Derivation of the main equations in *Motion of asymmetric bodies in two-dimensional shear flow*[*]

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2 Motion of particles in shear flow

The Stokes equations

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = -\boldsymbol{\nabla}\mathbf{p} + \mu \nabla^2 \boldsymbol{u} = \mathbf{0}, \tag{2.1}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2.2}$$

arise from placing the Navier-Stokes equation

$$\rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} \right) = -\nabla p + \mu \nabla^2 \boldsymbol{u} + \boldsymbol{f}$$

under low Reynolds number and incompressible flow conditions. Imcompressible flow implies flow velocity is divergence-free, $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$, which gives equation (2.2). The Reynolds number Re is defined as $Re = \frac{\rho u L}{\mu}$. The inertia term $\rho\left(\boldsymbol{u} \cdot \nabla \boldsymbol{u}\right)$ and the viscous term $\mu \nabla^2 \boldsymbol{u}$ of the Navier-Stokes equation have orders of magnitude equal to $\frac{\rho u^2}{l}$ and $\frac{\mu u}{L^2}$ respectively. Therefore, the order of magnitude of the ratio $\frac{\text{inertia term}}{\text{viscous term}}$ is equal to the Reynolds number. So when $Re << 1$, the inertia term will be much smaller than the viscous term, meaning $\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}\right) << -\nabla p + \mu \nabla^2 \boldsymbol{u} + \boldsymbol{f}$. As a result, we can obtain Equation (2.1) under low Reynolds number conditions.

[*]Numbering of equations and sections is consistent with that in the paper

2.1.2 A convenient form of the rate-of-strain flow resistance and mobility coefficients

Rate-of-strain $\boldsymbol{E}^{\infty}$ is a second-order tensor with nine components, but due to constraints of symmetry and incompressibility, only has five independent components. So one possible basis for a straining flow is

$$\boldsymbol{E}^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{E}^{(2)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{E}^{(3)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$
$$\boldsymbol{E}^{(4)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{E}^{(5)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{2.8}$$

and a straining flow can be expressed as

$$\begin{aligned}\boldsymbol{E}^{\infty} = \begin{bmatrix} E_{xx} & E_{xy} & E_{xz} \\ E_{yx} & E_{yy} & E_{yz} \\ E_{zx} & E_{zy} & E_{zz} \end{bmatrix} &= \frac{1}{2}(E_{xx}-E_{yy})\boldsymbol{E}^{(1)}+E_{xy}\boldsymbol{E}^{(2)}+\frac{1}{2}E_{xz}\boldsymbol{E}^{(3)}+E_{yz}\boldsymbol{E}^{(4)}+E_{zz}\boldsymbol{E}^{(5)} \\ &= \sum_{i=1}^{5} E_i^{\infty}\boldsymbol{E}^{(i)}.\end{aligned} \tag{2.9}$$

where each E_i^{∞} is a scalar coefficient. Then, the double dot product of third-rank resistance tensor $\tilde{\boldsymbol{G}}$ and $\boldsymbol{E}^{\infty}$ can be expressed as

$$\tilde{\boldsymbol{G}} : \boldsymbol{E}^{\infty} = \tilde{\boldsymbol{G}} : \left(\sum_{i=1}^{5} E_i^{\infty}\boldsymbol{E}^{(i)}\right) = \sum_{i=1}^{5} E_i^{\infty}\left(\tilde{\boldsymbol{G}} : \boldsymbol{E}^{(i)}\right).$$

Define

$$\boldsymbol{g}^{(i)} := \tilde{G} : \boldsymbol{E}^{(i)} \quad (i = 1, \ldots, 5).$$

From this, we can define the reduced-rank tensor

$$\hat{\boldsymbol{G}} = \begin{pmatrix} \boldsymbol{g}^{(1)} & \boldsymbol{g}^{(2)} & \cdots & g^{(5)} \end{pmatrix} \in \mathbb{R}^{3\times 5},$$

and write

$$\tilde{\boldsymbol{G}} : \boldsymbol{E}^{\infty} = \sum_{i=1}^{5} E_i^{\infty}\boldsymbol{g}^{(i)} = \hat{\boldsymbol{G}} \cdot \begin{bmatrix} E_1^{\infty} \\ E_2^{\infty} \\ \vdots \\ E_5^{\infty} \end{bmatrix}.$$

Repeating the same process for resistance tensor $\tilde{\boldsymbol{H}}$, we can express both of these reduced-rank tensors as follows: Using these facts it is convenient to define new second-rank resistance tensors $\hat{G}$ and $\hat{H}$ which relate the vector form of straining flow to the force and torque acting on a rigid particle. These new tensors are equivalent to the standard third-rank form,

$$\begin{bmatrix} \tilde{\boldsymbol{G}} \\ \tilde{\boldsymbol{H}} \end{bmatrix} \cdot \boldsymbol{E}^{\infty} = \begin{bmatrix} \hat{\boldsymbol{G}} \\ \hat{\boldsymbol{H}} \end{bmatrix} \cdot E^{\infty} = \begin{bmatrix} \hat{\boldsymbol{G}} \\ \hat{\boldsymbol{H}} \end{bmatrix} \cdot \begin{bmatrix} E_1^{\infty} & E_2^{\infty} & E_3^{\infty} & E_4^{\infty} & E_5^{\infty} \end{bmatrix}^T, \tag{2.10}$$

where reduced-rank tensors $\hat{\boldsymbol{G}}$ and $\hat{\boldsymbol{H}}$ both have dimensions of three by five.

2.2 Motion in two-dimensional shear flow

The mobility problem for a two-dimensional force-free and torque-free particle is written as

$$\begin{bmatrix} U_1 - U_1^{\infty} \\ U_2 - U_2^{\infty} \\ \mathit{\Omega}_3 - \frac{1}{2}\omega_3^{\infty} \end{bmatrix} = \begin{bmatrix} \hat{g}_{11} & \hat{g}_{12} \\ \hat{g}_{21} & \hat{g}_{22} \\ \hat{h}_{31} & \hat{h}_{32} \end{bmatrix} \cdot \begin{bmatrix} E_1^{\infty} \\ E_2^{\infty} \end{bmatrix}. \tag{2.15}$$

Two-dimensional extensional flow can be expressed as

$$\boldsymbol{E}^{\infty} = E_1^{\infty}\boldsymbol{E}^{(1)} + E_2^{\infty}\boldsymbol{E}^{(2)}. \tag{2.14}$$

The original paper has given the following definitions $\omega_3 = -\dot{\gamma}$, $E_1^{\infty} = \frac{1}{2}\dot{\gamma}\sin(2\theta)$ and $E_2^{\infty} = \frac{1}{2}\dot{\gamma}\cos(2\theta)$, where $\dot{\gamma}$ is the shear rate and θ is the angle between the laboratory frame and the particle frame. Using these definitions, we can rewrite the right-hand side of Equation (2.15) as:

$$\begin{bmatrix} U_1 - U_1^{\infty} \\ U_2 - U_2^{\infty} \\ \mathit{\Omega}_3 - \frac{1}{2}\omega_3^{\infty} \end{bmatrix} = \begin{bmatrix} U_1 - U_1^{\infty} \\ U_2 - U_2^{\infty} \\ \mathit{\Omega}_3 - \frac{1}{2}\omega_3^{\infty} \end{bmatrix} = \frac{\dot{\gamma}}{2} \begin{bmatrix} \hat{g}_{11} & \hat{g}_{12} \\ \hat{g}_{21} & \hat{g}_{22} \\ \hat{h}_{31} & \hat{h}_{32} \end{bmatrix} \cdot \begin{bmatrix} \sin(2\theta) \\ \cos(2\theta) \end{bmatrix}. \tag{1}$$

Under two-dimensional conditions, the angular velocity Ω_3 is the rotation rate $\frac{d\theta}{dt} = \dot{\theta}$, and in the particle reference frame, the flow velocity is $-(\tilde{\boldsymbol{U}} + \boldsymbol{U}^{\infty})$, where $\tilde{\boldsymbol{U}}$ describes the relative motion of the particle with respect the the background flow. So we can then rewrite the left-hand side of Equation (2.15) to get:

$$\begin{bmatrix} U_1 - U_1^{\infty} \\ U_2 - U_2^{\infty} \\ \mathit{\Omega}_3 - \frac{1}{2}\omega_3^{\infty} \end{bmatrix} = \begin{bmatrix} -\tilde{U}_2 - U_1^{\infty} \\ -\tilde{U}_2 - U_2^{\infty} \\ \dot{\theta} + \frac{\dot{\gamma}}{2} \end{bmatrix} = \frac{\dot{\gamma}}{2} \begin{bmatrix} \hat{g}_{11} & \hat{g}_{12} \\ \hat{g}_{21} & \hat{g}_{22} \\ \hat{h}_{31} & \hat{h}_{32} \end{bmatrix} \cdot \begin{bmatrix} \sin(2\theta) \\ \cos(2\theta) \end{bmatrix}. \tag{2.16}$$

3. Asymmetric bent slender rod

Slender-body theory assumes that the hydrodynamic force density $\boldsymbol{f}$ on a particle and velocity difference $\boldsymbol{U}^\infty - \boldsymbol{U}$ have a linear relationship,

$$\boldsymbol{f}(s) = \boldsymbol{R} \cdot (\boldsymbol{U}^\infty - \boldsymbol{U})$$

with the coefficient being the resistance tensor $\boldsymbol{R} = c_\perp \left(\boldsymbol{I} - \alpha \boldsymbol{e}_t \boldsymbol{e}_t\right)$. Here, $\boldsymbol{e}_t$ is the vector tangent to the particle's body, $c_\perp = 4\pi\mu/(\ln 1/\varepsilon)$ is the normal drag coefficient, and α is the unknown coefficient we want to derive. Additionally, there is also a drag coefficient along the tangential direction, $c_\parallel = 2\pi\mu/(\ln 1/\varepsilon)$. We can see that $c_\parallel = \frac{1}{2}c_\perp$.

We calculate the dot products of $\boldsymbol{R}$ along the tangential direction and normal direction respectively. Along the tangential direction,

$$\boldsymbol{R} \cdot \boldsymbol{e}_t = c_\perp \left(\boldsymbol{I} - \alpha \boldsymbol{e}_t \boldsymbol{e}_t\right) \cdot \boldsymbol{e}_t = c_\perp(\boldsymbol{e}_t - \alpha \boldsymbol{e}_t) = c_\perp(1-\alpha)\boldsymbol{e}_t.$$

For the normal direction, we first define a vector $\boldsymbol{v}_\perp$ such that $\boldsymbol{e}_t \cdot \boldsymbol{v}_\perp = 0$. Then,

$$\boldsymbol{R} \cdot \boldsymbol{v}_\perp = c_\perp \left(\boldsymbol{I} - \alpha \boldsymbol{e}_t \boldsymbol{e}_t\right) \cdot \boldsymbol{e}_\perp = c_\perp \boldsymbol{v}_\perp.$$

Since $\boldsymbol{R} \cdot \boldsymbol{e}_t = c_\parallel \boldsymbol{e}_t$, $c_\perp(1-\alpha) = c_\parallel = \frac{1}{2}c_\perp$, so $\alpha = \frac{1}{2}$. We then arrive at the equation below:

$$\boldsymbol{f}(s) = c_\perp \left(\boldsymbol{I} - \frac{1}{2}\boldsymbol{e}_t \boldsymbol{e}_t\right) \cdot (\boldsymbol{U}^\infty - \boldsymbol{U}). \tag{3.1}$$

We now derive the corresponding mobility tensor of Equation (3.1). The mobility tensor $\boldsymbol{M}$ satisfies the mobility problem:

$$(\boldsymbol{U}^\infty - \boldsymbol{U}) = \boldsymbol{M} \cdot \boldsymbol{f},$$

and is the inverse of the resistance tensor $\boldsymbol{R}$. We first define the matrix $\boldsymbol{A}$ as $\boldsymbol{A} = (\boldsymbol{I} - \alpha \boldsymbol{e}_t \boldsymbol{e}_t)$. We assume the inverse of $\boldsymbol{A}$ has the form $\boldsymbol{A}^{-1} = (\boldsymbol{I} + \beta \boldsymbol{e}_t \boldsymbol{e}_t)$, where β is some scalar coefficient. We take the dot product of $\boldsymbol{A}$ and $\boldsymbol{A}^{-1}$:

$$\boldsymbol{A} \cdot \boldsymbol{A}^{-1} = (\boldsymbol{I} - \alpha\, \boldsymbol{e}_t \boldsymbol{e}_t) \cdot (\boldsymbol{I} + \beta\, \boldsymbol{e}_t \boldsymbol{e}_t) = \boldsymbol{I} + (\beta - \alpha - \alpha\beta)\, \boldsymbol{e}_t \boldsymbol{e}_t.$$

For this to equal the identity tensor $\boldsymbol{I}$:

$$\beta - \alpha - \alpha\beta = 0 \quad \Rightarrow \quad \beta(1-\alpha) = \alpha \quad \Rightarrow \quad \beta = \frac{\alpha}{1-\alpha}$$

Therefore, we have:

$$(\boldsymbol{I} - \alpha\, \boldsymbol{e}_t \boldsymbol{e}_t)^{-1} = \boldsymbol{I} + \frac{\alpha}{1-\alpha}\, \boldsymbol{e}_t \boldsymbol{e}_t$$

For $\alpha = \frac{1}{2}$, which corresponds to the tangential drag coefficient $c_\parallel = c_\perp(1 - \alpha) = \frac{1}{2}c_\perp$, we obtain:

$$\left(\boldsymbol{I} - \frac{1}{2}\, \boldsymbol{e}_t \boldsymbol{e}_t\right)^{-1} = \boldsymbol{I} + \boldsymbol{e}_t \boldsymbol{e}_t$$

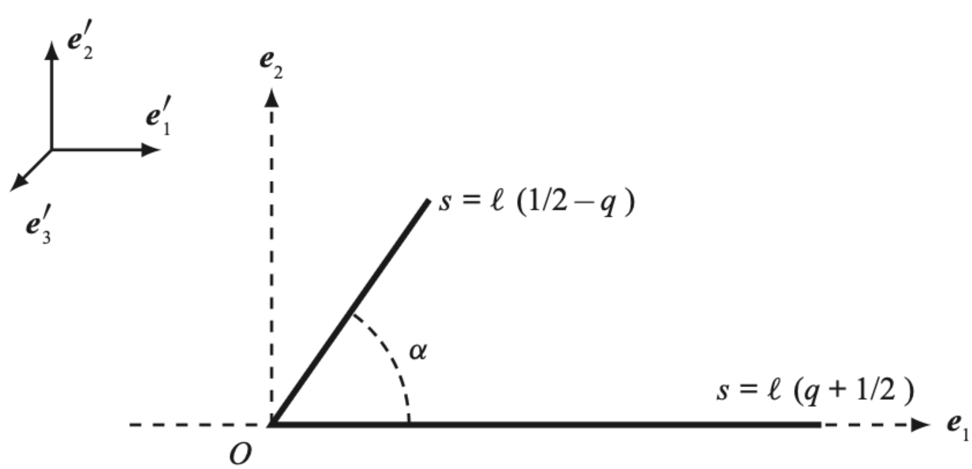
Finally, the mobility tensor becomes:

$$\boldsymbol{M} = \frac{1}{c_\perp} \left(\boldsymbol{I} + \boldsymbol{e}_t \boldsymbol{e}_t\right).$$

3.1 Resistance coefficients

Equation 3.4

In this section, we consider a slender body depicted by the figure below:



where the left and right coordinate frames depict the particle frame and laboratory frame respectively.

Here, l is the length of the particle, $q \,\epsilon\, [0, \frac{1}{2}]$ is the asymmetry parameter, $\alpha \,\epsilon\, [0, \pi]$ is the angle between the short and long arms of the slender body, and $s \,\epsilon\, [l(\frac{1}{2} - q), l(\frac{1}{2} + q)]$ is the arc length. The position and tangent vectors of this rod are given respectively by Equations (3.2) and (3.3):

$$\boldsymbol{r}(s) = \begin{cases} -s(\cos\alpha\,\boldsymbol{e}_1 + \sin\alpha\,\boldsymbol{e}_2) & s < 0, \\ s\boldsymbol{e}_1 & s \geq 0 \end{cases} \tag{3.2}$$

$$\boldsymbol{e}_t(s) = \frac{d\boldsymbol{r}}{ds} = \begin{cases} -(\cos\alpha\,\boldsymbol{e}_1 + \sin\alpha\,\boldsymbol{e}_2) & s < 0, \\ \boldsymbol{e}_1 & s \geq 0. \end{cases} \tag{3.3}$$

We now non-dimensionalize Equation (3.1) from the previous section. The orders of magnitude of velocity difference $\boldsymbol{U}^\infty - \boldsymbol{U}$ and force density $\boldsymbol{f}$ are equal to $\dot{\gamma}l$ and $c_\perp\dot{\gamma}l$ respectively. Dimensionless velocity difference $\hat{\boldsymbol{U}}^\infty - \hat{\boldsymbol{U}}$ is then expressed as $\hat{\boldsymbol{U}}^\infty - \hat{\boldsymbol{U}} = \frac{\boldsymbol{U}^\infty - \boldsymbol{U}}{\dot{\gamma}l}$. If we use $\hat{\boldsymbol{U}}^\infty - \hat{\boldsymbol{U}}$ to express dimensionless force density $\hat{\boldsymbol{f}}$, we find that dimensionless force density is simply dimensionless velocity difference divided by the drag coefficient: $\hat{f} = \frac{\boldsymbol{U}^\infty - \boldsymbol{U}}{c_\perp}$. Therefore, the non-dimensionalized form of (3.1) is:

$$\hat{\boldsymbol{f}}(s) = \left(\boldsymbol{I} - \frac{1}{2}\boldsymbol{e}_t\boldsymbol{e}_t\right) \cdot (\hat{\boldsymbol{U}}^\infty - \hat{\boldsymbol{U}}).$$

We integrate this dimensionless equation to find the expression for the force acting on a particle with tangent vector $\boldsymbol{e}_t$. The authors of the original paper impose a test flow where the slender body is fixed $\boldsymbol{U} = \boldsymbol{0}$ in a uniform background flow $\boldsymbol{U}^\infty = u_1\boldsymbol{e}_1^{'}$. Therefore we can rewrite $\hat{\boldsymbol{f}}$ as

$$\hat{\boldsymbol{f}} = \left(\boldsymbol{I} - \frac{1}{2}\boldsymbol{e}_t\boldsymbol{e}_t\right) \cdot u_1\boldsymbol{e}_1^{'} = u_1\left(\boldsymbol{e}_1' - \frac{1}{2}(\boldsymbol{e}_t \cdot \boldsymbol{e}_1')\boldsymbol{e}_t\right).$$

The lab frame $\boldsymbol{e}_1'$ relates to the particle frame $\boldsymbol{e}_1, \boldsymbol{e}_2$ through the rotation angle θ:

$$\boldsymbol{e}_1' = \cos\theta\,\boldsymbol{e}_1 - \sin\theta\,\boldsymbol{e}_2.$$

However, because the particle is fixed, $\theta = 0$ (since $\boldsymbol{e}_1$ is parallel to $\boldsymbol{e}_1'$), so:

$$\boldsymbol{e}_1' = \boldsymbol{e}_1.$$

Thus, $\boldsymbol{e}_t \cdot \boldsymbol{e}_1' = \boldsymbol{e}_t \cdot \boldsymbol{e}_1$. Now, according to (3.3), we consider two cases: when $s < 0$ and when $s \geq 0$. From the definition of $\boldsymbol{e}_t(s)$:

- For $s \geq 0$ (right arm):

$$\boldsymbol{e}_t \cdot \boldsymbol{e}_1 = \boldsymbol{e}_1 \cdot \boldsymbol{e}_1 = 1.$$

- For $s < 0$ (left arm):

$$\boldsymbol{e}_t \cdot \boldsymbol{e}_1 = -(\cos\alpha\, \boldsymbol{e}_1 + \sin\alpha\, \boldsymbol{e}_2) \cdot \boldsymbol{e}_1 = -\cos\alpha.$$

So the (dimensionless) piecewise force density for the right and left arms are as follows:

- For $s \geq 0$:

$$\boldsymbol{f}(s) = u_1 \left(\boldsymbol{e}_1 - \frac{1}{2}(1)\boldsymbol{e}_1 \right) = \frac{u_1}{2}\boldsymbol{e}_1.$$

- For $s < 0$:

$$\boldsymbol{f}(s) = u_1 \left(\boldsymbol{e}_1 - \frac{1}{2}(-\cos\alpha)(-\cos\alpha\, \boldsymbol{e}_1 - \sin\alpha\, \boldsymbol{e}_2) \right).$$

  Expanding:

$$\boldsymbol{f}(s) = u_1 \left(\left(1 - \frac{\cos^2\alpha}{2} \right) \boldsymbol{e}_1 - \frac{\sin 2\alpha}{4} \boldsymbol{e}_2 \right).$$

The total force $\boldsymbol{F}$ is the integral of $\boldsymbol{f}(s)$ over $s \in [q - 1/2, q + 1/2]$. Note:

- Left arm interval: $s \in [q - 1/2, 0]$, length $\frac{1}{2} - q$.
- Right arm interval: $s \in [0, q + 1/2]$, length $\frac{1}{2} + q$.

We calculate the contributions of the $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ components individually. For the $\boldsymbol{e}_1$ component,

- Right arm contribution:

$$\int_0^{q+1/2} \frac{u_1}{2}\, ds = \frac{u_1}{2} \left(q + \frac{1}{2} \right).$$

- Left arm contribution:

$$\int_{q-1/2}^{0} u_1 \left(1 - \frac{\cos^2\alpha}{2} \right)\, ds = u_1 \left(1 - \frac{\cos^2\alpha}{2} \right) \left(\frac{1}{2} - q \right).$$

- Combined:

$$F_1 = u_1 \left(\frac{q}{2} + \frac{1}{4} + \left(1 - \frac{\cos^2 \alpha}{2}\right) \left(\frac{1}{2} - q\right) \right).$$

Simplifying (using $\cos^2 \alpha = \frac{1+\cos 2\alpha}{2}$):

$$F_1 = \frac{u_1}{8} \left(5 - 2q + (2q-1)\cos 2\alpha\right).$$

For the $\boldsymbol{e}_2$ component, only the left arm contributes:

$$F_2 = \int_{q-1/2}^{0} \left(-\frac{u_1 \sin 2\alpha}{4}\right) \, ds = \frac{u_1(2q-1)\sin 2\alpha}{8}.$$

For the $\boldsymbol{e}_3$ component, there is no contribution from either arms: $F_3 = 0$. Combining the components, we arrive at Equation (3.4):

$$\mathbf{F} = \begin{bmatrix} \frac{1}{8}(5 - 2q + (2q-1)\cos 2\alpha \\ \frac{1}{8}(2q-1)\sin 2\alpha \\ 0 \end{bmatrix} u_1. \tag{3.4}$$

Equation 3.5

We now derive the components of the resistance tensor for an asymmetric slender particle in two-dimensional shear flow. Below is the general form for the grand resistance tensor:

$$\begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{T} \\ \boldsymbol{S} \end{bmatrix} = \mu \begin{bmatrix} \boldsymbol{A} & \tilde{\boldsymbol{B}} & \tilde{\boldsymbol{G}} \\ \boldsymbol{B} & \boldsymbol{C} & \tilde{\boldsymbol{H}} \\ \boldsymbol{G} & \boldsymbol{H} & \boldsymbol{M} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{U}^\infty - \boldsymbol{U} \\ \frac{1}{2}\boldsymbol{\omega}^\infty - \boldsymbol{\Omega} \\ \boldsymbol{E}^\infty \end{bmatrix}. \tag{2.5}$$

Below is the dimensionless form of the equation above.

$$\begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{T} \\ \boldsymbol{S} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \tilde{\boldsymbol{B}} & \tilde{\boldsymbol{G}} \\ \boldsymbol{B} & \boldsymbol{C} & \tilde{\boldsymbol{H}} \\ \boldsymbol{G} & \boldsymbol{H} & \boldsymbol{M} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{U}^\infty - \boldsymbol{U} \\ \frac{1}{2}\boldsymbol{\omega}^\infty - \boldsymbol{\Omega} \\ \boldsymbol{E}^\infty \end{bmatrix}. \tag{2.5}$$

All physical quantities used in calculations in this sections are dimensionless.

The original paper states several conditions that simplify the form of the resistance tensor. First, the grand resistance tensor is symmetric for second-rank tensors e.g. $B_{ij} = \tilde{B}_{ji}$. This means that $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, which are all three by three second-order tensors, are symmetric. Also, the particle motion we consider is two-dimensional, with the particle only revolving around the $\boldsymbol{e}_3 = \boldsymbol{e}_3^{'}$ axis. Additionally, under two-dimensional conditions, extensional flow $\boldsymbol{E}^\infty$ can be written as a linear combination of $\boldsymbol{E}^{(1)}$ and $\boldsymbol{E}^{(2)}$ (which are defined in (2.8)). Under these two conditions, many coefficients of the resistance tensor must be equal to zero: $\tilde{B}_{11} = \tilde{B}_{12} = \tilde{B}_{22} = \hat{G}_{3j} = \hat{H}_{1j} = \hat{H}_{2j} = 0$ for $j = 1, 2, 3, 4, 5$ ($\hat{\boldsymbol{G}}$ and $\hat{\boldsymbol{H}}$ are second-rank with dimensions of three by five).

The authors derive the remaining resistance coefficients by applying a series of test flows to the slender body.

The first test flow is the one used to derive (3.4):

$$\boldsymbol{U} = \boldsymbol{0}, \quad \boldsymbol{U}^\infty = u_1\boldsymbol{e}_1' = u_1\boldsymbol{e}_1,$$

which gives

$$\boldsymbol{F} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} (5 - 2q + (2q-1)\cos 2\alpha) \\ (2q-1)\sin 2\alpha \\ 0 \end{bmatrix}.$$

Here, the components of the resistance tensor A_{ij} satisfy:

$$\boldsymbol{F} = \begin{bmatrix} A_{11} \\ A_{21} \\ A_{31} \end{bmatrix} u_1 \quad \Rightarrow \quad \begin{bmatrix} A_{11} \\ A_{21} \\ A_{31} \end{bmatrix} = \frac{1}{8}\begin{bmatrix} (5 - 2q + (2q-1)\cos 2\alpha) \\ (2q-1)\sin 2\alpha \\ 0 \end{bmatrix}$$

The second test flow is

$$\boldsymbol{U} = \boldsymbol{0}, \quad \boldsymbol{U}^\infty = u_2\boldsymbol{e}_2' = u_2\boldsymbol{e}_2,$$

which gives

$$\boldsymbol{F} = \int_{-q+\frac{1}{2}}^{q+\frac{1}{2}} \boldsymbol{f}(s)\, ds = u_2(q + \tfrac{1}{2})\boldsymbol{e}_2 + (\tfrac{1}{2}\sin^2\alpha)\boldsymbol{e}_1 + (\tfrac{1}{2}\sin^2\alpha)\boldsymbol{e}_2$$

$$\Rightarrow \boldsymbol{F} = u_2(1 - \tfrac{1}{2}q)\sin^2\alpha\boldsymbol{e}_2$$
$$\Rightarrow A_{22} = 1 + \tfrac{1}{4}(2q-1)\sin^2\alpha.$$

The third test flow is

$$\boldsymbol{\Omega} = \boldsymbol{0}, \quad \boldsymbol{\omega}^\infty = \omega_3^\infty\boldsymbol{e}_3' = \omega_3^\infty\boldsymbol{e}_3.$$

The velocity vector $\boldsymbol{v}(s) = \boldsymbol{U}^\infty - \boldsymbol{U}$ is

$$\boldsymbol{v}(s) = \omega_3^\infty \boldsymbol{e}_3 \times \boldsymbol{r}(s) = \begin{cases} \omega_3^\infty \boldsymbol{e}_3 s(\sin\alpha \boldsymbol{e}_1 - \cos\alpha \boldsymbol{e}_2), & s < 0 \\ -\omega_3^\infty s \boldsymbol{e}_2, & s \geq 0. \end{cases}$$

When $s \geq 0$:

$$\boldsymbol{f}(s) = -\omega_3^\infty s \boldsymbol{e}_2 + \frac{\omega_3^\infty s}{2} \boldsymbol{e}_1 \boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = \omega_3^\infty s \boldsymbol{e}_2.$$

When $s < 0$:

$$\boldsymbol{f}(s) = \omega_3^\infty (\sin\alpha(1 - \frac{1}{2}\cos^2\alpha)\boldsymbol{e}_1 - \cos\alpha(1 - \frac{1}{2}\sin^2\alpha)\boldsymbol{e}_2).$$

So the torque on the particle for this test flow is given by

$$\boldsymbol{T}(s) = \int_{q-1/2}^{q+1/2} \boldsymbol{r}(s) \times \boldsymbol{f}(s)\, ds = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{8}(1-2q)^2 \sin\alpha \\ \frac{1}{8}((2q+1)^2 + (1-2q)^2 \cos\alpha \\ 0 \end{bmatrix} \omega_3^\infty = \begin{bmatrix} \tilde{B}_{13} \\ \tilde{B}_{23} \\ \tilde{B}_{33} \end{bmatrix} \omega_3^\infty$$

$$\Rightarrow \begin{bmatrix} \tilde{B}_{13} \\ \tilde{B}_{23} \\ \tilde{B}_{33} \end{bmatrix} = \begin{bmatrix} -\frac{1}{8}(1-2q)^2 \sin\alpha \\ \frac{1}{8}((2q+1)^2 + (1-2q)^2 \cos\alpha \\ 0 \end{bmatrix}.$$

The fourth test flow is

$$\boldsymbol{w}^\infty = \boldsymbol{0}, \quad \boldsymbol{\mathit{\Omega}} = \mathit{\Omega}_3 \boldsymbol{e}_3' = \mathit{\Omega}_3 \boldsymbol{e}_3.$$

The velocity vector is

$$\boldsymbol{v}(s) = \mathit{\Omega}_3 \boldsymbol{e}_3 \times \boldsymbol{r}(s) = \begin{cases} \mathit{\Omega}_3 \boldsymbol{e}_3 s(\sin\alpha \boldsymbol{e}_1 - \cos\alpha \boldsymbol{e}_2), & s < 0 \\ -\mathit{\Omega}_3 s \boldsymbol{e}_2, & s \geq 0. \end{cases}$$

When $s \geq 0$:

$$\boldsymbol{f}(s) = -\mathit{\Omega}_3 s \boldsymbol{e}_2 + \frac{\mathit{\Omega}_3 s}{2} \boldsymbol{e}_1 \boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = \mathit{\Omega}_3 s \boldsymbol{e}_2.$$

When $s < 0$:

$$\boldsymbol{f}(s) = \Omega_3(\sin\alpha(1 - \frac{1}{2}\cos^2\alpha)\boldsymbol{e}_1 - \cos\alpha(1 - \frac{1}{2}\sin^2\alpha)\boldsymbol{e}_2).$$

The torque is given by

$$\boldsymbol{T}(s) = \int_{q-1/2}^{q+1/2} \boldsymbol{r}(s) \times \boldsymbol{f}(s)\, ds = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{12} + q^2 \end{bmatrix} \Omega_3 = \begin{bmatrix} C_{31} \\ C_{32} \\ C_{33} \end{bmatrix} \Omega_3$$

$$\Rightarrow \begin{bmatrix} C_{31} \\ C_{32} \\ C_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{12} + q^2 \end{bmatrix}.$$

Because torque is expressed as $\boldsymbol{T} = \boldsymbol{C} \cdot (\frac{1}{2}\omega^\infty - \Omega)$ and the only nonzero torque component is $\frac{1}{12} + q^2$, the remaining components of $\boldsymbol{C}$ all equal zero: $C_{11} = C_{12} = C_{22} = 0$. Combining the results of these four test flows, we arrive at Equation (3.5):

$$\begin{bmatrix} A_{11} \\ A_{12} \\ A_{22} \\ \tilde{B}_{13} \\ \tilde{B}_{23} \\ C_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{8}\big(5 - 2q + (2q-1)\cos 2\alpha\big) \\ \frac{1}{8}(2q-1)\sin 2\alpha \\ 1 + \frac{1}{4}(2q-1)\sin^2\alpha \\ -\frac{1}{8}(1-2q)^2 \sin\alpha \\ \frac{1}{8}\big((2q+1)^2 + (1-2q)^2\cos\alpha\big) \\ 0 \end{bmatrix}. \tag{3.5}$$

Equation 3.6

The fifth test flow is

$$\boldsymbol{E}^\infty = \boldsymbol{E}^{(1)}, \quad \boldsymbol{E}^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

so the background velocity $\boldsymbol{U}^\infty$ is

$$\boldsymbol{U}^\infty(s) = \boldsymbol{E}^{(1)} \cdot \boldsymbol{r}(s) = \begin{cases} -s\cos\alpha\boldsymbol{e}_1 + s\sin\alpha\boldsymbol{e}_2, & s < 0 \\ s\boldsymbol{e}_1, & s \geq 0 \end{cases},$$

and the force density $\boldsymbol{f}(s)$ is

$$\boldsymbol{f}(s) = \begin{cases} -s\cos\alpha\boldsymbol{e}_1 + s\sin\cos\alpha\boldsymbol{e}_2 - \frac{1}{2}(-s\cos^2\alpha + \sin^2\alpha)\boldsymbol{e}_t, & s \geq 0 \\ \frac{s}{2}\boldsymbol{e}_1, & s < 0. \end{cases}$$

So the force $\boldsymbol{F}(s)$ is

$$\boldsymbol{F}(s) = \int_{q-1/2}^{q+1/2} \boldsymbol{f}(s)\, ds$$

$$= \begin{bmatrix} \frac{1}{32}(2(2q+1)^2 + 3(1-2q)^2\cos\alpha - (1-2q)^2\cos 3\alpha) \\ \frac{1}{8}(1-2q)\sin^3\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \hat{G}_{11} \\ \hat{G}_{12} \end{bmatrix} \cdot E_1^\infty = \begin{bmatrix} \hat{G}_{11} \\ \hat{G}_{12} \end{bmatrix} \cdot 1 = \begin{bmatrix} \hat{G}_{11} \\ \hat{G}_{12} \end{bmatrix}.$$

And the torque for this test flow around the $\boldsymbol{e}_3'$ axis is:

$$T_3 = \int_{q-1/2}^{q+1/2} (\boldsymbol{r}(s) \times \boldsymbol{f}(s))_3\, ds = \frac{1}{24}\left((2q+1)^3 - (2q-1)^3\cos 2\alpha\right) = \hat{H}_{31}.$$

The sixth test flow is

$$\boldsymbol{E}^\infty = \boldsymbol{E}^{(2)}, \quad \boldsymbol{E}^{(2)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and the background velocity $\boldsymbol{U}^\infty$ is

$$\boldsymbol{U}^\infty(s) = \boldsymbol{E}^{(2)} \cdot \boldsymbol{r}(s).$$

Using the same procedure as for the previous test flow, we calculate the corresponding force and torque to get

$$\begin{bmatrix} \hat{G}_{21} \\ \hat{G}_{22} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{16}(1-2q)^2(2+\cos 2\alpha)\sin\alpha \\ \frac{1}{32}\left(4(2q+1)^2 + 3(1-2q)^2\cos\alpha + (1-2q)^2\cos 3\alpha\right) \end{bmatrix}$$

and

$$T_3 = \int_{q-1/2}^{q+1/2} (\boldsymbol{r}(s) \times \boldsymbol{f}(s))_3\, ds = \frac{1}{24}\left((2q+1)^3 - (2q-1)^3\cos 2\alpha\right) = \hat{H}_{32}.$$

Together, the results of the fifth and sixth test flows give Equation (3.6):

$$\begin{bmatrix} \hat{G}_{11} \\ \hat{G}_{12} \\ \hat{G}_{21} \\ \hat{G}_{22} \\ \hat{H}_{31} \\ \hat{H}_{32} \end{bmatrix} = \begin{bmatrix} \frac{1}{32}\left(2(2q+1)^2 + 3(1-2q)^2\cos\alpha - (1-2q)^2\cos 3\alpha\right) \\ \frac{1}{8}(1-2q)^2\sin^3\alpha \\ -\frac{1}{16}(1-2q)^2(2+\cos 2\alpha)\sin\alpha \\ \frac{1}{32}\left(4(2q+1)^2 + 3(1-2q)^2\cos\alpha + (1-2q)^2\cos 3\alpha\right) \\ \frac{1}{24}(2q-1)^3\sin 2\alpha \\ \frac{1}{24}\left((2q+1)^3 - (2q-1)^3\cos 2\alpha\right) \end{bmatrix}. \tag{3.6}$$