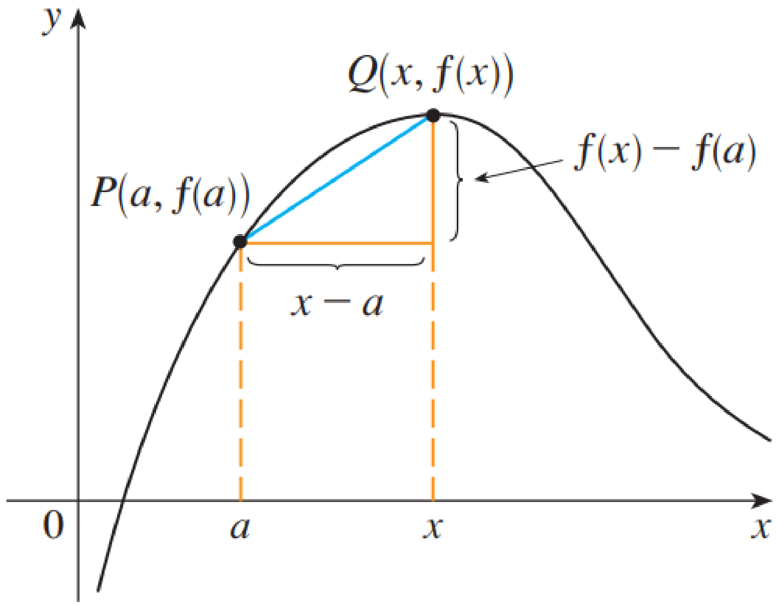
# DERIVATIVES

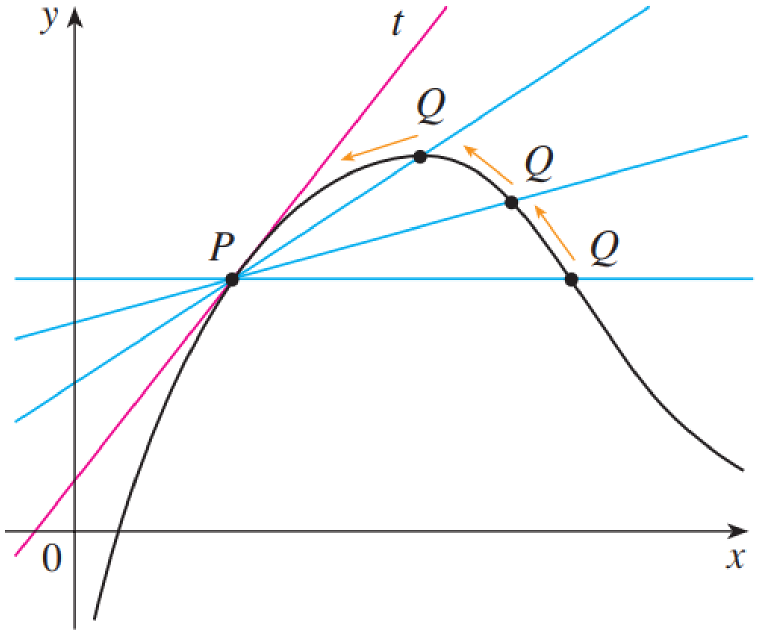
TANGENTS

If a curve *C* has equation *y = f(x)* and we want to find the tangent line *C* to at the point *P(a, f(a)),* then we consider a nearby point *Q(x, f(x)),* where $x \neq a,$ and compute the slope of the secant line *PQ* :

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Then we let *Q* approach *P* along the curve *C* by letting *x* approach *a*. If $M_{PQ}$ approaches a number *m*, then we define the *tangent t* to be the line through *P* with slope *m.*




[1] DEFINITION

The **tangent line** to the curve *y = f(x)* at the point *P(a, f(a))* is the line through *P* with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

EXAMPLE

Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$ .

SOLUTION

Here we have $a = 1$ and $f(x) = x^2$, so the slope is

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1}$$

$$= \lim_{x \to 1} (x + 1) = 1 + 1 = 2$$

Using the point-slope form of the equation of a line, we find that an equation of the tangent line at (1, 1) is

$$y - 1 = 2(x - 1) \qquad \text{or} \qquad y = 2x - 1$$

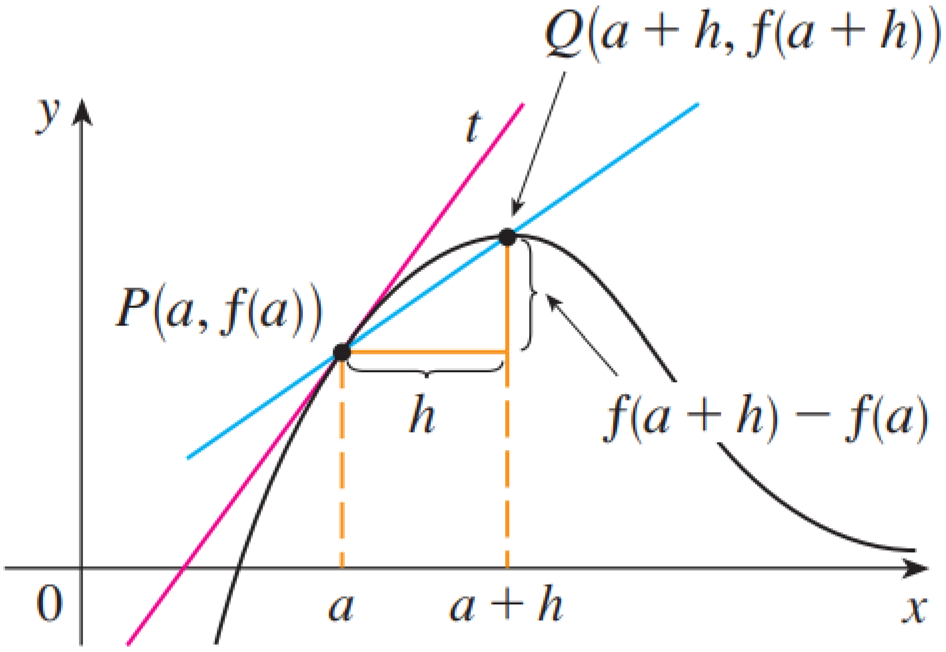
We sometimes refer to the slope of the tangent line to a curve at a point as the **slope of the curve** at the point.

There is another expression for the slope of a tangent line that is sometimes easier to use. If $h = x - a$, then $x = a + h$ and so the slope of the secant line $PQ$ is

$$m_{PQ} = \frac{f(a + h) - f(a)}{h}$$

Notice that as $x$ approaches $a$, $h$ approaches 0 (because $h = x - a$ ) and so the expression for the slope of the tangent line in Definition 1 becomes

[2]
$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



EXAMPLE

Find an equation of the tangent line to the hyperbola $y = 3/x$ at the point (3,1).

SOLUTION

Let $f(x) = 3/x$. Then the slope of the tangent at (3, 1) is

$$m = \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} = \lim_{h \to 0} \frac{\dfrac{3}{3 + h} - 1}{h} = \lim_{h \to 0} \frac{\dfrac{3 - (3 + h)}{3 + h}}{h}$$

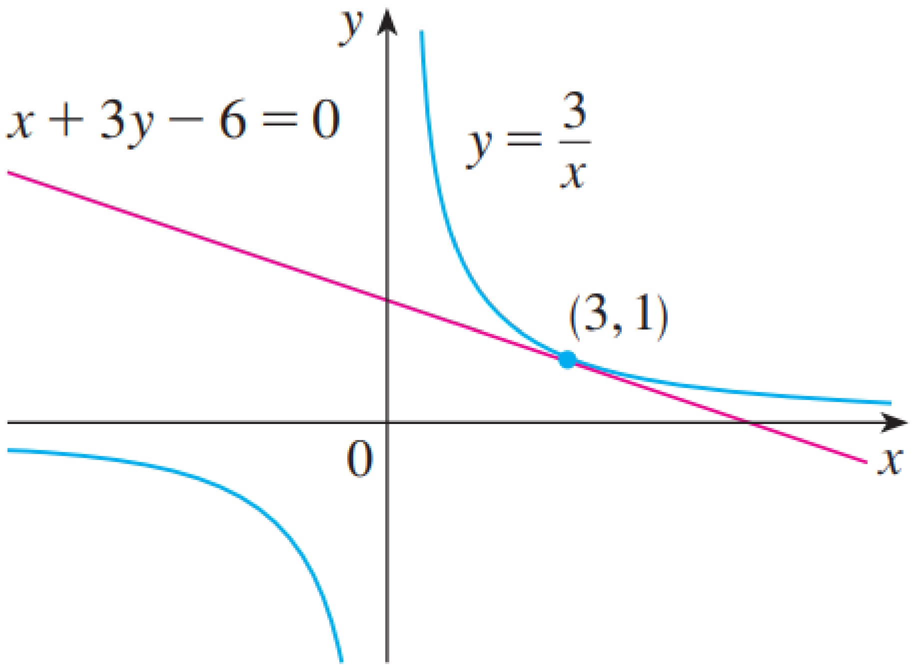
$$= \lim_{h \to 0} \frac{-h}{h(3 + h)} = \lim_{h \to 0} -\frac{1}{3 + h} = -\frac{1}{3}$$

Therefore an equation of the tangent at the point is

$$y - 1 = -\tfrac{1}{3}(x - 3)$$

which simplifies to

$$x + 3y - 6 = 0$$



DERIVATIVES

[4] DEFINITION

The **derivative of a function *f* at a number *a***, denoted by *f'(a), is*

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists

EXAMPLE

Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number *a*.

SOLUTION

From Definition 4 we have

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{[(a + h)^2 - 8(a + h) + 9] - [a^2 - 8a + 9]}{h}$$

$$= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h}$$

$$= \lim_{h \to 0} \frac{2ah + h^2 - 8h}{h} = \lim_{h \to 0} (2a + h - 8)$$

$$= 2a - 8$$

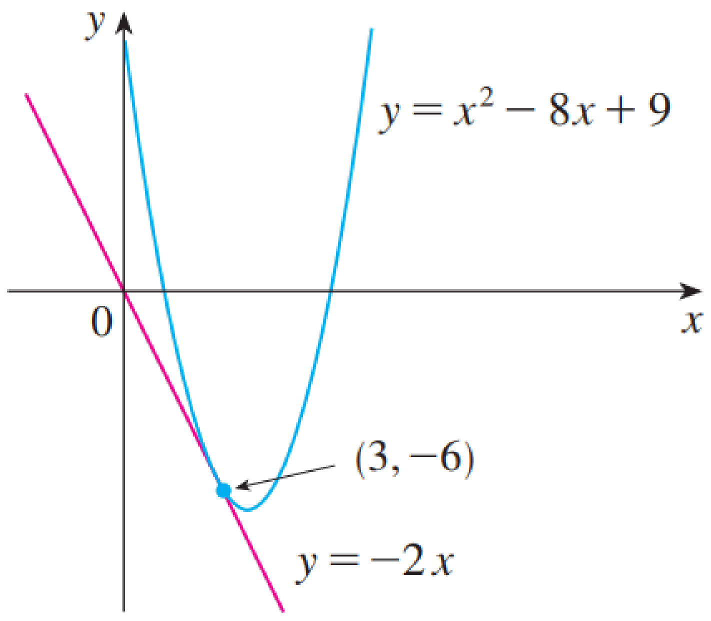
EXAMPLE

Find an equation of the tangent line to the parabola at $y = x^2 - 8x + 9$ the point (3, -6).

SOLUTION

We know that the derivative of $f(x) = x^2 - 8x + 9$ at the number a is $f'(a) = 2a - 8$. Therefore the slope of the tangent line at (3, -6) is $f'(3) = 2(3) - 8 = -2$. Thus an equation of the tangent line is

$$y - (-6) = (-2)(x - 3) \qquad \text{or} \qquad y = -2x$$



EXAMPLE

(a) If $f(x) = x^3 - x$, find a formula for $f'(x)$.
(b) Illustrate by comparing the graphs of $f$ and $f'$.

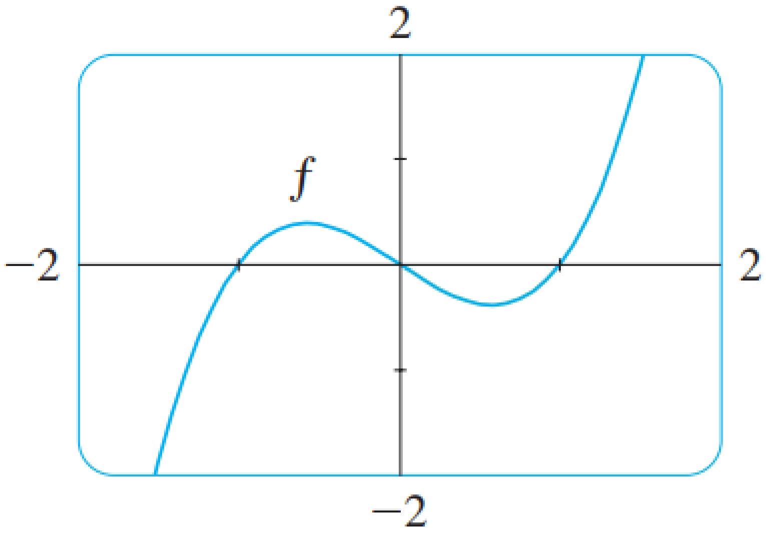
SOLUTION

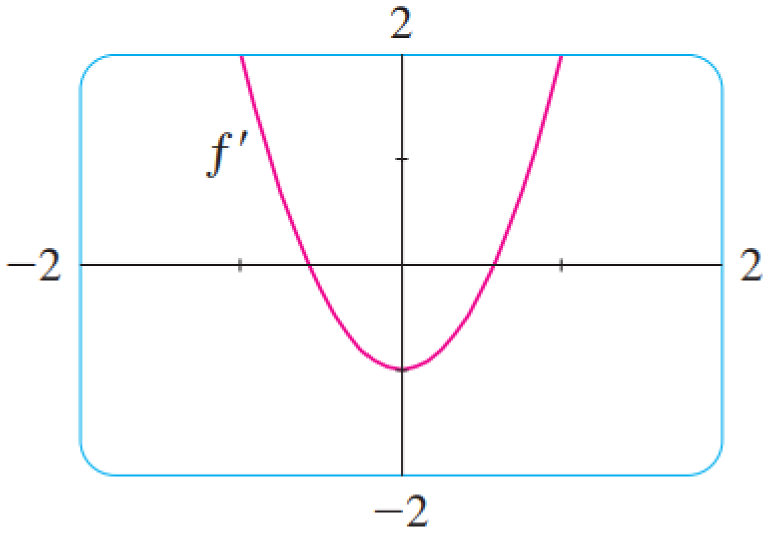
(a) $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x - h - x^3 + x}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} = \lim_{h \to 0} (3x^2 + 3xh + h^2 - 1) = 3x^2 - 1$$

(b) Notice that $f'(x) = 0$ when $f$ has horizontal tangents and $f'(x)$ is positive when the tangents have positive slope.




EXAMPLE

If $f(x) = \sqrt{x}$, find the derivative of $f$. State the domain of $f'$.

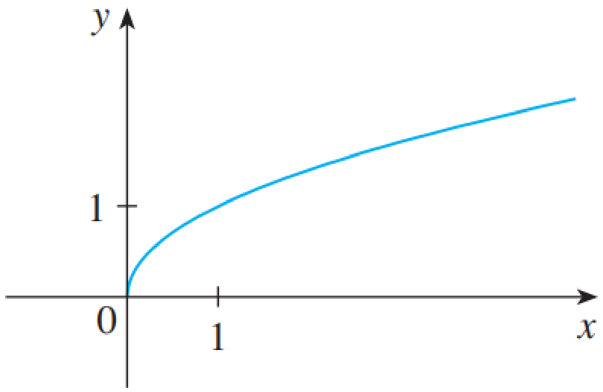
SOLUTION

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

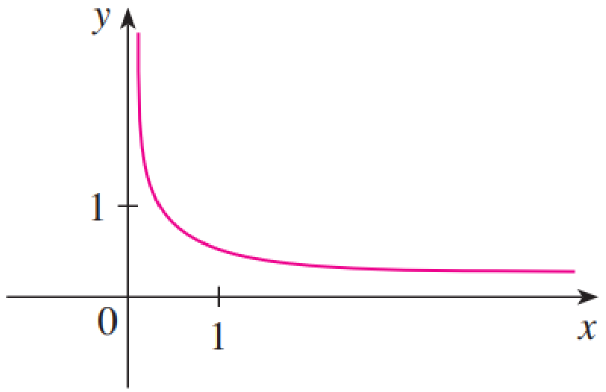
$$= \lim_{h \to 0} \left( \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right)$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$




(a) $f(x) = \sqrt{x}$




We see that *f'(x)* exists if *x > 0*, so the domain of *f'* is $(0, \infty)$. This is smaller than the domain of *f*, which is $[0, \infty)$.

[3] DEFINITION

A function *f* is **differentiable at a** if *f'(a)* exists. It is **differentiable on an open interval** $(a, b)$ [or $(a, \infty)$ or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

EXAMPLE

Where is the function $f(x) = |x|$ differentiable?

SOLUTION

If $x > 0$, then $|x| = x$ and we can choose *h* small enough that $x + h > 0$ and hence $|x + h| = x + h$. Therefore, for $x > 0$, we have

$$f'(x) = \lim_{h\to 0} \frac{|x+h| - |x|}{h}$$

$$= \lim_{h\to 0} \frac{(x+h) - x}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1$$

and so *f* is differentiable for any *x > 0*.

Similarly, for x < 0 we have $|x| = -x$ and *h* can be chosen small enough that *x + h < 0* and so $|x+h| = -(x+h)$. Therefore, for x < 0,

$$f'(x) = \lim_{h\to 0} \frac{|x+h| - |x|}{h}$$

$$= \lim_{h\to 0} \frac{-(x+h) - (-x)}{h} = \lim_{h\to 0} \frac{-h}{h} = \lim_{h\to 0} (-1) = -1$$

and so f is differentiable for any *x* < *0*.

For *x* = *0* we have

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{|0+h| - |0|}{h} \qquad \text{(if it exists)}$$

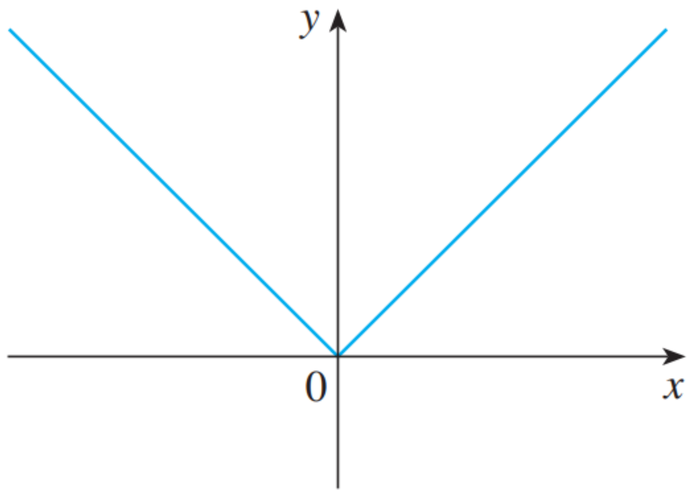
Computing the left and right limits separately:

$$\lim_{h\to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1$$

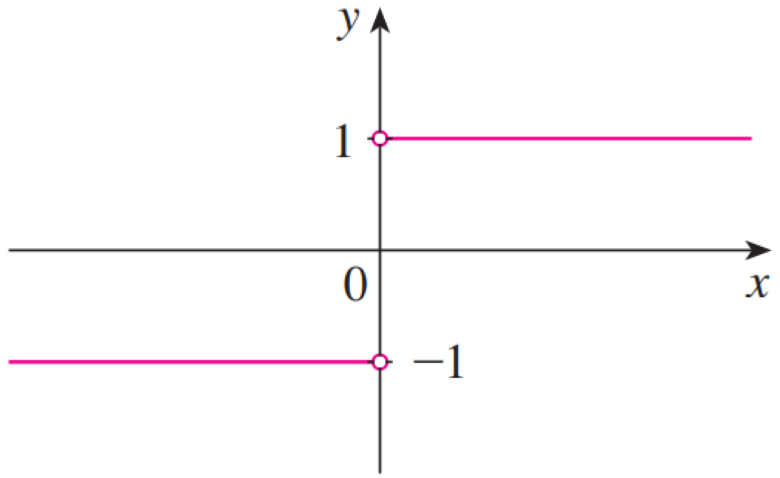
$$\lim_{h\to 0^-} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} (-1) = -1$$

Since these limits are different, *f'(0)* does not exist. Thus *f* is differentiable at all *x* except 0.

A formula for *f'* is given by $f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$

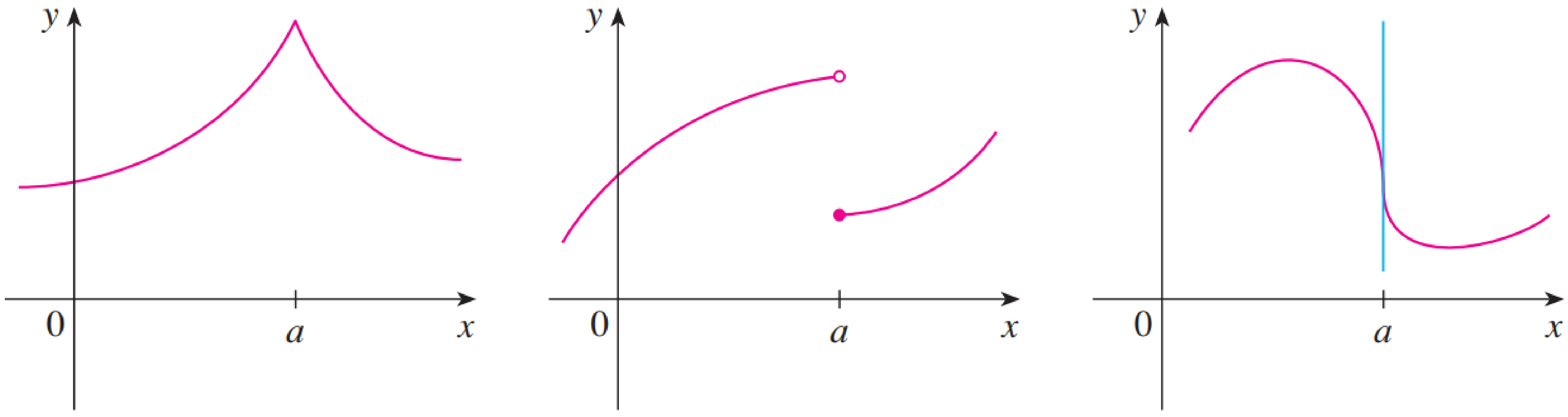



(a) $y = f(x) = |x|$




(b) $y = f'(x)$

# Three ways for *f* not to be differentiable at a




(a) A corner (b) A discontinuity (c) A vertical tangent