AN INTRODUCTION TO GENETIC ALGORITHMS FOR NUMERICAL OPTIMIZATION

Solutions for Exercises of Section 1

Exercise 1:

The simplex is remaining stuck on the rings of secondary maxima surrounding the central peak; the corresponding functions values can be found by plonking the values of r listed in eq. (1.2) of the tutorial into eqs. (1.1). Those values of r were obtained by differentiating (1.1) with respect to r, and solving the resulting nonlinear root finding problem using the bisection method, specifically the routine `rtbis` of Press *et al.*'s *Numerical Recipes* (§9.1 in 1992 edition).

I hope you enjoyed this one, because it is the easiest Exercise of the whole tutorial.

Exercise 2:

From eqs. (1.1) and (1.2) in the Tutorial, you can easily calculate that the height of the first ring of secondary maxima is

$$f(x,y) = 0.9216185;$$

we are this looking for the value of the radius r^* such that

$$0.9216185 = \cos^2(9\pi r^*)\exp(-(r^*)^2/0.15), \tag{1.1}$$

Which defines a little nonlinear rootfinding problem, for which the bisector method works well, because it's easy to set strict *a priori* bounds on r^*, for example $0 \leq r^* \leq r_{\min}(m=1)$, where $r_{\min}(m=1) = (18)^{-1}$ is the radius of the first ring of minima (see eq. [1.3] in tutorial). In this way you find

$$r^* = 9.5486 \times 10^{-3} \tag{1.2}$$

The corresponding surface area is

$$S = \pi(r^*)^2 = 2.8644 \times 10^{-4}$$

Now, the simplex method is *guaranteed* to end up on the global, central maximum if one of the starting vertices falls within S; the probability of this happening is simply S since the search spave has unit surface area; the probability of this *not* happening is $1-S$; so the probability of *none* of the the initial 3 vertices to land within S is $(1-S)^3$ (P1 is a 2-D problem, so the simplex is defined by three vertices); Therefore, the probability P of *any* one of three randomly positioned vertices to be within S is

$$P = 1 - (1-S)^3 = 8.591 \times 10^{-4} \tag{1.3}$$

This is significantly smaller than the probability $p_G = 0.0213$ inferred from running the simplex method (see Table 1 in tutorial), and represents evidence for the exploratory capabilities of the downhill simplex method. However, repeating the same analysis with $r^* = 0.110192$ (the radius of the first ring of minima) yields $P = 0.11$, *larger* than p_G; this indicates that sometimes, an initial vertex located on the slopes of the central peak (though lower than 0.921618) gets pulled away by the moving simplex. If running iterated simplex, based on eq. (1.3) one would predict that $P^{-1} = 1164$ trials would, on average, be enough to guarantee global convergence with high confidence.

Exercise 3:

As in Exercise 2, the first thing we need to do is compute the fractional area occuppied by the portion of the central peak having $f(w, x, y, z) \geq 0.921618$ in the 4-D landscape described by P3. This is clearly independent of dimensionality, so that eq. (1.2) from in Exercise 2 still holds: $r^* = 9.5386 \times 10^{-3}$. The next thing we need is the volume of a 4-D hypersphere of radius r^*. The general expression for the volume of a N-dimensional hypersphere of radius R is[1]:

$$V_N(R) = \frac{\pi^{N/2}}{(N/2)!} R^N, \qquad N = 1, 2, 3, ... \tag{1.4}$$

which gives $V = (\pi^2/2)R^4$ for $N = 4$. In case you ever need to compute the volume of a sphere in a space of odd-integer dimension (like our good old 3-D space), half-integer factorials are defined using the recursive properties of the gamma function $\Gamma(n)$ for non-integer n, which leads to

$$(N/2)! = \left(\frac{N}{2}\right)\left(\frac{N}{2} - 1\right) ... \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}, \qquad N = 1, 3, 5, ... \tag{1.5}$$

Anyway, the fractional area covered by the central peak for P3 is then $V = 4.08 \times 10^{-8}$. The probability of one of the five initial simplex vertices to land within V is computed in a manner entirely analogous to that described in Exercise 2:

$$P = 1 - (1 - V)^5 = 2.98 \times 10^{-7}$$

which is so close to zero that my pocket calculator has a hard time evaluating it. This is again smaller than the probability p_G listed in Table 1 in the tutorial, once again pointing to the exploratory capabilities of downhill simplex.

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[1] This expression can be found in R.K. Pathria's *Statistical Mechanics*, Appendix D (1972, Pergamon Press); my thanks to Tom Bogdan for the pointer.