THE CHAIN RULE

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If *g* is differentiable at *x* and *f* is differentiable at *g(x)*, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at *x* and *F'* is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if *y = f(u)* and *u = g(x)* are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

EXAMPLE

Find $F'(x)$ if $F(x) = \sqrt{x^2 + 1}$.

SOLUTION

If we let $u = x^2 + 1$ and $y = \sqrt{u}$, then

$$F'(x) = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2\sqrt{u}}(2x)$$

$$= \frac{1}{2\sqrt{x^2 + 1}}(2x) = \frac{x}{\sqrt{x^2 + 1}}$$

Note: In using the Chain Rule we work from the outside to the inside.

EXAMPLE

Differentiate (a) $y = \sin(x^2)$ and (b) $y = \sin^2 x$.

SOLUTION

(a) If $y = \sin(x^2)$, then the outer function is the sine function and the inner function is the squaring function, so the Chain Rule gives

$$\frac{dy}{dx} = \frac{d}{dx} \underbrace{\sin}_{\substack{\text{outer}\\ \text{function}}} \underbrace{(x^2)}_{\substack{\text{evaluated}\\ \text{at inner}\\ \text{function}}} = \underbrace{\cos}_{\substack{\text{derivative}\\ \text{of outer}\\ \text{function}}} \underbrace{(x^2)}_{\substack{\text{evaluated}\\ \text{at inner}\\ \text{function}}} \cdot \underbrace{2x}_{\substack{\text{derivative}\\ \text{of inner}\\ \text{function}}}$$

$$= 2x\cos(x^2)$$

(b) Note that $\sin^2 x = (\sin x)^2$. Here the outer function is the squaring function and the inner function is the sine function. So

$$\frac{dy}{dx} = \frac{d}{dx} \underbrace{(\sin x)}_{\substack{\text{inner} \\ \text{function}}}{}^2 = \underbrace{2}_{\substack{\text{derivative} \\ \text{of outer} \\ \text{function}}} \cdot \underbrace{(\sin x)}_{\substack{\text{evaluated} \\ \text{at inner} \\ \text{function}}} \cdot \underbrace{\cos x}_{\substack{\text{derivative} \\ \text{of inner} \\ \text{function}}}$$

The answer can be left as $2 \sin x \cos x$ or written as $\sin 2x$ (by a trigonometric identity known as the double-angle formula).

THE POWER RULE COMBINED WITH THE CHAIN RULE

If is any real number and is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Alternatively $$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

EXAMPLE

Differentiate $y = (x^3 - 1)^{100}$.

SOLUTION

Taking $u = g(x) = x^3 - 1$ and $n = 100$, we have

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - 1)^{100} = 100(x^3 - 1)^{99}\frac{d}{dx}(x^3 - 1)$$

$$= 100(x^3 - 1)^{99} \cdot 3x^2 = 300x^2(x^3 - 1)^{99}$$

EXAMPLE

Find $f'(x)$ if $f(x) = \dfrac{1}{\sqrt[3]{x^2 + x + 1}}$.

SOLUTION

First rewrite : $f(x) = (x^2 + x + 1)^{-1/3}$

Thus

$$f'(x) = -\tfrac{1}{3}(x^2 + x + 1)^{-4/3} \frac{d}{dx}(x^2 + x + 1)$$

$$= -\tfrac{1}{3}(x^2 + x + 1)^{-4/3}(2x + 1)$$

EXAMPLE

Find the derivative of the function $g(t) = \left(\frac{t-2}{2t+1}\right)^9$

SOLUTION

Combining the Power Rule, Chain Rule, and Quotient Rule, we get

$$g'(t) = 9\left(\frac{t-2}{2t+1}\right)^8 \frac{d}{dt}\left(\frac{t-2}{2t+1}\right)$$

$$= 9\left(\frac{t-2}{2t+1}\right)^8 \frac{(2t+1)\cdot 1 - 2(t-2)}{(2t+1)^2} = \frac{45(t-2)^8}{(2t+1)^{10}}$$

EXAMPLE

Differentiate $y = (2x + 1)^5(x^3 - x + 1)^4$.

SOLUTION

In this example we must use the Product Rule before using the Chain Rule:

$$\frac{dy}{dx} = (2x + 1)^5 \frac{d}{dx}(x^3 - x + 1)^4 + (x^3 - x + 1)^4 \frac{d}{dx}(2x + 1)^5$$

$$= (2x + 1)^5 \cdot 4(x^3 - x + 1)^3 \frac{d}{dx}(x^3 - x + 1)$$

$$+ (x^3 - x + 1)^4 \cdot 5(2x + 1)^4 \frac{d}{dx}(2x + 1)$$

$$= 4(2x + 1)^5(x^3 - x + 1)^3(3x^2 - 1) + 5(x^3 - x + 1)^4(2x + 1)^4 \cdot 2$$

Noticing that each term has the common factor $2(2x + 1)^4 (x^3 – x + 1)^3$, we could factor it out and write the answer as

$$\frac{dy}{dx} = 2(2x + 1)^4(x^3 - x + 1)^3(17x^3 + 6x^2 - 9x + 3)$$

EXAMPLE

Differentiate $y = e^{\sin x}$.

SOLUTION

Here the inner function is $g(x) = \sin x$ and the outer function is the exponential function $f(x) = e^x$. So, by the Chain Rule

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\sin x}\right) = e^{\sin x}\frac{d}{dx}(\sin x) = e^{\sin x}\cos x$$

We can use the Chain Rule to differentiate an exponential function with any base $a > 0$. Recall that $a = e^{\ln a}$. So

$$a^x = (e^{\ln a})^x = e^{(\ln a)x}$$

and the Chain Rule gives

$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{(\ln a)x}) = e^{(\ln a)x}\frac{d}{dx}(\ln a)x$$

$$= e^{(\ln a)x} \cdot \ln a = a^x \ln a$$

because ln a is a constant. So we have the formula

$$\frac{d}{dx}(a^x) = a^x \ln a$$

EXAMPLE

If $f(x) = \sin(\cos(\tan x))$, then

$$f'(x) = \cos(\cos(\tan x)) \frac{d}{dx} \cos(\tan x)$$

$$= \cos(\cos(\tan x)) [-\sin(\tan x)] \frac{d}{dx} (\tan x)$$

$$= -\cos(\cos(\tan x)) \sin(\tan x) \sec^2 x$$

EXAMPLE

Differentiate $y = e^{\sec 3\theta}$.

SOLUTION

The outer function is the exponential function, the middle function is the secant function and the inner function is the tripling function. So we have

$$\frac{dy}{d\theta} = e^{\sec 3\theta} \frac{d}{d\theta}(\sec 3\theta)$$

$$= e^{\sec 3\theta} \sec 3\theta \tan 3\theta \frac{d}{d\theta}(3\theta)$$

$$= 3e^{\sec 3\theta} \sec 3\theta \tan 3\theta$$