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MAT 2377
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Assignment #2

Q.1) (4 points) Let X be a discrete random variable. The following table shows its possible values x and the associated probabilities $P(X=x)=f(x)$.

x	-1	0	1	3
$f(x)$	$1/8$	$2/8$	$3/8$	$2/8$

(a) (1 point) Verify that $f(x)$ is a probability mass function.

(b) (1 point) Calculate $P(X<1)$, $P(X\leq 1)$ and $P(X<0.5 \text{ or } X>2)$

(c) (1 point) Find the cumulative distribution function of X.

(d) (1 point) Compute the mean and variance of X

S.1.a) $f(x)$ is a probability mass function if (i) $0\leq f(x)\leq 1$, (ii) $\sum_x f(x)=1$, and (iii) $P(X=x)=f(x)$.

(i) Because $P(X=x)=f(x)$, then $0\leq f(x)\leq 1 \Rightarrow 0\leq P(x)\leq 1$. Thus, since $0\leq P(-1)\leq 1 \Rightarrow 0\leq 1/8\leq 1$, $0\leq P(0)\leq 1 \Rightarrow 0\leq 2/8\leq 1$, $0\leq P(1)\leq 1\Rightarrow 0\leq 3/8\leq 1$, and $0\leq P(3)\leq 1 \Rightarrow 0\leq 2/8\leq 1$.

(ii) $\sum_x f(x) = f(-1)+f(0)+f(1)+f(3) = 1/8 + 2/8 + 3/8 + 2/8 = 8/8 = 1$

$\therefore f(x)$ is a probability mass function

b.) $P(X<1) = P(-1)+P(0) = 1/8 + 2/8 = 3/8$

$P(X\leq 1) = P(-1)+P(0)+P(1) = 1/8 + 2/8 + 3/8$

$P(X<0.5 \text{ or } X>2) = P(-1)+P(0)+P(3) = 1/8 + 2/8 + 3/8 = 6/8$

c.)

$$F(x) = \begin{cases} 0, & x<-1 \\ 1/8, & -1\leq x<0 \\ 3/8, & 0\leq x<1 \\ 6/8, & 1\leq x<3 \\ 1, & x\geq 3 \end{cases}$$

d.) Mean $=\mu = E[X] = \sum_{x\in R_X} x f(x) = (-1)(1/8)+(0)(2/8)+(1)(3/8)+3(2/8) = 1$

Variance $=\sigma^2 = V[X] = E[X^2]-\mu^2 = \left(\sum_{x\in R_X} x^2 f(x)\right) - \mu^2$, so $\sum_{x\in R_X} x^2 f(x) = (-1)^2(1/8)+(0)^2(2/8)+(1)^2(3/8)$

$+ (3)^2(2/8) = 11/4 = 2.75$, thus $V(X) = E[X^2] - \mu^2 = \left(\sum_{x \in R_X} x^2 f(x)\right) - \mu^2 = 2.75 - (1)^2 = 1.75$

Q.2.) (6 points) The operating time X (in hours) of a computer before its first failure is a continuous random variable whose probability density function is given by

$$f(x) = \begin{cases} ce^{-\frac{x}{125}}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(a) (1 point) Find the value of c.

(b) (1 point) Find the probability that the computer will work for at least 125 hours before its first failure

(c) (1 point) Find the probability that the operating time until the first failure is between 50 and 150 hours

(d) (1 point) Find the cumulative distribution $F(x)$ of the random variable X

(e) (2 points) Find the mean and variance of the random variable X

S.2.a) A probability density function (pdf) for a continuous random variable X, defined over the set of real numbers, has the following three properties: (i) $f(x) \geq 0$, for all $x \in \mathbb{R}$, (ii) $\int_{-\infty}^{\infty} f(x)dx = 1$, and (iii) $P(a < X < b) = \int_a^b f(x)dx$. Thus, we can find the value of c using (ii):

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_0^{\infty} f(x)dx \text{ (Additive Integral Property)}$$

$$1 = 0 + \int_0^{\infty} ce^{-\frac{x}{125}}dx$$

$$1 = c\int_0^{\infty} e^{-\frac{x}{125}}dx$$

$$1 = c\left[-125e^{-\frac{x}{125}}\right]_0^{\infty}$$

$$1 = -125c\left(e^{\frac{-\infty}{125}} - e^{\frac{-0}{125}}\right)$$

$$1 = -125c\left(\frac{1}{e^{\frac{\infty}{125}}} - \frac{1}{e^{\frac{0}{125}}}\right)$$

$$1 = -125c(0 - 1)$$

$$1 = 125c$$

$$\frac{1}{125} = c$$

$\therefore c = \frac{1}{125}$

b.) Using the property that states $P(a < X < b) = \int_a^b f(x)dx$ we note that $P(X > 125) = \frac{1}{125}\int_{125}^{\infty} e^{-\frac{x}{125}}dx = \frac{1}{125}\int_{125}^{\infty} e^{-\frac{x}{125}}dx = \frac{1}{125}\left[-125e^{-\frac{x}{125}}\right]_{125}^{\infty} = -1\left(e^{-\frac{\infty}{125}} - e^{-\frac{125}{125}}\right) = -1\left(0 - \frac{1}{e}\right) = \frac{1}{e} \cong 0.368$

c) $P(50<x<150) = \frac{1}{125}\int_{50}^{150} e^{-\frac{x}{125}}dx = \frac{1}{125}\left[-125e^{-\frac{x}{125}}\right]_{50}^{150} = -1\left(e^{-\frac{150}{125}} - e^{-\frac{50}{125}}\right) \cong 0.369$

d) $F(x) = \int_0^x f(y)dy = \int_0^x \frac{1}{125}e^{-\frac{y}{125}}dy = \frac{1}{125}\left[-125e^{-\frac{y}{125}}\right]_0^x = -1\left[e^{-\frac{x}{125}} - e^{-\frac{0}{125}}\right] = -e^{-\frac{x}{125}} - 1$

$$\therefore F(x) = \begin{cases} 0, & x<0 \\ -e^{-\frac{x}{125}} - 1, & x \geq 0 \end{cases}$$

e) Mean $= \mu = E[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_0^{\infty} 125xe^{-\frac{x}{125}}dx \Rightarrow u=x,\ dv = e^{-\frac{x}{125}}dx,\ du=dx,\ v=-125e^{-\frac{x}{125}}$
$\Rightarrow uv - \int vdu = -125xe^{-\frac{x}{125}} - \int_0^{\infty} -125e^{-\frac{x}{125}}dx = -125xe^{-\frac{x}{125}}\Big]_0^{\infty} + 125\int_0^{\infty} e^{-\frac{x}{125}}dx = [-125xe^{-\frac{\infty}{125}}$
$+ 125xe^{-}$

Q.3) (5 points) Let X and Y denote the lengths of life, in years, of two components in an electronic system. If the joint density function of these variables is

$$f(x,y) = \begin{cases} e^{-(x+y)}, & x>0,\ y>0 \\ 0, & \text{otherwise} \end{cases}$$

(a) (2.5 points) Find the marginal distribution of Y

(b) (2.5 points) Find $P(0<X<1 \mid Y=2)$

S.3.a) $h(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^1 e^{-(x+y)}dx = \int_0^1 e^{-x}/e^y dx = \frac{1}{e^y}\int_0^1 \frac{1}{e^x}dx = \frac{1}{e^y}\left[-e^{-x}\right]_0^1 = \frac{1}{e^y}\left[-\frac{1}{e} -\right.$
$\left.(-1)\right] = \frac{1}{e^y} - \frac{1}{e^{y+1}}$

b) $P(0<x<1 \mid Y=2) = \frac{f(x,y)}{h(y)}$