Derivations of the main equations in *Drift of elastic hinges in quasi-two-dimensional oscillating shear flows*[*]

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II. MOTION OF ELASTIC HINGES IN LINEAR FLOWS

B. Mobility of slender hinges

The coefficients of the resistance tensors in Equation (9),

$$\begin{bmatrix} F_1 \\ F_2 \\ T \end{bmatrix} = \underbrace{\frac{1}{8}\begin{bmatrix} 3-\cos 2\beta & -\sin 2\beta & -\sin\beta \\ -\sin 2\beta & 3+\cos 2\beta & \cos\beta \\ -\sin\beta & \cos\beta & \frac{1}{3} \end{bmatrix}}_{\boldsymbol{R}(\beta)} \begin{bmatrix} U_1^\infty - U_{p,1} \\ U_2^\infty - U_{p,2} \\ \frac{1}{2}\omega_3^\infty - \mathit{\Omega}_{p,3} \end{bmatrix}$$

$$+ \underbrace{\frac{1}{32}\begin{bmatrix} (3\cos\beta - \cos 3\beta) & 4\sin^3\beta \\ -(3\sin\beta + \sin 3\beta) & 4\cos^3\beta \\ -\frac{4}{3}\sin 2\beta & \frac{4}{3}\cos 2\beta \end{bmatrix}}_{\boldsymbol{Q}(\beta)} \begin{bmatrix} E_1^\infty \\ E_2^\infty \end{bmatrix}, \tag{9}$$

are different than of those in Equations (3.5) and (3.6) of the previous paper. This is due to differences in the geometries of the particles considered and the theories used to calculate the resistance coefficients. In this paper, the shape in consideration is a single straight rod with length $l/2$ titled at an angle β relative to the $\boldsymbol{e}_1$ axis of the body frame. In the previous paper, the shape considered was a bent rod made up of two segments with length $l/2$ and an angle α between them.

[*] Numbering of equations and sections is consistent with that in the paper

C. Dynamics of hinges in linear flows

1. Rigid hinge

In general, a symmetric hinge's configuration can be described by the state vector $\boldsymbol{X} = (x'_p, y'_p, \theta, \alpha)$. The hinge is required to be force-free and torque-free, meaning that the force and the torque on both arms of the hinge must sum to zero:

$$\boldsymbol{F}_1 + \boldsymbol{F}_2 = \boldsymbol{0} \tag{11}$$

$$T_1 + T_2 = 0. \tag{12}$$

α does not vary with time for a rigid hinge, so three dynamical equations are sufficient to describe the dynamics of a rigid hinge.

We define the total resistance tensors $\boldsymbol{R}_0$ and $\boldsymbol{Q}_0$ as the sum of the resistance on the top and bottom arms: $\boldsymbol{R}_0 = \boldsymbol{R}(-\frac{\alpha}{2}) + \boldsymbol{R}(\frac{\alpha}{2})$ and $\boldsymbol{Q}_0 = \boldsymbol{Q}(-\frac{\alpha}{2}) + \boldsymbol{Q}(\frac{\alpha}{2})$ $(\beta = \pm\frac{\alpha}{2})$. Substituting these definitions along with Equations (11) and (12) into (9) gives

$$-\boldsymbol{R}_0 \begin{bmatrix} U_{p,1} - U_1^\infty \\ U_{p,2} - U_2^\infty \\ \Omega_3 - \frac{1}{2}\omega_3^\infty \end{bmatrix} + \boldsymbol{Q}_0 \begin{bmatrix} E_1^\infty \\ E_2^\infty \end{bmatrix} = \boldsymbol{0}$$

$$\Rightarrow -\boldsymbol{R}_0 \begin{bmatrix} U_{p,1} - U_1^\infty \\ U_{p,2} - U_2^\infty \\ \Omega_3 - \frac{1}{2}\omega_3^\infty \end{bmatrix} = \boldsymbol{Q}_0 \begin{bmatrix} E_1^\infty \\ E_2^\infty \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} U_{p,1} - U_1^\infty \\ U_{p,2} - U_2^\infty \\ \Omega_3 - \frac{1}{2}\omega_3^\infty \end{bmatrix} = \boldsymbol{R}_0^{-1} \cdot \boldsymbol{Q}_0 \cdot \begin{bmatrix} E_1^\infty \\ E_2^\infty \end{bmatrix} = \begin{bmatrix} \hat{g}_{11} & \hat{g}_{12} \\ \hat{g}_{21} & \hat{g}_{22} \\ \hat{g}_{31} & \hat{g}_{32} \end{bmatrix} \cdot \begin{bmatrix} E_1^\infty \\ E_2^\infty \end{bmatrix}. \tag{13}$$

Calculating $\boldsymbol{R}_0^{-1} \cdot \boldsymbol{Q}_0$ using the coefficients from Equation (9) then gives the mobility coefficients for a rigid hinge,

$$\begin{bmatrix} \hat{g}_{11} & \hat{g}_{12} \\ \hat{g}_{21} & \hat{g}_{22} \\ \hat{g}_{31} & \hat{g}_{32} \end{bmatrix} = \frac{1}{2(\cos\alpha - 3)} \begin{bmatrix} \cos(\alpha/2)(\cos\alpha - 2) & 0 \\ 0 & -2\sin(\alpha/2)\sin\alpha \\ 0 & 12 - (\cos\alpha + 3)^2 \end{bmatrix}. \tag{14}$$

2. Elastic hinge

For an elastic hinge, all four of these variables vary with time, so we need four dynamical equations to describe their evolution. In regards to $\alpha(t)$, we consider the internal balance of torques acting at the hinge point O, which is expressed below,

$$T_1 - T_2 + 2\hat{\kappa}(\alpha - \alpha_0) = 0. \tag{15}$$

Here, $\hat{\kappa}$ is the dimensionless stiffness and α_0 is the equilibrium angle.

For an elastic hinge, the translational velocities of both segments are equal and can be expressed with a single vector $\boldsymbol{U}_p$, but the rotational velocities for each segment are different. The top segment rotates with velocity $\Omega_{3,p} = \frac{d\theta}{dt} + \frac{1}{2}\frac{d\alpha}{dt}$ and the bottom segment rotates with velocity $\Omega_{3,p} = \frac{d\theta}{dt} - \frac{1}{2}\frac{d\alpha}{dt}$.

Combining Equations (11) and (12) with (9) gives

$$-\boldsymbol{R}(\pm\frac{\alpha}{2})\begin{bmatrix}\boldsymbol{U}_p-\boldsymbol{U}^\infty\\ \Omega_3^{(\pm)}+\frac{1}{2}\omega_3\end{bmatrix}+\boldsymbol{Q}(\pm\frac{\alpha}{2})\begin{bmatrix}E_1^\infty\\ E_2^\infty\end{bmatrix}=\boldsymbol{0}$$

$$\Rightarrow -[\boldsymbol{R}(\frac{\alpha}{2})\begin{bmatrix}\boldsymbol{U}_p-\boldsymbol{U}^\infty\\ \Omega_3^{(+)}+\frac{1}{2}\omega_3\end{bmatrix}+\boldsymbol{R}(-\frac{\alpha}{2})\begin{bmatrix}\boldsymbol{U}_p-\boldsymbol{U}^\infty\\ \Omega_3^{(-)}+\frac{1}{2}\omega_3\end{bmatrix}]+\boldsymbol{Q}_0\begin{bmatrix}E_1^\infty\\ E_2^\infty\end{bmatrix}=\boldsymbol{0}$$

Combining the definitions for the rotational velocity component, we get

$$-[\boldsymbol{R}\left(\frac{\alpha}{2}\right)\begin{bmatrix}\boldsymbol{U}_p-\boldsymbol{U}^\infty\\ \frac{d\theta}{dt}+\frac{1}{2}\frac{d\alpha}{dt}+\frac{1}{2}\omega_3^\infty\end{bmatrix}+\boldsymbol{R}\left(-\frac{\alpha}{2}\right)\begin{bmatrix}\boldsymbol{U}_p-\boldsymbol{U}^\infty\\ \frac{d\theta}{dt}-\frac{1}{2}\frac{d\alpha}{dt}+\frac{1}{2}\omega_3^\infty\end{bmatrix}]+\boldsymbol{Q}_0\begin{bmatrix}E_1^\infty\\ E_2^\infty\end{bmatrix}=\boldsymbol{0}.$$

This equation can be split into two parts: one involving $\boldsymbol{U}_p-\boldsymbol{U}^\infty$ and $\frac{d\theta}{dt}$, and one involving $\frac{d\alpha}{dt}$:

$$-\left[\boldsymbol{R}\left(\tfrac{\alpha}{2}\right)+\boldsymbol{R}\left(-\tfrac{\alpha}{2}\right)\right]\begin{bmatrix}\boldsymbol{U}_p-\boldsymbol{U}^\infty\\ \frac{d\theta}{dt}+\frac{1}{2}\omega_3^\infty\end{bmatrix}+\boldsymbol{Q}_0\begin{bmatrix}E_1^\infty\\ E_2^\infty\end{bmatrix}-\left[\boldsymbol{R}\left(\tfrac{\alpha}{2}\right)-\boldsymbol{R}\left(-\tfrac{\alpha}{2}\right)\right]\begin{bmatrix}\boldsymbol{0}\\ \frac{1}{2}\frac{d\alpha}{dt}\end{bmatrix}=\boldsymbol{0}.$$

Using the definition of $\boldsymbol{R}_0$, we isolate $\boldsymbol{U}_p-\boldsymbol{U}^\infty$ and $\frac{d\theta}{dt}-\frac{1}{2}\omega_3^\infty$ to arrive at equation (16),

$$\begin{bmatrix}\boldsymbol{U}_p-\boldsymbol{U}^\infty\\ \Omega_3-\frac{1}{2}\omega_3^\infty\end{bmatrix}=\boldsymbol{R}_0^{-1}\left[\boldsymbol{Q}_0\begin{bmatrix}E_1^\infty\\ E_2^\infty\end{bmatrix}-\left(\boldsymbol{R}\left(\tfrac{\alpha}{2}\right)-\boldsymbol{R}\left(-\tfrac{\alpha}{2}\right)\right)\begin{bmatrix}\boldsymbol{0}\\ \frac{1}{2}\frac{d\alpha}{dt}\end{bmatrix}\right]. \tag{16}$$

Equation (16) gives three dynamical equations. To arrive at the fourth, we combine Equations (15) and (9). First, we use (9) to express the torque acting on the top and bottom arms. For the upper arm ($\beta=-\alpha/2$),

$$T_1=\begin{bmatrix}R_{31}\left(-\frac{\alpha}{2}\right) & R_{32}\left(-\frac{\alpha}{2}\right) & R_{33}\left(-\frac{\alpha}{2}\right)\end{bmatrix}\begin{bmatrix}U_1^\infty-U_{p,1}\\ U_2^\infty-U_{p,2}\\ \frac{1}{2}\omega_3^\infty-\left(\frac{d\theta}{dt}-\frac{1}{2}\frac{d\alpha}{dt}\right)\end{bmatrix}$$

$$+\begin{bmatrix}Q_{31}\left(-\frac{\alpha}{2}\right) & Q_{32}\left(-\frac{\alpha}{2}\right)\end{bmatrix}\begin{bmatrix}E_1^\infty\\ E_2^\infty\end{bmatrix}.$$

For the lower arm ($\beta=\alpha/2$),

$$T_2=\begin{bmatrix}R_{31}\left(\frac{\alpha}{2}\right) & R_{32}\left(\frac{\alpha}{2}\right) & R_{33}\left(\frac{\alpha}{2}\right)\end{bmatrix}\begin{bmatrix}U_1^\infty-U_{p,1}\\ U_2^\infty-U_{p,2}\\ \frac{1}{2}\omega_3^\infty-\left(\frac{d\theta}{dt}+\frac{1}{2}\frac{d\alpha}{dt}\right)\end{bmatrix}$$

$$+\begin{bmatrix}Q_{31}\left(\frac{\alpha}{2}\right) & Q_{32}\left(\frac{\alpha}{2}\right)\end{bmatrix}\begin{bmatrix}E_1^\infty\\ E_2^\infty\end{bmatrix}.$$

Subtracting T_2 from T_1 gives

$$T_1-T_2$$

$$=\begin{bmatrix}R_{31}(-\frac{\alpha}{2})-R_{31}(\frac{\alpha}{2}) & R_{32}(-\frac{\alpha}{2})-R_{32}(\frac{\alpha}{2}) & R_{33}(-\frac{\alpha}{2})-R_{33}(\frac{\alpha}{2})\end{bmatrix}\begin{bmatrix}U_1^\infty-U_{p,1}\\ U_2^\infty-U_{p,2}\\ \frac{1}{2}\omega_3^\infty-\frac{d\theta}{dt}\end{bmatrix}$$

$$-\frac{1}{2}\frac{d\alpha}{dt}\left(R_{33}(-\frac{\alpha}{2})+R_{33}(\frac{\alpha}{2})\right)$$
$$+\begin{bmatrix} Q_{31}(-\frac{\alpha}{2})-Q_{31}(\frac{\alpha}{2}) & Q_{32}(-\frac{\alpha}{2})-Q_{32}(\frac{\alpha}{2}) \end{bmatrix}\begin{bmatrix} E_1^\infty \\ E_2^\infty \end{bmatrix}.$$

We substitute this expression back into (15),

$$\begin{bmatrix} R_{31}(-\frac{\alpha}{2})-R_{31}(\frac{\alpha}{2}) & R_{32}(-\frac{\alpha}{2})-R_{32}(\frac{\alpha}{2}) & R_{33}(-\frac{\alpha}{2})-R_{33}(\frac{\alpha}{2}) \end{bmatrix}\begin{bmatrix} \boldsymbol{U}^\infty-\boldsymbol{U}_p \\ \frac{1}{2}\omega_3^\infty-\frac{d\theta}{dt} \end{bmatrix}$$
$$-\frac{1}{2}\frac{d\alpha}{dt}\left(R_{33}(-\frac{\alpha}{2})+R_{33}(\frac{\alpha}{2})\right)$$
$$+\begin{bmatrix} Q_{31}(-\frac{\alpha}{2})-Q_{31}(\frac{\alpha}{2}) & Q_{32}(-\frac{\alpha}{2})-Q_{32}(\frac{\alpha}{2}) \end{bmatrix}\begin{bmatrix} E_1^\infty \\ E_2^\infty \end{bmatrix}$$
$$=-2k(\alpha-\alpha_0).$$

Rewriting in matrix form gives Equation (17),

$$2\hat{\kappa}(\alpha-\alpha_0)=\boldsymbol{e}_3\cdot\left\{\left[\boldsymbol{R}\left(\frac{\alpha}{2}\right)-\boldsymbol{R}\left(-\frac{\alpha}{2}\right)\right]\cdot\begin{bmatrix} \boldsymbol{U}^\infty-\boldsymbol{U}_p \\ \frac{1}{2}\omega_3^\infty-\frac{d\theta}{dt} \end{bmatrix}\right.$$
$$\left.+\left[\boldsymbol{Q}\left(\frac{\alpha}{2}\right)-\boldsymbol{Q}\left(-\frac{\alpha}{2}\right)\right]\cdot\begin{bmatrix} E_1^\infty \\ E_2^\infty \end{bmatrix}-\left[\boldsymbol{R}\left(\frac{\alpha}{2}\right)+\boldsymbol{R}\left(-\frac{\alpha}{2}\right)\right]\cdot\begin{bmatrix} \boldsymbol{0} \\ \frac{1}{2}\frac{d\alpha}{dt} \end{bmatrix}\right\}, \qquad (17)$$

where $\boldsymbol{e}_3\cdot$ represents taking the third row (torque component) of the matrix expressions.

Now we can write out the four dynamical equations for an elastic hinge. First we use Equation (9) to calculate the specific expressions for $\boldsymbol{R}_0$ and $\boldsymbol{Q}_0$, arriving at

$$\boldsymbol{R}_0=\boldsymbol{R}\left(\frac{\alpha}{2}\right)+\boldsymbol{R}\left(-\frac{\alpha}{2}\right)=\frac{1}{4}\begin{bmatrix} 3-\cos\alpha & 0 & 0 \\ 0 & 3+\cos\alpha & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix},$$

and

$$\boldsymbol{Q}_0=\boldsymbol{Q}\left(\frac{\alpha}{2}\right)+\boldsymbol{Q}\left(-\frac{\alpha}{2}\right)=\frac{1}{16}\begin{bmatrix} 3\cos(\alpha/2)-\cos(3\alpha/2) & 0 \\ 0 & 4\cos^3(\alpha/2) \\ 0 & \frac{4}{3}\cos\alpha \end{bmatrix}.$$

The expression for $\boldsymbol{R}_0^{-1}\boldsymbol{Q}_0$ is

$$\boldsymbol{R}_0^{-1}\boldsymbol{Q}_0=\frac{1}{4}\begin{bmatrix} \frac{3\cos(\alpha/2)-\cos(3\alpha/2)}{3-\cos\alpha} & 0 \\ 0 & \frac{4\cos^3(\alpha/2)}{3+\cos\alpha} \\ 0 & \frac{16}{3}\cos\alpha \end{bmatrix}.$$

The components of Equation (16) can be expanded to get

$$U_{p,1}-U_1^\infty=\frac{3\cos(\alpha/2)-\cos(3\alpha/2)}{4(3-\cos\alpha)}E_1^\infty+\frac{\sin(\alpha/2)}{2(3-\cos\alpha)}\frac{d\alpha}{dt},$$

$$U_{p,2} - U_2^\infty = \frac{\cos^3(\alpha/2)}{3+\cos\alpha} E_2^\infty,$$

and

$$\frac{d\theta}{dt} - \frac{1}{2}\omega_3^\infty = \frac{4}{3}\cos\alpha \cdot E_2^\infty.$$

Similarly, Equation (17) can be expanded as

$$2\hat{\kappa}(\alpha-\alpha_0) = -\frac{\sin(\alpha/2)}{4}(U_1^\infty - U_{p,1}) + \frac{\cos(\alpha/2)}{4}\left(\frac{1}{2}\omega_3^\infty - \frac{d\theta}{dt}\right) - \frac{\sin\alpha}{3}E_1^\infty - \frac{1}{6}\frac{d\alpha}{dt}.$$

We then solve these two equations simultaneously to arrive at Equations (18a)-(18d),

$$\frac{d\theta}{dt} = \frac{3(E_2^\infty - \omega_3^\infty) - [E_2^\infty \cos\alpha + 6E_2^\infty - \omega_3^\infty]\cos\alpha}{2(\cos\alpha - 3)}, \tag{18a}$$

$$\frac{d\alpha}{dt} = \frac{E_1^\infty \sin\alpha(\cos\alpha - 6) - \kappa(\cos\alpha - 3)(\alpha_0 - \alpha)}{\cos\alpha + 3}, \tag{18b}$$

$$U_{p,1} - U_1^\infty = \frac{E_1^\infty[\cos\left(\frac{3\alpha}{2}\right) - 3\cos\left(\frac{\alpha}{2}\right)] - 2\sin\left(\frac{\alpha}{2}\right)\frac{d\alpha}{dt}}{4(\cos\alpha - 3)}, \tag{18c}$$

$$U_{p,2} - U_2^\infty = -\frac{E_2^\infty \sin\left(\frac{\alpha}{2}\right)\sin\alpha}{\cos\alpha - 3}. \tag{18d}$$

Here, $\kappa = 96\hat{\kappa}$ is another non-dimensional stiffness constant that is included for algebraic convenience.

III. ELASTIC HINGES IN A QUIESCENT FLUID

When there is no background flow, $\boldsymbol{u}^\infty = \boldsymbol{0}$, the non-dimensional parameter κ become $\kappa = 1$, and setting $\theta = 0$ simplifies the dynamical equations to

$$\frac{d\alpha}{dt} = (\alpha_0 - \alpha)\frac{3-\cos\alpha}{3+\cos\alpha}, \tag{19a}$$

$$\frac{dx}{dt} = \frac{1}{2}\left(\frac{\sin\left(\frac{\alpha}{2}\right)}{3-\cos\alpha}\frac{d\alpha}{dt}\right), \tag{19b}$$

$$\frac{dy}{dt} = 0. \tag{19c}$$

The paper examines what happens when the initial hinge angle is set to some angle that varies slightly from the equilibrium angle α_0. The authors consider a hinge with $\alpha_0 = \pi/2$ and $\alpha(t=0) = \alpha_0 \pm \pi/4$. The case with $\alpha(t=0) > \alpha_0$ is initially open and the case with $\alpha(t=0) < \alpha_0$. In both cases, the hinge angle eventually relaxes to the equilibrium, but the dynamics of relaxation are not symmetric, with the initially open hinge dissipating its stored elastic energy, $E_k = 1/2(\alpha - \alpha_0)$, quicker than the closed hinge.

The authors examine this symmetry breaking by expanding Equation (19a) in the limit of small initial deflection $\Delta\alpha$, where $\alpha(t=0)=\alpha_0+\Delta\alpha$. We define $\alpha(t)=\alpha_0+A_1(t)\Delta\alpha+A_2(t)(\Delta\alpha)^2+O[(\Delta\alpha^3)]$, substitute into (19a), and collect terms of like order to find

$$
\begin{aligned}
O(\Delta\alpha): \quad & \frac{dA_1}{dt} = A_1\left(\frac{\cos\alpha_0-3}{\cos\alpha_0+3}\right), \\
O[(\Delta\alpha)^2]: \quad & \frac{dA_2}{dt} = A_2\left(\frac{\cos\alpha_0-3}{\cos\alpha_0+3}\right) - \frac{6A_1^2\sin\alpha_0}{(\cos\alpha_0+3)^2}.
\end{aligned}
$$

For convenience, we also define the quantity $\hat{\beta}=\frac{3-\cos\alpha_0}{3+\cos\alpha_0}$ (with a range of 1/2 to 2). The initial conditions for (20a) and (20b) are $A_1(0)=1$ and $A_2(0)=0$. Solving for A_1 and A_2 gives

$$A_1(t)=e^{-\hat{\beta}t}, \tag{21a}$$

$$A_2(t)=-\frac{6e^{-2\hat{\beta}t}(e^{\hat{\beta}t}-1)\sin\alpha_0}{\hat{\beta}(\cos\alpha_0+3)^2}. \tag{21b}$$

We define $\alpha^+(t)$ as the solution for $\Delta\alpha>0$ (initially open) and $\Delta\alpha>0$ as the solution for $\Delta\alpha<0$ (intially closed). At order $O(\Delta\alpha)$, $\alpha^+-\alpha_0=\Delta\alpha A_1(t)$ and $\alpha^--\alpha_0=-\Delta\alpha A_1(t)$, so $\alpha^+-\alpha_0=-(\alpha^--\alpha_0)$. Therefore the solution is is symmetric with respect to $\Delta\alpha$. However at order $O(\Delta\alpha)^2$, this condition for symmetry is no longer satisfied.

Since $\sin\alpha_0>0$ ($\alpha_0\in(0,\pi)$), $\hat{\beta}>0$, and the denominator of $A_2(t)$ is positive,

$$A_2(t)=-\frac{6\sin\alpha_0(1-e^{-\hat{\beta}t})e^{-\hat{\beta}t}}{\hat{\beta}(\cos\alpha_0+3)^2}<0 \quad \forall t>0.$$

For initial conditions $\Delta\alpha>0$ and $\Delta\alpha<0$:

$$
\begin{aligned}
\alpha^+(t)-\alpha_0 &= |\Delta\alpha|e^{-\hat{\beta}t} - \frac{6\sin\alpha_0}{\hat{\beta}(\cos\alpha_0+3)^2}(1-e^{-\hat{\beta}t})e^{-\hat{\beta}t}(\Delta\alpha)^2, \\
\alpha^-(t)-\alpha_0 &= -|\Delta\alpha|e^{-\hat{\beta}t} - \frac{6\sin\alpha_0}{\hat{\beta}(\cos\alpha_0+3)^2}(1-e^{-\hat{\beta}t})e^{-\hat{\beta}t}(\Delta\alpha)^2.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
|\alpha^+(t)-\alpha_0| &= e^{-\hat{\beta}t}\left[|\Delta\alpha| - \frac{6\sin\alpha_0}{\hat{\beta}(\cos\alpha_0+3)^2}(1-e^{-\hat{\beta}t})|\Delta\alpha|^2\right], \\
|\alpha^-(t)-\alpha_0| &= e^{-\hat{\beta}t}\left[|\Delta\alpha| + \frac{6\sin\alpha_0}{\hat{\beta}(\cos\alpha_0+3)^2}(1-e^{-\hat{\beta}t})|\Delta\alpha|^2\right],
\end{aligned}
$$

which gives

$$|\alpha^+(t)-\alpha_0|<|\alpha^-(t)-\alpha_0| \quad \forall t>0.$$

Therefore the symmetry condition $\alpha^+-\alpha_0=-(\alpha^--\alpha_0)$ is clearly not satisfied.

The authors then explore the origins behind a drifting motion. The series expansion solution for $\alpha(t)$ allow us to write Equation (19b) as

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{2}\left(\frac{\sin\frac{\alpha_0}{2}}{3-\cos\alpha_0}\frac{dA_1}{dt}\right)\Delta\alpha \\ &+ \frac{1}{2}\left(\frac{\cos^3\frac{\alpha_0}{2}}{2(\cos\alpha_0-3)^2}\frac{dA_1^2}{dt} + \frac{\sin\frac{\alpha_0}{2}}{3-\cos\alpha_0}\frac{dA_2}{dt}\right)(\Delta\alpha)^2 + O[(\Delta\alpha)^3]. \quad (22)\end{aligned}$$

which can be integrated to directly given an expression for translation $x(t)$,

$$\begin{aligned}x(t) &= \frac{1}{2}\left(\frac{\sin\frac{\alpha_0}{2}}{3-\cos\alpha_0}[A_1(t)-1]\right)\Delta\alpha \\ &+ \frac{1}{2}\left(\frac{\cos^3\frac{\alpha_0}{2}}{2(\cos\alpha_0-3)^2}[A_1(t)^2-1] + \frac{\sin\frac{\alpha_0}{2}}{3-\cos\alpha_0}A_2(t)\right)(\Delta\alpha)^2 + O[(\Delta\alpha)^3]. \quad (23)\end{aligned}$$

Again, x^+ is defined as the solution for $\Delta\alpha > 0$ and x^- as the solution for $\Delta\alpha < 0$. At $O(\Delta\alpha)$, $x^+ = -x^-$ for all t. At $O(\Delta\alpha^2)$,

- The term $(\Delta\alpha)^2$ is always positive regardless of the sign of $\Delta\alpha$.
- The coefficient contains $A_2(t) < 0$ (from (21b)), making the net translation

$$x_2^+(t) = x_2^-(t) < 0.$$

- This leads to $|x^+(t)| > |x^-(t)|$ because

$$x^+(t) = (\text{negative linear}) + (\text{negative quadratic})$$

$$x^-(t) = (\text{positive linear}) + (\text{negative quadratic}).$$

If we take the steady-state limit ($t \to \infty$) of (23), we find,

- All transient terms vanish ($A_1(t), A_2(t) \to 0$).
- The remaining terms are

$$\lim_{t\to\infty} x(t) = -\frac{1}{2}\left(\frac{\sin\frac{\alpha_0}{2}}{3-\cos\alpha_0}\right)\Delta\alpha - \frac{1}{2}\left(\frac{\cos^3\frac{\alpha_0}{2}}{2(\cos\alpha_0-3)^2}\right)(\Delta\alpha)^2, \quad (24)$$

where terms of order $(\Delta\alpha)^3$ and above are omitted.

For $\Delta\alpha > 0$,

$$\lim_{t\to\infty} x^+(t) = -\frac{C_1}{2}\Delta\alpha - \frac{C_2}{4}(\Delta\alpha)^2,$$

and $\Delta\alpha < 0$,

$$\lim_{t\to\infty} x^-(t) = \frac{C_1}{2}\Delta\alpha - \frac{C_2}{4}(\Delta\alpha)^2,$$

where,

$$C_1 = \frac{\sin\frac{\alpha_0}{2}}{3-\cos\alpha_0}, \quad C_2 = \frac{\cos^3\frac{\alpha_0}{2}}{(\cos\alpha_0 - 3)^2}.$$

Taking absolute values and subtracting, we arrive at Equation (25),

$$|x^+| = \frac{C_1|\Delta\alpha|}{2} + \frac{C_2(\Delta\alpha)^2}{4}, \quad |x^-| = \frac{C_1|\Delta\alpha|}{2} - \frac{C_2(\Delta\alpha)^2}{4}$$

$$\Rightarrow \lim_{t\to\infty}(|x^+| - |x^-|) = \frac{C_2}{2}(\Delta\alpha)^2 = \frac{1}{2}\frac{\cos^3\frac{\alpha_0}{2}}{(\cos\alpha_0 - 3)^2}(\Delta\alpha)^2. \tag{25}$$

This difference in distance traveled is called the drift.

IV. HINGES IN TIME-VARYING SHEAR FLOWS

A. Hinge equations of motion in shear flows

1. Rigid hinge

We now derive the evolution equation for θ directly from Equation (13). First, a simple shear flow is defined as $\boldsymbol{u}^\infty(\boldsymbol{x}) = \dot\gamma y' \boldsymbol{e}_1'$, where $\dot\gamma(t) = \dot\gamma_0 \dot\Gamma(t)$. We also have $\omega_3^\infty = -\dot\Gamma(t)$ and

$$\begin{bmatrix} E_1^\infty \\ E_2^\infty \end{bmatrix} = \frac{\dot\Gamma(t)}{2}\begin{bmatrix} \sin(2\theta) \\ \cos(2\theta) \end{bmatrix}. \tag{27}$$

We can directly write out the equation for $\frac{d\theta}{dt}$ as,

$$\frac{d\theta}{dt} = -\dot\Gamma(t)\left(1 + A\cos[2(\theta-\psi)]\right),$$

where $A = \sqrt{\hat h_{31}^2 + \hat h_{32}^2}$ and $\psi = \frac{1}{2}\tan^{-1}(\frac{\hat h_{31}}{\hat h_{32}})$. We rewrite this equation as

$$\frac{d\theta}{1 + A\cos[2(\theta-\psi)]} = -\dot\Gamma(t)dt,$$

and then make the substitution $\phi = \theta - \psi$ to get

$$\frac{d\phi}{1 + A\cos 2\phi} = -\dot\Gamma(t)dt.$$

Using the integral identity

$$\int \frac{d\phi}{1 + A\cos 2\phi} = \frac{1}{\sqrt{1-A^2}}\tan^{-1}\left(\frac{\sqrt{1-A^2}\tan\phi}{1+A}\right) + C,$$

we get

$$\frac{1}{\sqrt{1-A^2}}\tan^{-1}\left(\frac{\sqrt{1-A^2}\tan\phi}{1+A}\right) = -G(t) + C.$$

where

$$G(t) = \int_0^t \dot{\Gamma}(t')dt'. \tag{30}$$

Solving for $\tan\phi$, we get

$$\tan(\phi) = \frac{1+A}{\sqrt{1-A^2}} \tan\Big(\sqrt{1-A^2}(-G(t)+C)\Big).$$

Re-substituting $\phi = \theta - \psi$ and re-arranging terms gives Equation (29),

$$(1+A)\tan(\psi-\theta) = \sqrt{1-A^2}\tan\Big(\sqrt{1-A^2}(G(t)-C)\Big). \tag{29}$$

Setting $\theta = \theta_0$ at $t = 0$ and substituting into (29) then gives

$$C = -\frac{1}{\sqrt{1-A^2}} \tan^{-1}\left(\frac{1+A}{\sqrt{1-A^2}} \tan(\psi - \theta_0)\right). \tag{31}$$

Next, to derive the evolution equations for $\frac{dx_p'}{dt} = U_{p,1}'$ and $\frac{dy_p'}{dt} = U_{p,2}'$, we first bring Equation (27) back into Equation (13) to get

$$U_{p,1} - U_1^\infty = \hat{g}_{11}E_1^\infty = \frac{\dot{\Gamma}}{2}\hat{g}_{11}\sin(2\theta)$$

and

$$U_{p,2} - U_2^\infty = \hat{g}_{22}E_2^\infty = \frac{\dot{\Gamma}}{2}\hat{g}_{22}\cos(2\theta).$$

We then get

$$\frac{dx_p'}{dt} = (U_{p,1} - U_1^\infty)\cos\theta - (U_{p,2} - U_2^\infty)\sin\theta + \dot{\Gamma}y_p'$$

and

$$\frac{dy_p'}{dt} = (U_{p,1} - U_1^\infty)\sin\theta + (U_{p,2} - U_2^\infty)\cos\theta.$$

From Equation (14),

$$\hat{g}_{11} = \frac{\cos(\alpha/2)(\cos\alpha - 2)}{2(\cos\alpha - 3)}, \quad \hat{g}_{22} = \frac{-2\sin(\alpha/2)\sin\alpha}{2(\cos\alpha - 3)},$$

so $\hat{g}_{22} < 0$, therefore the lab-frame velocities become

$$\frac{dx_p'}{dt} - \dot{\Gamma}y_p' = \frac{\dot{\Gamma}}{2}(\hat{g}_{11}\sin 2\theta\cos\theta + \hat{g}_{22}\cos 2\theta\sin\theta), \tag{33a}$$

$$\frac{dy_p'}{dt} = \frac{\dot{\Gamma}}{2}(\hat{g}_{22}\cos 2\theta\cos\theta - \hat{g}_{11}\sin 2\theta\sin\theta). \tag{33b}$$

2. Elastic hinge

Similar to the derivation process for a rigid hinge, substituting Equation (27) and $\omega_3^\infty = -\dot{\Gamma}(t)$ into (18a) and (18b) directly gives Equations (35a), (35b), (36a), and (36b).

For (35a), we have

$$\begin{aligned}\frac{d\theta}{dt} &= \frac{3\left(\frac{1}{2}\dot{\Gamma}\cos(2\theta)+\dot{\Gamma}\right)-\left(\frac{1}{2}\dot{\Gamma}\cos(2\theta)\cos\alpha+3\dot{\Gamma}\cos(2\theta)+\dot{\Gamma}\right)\cos\alpha}{2(\cos\alpha-3)} \\ &= \frac{\dot{\Gamma}\left[\frac{3}{2}\cos(2\theta)+3-\left(\frac{7}{2}\cos(2\theta)+1\right)\cos\alpha\right]}{2(\cos\alpha-3)}. \\ &= \frac{\dot{\Gamma}\left[\cos(2\theta)\left(\frac{3}{2}-\frac{7}{2}\cos\alpha\right)+3-\cos\alpha\right]}{2(\cos\alpha-3)} \\ &= -\frac{\dot{\Gamma}}{8}\left(\frac{(12\cos\alpha+\cos2\alpha-5)\cos(2\theta)}{\cos\alpha-3}+4\right). \end{aligned} \tag{35a}$$

A similar derivation process gives (35b):

$$\frac{d\alpha}{dt} = \frac{\dot{\Gamma}(\sin(2\alpha)-12\sin\alpha)\sin(2\theta)-4\kappa(\cos\alpha-3)(\alpha_0-\alpha)}{4(\cos\alpha+3)}. \tag{35b}$$

For (36a) and (36b), we have

$$\begin{aligned} U_{p,1}-U_1^\infty &= \frac{E_1^\infty\left[\cos\left(\frac{3\alpha}{2}\right)-3\cos\left(\frac{\alpha}{2}\right)\right]-2\sin\left(\frac{\alpha}{2}\right)\frac{d\alpha}{dt}}{4(\cos\alpha-3)} \\ &= \frac{\dot{\Gamma}\sin(2\theta)\left[\cos\left(\frac{3\alpha}{2}\right)-3\cos\left(\frac{\alpha}{2}\right)\right]-2\sin\left(\frac{\alpha}{2}\right)\frac{d\alpha}{dt}}{4(\cos\alpha-3)} \\ &= \frac{\dot{\Gamma}\cos\left(\frac{\alpha}{2}\right)(\cos\alpha-2)\sin(2\theta)-2\sin\left(\frac{\alpha}{2}\right)\frac{d\alpha}{dt}}{4(\cos\alpha-3)}, \end{aligned} \tag{36a}$$

and

$$\begin{aligned} U_{p,2}-U_2^\infty &= -\frac{E_2^\infty\sin\left(\frac{\alpha}{2}\right)\sin\alpha}{\cos\alpha-3} = -\frac{\dot{\Gamma}\cos(2\theta)\sin\left(\frac{\alpha}{2}\right)\sin\alpha}{2(\cos\alpha-3)} \\ &= -\frac{\dot{\Gamma}\cos(2\theta)\sin\left(\frac{\alpha}{2}\right)\cdot2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)}{2(\cos\alpha-3)} \\ &= -\frac{\dot{\Gamma}\cos\left(\frac{\alpha}{2}\right)\sin^2\left(\frac{\alpha}{2}\right)\cos(2\theta)}{\cos\alpha-3}. \end{aligned} \tag{36b}$$