DTU Compute
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Free surface simulation using the Incompressible Navier-Stokes Equations

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1 Introduction

This poster includes some of the work done for my thesis project. It concerns efficient simulation of free surface water waves using the incompressible Navier-Stokes equations. To get going, I've started with the linearized Incompressible Navier-Stokes Equations, which model the fluid flow of the water.

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1a}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\frac{1}{\rho}\nabla p + \boldsymbol{g} + \nu \nabla^2 \boldsymbol{u}. \tag{1b}$$

The tricky thing about free surfaces is that they move, and thus the position of the domain boundary must be coupled with the NSEs. The kinematic free surface condition for the linearized model is

$$\frac{\partial \eta(x,t)}{\partial t} = w|_{z=\eta}. \tag{2}$$

Furthermore, the vertical boundaries are assumed periodic, and the last boundary conditions are

$$p_D = 0 \quad \text{on } \eta, \tag{3a}$$

$$\boldsymbol{n} \cdot \boldsymbol{u} = 0 \quad \text{on } \Gamma_{bottom}. \tag{3b}$$

The aim is to use a higher-order time integration method combined with a Spectral Element Method to simulate free surface waves with high precision, high fidelity, and geometric flexibility.

2 Time integration and mass conservation

I have used Rothe's method, where I first employ an explicit Runge-Kutta method and then spatial discretization. In the Runge-Kutta scheme, one must simultaneously solve both for pressure and velocity field. This is done by applying the divergence operator to an RK stage and imposing the mass-conservation condition. This results in a Poisson boundary problem of the following form:

$$\nabla^2 p_D = \nabla \cdot \left(-\nabla p_S + \rho \nu \nabla^2 \boldsymbol{u}^{(k-1)} \right). \tag{4}$$

Solving for p_D with appropriate boundary conditions. The bottom condition is

$$\boldsymbol{n} \cdot \nabla p_D = \frac{\rho}{\beta_{kk}\Delta t}\boldsymbol{n} \cdot \sum_{j=1}^{k-1} \beta_{kj} \Delta t f(\boldsymbol{q}^{j-1}) + \boldsymbol{n} \cdot \left(\rho \nu \nabla^2 \boldsymbol{u}^{(k-1)} - \nabla p_S + \rho \boldsymbol{g} \right). \tag{5}$$

Then one can take an RK stage step with the found dynamic pressure to ensure the conservation of mass. Hereafter one can with the RK method update the position of the free surface using the free surface condition.

3 Nodal Spectral Element Discretization

All the operators are discretized using a nodal Spectral Element Method. So in general the three-stage rocket presented in 02689 is followed. We choose a weak multi-domain formulation of the problem, a local Legendre basis for both solution and test functions in a nodal representation, and we impose the necessary conditions to set up and solve an algebraic system of equations. This of course involves a lot of details on these operators, but I will spare you for these.

4 Convergence of method

To test whether the method arrived at works, the solution is compared to an analytical solution. There are however no analytical solutions to the NSEs, so we test the core functionalities using analytical solutions for potential flow problems. These do however not include viscous effects, and they are therefore neglected.

4.1 Convergence of the RK method

A convergence study is then carried out with four different RK methods of order one to four. The result is presented below.

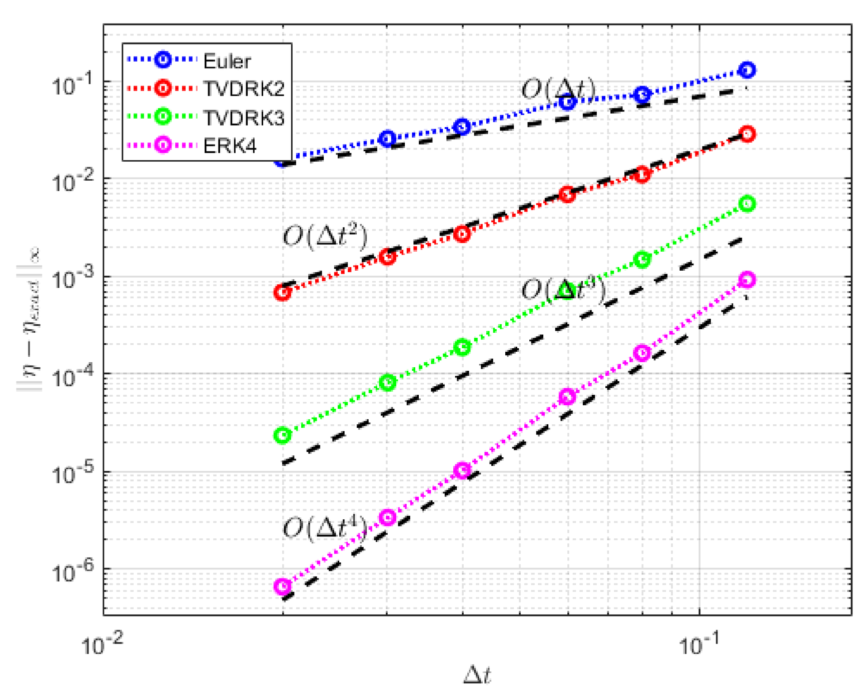

Figure 1: Convergence study of time stepping methods.

Here, we clearly see the expected convergence for each of the explicit RK methods.

4.2 Spatial convergence

Similarly, one can show the convergence with respect to the spatial discretization by using sufficiently small step sizes such that the error from the spatial discretization will dominate.

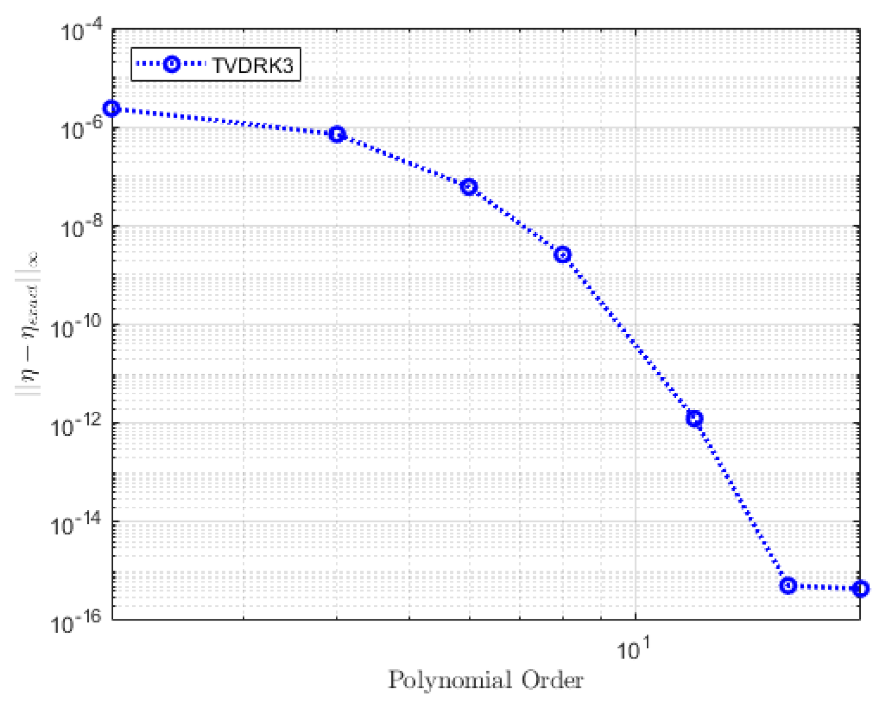

Figure 2: Convergence study of spatial discretization.

Here we clearly see the desired spectral convergence of the method.

4.3 Stability

Finally, we can show that the method is stable, as we can simulate for many wave periods and still get accurate results.

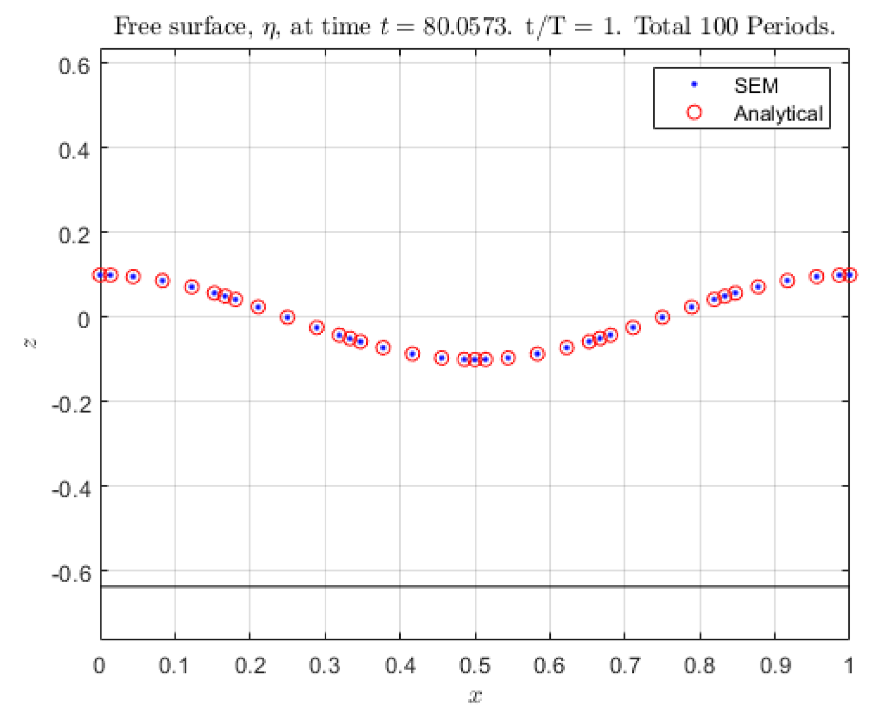

Figure 3: Free surface of SEM solution compared to analytical solution after 100 wave periods.

5 Current and future work

- Include nonlinear effects.
- Parallelization for large-scale computations.
- Efficient solvers for Poisson Problem.
  - p-multigrid.
- Stabilization techniques.

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