Extension of *Drift of elastic hinges in quasi-two-dimensional oscillating shear flows* to n-arm case*

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The original paper considers the case of a two-arm elastic hinge. In this section, I extend the paper's results to the case of an n-arm elastic hinge. I assume that each arm has length $l/2$, and that the equilibrium angle α_0 and stiffness $\hat{\kappa}$ between each adjacent pair of arms are the same throughout the entire hinge.

The resistance coefficients for the two-arm case are given in Equation (9) of the paper as

$$\begin{bmatrix} F_1 \\ F_2 \\ T \end{bmatrix} = \underbrace{\frac{1}{8}\begin{bmatrix} 3-\cos 2\beta & -\sin 2\beta & -\sin\beta \\ -\sin 2\beta & 3+\cos 2\beta & \cos\beta \\ -\sin\beta & \cos\beta & \frac{1}{3} \end{bmatrix}}_{\boldsymbol{R}(\beta)} \begin{bmatrix} U_1^\infty - U_{p,1} \\ U_2^\infty - U_{p,2} \\ \frac{1}{2}\omega_3^\infty - \Omega_{p,3} \end{bmatrix}$$

$$+ \underbrace{\frac{1}{32}\begin{bmatrix} (3\cos\beta - \cos 3\beta) & 4\sin^3\beta \\ -(3\sin\beta + \sin 3\beta) & 4\cos^3\beta \\ -\frac{4}{3}\sin 2\beta & \frac{4}{3}\cos 2\beta \end{bmatrix}}_{\boldsymbol{Q}(\beta)} \begin{bmatrix} E_1^\infty \\ E_2^\infty \end{bmatrix}, \tag{9}$$

For a n-arm hinge, we need to know two things to describe the entire hinge: the position of the 1st arm and the n-1 angular displacements α_i $(\alpha = 1, ..., n-1)$ between each pair of neighboring arms.

I define the absolute position of each arm using the following method. First, I assume that for any hinge, we can rotate it so that its first arm aligns with the $\boldsymbol{e}_1$ axis of the body-fixed frame. Therefore, the orientation angle β_1 of the first arm relative to the $\boldsymbol{e}_1$ axis is $0°$. The angular displacement between the first and second arms is α_1. Therefore, the orientation of the second arm can then be expressed as the orientation of the first arm plus the angular displacement: $\beta_2 = \beta_1 + \alpha_1 = \alpha_1$. Similarly, the orientation of the third arm can be expressed using the second orientation angle and the angular displacement between the

*For the equations that are from the original paper, the numbering is consistent with that in the paper.

second and the third arm: $\beta_3 = \beta_2 + \alpha_2 = \beta_1 + \alpha_1 + \alpha_2 = \alpha_1 + \alpha_2$. As a result, the orientation of the k-th arm ($k > 1$) can be expressed as

$$\beta_k = \sum_{i=1}^{k-1} \alpha_i, \quad k = 2, ..., n.$$

For each kth arm, the length and shape of each arm are assumed to be the same, which means that the bounds for the definite integral used to calculate the resistance coefficients do not change. Therefore, Equation (9) can be rewritten as

$$\begin{bmatrix} F_{k,1} \\ F_{k,2} \\ T_k \end{bmatrix} = \underbrace{\frac{1}{8}\begin{bmatrix} 3-\cos 2\beta_k & -\sin 2\beta_k & -\sin\beta_k \\ -\sin 2\beta_k & 3+\cos 2\beta_k & \cos\beta_k \\ -\sin\beta_k & \cos\beta_k & \frac{1}{3} \end{bmatrix}}_{\boldsymbol{R}(\beta_k)} \begin{bmatrix} U_1^\infty - U_{p,1} \\ U_2^\infty - U_{p,2} \\ \frac{1}{2}\omega_3^\infty - \Omega_{p,k} \end{bmatrix}$$
$$+ \underbrace{\frac{1}{32}\begin{bmatrix} (3\cos\beta_k - \cos 3\beta_k) & 4\sin^3\beta_k \\ -(3\sin\beta_k + \sin 3\beta_k) & 4\cos^3\beta_k \\ -\frac{4}{3}\sin 2\beta_k & \frac{4}{3}\cos 2\beta_k \end{bmatrix}}_{\boldsymbol{Q}(\beta_k)} \begin{bmatrix} E_1^\infty \\ E_2^\infty \end{bmatrix}. \tag{9a}$$

Here, $F_{k,1}$ and $F_{k,2}$ represent the first and second components respectively of the force exerted on the kth arm by the fluid, T_k represents the torque exerted on the kth arm by the fluid, and $\Omega_{p,k}$ represents the rotation velocity of the kth arm revolving around the $\boldsymbol{e}_3$ axis. That is, $\Omega_{p,k} = \frac{d}{dt}(\theta + \beta_k) = \frac{d\theta}{dt} + \sum_{i=1}^{k-1} \dot{\alpha}_i$.

Just like in the two-arm case, I require the n-arm hinge to be force-free and torque-free:

$$\sum_{k=1}^{n} \boldsymbol{F}_k = \boldsymbol{0},$$
$$\sum_{k=1}^{n} T_k = 0.$$

So applying these conditions to Equation (9a) gives

$$\sum_{k=1}^{n} \left[\boldsymbol{R}(\beta_k) \cdot \begin{bmatrix} U_1^\infty - U_{p,1} \\ U_2^\infty - U_{p,2} \\ \frac{1}{2}\omega_3^\infty - \Omega_{p,k} \end{bmatrix} + \boldsymbol{Q}(\beta_k) \cdot \begin{bmatrix} E_1^\infty \\ E_2^\infty \end{bmatrix} \right] = \boldsymbol{0}. \tag{10}$$

Similarly, the internal balance of torques at each hinge point also has to be considered. At the jth hinge point, $j = 1, ..., n-1$ we have

$$T_j - T_{j+1} + 2\hat{\kappa}(\alpha_j - \alpha_0) = 0. \tag{11}$$

The following goal is then to rewrite Equations (10) and (11) in the form of a linear matrix system $\boldsymbol{MX} = \boldsymbol{B}$, where $\boldsymbol{X}$ is a $(n+2) \times 1$ vector containing all the unknowns:

$$\boldsymbol{X} = \begin{bmatrix} U_{p,1} \\ U_{p,2} \\ \frac{d\theta}{dt} \\ \dot{\alpha_1} \\ \vdots \\ \dot{\alpha}_{n-1} \end{bmatrix}.$$

Rearranging Equation (10), I get

$$\sum_{k=1}^{n} \boldsymbol{R}(\beta_k) \cdot \begin{bmatrix} U_{p,1} \\ U_{p,2} \\ \Omega_{p,k} \end{bmatrix} = \sum_{k=1}^{n} \left(\boldsymbol{R}(\beta_k) \cdot \begin{bmatrix} U_1^\infty \\ U_2^\infty \\ \frac{1}{2}\omega_3^\infty \end{bmatrix} + \boldsymbol{Q}(\beta_k) \cdot \begin{bmatrix} E_1^\infty \\ E_2^\infty \end{bmatrix} \right). \tag{12}$$

I define the right side as

$$\boldsymbol{B}_{total} = \sum_{k=1}^{n} \left(\boldsymbol{R}(\beta_k) \cdot \begin{bmatrix} U_1^\infty \\ U_2^\infty \\ \frac{1}{2}\omega_3^\infty \end{bmatrix} + \boldsymbol{Q}(\beta_k) \cdot \begin{bmatrix} E_1^\infty \\ E_2^\infty \end{bmatrix} \right). \tag{13}$$

I can then rewrite each of the three rows of Equation (12). Using the first row as an example, (12) can be rewritten as

$$\sum_{k=1}^{n} (R_{11}(\beta_k) U_{p,1} + R_{12}(\beta_k) U_{p,2} + R_{13}(\beta_k)\Omega_{p,k}) = \boldsymbol{B}_{total,1}.$$

Now using the definition of $\Omega_{p,k}$, I can rewrite the last term on the left side of each row. Again using the first row as an example, the last term becomes

$$\sum_{k=1}^{n} R_{13}(\beta_k)\Omega_{p,k} = \sum_{k=1}^{n} R_{13}(\beta_k)(\frac{d\theta}{dt} + \sum_{i=1}^{k-1} \dot{\alpha}_i)$$

$$= \left(\sum_{k=1}^{n} R_{13}(\beta_k) \right) \frac{d\theta}{dt} + \sum_{k=1}^{n} \left(R_{13}(\beta_k) \sum_{i=1}^{k-1} \dot{\alpha}_i \right).$$

$$= \left(\sum_{k=1}^{n} R_{13}(\beta_k) \right) \frac{d\theta}{dt} + \sum_{i=1}^{n-1} \left(\sum_{k=i+1}^{n} R_{13}(\beta_k) \right) \dot{\alpha}_i.$$

This allows me to rewrite the first row (and subsequent rows) of Equation (12) as a linear combination of the unknown variables:

$$\left(\sum_{k=1}^{n} R_{11}(\beta_k) \right) U_{p,1} + \left(\sum_{k=1}^{n} R_{12}(\beta_k) \right) U_{p,2} + \left(\sum_{k=1}^{n} R_{13}(\beta_k) \right) \frac{d\theta}{dt}$$
$$+ \sum_{i=1}^{n-1} \left(\sum_{k=i+1}^{n} R_{13}(\beta_k) \right) \dot{\alpha}_i = \boldsymbol{B}_{total,1}. \tag{14}$$

Now, regarding Equation (11), I can write T_j $(j = 1, ..., n-1)$ as

$$T_j = \boldsymbol{R}_3(\beta_j) \cdot \begin{bmatrix} U_1^\infty - U_{p,1} \\ U_2^\infty - U_{p,2} \\ \frac{1}{2}\omega_3^\infty - \Omega_{p,j} \end{bmatrix} + \boldsymbol{Q}_3(\beta_j) \cdot \begin{bmatrix} E_1^\infty \\ E_2^\infty \end{bmatrix},$$

so $T_{j+1} - T_j$ is

$$T_{j+1} - T_j = \boldsymbol{R}_3(\beta_{j+1}) \begin{bmatrix} U_1^\infty - U_{p,1} \\ U_2^\infty - U_{p,2} \\ \frac{1}{2}\omega_3^\infty - \Omega_{p,j+1} \end{bmatrix} - \boldsymbol{R}_3(\beta_j) \cdot \begin{bmatrix} U_1^\infty - U_{p,1} \\ U_2^\infty - U_{p,2} \\ \frac{1}{2}\omega_3^\infty - \Omega_{p,j} \end{bmatrix}$$
$$+(\boldsymbol{Q}_3(\beta_{j+1}) - \boldsymbol{Q}_3(\beta_j)) \cdot \begin{bmatrix} E_1^\infty \\ E_2^\infty \end{bmatrix} = 2\hat{\kappa}(\alpha_j - \alpha_0). \tag{15}$$

Similar to how I derived Equation (13), I want to rewrite this equation in the form of *linear combination of unknown variables = some combination of all the known background flow variables.* This means that I need to expand

$$\boldsymbol{R}_3(\beta_{j+1}) \cdot \begin{bmatrix} U_1^\infty - U_{p,1} \\ U_2^\infty - U_{p,2} \\ \frac{1}{2}\omega_3^\infty - \Omega_{p,j+1} \end{bmatrix} - \boldsymbol{R}_3(\beta_j) \cdot \begin{bmatrix} U_1^\infty - U_{p,1} \\ U_2^\infty - U_{p,2} \\ \frac{1}{2}\omega_3^\infty - \Omega_{p,j} \end{bmatrix}$$

into a linear combination. Using the same process as the derivation for Equation (13), the expanded form of Equation (15) is

$$\begin{gathered}
[R_{31}(\beta_{j+1}) - R_{31}(\beta_j)]U_{p,1} + [R_{32}(\beta_{j+1}) - R_{32}(\beta_j)]U_{p,2} \\
+ [R_{33}(\beta_{j+1}) - R_{33}(\beta_j)]\frac{d\theta}{dt} + [R_{33}(\beta_{j+1}) - R_{33}(\beta_j)]\dot{\alpha}_1 + [R_{33}(\beta_{j+1}) - R_{33}(\beta_j)\dot{\alpha}_2 \\
+ \cdots + [R_{33}(\beta_{j+1}) - R_{33}(\beta_j)]\dot{\alpha}_{j-1} + [R_{33}(\beta_{j+1})]\dot{\alpha}_j \\
= [R_{31}(\beta_{j+1}) - R_{31}(\beta_j)]U_1^\infty + [R_{32}(\beta_{j+1}) - R_{32}(\beta_j)]U_2^\infty \\
+ [R_{33}(\beta_{j+1}) - R_{33}(\beta_j)]\frac{1}{2}\omega_3^\infty \\
+ [Q_{31}(\beta_{j+1}) - Q_{31}(\beta_j)]E_1^\infty + [Q_{32}(\beta_{j+1}) - Q_{32}(\beta_j)]E_2^\infty - 2\hat{\kappa}(\alpha_j - \alpha_0).
\end{gathered} \tag{16}$$

Finally, combining the information from Equations (13), (14), and (16), I can write the tensors $\boldsymbol{M}$ and $\boldsymbol{B}$ from $\boldsymbol{MX} = \boldsymbol{B}$ to obtain a complete linear system of equations.

$\boldsymbol{B}$ is a $(n+2) \times 1$ vector:

$$\boldsymbol{B} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_{j+3} \end{bmatrix},$$

where

1. **First 3 rows from Equation (13)**:

$$B_1 = \sum_{k=1}^{n} \left(R_{11}(\beta_k) U_1^\infty + R_{12}(\beta_k) U_2^\infty + R_{13}(\beta_k) \frac{1}{2} \omega_3^\infty + Q_{11}(\beta_k) E_1^\infty + Q_{12}(\beta_k) E_2^\infty \right)$$
$$B_2 = \sum_{k=1}^{n} \left(R_{21}(\beta_k) U_1^\infty + R_{22}(\beta_k) U_2^\infty + R_{23}(\beta_k) \frac{1}{2} \omega_3^\infty + Q_{21}(\beta_k) E_1^\infty + Q_{22}(\beta_k) E_2^\infty \right)$$
$$B_3 = \sum_{k=1}^{n} \left(R_{31}(\beta_k) U_1^\infty + R_{32}(\beta_k) U_2^\infty + R_{33}(\beta_k) \frac{1}{2} \omega_3^\infty + Q_{31}(\beta_k) E_1^\infty + Q_{32}(\beta_k) E_2^\infty \right).$$

2. **Rows 4 to $n+2$ from Equation (16),** $j = 1, \ldots, n-1$:

$$\begin{aligned} B_{j+3} = & \left[R_{31}(\beta_{j+1}) - R_{31}(\beta_j) \right] U_1^\infty \\ & + \left[R_{32}(\beta_{j+1}) - R_{32}(\beta_j) \right] U_2^\infty \\ & + \left[R_{33}(\beta_{j+1}) - R_{33}(\beta_j) \right] \frac{1}{2} \omega_3^\infty \\ & + \left[Q_{31}(\beta_{j+1}) - Q_{31}(\beta_j) \right] E_1^\infty \\ & + \left[Q_{32}(\beta_{j+1}) - Q_{32}(\beta_j) \right] E_2^\infty \\ & - 2\kappa(\alpha_j - \alpha_0). \end{aligned}$$

$\boldsymbol{M}$ is a $(n+2) \times (n+2)$ matrix:

1. **First 3 rows from Equation (12)**:

First row:

$$M_{1,1} = \sum_{k=1}^{n} R_{11}(\beta_k) \qquad M_{1,2} = \sum_{k=1}^{n} R_{12}(\beta_k)$$
$$M_{1,3} = \sum_{k=1}^{n} R_{13}(\beta_k) \qquad M_{1,3+i} = \sum_{k=i+1}^{n} R_{13}(\beta_k) \quad \text{for } i = 1, \ldots, n-1,$$

Second row:

$$M_{2,1} = \sum_{k=1}^{n} R_{21}(\beta_k) \qquad M_{2,2} = \sum_{k=1}^{n} R_{22}(\beta_k)$$
$$M_{2,3} = \sum_{k=1}^{n} R_{23}(\beta_k) \qquad M_{2,3+i} = \sum_{k=i+1}^{n} R_{23}(\beta_k) \quad \text{for } i = 1, \ldots, n-1,$$

Third row:

$$M_{3,1} = \sum_{k=1}^{n} R_{31}(\beta_k) \qquad M_{3,2} = \sum_{k=1}^{n} R_{32}(\beta_k)$$
$$M_{3,3} = \sum_{k=1}^{n} R_{33}(\beta_k) \qquad M_{3,3+i} = \sum_{k=i+1}^{n} R_{33}(\beta_k) \quad \text{for } i = 1, \ldots, n-1.$$

2. **Rows 4 to $n + 2$ from Equation (16),** $j = 1, \ldots, n-1$:

$$
\begin{aligned}
M_{j+3,1} &= R_{31}(\beta_{j+1}) - R_{31}(\beta_j) \\
M_{j+3,2} &= R_{32}(\beta_{j+1}) - R_{32}(\beta_j) \\
M_{j+3,3} &= R_{33}(\beta_{j+1}) - R_{33}(\beta_j).
\end{aligned}
$$

For $l = 4$ to $n + 2$:

$$
M_{j+3,l} = \begin{cases} R_{33}(\beta_{j+1}) - R_{33}(\beta_j), & \text{if } l - 3 < j \\ R_{33}(\beta_{j+1}), & \text{if } l - 3 = j. \end{cases}
$$

This information allows the dynamical equations embedded in the vector $\boldsymbol{X}$ to be expressed implicitly.