# Extension of *Drift of elastic hinges in quasi-two-dimensional oscillating shear flows* to $n$-arm case*

Julia Liu

August 17, 2025

The original paper considers the case of a two-arm elastic hinge. In this section, we extend the paper's results to the case of an $n$-arm elastic hinge. We assume that each arm has length $l/2$, and that the equilibrium angle $\alpha_0$ and stiffness $\hat{\kappa}$ between each adjacent pair of arms are the same throughout the entire hinge.

The resistance coefficients for the two-arm case are given in the paper as

$$\begin{bmatrix} F_1 \\ F_2 \\ T \end{bmatrix} = \underbrace{\frac{1}{8}\begin{bmatrix} 3-\cos 2\beta & -\sin 2\beta & -\sin\beta \\ -\sin 2\beta & 3+\cos 2\beta & \cos\beta \\ -\sin\beta & \cos\beta & \frac{1}{3} \end{bmatrix}}_{\boldsymbol{R}(\beta)} \begin{bmatrix} U_1^\infty - U_{p,1} \\ U_2^\infty - U_{p,2} \\ \frac{1}{2}\omega_3^\infty - \Omega_{p,3} \end{bmatrix}$$

$$+ \underbrace{\frac{1}{32}\begin{bmatrix} (3\cos\beta - \cos 3\beta) & 4\sin^3\beta \\ -(3\sin\beta + \sin 3\beta) & 4\cos^3\beta \\ -\frac{4}{3}\sin 2\beta & \frac{4}{3}\cos 2\beta \end{bmatrix}}_{\boldsymbol{Q}(\beta)} \begin{bmatrix} E_1^\infty \\ E_2^\infty \end{bmatrix}, \tag{9}$$

For a $n$-arm hinge, we need to know two things to describe the entire hinge: the position of the 1st arm and the $n$-1 angular displacements $\alpha_i$ $(\alpha = 1, ..., n-1)$ between each pair of neighboring arms.

We define the absolute position of each arm using the following method. First, we assume that for any hinge, we can rotate it such that its first arm aligns with the $\boldsymbol{e}_1$ axis of the body-fixed frame. Therefore, the orientation angle $\beta_1$ of the first arm relative to the $\boldsymbol{e}_1$ axis is $0°$. The angular displacement between the first and second arm is $\alpha_1$. Therefore, the orientation of the second arm can then be expressed as the orientation of the first arm plus the angular displacement: $\beta_2 = \beta_1 + \alpha_1 = \alpha_1$. Similarly, the orientation of the third arm can be expressed using the second orientation angle and the angular displacement between the second and the third arm: $\beta_3 = \beta_2 + \alpha_2 = \beta_1 + \alpha_1 + \alpha_2 = \alpha_1 + \alpha_2$.

*Numbering of equations and sections is consistent with that in the paper

As a result, the orientation of the $k$-th arm ($k > 1$) can be expressed as

$$\beta_k = \sum_{i=1}^{k-1} \alpha_i, \quad k = 2, ..., n.$$

For each $k$th arm, the length and shape of each arm is assumed to be the same, which means that the bounds for the definite integral used to calculate the resistance coefficients do not change. Therefore, we can directly rewrite equation (9) as

$$\begin{bmatrix} F_{k,1} \\ F_{k,2} \\ T_k \end{bmatrix} = \underbrace{\frac{1}{8} \begin{bmatrix} 3 - \cos 2\beta_k & -\sin 2\beta_k & -\sin \beta_k \\ -\sin 2\beta_k & 3 + \cos 2\beta_k & \cos \beta_k \\ -\sin \beta_k & \cos \beta_k & \frac{1}{3} \end{bmatrix}}_{\boldsymbol{R}(\beta_k)} \begin{bmatrix} U_1^\infty - U_{p,1} \\ U_2^\infty - U_{p,2} \\ \frac{1}{2}\omega_3^\infty - \Omega_{p,k} \end{bmatrix}$$

$$+ \underbrace{\frac{1}{32} \begin{bmatrix} (3\cos \beta_k - \cos 3\beta_k) & 4\sin^3 \beta_k \\ -(3\sin \beta_k + \sin 3\beta_k) & 4\cos^3 \beta_k \\ -\frac{4}{3}\sin 2\beta_k & \frac{4}{3}\cos 2\beta_k \end{bmatrix}}_{\boldsymbol{Q}(\beta_k)} \begin{bmatrix} E_1^\infty \\ E_2^\infty \end{bmatrix}. \tag{9a}$$

Here, $F_{k,1}$ and $F_{k,2}$ represent the first and second components respectively of the force exerted on the $k$th arm by the fluid, $T_k$ represents the torque exerted on the $k$th arm by the fluid, and $\Omega_{p,k}$ represents the rotation velocity of the $k$th arm revolving around the $\boldsymbol{e}_3$ axis. That is, $\Omega_{p,k} = \frac{d}{dt}(\theta + \beta_k) = \frac{d\theta}{dt} + \sum_{i=1}^{k-1} \dot{\alpha}_i$.

Just like in the two-arm case, we require the $n$-arm hinge to be force-free and torque-free:

$$\sum_{k=1}^{n} \boldsymbol{F}_k = \boldsymbol{0},$$

$$\sum_{k=1}^{n} T_k = 0.$$

So applying these conditions to Equation (9a) gives

$$\sum_{k=1}^{n} \left[ \boldsymbol{R}(\beta_k) \cdot \begin{bmatrix} U_1^\infty - U_{p,1} \\ U_2^\infty - U_{p,2} \\ \frac{1}{2}\omega_3^\infty - \Omega_{p,k} \end{bmatrix} + \boldsymbol{Q}(\beta_k) \cdot \begin{bmatrix} E_1^\infty \\ E_2^\infty \end{bmatrix} \right] = \boldsymbol{0}. \tag{10}$$

Similarly, we also have to consider the internal balance of torques at each hinge point. At the $j$th hinge point, $j = 1, ..., n-1$ we have

$$T_j - T_{j+1} + 2\hat{\kappa}(\alpha_j - \alpha_0) = 0. \tag{11}$$

Our following goal is then to rewrite Equations (10) and (11) in the form of a linear matrix system $\boldsymbol{MX} = \boldsymbol{B}$, where $\boldsymbol{X}$ is a $(n+2) \times 1$ vector containing all the unknowns:

$$\boldsymbol{X} = \begin{bmatrix} U_{p,1} \\ U_{p,2} \\ \frac{d\theta}{dt} \\ \dot{\alpha}_1 \\ \vdots \\ \dot{\alpha}_{n-1} \end{bmatrix}.$$

Rearranging Equation (10), we get

$$\sum_{k=1}^{n} \boldsymbol{R}(\beta_k) \cdot \begin{bmatrix} U_{p,1} \\ U_{p,2} \\ \Omega_{p,k} \end{bmatrix} = \sum_{k=1}^{n} \left( \boldsymbol{R}(\beta_k) \cdot \begin{bmatrix} U_1^\infty \\ U_2^\infty \\ \frac{1}{2}\omega_3^\infty \end{bmatrix} + \boldsymbol{Q}(\beta_k) \cdot \begin{bmatrix} E_1^\infty \\ E_2^\infty \end{bmatrix} \right). \tag{12}$$

We define the right side as

$$\boldsymbol{B}_{total} = \sum_{k=1}^{n} \left( \boldsymbol{R}(\beta_k) \cdot \begin{bmatrix} U_1^\infty \\ U_2^\infty \\ \frac{1}{2}\omega_3^\infty \end{bmatrix} + \boldsymbol{Q}(\beta_k) \cdot \begin{bmatrix} E_1^\infty \\ E_2^\infty \end{bmatrix} \right). \tag{13}$$

This allows to rewrite each of the three rows of Equation (12). Using the first row as an example, we can rewrite (12) as

$$\sum_{k=1}^{n} (R_{11}(\beta_k) U_{p,1} + R_{12}(\beta_k) U_{p,2} + R_{13}(\beta_k) \Omega_{p,k}) = \boldsymbol{B}_{total,1}.$$

Now using the definition of $\Omega_{p,k}$, we can rewrite the last term on the left side of each row. Again using the first row as an example, the last term becomes

$$\sum_{k=1}^{n} R_{13}(\beta_k) \Omega_{p,k} = \sum_{k=1}^{n} R_{13}(\beta_k) (\frac{d\theta}{dt} + \sum_{i=1}^{k-1} \dot{\alpha}_i)$$

$$= \left( \sum_{k=1}^{n} R_{13}(\beta_k) \right) \frac{d\theta}{dt} + \sum_{k=1}^{n} \left( R_{13}(\beta_k) \sum_{i=1}^{k-1} \dot{\alpha}_i \right).$$

$$= \left( \sum_{k=1}^{n} R_{13}(\beta_k) \right) \frac{d\theta}{dt} + \sum_{i=1}^{n-1} \left( \sum_{k=i+1}^{n} R_{13}(\beta_k) \right) \dot{\alpha}_i.$$

This allows us to rewrite the first row (and subsequent rows) of Equation (12) as a linear combination of the unknown variables:

$$\left( \sum_{k=1}^{n} R_{11}(\beta_k) \right) U_{p,1} + \left( \sum_{k=1}^{n} R_{12}(\beta_k) \right) U_{p,2} + \left( \sum_{k=1}^{n} R_{13}(\beta_k) \right) \frac{d\theta}{dt}$$
$$+ \sum_{i=1}^{n-1} \left( \sum_{k=i+1}^{n} R_{13}(\beta_k) \right) \dot{\alpha}_i = \boldsymbol{B}_{total,1}. \tag{14}$$

Now, regarding Equation (11), we can write $T_j$ $(j = 1, ..., n-1)$ as

$$T_j = \boldsymbol{R}_3(\beta_j) \cdot \begin{bmatrix} U_1^\infty - U_{p,1} \\ U_2^\infty - U_{p,2} \\ \frac{1}{2}\omega_3^\infty - \Omega_{p,j} \end{bmatrix} + \boldsymbol{Q}_3(\beta_j) \cdot \begin{bmatrix} E_1^\infty \\ E_2^\infty \end{bmatrix},$$

so $T_{j+1} - T_j$ is

$$T_{j+1} - T_j = \boldsymbol{R}_3(\beta_{j+1}) \begin{bmatrix} U_1^\infty - U_{p,1} \\ U_2^\infty - U_{p,2} \\ \frac{1}{2}\omega_3^\infty - \Omega_{p,j+1} \end{bmatrix} - \boldsymbol{R}_3(\beta_j) \cdot \begin{bmatrix} U_1^\infty - U_{p,1} \\ U_2^\infty - U_{p,2} \\ \frac{1}{2}\omega_3^\infty - \Omega_{p,j} \end{bmatrix}$$
$$+(\boldsymbol{Q}_3(\beta_{j+1}) - \boldsymbol{Q}_3(\beta_j)) \cdot \begin{bmatrix} E_1^\infty \\ E_2^\infty \end{bmatrix} = 2\hat{\kappa}(\alpha_j - \alpha_0). \tag{15}$$

Similar to the how we derived Equation (13), we want to rewrite this equation in the form of *linear combination of unknown variables = some combination of all the known background flow variables*. This means that we need to expand

$$\boldsymbol{R}_3(\beta_{j+1}) \cdot \begin{bmatrix} U_1^\infty - U_{p,1} \\ U_2^\infty - U_{p,2} \\ \frac{1}{2}\omega_3^\infty - \Omega_{p,j+1} \end{bmatrix} - \boldsymbol{R}_3(\beta_j) \cdot \begin{bmatrix} U_1^\infty - U_{p,1} \\ U_2^\infty - U_{p,2} \\ \frac{1}{2}\omega_3^\infty - \Omega_{p,j} \end{bmatrix}$$

into a linear combination. Using the same process as the derivation for Equation (13), the expanded form of Equation (15) is

$$\begin{aligned}
&[R_{31}(\beta_{j+1}) - R_{31}(\beta_j)]U_{p,1} + [R_{32}(\beta_{j+1}) - R_{32}(\beta_j)]U_{p,2} \\
&+ [R_{33}(\beta_{j+1}) - R_{33}(\beta_j)]\frac{d\theta}{dt} + [R_{33}(\beta_{j+1}) - R_{33}(\beta_j)]\dot{\alpha}_1 + [R_{33}(\beta_{j+1}) - R_{33}(\beta_j)\dot{\alpha}_2 \\
&+ \cdots + [R_{33}(\beta_{j+1}) - R_{33}(\beta_j)]\dot{\alpha}_{j-1} + [R_{33}(\beta_{j+1})]\dot{\alpha}_j \\
&= [R_{31}(\beta_{j+1}) - R_{31}(\beta_j)]U_1^\infty + [R_{32}(\beta_{j+1}) - R_{32}(\beta_j)]U_2^\infty \\
&+ [R_{33}(\beta_{j+1}) - R_{33}(\beta_j)]\frac{1}{2}\omega_3^\infty \\
&+ [Q_{31}(\beta_{j+1}) - Q_{31}(\beta_j)]E_1^\infty + [Q_{32}(\beta_{j+1}) - Q_{32}(\beta_j)]E_2^\infty - 2\hat{\kappa}(\alpha_j - \alpha_0).
\end{aligned} \tag{16}$$

Finally, combining the information from Equations (13), (14), and (16), we can write out the tensors $\boldsymbol{M}$ and $\boldsymbol{B}$ from $\boldsymbol{MX} = \boldsymbol{B}$ to get a complete linear system of equations.

$\boldsymbol{B}$ is a $(n+2) \times 1$ vector:

$$\boldsymbol{B} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_{j+3} \end{bmatrix},$$

where

1. **First 3 rows from Equation (13)**:

$$B_1 = \sum_{k=1}^{n} \left( R_{11}(\beta_k)U_1^\infty + R_{12}(\beta_k)U_2^\infty + R_{13}(\beta_k)\frac{1}{2}\omega_3^\infty + Q_{11}(\beta_k)E_1^\infty + Q_{12}(\beta_k)E_2^\infty \right)$$
$$B_2 = \sum_{k=1}^{n} \left( R_{21}(\beta_k)U_1^\infty + R_{22}(\beta_k)U_2^\infty + R_{23}(\beta_k)\frac{1}{2}\omega_3^\infty + Q_{21}(\beta_k)E_1^\infty + Q_{22}(\beta_k)E_2^\infty \right)$$
$$B_3 = \sum_{k=1}^{n} \left( R_{31}(\beta_k)U_1^\infty + R_{32}(\beta_k)U_2^\infty + R_{33}(\beta_k)\frac{1}{2}\omega_3^\infty + Q_{31}(\beta_k)E_1^\infty + Q_{32}(\beta_k)E_2^\infty \right).$$

2. **Rows 4 to $n+2$ from Equation (16),** $j = 1, \ldots, n-1$:

$$\begin{aligned} B_{j+3} = &\ [R_{31}(\beta_{j+1}) - R_{31}(\beta_j)]\, U_1^\infty \\ &+ [R_{32}(\beta_{j+1}) - R_{32}(\beta_j)]\, U_2^\infty \\ &+ [R_{33}(\beta_{j+1}) - R_{33}(\beta_j)]\, \frac{1}{2}\omega_3^\infty \\ &+ [Q_{31}(\beta_{j+1}) - Q_{31}(\beta_j)]\, E_1^\infty \\ &+ [Q_{32}(\beta_{j+1}) - Q_{32}(\beta_j)]\, E_2^\infty \\ &- 2\kappa(\alpha_j - \alpha_0). \end{aligned}$$

$\boldsymbol{M}$ is a $(n+2) \times (n+2)$ matrix:

1. **First 3 rows from Equation (12)**:

**First row**:

$$M_{1,1} = \sum_{k=1}^{n} R_{11}(\beta_k) \qquad M_{1,2} = \sum_{k=1}^{n} R_{12}(\beta_k)$$
$$M_{1,3} = \sum_{k=1}^{n} R_{13}(\beta_k) \qquad M_{1,3+i} = \sum_{k=i+1}^{n} R_{13}(\beta_k) \quad \text{for } i = 1, \ldots, n-1,$$

**Second row**:

$$M_{2,1} = \sum_{k=1}^{n} R_{21}(\beta_k) \qquad M_{2,2} = \sum_{k=1}^{n} R_{22}(\beta_k)$$
$$M_{2,3} = \sum_{k=1}^{n} R_{23}(\beta_k) \qquad M_{2,3+i} = \sum_{k=i+1}^{n} R_{23}(\beta_k) \quad \text{for } i = 1, \ldots, n-1,$$

**Third row**:

$$M_{3,1} = \sum_{k=1}^{n} R_{31}(\beta_k) \qquad M_{3,2} = \sum_{k=1}^{n} R_{32}(\beta_k)$$
$$M_{3,3} = \sum_{k=1}^{n} R_{33}(\beta_k) \qquad M_{3,3+i} = \sum_{k=i+1}^{n} R_{33}(\beta_k) \quad \text{for } i = 1, \ldots, n-1.$$

2. **Rows 4 to $n + 2$ from Equation (16),** $j = 1, \ldots, n-1$:

$$\begin{aligned} M_{j+3,1} &= R_{31}(\beta_{j+1}) - R_{31}(\beta_j) \\ M_{j+3,2} &= R_{32}(\beta_{j+1}) - R_{32}(\beta_j) \\ M_{j+3,3} &= R_{33}(\beta_{j+1}) - R_{33}(\beta_j). \end{aligned}$$

For $l = 4$ to $n + 2$:

$$M_{j+3,l} = \begin{cases} R_{33}(\beta_{j+1}) - R_{33}(\beta_j), & \text{if } l - 3 < j \\ R_{33}(\beta_{j+1}), & \text{if } l - 3 = j. \end{cases}$$

This information allows us to express the dynamical equations embedded in the vector $\boldsymbol{X}$ implicitly.