Derivaition: Panofsky-Wenzel

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DERIVATION → PANOFSKY WENZEL

① $\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$ ② $\vec{B} = \nabla \times \vec{A}$

grad	$\nabla\phi$
div	$\nabla\cdot\bar{a}$
curl	$\nabla\times\bar{a}$

③ $\nabla(\bar{a}\cdot\bar{b}) = \bar{a}\times(\nabla\times\bar{b}) + \bar{b}\times(\nabla\times\bar{a}) + (\bar{a}\cdot\nabla)\bar{b} + (\bar{b}\cdot\nabla)\bar{a}$

↳ Vector definition

1) $\bar{v}\times\bar{B}$ where $\bar{v}$ is only considered in the direction of particle propagation

i.e. $\bar{v} = \begin{pmatrix}0\\0\\v_0\end{pmatrix} = v_0\hat{z}$

from ② $\bar{B} = \nabla\times\bar{A}$

$\therefore \bar{v}\times\bar{B} = \bar{v}\times(\nabla\times\bar{A})$

rearrange ③ for the first term on the RHS

$\bar{a}\times(\nabla\times\bar{b}) = \nabla(\bar{a}\cdot\bar{b}) - \bar{b}\times(\nabla\times\bar{a}) - (\bar{a}\cdot\nabla)\bar{b} - (\bar{b}\cdot\nabla)\bar{a}$

$\therefore \bar{v}\times(\nabla\times\bar{A}) = \nabla(\bar{v}\cdot\bar{A}) - \bar{A}\times(\nabla\times\bar{v}) - (\bar{v}\cdot\nabla)\bar{A} - (\bar{A}\cdot\nabla)\bar{v}$

$= v_0\left[\nabla(\hat{z}\cdot\bar{A}) - \cancel{\bar{A}\times(\nabla\times\hat{z})}^{0} - (\hat{z}\cdot\nabla)\bar{A} - \cancel{(\bar{A}\cdot\nabla)\hat{z}}^{0}\right]$

as
$\nabla\times\hat{z} = \begin{pmatrix}\partial x\\\partial y\\\partial z\end{pmatrix}\times\begin{pmatrix}0\\0\\1\end{pmatrix}$

$= \begin{vmatrix} i & \partial x & 0\\ j & \partial y & 0\\ k & \partial z & 1\end{vmatrix} = \hat{i}(\partial y) - \partial x() + 0 = 0$

$= v_0\left[\nabla(\hat{z}\cdot\bar{A}) - (\hat{z}\cdot\nabla)\bar{A}\right]$

$\hat{z}\cdot\bar{A} = A_z$ $\hat{z}\cdot\nabla = \partial z$

$\partial_z\bar{A} = \frac{\partial\bar{A}}{\partial z}$

$\therefore \bar{v}\times(\nabla\times\bar{A}) = v_0\left[\nabla A_z - \frac{\partial\bar{A}}{\partial z}\right]$

$$\bar{v} \times \bar{B} \Rightarrow$$

$$\bar{v} \times (\nabla \times \bar{A}) = v_0 \left[\nabla A_z - \frac{\partial \bar{A}}{\partial z} \right]$$

2) Looking at the Lorentz Force: WE HAVE $\vec{E}$ ① WE SOLVED $\bar{v} \times \bar{B}$

$$F = q(\vec{E} + \bar{v} \times \bar{B})$$

$$= q\left(-\nabla\phi - \frac{\partial \bar{A}}{\partial t} + v_0 \left[\nabla A_z - \frac{\partial \bar{A}}{\partial z}\right]\right)$$

$$= q\left(-\nabla\phi + v_0 \nabla A_z - \frac{\partial \bar{A}}{\partial t} - v_0 \frac{\partial \bar{A}}{\partial z}\right)$$

... since the particle is only moving in z and in t

$$\frac{dA}{dt} = \frac{\partial \bar{A}}{\partial z}\underbrace{\frac{\delta z}{dt}}_{v_0} + \frac{\partial \bar{A}}{\partial t}\frac{\delta t}{dt}$$

$$F = q\left(-\nabla\phi + v_0 \nabla A_z - \frac{dA}{dt}\right)$$

④ $\bar{v} = \dfrac{\int F \cdot dz}{q}$

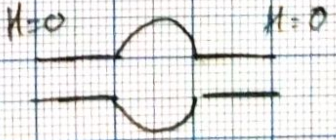
$$\bar{v} = \int -\nabla\phi + v_0 \nabla A_z - \frac{dA}{dt} \cdot dz$$

$$\downarrow$$

$$-\frac{dA}{dz} \cdot \frac{dz}{dt}$$

$$-\frac{dA}{dz} v_0$$

$$\therefore \bar{v} = \int -\nabla\phi + v_0 \nabla A_z - v_0 \frac{dA}{dr} \cdot dz \longrightarrow \int_{z_{min}}^{z_{max}} \frac{dA}{dz} \cdot dz = \Big[A_{(z)}\Big]_{z_{min}}^{z_{max}} = 0$$



$$\bar{v} = \int -\nabla\phi + v_0 \nabla A_z \cdot dz$$

$$= \nabla \int -\phi + v_0 A_z$$

⑤ Vec Der $\nabla \times \nabla \bar{c} = 0$

$$\nabla \times \bar{v} = 0$$

$\nabla \times \bar{v} = 0$

$\therefore \begin{vmatrix} i & \partial x & V_x \\ j & \partial y & V_y \\ k & \partial z & V_z \end{vmatrix} = 0 = \begin{pmatrix} \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \\ -\frac{\partial V_z}{\partial x} + \frac{\partial V_x}{\partial z} \\ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \end{pmatrix}$

... looking at the relationships between transverse and longitudinal:

$$\left.\begin{aligned} \partial y\, V_z &= \partial z\, V_y \\ \partial x\, V_z &= \partial z\, V_x \end{aligned}\right\} \boxed{\nabla_\perp V_z = \partial z\, \overline{V_\perp}}$$

... leading to:

$$\nabla_\perp V_z = \frac{\partial t}{\partial z} \frac{\partial}{\partial t} \overline{V_\perp}$$

$$= \frac{1}{c} \frac{\partial}{\partial t} \overline{V_\perp} \longrightarrow \text{as } \bar{A} = |A|\, e^{j\omega t}$$

$$\text{so } \frac{\partial}{\partial t} \bar{A} = j\omega \bar{A}$$

$$= j\omega \frac{1}{c} \overline{V_\perp}$$

$$\overline{V_\perp} = \nabla_\perp V_z \frac{c}{j\omega}$$

$$\boxed{\overline{V_\perp} = -\nabla_\perp V_z \frac{c}{j\omega}} \qquad \text{as } V_z = \int E_z \cdot dz$$

$$\boxed{\overline{V_\perp} = -\frac{jc}{\omega} \int \nabla_\perp E_z \cdot dz}$$