1 Mathematical Tools

1.1 VECTORS AND TENSORS

This book assumes that readers are familiar with the fundamental principles of vector calculus. This section briefly summarizes some of the results pertinent to applications discussed in later chapters. This overview is based on Chapter 1 in Joos (1934).

Quantities such temperature and mass that are characterized by the assignment of a single number (for example, 15°C and 243 kg) are called *scalars*. Other physical quantities, most notably displacement and velocity, cannot be completely defined by specifying a single number only. Consider, for example, the change in position of a stake placed in the ice surface. If the magnitude of the displacement of the stake is 100 m in a certain time interval, then without additional information, the new position of the stake may be anywhere on a circle with a radius of 100 m and centered on the original position. To completely describe the new position of the stake, the direction of displacement also needs to be given. In other words, displacement is defined by two numbers, namely, magnitude and direction. Quantities with this characteristic are called *vectors* and written as $\vec{u}$, with the arrow above indicating the vector quantity. Another frequently used convention is to use boldface (**u**) when referring to vectors. Graphically, vectors are shown as arrows, as in Figure 1.1.

Vectors can be subject to mathematical operations such as addition or subtraction of two or more vectors, or multiplication with a scalar. Given two vectors, $\vec{u}$ and $\vec{v}$, the sum is the vector that joins the initial point of $\vec{u}$ with the end point of $\vec{v}$ after putting the initial point of $\vec{v}$ at the end point of $\vec{u}$, as illustrated in Figure 1.1. From this figure it can be seen that vector addition is commutative, and

$$\vec{u} + \vec{v} = \vec{v} + \vec{u}. \tag{1.1}$$

Further, vector addition is associative and

$$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}) = \vec{u} + \vec{v} + \vec{w}. \tag{1.2}$$

Defining the vector $-\vec{v}$ as the vector with the same magnitude as $\vec{v}$ but with the opposite direction, vector subtraction, $\vec{u} - \vec{v}$, is equivalent to the addition, $\vec{u} + (-\vec{v})$. Finally, the product of a vector, $\vec{u}$, and a scalar, m, is a vector with the same direction of $\vec{u}$ (if $m > 0$) or pointing in the opposite direction (if $m < 0$), and with a magnitude that is equal to the absolute value of m times the magnitude of $\vec{u}$.

To define a vector, a coordinate system must be selected. Several such systems exist including the geographic coordinate system in which any point on the earth's surface is characterized by latitude and longitude, with sometimes a third coordinate included,

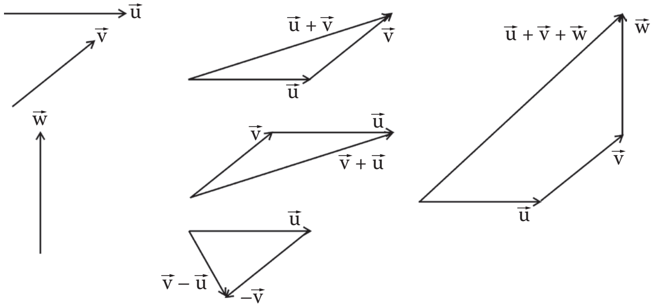

FIGURE 1.1 Illustrating addition and subtraction of three arbitrary vectors.

namely, height above sea level. In mathematics and physics, a commonly used system is the Cartesian coordinate system, which is used throughout most of this book. This system, shown in Figure 1.2, consists of three orthogonal (perpendicular) axes, referred to as the x-, y-, and z-axis. Any point in space is then uniquely defined by three coordinates, written as (x, y, z). Similarly, a vector is defined by three components. For example, velocity can be written as $\vec{u} = (u_x, u_y, u_z)$ or $\vec{u} = (u, v, w)$, with $= u_x = u$ the velocity component in the x-direction, and similar for the other two components. For the most part, this notation is used throughout this book. In a few instances, however, variable subscripts are used to indicate the three orthogonal directions as this allows for compact formulation of equations. For example, x_i with i = 1, 2, or 3 refers to the three orthogonal directions, with $x_1 = x$, $x_2 = y$, and $x_3 = z$. Similarly, $u_1 = u_x = u$, and so forth.

The coordinate system shown in Figure 1.2 is defined by three unit vectors, $\vec{i}$, $\vec{j}$, and $\vec{k}$, whose length equals unity and are oriented in the x-, y-, and z-direction, respectively. The velocity vector can then also be written as the sum of three vectors, each oriented in one of the three Cartesian directions. That is,

$$\vec{u} = u\vec{i} + v\vec{j} + w\vec{k}. \tag{1.3}$$

This notational convention invokes the additive properties of vectors, illustrated in Figure 1.1. Two or more vectors may be combined into a single vector by summing the corresponding components. Let $\vec{u}_1 = (u_1, v_1, w_1)$ and $\vec{u}_2 = (u_2, v_2, w_2)$ be

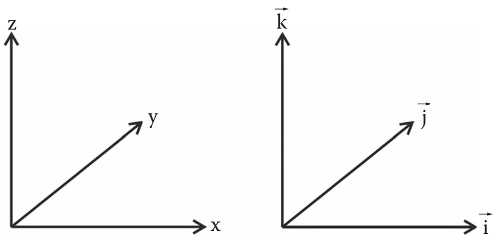

FIGURE 1.2 Cartesian coordinate system with the z-axis directed vertically upward (left panel). Each direction is defined by a unit vector (right panel).

the velocity of a point on a glacier measured over two consecutive years. The total velocity over the two-year interval is then

$$\vec{u}_t = \vec{u}_1 + \vec{u}_2 = (u_1 + u_2, v_1 + v_2, w_1 + w_2). \tag{1.4}$$

When discussing glacier flow, it is most convenient to take the z-axis in the vertically upward direction or, equivalently, perpendicular to the local geoid. Then the plane defined by the x- and y-axes corresponds to the (local) horizontal plane. The advantage of choosing this orientation is that the direction of the gravitational force coincides with the z-axis (but pointing in the opposite direction). This convention is opposite of that followed in many geophysics textbooks where the vertical axis points downward (see, for example, Turcotte and Schubert, 2002). The main difference is that gravity is negative when the vertical axis points upward and positive when this axis points downward.

In most glacier applications, the vertical velocity is small compared with the horizontal velocity components, and only the two-dimensional horizontal velocity vector, $\vec{u} = (u, v)$ needs to be considered. This vector is characterized by two quantities: magnitude, $|\vec{u}|$, and direction, ϕ, defined with respect to the x-axis. Referring to Figure 1.3, these quantities follow from trigonometry and

$$|\vec{u}| = \sqrt{u^2 + v^2}, \tag{1.5}$$

$$\tan\phi = \frac{v}{u}. \tag{1.6}$$

The velocity magnitude always has the same value, irrespective of the orientation of the (x, y) coordinate system. The direction, however, is defined as the angle between the velocity vector and the x-axis, and this angle will change if the orientation of the coordinate system changes.

Consider a Cartesian coordinate system (p, q) rotated over an angle α relative to the (x, y) system (both systems are in the horizontal plane so that the vertical z-axis is the rotation axis). To obtain the velocity components in the (p, q) system, the velocity components in the (x, y) system are rotated separately. From the geometry shown

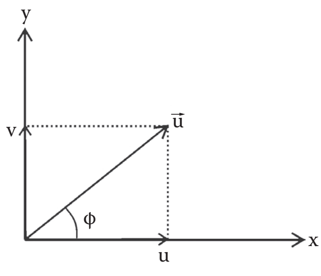

FIGURE 1.3 Components of the horizontal velocity vector, $\vec{u} = (u, v)$.

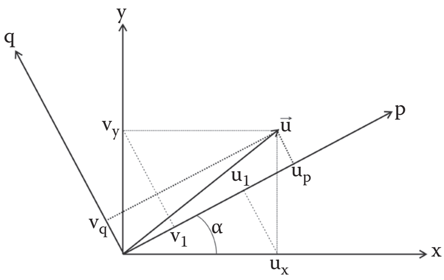

FIGURE 1.4 Illustrating vector rotation.

in Figure 1.4, it follows that cos $\alpha = u_1/u$ and sin $\alpha = v_1/v$. The total component of velocity in the p-direction is $u_p = u_1 + v_1$ and

$$u_p = u_x \cos\alpha + v_y \sin\alpha. \tag{1.7}$$

Similarly,

$$v_q = -u_x \sin\alpha + v_y \cos\alpha. \tag{1.8}$$

In geophysical applications, there are two ways that vectors can be multiplied, namely, the scalar and the vector products. The scalar product gives a number (scalar) that is equal to the product of the magnitudes of the two vectors multiplied by the cosine of the angle between them (Figure 1.5). That is,

$$\vec{F}\vec{d} = |\vec{F}|\,|\vec{d}| \cos\alpha, \tag{1.9}$$

where α represents the angle between the two vectors. It can be readily verified that this equation is equivalent to the sum of the corresponding components,

$$\vec{F}\vec{d} = F_x d_x + F_y d_y + F_z d_z. \tag{1.10}$$

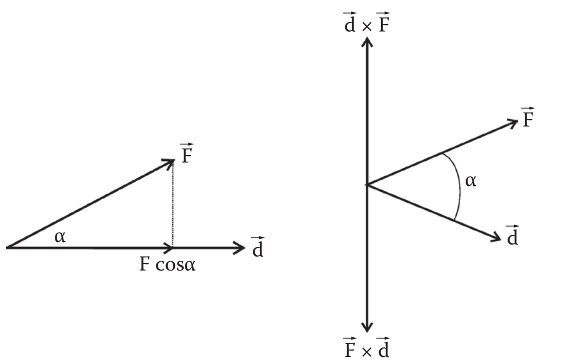

FIGURE 1.5 Illustrating the scalar product (left) and vector product (right) of two vectors.

Note that the scalar product of two vectors that are perpendicular is zero. To understand the physical meaning of the scalar product, let $\vec{F}$ be a constant force resulting in a displacement, $\vec{d}$. The scalar product then represents the work done by this force.

The vector product of two vectors $\vec{F}$ and $\vec{d}$ is itself a vector that is perpendicular to the plane formed by $\vec{F}$ and $\vec{d}$, and whose magnitude is equal to the area of the parallelogram formed by these two vectors. That is,

$$|\vec{F} \times \vec{d}| = |\vec{F}|\,|\vec{d}|\ \sin\alpha. \tag{1.11}$$

The resulting vector is given by

$$\vec{F} \times \vec{d} = (F_y d_z - F_z d_y)\vec{i} - (F_x d_z - F_z d_x)\vec{j} + (F_x d_y - F_y d_x)\vec{k}, \tag{1.12}$$

which can also be written in the form of a determinant

$$\vec{F} \times \vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ F_x & F_y & F_z \\ d_x & d_y & d_z \end{vmatrix}. \tag{1.13}$$

Note that the order in which the vectors appear in the vector product is significant and $\vec{F} \times \vec{d} = -\vec{d} \times \vec{F}$ (Figure 1.5). In physical applications, the vector product is used to calculate torque (vector product of radius and applied force) and magnetic force (vector product of magnetic field and velocity of a charge moving through this field).

An important mathematical operation used throughout this book is taking the derivative to evaluate how a quantity varies over time and in space. In the following, only space derivatives are considered. For a scalar quantity such as temperature, the derivative is a vector called the *gradient*, written as

$$\mathrm{grad}T = \frac{\partial T}{\partial x}\vec{i} + \frac{\partial T}{\partial y}\vec{j} + \frac{\partial T}{\partial z}\vec{k}, \tag{1.14}$$

or, equivalently

$$\mathrm{grad}T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right). \tag{1.15}$$

Each component of this vector describes how, at a particular location, the temperature changes in each of the three orthogonal directions.

Inspection of these equations shows that the temperature gradient is the product of a scalar, T, with the symbolical vector ∇, defined as

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}. \tag{1.16}$$

This vector, called *nabla,* was introduced by Sir William Hamilton in 1837 and named after an ancient Assyrian harp that has a similar shape as does the symbol. Note that while ∇ represents a vector, it usually does not have an arrow placed above it. This operator can also be applied to a vector. First, taking the scalar product of ∇ and the velocity $\vec{u} = (u, v, w)$ gives

$$\nabla \vec{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}. \tag{1.17}$$

This scalar quantity is called the *divergence* of the velocity, sometimes written as $\operatorname{div} \vec{u}$. Taking the vector product of ∇ and $\vec{u}$ produces another vector, called the *curl* of the velocity, which essentially describes rotation present in the velocity field. Analogous to equation (1.13),

$$\operatorname{curl} \vec{u} = \nabla \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}. \tag{1.18}$$

Next consider the space derivative of a vector. This involves calculation of the gradient of each of the three velocity components resulting in a set of three vectors defined by nine scalars. It can be shown that these nine components form a second-order *tensor.* Symbolically, a second-order tensor, $\vec{A}$, can be represented by a 3 × 3 matrix as

$$\vec{A} = \begin{pmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{yx} & A_{yy} & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} \end{pmatrix}. \tag{1.19}$$

Two tensors that feature prominently in glacier dynamics are the stress tensor, describing the components of stress acting on an ice cube, and the strain-rate tensor, describing the rate at which the cube deforms as a result of these stresses. Both tensors are introduced in the next section, but to conclude this introductory discussion, two important tensor properties that will be used in later chapters need to be addressed, namely, invariants and tensor rotation.

As noted above, the magnitude of a vector (the length of the arrow) does not change if the coordinate system is rotated. A second-order tensor has three such scalar quantities whose values are independent of the orientation of the coordinate axes; these quantities are called *invariants* and represent the solutions of the so-called characteristic equation

$$\det(\vec{A} - \lambda\vec{E}) = \begin{vmatrix} A_{xx} - \lambda & A_{xy} & A_{xz} \\ A_{yx} & A_{yy} - \lambda & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} - \lambda \end{vmatrix} = 0, \tag{1.20}$$

where $\vec{E}$ represents the identity tensor $(E_{xx} = E_{yy} = E_{zz} = 1,$ and all other elements are equal to 0). Omitting the algebra, it can be shown that three solutions exist, namely,

$$\lambda_1 = A_{xx} + A_{yy} + A_{zz}, \tag{1.21}$$

$$\lambda_2 = A_{xx}^2 + A_{yy}^2 + A_{zz}^2 + A_{xy}^2 + A_{xz}^2 + A_{yx}^2 + A_{yz}^2 + A_{zx}^2 + A_{zz}^2, \tag{1.22}$$

$$\begin{aligned}\lambda_3 &= \det(\vec{A}) = \\ &= A_{xx}(A_{yy}A_{zz} - A_{yz}A_{zy}) - A_{xy}(A_{yx}A_{zz} - A_{yz}A_{zx}) + A_{xz}(A_{yx}A_{zy} - A_{yy}A_{zx}).\end{aligned} \tag{1.23}$$

When considering the stress and strain-rate tensors, these invariants have a clear physical meaning, as discussed in the next section. A tensor for which the first invariant vanishes $(\lambda_1 = 0)$ is called a *deviator* (although in glaciology, use of the term *deviator* is restricted to the stress tensor).

The value of the nine tensor components depends on the orientation of the coordinate system, and thus, formulas for rotating tensors are needed equivalent to equations (1.7) and (1.8) for vector rotation. For any arbitrary rotation in three-dimensional space, these formulas become rather lengthy, but for most glaciological applications it is sufficient to consider rotation in the horizontal plane only (that is, rotation around the vertical z-axis). Further, when considering the stress or strain-rate tensor, only six independent coefficients need to be considered because both tensors are symmetric, with $A_{ij} = A_{ji}$. Let $\vec{B}$ be the symmetric 2×2 tensor with components

$$\vec{B} = \begin{pmatrix} B_{xx} & B_{xy} \\ B_{xy} & B_{yy} \end{pmatrix}. \tag{1.24}$$

As in Figure 1.3 the angle between the two coordinate systems is α. The tensor components in the (p, q) coordinate system are then related to those in the (x, y) system as (Jaeger, 1969, p. 7)

$$B_{pp} = B_{xx}\cos^2\alpha + B_{yy}\sin^2\alpha + 2B_{xy}\sin\alpha\cos\alpha, \tag{1.25}$$

$$B_{qq} = B_{xx}\sin^2\alpha + B_{yy}\cos^2\alpha - 2B_{xy}\sin\alpha\cos\alpha, \tag{1.26}$$

$$B_{pq} = (B_{yy} - B_{xx})\sin\alpha\cos\alpha + B_{xy}(\cos^2\alpha - \sin^2\alpha). \tag{1.27}$$

From equation (1.27) it follows that there is an angle ϕ for which $B_{pq} = 0$, namely, when

$$\tan 2\phi = \frac{2B_{xy}}{B_{xx} - B_{yy}}. \tag{1.28}$$

If $\vec{B}$ represents the stress tensor, this angle gives the direction of the two principal stresses in the horizontal plane.

1.2 STRESS AND STRAIN

Glacier flow is commonly described using concepts from continuum mechanics. The basic premise of continuum mechanics is that the material under consideration is continuous and that physical quantities such as mass and momentum associated with a small volume are distributed uniformly over that volume (Batchelor, 1967, p. 5). In that case, the response to applied stress can be described by a single constitutive relation or flow law. To model the flow of glaciers, such an approach works well and there is no need to consider deformation of each ice crystal individually. An obvious exception to this is when modeling the development of fabric patterns in ice under stress, as discussed in Section 2.4.

The goal of continuum mechanics is to describe how a material (in this case, glacier ice) deforms when subjected to force. This involves the constitutive relation, or flow law, which links the stress to the rate of deformation. Deformation of ice is discussed in Chapter 2, but first it may be instructive to have a closer look at the two central quantities involved, namely, stress and strain. A more extensive discussion on this topic can be found in many textbooks, for example, Means (1976) and Turcotte and Schubert (2002).

Stresses are forces per unit area. There are two types of stresses, namely, shear and normal stresses. A shear stress (also referred to as traction) acts along a surface (for example, friction generated by a block gliding over a rough surface); a normal stress acts perpendicular to the surface. In three-dimensional space, three perpendicular surfaces through a point are needed to describe all stresses acting at that point (Figure 1.6). For each surface, there are three stress components, namely, one normal stress acting perpendicular to the surface and two shear stresses acting along the two perpendicular directions in the surface. So, altogether there are nine stress components acting at each point. The stress components are written as σ_{ij}, with i, j = x, y, z, the three axes of the Cartesian coordinate system. The second subscript, j, denotes the direction normal to the surface on which the stress acts, while the first subscript, i, specifies the direction of the stress. For example, the shear stress σ_{xz} acts in the direction of the x-axis, along the plane normal to the z-axis

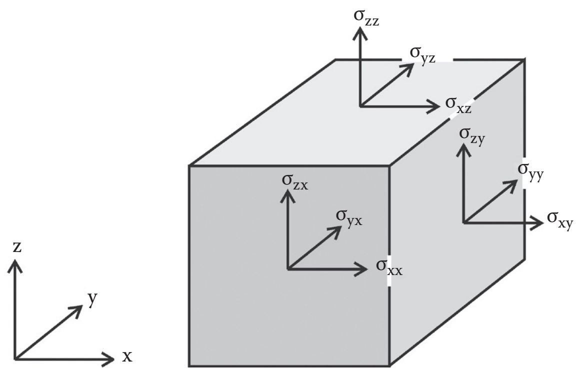

FIGURE 1.6 Stresses acting on the three perpendicular faces of a unit cube that is homogeneously stressed.

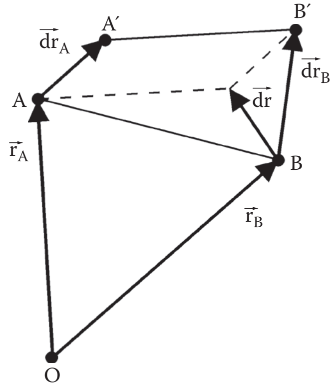

FIGURE 1.7 Deformation of a line segment.

(that is, the xy plane). It can be shown that balance of angular momentum requires that the stress tensor be symmetric and $\sigma_{ij} = \sigma_{ji}$ (for example, Jaeger, 1969, p. 6; Hutter, 1983, p.14; Greve and Blatter, 2009, Section 3.3.6). So, the distribution of stress is described completely by six independent components (three shear stresses and three normal stresses).

Under the combined actions of all stresses acting on an ice volume, deformation takes place. To see how this can be described, consider two neighboring points A and B. After deformation, their positions are A′ and B′, respectively (Figure 1.7). The deformation of the line segment AB is then A′B′ – AB. Using the notation of Figure 1.7, the deformation vector is

$$\begin{aligned} \overrightarrow{dr} &= \overrightarrow{A'B'} - \overrightarrow{AB} = \\ &= \left(\overrightarrow{r_B} + \overrightarrow{dr_B}\right) - \left(\overrightarrow{r_A} + \overrightarrow{dr_A}\right) - \left(\overrightarrow{r_B} - \overrightarrow{r_A}\right) = \\ &= \overrightarrow{dr_B} - \overrightarrow{dr_A}. \end{aligned} \tag{1.29}$$

For a Cartesian coordinate system x_j (j = 1, 2, 3), first-order Taylor expansion gives

$$\overrightarrow{dr} = \frac{\partial \vec{r}}{\partial x_j} dx_j, \tag{1.30}$$

or, for each component of the displacement vector

$$dr_i = \frac{\partial r_i}{\partial x_j} dx_j, \tag{1.31}$$

where summation over repeated indices is implied.

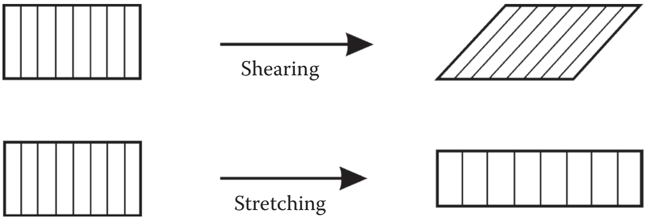

FIGURE 1.8 Illustrating the two modes of deformation.

The nine quantities, $\partial r_i/\partial x_j$, form a second-order tensor (a Jacobian), which can be written as the sum of a symmetric and an antisymmetric part

$$\frac{\partial r_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial r_i}{\partial x_j} + \frac{\partial r_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial r_i}{\partial x_j} - \frac{\partial r_j}{\partial x_i}\right). \tag{1.32}$$

The second term on the right-hand side represents a rigid-body rotation, while the first term actually describes how the segment is deformed. This is called the strain tensor, with elements given by

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial r_i}{\partial x_j} + \frac{\partial r_j}{\partial x_i}\right). \tag{1.33}$$

As illustrated in Figure 1.8, there are two types of deformation, namely, stretching (described by the diagonal elements of the strain tensor, with $i = j$) and shearing ($i \neq j$, the off-diagonal elements).

As ice deforms, it tries to reach a steady state in which individual ice crystals are still being deformed, but at a (locally) constant rate. In other words, a stationary stress- and velocity-field is established. Therefore, strain rates, rather than strains, are commonly used in glaciology. In the usual notation, strain rates are denoted by a dot above the elements of the strain tensor (referring to the time derivation). Then, by definition

$$\begin{aligned} \dot{\varepsilon}_{ij} &= \frac{d}{dt}(\varepsilon_{ij}) = \\ &= \frac{1}{2}\left[\frac{\partial}{\partial x_j}\left(\frac{dr_i}{dt}\right) + \frac{\partial}{\partial x_i}\left(\frac{dr_j}{dt}\right)\right] = \\ &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \end{aligned} \tag{1.34}$$

where the u_i represent the three components of ice velocity in the three orthogonal directions, x_i. There are six independent strain-rate components, and the strain-rate tensor exhibits the same symmetry as does the stress tensor.

The experiments of Rigsby (1958) suggest that the deformation of ice is practically independent of the hydrostatic pressure, provided that the difference between the ice temperature and the pressure-melting temperature is kept constant. Then the constitutive relation, linking the rate of deformation to applied stress, must be independent of the hydrostatic pressure. To achieve this, deviatoric stresses, rather than full stresses, are used to describe the rheological properties of glacier ice.

Let σ_{ij} denote the components of the full stress tensor (sometimes referred to as the Cauchy stress). The hydrostatic pressure is the sum of the three normal stresses, $P = (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})/3$. The full stress tensor is now partitioned as follows

$$\begin{pmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{pmatrix} =$$

$$= \begin{pmatrix} (2\sigma_{xx} - \sigma_{yy} - \sigma_{zz})/3 & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & (2\sigma_{yy} - \sigma_{xx} - \sigma_{zz})/3 & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & (2\sigma_{zz} - \sigma_{xx} - \sigma_{yy})/3 \end{pmatrix} +$$

$$+ \begin{pmatrix} (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})/3 & 0 & 0 \\ 0 & (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})/3 & 0 \\ 0 & 0 & (\sigma_{xx} - \sigma_{yy} - \sigma_{zz})/3 \end{pmatrix} \tag{1.35}$$

The first tensor on the right-hand side is called the stress deviator. In short, its components are given by

$$\tau_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}, \tag{1.36}$$

where δ_{ij} denotes the Kronecker delta ($\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$), and summation over repeat indexes is implied in equation (1.36). Thus, a deviatoric normal stress is defined as the full normal stress minus the hydrostatic pressure. The shear stresses are unaffected by this partitioning.

A few words on the notation commonly used in the glaciological literature may be helpful. In the literature, the components of stress are variably denoted by σ_{ij} or by τ_{ij}. Sometimes, σ_{ij} is reserved for normal stress ($i = j$) and τ_{ij} for shear stress ($i \neq j$). Deviatoric stresses are commonly denoted by a prime, for example, σ'_{ij}. In this book, in an effort to keep the notation simple and consistent, σ_{ij} refers to full stresses and τ_{ij} to deviatoric stresses.

As discussed in Section 1.1, a second-order tensor has three invariants. For the full stress tensor, the first invariant is the sum of the three normal stresses and equals three times the hydrostatic pressure, defined as the average pressure (or the average of the three normal stresses; hence the factor 1/3). For the deviatoric stress tensor, the first invariant is zero. The second invariant of this tensor gives the *effective stress*, τ_e, defined as

$$2\tau_e^2 = \tau_{xx}^2 + \tau_{yy}^2 + \tau_{zz}^2 + 2\left(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2\right). \tag{1.37}$$

For the strain-rate tensor, the first invariant is

$$\lambda_1 = \dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy} + \dot{\varepsilon}_{zz}. \tag{1.38}$$

For incompressible ice the density is constant and this first invariant is zero, and conservation of volume is expressed as

$$\dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy} + \dot{\varepsilon}_{zz} = 0. \tag{1.39}$$

The effective strain rate is defined similar to the effective stress and related to the second invariant of the strain-rate tensor. The third invariants of the stress and strain-rate tensors are usually not considered in glacier modeling, with the exception of a few studies that considered normal stress effects or dilatancy (cf. Section 2.3).

1.3 ERROR ANALYSIS

The primary objective of this book is to provide the mathematical and theoretical framework necessary for modeling glacier flow and for quantitatively interpreting measurements on glaciers. One such application involves the force-budget technique, in which measured surface velocities are used to estimate forces acting on the glacier. Equally important as estimating the magnitude of the various resistive forces is to assign uncertainties or errors to these calculations. These uncertainties arise because of errors in the measurements. This section provides a brief overview of error analysis and how errors propagate through calculations. Taylor (1997) gives a more extensive introductory discussion of the topic.

In science, the word *error* refers to the uncertainty inherent when measuring physical quantities, and not to the colloquial meaning of *mistake*. How these errors are determined or estimated depends on the type of measurements. For the following discussion, these so-called input errors are assumed to be known and denoted by Δ. Now consider a function $F(\alpha_i)$ with variables α_i; the respective errors are denoted by ΔF and $\Delta\alpha_i$. If the errors in the input variables are independent, the error in the function is estimated from

$$(\Delta F)^2 = \sum_i \left(\frac{\partial F}{\partial \alpha_i} \Delta\alpha_i \right)^2. \tag{1.40}$$

To illustrate how this formula is used, along-flow force balance is considered. The relevant equations are derived in Chapter 3 and readers unfamiliar with these equations may want to return to this section at a later stage.

If the x-axis is taken to coincide with the direction of flow, balance of forces is expressed by equation (3.22). Making the assumption that the resistive stresses, R_{xx} and R_{xy}, are constant throughout the ice thickness, this equation becomes

$$\tau_{dx} = \tau_{bx} - \frac{\partial}{\partial x}(HR_{xx}) - \frac{\partial}{\partial y}(HR_{xy}). \tag{1.41}$$

The second and third terms on the right-hand side are estimated from strain rates, which, in turn, are estimated from measured velocities.

Let U_1 and U_2 denote the velocities at two neighboring stations, located at x_1 and x_2, respectively. The average stretching rate between these stations is then

$$\dot{\varepsilon}_{xx} = \frac{U_2 - U_1}{x_2 - x_1}. \tag{1.42}$$

The error in the positions is neglected, and both velocity errors are equal to ΔU. The error in the strain rate is then

$$\begin{aligned}(\Delta\dot{\varepsilon}_{xx})^2 &= \frac{(\Delta U_2)^2 + (\Delta U_1)^2}{(x_2 - x_1)^2} = \\ &= \frac{2(\Delta U)^2}{(x_2 - x_1)^2},\end{aligned} \tag{1.43}$$

or

$$\Delta\dot{\varepsilon}_{xx} = \frac{\Delta U}{x_2 - x_1}\sqrt{2}. \tag{1.44}$$

This equation shows that the error in strain rate is inversely proportional to the distance over which the velocity gradient is calculated. This is important to keep in mind because it allows the strain-rate error to be minimized by increasing the differencing interval. Of course, increasing the differencing interval goes at the expense of spatial resolution, and strain-rate variations at a scale smaller than this interval will not be resolved. Depending on the application, careful consideration should be given to this tradeoff between minimizing errors and spatial resolution.

As an example, consider a glacier on which the average velocity increases by 1 m/yr for every km in the downflow direction. The average strain rate is then $\dot{\varepsilon}_{xx} = 10^{-3}$ yr^{-1}. If velocities are measured every 10 m, small-scale variations in strain rate can be calculated by differencing neighboring velocities. If the velocity error is 10 cm/yr, this would give an error in strain rate $\Delta\dot{\varepsilon}_{xx} = 0.014$ yr^{-1}, or an order of magnitude greater than the average strain rate itself. To reduce this error to, say 10% of the average strain rate, the differencing interval should be increased to 1 km or greater. Thus, the velocity error essentially dictates the spatial scale over which strain rates can be meaningfully assessed. Because the strain rate equals the velocity gradient, the actual magnitude of the velocity is irrelevant and the preceding conclusion applies equally to glaciers that move at speeds of a few m/yr and those that move at speeds of several km/yr.

The next step is to estimate the error in the corresponding resistive stress, R_{xx}. This stress is related to the stretching rate through the flow law (3.48). To avoid getting lost in algebra and to keep the discussion clear, the simplifying assumption is made that $\dot{\varepsilon}_{xx}$ is the dominant strain rate contributing to the effective strain rate and that lateral spreading is zero. Then

$$R_{xx} = 2B\dot{\varepsilon}_{xx}^{1/n}. \tag{1.45}$$

If the error in the viscosity parameter, B, and flow-law exponent, n, may be neglected, applying equation (1.40) gives

$$(\Delta R_{xx})^2 = \left(\frac{1}{n}2B\dot{\varepsilon}_{xx}^{1/n-1}\ \Delta\dot{\varepsilon}_{xx}\right)^2, \tag{1.46}$$

which can be rewritten in terms of fractional uncertainty

$$\left|\frac{\Delta R_{xx}}{R_{xx}}\right| = \left|\frac{\Delta\dot{\varepsilon}_{xx}}{\dot{\varepsilon}_{xx}}\right|, \tag{1.47}$$

where the vertical bars denote the absolute value. An intermediate value for the viscosity parameter is B = 500 kPa yr$^{1/3}$ for n = 3 (Figure 2.4). For the numerical example discussed above, the average resistive stress is 100 kPa. Assuming velocity gradients are calculated over a horizontal distance of 1 km, the error in strain rate is 1.4×10^{-4} yr^{-1} and, from equation (1.47), the error in the resistive stress, ΔR_{xx}, is 14 kPa.

The next step is to estimate the error in the longitudinal stress gradient term (the second term on the right-hand side of the balance equation (1.41)). Resistance to flow from gradients in longitudinal stress is written as

$$F_{lon} = \frac{\partial}{\partial x}(HR_{xx}). \tag{1.48}$$

First consider the term in parenthesis on the right-hand side. This product of ice thickness, H, and resistive stress, R_{xx} has an error that can be estimated using equation (1.40). Similar to the derivation leading to equation (1.47) the fractional error is given by

$$\left(\frac{\Delta(HR_{xx})}{HR_{xx}}\right)^2 = \left(\frac{\Delta H}{H}\right)^2 + \left(\frac{\Delta R_{xx}}{R_{xx}}\right)^2. \tag{1.49}$$

For an average ice thickness of 1500 m with measurement uncertainty of 50 m, and continuing with the numerical example, this gives $\Delta(HR_{xx}) = 2.2 \times 10^4$ kPa m. It can be readily verified that most of this uncertainty is due to the error in R_{xx} and that the error in ice thickness is relatively unimportant. Similar to the error in strain rate (equation (1.44)), the error in the longitudinal stress gradient is

$$\Delta F_{lon} = \frac{\Delta(HR_{xx})}{x_2 - x_1}\sqrt{2}. \tag{1.50}$$

Again, taking a differencing distance of 1 km, this error is about 31 kPa. Typical values for the driving stress are in the range of 50 to 150 kPa, and consequently, the error in F_{lon} could be a significant fraction of the driving stress, rendering any conclusion about the role of longitudinal stress gradients in the balance of forces rather uncertain. The only way to reduce this uncertainty is by increasing the spatial interval to, say, 10 km.

This numerical example shows that small errors in measured velocities can result in large errors in the force balance terms. The reason for this is that the last two terms on the right-hand side of the balance equation (1.41) are proportional to the second derivative of the ice velocity or, equivalently, to the curvature of the velocity profile in the x- and y-directions, respectively. Small errors in the velocity are amplified by taking the derivative, and to reduce the resulting uncertainty, spatial derivatives must be calculated over greater distances. This may not always be the best option, however. For example, the width of the fast-moving part of Jakobshavn Isbræ in West Greenland is a little more than 3 km wide (Van der Veen et al., 2011), and to obtain an accurate estimating of the role of lateral drag from transverse velocity gradients, an approach involving linear regression is more appropriate. This procedure is discussed in Section 11.4.

1.4 PARAMETRIC UNCERTAINTY ANALYSIS

The analysis of errors discussed in Section 1.3 follows standard treatment of errors as can be found in, for example, introductory texts on experimental physics (Squires, 2001) or textbooks on error analysis (Taylor, 1997). Formula (1.40) for error propagation applies to any quantity that is a function of several variables, but its practical use may be limited if analytical expressions for the derivatives cannot be easily derived. In such cases, a parametric uncertainty analysis may provide an error estimate. In this procedure, input variables, α_i, are assigned values selected randomly from their probability density distributions, and the resulting value of F is calculated. This procedure is repeated many times to obtain a distribution for the dependent variable, F, from which the standard deviation or error can be estimated. For example, Van der Veen (2002b) applies this method to estimate the range of projected contributions from Greenland and Antarctica to global sea level change in 2100 AD, given uncertainties in parameter values and in warming scenarios. To illustrate this so-called Monte Carlo approach, consider the calculation of strain rates.

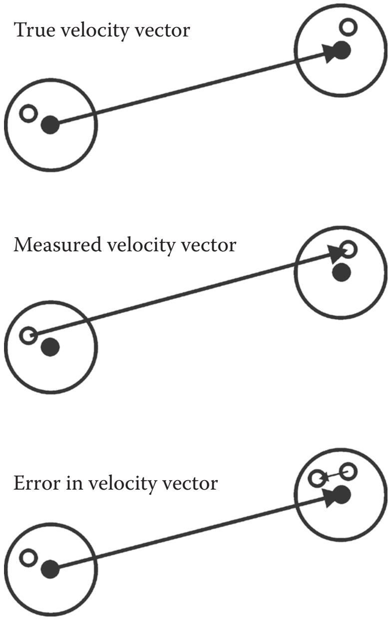

FIGURE 1.9 Illustrating the source of uncertainty in measured velocities. Filled black dots represent the true position of a location at two different times. Large circles indicate the uncertainty in measuring the true positions. Small open circles represent the actual measurements from which the measured velocity is derived (middle panel). The error in velocity is represented by the difference between the true velocity (upper panel) and the measured velocity and corresponds to the small vector connecting both measurement errors (lower panel).

Strain rates are linked to velocity gradients. If the x-direction is taken in the (approximate) flow direction, and U denotes the velocity in this direction, then the along-flow stretching rate is

$$\dot{\varepsilon}_{xx} = \frac{\partial U}{\partial x}. \tag{1.51}$$

Typically, the velocity is determined by measuring the displacement of a visible marker on the glacier surface. This could be a global positioning system (GPS) antenna anchored in the surface and surveyed repeatedly, or a visible feature such as a crevasse identified on successive satellite images. Referring to Figure 1.9 (upper panel), the true velocity vector is given by the change in position of the true positions over the interval of measurements. However, due to measurement uncertainties such as image pixel size, recognizing exactly the same feature on successive images, uncertainties in GPS positions, and other factors, positional errors are introduced and the measured velocity is the displacement of two points that are each displaced from their true positions (middle panel). The error in velocity is then given by the vector connecting the measured positions after subtracting the true displacement (lower panel). There is another error introduced because the measured velocity may be assigned to

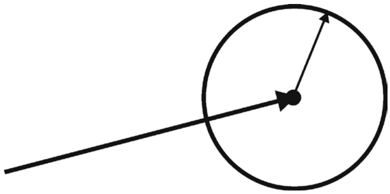

FIGURE 1.10 The error in measured velocity can be represented by the small vector starting at the end point of the velocity vector. The end point of this error vector can be located anywhere in the circle centered on the end point of the measured velocity vector. The radius of this circle is determined by the error in position and the time interval over which displacements are measured.

the wrong location, but that error is ignored here. Thus, the velocity can be represented by the measured velocity, plus some error that can be represented graphically as another vector to be added to the measured velocity (Figure 1.10). The magnitude of the error vector is determined by the error in locations. The angle can be anywhere between –180 and +180 degrees. Further, this error is independent of the magnitude and direction of the average measured velocity. It is essentially the difference between the errors in the measured positions after subtracting the average displacement.

Because interest is in the velocity error, the actual distance between the two true points does not matter and both are assigned the coordinates (0, 0). The two location errors can then be represented by two points (x_1, y_1) and (x_2, y_2). Assuming a standard deviation of 10 m for the position error, and assuming the position error is Gaussian distributed, the four coordinates are randomly selected and the two components of the velocity error vector calculated as

$$U = \frac{x_2 - x_1}{dt}, \tag{1.52}$$

$$V = \frac{y_2 - y_1}{dt}, \tag{1.53}$$

where dt = 1 year represents the time interval between successive position determinations. This process of randomly selecting the four coordinates of the two points is repeated 10,000 times to yield the probability distributions shown in Figure 1.11. As was to be expected, the average for both velocity components is zero, while the standard deviation is 14 m/yr, corresponding to the value predicted by equation (1.40).

The corresponding error in stretching rate can be estimated from a similar procedure. Consider two points at a nominal distance of 1 km apart. That is, the distance between the true locations is 1 km. At each location, the error velocity is determined as for Figure 1.11. The error in stretching rate is then estimated from

$$\dot{\varepsilon}_{xx} = \frac{U_2 - U_1}{dx}, \tag{1.54}$$

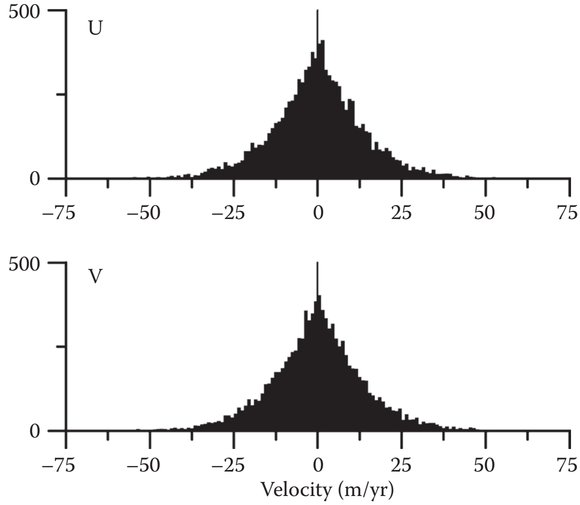

FIGURE 1.11 Probability distributions for the two components of the velocity error vector determined from a Monte Carlo simulation as explained in the text. For both distributions the mean (indicated by the vertical line) is zero and the standard deviation is 14 m/yr, corresponding to a positional error of 10 m and an interval of 1 year over which displacements are measured.

where dx represents the distance between the two velocity points. As noted, for this calculation, the nominal distance is 1 km, but the actual distance varies because of the positional error. Again, this procedure is repeated 10,000 times. The resulting probability distribution for the error in strain rate is shown in the upper panel of Figure 1.12 and corresponds to a Gaussian distribution with zero mean and standard deviation equal to 0.02 yr^{-1}. Interestingly, this standard deviation is nearly the same as that predicted by equation (1.42), in which the error in positions are neglected. The reason for this is that the relative error in distance, dx, between the two locations is much smaller than the relative error in the two velocities.

An alternative method for estimating strain rates is to use relative length changes. Consider a line segment connecting two points, A and B, with an initial length, L, aligned in the x-direction. After some time, δt, the length of this segment has changed by an amount δL. The corresponding strain is then

$$\varepsilon_{xx} = \frac{\delta L}{L}, \tag{1.55}$$

and the strain rate is

$$\dot{\varepsilon}_{xx} = \frac{\delta L}{L\delta t}. \tag{1.56}$$

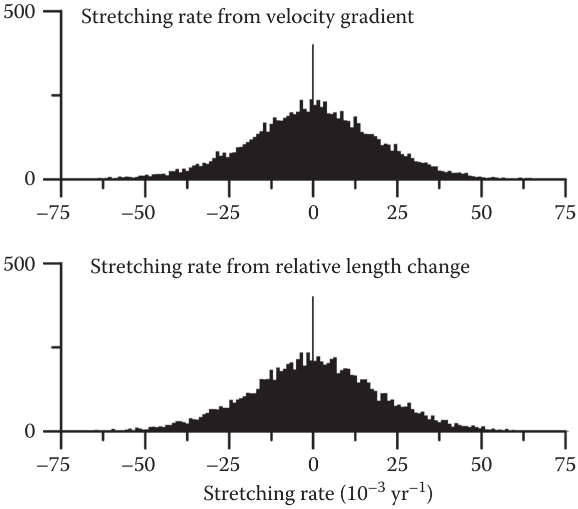

FIGURE 1.12 Probability distributions for error in along-flow stretching rate determined from a Monte Carlo simulation as explained in the text. The upper panel corresponds to stretching rate calculated from velocity gradients, and the lower panel to the stretching rate calculated from relative displacement. For both distributions the mean (indicated by the vertical line) is zero and the standard deviation is 0.02 yr^{-1}, corresponding to a positional error of 10 m and an interval of 1 year over which displacements are measured.

To demonstrate that this expression is similar to the definition of strain rate in terms of velocity gradient, note that the entire line segment is moving in the x-direction and any change in length must result from a velocity difference at the end points, A and B. That is

$$\frac{\delta L}{\delta t} = U_B - U_A, \tag{1.57}$$

and

$$\dot{\varepsilon}_{xx} = \frac{\delta L}{L\delta t} = \frac{U_B - U_A}{L} = \frac{\partial U}{\partial x}. \tag{1.58}$$

To allow for comparison with the error in strain rate determined from velocity errors, equation (1.56) is applied to two points separated by a nominal distance of 1 km and with a positional error with a standard deviation of 10 m, and the change in length is assumed to occur over a period of 1 year. Using the same Monte Carlo technique, the probability distribution of the error in stretching rate shown in the lower panel in Figure 1.12 is obtained. The mean and standard deviation are the same as for the distribution shown in the upper panel.

These examples illustrate how a parametric uncertainty analysis can be used to estimate the error in derived quantities. Admittedly, these examples are sufficiently trivial to allow for application of the error analysis discussed in Section 1.3. For dependent variables that are more complex functions of input parameters, the procedure for estimating the error is essentially the same as discussed in this section.

1.5 CALCULATING STRAIN RATES

Strain rates are related to velocity gradients, and most often, measured surface velocities are used to estimate surface strain rates. When maps of surface strain rates on a glacier are produced, the usual procedure is to use velocities interpolated to a regular grid with Cartesian axes (as in the example of Byrd Glacier, Antarctica, discussed in Section 11.2). This is an expedient procedure, but a concern is that artifacts could be introduced into the calculated strain rates. This is because velocity measurements usually are irregularly spaced, and the gridding may introduce some noise or irregularities, especially in regions where data density is low. Such noise may be inconsequential for the velocities themselves but becomes amplified when taking derivatives for strain rates. In calculating the resistive terms in the balance of forces, this noise may be further amplified because these resistive terms involve the spatial gradient of strain rates (or, equivalently, the second derivative of the velocity). No study has been conducted to evaluate whether this concern is warranted. Of course, if only gridded velocity data are made available, there is no other recourse than to use these products. Ideally, however, strain rates should be determined from irregularly spaced actual velocity measurements and then, if need be, interpolated to a regular grid.

A second instance where strain rates must be estimated from a sparse set of velocity determinations is when velocities are measured from a small number of GPS stations covering an area of interest. For example, Bassis et al. (2007) deployed 12 GPS stations around the tip of a rift on the Amery Ice Shelf, with the objective to estimate stresses acting on the rift. In such cases, calculating strain rates is slightly more involved than when gridded velocities are available. This section outlines how strain rates can be estimated using three stations whose position is tracked over time (either using GPS, feature tracking on repeat imagery, or measuring distances between the stations using electronic measurement devices).

In the horizontal (x, y) plane, there are three strain rates that describe the deformation. Following the convention adopted in this book, the x-axis is chosen to align with the average direction of ice flow. Then, $\dot{\varepsilon}_{xx}$ represents stretching in the flow direction, $\dot{\varepsilon}_{xy}$ represents lateral shearing, and $\dot{\varepsilon}_{yy}$ represents convergence or divergence in the transverse direction. To calculate these three components, three stations are required from which strain rates in three independent directions can be estimated.

The geometry for calculating strain rates is shown in Figure 1.13. Three survey stations are considered, A, B, and C. Recalling that strain rates are estimated from relative velocities or displacements, the position of the first station, A, may be considered fixed in time. The filled black circles in this figure represent the locations at some initial time, and the open circles labeled B′ and C′ represent the position of stations

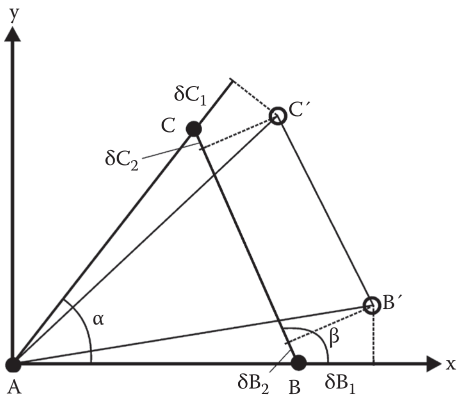

FIGURE 1.13 Geometry used to calculate strain rates from the relative displacements of three survey stations.

B and C (relative to station A) after some time, δt. Further, without loss of generality, the two stations A and B are positioned to align with the (initial) direction of flow, and thus with the x-axis (the results given below can be readily transformed to any arbitrary coordinate system using the tensor rotation formulas given in Section 1.1).

The initial survey gives the three distances, AB, AC, and BC. Repeating the survey after some time, δt, the position of stations B and C, relative to station A (which also has moved, but that does not enter into the calculation of strain rates), are given by B′ and C′, respectively. The displacements of these two stations can be projected onto the initial three directions corresponding to the sides of the triangle ABC. These displacements are denoted by δB_1 for the displacement of station B in the AB direction, δB_2 for the displacement of station B along the BC direction, and similarly for station C (Figure 1.13). Note that in this example, both δB_2 and δC_2 are negative, indicating shortening of the distance BC.

The three strain rates determined from the displacements are

$$\dot{\varepsilon}_{AB} = \frac{\delta B_1}{|AB|\,\delta t}, \tag{1.59}$$

$$\dot{\varepsilon}_{AC} = \frac{\delta C_1}{|AC|\,\delta t}, \tag{1.60}$$

$$\dot{\varepsilon}_{BC} = \frac{\delta B_2 + \delta C_2}{|BC|\,\delta t}. \tag{1.61}$$

Equivalently, these strain rates can be estimated from the gradients in velocity components in each of the three directions, AB, AC, and BC. This gives the same result as equations (1.59)–(1.61).

The three measured strain rates can be related to the strain rates defined in the (x, y) coordinate system using the tensor rotation formulas (1.25–1.27). The AB direction is aligned with the x-axis, and the angles between the AC and BC directions and the x-direction are α and β, respectively. Then

$$\dot{\varepsilon}_{AB} = \dot{\varepsilon}_{xx}, \tag{1.62}$$

$$\dot{\varepsilon}_{AC} = \dot{\varepsilon}_{xx}\cos^2\alpha + \dot{\varepsilon}_{yy}\sin^2\alpha + 2\dot{\varepsilon}_{xy}\sin\alpha\cos\alpha, \tag{1.63}$$

$$\dot{\varepsilon}_{BC} = \dot{\varepsilon}_{xx}\cos^2\beta + \dot{\varepsilon}_{yy}\sin^2\beta + 2\dot{\varepsilon}_{xy}\sin\beta\cos\beta. \tag{1.64}$$

After some tedious algebra, this set of equations can be solved for the three strain rate components (Turcotte and Schubert, 2002, p. 98):

$$\dot{\varepsilon}_{xx} = \dot{\varepsilon}_{AB}, \tag{1.65}$$

$$\dot{\varepsilon}_{yy} = \frac{1}{\tan\beta - \tan\alpha}[\dot{\varepsilon}_{AB}(\text{ctn}\alpha - \text{ctn}\beta) - \dot{\varepsilon}_{AC}(\text{sea}\alpha\ \text{csa}\alpha) + \dot{\varepsilon}_{BC}(\sec\beta\ \csc\beta)], \tag{1.66}$$

$$\dot{\varepsilon}_{xy} = \frac{1}{2(\text{ctn}\beta - \text{ctn}\alpha)}\left[\dot{\varepsilon}_{AB}(\text{ctn}^2\alpha - \text{ctn}^2\beta) - \dot{\varepsilon}_{AC}\csc^2\alpha + \dot{\varepsilon}_{BC}\csc^2\beta\right], \tag{1.67}$$

with $\sec\alpha = 1/\cos\alpha$, $\csc\alpha = 1/\sin\alpha$, and $\text{ctn}\alpha = 1/\tan\alpha$.

As an example, consider the displacements of three survey stations on Helheim Glacier, east Greenland, whose positions were continuously tracked over a 20-day period in June 2007, using GPS (data provided by Leigh Stearns). GPS positions are in latitude and longitude and elevation above the geoid. Here, only the horizontal positions are considered. The first step is to convert the horizontal positions to a local Cartesian coordinate system. The most often used Cartesian coordinate system for regional studies is the Universal Transverse Mercator (UTM) system developed by the U.S. Army Corps of Engineers in the 1940s. Figure 1.14 shows the displacements of the three stations in this coordinate system; Table 1.1 gives the start and end positions of the three stations, both in absolute UTM coordinates and relative to the initial position of station A in the UTM coordinate system.

As in Figure 1.13, a more convenient Cartesian coordinate system is introduced with the x-axis in the direction of the line connecting the initial positions of stations A and B. This requires coordinate transformation using formulas equivalent to equations (1.7) and (1.8). The rotation angle follows from the coordinates of station B relative to station A and equals –22°. This coordinate transformation gives the positions relative to the initial position of station A listed in Table 1.2. As noted

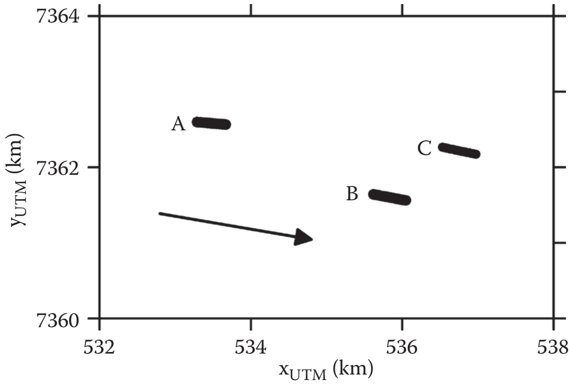

FIGURE 1.14 Displacement of three survey stations on Helheim Glacier, east Greenland, in the UTM (zone 24) coordinate system. Positions were measured over a 20-day period in June 2007. The arrow indicates the direction of the principal tensile strain. (Data provided by Leigh Stearns.)

above, strain rates are estimated from relative displacements and the position of station A may be considered fixed in time. Keeping this station stationary gives the relative end positions of stations B and C listed in Table 1.2 under the second set of coordinates.

The next step is to determine the four displacements, δB_1, δB_2, δC_1, and δC_2, in the initial three directions of the sides of the triangle ABC (Figure 1.13). This

TABLE 1.1
Start and End Positions of Three Survey Stations on Helheim Glacier, East Greenland, in the UTM Coordinate System

Station	Start Position		End Position	
	x_{UTM} (m)	y_{UTM} (m)	x_{UTM} (m)	y_{UTM} (m)
A	533,287.61	7,362,596.78	533,667.96	7,362,564.84
B	535,618.03	7,361,641.36	536,046.75	7,361,561.02
C	536,530.00	7,362,265.00	536,974.00	7,362,172.00
	Relative Start Position		**Relative End Position**	
A	0	0	380.35	–31.94
B	2,330.42	–955.42	2,759.13	–1,035.76
C	3,242.39	–331.78	3,686.39	–424.78

Source: Data provided by Leigh Stearns.
Note: The first set of coordinates gives positions in absolute coordinates, and the second set of coordinates gives positions relative to the initial position of survey station A.

TABLE 1.2
Start and End Positions of Three Survey Stations on Helheim Glacier, East Greenland, in the Local Coordinate System with the X-Axis Directed along the Line Connecting the Initial Positions of Stations A and B

Station	Relative Start Position		Relative End Position	
	x (m)	y (m)	x (m)	y (m)
A	0	0	364.04	114.72
B	2,518.66	0	2,945.82	88.29
C	3,125.90	922.97	3,571.99	1,005.34
	Relative Start Position		**Relative End Position**	
A	0	0	0	0
B	2,518.66	0	2,581.78	–26.44
C	3,125.90	922.97	3,207.96	890.62

Note: The first set of coordinates gives positions in relative coordinates, and the second set of coordinates gives relative end positions assuming that survey station A remained stationary.

is a rather tedious procedure involving determining many relative angles. For the example under consideration, equations (1.59)–(1.61) for the three strain rates may be approximated as

$$\dot{\varepsilon}_{AB} = \frac{|AB'| - |AB|}{|AB|\,\delta t}, \tag{1.68}$$

$$\dot{\varepsilon}_{AC} = \frac{|AB'| - |AC|}{|AC|\,\delta t}, \tag{1.69}$$

$$\dot{\varepsilon}_{BC} = \frac{|B'C'| - |BC|}{|BC|\,\delta t}. \tag{1.70}$$

It is left to the reader to check that, for this example, using these approximations results in calculated strain rates differing by a few percent from the values calculated using the exact formulas.

Changes in the length of line segments are readily calculated from the relative coordinates given in Table 1.2. Applying equations (1.68)–(1.70) gives the following values for the three strain rates: $\dot{\varepsilon}_{AB} = 0.458\ \text{yr}^{-1}$, $\dot{\varepsilon}_{AC} = 0.392\ \text{yr}^{-1}$, and $\dot{\varepsilon}_{BC} = 0.093\ \text{yr}^{-1}$. Using the transformation equations (1.65)–(1.67), strain rates in the orthogonal (x, y) coordinate system are found to be $\dot{\varepsilon}_{xx} = 0.458\ \text{yr}^{-1}$, $\dot{\varepsilon}_{yy} = -1.060\ \text{yr}^{-1}$, and $\dot{\varepsilon}_{xy} = 0.267\ \text{yr}^{-1}$. As was to be expected, the angle of the principal tensile strain rate is 9.7° (using equation (1.28)), which corresponds to the flow direction of the three stations.