DERIVATIVES OF LOGARITHMIC FUNCTIONS

[1] $$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

PROOF Let $y = log_a x$. Then $a^y = x$

Differentiating this equation implicitly with respect to x, we get

$$a^y(\ln a)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}$$

If we put in [1], then the factor ln *a* on the right side becomes ln *e* = 1 and we get the formula for the derivative of the natural logarithmic function $log_e x = ln x$:

[2] $$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

EXAMPLE

Differentiate $y = \ln(x^3 + 1)$.

SOLUTION

To use the Chain Rule, we let $u = x^3 + 1$. Then $y = \ln u$, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}\frac{du}{dx} = \frac{1}{x^3 + 1}(3x^2) = \frac{3x^2}{x^3 + 1}$$

In general, if we combine [2] with the Chain Rule as in Example 1, we get

[3] $$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$$

EXAMPLE Find $\frac{d}{dx}\ln(\sin x)$.

SOLUTION Using [3], we have

$$\frac{d}{dx}\ln(\sin x) = \frac{1}{\sin x}\frac{d}{dx}(\sin x) = \frac{1}{\sin x}\cos x = \cot x$$

EXAMPLE Differentiate $f(x) = \log_{10}(2 + \sin x)$.

SOLUTION Using [1] with a = 10, we have

$$f'(x) = \frac{d}{dx} \log_{10}(2 + \sin x) = \frac{1}{(2 + \sin x) \ln 10} \frac{d}{dx} (2 + \sin x)$$

$$= \frac{\cos x}{(2 + \sin x) \ln 10}$$

EXAMPLE Find $\dfrac{d}{dx} \ln \dfrac{x + 1}{\sqrt{x - 2}}$.

SOLUTION 1

$$\frac{d}{dx} \ln \frac{x + 1}{\sqrt{x - 2}} = \frac{1}{\dfrac{x + 1}{\sqrt{x - 2}}} \frac{d}{dx} \frac{x + 1}{\sqrt{x - 2}}$$

$$= \frac{\sqrt{x - 2}}{x + 1} \frac{\sqrt{x - 2} \cdot 1 - (x + 1)\left(\frac{1}{2}\right)(x - 2)^{-1/2}}{x - 2}$$

$$= \frac{x - 2 - \frac{1}{2}(x + 1)}{(x + 1)(x - 2)} = \frac{x - 5}{2(x + 1)(x - 2)}$$

SOLUTION 2 If we first simplify the given function using the laws of logarithms, then the differentiation becomes easier:

$$\frac{d}{dx} \ln \frac{x + 1}{\sqrt{x - 2}} = \frac{d}{dx} \left[\ln(x + 1) - \tfrac{1}{2} \ln(x - 2)\right]$$

$$= \frac{1}{x + 1} - \frac{1}{2}\left(\frac{1}{x - 2}\right)$$

EXAMPLE Find $f'(x)$ if $f(x) = \ln|x|$.

SOLUTION Since $f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$

it follows that $f'(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\ \dfrac{1}{-x}(-1) = \dfrac{1}{x} & \text{if } x < 0 \end{cases}$

Thus $f'(x) = 1/x$ for all $x \neq 0$.

$$\frac{d}{dx} \ln|x| = \frac{1}{x}$$

[4]

LOGARITHMIC DIFFERENTIATION

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The method used in the following example is called **logarithmic differentiation**.

EXAMPLE Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2 + 1}}{(3x + 2)^5}$.

SOLUTION We take logarithms of both sides of the equation and use the Laws of Logarithms to simplify:

$$\ln y = \tfrac{3}{4}\ln x + \tfrac{1}{2}\ln(x^2 + 1) - 5\ln(3x + 2)$$

Differentiating implicitly with respect to *x* gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} - 5\cdot\frac{3}{3x+2}$$

Solving for *dy/dx* , we get

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

Because we have an explicit expression for *y*, we can substitute and write

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

STEPS IN LOGARITHMIC DIFFERENTIATION

1. Take natural logarithms of both sides of an equation *y = f(x)* and use the Laws of Logarithms to simplify.
2. Differentiate implicitly with respect to *x*.
3. Solve the resulting equation for *y'*.

If *f(x) < 0* for some values of *x*, then *ln f(x)* is not defined, but we can write $|y| = |f(x)|$ and use [4].

THE POWER RULE If *n* is any real number and f(x) = x^n, then

$$f'(x) = nx^{n-1}$$

PROOF

Let $y = x^n$ and use logarithmic differentiation

$$\ln|y| = \ln|x|^n = n \ln|x| \qquad x \neq 0$$

Therefore
$$\frac{y'}{y} = \frac{n}{x}$$

Hence
$$y' = n\frac{y}{x} = n\frac{x^n}{x} = nx^{n-1}$$

In general there are four cases for exponents and bases:

1. $\dfrac{d}{dx}(a^b) = 0 \qquad (a \text{ and } b \text{ are constants})$

2. $\dfrac{d}{dx}[f(x)]^b = b[f(x)]^{b-1}f'(x)$

3. $\dfrac{d}{dx}[a^{g(x)}] = a^{g(x)}(\ln a)g'(x)$

4. To find $(d/dx)[f(x)]^{g(x)}$, logarithmic differentiation can be used

EXAMPLE Differentiate $y = x^{\sqrt{x}}$.

SOLUTION 1 Using logarithmic differentiation, we have

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x} \ln x$$

$$\frac{y'}{y} = \sqrt{x} \cdot \frac{1}{x} + (\ln x) \frac{1}{2\sqrt{x}}$$

$$y' = y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right)$$

SOLUTION 2 Another method is to write $x^{\sqrt{x}} = (e^{\ln x})^{\sqrt{x}}$:

$$\frac{d}{dx}\left(x^{\sqrt{x}}\right) = \frac{d}{dx}\left(e^{\sqrt{x}\ln x}\right) = e^{\sqrt{x}\ln x}\frac{d}{dx}\left(\sqrt{x}\ln x\right)$$

$$= x^{\sqrt{x}}\left(\frac{2+\ln x}{2\sqrt{x}}\right)$$

THE NUMBER *e* AS A LIMIT

We have shown that if $f(x) = \ln x$, then $f'(x) = 1/x$. Thus $f'(1) = 1$.
We now use this fact to express the number *e* as a limit.

From the definition of a derivative as a limit, we have

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{x \to 0} \frac{f(1+x) - f(1)}{x}$$

$$= \lim_{x \to 0} \frac{\ln(1+x) - \ln 1}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1+x)$$

$$= \lim_{x \to 0} \ln(1+x)^{1/x}$$

Because $f'(1) = 1$, we have $\lim_{x \to 0} \ln(1 + x)^{1/x} = 1$

$$e = e^1 = e^{\lim_{x \to 0} \ln(1+x)^{1/x}} = \lim_{x \to 0} e^{\ln(1+x)^{1/x}} = \lim_{x \to 0} (1 + x)^{1/x}$$

[5] $$e = \lim_{x \to 0} (1 + x)^{1/x}$$

If we put $n = 1/x$ in [5], then $n \to \infty$ as $x \to 0^+$ and so an alternative expression for e is

[6] $$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$