Kapitel II

Valuation theory

§ 7 Discrete valuations

Example 7.1 Let $P \in \mathbb{N}$ prime. For $x \in \mathbb{Z}\backslash\{0\}$ let

$$\nu_p(x) = \max\{k \in \mathbb{N} \mid p^k \mid x\}.$$

Then $p^{\nu_p(x)} \mid x, \quad p^{\nu_p(x)+1} \nmid x$. Example: $\nu_2(12) = 2$. Write $x = p^{\nu_p(x)} \cdot x'$ where $p \nmid x'$. For $\frac{x}{y} \in \mathbb{Q}^\times$ define

$$\nu_p\left(\frac{x}{y}\right) = \nu_p(x) - \nu_p(y).$$

This defines a map $\nu_p : \mathbb{Q} \longrightarrow \mathbb{Z}$, such that

(i) $v_p(ab) = \nu_p(a) + \nu_p(b)$ (clear)

(ii) $v_p(a+b) \geqslant \min\{\nu_p(a), \nu_p(b)\}$, since: Write $a = p^{\nu_p(a)} \cdot a', b = p^{\nu_p(b)} \cdot b'$. Let w.l.o.g $\nu_p(b) \leqslant \nu_p(a)$. Then we have

$$a + b = p^{\nu_p(a)} \cdot a' + p^{\nu_p(b)} \cdot b' = p^{\nu_p(b)} \cdot \left(b' + a' \cdot p^{\nu_p(a) - \nu_p(b)}\right).$$

Hence $p^{\nu_p(b)} \mid a + b$ and thus $\nu_p(a+b) \geqslant \nu_p(b) = \min\{\nu_p(a), \nu_p(b)\}$.

Definition 7.2 Let k be afield. A *discrete valuation* on k is a surjectove group homomorphism $\nu_k^\times \longrightarrow (\mathbb{Z}, +)$ satisfying

$$\nu(x+y) \geqslant \min\{\nu(x), \nu(y)\} \qquad \text{for all } x, y \in k^\times, \ x \neq -y.$$

Remark 7.3 *Let R be a factorial domain, $k = \mathrm{Quot(R)}$. Let further be $p \in R\backslash\{0\}$ be a prime element. Then $\nu_p : k^\times \longrightarrow \mathbb{Z}$ can be defined as in Example 7.1: Write*

$$x = e \cdot \prod_{p \in \mathbb{P}} p^{\nu_p(x)}, \qquad e \in R^\times$$

where $\mathbb{P}$ denotes set of representatives of prime elements of R. Then ν_p is a discrete valuation on k.

Example 7.4 Let k be a field, $a \in k$, $R = k[X]$ and $p_a = X - a \in k[X]$. For $f \in k[X]$ define $\nu_{p_a}(f) = n$ if f has an n-fold root in a, i.e. $f = (X-a)^n \cdot g$ for some $0 \neq g \in k[X]$. Then ν_{p_a} is a discrete valuation on $k(X) = \mathrm{Quot}(\mathrm{k}[X])$ satisfying $\nu_p|_k = 0$.

Remark 7.5 *There is no discrete valuation on $\mathbb{C}$.*

proof. Assume there exists a discrete valuation on $\mathbb{C}$, say $\nu : \mathbb{C}^\times \longrightarrow \mathbb{Z}$. Since ν is surjective, there exists $z \in \mathbb{C}^\times$ such that $\nu(z) = 1$.
Let now $y \in \mathbb{C}^\times$ such that $y^2 = z$. Then we have

$$1 = \nu(z) = \nu(y^2) = \nu(y \cdot y) = \nu(y) + \nu(y) = 2\nu(y) \quad \Longleftrightarrow \quad \nu(y) = \frac{1}{2} \notin \mathbb{Z}$$

which is a contradiction. □

Example 7.6 Let $\nu : \mathbb{Q}^\times \longrightarrow \mathbb{Z}$ be a nontrivial discrete valuation. Then there exists $a \in \mathbb{Z}$ such that $\nu(a) \neq 0$ and hence we find $p \in \mathbb{P}$: $\nu(p) \neq 0$.
If $\nu(q) = 0$ for all $q \in \mathbb{P}$, then $\nu = \nu_p$.
Assume we have $\nu(p) \neq 0 \neq \nu(q)$ for some $p \neq q \in \mathbb{P}$ and write $1 = ap + bq$ for suitable $a, b \in \mathbb{Z}$. Then

$$0 = \nu(1) = \nu(ap{+}bq) \geqslant \min\{\nu(ap), \nu(bq)\} = \min\{\underbrace{\nu(a)}_{\geqslant 0\ (*)} + \nu(p), \underbrace{\nu(b)}_{\geqslant 0\ (*)} + \nu(q)\} \geqslant \min\{\nu(p), \nu(q)\} > 0$$

Hence a contradiction, i.e. we have $\nu(p) \neq 0$ for at most one $p \in \mathbb{P}$, thus $\nu = \nu_p$.
$(*)$ obtain that we have $\nu(1) = \nu(1 \cdot 1) = \nu(1) + \nu(1) \ \Rightarrow \nu(1) = 0$ and by induction

$$\nu(a) = \nu(1 + (a-1)) \geqslant \min\{\nu(1), \nu(a-1)\} \geqslant 0$$

Proposition 7.7 *Let k be a field and $\nu : k^\times \longrightarrow \mathbb{Z}$ be a discrete valuation on k.*

(i) $\nu(1) = \nu(-1) = 0$.

(ii) $\mathcal{O}_\nu := \{x \in k^\times \mid \nu(x) \geqslant 0\} \cup \{0\}$ *is a ring, called the valuation ring of* ν.

(iii) $\mathfrak{m}_\nu := \{x \in k^\times \mid \nu(x) > 0\} \cup \{0\} \lhd \mathcal{O}_\nu$ *is an ideal in* $\mathcal{O}_\nu$*, called the valuation ideal of* ν. *More precisely,* $\mathfrak{m}_\nu$ *is the only maximal ideal in* $\mathcal{O}_\nu$*, i.e.* $\mathcal{O}_\nu$ *is a local ring.*

(iv) $\mathfrak{m}_\nu$ *is a principal ideal.*

(v) $\mathcal{O}_\nu$ *is a principal ideal domain. More precisely, any ideal* $I \neq \{0\}$ *in* $\mathcal{O}_\nu$ *is of the form* $I = (t^d)$ *for some* $d \in \mathbb{N}$ *and* $t \in \mathfrak{m}_\nu$ *with* $\nu(t) = 1$.

(vi) *We have* $k = \mathrm{Quot(R)}$ *and for* $x \in k^\times$*:* $x \in \mathcal{O}_\nu$ *or* $\frac{1}{x} \in \mathcal{O}_\nu$.

proof. (ii) This is strict calculating, which may be verified by the reader.

(iii) $\mathfrak{m}_\nu$ is an ideal, since for $x, y \in \mathfrak{m}_\nu, \alpha \in \mathcal{O}_\nu$ we have

$$\nu(x+y) \geqslant \min\{\nu(x), \nu(y)\} > 0, \qquad \nu(\alpha x) = \underbrace{\nu(\alpha)}_{\geqslant 0} + \nu(x) \geqslant \nu(x) > 0.$$

Let now $x \in \mathcal{O}_\nu$ with $\nu(x) = 0$. Then

$$\nu\left(\frac{1}{x}\right) = \nu(1) - \nu(x) = -\nu(x) = 0,$$

hence $x \in \mathcal{O}_\nu^\times$. Thus we have $\mathfrak{m}_\nu = \mathcal{O}_\nu \backslash \mathcal{O}_\nu^\times$ and the claim follows.

(iv) Let $t \in \mathfrak{m}_\nu$ such that $\nu(t) = 1$. Then for $x \in \mathfrak{m}_\nu$ let $\nu(x) = d > 0$. Then we have

$$\nu\left(x \cdot t^{-d}\right) = \nu(x) + \nu\left(\frac{1}{t^d}\right) = d + 0 - d = 0$$

Define $e := x \cdot t^{-d} \in \mathcal{O}_\nu^\times$. Then $x = e \cdot t^d$, hence $\mathfrak{m}_\nu = (t)$.

(v) Let $\{0\} \neq I \neq \mathcal{O}_\nu$ be an ideal in $\mathcal{O}_\nu$. Let $d := \min\{\nu(x) \mid x \in I\backslash\{0\}\} > 0$.

'$\supseteq$' Let $x \in I$ such that $\nu(x) = d$. By part (iv) we have $x = e \cdot t^d$ for some $e \in \mathcal{O}_\nu^\times$, hence we have $t^d \in I$; thus $(t^d) \subseteq I$.

'$\subseteq$' Let now $y \in I\backslash\{0\}$ and write $y = e \cdot t^{\nu(y)}$ for some $e \in \mathcal{O}_\nu^\times$ and $\nu(y) > d$. Then $y = t^d \cdot e \cdot t^{\nu(y)-d}$, hence $y \in (t^d)$ and thus $I \subseteq (t^d)$.

(vi) If $\nu(x) \geqslant 0$, then $x \in \mathcal{O}_\nu$. If $\nu(x) < 0$, we have

$$\nu\left(\frac{1}{x}\right) = \nu(1) - \nu(x) = -\nu(x) > 0, \qquad \text{hence } \frac{1}{x} \in \mathfrak{m}_\nu \subseteq \mathcal{O}_\nu,$$

which we wanted to show. □

Definition 7.8 An integral domain R is called a *discrete valuation ring*, if there exists a discrete valuation ν of $k = \mathrm{Quot(R)}$ such that $R = \mathcal{O}_\nu$.

Proposition 7.9 *Let R be a lokal integral domain. Then the following statements are equivalent.*

(i) R is a discrete valuation ring.

(ii) R is a principal ideal domain.

(iii) There exists $t \in R\backslash\{0\}$ such that every $x \in R\backslash\{0\}$ can uniquely be written in the form

$$x = e \cdot t^d \qquad \text{for some } e \in R^\times,\ d \geqslant 0$$

proof. '(i) $\Rightarrow$ (ii)' This follows by 7.7.

'(ii) $\Rightarrow$ (iii)' We know that principal ideal domains are factorial. Let $t \in R$ be a generator of the maximal ideal $\mathfrak{m}$ of R. Then t is prime, since any maximal ideal is also prime. Let now $p \in R\backslash\{0\}$ a prime element. Then $p \notin R^\times$, hence $p \in \mathfrak{m}$, thus we can write $p = t \cdot x$ for some $x \in R$. Since p is prime, hence irreducible, we have $x \in R^\times \ \Rightarrow (p) = (t)$. Thus we have $p = t$ and we have only one prime element in R. The unique prime factorization in factorial domains gives us $x = e \cdot t^d$ for some $e \in R^\times$ and $d \geqslant 0$.

'(iii)⇒(i)' For $x = e \cdot t^d \in R\backslash\{0\}$, $e \in R^\times, d \geqslant 0$ define $\nu(x) = d$. We claim that ν is discrete valuation. We have

$$\nu(xy) = \nu\left(et^d \cdot e't^{d'}\right) = \nu\left(ee't^{d+d'}\right) = \nu\left(e''t^{d+d'}\right) = d + d'.$$

Let w.l.o.g. $d \leqslant d'$. Then

$$\nu(x+y) = \nu\left(et^d + e't^{d'}\right) = \nu\left(t^d\left(e + e't^{d'-d}\right)\right) \geqslant d = \min\{d, d'\}$$

which we extend to

$$\nu : k^\times \longrightarrow \mathbb{Z}, \qquad \nu\left(\frac{x}{y}\right) = \nu(x) - \nu(y).$$

This is well defined: For $\frac{x}{y} = \frac{x'}{y'}$ we have $xy' = x'y$ and $\nu(x'y) = \nu(x)+\nu(y') = \nu(x')+\nu(y)$, thus

$$\nu\left(\frac{x}{y}\right) = \nu(x) - \nu(y) = \nu(x') - \nu(y') = \nu\left(\frac{x'}{y'}\right).$$

Finally we have $\nu(t) = 1$, hence $\nu : k^\times \longrightarrow \mathbb{Z}$ is surjective. Thus ν is a discrete valuation on k and $R = \mathcal{O}_\nu$. □

Definition + proposition 7.10 Let R be a local ring with maximal ideal $\mathfrak{m}$.

(i) $k := R/\mathfrak{m}$ is called the *residue field* of R.

(ii) $\mathfrak{m}/\mathfrak{m}^2$ has a structure of a k-vector space.

(iii) If R is a discrete valuation ring, then $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1$.

proof. (ii) For $a \in R$, $x \in \mathfrak{m}$ define $\overline{a}\overline{x} = \overline{ax}$, where $\overline{a}, \overline{x}$ are the images of a, x in k.

This is well defined: Let $a' \in R$ with $\overline{a'} = \overline{a}$ and $x' \in \mathfrak{m}$ with $\overline{x'} = \overline{x}$. We have to show that

$$\overline{a'x'} = \overline{ax} \iff a'x' - ax \in \mathfrak{m}^2$$

We have $\overline{a'} = \overline{a}$, hence $a' = a + y$ for some $y \in \mathfrak{m}$. Analogously we have $\overline{x'} = \overline{x}$, hence $x' = x+$ for some $z \in \mathfrak{m}^2$. Thus we have

$$a'x' = (a+y)(b+z) = ax + az + xy + yz \equiv ax \mod \mathfrak{m}^2,$$

which finishes the proof. □

§ 8 The Gauß Lemma

Let R be a UFD (unique factorization domain), $\mathbb{P}$ a set of representatives of the primes in R with respect to *associateness*, i.e. $x \sim y \Leftrightarrow y = u \cdot x$ for some $u \in R^\times$. Every $x \in R\backslash\{0\}$ has a unique factorization

$$x = u \cdot \prod_{p \in \mathbb{P}} p^{\nu_p(x)}, \qquad \nu_p(x) \geqslant 0 \text{ for } p \in \mathbb{P},\ u \in R^\times$$

where $\nu_p : k^\times \longrightarrow \mathbb{Z}$ is a discrete valuation on $k = \mathrm{Quot}(R)$.

Definition + proposition 8.1 Let R be a factorial domain, $k = \mathrm{Quot}(R)$ and

$$f = \sum_{i=0}^{n} a_i X^i \in k[X]\backslash\{0\}, \qquad a_n \neq 0.$$

(i) For $p \in \mathbb{P}$ let $\nu_p(f) = \min\{\nu_p(a_i) \mid 0 \leqslant i \leqslant n\}$.
(ii) f is called *primitive*, if $\nu_p(f) = 0$ for all $p \in \mathbb{P}$.
(iii) If f is primitive, then $f \in R[X]$.
(iv) If $f \in R[X]$ is monic, i.e. $a_n = 1$, then f is primitive.
(v) There exists $c \in k^\times$ such that $c \cdot f$ is primitive.

proof. (iii) If f is primitive, we have $\min_{1\leqslant i\leqslant n}\{\nu_p(a_i)\} = 0$, i.e. $\nu_p(a_i) \geqslant 0$ for all $1 \leqslant i \leqslant n$. Thus $a_i \in R$ and $f \in R[X]$.

(iv) If $a_i \in R$ we have $\nu_p(a_i) \geqslant 0$ for all $1 \leqslant i \leqslant n$. Moreover $\nu_p(a_n) = \nu_p(1) = 0$, hence $\nu_p(f) = \min_{1\leqslant i\leqslant n}\{\nu_p(a_i)\} = 0$. thus f is primitive.

(v) For $\nu_p(f) := d$ choose $c := p^{-d} \in k^\times$. Then

$$\nu_p(c \cdot f) = \nu_p(c) + \nu_p(f) = \nu_p(p^{-d}) + d = -d + d = 0,$$

thus $c \cdot f$ is primitive. □

Proposition 8.2 (*Gauß-Lemma*) *For $f, g \in k[X]$ and $p \in \mathbb{P}$ we have*

$$\nu_p(f \cdot g) = \nu_p(f) + \nu_p(g).$$

proof. Write

$$f = \sum_{i=0}^{n} a_i X^i, \qquad g = \sum_{j=0}^{m} b_j X^j, \qquad f \cdot g = \sum_{k=0}^{m+n} c_k X^k, \quad c_k = \sum_{i=0}^{k} a_i b_{k-i}$$

case 1 Assume $m = 0$, i.e. $g = b_0 \in k^\times$. Then $c_k = a_k \cdot b_0$, hence

$$\nu_p(c_k) = \nu_p(a_k) + \nu_p(b_0).$$

Then we obtain

$$\nu_p(f\cdot g) \;=\; \min_{0\leqslant k\leqslant n} \nu_p(c_k) = \min_{0\leqslant k\leqslant n}\{\nu_p(a_k)+\nu_p(b_0)\} = \nu_p(b_0)+\min_{0\leqslant k\leqslant n}\{\nu_p(a_k)\} \;=\; \nu_p(g)+\nu_p(f)$$

case 2 Assume $\nu_p(f) = 0 = \nu_p(g)$, i.e. f, g are primitive. Clearly $\nu_p(fg) \geqslant 0$. We have to show: $\nu_p(fg) = 0$. Let $i_0 := \max\{i \mid \nu_p(a_i) = 0\}$ and $j_0 := \max\{j \mid \nu_p(b_j) = 0\}$. Then

$$c_{i_0+j_0} = \sum_{i=0}^{i_0+j_0} a_i b_{i_0+j_0-i} = \underbrace{\sum_{i=0}^{i_0-1} a_i b_{i_0+j_0-i}}_{(A)} + a_{i_0+j_0} + \underbrace{\sum_{i=i_0+1}^{i_0+j_0} a_i b_{i_0+j_0-i}}_{(B)}$$

We have $\nu_p(a_{i_0}b_{j_0}) = \nu_p(a_{i_0}) + \nu_p(b_{j_0}) = 0$. We have $i_0 + j_0 - i > j_0$, hence $\nu_p(b_{i_0+j_0-i}) \geqslant 1$ for $0 \leqslant i \leqslant i_0 - 1$. Then

$$\begin{aligned}
\nu_p(A) &= \nu_p\left(\sum_{i=0}^{i_0-1} a_i b_{i_0+j_0-i}\right) &\geqslant& \min_{0\leqslant i\leqslant i_0-1}\{\nu_p(a_i b_{i_0+j_0-1})\} \\
&&=& \min_{0\leqslant i\leqslant i_0-1}\{v_p(a_i) + \nu_p(b_{i_0+j_0-1})\} \\
&&\geqslant& \min_{0\leqslant i\leqslant i_0-1}\{\nu_p(b_{i_0+j_0-1})\} \\
&&\geqslant& 1 \\
\nu_p(B) &= \nu_p\left(\sum_{i=i_0+1}^{i_0+j_0} a_i b_{i_0+j_0-i}\right) &\geqslant& 1.
\end{aligned}$$

Since we have

$$0 = \nu_p(a_{i_0}b_{j_0}) \geqslant \min\{\nu_p(c_{i_0+j_0}), \nu_p(A), \nu_p(B)\} = \nu_p(c_{i_0+j_0}) = 0$$

we get $\nu_p(c_{i_0+j_0}) = 0$. Hence we obtain

$$\nu_p(fg) = \min\{\nu_p(c_i) \mid 0 \leqslant i \leqslant m+n\} = \nu_p(c_{i_0+j_0}) = 0.$$

case 3 Consider now the general case, i.e. f, g are arbitrary. Multiply f and g by suitable constants a and b, such that $\tilde{f} := af$ and $\tilde{g} := bg$ are primitive. Then by the first two cases we have

$$\begin{aligned}
\nu_p(fg) &= \nu_p\left(\frac{1}{a}\frac{1}{b}\tilde{f}\tilde{g}\right) \overset{1}{=} \nu_p\left(\frac{1}{a}\frac{1}{b}\right) + \nu_p(\tilde{f}\tilde{g}) \overset{2}{=} \nu_p\left(\frac{1}{a}\right) + \nu_p\left(\frac{1}{b}\right) + \underbrace{\nu_p(\tilde{f})}_{=0} + \underbrace{\nu_p(\tilde{g})}_{=0} \\
&= \nu_p\left(\frac{1}{a}\right) + \nu_p(\tilde{f}) + \nu_p\left(\frac{1}{b}\right) + \nu_p(\tilde{g}) = \nu_p\left(\frac{1}{a}\tilde{f}\right) + \nu_p\left(\frac{1}{b}\tilde{g}\right) \\
&= \nu_p(f) + \nu_p(g),
\end{aligned}$$

which finishes the proof. □

Theorem 8.3 (*Eisenstein's criterion for irreducibility*) *Let R be a factorial domain, $p \in \mathbb{P}$ and*

$$f = \sum_{i=0}^{n} a_i X^i \quad \in R[X] \backslash \{0\}$$

Assume that f is primitive and we have

(i) $\nu_p(a_0) = 1$,

(ii) $\nu_p(a_i) \geqslant 1$ *or* $a_i = 0$ *for* $1 \leqslant i \leqslant n-1$ *and*

(iii) $\nu_p(a_n) = 0$

Then f is irreducible over $R[X]$.

proof. Assume that $f = g \cdot h$ with some $g, h \in R[X]$. Write

$$g = \sum_{i=0}^{r} b_i X^i, \qquad h = \sum_{j=0}^{s} c_i X^j, \qquad \text{with } r+s=n$$

Then we have $a_0 = b_0 c_0$. W.l.o.g. $\nu_p(b_0) = 1$ and $\nu_p(c_0) = 0$. Further $a_n = b_r c_s$, thus we must have $\nu_p(b_r) = \nu_p(c_s) = 0$ for $\nu_p(a_n) = 0$. Let now

$$d := \max\{i \mid \nu_p(b_j) \geqslant 1 \text{ for } 0 \leqslant j \leqslant i\}$$

Obviously $0 \leqslant d \leqslant r-1$. Consider

$$a_{d+1} = \underbrace{b_{d+1} c_0}_{=:A} + \underbrace{\sum_{i=0}^{d} b_i c_{d+1-i}}_{=:B}.$$

We have

$$\nu_p(A) = \nu_p(b_{d+1}) + \nu_p(c_0) = 0 + 0 = 0,$$

$$\nu_p(B) \geqslant \min_{0 \leqslant i \leqslant d} \{\nu_p(b_i c_{d+1-1}) \geqslant 1$$

and thus $\nu_p(a_{d+1}) = 0$. But this implies $d+1 = n \Leftrightarrow n-1 = d \leqslant r-1 \Rightarrow n \leqslant r \Rightarrow n = r$. Then we have $s = 0$, thus $h = c_0$ is constant. Further for $q \in \mathbb{P}$ we have

$$0 = \nu_q(f) = \nu_q(gc_o) = \underbrace{\nu_q(g)}_{\geqslant 0} + \nu_q(c_0)$$

i.e. $\nu_q(c_0) = 0$, hence $c_0 \in R^\times$ and f is irreducible. □

Theorem 8.4 (*Gauß*) *Let R be a factorial domain. Then $R[X]$ is factorial.*

proof. Let $f \in R[X] \backslash \{0\} \subseteq k[X]$ where $k = \mathrm{Quot}(R)$. Since $k[X]$ is factorial, we can write

$$f = c \cdot f_1 \cdots f_n, \quad f_i \in k[X] \text{ prime }, \ c \in k^\times$$

W.l.o.g the. f_i are primitive, otherse multiply them by suitable constants. In particular we have $f_i \in R[X]$. Note that $c \in R$: For $p \in \mathbb{P}$, we have

$$0 = \nu_p(f) = \nu_p(c) + \sum_{i=1}^{n} \nu_p(f_i) = \nu_p(c).$$

Write $c = \epsilon \cdot p_1 \cdots p_r$ with some $\epsilon \in R^\times$ and $p_i \in \mathbb{P}$.Then by
Claim (a) $f_i \in R[X]$ are prime for $1 \leqslant i \leqslant n$.
Claim (b) $p_i \in R[X]$ are prime for $1 \leqslant i \leqslant r$.
we have found a factorization of f into prime elements and hence $R[X]$ is factorial. Now prove the claims.

(a) Let $g, h \in R[X]$ such that $gh \in (f_i) = f_i R[X]$.
May assume that $g \in f_i k[X]$, i.e. $g = f_i \tilde{g}$ for some $\tilde{g} \in k[X]$. For $p \in \mathbb{P}$ we obtain

$$0 \leqslant \nu_p(g) = \underbrace{\nu_p(f_i)}_{=0} + \nu_p(\tilde{g}) = \nu_p(\tilde{g}).$$

Thus we get $\tilde{g} \in R[X]$, which implies $g = f_i \tilde{g} \in f_i R[X] = (f_i)$.

(b) Since $\pi : R \longrightarrow R/(p)$ induces a map $\psi : R[X] \longrightarrow R/(p)[X]$ with $\ker(\psi) = pR[X]$ we have

$$R[X]/pR[X] \cong R/pR[X].$$

Since R/pR is an integral domain, (p) is prime. □

Corollary 8.5 *Let k be a field. Then $k[X_1, \ldots X_n]$ is factorial for any $n \in \mathbb{N}$.*

Corollary 8.6 *Let R be a factorial domain, $k = Quot(R)$. If $f \in R[X]$ is irreducible over $R[X]$, then f is irreducible over $k[X]$.*

proof. Let $0 \neq f = c \cdot f_1 \cdots f_n$ be decomposition of f in $k[X]$, i.e. $c \in k^\times$ and $f_i \in k[X]$ irreducible for $1 \leqslant i \leqslant n$. We may assume that the f_i are primitive, hence contained in $R[X]$, since we can multiply them by suitable constants. We still have to show $c \in R$. Since $f \in k[X]$, i.e. $\nu_p(f) \geqslant 0$ we have

$$\nu_p(f) = \nu_p(c \cdot f_1 \cdots f_n) = \nu_p(c) + \sum_{i=1}^{n} \underbrace{\nu_p(f_i)}_{=0} = \nu_p(c) \overset{!}{\geqslant} 0$$

Thus $c \in R$. Then the decomposition from above is in R - but since f is irreducible in R, we have $n = 1$ and $c \in R^\times$. □

§ 9 Absolute values

Definition 9.1 Let k be a field. A map

$$|\cdot| : k \longrightarrow \mathbb{R}_{\geqslant 0}$$

is called an *absolute value*, if

(i) *positive definiteness:* $|x| = 0 \iff x = 0$

(ii) *multiplicativeness:* $|xy| = |x| \cdot |y|$ for all $x, y \in k$.

(iii) *triangle inequality:* $|x + y| \leqslant |x| + |y|$ for all $x, y \in k$.

Example 9.2 (i) The 'normal' absolute value $|\cdot|_\infty$ on $\mathbb{C}$ and on any of its subfields denotes an absolute value.

(ii) Let $\nu_k^\times \longrightarrow \mathbb{Z}$ be a discrete valuation, $\rho \in (0, 1)$. Then

$$|\cdot|_\nu : k \longrightarrow \mathbb{R},\ x \mapsto \begin{cases} \rho^{\nu(x)} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is an absolute value on k, since

(1) Trivial, since $|0| = 0$ and $\rho^x \neq 0$ for any $x \in \mathbb{Z}$.

(2) Clearly $|xy|_\nu = \rho^{\nu(xy)} = \rho^{\nu(x)+\nu(y)} = \rho^{\nu(x)}\rho^{\nu(y)} = |x|_\nu |y|_\nu$.

(3) Further

$$|x+y|_\nu \ = \ \rho^{\nu(x+y)} \leqslant \rho^{\min\{\nu(x),\nu(y)\}} = \max\{\rho^{\nu(x)}, \rho^{\nu(y)}\} = \max\{|x|_\nu, |y|_\nu\} \ \leqslant \ |x|_\nu + |y|_\nu$$

(iii) For the p-adic valuation ν_p on $\mathbb{Q}$ we choose $\rho := \frac{1}{p}$. Then $|x|_p = p^{-\nu_p(x)}$ is an absolute value.

Remark + definition 9.3 Let k be a field, $|\cdot|$ an absolute value on k.

(i) $|1| = |-1| = 1$ and $|x| = |-x|$ for all $x \in k$.

(ii) The absolute value is called trivial, if $|x| = 1$ for all $x \in k$.

proof. We have $|1| = |1 \cdot 1| = |1| \cdot |1|$, hence $|1| = 1$. Moreover $|-1| = |1 \cdot (-1)| = |1| \cdot |-1|$, hence $|-1| = 1$. For $x \in k$ we have $|-x| = |(-1) \cdot x| = |-1| \cdot |x| = |x|$. □

Proposition + definition 9.4 Let k be a field with $\mathrm{char}(k) = 0$, i.e. $k \supseteq \mathbb{Q}$ and $|\cdot|$ an absolute value on k.

(i) $|\cdot|$ is called *archimedean*, if $|n| > 1$ for all $n \in \mathbb{Z}\backslash\{-1, 0, 1\}$.

(ii) $|\cdot|$ is called *nonarchimedean*, if $|n| \leqslant 1$ for all $n \in \mathbb{Z}$.

(iii) $|\cdot|$ is either archimedean or nonarchimedean.

(iv) The p-adic absolute value on $\mathbb{Q}$ is nonarchimedean.

proof of (iii). Since $|n| = |-n|$, it suffices to check $n \in \mathbb{N}$. Let $a \in \mathbb{N} \subseteq k$ with $|a| > 1$. Assume there exists $b \in \mathbb{N}_{>1}$ with $|b| \leqslant 1$. Write

$$a = \sum_{i=0}^{N} \alpha_i b^i \qquad \alpha_i \in \{0, \dots b-1\}, \ |N| = \lfloor \log_b(a) \rfloor.$$

Then we have

$$|a| \ \leqslant \ \sum_{i=0}^{\lfloor \log_b(a) \rfloor} |\alpha_i||b|^i \ \leqslant \ \log_b(a) \cdot \max_{0 \leqslant i \leqslant \lfloor \log_b(a) \rfloor} \{|\alpha_i|\} \ =: \ \log_b(a) \cdot c,$$

$$|a^n| \leqslant \log_b(a^n) \cdot c \ = \ n \cdot \log_b(a) \cdot c$$

and $|a^n|$ grows linearly in n. Likewise we get for $n \in \mathbb{N}$

$$a^n = \sum_{i=0}^{\lfloor \log_b(a^n) \rfloor} \alpha_i^{(n)} b^i, \qquad \alpha_i^{(n)} \in \{0, \dots b-1\},$$

$$|a^n| = |a|^n \ \leqslant \ (\log_b(a) \cdot c)^n$$

which grows exponentially in n, which is a contradiction. Hence the claim follows. □

Remark 9.5 *An absolute value $|\cdot|$ on a field k induces a metric*

$$d(x,y) := |x-y|, \qquad x, y \in k$$

Therefore, k as a topology and aspects as 'convergence' and 'cauchy sequences' are meaningful.

Definition + remark 9.6 (i) Two absolute values $|\cdot|_1, |\cdot|_2$ on k are called *equivalent*, if there exists $s \in \mathbb{R}$, such that $|x|_1 = |x|_2^s$ for all $x \in k$. In this case, we write $|\cdot|_1 \sim |\cdot|_2$.

(ii) Two absolutes values $|\cdot|_1, |\cdot|_2$ are equivalent if and only if the induce the same topology on k.

proof. Is left for the reader as an exercise.

Example 9.7 The p-adic absolute values on $\mathbb{Q}$ are not equivalent for $p \neq q \in \mathbb{P}$. Consider

$$|p^n|_p = p^{-n} \xrightarrow{n \to \infty} 0, \qquad |p^n|_q = 1 \ \text{ for all } n \in \mathbb{N}$$

Moreover we have $|\cdot|p \nsim |\cdot|_\infty$, since by the transittivity of equivalence of absolute values, we have

$$|\cdot|_p \sim |\cdot|_\infty \sim |\cdot|_q$$

which is not true.

Theorem 9.8 (*Ostrowski*) *Any nontrivial absolute value $|\cdot|$ on $\mathbb{Q}$ is equivalent either to the standard absolute value $|\cdot|_\infty$ on $\mathbb{Q}$ or to a p-adic absolute value $|\cdot|_p$ for some $p \in \mathbb{P}$.*

proof. **case 1** Assume $|\cdot|$ is nonarchimedean. We want to show, that in this case $|\cdot| \sim |\cdot|_p$ for some $p \in \mathbb{P}$. Since $|\cdot|$ is non-trivial, there exists $x \in \mathbb{N}$ such that

$$|x| = \left|\prod_{p\in\mathbb{P}} p^{\nu_p(x)}\right| = \prod_{p\in\mathbb{P}} |p|^{\nu_p(x)} \neq 1$$

for at least one $x \in \mathbb{Q}$, hence, we have $|p| \neq 1$ for at least one $p \in \mathbb{P}$, i.e. $|p| < 1$. Assume there is another prime $q \neq p$ with $|q| < 1$. Then we find $N \in \mathbb{N}$, such that

$$|p|^N \lneq \frac{1}{2}, \qquad |q|^N \lneq \frac{1}{2}.$$

Moreover, since p^N, q^N are coprime, we can write

$$1 = a \cdot p^N + b \cdot q^N \qquad \text{for suitable } a, b \in \mathbb{Z}.$$

So the contradiction follows by

$$1 = |1| = \left|ap^N + bq^N\right| \leqslant \underbrace{|a|}_{\leqslant 1}\underbrace{\left|p^N\right|}_{<\frac{1}{2}} + \underbrace{|b|}_{\leqslant 1}\underbrace{\left|q^N\right|}_{<\frac{1}{2}} < 1,$$

hence we have $|q| = 1$ for any $q \neq p \in \mathbb{P}$. Let now $s := -\log_p |p|$. For $x \in \mathbb{Q}^\times$ we obtain

$$|x| = \left|\prod_{\tilde{p}\in\mathbb{P}} \tilde{p}^{\nu_{\tilde{p}}(x)}\right| = \prod_{\tilde{p}\in\mathbb{P}} |\tilde{p}|^{\nu_{\tilde{p}}(x)} = |p|^{\nu_p(x)} = p^{-s\cdot\nu_p(x)} = \left(p^{-\nu_p(x)}\right)^s = |x|_p^s$$

thus we have $|\cdot| \sim |\cdot|_p$.

case 2 Let now $|\cdot|$ be archimedean. We now have to show $|\cdot| \sim |\cdot|_\infty$. For $n \in \mathbb{N}_{\geqslant 2}$ we have

$$1 < |n| = \left|\sum_{i=1}^{n} 1\right| \leqslant \sum_{i=1}^{n} |1| = n.$$

For any $a \in \mathbb{N}_{\geqslant 2}$ we find $s := s(a) \in \mathbb{R}_{<0}$ such that

$$|a| = |a|_\infty^s = a^s$$

namely

$$s = \log_a(|a|) = \frac{\log(|a|)}{\log(a)}.$$

Claim (a) We have

$$\frac{\log(|a|)}{\log(a)} = \frac{\log(|2|)}{\log(2)}.$$

Since now s is independent of a, we have $|\cdot| \sim |\cdot|_\infty$. Prove now the claim:

(a) For $n \in \mathbb{N}$ write

$$2^n = \sum_{i=0}^{N} \alpha_i a^i \quad \text{with } \alpha_i \in \{0, \dots a-1\} \text{ and } N \leqslant \log_a 2^n = n \cdot \frac{\log(2)}{\log(a)}.$$

Then we have

$$|2|^n = |2^n| \leqslant \sum_{i=0}^{N} \underbrace{|\alpha_i|}_{\leqslant \alpha_i < a} \overbrace{|a|^i} \leqslant |a|^N \leqslant (N+1)\dot{}a \cdot |a|^N,$$

hence we get

$$\begin{aligned} n \cdot \log(|2|) &\leqslant \log(N+1) + \log(a) + N\log(|a|) \\ &\leqslant \log\left(n \cdot \frac{\log(2)}{\log(a)} + 1\right) + \log(a) + n \cdot \frac{\log(2)}{\log(a)} \cdot \log(|a|). \end{aligned}$$

Multiplying the equation by $\frac{1}{n} \cdot \frac{1}{\log(2)}$ gives us

$$\frac{\log(|2|)}{\log(2)} \leqslant \frac{1}{n} \cdot \log\left(n \cdot \frac{\log(2)}{\log(a)} + 1\right) + \frac{\log(|a|)}{\log(a)}$$

and thus

$$\frac{\log(|2|)}{\log(2)} \leqslant \frac{\log(|a|)}{\log(a)}.$$

Swapping the roles of a and 2 in the equation above gives us the other inequality. Hence we have equality, which proves the claim. □

Proposition 9.9 *Let $|\cdot|$ be a nonarchimedean absolute value on a field k.*

(i) $|x+y| \leqslant \max\{|x|, |y|\}$ *for all* $x, y \in k$.

(ii) *If* $|x| \neq |y|$, *then equality holds in (i).*

proof. (i) If $x = 0$, we have $|y+x| = |y| \leqslant \max\{0, |y|\} = \max\{|x|, |y|\}$. Thus assume $x \neq 0$. We have $|x+y| = |x|\left|1+\frac{y}{x}\right|$. It suffices to show $|x+1| \leqslant \max\{1, |x|\}$. Then we get

$$|x+y| = |y| \cdot \left|1 + \frac{x}{y}\right| \leqslant |y| \cdot \max\left\{\left|\frac{x}{y}\right|, |1|\right\} \leqslant \max\{|x|, |y|\}$$

For $n \in \mathbb{N}$ we have

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Then we have

$$|x+1|^n \;=\; |(x+1)^n| \;=\; \left|\sum_{k=0}^{n}\binom{n}{k}x^k\right| \;\leqslant\; \sum_{k=0}^{n}\underbrace{\left|\binom{n}{k}\right|}_{\leqslant 1}\underbrace{|x|}_{\leqslant 1}{}^{k} \;\leqslant\; n+1,$$

hence

$$|x+1| \leqslant \sqrt[n]{n+1} \qquad \text{for all } n \in \mathbb{N}.$$

Thus $|1+x| \leqslant 1$. Since we clearly have $|x+1| \leqslant |x|$, we all in all have

$$|x+1| \leqslant \max|\{|x|, 1\}.$$

(ii) Let $z = x + y$ and assume $|x| < |y|$. We have to show $|z| = |y|$. Assume $|z| < |y|$. Then

$$|y| = |z - x| \overset{(i)}{\leqslant} \max\{|z|, |-x|\} < |y| \quad ↯$$

and the proof is done. $\square$

Proposition 9.10 *Let $|\cdot|$ be an a nonarchimedean absolute value on a field k. Then*

(i) We have a local ring

$$\overline{\mathcal{B}}_1(0) := \{x \in k \big| |x| \leqslant 1\} =: \mathcal{O}_k$$

with maximal ideal

$$\mathcal{B}_1(0) := \{x \in k \big| |x| < 1\} =: \mathfrak{m}_k$$

(ii) Every point in ball is its center.

(iii) Balls are either disjoint or one of them is contained in the other one.

(iv) All triangles are isosceles.

proof. (i) By 9.8(i), $\overline{\mathcal{B}}_1(0)$ is closed under Addition. The remaining is calculating.

(ii) Let $z \in \overline{\mathcal{B}}_r(x)$. To show: $\overline{\mathcal{B}}_r(z) = \overline{\mathcal{B}}_r(x)$.

'$\subseteq$' Let $y \in \overline{\mathcal{B}}_r(z)$, i.e. we have $|y - z| \leqslant r$. Then

$$|y - x| = |y - z + z - x| \leqslant \max\{|y - z|, |z - x|\} \leqslant r \quad \Rightarrow \quad y \in \overline{\mathcal{B}}_r(x).$$

Thus we have $\overline{\mathcal{B}}_r(z) \subseteq \overline{\mathcal{B}}_r(x)$.

'$\supseteq$' Follows by symmetry.

(iii) Let $\mathcal{B} := \overline{\mathcal{B}}_r(x)$, $\mathcal{B}' := \overline{\mathcal{B}}_{r'}(x')$ and $y \in \mathcal{B} \cap \mathcal{B}'$. W.l.o.g. $r \leqslant r'$.

Then for $z \in \mathcal{B}$ we have

$$|z - x'| = |z - x + x - y + y - x'| \leqslant \max\{|z - x|, |x - y|, |y - x'|\} = \max\{r, r, r'\} = r'$$

which implies $z \in \mathbb{B}'$. Hence we have $\mathcal{B} \subseteq \mathcal{B}'$.

(iv) Follows from 9.8(ii). $\square$

Corollary 9.11 *Let k be a field, $|\cdot|$ a nonarchimedean absolute value on k.*

(i) All balls are closed and open, considering the topology on k induced by the metric $d(x, y) = |x - y|$.

(ii) k is totally disconnected, i.e. no subset of k containing more than on element is connected.

proof. (i) Let $\mathcal{B} := \overline{\mathcal{B}}_r(x)$ be a closed ball for some $x \in k$, $r \in \mathbb{R}_{\geqslant 0}$. Then $\mathcal{B}$ topologically clearly is closed . Let now $y \in \mathcal{B}$. Then $\mathcal{B}_r(y) \subseteq \mathcal{B}$ by 9.9(ii), i.e. $\mathcal{B}$ is open.

Let now $\mathcal{B} := \mathcal{B}_r(x)$ be an open ball and $y \in k$ a boundary point. Thus for all $s > 0$ we find $z \in \mathcal{B}_s(x) \cap \mathcal{B}_r(x)$. Choose $s \leqslant r$. Then

$$d(x, y) \leqslant \max\{d(y, z), d(x, z)\} < \max\{s, r\} = r.$$

Thus $y \in \mathcal{B}_r(x)$, hence $\mathcal{B}_r(x)$ is contains its boundary and is closed.

(ii) Let $X \subseteq k$ be a subset with $x \neq y \in X$. Then for $r := |x - y| > 0$ we get

$$X = \left(\overline{\mathcal{B}}_{\frac{r}{2}}(x) \cap X\right) \cup \left(X \backslash \overline{\mathcal{B}}_{\frac{r}{2}}(x)\right)$$

which is a decomposition of X into two nonempty, disjoint open subset, i.e. the claim follows.

Example 9.12 (*Geometry on* $(\mathbb{Q}, |\cdot|_p)$*)* The unit disc in $(\mathbb{Q}, |\cdot|_p)$ is

$$\left\{\frac{a}{b} \in \mathbb{Q} \mid p \nmid b\right\} =: \mathbb{Z}_{(p)}$$

The maximal ideal is

$$\left\{\frac{a}{b} \in \mathbb{Q} \mid p \nmid b, p \mid a\right\} = p \cdot \mathbb{Z}_{(p)} = \overline{\mathcal{B}}_{\frac{1}{p}}(0)$$

We have

$$\left\{x \in \mathbb{Q} \mid |x|_p < 1\right\} = \left\{x \in \mathbb{Q} \mid |x|_\infty < \frac{1}{p}\right\}$$

Moreover

$$\mathbb{Z}_{(p)} \Big/ p\mathbb{Z}_{(p)} \cong \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p = \{\overline{0}, \overline{1}, \ldots, \overline{p-1}\}$$

$\overline{\mathcal{B}}_1(0)$ is the disjoint union of the $\overline{\mathcal{B}}_{\frac{1}{p}}(i)$ for $0 \leqslant i \leqslant p - 1$, where $\overline{\mathcal{B}}_{\frac{1}{p}}(i) = i + p\mathbb{Z}_{(p)}$.

§ 10 Completions, p-adic numbers and Hensel's Lemma

Remark 10.1 *Let $|\cdot|$ be an absolute value on a field k. Let*

$$\mathcal{C} := \{(a_n)_{n\in\mathbb{N}} \mid (a_n) \text{ is Cauchy sequence in } (k, |\cdot|)\}$$

be th ring (!) of Cauchy sequences in k and

$$\mathcal{N} := \left\{(a_n)_{n\in\mathbb{N}} \mid \lim_{n\to\infty} a_n = 0\right\} \trianglelefteq \mathcal{C}$$

the ideal (!) of Cauchy sequences converging to 0. Then

(i) $\mathcal{N}$ is a maximal ideal.

(ii) $k' := \mathcal{C}/\mathcal{N}$ is a field extension of k.

(iii) $|\overline{(a_n)_{n\in\mathbb{N}}}| := \lim_{n\to\infty}(a_n) \in \mathbb{R}_{\geqslant 0}$ is an absolute value on k' extending $|\cdot|$.

(iv) k' is complete with respect to $|\cdot|$.

Remark 10.2 *If $|\cdot|$ is nonarchimedean, for every Cauchy sequence $(a_n)_{n\in\mathbb{N}} \notin \mathcal{N}$ we have $|a_m| = |a_n|$ for all $m, n \gg 0$.*

proof. Since $(a_n) \notin \mathcal{N}$, 0 is not an accumulation point of (a_n). $\Longrightarrow |a_n| \geqslant \epsilon$ for some $\epsilon > 0$ and all $n \geqslant n_0(\epsilon) =: n_0$. Thus for $n, m \geqslant n_0$ we have $|a_n - a_m| < \epsilon$. This implies by 9.8 (ii)

$$|a_n - a_m| \lneqq \max\{|a_n|, |a_m|\} \implies |a_n| = |a_m|,$$

which was the claim. □

Definition 10.3 Let $k = \mathbb{Q}$, $|\cdot| = |\cdot|_p$ for some $p \in \mathbb{P}$. Then the field k' on 10.1 is called the field of *p-adic numbers* and denoted by $\mathbb{Q}_p$. The valuation ring is called the ring of *p-adic integers* and is denoted by $\mathbb{Z}_p$.

Remark 10.4 *(i) $\mathbb{Z} \subset \mathbb{Z}_{(p)} \subset \mathbb{Z}_p$.*

(ii) The maximal ideal in $\mathbb{Z}_p$ is $p\mathbb{Z}_p$.

(iii) $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$.

(iv) $\mathbb{Z}_p$ is a discrete valuation ring.

proof. (i) The first inclusion is clear. For the second one consider $x = \frac{r}{s} \in \mathbb{Z}_{(p)}$. Then by definition of localization we have $p \nmid s$ and hence

$$|x| = \left|\frac{r}{s}\right| = \frac{|r|}{|s|} = |r| \leqslant 1$$

and thus $x \in \mathbb{Z}_p$. Now prove that $\mathbb{Z}$ is dence in $\mathbb{Z}_p$: Let $x \in \mathbb{Z}_p$ with p-adic expansion

$$x = \sum_{i=0}^{\infty} a_i p^i, \qquad a_i \in \{0, 1, \ldots, p-1\}.$$

Define a sequence $(x_n)_{n\in\mathbb{N}}$ by

$$x_n := \sum_{i=0}^{n} a_i p^i \in \mathbb{Z}.$$

Then we have

$$|x - x_n| = \Big| \sum_{i=n+1}^{\infty} \Big| = \max_{i \geqslant n+1} \{|p^i|\} = \left|p^{n+1}\right| = p^{-(n+1)} \xrightarrow{n\to\infty} 0$$

and hence $\mathbb{Z}$ is dence in $\mathbb{Z}_p$.

(ii) Recall that the maximal ideal is given by

$$\mathfrak{m} = \{x \in \mathbb{Z}_p \mid |x| < 1\} \overset{!}{=} p\mathbb{Z}_p$$

'$\subseteq$' Let $x \in \mathfrak{m}$, i.e. $|x| < 1$. Thus we have $|x| < \left|\frac{1}{p}\right|$. This implies

$$|p^{-1}x| \leqslant 1 \iff p^{-1}x \in \mathbb{Z}_p.$$

and thus $p^{-1}x = y$ for some $y \in \mathbb{Z}_p$. Then we have $x = py \in p\mathbb{Z}_p$.

'$\supseteq$' Let $x \in p\mathbb{Z}_p$, i.e. we can write $x = py$ for some $y \in \mathbb{Z}_p$. Then $|x| = |py| = |p||y| < 1$ and hence $x \in \mathfrak{m}$.

(iii) Consider the surjective homomorphism

$$\psi_p : \mathbb{Z}_p \longrightarrow \mathbb{Z}/p\mathbb{Z}, \quad x = \sum_{i=0}^{n} a_i p^i \mapsto a_0.$$

We have

$$\ker(\psi_p) = \{x \in \mathbb{Z}_p \mid a_0 \equiv 0 \mod p\} = p\mathbb{Z}_p,$$

thus we get $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$ by homomorphism theorem.

(iv) The absolute value $|\cdot| = |\cdot|_p$ on $\mathbb{Q}_p$ induces a discrete valuation ν on $\mathbb{Q}_p^\times$. With respect to this valuation we have

$$\mathcal{O}_\nu = \{x \in \mathbb{Q}_p \mid \nu(x) \geqslant 0\} \cup \{0\} = \{x \in \mathbb{Q}_p \mid |x| \leqslant 1\} = \mathbb{Z}_p,$$

which finishes the proof. □

Proposition 10.5 *(i) Any $x \in \mathbb{Z}_p$ can uniquely be written in the form*

$$x = \sum_{i=0}^{\infty} a_i p^i, \qquad a_i \in \{0, 1, \dots, p-1\}.$$

(ii) Any $x \in \mathbb{Q}_p$ can uniquely be written in the form

$$x = \sum_{i=-m}^{\infty} a_i p^i, \qquad m \in \mathbb{Z},\ a_i \in \{0, 1, \dots, p-1\},\ a_m \neq 0.$$

proof. (i) We first obtain, that any series

$$\sum_{i=0}^{\infty} a_i p^i, \qquad a_i \in \{0, \dots, p-1\}$$

converges, since for $n > m$ we have

$$\left| \sum_{i=0}^{n} a_i p^i - \sum_{i=0}^{m} a_i p^i \right| = \left| \sum_{i=n+1}^{m} a_i p^i \right| = \left|p^{m+1}\right| \underbrace{\left| \sum_{i=n+1}^{m} a_i p^{i-(m+1)} \right|}_{\leqslant 1} \leqslant \left|p^{m+1}\right|.$$

uniqueness Let

$$x = \sum_{i=0}^{\infty} a_i p^i = \sum_{i=0}^{\infty} b_i p^i, \qquad a_i, b_i \in \{0, 1, \dots, p-1\}$$

representations of $x \in \mathbb{Q}_p$. Assume them to be different and define $i_o := \min\{i \in \mathbb{N}_0 \mid a_i \neq b_i\}$. Then

$$0 = \left| \sum_{i=0}^{\infty} a_i p^i - \sum_{i=0}^{\infty} b_i p^i \right| = \left| \underbrace{p^{i_0}(a_{i_0} - b_{i_0})}_{=:A} + p^{i_0+1} \cdot \underbrace{\left(\sum_{i=i_0+1}^{\infty} a_i p^{i-(i_0+1)} - \sum_{i=i_0+1}^{\infty} b_i p^{i-(i_0+1)} \right)}_{=:B} \right|.$$

We obtain $\nu_p(A) = p^{-i_0}$ and

$$B \in \mathbb{Z}_p, \quad \nu_p\left(p^{i_0+1} \cdot B\right) = \nu_p\left(p^{i_0+1}\right) \underbrace{\nu_p(B)}_{\leqslant 1} \leqslant \nu_p\left(p^{i_0+1}\right) = p^{-(i_0+1)},$$

so all in all

$$0 = \left|A + p^{i_0+1} \cdot B\right| \overset{9.8(ii)}{=} \max\{p^{-i_0}, p^{-(i_0+1)}\} = p^{-i_0} \ \lightning.$$

existence Look at $\overline{x} \in \mathbb{Z}_p / p\mathbb{Z}_p = \mathbb{F}_p$.

Let a_0 be the representative of x in $\{0, 1, \dots, p-1\}$. Then we have

$$|x - a_0| < 1 \ \Leftrightarrow \ |x - a_0| \leqslant \frac{1}{p}.$$

In the next step, let a_1 be the representative of $\frac{1}{p}(x-a_0)$ in $\{0,1,\dots,p-1\}$. Then

$$\left|\frac{1}{p}(x-a_0)-a_1\right| = \left|\frac{1}{p}\right| |x-a_0-a_1p| \leqslant \frac{1}{p}$$

and thus $|x-a_0-a_1p| \leqslant \frac{1}{p^2}$. Inductively we let a_n be the representative of

$$\frac{1}{p^n}(x-a_0-a_1p-\ldots-a_{n-1}p^{n-1}) = \frac{1}{p^n}\left(x-\sum_{i=0}^{n-1} a_ip^i\right)$$

in $\{0,1,\dots,p-1\}$. Then we have

$$\left|x-\sum_{i=0}^{n-1} a_ip^i\right| \leqslant \frac{1}{p^{n+1}}.$$

and finally

$$\lim_{n\to\infty}\left|x-\sum_{i=0}^{n-1} a_ip^i\right| \leqslant \lim_{n\to\infty}\frac{1}{p^{n+1}} = 0 \implies x = \sum_{i=0}^{\infty} a_ip^i.$$

(ii) If $|x| = p^m$ for some $m \in \mathbb{Z}$, we have

$$|x\cdot p^m| = |d|\cdot|p^m| = p^m\cdot p^{-m} = 1, \qquad \text{i.e. } x\cdot p^m \in \mathbb{Z}_p^\times$$

By part (i) we conclude

$$x\cdot p^m = \sum_{i=0}^{\infty} a_ip^i, \quad a_0 \neq 0.$$

Thus we have

$$x = \frac{1}{p^m}\cdot x\cdot p^m = \frac{1}{p^m}\cdot\sum_{i=0}^{\infty} a_ip^i = \sum_{i=-m}^{\infty} a_{i+m}p^i,$$

which was the assertion. □

Remark 10.6 *What is -1 in $\mathbb{Q}_p$? We have $a_0 = p-1$, since $\overline{p-1}-\overline{(-a)} = \overline{p} = 0$. a_1 is the representative of $\frac{1}{p}(-1-(p-1)) = -1$, i.e. $a_1 = p-1$. a_2 is the representative of $\frac{1}{p^2}(-1-(p-1)-(p-1)p) = -1$, i.e. $a_2 = p-1$. Inductively we have $a_n = p-1$ for all $n \in \mathbb{N}_0$, so we get*

$$-1 = \sum_{i=0}^{\infty} a_ip^i = \sum_{i=0}^{\infty}(p-1)p^i.$$

Moreover we obtain

$$\sum_{i=0}^{\infty}(p-1)p^i = (p-1)\sum_{i=0}^{\infty} p^i = (p-1)\cdot\frac{1}{1-p} = \frac{p-1}{1-p} = -1.$$

Remark 10.7 *Let*

$$x = \sum_{i=0}^{\infty} a_i p^i, \qquad y = \sum_{i=0}^{\infty} b_i p^i$$

p-adic integers. Then

$$x + y = \sum_{i=0}^{\infty} c_i p^i$$

with coefficients

$$c_0 = \begin{cases} a_0 + b_0 & \text{if } a_0 + b_0 < p \\ a_0 + b_0 - p & \text{if } a_0 + b_0 \geqslant p \end{cases}$$

$$c_1 = \begin{cases} a_1 + b_1 & \text{if } a_0 + b_0 < p \text{ and } a_1 + b_1 < p \\ a_1 + b_1 - p & \text{if } a_0 + b_0 < p \text{ and } a_1 + b_1 \geqslant p \\ a_1 + b_1 + 1 & \text{if } a_0 + b_0 \geqslant p \text{ and } a_1 + b_1 + 1 < p \\ a_1 + b_1 + 1 - p & \text{if } a_0 + b_0 \geqslant p \text{ and } a_1 + b_1 + 1 \geqslant p \end{cases}$$

Inductively let

$$\epsilon_0 := 0, \qquad \epsilon_i := \begin{cases} 0 & \text{if } a_i + b_i + \epsilon_{i-1} < p \\ 1 & \text{if } a_i + b_i + \epsilon_{i-1} \geqslant p \end{cases} \qquad \text{for } i \geqslant 1$$

Then we have

$$c_i = \begin{cases} a_i + b_i + \epsilon_i & \text{if } a_i + b_i + \epsilon_i < p \\ a_i + b_i + \epsilon_i - p & \text{if } a_i + b_i + \epsilon_i \geqslant p \end{cases}$$

Remark 10.8 *(i)* $\sqrt{p} \notin \mathbb{Q}_p$, *since* $|\sqrt{p}| = \sqrt{|p|} = \sqrt{\frac{1}{p}} \in \left(\frac{1}{p}, 1\right)$, *which is not possible.*

(ii) Let $a \in \mathbb{Z}_p^\times$ *with image* $\overline{a} \in \mathbb{F}_p^\times \backslash \mathbb{F}_p^{\times 2}$, *where*

$$\mathbb{F}_p^{\times 2} = \{x \in \mathbb{F}_p \mid \text{ there exists } y \in \mathbb{F}_p : y^2 = x\}$$

denotes the set of squares. Then $\sqrt{a} \notin \mathbb{Q}_p$. *Assume* a *is a aquare, i.e.* $b^2 = a$. *Then*

$$|b| = \sqrt{|a|} = 1 \quad \Rightarrow \quad b \in \mathbb{Z}_p^\times$$

But then $\overline{b} \in \mathbb{F}_p$ *satisfies* $\overline{b}^2 \equiv a$, *which is a contradiction, since* $a \notin \mathbb{F}_p^{\times 2}$.

(iii) Let now $\overline{\mathbb{Q}}_p$ *be the algebraic closure of* $\mathbb{Q}_p$ *with valuation ring* $\overline{\mathbb{Z}}_p$ *and maximal ideal* $\overline{\mathfrak{m}}_p$. *Then* $\overline{\mathbb{Z}}_p/\overline{\mathfrak{m}}$ *is algebraically closed. Moreover* $\mathbb{Q}_p$ *is complete with respect to* $|\cdot|_p$. *The completion* $\mathbb{C}_p$ *of* $\overline{\mathbb{Q}}_p$ *is complete and algebraically closed, but:*

(1) $|\cdot|_p$ *is not a discrete valuation.*

(2) $\overline{\mathbb{Z}}_p$ *is not a discrete valuation ring.*

(3) $\overline{\mathfrak{m}}_p$ *is not a principal ideal.*

Theorem 10.9 (*Hensel's Lemma*) *Let*

$$f = \sum_{i=0}^{n} a_i X^i \in \mathbb{Z}_p[X], \qquad \overline{f} = \sum_{i=0}^{n} \overline{a_i} X^i \in \mathbb{F}[X]$$

where $\overline{f}$ is the reduction of f in $\mathbb{F}[X]$. Suppose that $\overline{f} = f_1 \cdot f_2$ with $f_1, f_2 \in \mathbb{F}_p[X]$ relatively prime. Then there exist $g, h \in \mathbb{Z}_p[X]$, such that

$$f = g \cdot h, \quad \overline{g} = f_1, \overline{h} = f_2, \quad \deg(f_1) = \deg(g)$$

proof. Let $d := \deg(f), m := \deg(f_1)$. Then $\deg(f_2) \leqslant d - m$. Choose $g_0, h_0 \in \mathbb{Z}_p[X]$ such that $\overline{g_0} = f_1, \overline{h_0} = f_2, \deg(g_0) = m, \deg(h_0) = d - m$. *Strategy:* Find $g_1 = g_0 + pc_1$, $h_1 = h_0 + pd_1$ with some $c_1, d_1 \in \mathbb{Z}_p[X]$, such that

$$f - g_1 h_1 \quad \in p^2 \mathbb{Z}_p[X].$$

Therefore we have a

Claim (a) For $n \geqslant 1$ there exists $c_n, d_n \in \mathbb{Z}_p[X]$ with $\deg(c_n) \leqslant m, \deg(d_n) \leqslant d - m$ and

$$f - g_n h_n \in p^{n+1} \mathbb{Z}_p[X], \qquad \text{where } g_n = g_{n-1} + p^n c_n, \quad h_n = h_{n-1} + p^n d_n.$$

Assuming (a), write

$$g_n = \sum_{i=0}^{m} g_{n,i} X^i, \qquad h_n = \sum_{i=0}^{d-m} h_{n,i} X^i.$$

By construction, the $(g_{n,i})$ converge to some $\alpha_i \in \mathbb{Z}_p$ and the $(h_{n,i})$ converge to some $\beta_i \in \mathbb{Z}_p$. Let

$$g := \sum_{i=0}^{m} \alpha_i X^i, \qquad h := \sum_{i=0}^{d-m} \beta_i X^i.$$

Observe, that $\deg(g) = m, \deg(h) = d - m$. Obviously we have

$$f = g \cdot h.$$

It remains to show the claim.

(a) c_n, d_n have to satisfy

$$\begin{aligned} f - g_n h_n &= f - (g_{n-1} + p^n c_n) \cdot (h_{n-1} + p^n d_n) \\ &= f - g_{n-1} h_{n-1} - p^n \cdot (g_{n-1} d_n + h_{n-1} c_n + p^n c_n d_n) \\ &\overset{!}{\in} p^{n+1} \mathbb{Z}_p[X] \end{aligned}$$

where $f - g_{n-1}h_{n-1} \in p^n\mathbb{Z}_p[X]$ by hypothesis. We get

$$\tilde{f}_n := \frac{1}{p^n}(f - g_{n-1}h_{n-1}) \equiv c_nh_{n-1} + d_ng_{n-1} \mod p \ (*)$$

Since f_1, f_2 are relatively prime and $g_j \equiv g_k \mod p$ for any j, k, we find integers $a, b \in \mathbb{Z}$, such that

$$af_1, bf_2 = 1 \implies ag_{n-1} + bh_{n-1} \equiv 1 \mod p.$$

Multiplying the equation by $\tilde{f}_n$ gives us

$$\tilde{f}_n \equiv \underbrace{a\tilde{f}_n}_{=:\tilde{d}_n} g_{n-1} + \underbrace{b\tilde{f}_n}_{=:\tilde{c}_n} h_{n-1} \mod p \ (**).$$

Further $\mathbb{Z}_p[X]$ is euclidean, thus we can choose $q_n, r_n \in \mathbb{Z}_p[X]$, $\deg(r_n) < m$ such that

$$b\tilde{f}_n = q_ng_{n-1} + r_n.$$

By $(**)$ we have

$$g_{n-1}\left(a\tilde{f}_n + q_nh_{n-1}\right) + r_n \equiv \tilde{f}_n \mod p.$$

Let now $c_n = r_n, d_n = a\tilde{f}_n + q_nh_{n-1}$. All the terms are divisible by p. Then

$$d_n \equiv a\tilde{f}_n + q_nh_{n-1} \mod p.$$

Thus $(*)$ holds and we have

$$\deg(d_n) = \deg(\overline{d_n}) \leqslant \deg\left(\underbrace{\overbrace{\overline{\tilde{f}}_n}^{\leqslant d} - \overbrace{\overline{c}_n}^{<m}\overbrace{\overline{h}_{n-1}}^{<d-m}}_{\leqslant d}\right) - \underbrace{\deg(\overline{g}_{n-1})}_{=m} \leqslant d - m$$

since $\overline{d}_n\overline{g}_{n-1} = \overline{\tilde{f}}_n - \overline{c}_n\overline{h}_{n-1}$. Thus, the claim is proved. □

Corollary 10.10 *Let $p \in \mathbb{P}$ odd. Then $a \in \mathbb{Z}_p^\times$ is a square if and only if $\overline{a} \in \mathbb{F}_p^\times$ is a square.*

Proposition 10.11 *$a \in \mathbb{Q}$ is a square if and only if $a > 0$ and a is a square in $\mathbb{Q}_p$ for all $p \in \mathbb{P}$.*
Remark: This is a special case of the 'Hasse-Minkowski-Theorem'.