2016 E1 Pr1

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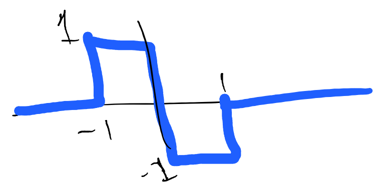
f(t) est périodique, donc sa transformée de Fourier est

$$f_p(t) \Leftrightarrow 2\pi \sum_{n=-\infty}^{+\infty} F_{Série}(n)\delta(\omega - n\omega_0)$$

Pour trouver les coefficients $F(n)$, nous utilisons le méthode de restriction.

$$f_r(t) = \begin{cases} f_p(t) & -1<t<1 \\ 0 & \text{ailleurs} \end{cases}$$

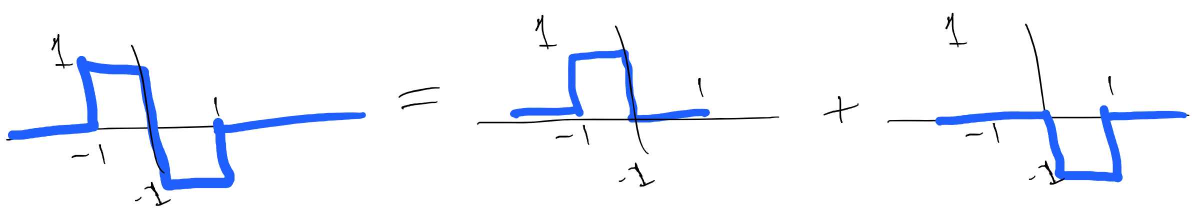


$$F(n) = \left.\frac{F_r(\omega)}{T_0}\right|_{n\omega_0}$$

$T_0 = 5 \qquad \omega_0 = 2\pi/5 \qquad F(n) = \frac{1}{5} F_r\left(\frac{2\pi}{5}\cdot n\right)$

Il y a plusieurs méthodes pour trouver $F_r(\omega)$; notons $F_r(0) = 0$.

Methode A



$\mathrm{Rect}(t+\frac{1}{2}) \qquad - \mathrm{Rect}(t-\frac{1}{2})$

$$f_r(t) = \mathrm{Rect}(t+\tfrac{1}{2}) - \mathrm{Rect}(t-\tfrac{1}{2})$$

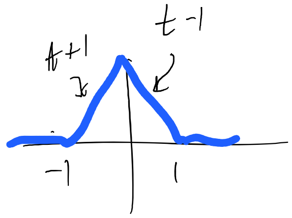
$$\text{Rect}\,(t+1/2) \Leftrightarrow e^{j\omega/2}\cdot Sa\,(\omega/2) \qquad \text{Rect}\,(t-1/2) \Leftrightarrow e^{-j\omega/2}\, Sa\,\omega/2$$

$$F_r(\omega) = e^{j\omega/2}\, Sa\,\frac{\omega}{2} - e^{-j\omega/2}\, Sa\,\frac{\omega}{2} = \left(e^{j\omega/2} - e^{-j\omega/2}\right) Sa\,\omega/2$$

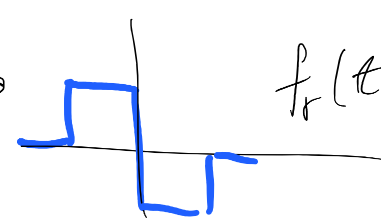
$$= 2j\sin\frac{\omega}{2}\, Sa\,\frac{\omega}{2} = 2j\,\frac{\sin^2\omega/2}{\omega/2} = 4j\,\frac{\sin^2\omega/2}{\omega}$$

Methode B

$\frac{d}{dt}$ Trit:



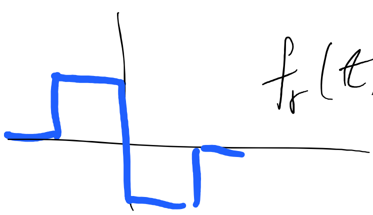
$\xrightarrow{\frac{d}{dt}}$



$f_r(t)$!

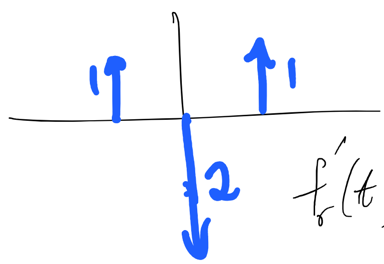
$$\frac{d}{dt}\,\text{Tri}\,t \Leftrightarrow j\omega\, Sa^2\frac{\omega}{2} = j\omega\,\frac{\sin^2\omega/2}{(\omega/2)^2} = j4\,\frac{\sin^2\frac{\omega}{2}}{\omega} \quad \checkmark$$

Méthode C



$f_r(t)$

$\xRightarrow{\frac{d}{dt}}$



$$f_r'(t) = \delta(t-1) + \delta(t+1) - 2\delta(t)$$

$$j\omega F(\omega) = e^{j\omega} + e^{-j\omega} - 2$$

$$= 2\cos\omega - 2 = 2(\cos\omega - 1) = -4\sin^2\omega/2$$

notons $2\cos^2\theta = 1+\cos 2\theta \Rightarrow 2(1-\sin^2\theta) = 1+\cos 2\theta;\ 1-\cos 2\theta = 2\sin^2\theta$

$$F_r(\omega) = -4\sin^2\omega/2 = 4j\sin^2\omega/2 \quad \checkmark \qquad F_r(0) = 0$$

$$F_r(\omega) = \frac{-4\sin^2 \omega/2}{j\omega} = 4j\frac{\sin^2 \omega/2}{\omega} \checkmark \qquad F_r(0) = 0$$

Méthode D

$$f_r(t) = \begin{cases} +1 & -1<t<0 \\ -1 & 0<t<1 \\ 0 & \text{ailleurs} \end{cases} \qquad F_r(\omega) = 0$$

$$F_r(\omega) = \int_{-1}^{0} e^{-j\omega t}\,dt + \int_{0}^{1} (-1)e^{-j\omega t}\,dt$$

$$= \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-1}^{0} - \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{0}^{1}$$

$$= \frac{1-e^{+j\omega}}{-j\omega} - \frac{e^{-j\omega}-1}{-j\omega}$$

$$= \frac{1}{j\omega}\left[e^{-j\omega}-1-1+e^{+j\omega}\right] = \frac{1}{j\omega}\left[e^{j\omega}+e^{-j\omega}-2\right]$$

$$= \frac{2\cos\omega - 2}{j\omega} = \frac{2j}{\omega}\left[1-\cos\omega\right] = \frac{4j}{\omega}\sin^2\frac{\omega}{2}$$

$$F(n) = \left.\frac{F_r(\omega)}{T_0}\right|_{n\omega_0} = \frac{2\cos n\omega_0 - 2}{jn\omega_0 T_0} = \frac{2}{2\pi jn}(\cos n\omega_0 - 1)$$

...

$$= \frac{j}{n\pi}\left(1 - \cos n\omega_0\right)$$

$T_0 = 5 \quad \omega_0 = 2\pi/5 \qquad \omega_0 T_0 = 2\pi$

$$F(n) = \frac{2j}{n\pi} \sin^2 \frac{2\pi}{5\cdot 2} = \frac{2j}{n\pi} \sin^2 \frac{\pi}{5} n \qquad F(0) = 0$$

$$F(\omega) = 2\pi \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{2j}{n\pi} \sin^2 \frac{n\pi}{5} \cdot \delta(\omega - n\pi/5)$$

$$= 4j \sum_{\substack{n=-\infty \\ n\neq 0}} \frac{\sin^2 \frac{\pi}{5} n}{n} \delta(\omega - n\pi/5)$$

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ou directement ...

$$F(n) = \frac{1}{T_0}\int_{-1}^{0} 1 \cdot e^{-jn\omega_0 t}\, dt - \frac{1}{T_0}\int_0^1 e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T_0}\left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\Big|_{-1}^{0} - \frac{e^{-jn\omega_0 t}}{-jn\omega_0}\Big|_0^1 \right]$$

$$= \frac{1}{T_0 jn\omega_0}\left[e^{jn\omega_0} - 1 - \left(1 - e^{-jn\omega_0}\right)\right]$$

$$= \frac{1}{2\pi nj}\left[e^{jn\omega_0} + e^{-jn\omega_0} - 2\right]$$

$$= \frac{j}{n\pi}\left[1 - \cos n\omega_0\right] = \frac{2j}{\pi n}\sin^2\frac{n\omega_0}{2}$$

2016 E1 Pr2

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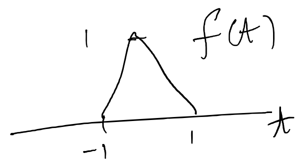
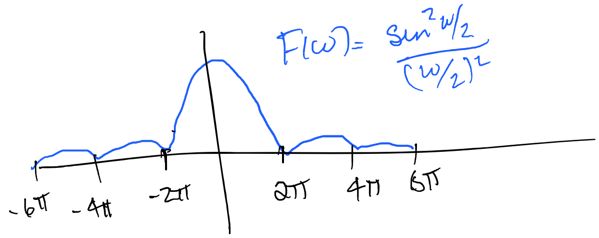
$$f(t) = \text{Tri}(t)$$



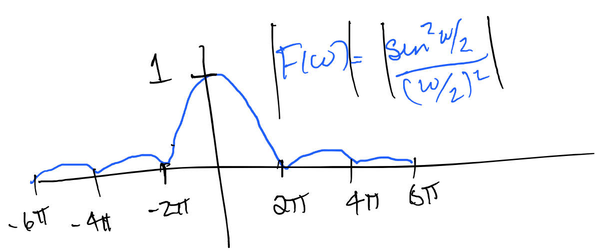
$$F(\omega) = Sa^2\frac{\omega}{2}$$

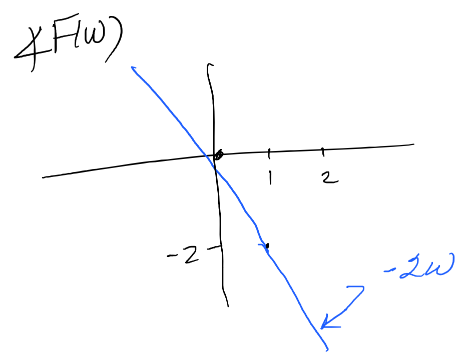


$$f(t-2) \Leftrightarrow e^{-j2\omega} F(\omega)$$

$$\text{Tri}(t-2) \Leftrightarrow e^{-2j\omega} Sa^2\frac{\omega}{2}$$

amplitude inchangé
phase maintenant linéaire





$$\sin(50t)\, f(t)$$

$$= \frac{1}{2j}\left(e^{j50t} - e^{-j50t}\right) f(t)$$

$$= \frac{1}{2j} e^{j50t} f(t) - \frac{1}{2j} e^{-j50t} f(t)$$

$$e^{j50t} f(t) \Leftrightarrow F(\omega - 50)$$

$$e^{j50t} f(t) \Leftrightarrow F(\omega-50)$$

$$e^{-j50t} f(t) \Leftrightarrow F(\omega+50)$$

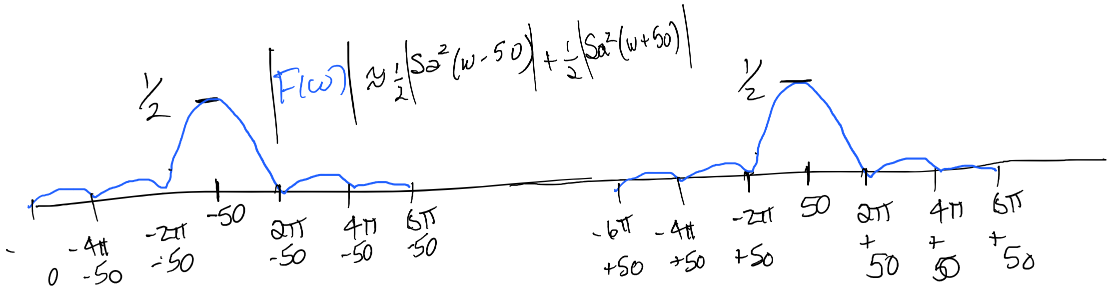
$$\sin 50t \ Tri(t) \Leftrightarrow \frac{1}{2j} Sa^2\left(\frac{\omega-50}{2}\right) - \frac{1}{2j} Sa^2\left(\frac{\omega+50}{2}\right)$$

$$\Leftrightarrow \frac{j}{2}\left[Sa^2\left(\frac{\omega+50}{2}\right) - Sa^2\left(\frac{\omega-50}{2}\right)\right]$$

Notons pour $\omega << 0$ $\quad Sa^2\left(\frac{\omega-50}{2}\right) >> Sa^2\left(\frac{\omega+50}{2}\right)$

$\omega >> 0 \quad Sa^2\left(\frac{\omega+50}{2}\right) >> Sa^2\left(\frac{\omega-50}{2}\right)$

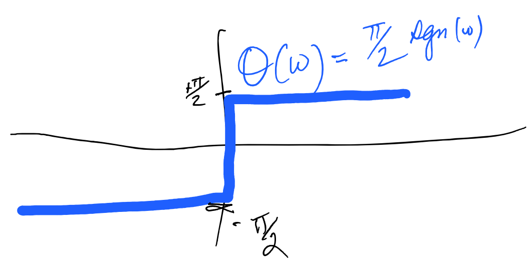
$$\frac{1}{2}\left| Sa^2\left(\frac{\omega-50}{2}\right) - Sa^2\left(\frac{\omega+50}{2}\right)\right| \approx \frac{1}{2}\left| Sa^2\left(\frac{\omega-50}{2}\right)\right| + \frac{1}{2}\left| Sa^2 \frac{\omega+50}{2}\right|$$



$F(\omega)$ est imaginaire pur, donc la phase est $+\frac{\pi}{2}$ ou $-\frac{\pi}{2}$. $\quad Im F(\omega) < 0 \Rightarrow \angle F(\omega) = -\frac{\pi}{2}$, $\quad Im F(\omega) > 0 \Rightarrow \angle F(\omega) = \frac{\pi}{2}$

pour $\omega > 0 \quad Sa^2\left(\frac{\omega-50}{2}\right) > Sa^2\left(\frac{\omega+50}{2}\right) \Rightarrow Im F(\omega) < 0 \Rightarrow \angle = -\frac{\pi}{2}$

pour $\omega < 0 \quad Sa^2\left(\frac{\omega-50}{2}\right) < Sa^2\left(\frac{\omega+50}{2}\right) \Rightarrow Im F(\omega) > 0 \Rightarrow \angle = +\frac{\pi}{2}$



$$\frac{d}{dt} f(t) \Longleftrightarrow (j\omega) F(\omega)$$

$$\frac{d}{dt} Tri(t) \Longleftrightarrow j\omega \cdot Sa^2\left(\frac{\omega}{2}\right) = j\omega \frac{\sin^2 \omega/2}{(\omega/2)^2} = \frac{\sin^2\left(\frac{\omega}{2}\right)}{\omega} \cdot j4$$

Notons que nous avons le même résultat que le problème un. La fonction dans le problème un est la dérivé du triangle.

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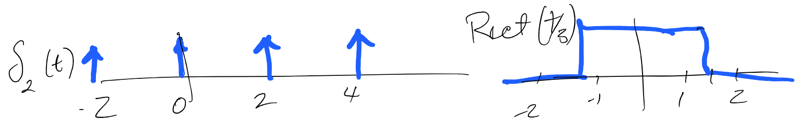
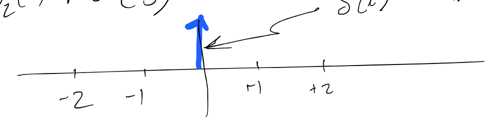
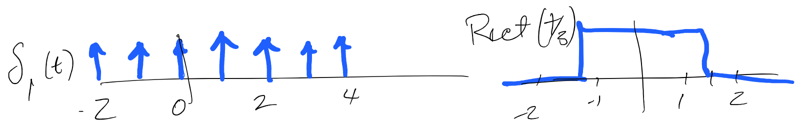


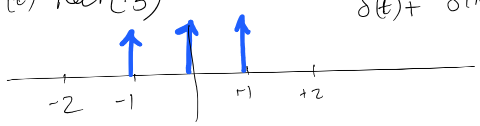
$\delta_2(t)\cdot \mathrm{Rect}(t/3)$

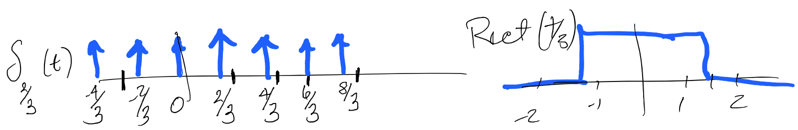




$\delta_1(t)\cdot \mathrm{Rect}(t/3)$

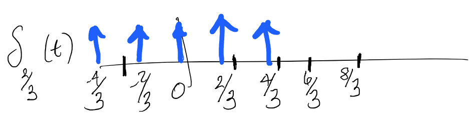
$\delta(t) + \delta(t-1) + \delta(t+1)$





$\delta_{2/3}(t)\cdot \mathrm{Rect}(t/3)$

$\delta(t) + \delta(t+2/3) + \delta(t-2/3)$
$+ \delta(t+4/3) + \delta(t-4/3)$



$$TF\{\delta_2(t)\mathrm{Rect}\,t/3\} = TF\{\delta(t)\} = 1$$

$$TF\{\delta_1(t)\mathrm{Rect}\,t/3\} = TF\{\delta(t) + \delta(t-1) + \delta(t+1)\}$$
$$= 1 + e^{jw} + e^{-jw}$$
$$= 1 + 2\cos w$$

$$TF\left\{\delta_{\frac{2}{3}}(t)\,\mathrm{Rect}\,\tfrac{t}{3}\right\} = TF\left\{\delta(t) + \delta(t-\tfrac{2}{3}) + \delta(t+\tfrac{2}{3}) + \delta(t-\tfrac{4}{3}) + \delta(t+\tfrac{4}{3})\right\}$$

$$= 1 + e^{j\frac{2}{3}\omega} + e^{-j\frac{2}{3}\omega} + e^{j\frac{4}{3}\omega} + e^{-j\frac{4}{3}\omega}$$

$$= 1 + 2\cos\tfrac{2}{3}\omega + 2\cos\tfrac{4}{3}\omega$$

2016 E2 Pr4

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$f(t) = 1/t^2$

Par dualité

$$\operatorname{sgn} t \Leftrightarrow \frac{2}{j\omega}$$

donc

$$\frac{2}{jt} \Leftrightarrow 2\pi \operatorname{sgn}(-\omega)$$

$$\frac{1}{t} \Leftrightarrow \frac{j}{2} \cdot 2\pi \operatorname{sgn}(-\omega) = j\pi \operatorname{sgn}(-\omega) = -j\pi \operatorname{sgn} \omega$$

En exploitant

$$\frac{d}{dt} f(t) \Leftrightarrow j\omega F(\omega)$$

nous avons

$$\frac{d}{dt} \frac{1}{t} \Leftrightarrow j\omega \cdot (-j\pi) \operatorname{sgn} \omega$$

$$-1/t^2 \Leftrightarrow \omega\pi \operatorname{sgn} \omega$$

$$1/t^2 \Leftrightarrow -\omega\pi \operatorname{sgn} \omega$$