Agregation in Statistical Learning

On-Line Estimation with the Multivariate Gaussian Distribution

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Introduction

- Article from 2007
- Estimate a distribution with On-Line learning
- Gaussian Distribution in 1-D and d-D
- Learner gives mean μ and covariance Σ, creates loss $\ell_t(\mu_t, \Sigma_t)$
- Literature didn't study before estimation of arbitrary covariance

- Loss function in One-Dimension :

$$\ell_t(\mu, \sigma^2) \triangleq \frac{(x_t - \mu)^2}{2\sigma^2} + \frac{1}{2}\ln(\sigma^2)$$

- In d-Dimension :

$$\ell_t(\mu, \Sigma) \triangleq \frac{1}{2}(x_t - \mu)^\top \Sigma^{-1}(x_t - \mu) + \frac{1}{2}\ln|\Sigma|$$

- Regret :

$$R_T(S) \triangleq \underbrace{\sum_{t=1}^{T} \ell_t(\mu_t, \sigma_t^2)}_{L_T(S)} - \inf_{\mu \in \mathbb{R}, \sigma^2 \in \mathbb{R}_{\geq>0}} L_T(\mu, \sigma^2)$$

From there, we have to calculate μ, σ^2 and also the Regret with T trials.

Degeneracies

To avoid degeneracies, we need to :

- Bound x_t : $|x_t| \leq r$, for $r \geq 0$
- Create a zeroth instance to avoid null-variance if first T trials are the same or very close to each other
  - So we fix $\tilde{\sigma}^2$ and incur a loss of :

$$\ell_0(\mu, \sigma^2) = \frac{\mu^2 + \tilde{\sigma}^2}{2\sigma^2} + \frac{1}{2}\ln(\sigma^2)$$

  - Not mentioned in the article but this is like putting a prior in a Bayesian context

Regret in One Dimension

$$L_T(\mu, \sigma^2) = \frac{\mu^2 + \tilde{\sigma}^2}{2\sigma^2} + \sum_{t=1}^{T} \left[\frac{(x_t - \mu)^2}{2\sigma^2} + \frac{1}{2} \ln \sigma^2 \right]$$

Let's find its infemum!

$$\frac{\partial}{\partial \mu} L_T(\bar{\mu}_T, \bar{\sigma}_T^2) = \frac{\bar{\mu}_T}{\bar{\sigma}_T^2} + \sum_{t=1}^{T} \frac{\bar{\mu}_T - x_t}{\bar{\sigma}_T^2} = 0$$

$$\frac{\partial}{\partial \sigma^2} L_T(\bar{\mu}_T, \bar{\sigma}_T^2) = -\frac{\bar{\mu}_T^2 + \tilde{\sigma}_T^2}{2\bar{\sigma}_T^4} - \sum_{t=1}^{T} \frac{(x_t - \bar{\mu}_T)^2}{2\bar{\sigma}_T^4} + \frac{1}{2\bar{\sigma}_T^2} = 0$$

Regret in One Dimension

This leads to the following expressions:

$$\bar{\mu}_T = \frac{\sum_{t=1}^T x_t}{T+1} \quad \text{and} \quad \bar{\sigma}_T^2 = \frac{1}{T+1}\left[\tilde{\sigma}^2 + \sum_{t=1}^T x_t^2\right] - \bar{\mu}_T^2$$

Injected in $L_T(\mu, \sigma^2)$:

$$\begin{aligned} L_T(\bar{\mu}_T, \bar{\sigma}_T^2) &= \frac{\bar{\mu}_T^2 + \tilde{\sigma}^2}{2\bar{\sigma}_T^2} + \frac{1}{2}\ln(\bar{\sigma}_T^2) + \sum_{t=1}^T \frac{(x_t - \bar{\mu}_T)^2}{2\bar{\sigma}_T^2} + \frac{1}{2}\ln(\bar{\sigma}_T^2) \\ &= \frac{T+1}{2} + \frac{T+1}{2}\ln\bar{\sigma}_T^2 \\ &\to \end{aligned}$$

Implementing the FTL strategy

Recursive update :

$$\begin{cases} \mu_{t+1} = \mu_t + \frac{1}{t+1}(x_t - \mu_t) \\ \sigma_{t+1}^2 = \frac{t}{t+1}\sigma_t^2 + \frac{t}{(t+1)^2}(x_t - \mu_t)^2 \end{cases}$$

Bounds :

$$\begin{cases} R_T \leq \sum_{t=1}^{T} \frac{1}{4(t+1)} \left[\frac{(x_t-\mu_t)^2}{\sigma_t^2}\right]^2 + \frac{1}{4}\ln(T+1) + \frac{1}{12} \\ \sum_{t=1}^{T} \frac{1}{4(t+1)} \left[\frac{(x_t-\mu_t)^2}{\sigma_t^2}\right]^2 - \frac{1}{6(t+1)^2} \left[\frac{(x_t-\mu_t)^2}{\sigma_t^2}\right]^3 + \frac{1}{4}\ln(T+1) \leq R_T \end{cases}$$

So if : $\liminf \sigma_t^2 > 0$ then for $T > T_0$, the Regret is $O(T_0 + \log(T/T_0))$

Proof of the Bounds

- We consider $R_t - R_{t-1}$ and do a Taylor Expansion of $\ln(1+z)$[1]
- Then using a telescoping sum we get the bounds

[1] $\forall z \geq 0, z - z^2/2 \leq \ln(1+z) \leq z - z^2/2 + z^3/3$

More general bounds

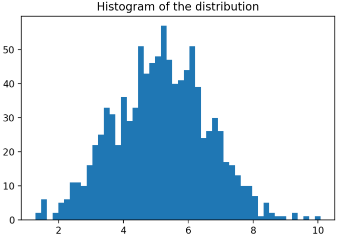
Theorem 1. For $r \leq \tilde{\sigma}^2$, there exists a sequence (x_t) such that the regret of FTL after T trials is

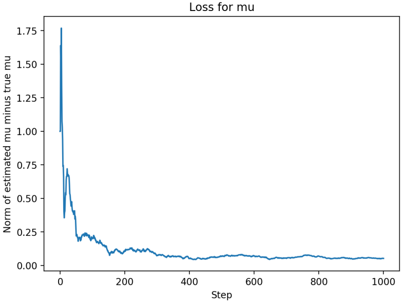
$$R_T \geq \frac{1}{12}\left(\frac{r}{\tilde{\sigma}}\right)^4 \frac{T^2}{T+1} + \frac{1}{4}\ln(T+1)$$

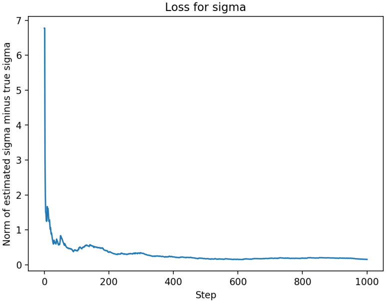
Theorem 3. For any $T \geq 1$ and any sequence (x_t), the regret of FTL after T trials is:

$$R_T \leq \left(\left(\frac{2r}{\tilde{\sigma}}\right)^4 + \left(\frac{2r}{\tilde{\sigma}}\right)^2\right)(T+1) + \frac{1}{4}\ln(T+1) + \frac{1}{12}$$

A correct estimation





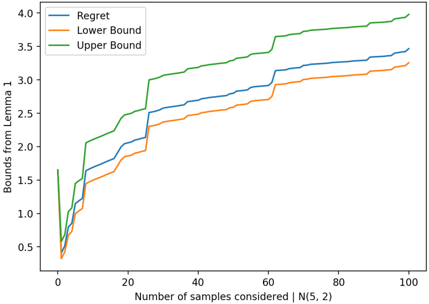


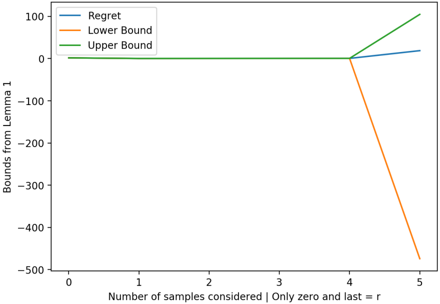
- Example of sequence :

$$\begin{cases} x_1 = x_2 = \cdots = x_{T-1} = 0 \\ x_T = r \end{cases}$$

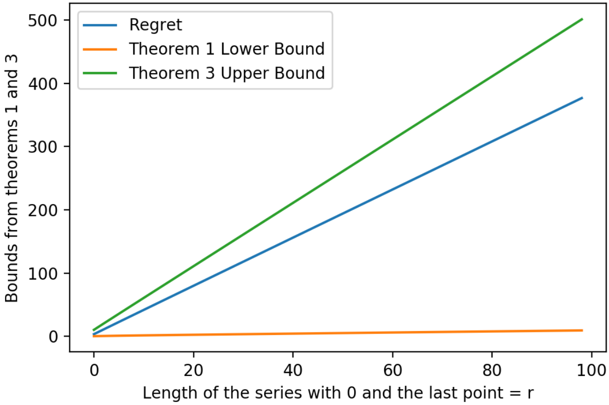
- The learner becomes confident, and the difference of regret only "explodes" in trial T. So the last contribution to the to the Lower Bound Regret is linear in T
- We'll see we can get a linear regret up to a trial T where it becomes logarithmic

The bound from Lemma 1





Linear Bounds



Differences with One Dimension

- Since x_t is a vector, we need $||x_t|| \leq r$
- For $\tilde{\sigma}^2$,

$$\ell_0(\mu, \Sigma) \triangleq \mathbb{E}_v \left[\frac{1}{2}(v-\mu)^\top \Sigma^{-1}(v-\mu) + \frac{1}{2} \ln |\Sigma| \right]$$

  where $\mathbb{E}[vv^\top] = \tilde{\sigma}^2 I_d$
- $\mu_{t+1} = \mu_t + \frac{1}{t+1}(x_t - \mu_t)$ and
  $(t+1)\Sigma_{t+1} = t\Sigma_t + (x_t - \mu_t)(x_t - \mu_t)^\top \frac{t}{t+1}$
- Extra factor **d** in all formulas

More general bounds

Theorem 5. For $r \leq \tilde{\sigma}^2$, there exists a sequence (x_t) such that the regret of FTL after T trials is

$$R_T \geq \frac{\mathbf{d}}{12}\left(\frac{r}{\tilde{\sigma}}\right)^4\left(T - \frac{\mathbf{d}}{2} + \frac{1}{2}\right)\left(\frac{T - \mathbf{d} + 1}{T - \mathbf{d} + 2}\right)$$

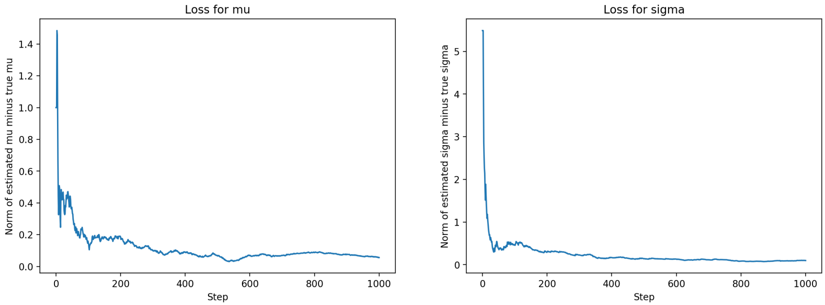
$$\times\left(1 - \frac{\mathbf{d} - 1}{(T - \mathbf{d} + 1)(T - \mathbf{d} + 2)}\right)^2 + \frac{\mathbf{d}}{4}\ln(T + 1)$$

Theorem 7. For any $T \geq 1$ and any sequence (x_t), the regret of FTL after T trials is:

$$R_T \leq \frac{\mathbf{d}}{4}\left(\left(\frac{2r}{\tilde{\sigma}}\right)^4 + \left(\frac{2r}{\tilde{\sigma}}\right)^2\right)(T + 1) + \frac{\mathbf{d}}{4}\ln(T + 1) + \frac{\mathbf{d}}{12}$$

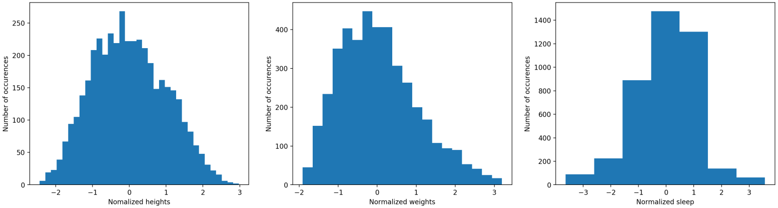
Convergence on generated data

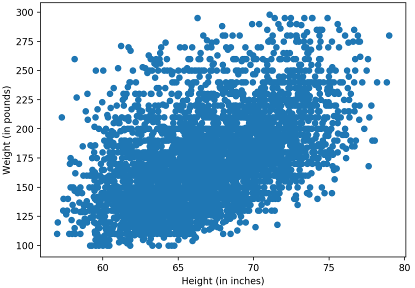
1000 samples, $\mu = (1, 0, 0)$, $\Sigma = I_3$



Test on a real dataset

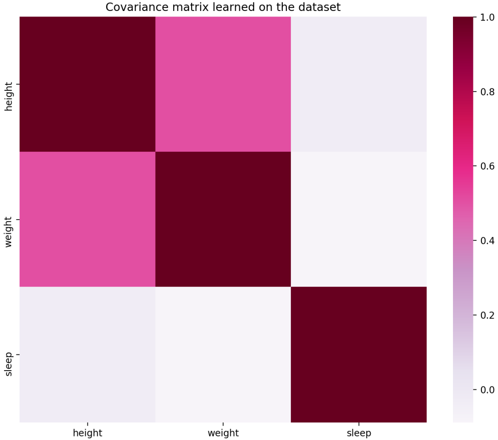
Dataset with: height, weight and sleep in America in 2007





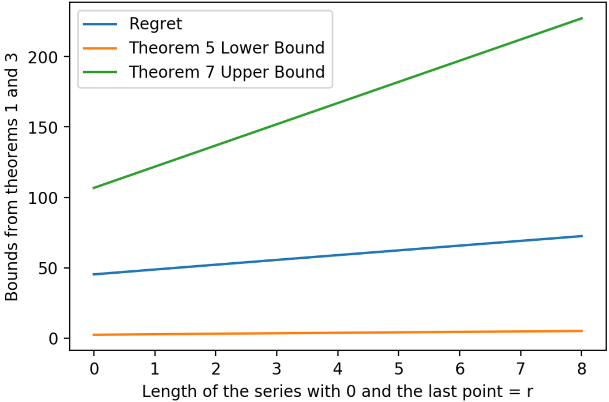
Test on a real dataset

Mean is the same as mean of data, and covariance seems reasonable

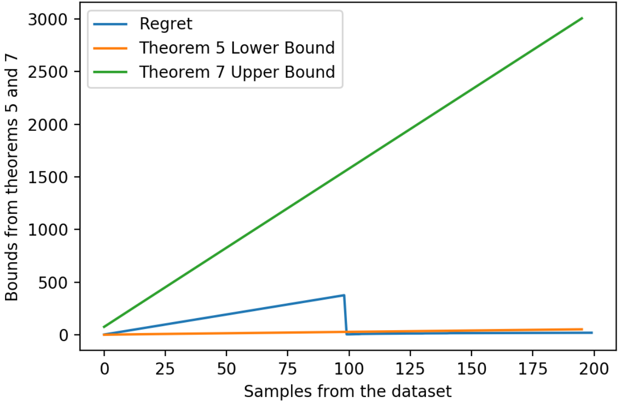


Weight and height are correlated

Bounds on a specific series (x_t)



Bounds on a real data set



Conclusion

- This theoretical framework can be verified in practice through simulations
- It does not give satisfactory results as:
  - Some bounds are given only on specific series
  - These series are not well characterized
- It is still limited: further work on gaussian mixtures, etc.

References I

Dasgupta S., Hsu D. (2007) On-Line Estimation with the Multivariate Gaussian Distribution. In: Bshouty N.H., Gentile C. (eds) Learning Theory. COLT 2007. Lecture Notes in Computer Science, vol 4539. Springer, Berlin, Heidelberg