Extension of *Drift of elastic hinges in quasi-two-dimensional oscillating shear flows* to n-arm case (2)

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1 Verification for $n = 2$ case

The resistance coefficients for the two-arm case are given by

$$\begin{bmatrix} F_{k,1} \\ F_{k,2} \\ T_k \end{bmatrix} = \underbrace{\frac{1}{8}\begin{bmatrix} 3-\cos 2\beta_k & -\sin 2\beta_k & -\sin\beta_k \\ -\sin 2\beta_k & 3+\cos 2\beta_k & \cos\beta_k \\ -\sin\beta_k & \cos\beta_k & \frac{1}{3} \end{bmatrix}}_{\boldsymbol{R}(\beta_k)} \begin{bmatrix} U_1^\infty - U_{p,1} \\ U_2^\infty - U_{p,2} \\ \frac{1}{2}\omega_3^\infty - \Omega_{p,k} \end{bmatrix} + \underbrace{\frac{1}{32}\begin{bmatrix} (3\cos\beta_k - \cos 3\beta_k) & 4\sin^3\beta_k \\ -(3\sin\beta_k + \sin 3\beta_k) & 4\cos^3\beta_k \\ -\frac{4}{3}\sin 2\beta_k & \frac{4}{3}\cos 2\beta_k \end{bmatrix}}_{\boldsymbol{Q}(\beta_k)} \begin{bmatrix} E_1^\infty \\ E_2^\infty \end{bmatrix}, \quad (1)$$

where $k = 1, 2$. In the paper, the two orientation angles are defined as and $\beta_1 = -\alpha/2$ and $\beta_2 = \alpha/2$, where α is the angular displacement between the two arms.

Bringing $n = 2$ into the system of equations I previously defined, the unknowns vector $\boldsymbol{X}$ becomes

$$\boldsymbol{X} = \begin{bmatrix} U_{p,1} \\ U_{p,2} \\ \frac{d\theta}{dt} \\ \dot{\alpha}_1 \end{bmatrix}. \quad (2)$$

$\boldsymbol{M}$ and $\boldsymbol{B}$ become 4×4 and 4×1 respectively. The four rows of $\boldsymbol{B}$ are

$$B_1 = \sum_{k=1}^{2} \left(R_{11}(\beta_k) U_1^{\infty} + R_{12}(\beta_k) U_2^{\infty} + R_{13}(\beta_k) \frac{1}{2} \omega_3^{\infty} + Q_{11}(\beta_k) E_1^{\infty} + Q_{12}(\beta_k) E_2^{\infty} \right),$$

$$B_2 = \sum_{k=1}^{2} \left(R_{21}(\beta_k) U_1^{\infty} + R_{22}(\beta_k) U_2^{\infty} + R_{23}(\beta_k) \frac{1}{2} \omega_3^{\infty} + Q_{21}(\beta_k) E_1^{\infty} + Q_{22}(\beta_k) E_2^{\infty} \right),$$

$$B_3 = \sum_{k=1}^{2} \left(R_{31}(\beta_k) U_1^{\infty} + R_{32}(\beta_k) U_2^{\infty} + R_{33}(\beta_k) \frac{1}{2} \omega_3^{\infty} + Q_{31}(\beta_k) E_1^{\infty} + Q_{32}(\beta_k) E_2^{\infty} \right),$$

$$B_4 = \left[R_{31}(\beta_2) - R_{31}(\beta_1) \right] U_1^{\infty} + \ \left[R_{32}(\beta_2) - R_{32}(\beta_1) \right] U_2^{\infty} + \left[R_{33}(\beta_3) - R_{33}(\beta_1) \right] \frac{1}{2} \omega_3^{\infty}$$
$$+ \ \left[Q_{31}(\beta_2) - Q_{31}(\beta_1) \right] E_1^{\infty} + \left[Q_{32}(\beta_2) - Q_{32}(\beta_1) \right] E_2^{\infty} - 2\kappa(\alpha_j - \alpha_0).$$

The elements of the four rows of $\boldsymbol{M}$ are

First row:

$$M_{1,1} = \sum_{k=1}^{2} R_{11}(\beta_k) = R_{11}(-\alpha/2) + R_{11}(\alpha/2)$$
$$M_{1,2} = \sum_{k=1}^{2} R_{12}(\beta_k) = R_{12}(-\alpha/2) + R_{12}(\alpha/2)$$
$$M_{1,3} = \sum_{k=1}^{2} R_{13}(\beta_k) = R_{13}(-\alpha/2) + R_{13}(\alpha/2)$$
$$M_{1,4} = \sum_{k=2}^{2} R_{13}(\beta_k) = R_{13}(\alpha/2),$$

Second row:

$$M_{2,1} = \sum_{k=1}^{2} R_{21}(\beta_k) = R_{21}(-\alpha/2) + R_{21}(\alpha/2)$$
$$M_{2,2} = \sum_{k=1}^{2} R_{22}(\beta_k) = R_{22}(-\alpha/2) + R_{22}(\alpha/2)$$
$$M_{2,3} = \sum_{k=1}^{2} R_{23}(\beta_k) = R_{23}(-\alpha/2) + R_{23}(\alpha/2)$$
$$M_{2,4} = \sum_{k=2}^{2} R_{23}(\beta_k) = R_{23}(\alpha/2),$$

Third row:

$$M_{3,1} = \sum_{k=1}^{2} R_{31}(\beta_k) = R_{31}(-\alpha/2) + R_{31}(\alpha/2)$$
$$M_{3,2} = \sum_{k=1}^{2} R_{32}(\beta_k) = R_{32}(-\alpha/2) + R_{32}(\alpha/2)$$
$$M_{3,3} = \sum_{k=1}^{2} R_{33}(\beta_k) = R_{33}(-\alpha/2) + R_{33}(\alpha/2)$$
$$M_{3,4} = \sum_{k=2}^{2} R_{33}(\beta_k) = R_{33}(\alpha/2),$$

Fourth row:

$$M_{4,1} = R_{31}(\beta_2) - R_{31}(\beta_1) = R_{31}(\alpha/2) - R_{31}(-\alpha/2)$$
$$M_{4,2} = R_{32}(\beta_2) - R_{32}(\beta_1) = R_{32}(\alpha/2) - R_{32}(-\alpha/2)$$
$$M_{4,3} = R_{33}(\beta_2) - R_{33}(\beta_1) = R_{33}(\alpha/2) - R_{33}(-\alpha/2)$$
$$M_{4,4} = R_{33}(\beta_2) = R_{33}(\alpha/2).$$

I then bring the resistance coefficients from Equation (1) into the expressions for the elements of $\boldsymbol{M}$ and $\boldsymbol{B}$. The elements of $\boldsymbol{M}$ simplify to

$$\begin{aligned}
&M_{11} = \frac{3-\cos\alpha}{4}, && M_{12} = 0, && M_{13} = 0, && M_{14} = -\tfrac{1}{8}\sin\left(\tfrac{\alpha}{2}\right), \\
&M_{21} = 0, && M_{22} = \frac{3+\cos\alpha}{4}, && M_{23} = \tfrac{1}{4}\cos\left(\tfrac{\alpha}{2}\right), && M_{24} = \tfrac{1}{8}\cos\left(\tfrac{\alpha}{2}\right), \\
&M_{31} = 0, && M_{32} = \tfrac{1}{4}\cos\left(\tfrac{\alpha}{2}\right), && M_{33} = \tfrac{1}{12}, && M_{34} = \tfrac{1}{24}, \\
&M_{41} = -\tfrac{1}{4}\sin\left(\tfrac{\alpha}{2}\right), && M_{42} = 0, && M_{43} = 0, && M_{44} = \frac{1}{24},
\end{aligned}$$

and the elements of $\boldsymbol{B}$ simplify to

$$B_1 = \frac{3-\cos\alpha}{4} U_1^\infty + \frac{1}{16}\left(3\cos\tfrac{\alpha}{2} - \cos\tfrac{3\alpha}{2}\right) E_1^\infty,$$
$$B_2 = \frac{3+\cos\alpha}{4} U_2^\infty + \frac{1}{8}\cos\left(\tfrac{\alpha}{2}\right)\omega_3^\infty + \frac{1}{4}\cos^3\left(\tfrac{\alpha}{2}\right) E_2^\infty,$$
$$B_3 = \frac{1}{4}\cos\left(\tfrac{\alpha}{2}\right) U_2^\infty + \frac{1}{24}\omega_3^\infty + \frac{1}{12}\cos\alpha\, E_2^\infty,$$
$$B_4 = -\frac{1}{4}\sin\left(\tfrac{\alpha}{2}\right) U_1^\infty - \frac{1}{12}\sin\alpha\, E_1^\infty - 2\kappa(\alpha-\alpha_0).$$

Using the expressions above, I can solve for $\boldsymbol{X}$ using Python to get

$$
\begin{aligned}
U_{p,1} - U_1^\infty = -\frac{1}{2(\cos\alpha + 3)}\Bigg(& 4E_1^\infty \sin\left(\frac{\alpha}{2}\right)\sin\alpha - 3E_1^\infty \cos\left(\frac{\alpha}{2}\right) \\
& + E_1^\infty \cos\left(\frac{3\alpha}{2}\right) + 12U_1^\infty \sin^2\left(\frac{\alpha}{2}\right) \\
& + 6U_1^\infty \cos\alpha - 6U_1^\infty + 96\alpha\kappa \sin\left(\frac{\alpha}{2}\right) \\
& - 96\alpha_0\kappa \sin\left(\frac{\alpha}{2}\right)\Bigg)
\end{aligned} \tag{3}
$$

$$
U_{p,2} - U_2^\infty = \frac{E_2^\infty(\cos\alpha - 1)\sin\alpha}{-7\sin\left(\frac{\alpha}{2}\right) + \sin\left(\frac{3\alpha}{2}\right)} \tag{4}
$$

$$
\begin{aligned}
\frac{d\theta}{dt} = \frac{1}{4(\cos(2\alpha) - 17)}\Bigg(& -73E_1^\infty \sin\alpha + 18E_1^\infty \sin(2\alpha) - E_1^\infty \sin(3\alpha) \\
& - 63E_2^\infty \cos\alpha - 18E_2^\infty \cos(2\alpha) - E_2^\infty \cos(3\alpha) + 18E_2^\infty \\
& + 2304\alpha\kappa\cos\alpha - 192\alpha\kappa\cos(2\alpha) - 3648\alpha\kappa \\
& - 2304\alpha_0\kappa\cos\alpha + 192\alpha_0\kappa\cos(2\alpha) + 3648\alpha_0\kappa \\
& + 2\omega_3^\infty\cos(2\alpha) - 34\omega_3^\infty\Bigg)
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
\frac{d\alpha}{dt} = \frac{1}{2(\cos\alpha + 3)}\Bigg(& -12E_1^\infty \sin\alpha + E_1^\infty \sin(2\alpha) \\
& + 192\alpha\kappa\cos\alpha - 576\alpha\kappa \\
& - 192\alpha_0\kappa\cos\alpha + 576\alpha_0\kappa\Bigg)
\end{aligned} \tag{6}
$$

These expressions are much more complicated than Equations (18a)-(18d) in the original paper, so I will use Python to plug in certain values for background flow and stiffness κ, and then see if the differences between Equations (5)-(8) and the corresponding Equations (18a)-(18d) come out to be zero. The hinge under consideration is elastic, so the stiffness cannot be too high. I first set $\kappa = 0.1$ and then vary the background flow variables:

U_1^∞	U_2^∞	ω_3^∞	E_1^∞	E_2^∞	α	α_0
0.5	-0.2	0.5	1	1	$\pi/4$	$\pi/4$
1	1	1	0.5	0.2	$\pi/6$	$\pi/4$
2	3	0.01	4	6	$\pi/2$	$\pi/8$

κ	$\Delta(\mathrm{d}\theta/\mathrm{d}t)$	$\Delta(\mathrm{d}\alpha/\mathrm{d}t)$	$\Delta(U_{p,1} - U_1^\infty)$	$\Delta(U_{p,2} - U_2^\infty)$
0.1	80.124661%	0.000000%	0.000000%	0.000000%
0.1	82.695108%	0.000000%	0.000000%	0.000000%
0.1	322.366169%	0.000000%	0.000000%	0.000000%

Table 1: Some background flows and corresponding percentage error

All the other equations have zero error, but the error for the $\frac{d\theta}{dt}$ equation is very large. However, it is worth observing how this error varies with background vorticity ω_3^∞.

Below is how the error for the $\frac{d\theta}{dt}$ equation varies with ω_3^∞ (other variables fixed).

U_1^∞	U_2^∞	ω_3^∞	E_1^∞	E_2^∞	α	α_0	κ	$\Delta(\mathrm{d}\theta/\mathrm{d}t)$
1	1	2	0.5	0.2	$\pi/6$	$\pi/4$	0.1	46.363477%
1	1	5	0.5	0.2	$\pi/6$	$\pi/4$	0.1	20.001214%
1	1	10	0.5	0.2	$\pi/6$	$\pi/4$	0.1	10.269323%
1	1	90	0.5	0.2	$\pi/6$	$\pi/4$	0.1	1.168956%
1	1	1000	0.5	0.2	$\pi/6$	$\pi/4$	0.1	0.105500%

Table 2: Different values of ω_3^∞ and corresponding error for the $\frac{d\theta}{dt}$ equation

So, it can be seen that as $\omega_3^\infty \to \infty$, the error for the $\frac{d\theta}{dt}$ equation approaches zero.