Notes on smoothing bathymetries using Shapiro filters

Michael J. Bell and David Storkey

Met Office, Fitzroy Rd, Exeter, UK

Abstract

The options for using Shapiro filters to smooth bathymetries are summarised. A technique for ensuring that the smoothed bathymetric depth is greater than a prescribed minimum is described. This technique could also be used to control the depths of sills and channels.

1. Introduction

These are informal notes on the form and properties of the Shapiro filter used to smooth the global bathymetries and the successive correction method used to ensure that the coastline is unchanged by the smoothing operator. The description of the Shapiro filter draws on Francis (1975) [1]. The form of the Shapiro described here and the successive correction method have been coded in a modified version of the *cdfsmooth* module in CDFTools.

2. Notes on Shapiro filters

A 1D Shapiro filter uses repeated applications of the operator

$$T\phi_i \equiv \tfrac{1}{4}(-\phi_{i+1} + 2\phi_i - \phi_{i-1}). \tag{1}$$

The filtering properties of T are easily found. If

$$\phi_i = \cos(mi\Delta x) \tag{2}$$

then

$$T\phi_i = D_m\phi_i \quad ; \quad D_m = \sin^2(\frac{m\Delta x}{2}). \tag{3}$$

For 2 grid-length waves, $D_m = 1$ and for very long waves $D_m \downarrow 0$.

We can write

$$\phi = (1-T)(1-T)^{-1}\phi = (1-T)(1+T+T^2+T^3+...)\phi, \qquad (4)$$

and expect the series to "converge" if $|T| < 1$. It is important to note that the operator T in (1) does indeed satisfy $|T| < 1$ and that the first term $(1-T)$ on the rhs of (4) will eliminate any amplitude in the 2 grid-length wave. So the series converges to a field which has no 2 grid-length wave.

Direct calculation of terms shows that the filter obtained by truncating the series after k terms,

$$F^{(k)} \equiv (1-T)(1+T+T^2+...+T^k) = 1 - T^{k+1}, k \geq 0. \qquad (5)$$

Using (3) in (5), the spectral response of the filter, $D_m^{(k)}$ is given by

$$D_m^{(k)} = 1 - \sin^{2(k+1)}(\frac{m\Delta x}{2}), k > 0. \qquad (6)$$

The $k+1$ order Shapiro filter is obtained by performing $k+1$ passes of the operator T and subtracting the result from the original field.

Francis(1975) shows that the filter properties $D_n^{(k)}$ of this filter become very sharp. For k+1 = 8, even the 3 grid-length wave is only reduced in amplitude by 10% and the 4 grid-length wave is reduced by less than 1%.

The filter with $k = 0$ is $F^{(0)} = 1 - T$. This reduces the amplitude of the 2 grid-length wave to zero, but reduces the amplitude of the 3 and 4 grid-length waves by factors of 75% and 25% respectively. Its one advantage is that it does not increase the range of the field. This means that bathymetries calculated using it will have positive depths throughout the domain.

There are two options for applying the Shapiro filter to a 2D field $\phi(i,j)$. One could filter in the i dimension then filter in the j dimension. Or one could use a four-point Laplacian for the filtering. Let us denote 1D filtering in the i direction by U and 1D filtering in the j direction by V. Then the first 2D filter

$$T_L \equiv VU. \qquad (7)$$

For the Laplacian filter it is essential to ensure that $|T| < 1$ so we take

$$T_P \equiv \tfrac{1}{2}(U + V). \tag{8}$$

The L subscript stands for "lines" as in the method of lines and the P stands for the "P" in Laplacian. The spectral filtering by T_L is the product of the 1D spectra whereas the spectral filtering by the Laplacian is $(m^2 + n^2)^k$ where n is the wave-number on the second axis. This filtering is much less focused on the diagonal in spectral space and is much the preferable of the two. Both T_L and T_P have been coded up but T_L should not normally be used.

3. Notes on constraining the smoothed fields

The impact of the filter on a field that is equal to zero except at a single point is easily calculated. The filter smooths out this delta function, but only locally. The response function has weak negative side lobes and significant amplitude at the central point. Its details depend of course on the order of the filter. The filter is linear in the field, so its response to changes in the input field are easily predicted.

If the smoothed field is negative at some point, one can calculate a corrected field to use for input to the filter. The corrected field is the original field except at points where the smoothed field is negative. At these points positive corrections are added to the original field that are somewhat larger than the negative value in the smoothed field at that point. By this means it is relatively easy to generate output fields from the filter that are positive at all sea points and close to the original fields in a root mean square difference sense.

References

[1] Francis, P. E.: The use of a multipoint filter as a dissipative mechanism in a numerical model of the general circulation of the atmosphere, Quarterly Journal of the Royal Meteorological Society, 101, 567–582, https://doi.org/https://doi.org/10.1002/qj.49710142913, 1975.