3 Mechanics of Glacier Flow

3.1 FORCE BALANCE

Glaciers and ice sheets generally move very slowly, with speeds up to a few tens of meters per day at most. This suggests that in good approximation accelerations may be neglected so that Newton's second law reduces to an equilibrium of forces. Forces applied to the surface of a volume element must balance the body force (gravity) that acts on the entire volume. To arrive at the balance equations, consider a small volume element in the glacier, centered at (x, y, z) with dimensions ∂x, ∂y, and ∂z as shown in Figure 3.1. It is convenient to choose the Cartesian coordinate system such that the z-axis is vertical and positive upward.

There are three components of stress acting in the x-direction, namely the normal stress, σ_{xx}, on the (y, z) faces, and the two shear stresses, σ_{zx} and σ_{yx}, on the (x, y) and (x, z) faces, respectively. The normal stress acting on the right face of the volume element is approximately

$$\sigma_{xx}(x) + \frac{\partial \sigma_{xx}}{\partial x}\frac{\partial x}{2}, \tag{3.1}$$

and that acting on the left face is

$$\sigma_{xx}(x) - \frac{\partial \sigma_{xx}}{\partial x}\frac{\partial x}{2}. \tag{3.2}$$

The net normal force in the x-direction acting on the volume element is the difference between the force on the right face and on the left face, multiplied by the area of each face:

$$\left(\sigma_{xx}(x) + \frac{\partial \sigma_{xx}}{\partial x}\frac{\partial x}{2}\right)\partial y\,\partial z - \left(\sigma_{xx}(x) - \frac{\partial \sigma_{xx}}{\partial x}\frac{\partial x}{2}\right)\partial y\,\partial z = \\ = \frac{\partial \sigma_{xx}}{\partial x}\partial x\,\partial y\,\partial z. \tag{3.3}$$

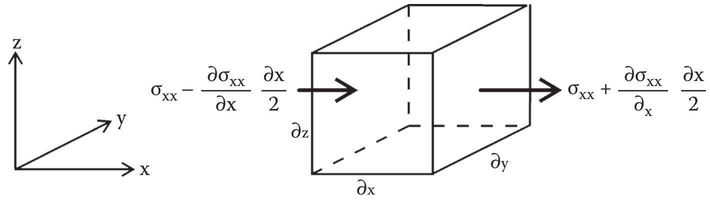

FIGURE 3.1 Normal stress acting on two opposing sides of a small volume element.

The shear stresses acting on the top and bottom faces are, respectively,

$$\sigma_{xz}(z) + \frac{\partial \sigma_{xz}}{\partial z} \frac{\partial z}{2}, \tag{3.4}$$

$$\sigma_{xz}(z) - \frac{\partial \sigma_{xz}}{\partial z} \frac{\partial z}{2}, \tag{3.5}$$

leading to a net force in the x-direction

$$\frac{\partial \sigma_{zx}}{\partial z} \partial x \, \partial y \, \partial z. \tag{3.6}$$

Similarly, the net force associated with shear stresses acting on the front and back face is given by

$$\frac{\partial \sigma_{yx}}{\partial y} \partial x \, \partial y \, \partial z. \tag{3.7}$$

With the x-axis chosen in the horizontal direction, there is no component of gravity acting in this direction. Summing the three net forces given by (3.3), (3.6), and (3.7) and dividing by the volume of the element $\partial x \cdot \partial y \cdot \partial z$, balance of forces in the x-direction is expressed by

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} = 0. \tag{3.8}$$

For the other horizontal direction (the y-direction) the balance equation is similar:

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} = 0. \tag{3.9}$$

The z-axis is chosen vertically, and positive upward, so that the gravitational body force acting per unit volume equals $-\rho g$, where ρ denotes the density of ice and g the acceleration due to gravity. Force balance for the vertical direction is then given by

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho g. \tag{3.10}$$

The stress tensor is symmetric, and there are six independent components of stress (Section 1.2). Because there are only three force-balance equations, the stress distribution cannot be determined without additional equations or simplifying assumptions. By invoking the constitutive relation, the number of unknowns can be reduced to three, namely, the three components of velocity. In principle, this would allow the velocity distribution, and hence the stresses, to be calculated. However, because the flow law for glacier ice is nonlinear, analytical solutions can be derived for only a limited number of very simple cases. These solutions are discussed in Chapter 4.

The force-balance equations derived above describe local conditions that apply to small volumes at any location within the glacier. From the perspective of modeling glacier flow, it is often necessary to extend the analysis to apply to the entire thickness to allow identification of (potential) sites of resistance to glacier flow, such as drag at the glacier bed or lateral drag at fjord walls. Budget of forces in a section of glacier has been discussed many times before (for example, Budd, 1970a, b; Echelmeyer and Kamb, 1986; Hutter, 1983, p. 265; Kamb, 1986; Kamb and Echelmeyer, 1986a, b; Nye, 1957, 1969b; Paterson, 1994, p. 263). The usual procedure is to assume plane flow in the x-direction (so that derivatives with respect to the other horizontal y-direction are zero) and integrate the balance equations (3.8) and (3.9) over the ice thickness, subject to the boundary conditions that the upper surface must be stress free. The resulting equation is

$$\tau_b \approx -\rho g H \left|\frac{\partial h}{\partial x}\right| + 2G - T, \tag{3.11}$$

where τ_b represents the shear stress at the base of the glacier, H the ice thickness, and h the elevation of the surface. The second and third terms on the right-hand side are defined respectively as

$$G = \frac{\partial}{\partial x} \int_{h-H}^{h} \tau_{xx}\, dz, \tag{3.12}$$

and

$$T = \int_{h-H}^{h} \int_{h-H}^{z} \frac{\partial^2 \tau_{xz}}{\partial x^2}\, d\bar{z}\, dz. \tag{3.13}$$

In equation (3.12), $\tau_{xx} = (\sigma_{xx} - \sigma_{zz})/2$ represents the longitudinal stress deviator for the case of plane strain.

The balance equation (3.11) is difficult to interpret because the physical meanings of G and T are not immediately clear. It is often argued that when averaged over horizontal distances 20 times the ice thickness or more, both terms may be neglected, while on the intermediate scale, with averages taken over about four ice thicknesses, T is negligible but G may be important. On smaller scales, all terms in equation (3.11) must be taken into account. This averaging scheme is based on observations and scaling arguments (for example, Kamb and Echelmeyer, 1986b; Greve and Blatter, 2009). However, the physical significance of neglecting G and/or T is not obvious. This confusion can be avoided if resistive stresses, rather than deviatoric stresses, are used to formulate the balance equations. By using resistive stresses, the physical significance of G and T becomes more clear and the implications of neglecting either or both become more evident.

The partitioning of full stresses into lithostatic and resistive components is based on the notion that glacier flow is driven by gravity and opposed by resistive forces such as basal drag (Whillans, 1987). The lithostatic stress at some depth in the ice is equal to the weight of the ice above that depth, and horizontal gradients in this stress drive the glacier flow. The resistive stresses are the difference between full stress and the lithostatic component, and they usually impede the flow of the glacier. Thus, a separation is made between action, or gravitational forces, and reaction, or resistive forces. In geophysical applications, the resistive stress is often referred to as the *tectonic stress* (Turcotte and Schubert, 2002, p. 77). Engelder (1993, p. 10) defines tectonic stresses as “components of the in situ stress field which are a deviation from a reference state.” A convenient reference state is the lithostatic stress (Turcotte and Schubert, 2002, p. 77).

The lithostatic stress is defined as the weight of ice above a level:

$$L = -\rho g (h - z) . \tag{3.14}$$

Full stresses are now written as

$$\sigma_{ij} = R_{ij} + \delta_{ij} L, \tag{3.15}$$

where δ_{ij} represents the Kronecker delta ($\delta_{ij} = 1$ for $i = j$; $\delta_{ij} = 0$ for $i \neq j$) and R_{ij} denotes the components of the resistive stress tensor. As with the partitioning of full stresses in stress deviator and hydrostatic pressure (equation (1.36)), only the normal components of the full stress tensor are affected by the partitioning (3.15). To arrive at the force-balance equations in terms of resistive stresses, the analysis of Van der Veen and Whillans (1989a) is followed.

Using the separation of full stresses into resistive and lithostatic components, equation (3.8) expressing force balance in the x-direction becomes

$$\frac{\partial [R_{xx} - \rho g (h - z)]}{\partial x} + \frac{\partial R_{xy}}{\partial y} + \frac{\partial R_{xz}}{\partial z} = 0. \tag{3.16}$$

Integrating from the base of an ice column (z = h – H) to the surface (z = h) gives

$$\int_{h-H}^{h} \frac{\partial R_{xx}}{\partial x}\, dz - \rho g H \frac{\partial h}{\partial x} + \int_{h-H}^{h} \frac{\partial R_{xy}}{\partial y}\, dz + R_{xz}(h) - R_{xz}(h-H) = 0. \tag{3.17}$$

The order of integration and differentiation can be switched by applying Leibnitz's rule:

$$\begin{aligned} &\frac{\partial}{\partial x}\int_{h-H}^{h} R_{xx} dz - R_{xx}(h)\frac{\partial h}{\partial x} + R_{xx}(h-H)\frac{\partial (h-H)}{\partial x} + \\ &+\frac{\partial}{\partial y}\int_{h-H}^{h} R_{xy}\, dz - R_{xy}(h)\frac{\partial h}{\partial y} + R_{xy}(h-H)\frac{\partial (h-H)}{\partial y} + \\ &-\rho g H\frac{\partial h}{\partial x} + R_{xz}(h) - R_{xz}(h-H) = 0. \end{aligned} \tag{3.18}$$

This equation can be greatly simplified by applying the surface boundary condition. The upper surface must be stress free; that is, the shear stress parallel to the surface must be zero. Using standard tensor transformation (for example, Jaeger, 1969), this condition reads, in terms of resistive stresses

$$R_{xx}(h)\frac{\partial h}{\partial x} + R_{xy}(h)\frac{\partial h}{\partial y} - R_{xz}(h) = 0. \tag{3.19}$$

Next, basal drag is defined to include all basal resistance

$$\tau_{bx} = R_{xz}(h-H) - R_{xx}(h-H)\frac{\partial (h-H)}{\partial x} - R_{xy}(h-H)\frac{\partial (h-H)}{\partial y}. \tag{3.20}$$

The driving stress is given by the familiar formula involving ice thickness and surface slope

$$\tau_{dx} = -\rho g H \frac{\partial h}{\partial x}. \tag{3.21}$$

The balance equation (3.18) then reduces to

$$\tau_{dx} = \tau_{bx} - \frac{\partial}{\partial x}\int_{h-H}^{h} R_{xx}\, dz - \frac{\partial}{\partial y}\int_{h-H}^{h} R_{xy}\, dz. \tag{3.22}$$

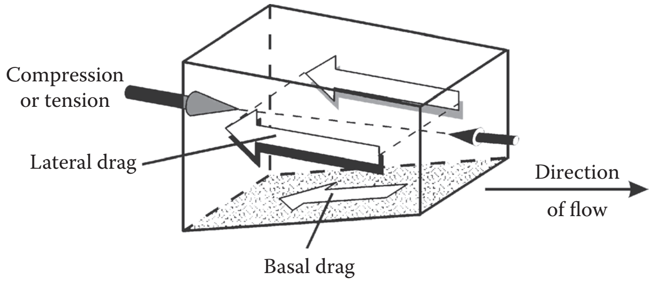

FIGURE 3.2 Resistive stresses opposing the driving stress (not shown), directed in the direction of decreasing surface slope. Flow resistance is associated with gradients in longitudinal stress (when compression from upglacier is larger or smaller than compression from downglacier), with friction generated at the sides of the glacier, and with drag at the bed.

Similarly, force balance for the second horizontal y-direction is given by

$$\tau_{dy} = \tau_{by} - \frac{\partial}{\partial x}\int_{h-H}^{h} R_{xy}\,dz - \frac{\partial}{\partial y}\int_{h-H}^{h} R_{yy}\,dz. \tag{3.23}$$

The balance equation for the vertical direction is discussed later in Section 3.4.

Each of the terms in equations (3.22) and (3.23) has a clear physical meaning as illustrated in Figure 3.2. Considering the balance equation for the x-direction, the driving stress represents the action, making the glacier flow. In most cases, this action is resisted by drag at the glacier base (first term on the right-hand side), by the difference between the normal forces acting on the right and left faces of an ice column (second term), and by lateral drag (third term). In some cases, however, normal stresses and lateral drag may act in accordance with the driving stress. Because each term is associated with a physical process, the implications of omitting one or more terms become more clear than when the balance equation (3.11) is used.

The main advantage of using the balance equations (3.22) and (3.23) is that these equations make a clear distinction between the various sources of flow resistance. Thus, they provide a useful tool for studying the mechanics of glaciers. Where velocity measurements are available, the importance of each of the resistive terms can be assessed, using the constitutive relation to link resistive stresses to strain rates. Determining the location and magnitude of forces opposing the flow of a glacier is the first step in developing models to describe the flow of glaciers.

In retrospect, the term "resistive" stress is a somewhat unfortunate choice because, as noted, these stresses do not always offer resistance to flow. Gradients in longitudinal stress (the second term on the right-hand side of equation (3.22)) can act in cooperation with the driving stress and pull the ice forward. Similarly, slower-moving ice outboard of fast-moving ice streams and outlet glaciers can be dragged in the downflow direction (Van der Veen et al., 2009). Perhaps a more appropriate term would be *flow stress* or, following geophysical terminology, *tectonic stress*. The R_{ij}

represent the stresses acting on the glacier that are associated with glacier flow and deformation as opposed to the driving stress, which describes the action of gravity. Nevertheless, the terminology appears to have made its way into the glaciological literature (for example, Cuffey and Paterson, 2010, Section 8.2.2), and a name change at this stage would likely introduce more confusion.

3.2 INTERPRETING FORCE BALANCE

Glacier flow is driven by the downslope component of gravity. This action is described by horizontal gradients in the lithostatic stress or, equivalently, by the driving stress as given by equation (3.11) and a similar expression for the second horizontal direction. The driving stress is balanced by friction between the moving ice and the bed, by lateral drag where the glacier is laterally bounded by rock or slower-moving ice, and by gradients in longitudinal tension and/or compression. The formal derivation of the balance equations given in the preceding section may obscure the physical meaning of the various terms. Therefore, this section provides a more intuitive derivation of the depth-integrated force-balance equation based on the geometric approach of Van der Veen and Payne (2003).

Figure 3.3 shows the lithostatic stresses acting on a section of glacier extending from the bed to the surface. Summing the gravitational forces acting on the column gives the component of driving stress in the x-direction. At each depth in the column, the lithostatic stress equals the weight of the ice above and, neglecting low-density firn near the surface, the lithostatic stress increases linearly with depth:

$$L(z) = -\rho g (h - z), \tag{3.24}$$

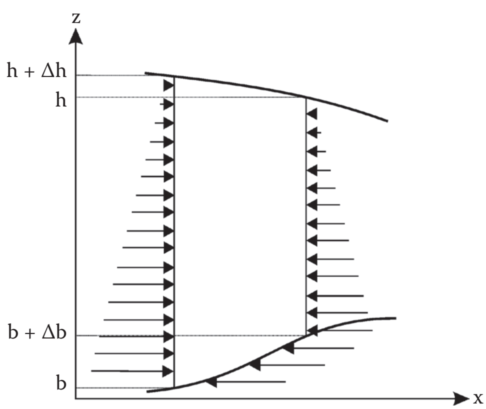

FIGURE 3.3 Vertical column through a glacier showing lithostatic stresses. Only those acting on the column are shown and summed to give the driving stress. (Reprinted from Van der Veen, C. J., and A. J. Payne, in *Mass Balance of the Cryosphere: Observations and Modeling of Contemporary and Future Changes*, Cambridge University Press, Cambridge, 2003. With permission of Cambridge University Press.)

where z = h represents the surface elevation; z = b denotes the elevation of the glacier bed. Integrating over the full depth gives the total lithostatic force acting on the left-hand face of the column:

$$\int_b^h \rho g(h - z)\, dz = \frac{1}{2}\rho g(h - b)^2. \tag{3.25}$$

At the right-hand side of the column, the surface and bed elevations are (h + Δh) and (b + Δb), respectively, and the force acting on that face is

$$\int_{b+\Delta b}^{h+\Delta h} \rho g(h - z)\, dz = \frac{1}{2}\rho g[(h+\Delta h) - (b+\Delta b)]^2. \tag{3.26}$$

Note that forces acting in the x-direction are taken positive so that the force acting on the right side is negative. To evaluate the lithostatic stress at the bed, the average column thickness may be used. This stress acts on a vertical step in the glacier sole with height Δb, giving the total lithostatic force at the bed:

$$-\rho g\left[\left(h + \frac{\Delta h}{2}\right) - \left(b + \frac{\Delta b}{2}\right)\right]\Delta b. \tag{3.27}$$

Summing the three forces acting on the column gives the net lithostatic force:

$$-\frac{1}{2}\rho g\left[\left(h + \frac{\Delta h}{2}\right) - \left(b + \frac{\Delta b}{2}\right)\right]\Delta h. \tag{3.28}$$

Denoting the average thickness by $\bar{H}$ and dividing by the length of the column, Δx, gives the driving force per unit map area,

$$-\rho g\bar{H}\frac{\Delta h}{\Delta x}. \tag{3.29}$$

Taking the limit $\Delta x \to 0$ yields the driving stress

$$\tau_{dx} = -\rho g H\frac{\partial h}{\partial x}. \tag{3.30}$$

A similar expression can be derived for the driving stress corresponding to the other horizontal direction.

The derivation of driving stress above differs from the conventional approach in which stresses in a plane sloping slab are considered (for example, Paterson, 1994, p. 240; Greve and Blatter, 2009, p. 83) and lamellar flow is assumed in which the driving stress is balanced by drag at the glacier base. From these definitions, it is not clear how the definition of driving stress can be extended to more complex geometries or to ice shelves on which basal drag is nonexistent. In the present approach, the term *driving stress* is reserved for the action due to gravity and it is given by equation (3.30) for any glacier geometry, irrespective of which resistive stress opposes this gravitational action.

Note that the slope of the base of the column is irrelevant for calculating the driving stress, and the lithostatic stress acting at the bed cancels out when adding the three forces acting on the column. While this may appear to be counterintuitive, why the bed slope does not enter into the definition of driving stress can be understood by considering a fluid of constant density confined in a cup. There is no tendency of the fluid to flow to the middle of the cup, despite the slope of the bottom of the cup. A nonzero surface slope is needed to drive the flow. The implication is that glaciers can flow "uphill," that is, against a reversed bed slope, provided the surface elevation decreases in the downflow direction. One could argue that for mountain glaciers on a steep slope, it is the bed slope that determines the driving stress. However, for such glaciers, the ice thickness generally is more or less constant in the flow direction and the surface slope is approximately equal to the bed slope. Equation (3.30), which involves only the surface slope, remains valid.

Resistive stresses are associated with glacier flow and, generally, act to oppose the driving stress. Consider first the horizontal normal stress, R_{xx}, acting on the vertical column, as shown in Figure 3.4. How this stress varies with depth is generally not known. Note that the contribution of the normal stress acting on the sloping bed is included in the definition of basal drag (equation (3.20)) and need not be considered

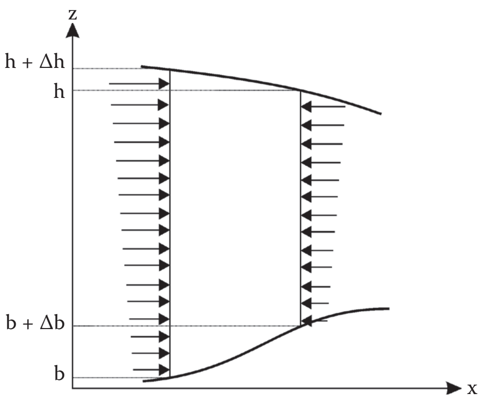

FIGURE 3.4 Horizontal resistive stresses acting on a vertical column through a glacier. The difference between the net force acting on the right and left faces gives the resistance to flow associated with longitudinal tension or compression. Note that the depth variation of these horizontal stresses is generally unknown. (Reprinted from Van der Veen, C. J., and A. J. Payne, in *Mass Balance of the Cryosphere: Observations and Modeling of Contemporary and Future Changes*, Cambridge University Press, Cambridge, 2003. With permission of Cambridge University Press.)

at this point. The longitudinal resistive force acting on the left face is obtained by integrating the resistive stress over the ice thickness:

$$\int_{b}^{h} R_{xx}(z)\,dz = H\bar{R}_{xx}, \tag{3.31}$$

where the overbar denotes the depth-averaged resistive stress. This force acts in the x-direction and is taken positive; the longitudinal force on the right face is oriented in the negative x-direction and therefore negative. Assuming the resistive stress on the right face differs from that on the left face by an amount ΔR_{xx}, integrating over the ice thickness gives

$$-\int_{b+\Delta b}^{h+\Delta h} \left[R_{xx}(z) + \Delta R_{xx}(z)\right] dz =$$

$$= -H\bar{R}_{xx} - \Delta H\bar{R}_{xx} - H\Delta\bar{R}_{xx} - \Delta H\Delta\bar{R}_{xx}. \tag{3.32}$$

Here $\Delta H = \Delta h - \Delta b$ is the difference in ice thickness at the two faces. Summing equations (3.31) and (3.32) and neglecting the higher-order term in the latter equation gives the net longitudinal force acting on the column

$$-\left[\frac{\Delta H}{\Delta x}\bar{R}_{xx} + H\frac{\Delta\bar{R}_{xx}}{\Delta x}\right]\Delta x. \tag{3.33}$$

Dividing by the length of the column gives the force per unit map area, and after taking the limit $\Delta x \to 0$, this net force is

$$-\frac{\partial H}{\partial x}\bar{R}_{xx} - H\frac{\partial \bar{R}_{xx}}{\partial x} = -\frac{\partial}{\partial x}(H\bar{R}_{xx}). \tag{3.34}$$

Expression (3.34) shows that resistance to flow is offered by *gradients* in resistive stress, rather than by the *magnitude* of the stress. This is readily understood by considering two people pushing on opposite ends of a large object; if both push equally hard, the object will not move because the net force is zero. Similarly, on glaciers, the longitudinal stress itself may be large compared with the driving stress, but the effect on force balance may be small when the gradients are much smaller than the driving stress. Note that gradients in longitudinal stress can oppose the driving stress but can also act in cooperation with the driving stress, where the ice is being pushed from upglacier or pulled from downglacier.

Where a glacier or ice stream is bounded by rock walls or slower-moving ice, lateral drag provides resistance to flow. This frictional resistance acts on vertical faces parallel to the x-direction and, following a derivation similar to that for the longitudinal force, is given by

$$-\frac{\partial}{\partial y}(H\bar{R}_{xy}), \tag{3.35}$$

where the horizontal y-axis is perpendicular to the x-axis.

The third source of resistance to flow is drag at the glacier bed, as expressed by basal drag, τ_{bx}. This drag may be due to frictional forces (skin drag) between the moving ice and the bed, and resistive stresses acting on a sloping bed (form drag). Formally, basal drag is defined as

$$\tau_{bx} = R_{xz}(b) - R_{xx}(b)\frac{\partial b}{\partial x} - R_{xy}(b)\frac{\partial b}{\partial y}. \tag{3.36}$$

The first term on the right-hand side describes skin friction and is associated with vertical shear in velocity at the glacier base.

Balance of forces requires the driving stress to be balanced by resistance to flow offered by gradients in longitudinal stress, lateral drag, and basal drag. Thus

$$\tau_{dx} = \tau_{bx} - \frac{\partial}{\partial x}(H\bar{R}_{xx}) - \frac{\partial}{\partial y}(H\bar{R}_{xy}). \tag{3.37}$$

A similar equation can be derived for the balance of forces in the other horizontal direction. These balance equations form the basis of the force-budget technique discussed in the next section.

3.3 THE FORCE-BUDGET TECHNIQUE

Neither the formal derivation in Section 3.1 nor the more intuitive derivation in Section 3.2 makes any assumption about glacier geometry or relative importance of resistance to flow, and the resulting force-balance equations apply to any glacier geometry and flow regime. A common practice is to apply scaling methods to these equations to determine the dominant terms in the balance equations (for example, Greve and Blatter, 2009). An alternative procedure is to determine the budget of forces where measurements of glacier geometry and (surface) velocity are available. The driving stress is calculated from the geometry (thickness and surface slope), while lateral drag and resistance associated with gradients in longitudinal stress can be estimated from measured velocities. Basal drag cannot be calculated directly; rather, this quantity is estimated from the force-balance equation as the value needed for all forces to sum up to zero.

Where surface velocity measurements are available, surface strain rates can be evaluated from the horizontal gradients in velocity. To relate these strain rates to stresses, the constitutive relation needs to be invoked. As discussed in Section 2.2, the constitutive relation links strain rates to deviatoric stresses. This means that to apply the constitutive relation to link resistive stresses to strain rates, it is necessary to write the resistive stresses in terms of stress deviators first. Comparing the two schemes for separating full stresses

$$\sigma_{ij} = \tau_{ij} + \delta_{ij}P, \tag{3.38}$$

and

$$\sigma_{ij} = R_{ij} + \delta_{ij}L, \tag{3.39}$$

where $P = (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})/3$ represents the spherical stress. Eliminating the full stress from these equations gives

$$\tau_{ij} + \delta_{ij}(P - L) = R_{ij}. \tag{3.40}$$

Taking i = j = z yields

$$\begin{aligned} P - L &= -\tau_{zz} + R_{zz} = \\ &= \tau_{xx} + \tau_{yy} + R_{zz}, \end{aligned} \tag{3.41}$$

because, by definition, the three normal deviatoric stresses sum to zero. The resistive stresses can now be written in terms of deviatoric stresses as follows:

$$R_{xx} = 2\tau_{xx} + \tau_{yy} + R_{zz}, \tag{3.42}$$

$$R_{yy} = 2\tau_{yy} + \tau_{xx} + R_{zz}, \tag{3.43}$$

$$R_{xy} = \tau_{yy}, \tag{3.44}$$

$$R_{xz} = \tau_{xz}, \tag{3.45}$$

$$R_{yz} = \tau_{yz}. \tag{3.46}$$

Applying the constitutive relation (2.12), the horizontal resistive stresses can be expressed in terms of strain rates:

$$R_{ij} = B\dot{\varepsilon}_e^{1/n-1}\dot{\varepsilon}_{ij}, \qquad i \neq j = x, y, z, \tag{3.47}$$

$$R_{ii} = B\dot{\varepsilon}_e^{1/n-1}(2\dot{\varepsilon}_{ii} + \dot{\varepsilon}_{jj}) + R_{zz}, \quad i = j = x,\ y. \tag{3.48}$$

These expressions contain the vertical resistive stress, R_{zz}. This stress can be calculated from consideration of the vertical force balance. This is discussed further in Section 3.4, but for now, it is sufficient to note that this stress is zero in most applications.

For the case in which the surface strain rates can be taken representative of those at depth, the theory for the force-budget technique is now complete. Measured surface velocities are used to determine strain rates, from which the resistive stresses at the surface are calculated (using equations (3.47) and (3.48), and neglecting the vertical resistive stress, R_{zz}). Taking these stresses constant through the ice thickness, the last two terms in the horizontal balance equations (3.22) and (3.23) can be estimated and, with the driving stress calculated from the glacier geometry, the remaining term, basal drag, follows from the requirement that the sum of all forces must be zero.

This so-called isothermal block-flow model has the great merit of being simple to carry out. Its drawback is that the viscous terms in the balance equations (the last two terms in (3.22) and (3.23)) tend to be overestimated because surface values are applied to the entire glacier thickness. Thus, the inferred value for basal drag represents a limiting value. The other extreme value for basal drag can be found by setting the viscous terms equal to zero; that is, by equating basal drag to the driving stress. The actual value for basal drag may be expected to fall in between these two extremes. An example illustrating the steps involved in the force-budget technique is discussed in Section 11.2.

The assumption of isothermal block flow need not be made, and Van der Veen and Whillans (1989a) describe a solution scheme that solves for the three components of velocity throughout a section of glacier (c.f. Van der Veen, 1989; Van der Veen, 1999b, Section 3.4). Calculations start at the glacier surface, where measured velocities are prescribed, and steadily progress downward by solving force balance for successive ice layers. The main difficulty with this downward calculation is that it is an inverse technique. That is, observed surface effects (surface velocities and surface slope) are used to determine the basal conditions that are causing these effects. However, as discussed in Section 4.6, the ice acts as a filter and the amplitude of horizontal variations decreases upward. This means that surface features reflect basal conditions, but the variations at the surface are much smaller than those at the glacier bed. Also, the extent of glacial filtering is strongly dependent on the wavelength of the feature and a basal feature with a horizontal scale of, say, one ice thickness may be difficult to detect at the surface, if at all. By using the downward calculation scheme, the filter works in the opposite way, and surface effects are amplified as the calculation proceeds downward. In particular, the small-scale surface features are amplified most. Typically, measured surface velocities contain relatively small errors that are amplified in the downward force-budget calculation. A more practical approach is to use a full-stress solver and find a solution that minimizes the difference between calculated and measured surface velocities in a least-squares sense.

3.4 BRIDGING EFFECTS

Where the weight of an ice column is fully supported by the bed underneath, the vertical normal stress at a depth (z – h) below the ice surface is equal to the weight of the ice above. That is

$$\sigma_{zz} = -\rho g(h - z), \tag{3.49}$$

and the vertical resistive stress, R_{zz}, is zero (equations (3.14) and (3.15)). For most applications, this is a valid approximation. Over horizontal distances that are short compared with the ice thickness, differences from lithostatic may become important. For example, in the lee of a subglacial hill, the basal ice may become separated from the bed if cavitation occurs. The cavity does not support the weight of the ice, leading to shear-stress gradients that effectively transfer the weight to surrounding areas where the ice is in contact with the bed (Section 7.2). This is similar to a bridge (hence the term *bridging effects*). The span of the bridge is not supported from below so that the full vertical normal stress is zero there and R_{zz} equals minus the lithostatic stress. The abutments carry the weight of the entire bridge, and the vertical normal stress is larger than the weight of the material directly above (Figure 3.5).

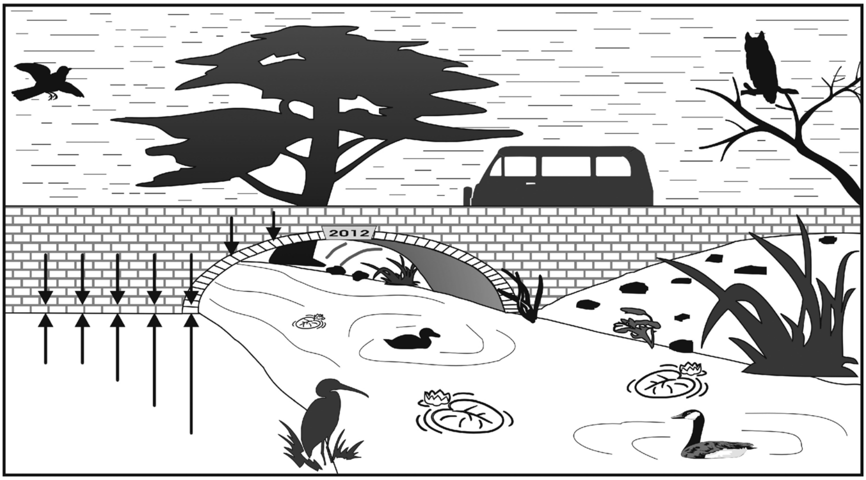

FIGURE 3.5 Illustrating bridging effects. The weight of the span of the bridge is not supported from below, but transferred to the abutments through variations in the shear stress; the abutments adjacent to the bridge support more than the weight of material directly above. (From Van der Veen, C. J., and I. M. Whillans, *J. Glaciol.*, 35, 53–60, 1989a. Reprinted from the *Journal of Glaciology* with permission of the International Glaciological Society and the authors.)

Force balance in the vertical direction is described by equation (3.10). Substituting the partitioning of full stresses into lithostatic and resistive components, this equation becomes

$$\frac{\partial R_{xz}}{\partial x} + \frac{\partial R_{yz}}{\partial y} + \frac{\partial R_{zz}}{\partial z} = 0. \tag{3.50}$$

Neglecting bridging effects ($R_{zz} \approx 0$) is thus equivalent to neglecting horizontal gradients in the vertical shear stresses, R_{xz} and R_{yz}. To estimate whether this approximation is reasonable, the balance equation is integrated over the full ice thickness to give

$$R_{zz}(b) = \int_b^h \frac{\partial R_{xz}}{\partial x}\, dz = \frac{\partial}{\partial x}\int_b^h R_{xz}\, dz + R_{xz}(b)\frac{\partial b}{\partial x}, \tag{3.51}$$

where $R_{zz}(h)$ and $R_{xz}(h)$ have been set to zero. The second term in equation (3.50) involving the transverse direction is omitted for brevity, but this has no effect on the following order-of-magnitude estimate.

In the first approximation, the vertical shear stress increases linearly with depth from zero at the surface to a maximum value equal to basal drag at the glacier base (c.f. Section 4.2). That is

$$R_{xz}(z) = \frac{h - z}{H}\tau_{bx}. \tag{3.52}$$

Substituting in equation (3.51) gives

$$R_{zz}(b) = \frac{1}{2} H \frac{\partial \tau_{bx}}{\partial x} + \tau_{bx} \frac{\partial b}{\partial x}. \tag{3.53}$$

The first term on the right-hand side may become important in the vicinity of *sticky spots* or subglacial lakes, while the second term may be significant where pronounced basal relief exists. Consider first a sticky spot with a length of 5 km and basal drag equal to 150 kPa (admittedly a high value; see Figure 4.3), surrounded by near-frictionless sediments. For a 1 km thick glacier, this gives

$$\frac{1}{2} H \frac{\partial \tau_{bx}}{\partial x} \approx \frac{1000}{2} \frac{150}{5000} = 15 \text{ kPa}. \tag{3.54}$$

Compared to the lithostatic stress at the bed, $L(b) = \rho g H \approx 9000$ kPa, the vertical resistive stress associated with spatial variations in basal drag is insignificant. Similarly, even if the bed slope is as large as 100 m/km, the associated vertical resistive stress is only 15 kPa for a basal drag equal to 150 kPa. This estimate of the magnitude of R_{zz} indicates that this stress generally will be small and may be set to zero when considering force balance or modeling glacier flow.

3.5 STOKES EQUATION APPLIED TO GLACIER FLOW

In fluid dynamics, flow of a viscous fluid is described by the Navier–Stokes equations (for example, Chorin and Marsden, 1992, p. 33; Pedlosky, 1982, p. 173). These equations express balance of forces in terms of velocity gradients and form the basis for many finite-element solvers. Given the increased application of finite-element routines to glaciological problems, this section illustrates how the formulation commonly used in fluid mechanics is equivalent to the balance equations (3.8)–(3.10).

For an incompressible fluid, the Navier–Stokes equations can be written in vector form as

$$-\rho \frac{\partial \vec{u}}{\partial t} - \rho (\vec{u} \bullet \nabla) \vec{u} = \nabla P - \nabla \bullet [\eta (\nabla \vec{u} + (\nabla \vec{u})^T] - \vec{F}, \tag{3.55}$$

where ρ represents density, $\vec{u} = (u, v, w)$ the velocity vector, P the pressure, η the dynamic viscosity, and $\vec{F}$ the volume or body force. The superscript T indicates the transpose of the velocity gradient tensor. For a 2 × 2 tensor with components a_{ij}, the transpose is defined as the tensor with components a_{ji}. Derivations of the Navier–Stokes equations for a viscous incompressible fluid can be found in Brown (1991, Section 6.3.3) and Greve and Blatter (2009, Section 5.1.1).

The terms on the left-hand side of equation (3.55) represent acceleration

$$\frac{d\vec{u}}{dt} = \frac{\partial \vec{u}}{\partial t} + \vec{u} \bullet \nabla \vec{u}. \tag{3.56}$$

For glacier flow, accelerations may be neglected and the basic equation becomes

$$\nabla P - \nabla \bullet [\eta(\nabla \vec{u} + (\nabla \vec{u})^T] = \vec{F}. \tag{3.57}$$

This simplification of the Navier–Stokes equation is referred to as the Stokes equation for an incompressible fluid.

The body force for glacier flow is gravity acting in the vertical z-direction. Then

$$F_z = -\rho g, \tag{3.58}$$

while the components in the x- and y-directions are zero. In this expression, $\rho = 920\ \mathrm{kg/m^3}$ represents the density of ice, and $g = 9.8\ \mathrm{m/s^2}$ the gravitational acceleration.

The above form (3.57) of the Stokes equation is in compact vector notation. To make the correspondence to the equations commonly used in glaciology more clear, the three corresponding equations (for each of the three orthogonal directions) are considered:

$$\frac{\partial P}{\partial x} - \frac{\partial}{\partial x}\left(2\eta\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(\eta\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) - \frac{\partial}{\partial z}\left(\eta\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right) = 0, \tag{3.59}$$

$$\frac{\partial P}{\partial y} - \frac{\partial}{\partial x}\left(\eta\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right) - \frac{\partial}{\partial y}\left(2\eta\frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial z}\left(\eta\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial z}\right)\right) = 0, \tag{3.60}$$

$$\frac{\partial P}{\partial z} - \frac{\partial}{\partial x}\left(\eta\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right) - \frac{\partial}{\partial y}\left(\eta\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right) - \frac{\partial}{\partial z}\left(2\eta\frac{\partial w}{\partial z}\right) = -\rho g. \tag{3.61}$$

To rewrite the Stokes equation applied to glacier flow in the more common force-balance form, the flow law for glacier ice needs to be considered. This relation links deviatoric stresses to strain rates (or velocity gradients) as (Section 2.2)

$$\tau_{ij} = B\dot{\varepsilon}_e^{1/n-1}\ \dot{\varepsilon}_{ij}. \tag{3.62}$$

In this expression, the effective strain rate is

$$\dot{\varepsilon}_e = \left[\frac{1}{2}\left(\dot{\varepsilon}_{xx}^2 + \dot{\varepsilon}_{yy}^2 + \dot{\varepsilon}_{zz}^2\right) + \left(\dot{\varepsilon}_{xy}^2 + \dot{\varepsilon}_{xz}^2 + \dot{\varepsilon}_{yz}^2\right)\right]^{1/2}. \tag{3.63}$$

Defining the effective viscosity as

$$\eta = \frac{1}{2}B\dot{\varepsilon}_e^{1/n-1}, \tag{3.64}$$

the flow law can be simplified to

$$\tau_{ij} = 2\eta\ \dot{\varepsilon}_{ij}. \tag{3.65}$$

In particular,

$$\tau_{xx} = 2\eta\frac{\partial u}{\partial x}, \tag{3.66}$$

$$\tau_{xy} = \eta\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{3.67}$$

$$\tau_{xz} = \eta\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right). \tag{3.68}$$

The Stokes equation (3.59) for the x-direction can then be written as

$$\frac{\partial P}{\partial x} - \frac{\partial}{\partial x}(\tau_{xx}) - \frac{\partial}{\partial y}(\tau_{xy}) - \frac{\partial}{\partial z}(\tau_{xz}) = 0. \tag{3.69}$$

In fluid dynamics, the pressure, P, is defined as

$$P = -\,(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})/3. \tag{3.70}$$

Note that this definition has the opposite sign from that commonly used in glaciology! Then, deviatoric stresses, τ_{ij}, are linked to full stresses, σ_{ij}, as

$$\tau_{ij} = \sigma_{ij} + \delta_{ij}P, \tag{3.71}$$

and equation (3.69) becomes

$$-\frac{\partial}{\partial x}(\sigma_{xx}) - \frac{\partial}{\partial y}(\sigma_{xy}) - \frac{\partial}{\partial z}(\sigma_{xz}) = 0. \tag{3.72}$$

This equation is the same as the force-balance equation (3.8) that serves as the starting equation for modeling glacier flow,

$$\frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{xy}) + \frac{\partial}{\partial z}(\sigma_{xz}) = 0. \tag{3.73}$$

The Stokes equations do not involve assumptions about relative magnitudes and can be solved using iterative procedures after supplementing with appropriate boundary conditions. Examples of studies applying this method include Johnson and Staiger (2007), Gagliardini et al. (2007), Gagliardini and Zwinger (2007), and Durand et al. (2009b). In these studies, finite-element codes are used to solve equation (3.55) by numerical iteration. Typically, plane flow along a glacier flowline is considered. Isothermal conditions may be assumed or the temperature throughout the glacier is calculated by simultaneously solving the thermodynamic equation. Anyone who has tried this knows that key to finding a correct or physically plausible solution is to impose appropriate boundary conditions at the four boundaries of the model domain (ice surface, ice/bed interface, and the upglacier and downglacier ends of the flowline).

The upper surface must be stress free, and both the shear stress parallel to the surface and the normal stress perpendicular to the surface must be zero. Equation (3.19) expresses the first of these conditions. The second condition enters when considering balance of forces in the vertical direction and is approximated as $R_{zz}(h) = 0$ in Section 3.4. In vector notation, the requirement for a stress-free surface is written as

$$\vec{n}_s \bullet (\vec{\sigma}_s \bullet \vec{n}_s) = 0, \tag{3.74}$$

(Durand et al., 2009b), or, equivalently,

$$P_s \vec{I} - \left[\eta\left(\nabla \vec{u}_s + (\nabla \vec{u}_s\right)^T \right] = 0, \tag{3.75}$$

(Johnson and Staiger, 2007). In these equations, the subscript s refers to surface values and $\vec{n}_s$ represents the unit vector outward and perpendicular to the ice surface; $\vec{\sigma}$ represents the full stress.

The basal boundary condition depends on what type of glacier is being modeled. For a grounded glacier frozen to the bed, the following no-slip condition is prescribed (Johnson and Staiger, 2007):

$$\vec{u}_b = 0. \tag{3.76}$$

On a grounded glacier subject to basal sliding, a sliding relation linking basal velocity to basal drag must be imposed. For example, Durand and others (2009b) use

$$\tau_b = \vec{t}_b \bullet (\vec{\sigma}_b \bullet \vec{n}_b) = C U_b^m, \tag{3.77}$$

and

$$\vec{u}_b \bullet \vec{n}_b = 0, \tag{3.78}$$

where $\vec{t}_b$ is the unit vector tangent to the bed, such that $\vec{n}_b \times \vec{t}_b = 0$. The second basal requirement states that the sliding velocity must be parallel to the bed. For the case of floating ice shelves, basal drag must vanish, and

$$\tau_b = \vec{t}_b \bullet (\vec{\sigma}_b \bullet \vec{n}_b) = 0. \tag{3.79}$$

Ignoring the effect of long-period ocean waves (c.f. Sergienko, 2010), the normal stress at the base of the ice shelf equals the water pressure, and

$$\vec{\sigma}_b \bullet \vec{n}_b = -\vec{n}_b (\rho g H). \tag{3.80}$$

For a flowline originating at the ice divide, the upglacier boundary conditions follow from the requirement of zero mass flux,

$$\vec{u}_d \bullet \vec{n}_d = 0, \tag{3.81}$$

and vanishing shear stress

$$\vec{t}_d\left[P_d\vec{I} - \left[\eta(\nabla\vec{u}_d + (\nabla\vec{u}_d)^T\right]\right] = 0, \tag{3.82}$$

(Johnson and Staiger, 2007). In these expressions, the subscript d refers to the (usually) vertical ice divide boundary and $\vec{n}_d$ is the unit vector perpendicular to this boundary, while $\vec{t}_d$ is the unit vector tangent to the divide boundary.

Where the lower end of the model domain coincides with the floating calving front, the normal stress is prescribed in terms of sea-water pressure as

$$\vec{\sigma}_c \bullet \vec{n}_c = -\vec{n}_{cb}\, P_w, \tag{3.83}$$

where

$$P_w = \begin{cases} 0 & \text{if} \quad \dfrac{\rho}{\rho_w}H \le z < H \\ \rho_w\, g\left(\dfrac{\rho}{\rho_w}H - z\right) & \text{if} \quad 0 \le z < \dfrac{\rho}{\rho_w}H \end{cases} \tag{3.84}$$

with z = 0 corresponding to sea level (Sergienko, 2010).

For applications in which the modeled flowline does not extend from the divide to the calving front, formulating boundary conditions for the up- and downglacier boundaries is less straightforward. Because these boundary conditions will depend on the particular geometry being modeled, no general expressions can be given here.

3.6 CREEP CLOSURE OF ENGLACIAL TUNNELS

The balance equations discussed in this chapter, together with the constitutive relation discussed in Chapter 2, form the basis for models of glacier flow. In most instances, simplifying assumptions are invoked to allow an analytical solution to be found. Examples are discussed in following chapters, and in particular Chapter 5. For most of these applications, a Cartesian coordinate system is used with the three coordinate axes mutually perpendicular. However, in some situations it is more convenient to use cylindrical coordinates (r, θ, z), with the coordinates of a point defined by the radius, r, the azimuth angle, θ, and elevation, z (Figure 3.6). This system is most suitable if the problem being modeled possesses rotational symmetry around the z-axis, such as an axi-symmetric ice sheet (discussed in Section 5.4). In this section, another example of the use of cylindrical coordinates in solving for the stress distribution is considered, namely, creep closure of an englacial drainage tunnel. Such tunnels are maintained by a balance between the rate at which the tunnels close due to creep of the overlying ice, and melting of the tunnel walls by

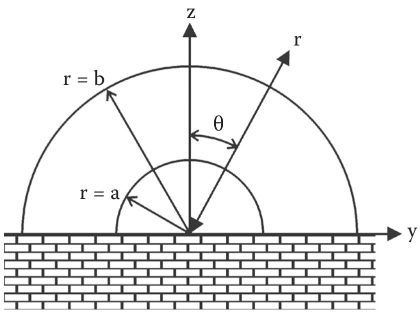

FIGURE 3.6 Geometry used in the determination of the creep closure rate of a semicircular tunnel. The radius of the tunnel is a; at sufficiently large distances from the tunnel (r = b >> a), the flow of the glacier is unperturbed by the tunnel and, in the x-direction, perpendicular to the plane of drawing.

the water flowing through them (c.f. Section 7.5). Tunnel closure was first discussed by Nye (1953) in connection with estimating the flow-law parameters from the observed rate of borehole closure. The analysis given below is somewhat different from that of Nye (1953) and demonstrates that even for a comparatively well-defined and simple problem, deriving an analytical solution may become rather involved.

To allow an analytical solution to be found, the shape of the englacial tunnel, incised in the basal ice, is assumed to be semicircular, with radius a as shown in Figure 3.6. The axis of the tunnel is taken to follow the main direction of ice flow and extending infinitely far in the x-direction, so that the problem becomes essentially two-dimensional. Additionally, there is no ice flow in the transverse y-direction, other than localized creep into the tunnel.

The Cartesian and cylindrical coordinates are related as

$$\begin{aligned} y &= r\cos\theta, \\ z &= r\sin\theta. \end{aligned} \tag{3.85}$$

The components of the stress tensor in the polar coordinates are found by using standard tensor transformation formulas

$$\sigma_{rr} = \frac{\sigma_{zz}+\sigma_{yy}}{2} + \frac{\sigma_{zz}-\sigma_{yy}}{2}\cos 2\theta + \sigma_{yz}\sin 2\theta, \tag{3.86}$$

$$\sigma_{\theta\theta} = \frac{\sigma_{zz}+\sigma_{yy}}{2} - \frac{\sigma_{zz}-\sigma_{yy}}{2}\cos 2\theta - \sigma_{yz}\sin 2\theta, \tag{3.87}$$

$$\sigma_{r\theta} = \sigma_{yz}\cos 2\theta - \frac{\sigma_{zz}-\sigma_{yy}}{2}\sin 2\theta. \tag{3.88}$$

The equations describing force balance in polar coordinates can be derived by considering the forces acting on a small element within the glacier (for example, Xu, 1992), similar to the derivation in Section 3.1. Summing all forces acting in the radial direction and setting the result equal to zero (because accelerations may be neglected) gives

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + F_r = 0. \tag{3.89}$$

Similarly, for the tangential direction, force balance is given by

$$\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{2\sigma_{r\theta}}{r} + F_\theta = 0. \tag{3.90}$$

In these expressions, F_r and F_θ represent the components of the body force in the radial and tangential directions, respectively. The only body force acting on the glacier is gravity, $-\rho g$, directed vertically downward. Thus,

$$\begin{aligned} F_r &= -\rho g \cos\theta, \\ F_\theta &= +\rho g \sin\theta. \end{aligned} \tag{3.91}$$

The stress solution to be derived must satisfy boundary conditions. At the wall of the tunnel ($r = a$) the radial normal stress must balance the water pressure within the tunnel, P_w, while the shear stress along the tunnel wall may be set to zero. Thus

$$\sigma_{rr}(a) = -P_w, \tag{3.92}$$

$$\sigma_{r\theta}(a) = 0. \tag{3.93}$$

The tunnel is assumed to be small compared with the dimensions of the glacier, and it may be expected that the tunnel only affects stresses within a circular area surrounding its walls. In other words, at radial distances much larger than the radius of the tunnel ($r >> a$), the stress solution should be unaltered by the presence of the tunnel.

The undisturbed stress solution (referenced to the Cartesian coordinate system) can be found as follows. From the assumption that the flow is independent of the along-flow x-direction, it follows that the stress deviator in this direction must be zero ($\tau_{xx} = 0$). Similarly, the stress deviator in the transverse direction must be zero ($\tau_{yy} = 0$). Applying these conditions, and using the definition of stress deviator, the two normal-stress components in both horizontal directions are found to be

$$\begin{aligned} \sigma_{xx} &= \sigma_{zz}, \\ \sigma_{yy} &= \sigma_{zz}. \end{aligned} \tag{3.94}$$

For the present discussion, σ_{xx} need not be considered. Neglecting bridging effects, the vertical normal stress is simply the weight of the ice above; thus,

$$\sigma_{zz} = -\rho g(H - z), \tag{3.95}$$

if z = 0 is chosen at the base of the glacier, and H represents the thickness of the overlying ice. The undisturbed shear stress, σ_{yz}, is zero and applying the transformation formulas (3.86)–(3.88) gives the undisturbed stresses referenced to the cylindrical coordinates

$$\sigma_{rr}(r) = -P_i + \rho g r \cos\theta, \tag{3.96}$$

$$\sigma_{\theta\theta}(r) = -P_i + \rho g r \cos\theta, \tag{3.97}$$

$$\sigma_{r\theta}(r) = 0, \tag{3.98}$$

with $P_i = \rho gH$ the ice overburden pressure.

The next step is to reformulate the flow law for strain rates and stresses referenced to cylindrical coordinates. In Cartesian coordinates, strain rates are linked to deviatoric stresses using the conventional flow law (Section 2.2)

$$\dot{\varepsilon}_{ij} = A\tau_e^{n-2}\tau_{ij}, \tag{3.99}$$

with A the rate factor and τ_e the effective stress. For the geometry under consideration, the effective stress is

$$\begin{aligned} \tau_e^2 &= \tau_{zz}^2 + \tau_{yz}^2 = \\ &= \frac{1}{4}(\sigma_{zz} - \sigma_{yy})^2 + \sigma_{yz}^2, \end{aligned} \tag{3.100}$$

or, in terms of stresses referred to the cylindrical coordinates

$$\tau_e^2 = \frac{1}{4}(\sigma_{rr} - \sigma_{\theta\theta})^2 + \sigma_{r\theta}^2. \tag{3.101}$$

The strain-rate tensor is subject to the same transformation formulas (3.86)–(3.88) as the stress tensor. Noting that $\dot{\varepsilon}_{xx}$ is zero so incompressibility requires $\dot{\varepsilon}_{yy} + \dot{\varepsilon}_{zz} = 0$, the strain-rate components in cylindrical coordinates are

$$\dot{\varepsilon}_{rr} = \frac{1}{2}(\dot{\varepsilon}_{zz} - \dot{\varepsilon}_{yy})\cos 2\theta + \dot{\varepsilon}_{yz}\sin 2\theta, \tag{3.102}$$

$$\dot{\varepsilon}_{\theta\theta} = -\dot{\varepsilon}_{rr}, \tag{3.103}$$

$$\dot{\varepsilon}_{r\theta} = \dot{\varepsilon}_{yz}\cos 2\theta - \frac{1}{2}(\dot{\varepsilon}_{zz} - \dot{\varepsilon}_{yy})\sin 2\theta. \tag{3.104}$$

Equation (3.103) can be interpreted as the incompressibility condition. Invoking the flow law (3.99) to express the Cartesian strain-rate components in terms of Cartesian stress deviators and applying the stress-transformation formulas gives the flow law for the cylindrical coordinate system:

$$\dot{\varepsilon}_{rr} = \frac{A}{2^n}\left[\left(\sigma_{rr} - \sigma_{\theta\theta}\right)^2 + 4\sigma_{r\theta}^2\right]^{(n-1)/2} (\sigma_{rr} - \sigma_{\theta\theta}), \tag{3.105}$$

$$\dot{\varepsilon}_{r\theta} = \frac{A}{2^{n-1}}\left[\left(\sigma_{rr} - \sigma_{\theta\theta}\right)^2 + 4\sigma_{r\theta}^2\right]^{(n-1)/2} \sigma_{r\theta}. \tag{3.106}$$

As in the Cartesian coordinate system, strain rates in the cylindrical coordinate system are linked to velocity gradients. The two components of velocity are u_r in the radial direction and u_θ in the tangential direction (in addition, there may be a component in the along-flow x-direction, but this velocity need not be considered in the present analysis). The strain-rate components are (for example, Xu, 1992)

$$\dot{\varepsilon}_{rr} = \frac{\partial u_r}{\partial r}, \tag{3.107}$$

$$\dot{\varepsilon}_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}, \tag{3.108}$$

$$\dot{\varepsilon}_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}. \tag{3.109}$$

Applying equation (3.103) gives

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0, \tag{3.110}$$

as the incompressibility condition expressed in terms of velocity gradients.

With the basic equations in place, the stress solution in the vicinity of the tunnel can be derived. According to the boundary conditions (3.93) and (3.98), the shear stress, $\sigma_{r\theta}$ is zero on both boundaries of the area affected by the tunnel ($a \le r \le b$). It may therefore be expected that this stress is small, if not zero, in the entire domain. Neglecting the shear stress, the force-balance equations (3.89) and (3.90) reduce to

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} - \rho g \cos\theta = 0, \tag{3.111}$$

$$\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \rho g \sin\theta = 0. \tag{3.112}$$

The second equation contains as the only unknown the tangential normal stress, $\sigma_{\theta\theta}$, and can thus be solved, subject to the boundary condition (3.97). Substituting the solution in the first balance equation (3.111) allows the other normal stress component to be found.

Integrating equation (3.112) with respect to the radial distance, r, gives

$$\sigma_{\theta\theta} = \rho g r \cos\theta + C(r), \tag{3.113}$$

in which the integration constant, C(r), may be a function of the radial distance, r. Applying the boundary condition (3.97) gives

$$C(r) = -P_i, \tag{3.114}$$

for r >> a. Any function of the radial distance that reduces to (3.114) for large values of r can be chosen for C. The simplest possible form, namely, $C = -P_i$ for all values of r, is selected here.

Substituting the solution (3.113) in the balance equation (3.111) gives

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr}}{r} - \frac{C}{r} - 2\rho g \cos\theta = 0. \tag{3.115}$$

The solution of the homogeneous part is found by setting C = 0 and is

$$\sigma_{rr}^{(h)} = \frac{B}{r}, \tag{3.116}$$

with B a constant. To find the solution of the full equation, assume that B = B(r). Substituting in equation (3.115) gives

$$\frac{\partial B}{\partial r} = -C + 2\rho g r \cos\theta. \tag{3.117}$$

This expression can be integrated to give

$$B(r) = -P_i r + \rho g r^2 \cos\theta + G, \tag{3.118}$$

in which G represents yet another integration constant that may be a function of the angle, θ.

The solution for the radial normal stress now becomes

$$\sigma_{rr}(r) = -P_i + pgr\cos\theta + \frac{G}{r}. \tag{3.119}$$

For large values of the radial distance, r, this expression reduces to the undisturbed solution (3.96). At the wall of the tunnel, the radial normal stress is

$$\sigma_{rr}(a) = -P_i + \rho g a \cos\theta + \frac{G}{a}, \tag{3.120}$$

and should balance the water pressure, P_w. This yields the integration constant

$$G = aP_e - \rho g a^2 \cos\theta, \tag{3.121}$$

in which $P_e = P_i - P_w$ represents the effective basal pressure. The radial normal stress thus becomes

$$\sigma_{rr}(r) = -P_i + \frac{a}{r}P_e + \rho g r\left(1 - \frac{a^2}{r^2}\right)\cos\theta. \tag{3.122}$$

An expression for the radial velocity, u_r, can be found by integrating expression (3.105) for the radial strain rate with respect to r. From the stress solutions (3.113) and (3.122), it follows that

$$\sigma_{rr} - \sigma_{\theta\theta} = \frac{a}{r}P_e - \rho g\frac{a^2}{r}\cos\theta, \tag{3.123}$$

giving the radial strain rate

$$\dot{\varepsilon}_{rr}(r) = \frac{A}{2^n}\left(aP_e - \rho g a^2 \cos\theta\right)^n \frac{1}{r^n}. \tag{3.124}$$

Integrating with respect to r yields

$$u_r(r) = \frac{A}{2^n}\left(aP_e - \rho g a^2 \cos\theta\right)^n \frac{-1}{(n-1)r^{n-1}}. \tag{3.125}$$

The integration constant has been set to zero to satisfy the condition that the radial velocity must vanish for large values of the radial distance.

The rate at which the tunnel closes is given by $-u_r(a)/a$, or, using equation (3.125),

$$S = \frac{A}{(n-1)2^n}\left(P_e - \rho g a \cos\theta\right)^n. \tag{3.126}$$

The effective pressure, P_e, is defined as the difference between the ice overburden pressure and the water pressure. Compared to the weight of the ice above (ρgH), the second term between the brackets may be neglected and

$$S = \frac{AP_e^n}{(n-1)2^n}. \tag{3.127}$$

This result is essentially the same as that derived by Nye (1953), except for the constants in the denominator. His derivation, however, is somewhat different from the one given above and, admittedly, much shorter.

Nye (1953) makes the assumption that the flow is purely radial, so that the tangential component of velocity vanishes. The incompressibility condition then reduces to

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} = 0, \tag{3.128}$$

which can be integrated to give

$$u_r(r) = -\frac{C}{r}, \tag{3.129}$$

with C a positive constant. Differentiating equation (3.129) with respect to r gives the radial strain rate

$$\dot{\varepsilon}_{rr}(r) = \frac{C}{a^2}. \tag{3.130}$$

Inverting the flow law (3.105), the effective stress is

$$\begin{aligned} \sigma_{rr} - \sigma_{\theta\theta} &= \frac{2}{A^{1/n}} \dot{\varepsilon}_{rr}^{1/n} = \\ &= \frac{2C^{1/n}}{A^{1/n}} \frac{1}{r^{2/n}}. \end{aligned} \tag{3.131}$$

Substituting this expression in the balance equation (3.111) gives

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{2C^{1/n}}{A^{1/n}} \frac{1}{r^{2/n+1}} + \rho g \cos\theta = 0. \tag{3.132}$$

Integrating with respect to r, and using the condition that the radial stress must be equal to the undisturbed solution (3.96) for large values of r to determine the integration constant, yields the stress solution

$$\sigma_{rr}(r) = -P_i + \rho g r \cos\theta + \frac{nC^{1/n}}{A^{1/n}} \frac{1}{r^{2/n}}. \tag{3.133}$$

The constant, C, can be determined from the condition that on the wall of the tunnel the radial normal stress must balance the water pressure, giving

$$C = \frac{A}{n^n} (P_e - \rho g a \cos\theta)^n a^n. \tag{3.134}$$

The rate of closure, $S = u_r(a)/a$, now becomes

$$S = \frac{A}{n^n}(P_e - \rho g a \cos\theta)^n. \qquad (3.135)$$

Comparison with equation (3.126) shows that both methods yield the same result, except for the constants. This difference can be attributed to neglecting the tangential component of velocity in Nye's analysis. Given the uncertainty in the rate factor, the difference is probably not very important.