HYPERBOLIC FUNCTION

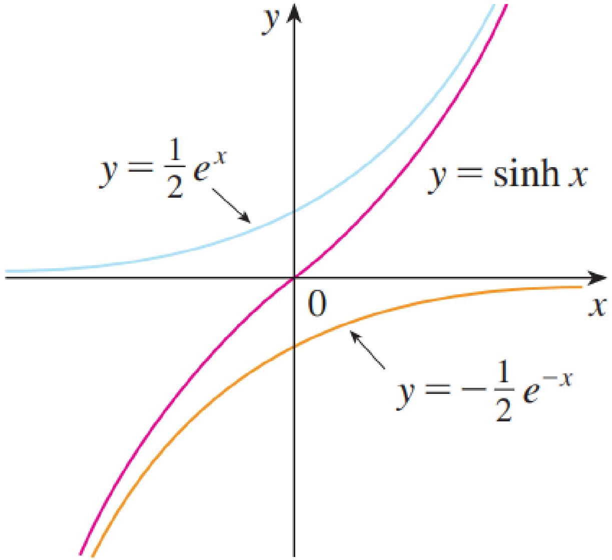
DEFINITION OF THE HYPERBOLIC FUNCTIONS

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

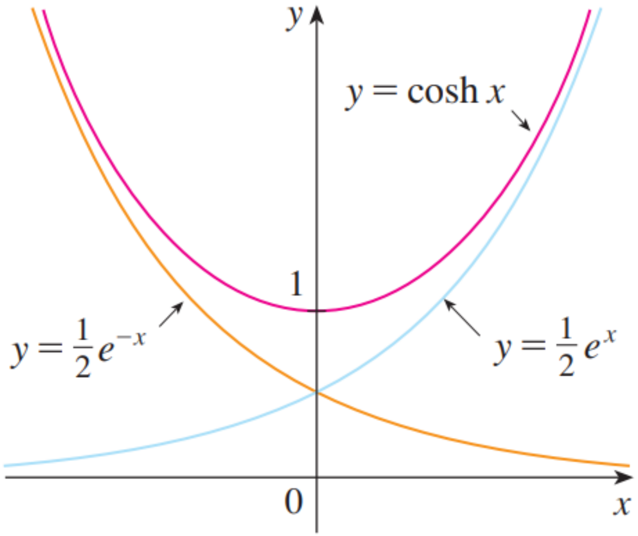
$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

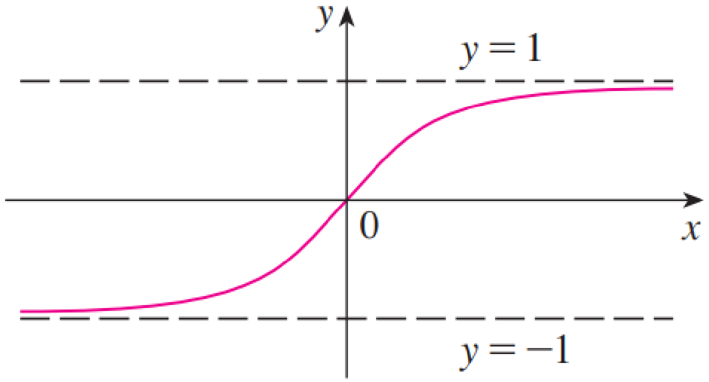
The graphs of hyperbolic sine and cosine can be sketched using graphical addition



$y = \sinh x = \frac{1}{2}e^{x} - \frac{1}{2}e^{-x}$



$y = \cosh x = \frac{1}{2}e^{x} + \frac{1}{2}e^{-x}$



$y = \tanh x$

Note that sinh has domain $\mathbb{R}$ and range $\mathbb{R}$, while cosh has domain $\mathbb{R}$ and range $[1, \infty)$. The function tanh has the horizontal asymptotes $y = \pm 1$.

The hyperbolic functions satisfy a number of identities that are similar to well-known trigonometric identities.

HYPERBOLIC IDENTITIES

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

EXAMPLE Prove (a) $\cosh^2 x - \sinh^2 x = 1$ and (b) $1 - \tanh^2 x = \operatorname{sech}^2 x$.

SOLUTION

(a) $$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{4}{4} = 1$$

(b) We start with the identity proved in part (a): $\cosh^2 x - \sinh^2 x = 1$

If we divide both sides by $\cosh^2 x$, we get: $$1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

or $$1 - \tanh^2 x = \operatorname{sech}^2 x$$

[1] DERIVATIVES OF HYPERBOLIC FUNCTIONS

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

INVERSE HYPERBOLIC FUNCTIONS

You can see from the graph from before that sinh and tanh are one-to-one functions and so they have inverse functions denoted by $\sinh^{-1}$ and $\tanh^{-1}$. The graph of cosh shows that is not one-to-one, but when restricted to the domain $[0, \infty)$ it becomes one-to-one. The inverse hyperbolic cosine function is defined as the inverse of this restricted function.

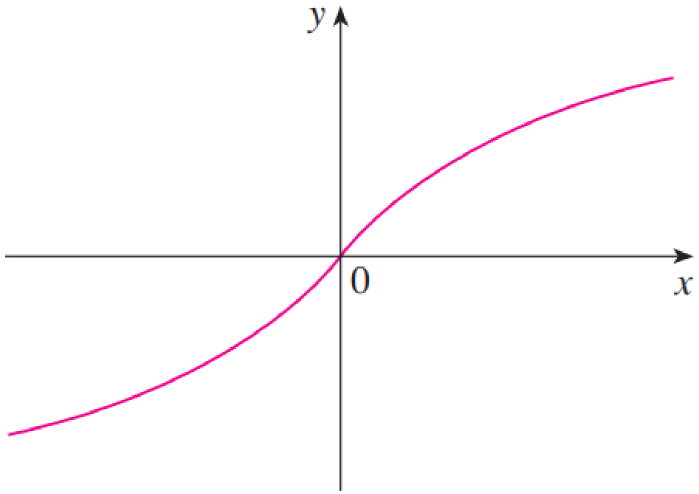
[2]

$$y = \sinh^{-1}x \iff \sinh y = x$$

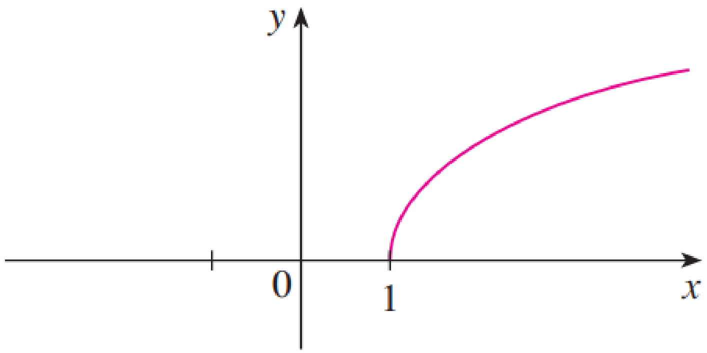
$$y = \cosh^{-1}x \iff \cosh y = x \quad \text{and} \quad y \geqslant 0$$

$$y = \tanh^{-1}x \iff \tanh y = x$$

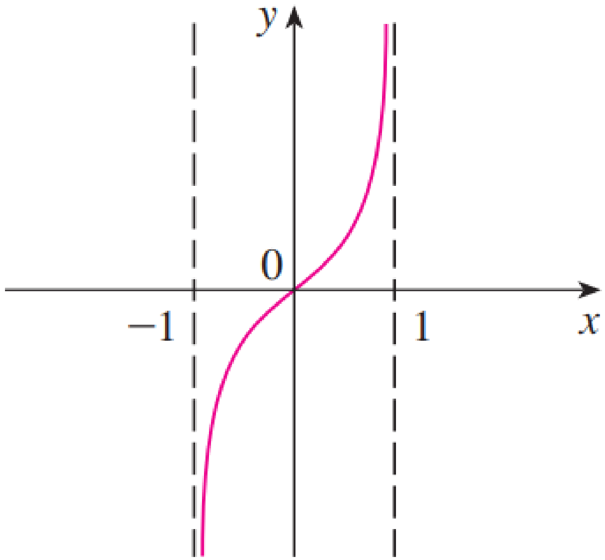
We can sketch the graphs of $\sinh^{-1}$, $\cosh^{-1}$, and $\tanh^{-1}$ by using the graph of sinh, cosh, and tanh.



$y = \sinh^{-1} x$



$y = \cosh^{-1} x$



$y = \tanh^{-1} x$

domain $= \mathbb{R}$ range $= \mathbb{R}$ domain $= [1, \infty)$ range $= [0, \infty)$ domain $= (-1, 1)$ range $= \mathbb{R}$

Since the hyperbolic functions are defined in terms of exponential functions, it can also be expressed in terms of logarithms. In particular, we have:

[3] $$\sinh^{-1}x = \ln\left(x + \sqrt{x^2 + 1}\right) \qquad x \in \mathbb{R}$$

[4] $$\cosh^{-1}x = \ln\left(x + \sqrt{x^2 - 1}\right) \qquad x \geqslant 1$$

[5] $$\tanh^{-1}x = \tfrac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right) \qquad -1 < x < 1$$

EXAMPLE Show that $\sinh^{-1}x = \ln\left(x + \sqrt{x^2 + 1}\right)$.

SOLUTION Let $y = \sinh^{-1}x$. Then

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

So

$$e^y - 2x - e^{-y} = 0$$

or, multiplying by e^y,

$$e^{2y} - 2xe^y - 1 = 0$$

This is really a quadratic equation in e^y:

$$(e^y)^2 - 2x(e^y) - 1 = 0$$

Solving by the quadratic formula, we get

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}$$

Note that $e^y > 0$, but $x - \sqrt{x^2 + 1} < 0$ (because $x < \sqrt{x^2 + 1}$).
Thus the minus sign is inadmissible and we have

$$e^y = x + \sqrt{x^2 + 1}$$

Therefore $$y = \ln(e^y) = \ln\left(x + \sqrt{x^2 + 1}\right)$$

[6] DERIVATIVES OF INVERSE HYPERBOLIC FUNCTIONS

$$\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1 + x^2}} \qquad \frac{d}{dx}(\operatorname{csch}^{-1}x) = -\frac{1}{|x|\sqrt{x^2 + 1}}$$

$$\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2 - 1}} \qquad \frac{d}{dx}(\operatorname{sech}^{-1}x) = -\frac{1}{x\sqrt{1 - x^2}}$$

$$\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1 - x^2} \qquad \frac{d}{dx}(\coth^{-1}x) = \frac{1}{1 - x^2}$$

Notice that the formulas for the derivatives of $\tanh^{-1} x$ and $\coth^{-1} x$ appear to be identical. But the domains of these functions have no numbers in common: $\tanh^{-1} x$ is defined for $|x| < 1,$ whereas $\coth^{-1} x$ is defined for $|x| > 1.$

EXAMPLE Prove that $\dfrac{d}{dx}(\sinh^{-1}x) = \dfrac{1}{\sqrt{1 + x^2}}$.

SOLUTION Let y = $\sinh^{-1} x$. Then sinh *y* = *x*. If we differentiate this equation implicitly with respect to *x*, we get

$$\cosh y \frac{dy}{dx} = 1$$

Since $\cosh^2 y - \sinh^2 y = 1$ and $\cosh y \geqslant 0$,
we have $\cosh y = \sqrt{1 + \sinh^2 y}$, so

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}$$

EXAMPLE Find $\dfrac{d}{dx}[\tanh^{-1}(\sin x)]$.

SOLUTION Using [6] and the Chain Rule, we have

$$\frac{d}{dx}[\tanh^{-1}(\sin x)] = \frac{1}{1 - (\sin x)^2}\frac{d}{dx}(\sin x)$$

$$= \frac{1}{1 - \sin^2 x}\cos x = \frac{\cos x}{\cos^2 x} = \sec x$$