathrm{d}v^3}\\
&+2v^2(v+1)^2(v-1)^2\left(18v^4+6v^3+\left[E^2-g^2-k^2-\frac{83}{4}\right]v^2-\frac{5}{2}v-E^2+g^2+\frac{13}{4}\right)\frac{\mathrm{d}^2\Theta_{11}}{\mathrm{d}v^2}\\
&+2v^3(v+1)^3(v-1)^3(6v^2+v-3)\frac{\mathrm{d}\Theta_{11}}{\mathrm{d}v}\\
&+v^4(v+1)^4(v-1)^4\Theta_{11}.
\end{aligned} \tag{77}$$

Hence $v = 0$ is also a regular singular point.

We next solve the indicial equations to find the exponents at each of the singular points. At the north pole, corresponding to local behaviour $\Theta_{11}(u) = (u-1)^\mu F(u)$, the indicial equation is

$$\mu(\mu-1)(\mu-2)(\mu-3)+4\mu(\mu-1)(\mu-2)+\left(-\frac{k^2}{2}+2\right)\mu(\mu-1)-\frac{k^2}{4}+\frac{k^4}{16}=0, \tag{78}$$

with solutions

$$\mu = \pm\frac{k}{2}, 1 \pm \frac{k}{2}. \tag{79}$$

At the south pole, corresponding to local behaviour $\Theta_{11}(u) = (u+1)^{\mu} G(u)$, the indicial equation is

$$\begin{aligned}\mu(\mu-1)(\mu-2)(\mu-3) + 2\mu(\mu-1)(\mu-2) + \left(-\frac{k^2}{2} - \frac{3}{2}\right)\mu(\mu-1)& \\ + \left(-\frac{1}{2}k^2 - 2k + \frac{3}{2}\right)\mu - \frac{15}{16} + 2k - \frac{9}{8}k^2 + \frac{1}{16}k^4 = 0,&\end{aligned} \tag{80}$$

with solutions

$$\mu = \frac{3}{2} - \frac{k}{2}, \frac{k}{2} - \frac{1}{2}, \frac{5}{2} + \frac{k}{2}, -\frac{k}{2} + \frac{1}{2}. \tag{81}$$

Finally, at complex infinity, corresponding to local behaviour $\Theta_{11}(u) = u^{-\mu} H(u)$, the indicial equation is

$$\begin{aligned}\mu(\mu-1)(\mu-2)(\mu-3) + 6\mu(\mu-1)(\mu-2) + \left(\frac{13}{2} - 2E^2 + 2g^2\right)\mu(\mu-1)& \\ + \left(\frac{1}{2} - 2E^2 + 2g^2\right)\mu + E^4 - \frac{(32g^2+8)}{16}E^2 - 2Eg + g^4 - \frac{3}{2}g^2 + \frac{1}{16} = 0,&\end{aligned} \tag{82}$$

with solutions

$$\mu = \pm\frac{\sqrt{4E^2 - 4g^2 + 4E + 4g + 1}}{2}, \pm\frac{\sqrt{4E^2 - 4g^2 - 4E - 4g + 1}}{2}. \tag{83}$$

Here, $F(u), G(u)$ and $H(u)$ are all Frobenius series in u.

The Fuchs relation says that the sum of all indices is $\frac{n(n-1)(r-1)}{2}$, where n is the order of the DE and r is the number of finite singularities. For this system, $n = 4$ and $r = 2$, so the indices should sum to 6. By inspecting the values above, we see that this is true: the south pole contributes $\frac{5}{2} + \frac{3}{2} = 4$ overall, the north pole contributes $1 + 1 = 2$ overall, and the pole at infinity gives no overall contribution.

4.2 Polynomial solutions

We conjecture that, for a given integer M, the possible energy levels E_f are the four values in the set

$$\left\{\frac{1}{2} \pm \frac{\sqrt{4g^2 + 4g + (2M+1)^2}}{2}, -\frac{1}{2} \pm \frac{4g^2 - 4g + (2M+1)^2}{2}\right\}, \tag{84}$$

and that M combined the generalised angular momentum and another quantum number which controls the highest degree of u that occurs in the corresponding polynomial parts of the solution for $\Theta_{11}(u)$. In particular, $M \geq |k|$.

Explicit solutions we have already found include constant order $k = 0$ solutions,

$$\Psi^{\pm}(t, \phi, u) = \mathrm{e}^{\mathrm{i}(g\pm1)t}\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\phi}\sqrt{1+u} \\ \sqrt{1-u} \\ \mp\sqrt{1-u} \\ \pm\mathrm{e}^{\mathrm{i}\phi}\sqrt{1+u}\end{pmatrix}. \tag{85}$$

Note that the apparent divergence due to winding at the north pole $u = 1$ is a co-ordinate effect, and is eliminated by the spinorial transition function. In the north chart,

$$\tilde{\Psi}^{\pm}(t,\phi,u) = \mathrm{e}^{\mathrm{i}(g\pm 1)t} \begin{pmatrix} -\mathrm{i}\sqrt{1+u} \\ -\mathrm{i}\mathrm{e}^{\mathrm{i}\phi}\sqrt{1-u} \\ \mp\mathrm{i}\mathrm{e}^{-\mathrm{i}\phi}\sqrt{1-u} \\ \pm\mathrm{i}\sqrt{1+u} \end{pmatrix}. \tag{86}$$

We might make the rather counter-intuitive claim that since $\Psi^{\dagger}\Psi$ is uniform on S^2, the fermion is actually localised by the totally uniform soliton!

We also have four linear order solutions (i.e. $M = 1$) for $k = 0$, taking the general form in the south chart,

$$\Psi^{\pm}(t,\phi,u) = \mathrm{e}^{-\mathrm{i}E_f t} \begin{pmatrix} \mathrm{e}^{-i\phi}\sqrt{1+u}\left(1 + a_1^{\pm}(1+u)\right) \\ \sqrt{1-u}\left(a_0^{\pm} + a_1^{\pm}(1+u)\right) \\ \mp\sqrt{1-u}\left(a_0^{\pm} + a_1^{\pm}(1+u)\right) \\ \pm\mathrm{e}^{\mathrm{i}\phi}\sqrt{1+u}\left(1 + a_1^{\pm}(1+u)\right) \end{pmatrix}, \tag{87}$$

where

$$a_1^{\pm} = \pm\frac{g(E^{\pm} - g \mp 1)}{(E^{\pm} - g)(E^{\pm} - g \mp 2)}, \tag{88}$$

$$a_0^{\pm} = 1 \mp \frac{E^{\pm} - g \mp 1}{E^{\pm} - g}, \tag{89}$$

and the energies $E^{\pm}$ satisfy $E^2 \mp E - g^2 \mp g - 2 = 0$ respectively on positive and negative branches. Here, the $\pm$ signs in components Θ_{21} and Θ_{22} are a consistent binary choice, independent of the energy branch, but all other $\pm$ signs correspond to the choice of energy branch. We plot the normalised spatial density of the positive energy branch solution in Figure 1.

The next energy levels for $k = 0$ with quadratic polynomial components ($M = 2$) seem to be determined by

$$E^2 \mp E - g^2 \mp g - 6 = 0, \tag{90}$$

and if our conjecture holds, the higher $k = 0$ energy levels are precisely

$$E^2 \mp E - g^2 \mp g - \frac{n(n+1)}{2} = 0. \tag{91}$$

[TBD: add all explicit solutions we've found so far, with energy spectra and plots of fermionic density to exhibit "localisation" for zero-crossing modes.

TBD: The rest of this!]

5 The decoupled massive fermion

[TBD: Repeat the Fuchsian analysis with $g = 0$ and $m_f \neq 0$. Classify solutions by spinor spherical harmonics (remarking that the isospinor components are decoupled) and compare to Abrikosov's results. We expect some difference owing to Abrikosov's treatment being Euclidean, whereas even though we seek static solutions, the model as a whole is Lorentzian.]

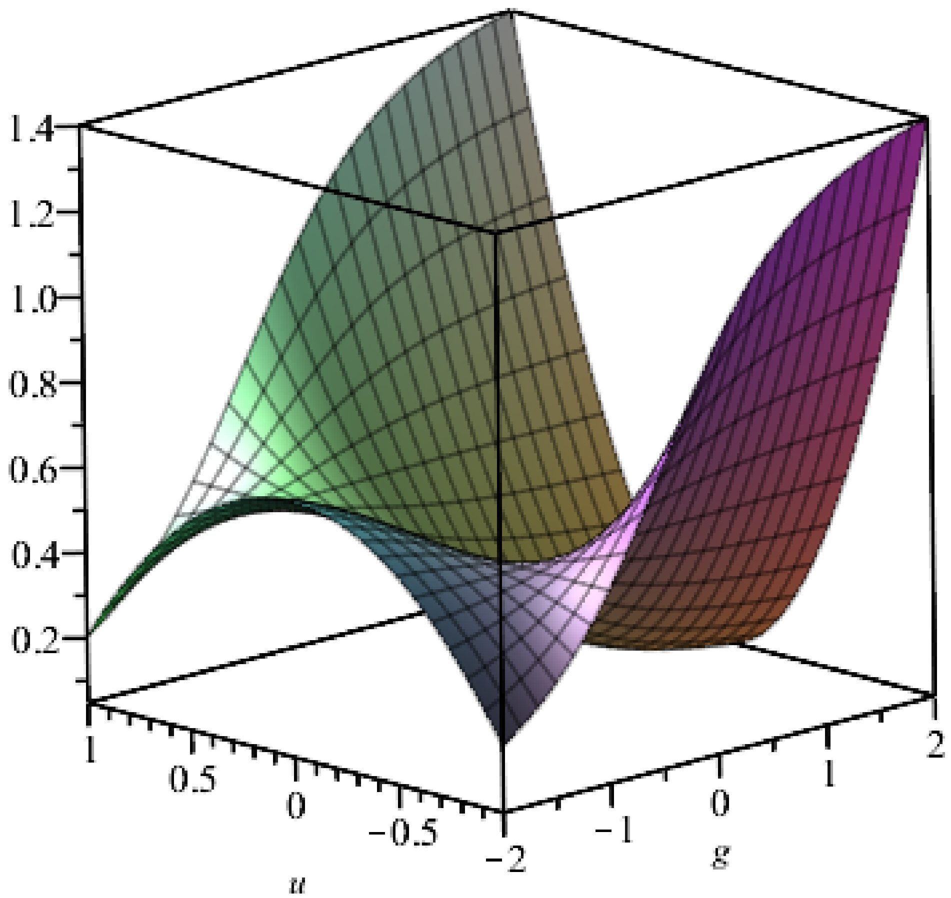

Figure 1: Normalised spatial density of the $k = 0$ linear, $\Theta_{12} = \Theta_{21}$, positive-energy branch solution

6 Conclusions and outlook

[Outstanding remarks for future investigation:

1. We want to numerically explore the non-Fuchsian case of massive coupled fermions.
2. What occurs in the limit $\rho \to \infty$?
3. Can we numerically model bringing the soliton out of the background, by introducing more terms to the Lagrangian? It is fairly obvious how to introduce a Dirichlet term weighted by an inertial mass, but we may also want to introduce the Skyrme term containing fourth-order coupling of derivatives, which itself has a free parameter, in order to match the baby Skyrme model more closely. (Similarly, we may want to introduce a potential term for the soliton.)

]

A Adjoints of differential operators for representations of fields

A.1 Fields with one component

We briefly recall how the adjoint of a linear differential operator is defined, with some common examples from quantum mechanics. A classical mechanical N-dimensional system is described by its $2N$ canonical coordinates and Poisson bracket. Quantisation of such a (bosonic) system entails finding a representation of this structure on a (usually infinite-dimensional) space of functions, whereby the canonical coordinates x_i, p_j become linear differential operators $\hat{x}_i, \hat{p}_j$, and the Poisson brackets are replaced by commutators, multiplied by a factor of $\mathrm{i}\hbar$. This yields the canonical commutation relations (attributed to Max Born, 1925):

$$\{x_i, p_j\} = \delta_{ij} \to [\hat{x}_i, \hat{p}_j] = \mathrm{i}\hbar\delta_{ij}, \tag{92}$$

where i and j run from 1 to N in all appearances. The canonical Poisson bracket for real functions of phase space $f(x_i, p_j), g(x_i, p_j)$ is

$$\{f, g\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x_i} \frac{\partial f}{\partial p_i} \right), \tag{93}$$

but it is often physically meaningful to work more generally in the complexified space of such functions; for example, in three dimensions, classical mechanical angular momentum $\mathbf{L} = \mathbf{x} \wedge \mathbf{p}$ can (after fixing an axis) be meaningfully manipulated and interpreted in terms of the raising and lowering elements $L_\pm = L_1 \pm \mathrm{i}L_2$. In the complexified space we describe, it is clear that the raising and lowering elements form a pair of complex conjugates.

The position representation, over wavefunctions $\psi = \psi(\mathbf{x})$ of position $\mathbf{x} = (x_1, \dots, x_N)$, is $\hat{x}_i = x_i$ and $\hat{p}_i = -\mathrm{i}\hbar\frac{\partial}{\partial x_i}$. The inner product on the function space is

$$\langle \psi, \varphi \rangle = \int \psi^*(\mathbf{x})\varphi(\mathbf{x})\mathrm{d}^N x, \tag{94}$$

where in this appendix we specifically use the asterisk * to mean complex conjugation. The domain of integration is the domain on which the space of functions is defined – often it is $\mathbb{R}^n$ and we admit only functions which vanish sufficiently quickly as $|\mathbf{x}| \to \infty$.

With respect to this inner product, the adjoint $A^\dagger$ of a linear operator A satisfies

$$\langle A^\dagger \psi, \varphi\rangle = \langle \psi, A\varphi\rangle. \tag{95}$$

We wish for our physically meaningful quantum operators to be self-adjoint. We may check by hand that momenta $\hat{p}_i$ is self-adjoint:

$$\begin{aligned}
\langle \psi, \hat{p}_i \varphi\rangle &= \int_{\mathbb{R}^N} \psi^*(\mathbf{x}) \left(-\mathrm{i}\hbar \frac{\partial}{\partial x_i}\right) \varphi(\mathbf{x})\, \mathrm{d}^N x \\
&= \int_{\mathbb{R}^{N-1}} \psi^*(\mathbf{x})(-\mathrm{i}\hbar)\varphi(\mathbf{x})\, \mathrm{d}^{N-1}x \Bigg|_{x_i\to-\infty}^{x_i\to+\infty} - \int_{\mathbb{R}^N} \left(\frac{\partial}{\partial x_i}\right) \psi^*(\mathbf{x})(-\mathrm{i}\hbar)\varphi(\mathbf{x})\, \mathrm{d}^N x \\
&= 0 + \int_{\mathbb{R}^N} \left(\mathrm{i}\hbar \frac{\partial}{\partial x_i}\right) \psi^*(\mathbf{x})\varphi(\mathbf{x})\, \mathrm{d}^N x \\
&= \int_{\mathbb{R}^N} \left[\left(-\mathrm{i}\hbar \frac{\partial}{\partial x_i}\right)\psi(\mathbf{x})\right]^* \varphi(\mathbf{x})\, \mathrm{d}^N x \\
&= \langle \hat{p}_i \psi, \varphi\rangle.
\end{aligned}$$

It is clear that the $\hat{x}_i$ are also self-adjoint, since they just introduce a scalar (i.e. rank-0 differential operator) factor of x_i into the integrand. Consequently, the coordinate angular momentum operators $\hat{L}_i$ are also self-adjoint - a similar integration by parts to the above will confirm as much. The raising and lowering operators $\hat{L}_\pm$ are not self-adjoint, but they do form an adjoint pair:

$$\langle \hat{L}_+\psi, \varphi\rangle = \langle \psi, \hat{L}_-\varphi\rangle \quad \text{and} \quad \langle \hat{L}_-\psi, \varphi\rangle = \langle \psi, \hat{L}_+\varphi\rangle. \tag{96}$$

It may be tempting, therefore, to think that in order to find the adjoints of the quantum *operators*, it is enough to quantise the complex conjugate of the function on phase space in the classical system. However, a closer examination reveals this not to be the case. For example, this would imply that self-adjoint operators are real: while the position operators $\hat{x}_i$ are real, it is clear that the momenta $\hat{p}_i$ are purely imaginary, despite being self-adjoint. Similarly, a computation of the expressions of the raising and lowering operators $\hat{L}_\pm$ shows that these do not form a pair of complex conjugates but suffer an additional factor of -1.

We now show this explicitly. The raising and lowering operators are expressed as follows:

$$\begin{aligned}
\hat{L}_\pm &= \hat{L}_1 \pm \mathrm{i}\hat{L}_2 \\
&= -\mathrm{i}\hbar \left[\left(x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}\right) \pm \mathrm{i}\left(x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3}\right)\right] \\
&= \hbar \left[\mathrm{i}x_3 \frac{\partial}{\partial x_2} - \mathrm{i}x_2 \frac{\partial}{\partial x_3} \pm x_3 \frac{\partial}{\partial x_1} \mp x_1 \frac{\partial}{\partial x_3}\right] \\
&= \hbar \left[\pm x_3 \frac{\partial}{\partial x_1} + \mathrm{i}x_3 \frac{\partial}{\partial x_2} + (\mp x_1 - \mathrm{i}x_2)\frac{\partial}{\partial x_3}\right]
\end{aligned} \tag{97}$$

So indeed $\hat{L}_+ = -\hat{L}_-^*$. It may again be easily checked that these operators form an adjoint pair: for example,

$$
\begin{aligned}
\langle\psi,\hat{L}_+\varphi\rangle &= \int_{\mathbb{R}^3}\psi^*\hbar\left[x_3\left(\frac{\partial}{\partial x_1}+\mathrm{i}\frac{\partial}{\partial x_2}\right)+\left(-x_1-\mathrm{i}x_2\right)\frac{\partial}{\partial x_3}\right]\varphi\,\mathrm{d}^3x \\
&= \int_{\mathbb{R}^3}\hbar\left[\psi^*x_3\frac{\partial\varphi}{\partial x_1}+\mathrm{i}\psi^*x_3\frac{\partial\varphi}{\partial x_2}-\psi^*x_1\frac{\partial\varphi}{\partial x_3}-\mathrm{i}\psi^*x_2\frac{\partial\varphi}{\partial x_3}\right]\mathrm{d}^3x \\
&= \int_{\mathbb{R}^3}\hbar\left[-\frac{\partial}{\partial x_3}\left(\psi^*x_3\right)\varphi-\mathrm{i}\frac{\partial}{\partial x_2}\left(\psi^*x_3\right)\varphi+\frac{\partial}{\partial x_3}\left(\psi^*x_1\right)\varphi+\mathrm{i}\frac{\partial}{\partial x_3}\left(\psi^*x_2\right)\varphi\right]\mathrm{d}^3x
\end{aligned}
$$

(where the boundary term has vanished)

$$
\begin{aligned}
&= \int_{\mathbb{R}^3}\hbar\left[-x_3\frac{\partial\psi^*}{\partial x_1}\varphi-\mathrm{i}x_3\frac{\partial\psi^*}{\partial x_2}\varphi+x_1\frac{\partial\psi^*}{\partial x_3}\varphi+\mathrm{i}x_2\frac{\partial\psi^*}{\partial x_3}\varphi\right]\mathrm{d}^3x \\
&= \int_{\mathbb{R}^3}\hbar\left[\left(-x_3\frac{\partial}{\partial x_1}\psi\right)^*\varphi+\left(\mathrm{i}x_3\frac{\partial}{\partial x_2}\psi\right)^*\varphi+\left(x_1\frac{\partial}{\partial x_3}\psi\right)^*\varphi+\left(-\mathrm{i}x_2\frac{\partial}{\partial x_3}\psi\right)^*\varphi\right]\mathrm{d}^3x \\
&= \int_{\mathbb{R}^3}\hbar\left[\left(-x_3\frac{\partial}{\partial x_1}+\mathrm{i}x_3\frac{\partial}{\partial x_2}+x_1\frac{\partial}{\partial x_3}-\mathrm{i}x_2\frac{\partial}{\partial x_3}\right)\psi\right]^*\varphi\,\mathrm{d}^3x \\
&= \int_{\mathbb{R}^3}\left[\hat{L}_-\psi\right]^*\varphi\,\mathrm{d}^3x \\
&= \langle\hat{L}_-\psi,\varphi\rangle.
\end{aligned}
$$

It is evident thus that the inner product and the complex structure interact in a non-trivial way, as far as taking adjoints of quantum operators is concerned. We will presently derive the general method of taking the adjoint of a quantum operator on $\mathbb{R}^3$. We will do so in spherical polar coordinates (r,θ,ϕ), as that is the coordinate system most useful in the treatment of the Skyrmion-fermion system – in fact, we reduce further to working only on the unit sphere where $r=1$. As a prologue, we repeat the previous demonstration that $\langle\psi,\hat{L}_+\varphi\rangle=\langle\hat{L}_-\psi,\varphi\rangle$, now in spherical polars. In these coordinates, the raising and lowering operators are

$$
\hat{L}_\pm = \pm\hbar\mathrm{e}^{\pm\mathrm{i}\phi}\left[\frac{\partial}{\partial\theta}\pm\mathrm{i}\cot\theta\frac{\partial}{\partial\phi}\right] \tag{98}
$$

and the inner product is

$$
\langle f,g\rangle_{\theta,\phi} = \int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}f^*g\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi. \tag{99}
$$

Note that we have not changed the inner product - the subscript just emphasises that we are now expressing it in spherical polars, to compare and contrast with the previous Cartesian expression. Now,

$$
\begin{aligned}
\langle f,\hat{L}_+g\rangle_{\theta,\phi} &= \int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}f^*\left(\hat{L}_+g\right)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}f^*\hbar\left(\mathrm{e}^{\mathrm{i}\phi}\frac{\partial g}{\partial\theta}+\mathrm{i}\mathrm{e}^{\mathrm{i}\phi}\cot\theta\frac{\partial g}{\partial\phi}\right)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}\hbar\left[f^*\mathrm{e}^{\mathrm{i}\phi}\sin\theta\frac{\partial g}{\partial\theta}+\mathrm{i}f^*\mathrm{e}^{\mathrm{i}\phi}\cos\theta\frac{\partial g}{\partial\phi}\right]\mathrm{d}\theta\,\mathrm{d}\phi
\end{aligned}
$$

Note that the boundary terms will be $\sim \sin\theta|_{\theta=0}^{\theta=\pi}$ and $\sim \mathrm{e}^{\mathrm{i}\phi}|_{\phi=0}^{\phi=2\pi}$, so they will vanish. Then, integrating by parts,

$$\begin{aligned}
\langle f, \hat{L}_+ g\rangle_{\theta,\phi} &= \int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi} \hbar\left[-\frac{\partial}{\partial\theta}\left(f^*\mathrm{e}^{\mathrm{i}\phi}\sin\theta\right)g - \mathrm{i}\frac{\partial}{\partial\phi}\left(f^*\mathrm{e}^{\mathrm{i}\phi}\cos\theta\right)g\right]\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi} \hbar\left[-\mathrm{e}^{\mathrm{i}\phi}\frac{\partial f^*}{\partial\theta}\sin\theta\, g - \mathrm{e}^{\mathrm{i}\phi}f^*\cos\theta\, g - \mathrm{i}\cos\theta\,\mathrm{e}^{\mathrm{i}\phi}\frac{\partial f^*}{\partial\phi}g - \mathrm{i}\cos\theta\, f^*(\mathrm{i}\mathrm{e}^{\mathrm{i}\phi}g)\right]\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi} \hbar\left[\left(-\mathrm{e}^{-\mathrm{i}\phi}\frac{\partial f}{\partial\theta}\right)^* g\sin\theta - \mathrm{e}^{\mathrm{i}\phi}f^*\cos\theta\, g + \left(\mathrm{i}\mathrm{e}^{-\mathrm{i}\phi}\frac{\cos\theta}{\sin\theta}\frac{\partial f}{\partial\phi}\right)^* g\sin\theta + \mathrm{e}^{\mathrm{i}\phi}f^*\cos\theta\, g\right]\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi} \left[\hbar\left(-\mathrm{e}^{-\mathrm{i}\phi}\frac{\partial}{\partial\theta} + \mathrm{i}\mathrm{e}^{-\mathrm{i}\phi}\cot\theta\frac{\partial}{\partial\phi}\right)f\right]^* g\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \langle \hat{L}_- f, g\rangle_{\theta,\phi}
\end{aligned}$$

We observe that in non-Cartesian coordinate systems, the behaviour of the Jacobian factor under integration by parts is essential to the notion of adjointness.

We now derive the general formula for the adjoint of a first-order differential operator on the unit 2-sphere, in spherical polars. Suppose we have some fields f, g in the usual spherical polars (θ, ϕ) defined on a patch of S^2 – in fact, we will allow them to be defined everywhere except possibly the north and south poles where $\theta = 0, \pi$, and require periodicity in the azimuthal coordinate, e.g. $f(\theta,\phi) = f(\theta, \phi + 2\pi)$ for all θ, ϕ. Suppose further a general operator takes the form

$$\hat{O} = \alpha(\theta,\phi)\frac{\partial}{\partial\theta} + \beta(\theta,\phi)\frac{\partial}{\partial\phi} + \gamma(\theta,\phi), \tag{100}$$

for some as-yet unspecified functions α, β, γ, and noting that we allow a contribution of a zero-order term in γ. Then, formally,

$$\begin{aligned}
\langle f, \hat{O}g\rangle &= \int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi} f^*\hat{O}g\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi} \left[f^*\sin\theta\,\alpha\frac{\partial g}{\partial\theta} + f^*\sin\theta\,\beta\frac{\partial g}{\partial\phi} + f^*\sin\theta\,\gamma g\right]\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \left[\int_{\phi=0}^{\phi=2\pi} f^*\sin\theta\,\alpha g\,\mathrm{d}\phi\right]_{\theta=0}^{\theta=\pi} + \left[\int_{\theta=0}^{\theta=\pi} f^*\sin\theta\,\beta g\right]_{\phi=0}^{\phi=2\pi} \\
&\quad + \int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}\left[-\frac{\partial}{\partial\theta}\left(f^*\sin\theta\,\alpha\right)g - \frac{\partial}{\partial\phi}\left(f^*\sin\theta\,\beta\right)g + f^*\sin\theta\,\gamma g\right]\,\mathrm{d}\theta\,\mathrm{d}\phi
\end{aligned}$$

For this to be well-defined, we can insist that the operator $\hat{O}$ obey at least the boundary conditions:

- $\alpha\sin\theta \to 0$ as $\theta \to 0, \pi$, and
- $\beta(\theta,\phi) = \beta(\theta, \phi + 2\pi)$ for all θ, π.

Certainly these are reasonable requirements both mathematically and physically. Then,

$$\begin{aligned}\langle f, \hat{O}g\rangle &= \int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}\Bigg[\left(-\alpha^*\frac{\partial f}{\partial\theta}\right)^* g\sin\theta - f^*\frac{\partial}{\partial\theta}\left(\alpha\sin\theta\right)g \\ &\qquad + \left(-\beta^*\frac{\partial f}{\partial\phi}\right)^* g\sin\theta - f^*\frac{\partial}{\partial\phi}\left(\beta\sin\theta\right)g + (\gamma f)^* g\sin\theta\Bigg]\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= \int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}\left[\left(-\alpha^*\frac{\partial}{\partial\theta} - \beta^*\frac{\partial}{\partial\phi} + \gamma^* - \frac{\partial\alpha^*}{\partial\theta} - \frac{\partial\beta^*}{\partial\phi} - \alpha^*\cot\theta\right)f\right]^* g\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\end{aligned}$$

By definition, the RHS is $\langle\hat{O}^\dagger f, g\rangle$. We have thus established that

$$\hat{O}^\dagger = -\alpha^*\frac{\partial}{\partial\theta} - \beta^*\frac{\partial}{\partial\phi} + \gamma^* - \frac{\partial\alpha^*}{\partial\theta} - \frac{\partial\beta^*}{\partial\phi} - \alpha^*\cot\theta \tag{101}$$

when $\hat{O}$ is as above, at least as long as α and β obey the boundary conditions we have insisted upon. It is easily seen that this formulation is valid for the operators $\hat{x}_i, \hat{p}_i, \hat{L}_i$ and $\hat{L}_\pm$ which we have seen so far; it is also so for the adjoint pair of operators H_a, H_b found in the fermionic Hamiltonian at (54).

We might also notice that there are a set of operators for which it is particularly easy to take their adjoints. Consider the case where α, β, γ satisfy

$$\begin{aligned}\gamma &= \frac{1}{2}\left(\frac{\partial\alpha}{\partial\theta} + \frac{\partial\beta}{\partial\phi} + \alpha\cot\theta\right) \\ \Leftrightarrow -\gamma^* &= \gamma^* - \frac{\partial\alpha^*}{\partial\theta} - \frac{\partial\beta^*}{\partial\phi} - \alpha^*\cot\theta\end{aligned} \tag{102}$$

For such an $\hat{O}$, the adjoint is simply $\hat{O}^\dagger = -\hat{O}^*$. $\hat{L}_\pm$ are of this form, as are the adjoint pair H_a, H_b appearing at (54).

A.2 Fields with multiple components

We now consider the case where the fields have multiple components, i.e. are vector-valued, so the operators for which we seek an adjoint have a matrix structure, as is the case with the Hamiltonian (54). It is not difficult with some simple linear algebra to clarify how to correctly take the adjoint of such an operator. We use the following notation in this section:

- An asterisk * on a scalar, vector or matrix indicates the element-wise complex conjugate, as in the previous section.
- A superscript ^{T} indicates the transpose of a matrix or vector.
- The Hermitian conjugate of a matrix or vector, the conjugate transpose, is indicated with a superscript ^{C}. This is to avoid confusion with the dagger † which, as in the previous section, we have reserved for the adjoint of an operator on a one-component function. Thus, for example, if A is a matrix, $A^C = (A^T)^* = (A^*)^T$.
- A double dagger ‡ indicates here the adjoint of a matrix-structured operator, for which we are seeking a general expression.

Vector functions $\mathbf{u}(\mathbf{x}), \mathbf{v}(\mathbf{x})$ have components $u_i, v_j,\ i, j = 1 \ldots N$. A matrix operator A acts component-wise on such functions as $(A\mathbf{v})_i = \sum_j A_{ij} v_j$. The components $A_{ij} = A_{ij}(\mathbf{x})$ may be composed of scalar functions or first-order differential operators.

We consider an inner product on the space of vector functions

$$\langle \mathbf{u}, \mathbf{v} \rangle = \int \mathbf{u}^C \mathbf{v} \, \mathrm{d}^N x = \int \sum_i u_i^* v_i \, \mathrm{d}^N x. \tag{103}$$

Introducing a matrix operator A,

$$\langle \mathbf{u}, A\mathbf{v} \rangle = \int \mathbf{u}^C A\mathbf{v} \, \mathrm{d}^N x = \int \sum_{i,j} u_i^* A_{ij} v_j \, \mathrm{d}^N x \tag{104}$$

In the case that the components A_{ij} are just scalars, then we see the Hermitian conjugate plays the usual role as the adjoint:

$$\begin{aligned} \int \sum_{i,j} u_i^* A_{ij} v_j \, \mathrm{d}^N x &= \int \sum_{i,j} \left(A_{ji}^* u_i\right)^{*T} v_j \, \mathrm{d}^N x \\ \Rightarrow \langle \mathbf{u}, A\mathbf{v} \rangle &= \langle A^C \mathbf{u}, \mathbf{v} \rangle. \end{aligned}$$

When the components A_{ij} are now allowed to be differential operators, we seek an adjoint operator $A^\ddagger$ which by definition satisfies

$$\langle \mathbf{u}, A\mathbf{v} \rangle = \langle A^\ddagger \mathbf{u}, \mathbf{v} \rangle. \tag{105}$$

Let us call that operator B. Then,

$$\begin{aligned} \langle \mathbf{u}, A\mathbf{v} \rangle &= \langle B\mathbf{u}, \mathbf{v} \rangle \\ \Rightarrow \int \mathbf{u}^C \left(A\mathbf{v}\right) \, \mathrm{d}^N x &= \int \left(B\mathbf{u}\right)^C \mathbf{v} \, \mathrm{d}^N x \\ \Rightarrow \int \sum_i u_i^* (A\mathbf{v})_i \, \mathrm{d}^N x &= \int \sum_k (B\mathbf{u})_k^* v_k \, \mathrm{d}^N x \\ \Rightarrow \int \sum_{i,j} u_i^* \left(A_{ij}[v_j]\right) \, \mathrm{d}^N x &= \int \sum_{k,l} \left(B_{kl}[u_l]\right)^* v_k \, \mathrm{d}^N x \end{aligned}$$

In fact by e.g. considering test functions $\mathbf{u}, \mathbf{v}$ with all but one component 0, we see that this must hold in each component independently, so

$$\langle u_i, A_{ij} v_j \rangle = \langle B_{ji} u_i, v_j \rangle \tag{106}$$

$$\Rightarrow B_{ij} = A_{ji}^\dagger, \tag{107}$$

where we emphasise the dagger † indicates the one-component operator adjoint derived in the previous section. Thus, the matrix operator adjoint is given by

$$A^\ddagger = (A^\dagger)^T. \tag{108}$$

We note that any and all necessary complex conjugation is contained in the element-wise adjoint operation †, and so this does indeed reduce to the usual Hermitian conjugate ^{C} when all the components A_{ij} are scalar. Thus armed, for example, it is finally clear that the Hamiltonian (54) is a self-adjoint operator in the correct sense.

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