

Extension of *Drift of elastic hinges in quasi-two-dimensional oscillating shear flows* to n -arm case (2)

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1 Verification for $n = 2$ case

The resistance coefficients for the two-arm case are given by

$$\begin{aligned} \begin{bmatrix} F_{k,1} \\ F_{k,2} \\ T_k \end{bmatrix} &= \underbrace{\frac{1}{8} \begin{bmatrix} 3 - \cos 2\beta_k & -\sin 2\beta_k & -\sin \beta_k \\ -\sin 2\beta_k & 3 + \cos 2\beta_k & \cos \beta_k \\ -\sin \beta_k & \cos \beta_k & \frac{1}{3} \end{bmatrix}}_{\mathbf{R}(\beta_k)} \begin{bmatrix} U_1^\infty - U_{p,1} \\ U_2^\infty - U_{p,2} \\ \frac{1}{2}\omega_3^\infty - \Omega_{p,k} \end{bmatrix} \\ &+ \underbrace{\frac{1}{32} \begin{bmatrix} (3 \cos \beta_k - \cos 3\beta_k) & 4 \sin^3 \beta_k \\ -(3 \sin \beta_k + \sin 3\beta_k) & 4 \cos^3 \beta_k \\ -\frac{4}{3} \sin 2\beta_k & \frac{4}{3} \cos 2\beta_k \end{bmatrix}}_{\mathbf{Q}(\beta_k)} \begin{bmatrix} E_1^\infty \\ E_2^\infty \end{bmatrix}, \quad (1) \end{aligned}$$

where $k = 1, 2$. In the paper, the two orientation angles are defined as and $\beta_1 = -\alpha/2$ and $\beta_2 = \alpha/2$, where α is the angular displacement between the two arms.

Bringing $n = 2$ into the system of equations I previously defined, the unknowns vector \mathbf{X} becomes

$$\mathbf{X} = \begin{bmatrix} U_{p,1} \\ U_{p,2} \\ \frac{d\theta}{dt} \\ \dot{\alpha}_1 \end{bmatrix}. \quad (2)$$

\mathbf{M} and \mathbf{B} become 4×4 and 4×1 respectively. The four rows of \mathbf{B} are

$$\begin{aligned}
B_1 &= \sum_{k=1}^2 \left(R_{11}(\beta_k) U_1^\infty + R_{12}(\beta_k) U_2^\infty + R_{13}(\beta_k) \frac{1}{2} \omega_3^\infty + Q_{11}(\beta_k) E_1^\infty + Q_{12}(\beta_k) E_2^\infty \right), \\
B_2 &= \sum_{k=1}^2 \left(R_{21}(\beta_k) U_1^\infty + R_{22}(\beta_k) U_2^\infty + R_{23}(\beta_k) \frac{1}{2} \omega_3^\infty + Q_{21}(\beta_k) E_1^\infty + Q_{22}(\beta_k) E_2^\infty \right), \\
B_3 &= \sum_{k=1}^2 \left(R_{31}(\beta_k) U_1^\infty + R_{32}(\beta_k) U_2^\infty + R_{33}(\beta_k) \frac{1}{2} \omega_3^\infty + Q_{31}(\beta_k) E_1^\infty + Q_{32}(\beta_k) E_2^\infty \right), \\
B_4 &= [R_{31}(\beta_2) - R_{31}(\beta_1)] U_1^\infty + [R_{32}(\beta_2) - R_{32}(\beta_1)] U_2^\infty + [R_{33}(\beta_2) - R_{33}(\beta_1)] \frac{1}{2} \omega_3^\infty \\
&\quad + [Q_{31}(\beta_2) - Q_{31}(\beta_1)] E_1^\infty + [Q_{32}(\beta_2) - Q_{32}(\beta_1)] E_2^\infty - 2\kappa(\alpha_j - \alpha_0).
\end{aligned}$$

The elements of the four rows of \mathbf{M} are

First row:

$$\begin{aligned}
M_{1,1} &= \sum_{k=1}^2 R_{11}(\beta_k) = R_{11}(-\alpha/2) + R_{11}(\alpha/2) \\
M_{1,2} &= \sum_{k=1}^2 R_{12}(\beta_k) = R_{12}(-\alpha/2) + R_{12}(\alpha/2) \\
M_{1,3} &= \sum_{k=1}^2 R_{13}(\beta_k) = R_{13}(-\alpha/2) + R_{13}(\alpha/2) \\
M_{1,4} &= \sum_{k=2}^2 R_{13}(\beta_k) = R_{13}(\alpha/2),
\end{aligned}$$

Second row:

$$\begin{aligned}
M_{2,1} &= \sum_{k=1}^2 R_{21}(\beta_k) = R_{21}(-\alpha/2) + R_{21}(\alpha/2) \\
M_{2,2} &= \sum_{k=1}^2 R_{22}(\beta_k) = R_{22}(-\alpha/2) + R_{22}(\alpha/2) \\
M_{2,3} &= \sum_{k=1}^2 R_{23}(\beta_k) = R_{23}(-\alpha/2) + R_{23}(\alpha/2) \\
M_{2,4} &= \sum_{k=2}^2 R_{23}(\beta_k) = R_{23}(\alpha/2),
\end{aligned}$$

Third row:

$$\begin{aligned}
M_{3,1} &= \sum_{k=1}^2 R_{31}(\beta_k) = R_{31}(-\alpha/2) + R_{31}(\alpha/2) \\
M_{3,2} &= \sum_{k=1}^2 R_{32}(\beta_k) = R_{32}(-\alpha/2) + R_{32}(\alpha/2) \\
M_{3,3} &= \sum_{k=1}^2 R_{33}(\beta_k) = R_{33}(-\alpha/2) + R_{33}(\alpha/2) \\
M_{3,4} &= \sum_{k=2}^2 R_{33}(\beta_k) = R_{33}(\alpha/2),
\end{aligned}$$

Fourth row:

$$\begin{aligned}
M_{4,1} &= R_{31}(\beta_2) - R_{31}(\beta_1) = R_{31}(\alpha/2) - R_{31}(-\alpha/2) \\
M_{4,2} &= R_{32}(\beta_2) - R_{32}(\beta_1) = R_{32}(\alpha/2) - R_{32}(-\alpha/2) \\
M_{4,3} &= R_{33}(\beta_2) - R_{33}(\beta_1) = R_{33}(\alpha/2) - R_{33}(-\alpha/2) \\
M_{4,4} &= R_{33}(\beta_2) = R_{33}(\alpha/2).
\end{aligned}$$

I then bring the resistance coefficients from Equation (1) into the expressions for the elements of \mathbf{M} and \mathbf{B} . The elements of \mathbf{M} simplify to

$$\begin{aligned}
M_{11} &= \frac{3 - \cos \alpha}{4}, & M_{12} &= 0, & M_{13} &= 0, & M_{14} &= -\frac{1}{8} \sin\left(\frac{\alpha}{2}\right), \\
M_{21} &= 0, & M_{22} &= \frac{3 + \cos \alpha}{4}, & M_{23} &= \frac{1}{4} \cos\left(\frac{\alpha}{2}\right), & M_{24} &= \frac{1}{8} \cos\left(\frac{\alpha}{2}\right), \\
M_{31} &= 0, & M_{32} &= \frac{1}{4} \cos\left(\frac{\alpha}{2}\right), & M_{33} &= \frac{1}{12}, & M_{34} &= \frac{1}{24}, \\
M_{41} &= -\frac{1}{4} \sin\left(\frac{\alpha}{2}\right), & M_{42} &= 0, & M_{43} &= 0, & M_{44} &= \frac{1}{24},
\end{aligned}$$

and the elements of \mathbf{B} simplify to

$$\begin{aligned}
B_1 &= \frac{3 - \cos \alpha}{4} U_1^\infty + \frac{1}{16} \left(3 \cos \frac{\alpha}{2} - \cos \frac{3\alpha}{2} \right) E_1^\infty, \\
B_2 &= \frac{3 + \cos \alpha}{4} U_2^\infty + \frac{1}{8} \cos\left(\frac{\alpha}{2}\right) \omega_3^\infty + \frac{1}{4} \cos^3\left(\frac{\alpha}{2}\right) E_2^\infty, \\
B_3 &= \frac{1}{4} \cos\left(\frac{\alpha}{2}\right) U_2^\infty + \frac{1}{24} \omega_3^\infty + \frac{1}{12} \cos \alpha E_2^\infty, \\
B_4 &= -\frac{1}{4} \sin\left(\frac{\alpha}{2}\right) U_1^\infty - \frac{1}{12} \sin \alpha E_1^\infty - 2\kappa(\alpha - \alpha_0).
\end{aligned}$$

Using the expressions above, I can solve for \mathbf{X} using Python to get

$$U_{p,1} - U_1^\infty = -\frac{1}{2(\cos \alpha + 3)} \left(4E_1^\infty \sin\left(\frac{\alpha}{2}\right) \sin \alpha - 3E_1^\infty \cos\left(\frac{\alpha}{2}\right) \right. \\ \left. + E_1^\infty \cos\left(\frac{3\alpha}{2}\right) + 12U_1^\infty \sin^2\left(\frac{\alpha}{2}\right) \right. \\ \left. + 6U_1^\infty \cos \alpha - 6U_1^\infty + 96\alpha\kappa \sin\left(\frac{\alpha}{2}\right) \right. \\ \left. - 96\alpha_0\kappa \sin\left(\frac{\alpha}{2}\right) \right) \quad (3)$$

$$U_{p,2} - U_2^\infty = \frac{E_2^\infty (\cos \alpha - 1) \sin \alpha}{-7 \sin\left(\frac{\alpha}{2}\right) + \sin\left(\frac{3\alpha}{2}\right)} \quad (4)$$

$$\frac{d\theta}{dt} = \frac{1}{4(\cos(2\alpha) - 17)} \left(-73E_1^\infty \sin \alpha + 18E_1^\infty \sin(2\alpha) - E_1^\infty \sin(3\alpha) \right. \\ \left. - 63E_2^\infty \cos \alpha - 18E_2^\infty \cos(2\alpha) - E_2^\infty \cos(3\alpha) + 18E_2^\infty \right. \\ \left. + 2304\alpha\kappa \cos \alpha - 192\alpha\kappa \cos(2\alpha) - 3648\alpha\kappa \right. \\ \left. - 2304\alpha_0\kappa \cos \alpha + 192\alpha_0\kappa \cos(2\alpha) + 3648\alpha_0\kappa \right. \\ \left. + 2\omega_3^\infty \cos(2\alpha) - 34\omega_3^\infty \right) \quad (5)$$

$$\frac{d\alpha}{dt} = \frac{1}{2(\cos \alpha + 3)} \left(-12E_1^\infty \sin \alpha + E_1^\infty \sin(2\alpha) \right. \\ \left. + 192\alpha\kappa \cos \alpha - 576\alpha\kappa \right. \\ \left. - 192\alpha_0\kappa \cos \alpha + 576\alpha_0\kappa \right) \quad (6)$$

These expressions are much more complicated than Equations (18a)-(18d) in the original paper, so I will use Python to plug in certain values for background flow and stiffness κ , and then see if the differences between Equations (5)-(8) and the corresponding Equations (18a)-(18d) come out to be zero. The hinge under consideration is elastic, so the stiffness cannot be too high. I first set $\kappa = 0.1$ and then vary the background flow variables:

	U_1^∞	U_2^∞	ω_3^∞	E_1^∞	E_2^∞	α	α_0
	0.5	-0.2	0.5	1	1	$\pi/4$	$\pi/4$
	1	1	1	0.5	0.2	$\pi/6$	$\pi/4$
	2	3	0.01	4	6	$\pi/2$	$\pi/8$
κ	$\Delta(d\theta/dt)$	$\Delta(d\alpha/dt)$	$\Delta(U_{p,1} - U_1^\infty)$	$\Delta(U_{p,2} - U_2^\infty)$			
0.1	80.124661%	0.000000%	0.000000%	0.000000%			
0.1	82.695108%	0.000000%	0.000000%	0.000000%			
0.1	322.366169%	0.000000%	0.000000%	0.000000%			

Table 1: Some background flows and corresponding percentage error

All the other equations have zero error, but the error for the $\frac{d\theta}{dt}$ equation is very large. However, it is worth observing how this error varies with background vorticity ω^∞ .

Below is how the error for the $\frac{d\theta}{dt}$ equation varies with ω^∞ (other variables fixed).

U_1^∞	U_2^∞	ω_3^∞	E_1^∞	E_2^∞	α	α_0	κ	$\Delta(d\theta/dt)$
1	1	2	0.5	0.2	$\pi/6$	$\pi/4$	0.1	46.363477%
1	1	5	0.5	0.2	$\pi/6$	$\pi/4$	0.1	20.001214%
1	1	10	0.5	0.2	$\pi/6$	$\pi/4$	0.1	10.269323%
1	1	90	0.5	0.2	$\pi/6$	$\pi/4$	0.1	1.168956%
1	1	1000	0.5	0.2	$\pi/6$	$\pi/4$	0.1	0.105500%

Table 2: Different values of ω_3^∞ and corresponding error for the $\frac{d\theta}{dt}$ equation

So, it can be seen that as $\omega_3^\infty \rightarrow \infty$, the error for the $\frac{d\theta}{dt}$ equation approaches zero.