Computanional Physics - Problem Sheet 3

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1 Stability of the Modified Euler Method

The modified Euler method has been defined in the lecture (Mazzarello CP lecture 3, slide 9) as

$$x(t_{n+1}) = x(t_n) + \tau f\left(x(t_n) + \frac{\tau}{2} f(x(t_n), t_n), t + \frac{\tau}{2}\right) \tag{1}$$

$$\dot{x}(t) = f(x, t) \tag{2}$$

The stability is analyzed for the differential equation

$$f(x, t) = \lambda x(t) \tag{3}$$

with initial value $x(t_0)$ where $\lambda \in \mathbb{C}$.

With the defines function $f(x, t)$, Eq. (1) reads

$$x(t_{n+1}) = x(t_n) + \tau\lambda\left(x(t_n) + \frac{\tau}{2}\lambda x(t_n)\right) \tag{4}$$

$$= \left(1 + \tau\lambda + \frac{(\tau\lambda)^2}{2}\right) x(t_n) \tag{5}$$

$$\Rightarrow \qquad x(t_n) = \left(1 + \tau\lambda + \frac{(\tau\lambda)^2}{2}\right)^n x(t_0) \tag{6}$$

Thus, the method is absolutely stable as long as

$$\left|1 + \tau\lambda + \frac{(\tau\lambda)^2}{2}\right| < 1 \tag{7}$$

Fig. 1 shows the region of absolute stability as a function of $\tau\lambda \in \mathbb{C}$. The figure was created with the "implicitplot" function of the software "Maple" (see Maple file "cp_ex3_p1.mw"). Compared to the regular Euler method where the region of stability is described by a circle with radius 1 centered at $\tau\lambda = -1$, the region here appears stretched into the direction of the imaginary axis. Hence, the region of stability is larger in latter case.

Restricting $0 > \lambda \in \mathbb{R}$ yields stability for

$$\tau < \frac{2}{|\lambda|} \tag{8}$$

as can be seen in the figure when considering the bounding values of $\tau\lambda$. This is the same criterion as for the regular Euler method with the same restriction to λ. As a result, the modified Euler method does not lead to an in improvement in terms of stability compared to the regular one.

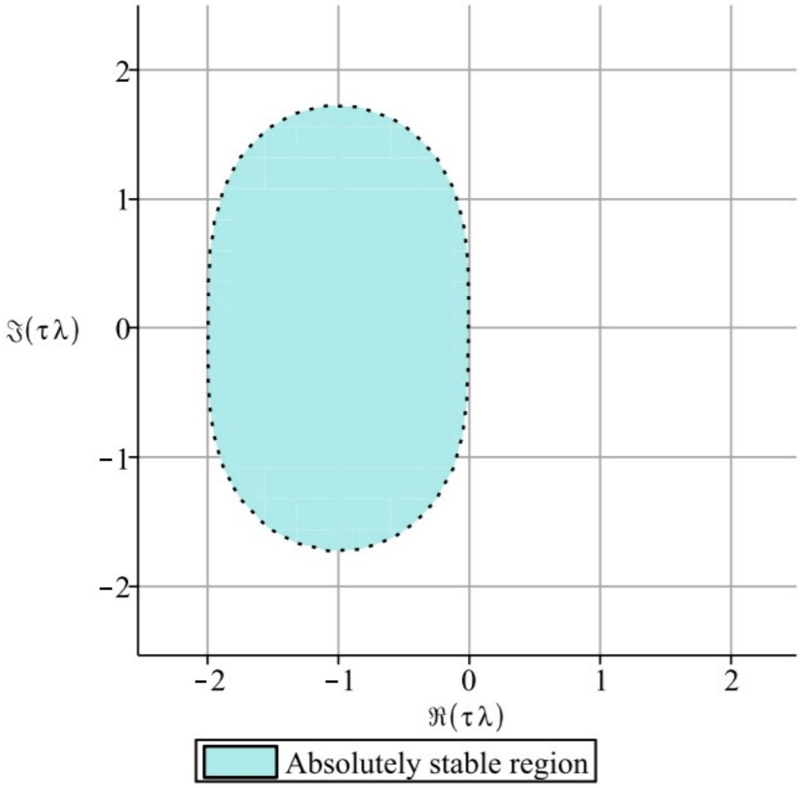

Figure 1: Region of stability for modified Euler method (Eq. (1)) and differential equation (3). The border (dotted line) is not part of the stable region.

2 Driven Pendulum

The code for this problem can be found in the file "cp_ex3_2-3.py" (ll. 1-113). Also provided is the file "newton.py" which contains a skeleton class ("newton_1D_sket") that was implemented for exercise 2 of this course. The class is inherited by classes that implement particular methods, e.g., the 4th-order Runge-Kutta in this case, for the solution Newtons equation in 1D. The daughter classes then, have to specify a function executing one step of their method only as well as additional parameters in their initializers.

The calculations were executed using the given parameters. The number of steps was set to $N = 2^{16}$ with problem 3 of this exercise in mind. For each Q, the $r(t)$ for $t \in [0, t_{8000}]$ as well as the phase space diagram for $t \in [t_{2000}, t_{8000}]$ are plotted. The confinement in latter case is to remove the points that emerge from the initial transient oscillation (see below).

Results for different Q

(i) $Q = 0.5$: The results of the test values at $t_{20} \approx 0.942$ were $r(t_{20}) \approx 0.737$ and $v(t_{20}) = -0.450$.

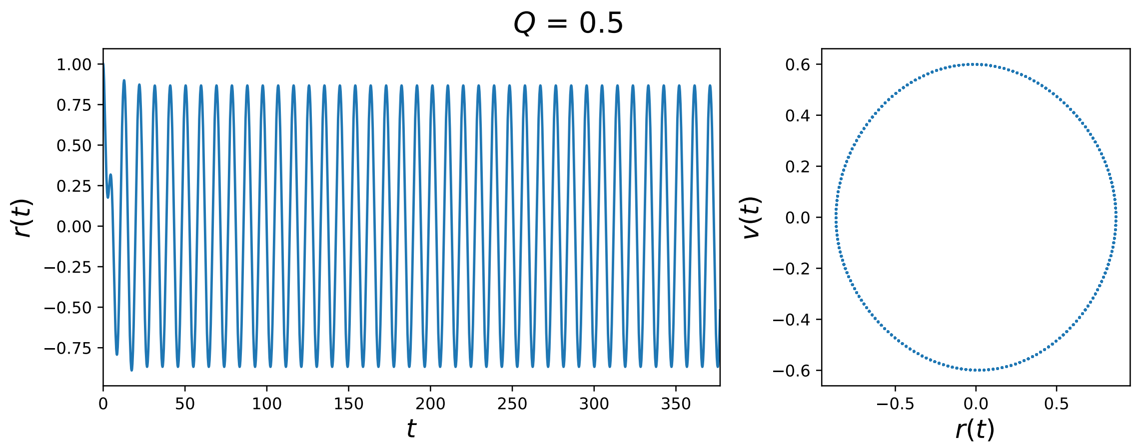

Figure 2: Calculated trajectory (left) and phase space diagram (right) for $Q = 0.5$.

(ii) $Q = 0.9$:

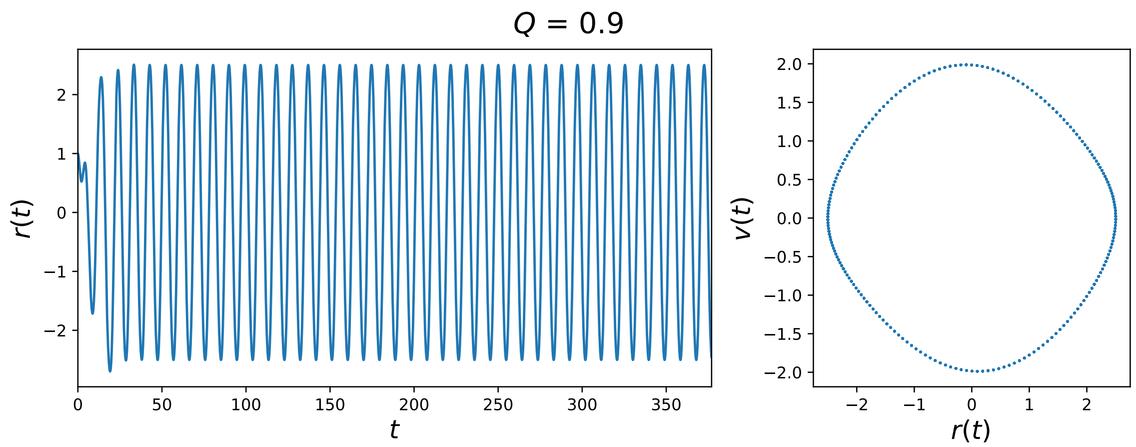

Figure 3: Calculated trajectory (left) and phase space diagram (right) for $Q = 0.9$.

(iii) $Q = 1.2$:

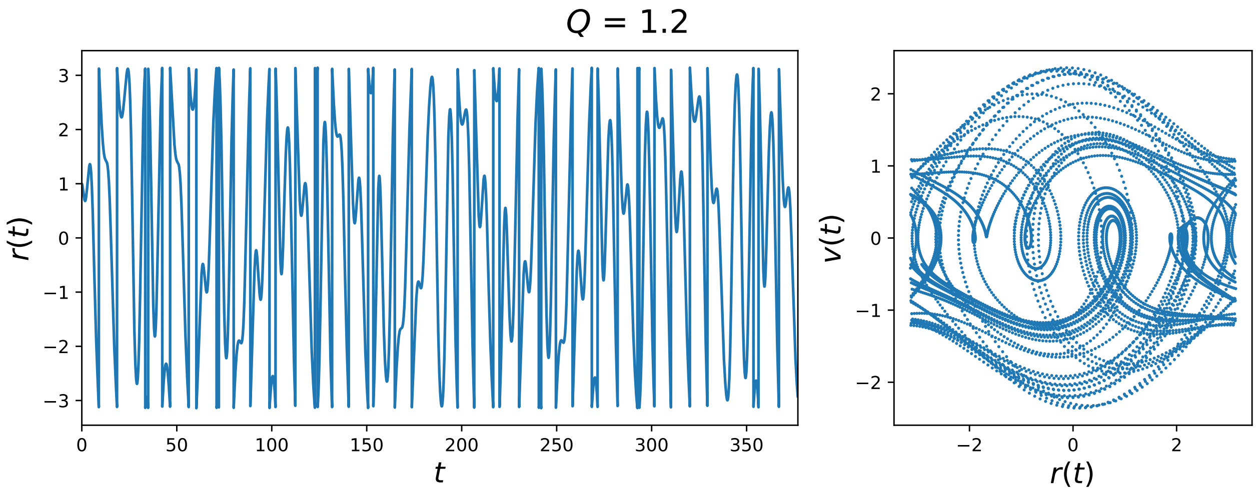

Figure 4: Calculated trajectory (left) and phase space diagram (right) for $Q = 1.2$.

Discussion In case (i), the motion turns into a periodic motion with frequency Ω after an initial transient period. The motion is to a high degree sinusoidal as can be seen from the almost elliptical phase space trajectory. The amplitude is by more than $\pi/2$ away from the domain boundaries for $r(t)$.

A periodic behaviour after an initial transient period is seen in case (ii) as well. The phase space trajectory reveals that the motion here is not sinusoidal since it deviates from an ellipsis.

Case (iii) shows chaotic behaviour. A periodicity is not recognizable anymore and the phase space trajectory does not converge to a certain orbit. Also, the domain boundary of $r(t)$ is reached multiple times as can be seen in the sudden jumps from $-\pi$ to π (and vice versa) that occur in the trajectory. These jumps are result of wrapping the trajectory into $[-\pi, \pi)$.

3 Power Spectrum of the Driven Pendulum

The code for this exercise can be found in the file "cp_ex3_p2-3.py" (ll. 114-140). The fast Fourier transformation is calculated for the different cases in problem 2 without wrapping the trajectories. The periodogram is calculated subsequently.

The power spectra approximated by periodograms for the three cases can be seen in Fig. 5 to 7. Also shown is the spectrum for case (iii) of problem 2 after wrapping the trajectory into the interval $[\pi, \pi)$ The parameters were kept at the same values as in problem 2.

In Fig. 5 and 6 a prominent peak at the driving force's frequency is present. Also visible are peaks at odd multiples of this frequency. The amplitudes of these peaks are more than 4 orders of magnitude smaller for the $Q = 0.5$ case which explains why the motion appears to be sinusoidal-like. The valley at $\nu/\nu_0 \approx 1.3$ does not seem to have a physical origin since it disappears when the the number of steps is doubled. In the $Q = 0.9$ case, the amplitudes of the first peak and of the peak at 3 and 5 times the driving frequency differ by only approximately 2 and 4 orders of magnitude. This means that higher harmonics are added to the motion resulting in a non-sinusoidal trajectory as was observed in the phase space diagram before. The regions without sharp peaks accumulate some disturbances when compared with the regions in Fig. 5. These disturbances also varied, when the number of steps was increased to 2^{17}. An explanation for this behaviour could not be found.

The spectrum in Fig. 7 for the $Q = 1.2$ case shows an irregular pattern which matches the expectation for a chaotic motion with no prominently visible periodicity. The increase occurring for $\nu/\nu_0 \to 0$ most likely stems from the fact that the trajectory leaves the interval $[-\pi, \pi)$ since it disappears when the wrapped trajectory is used as can be seen in Fig. 8. Here, the signals at one and three times the driving force's frequency are still visible suggesting that the periodicity is persistent to some extend.

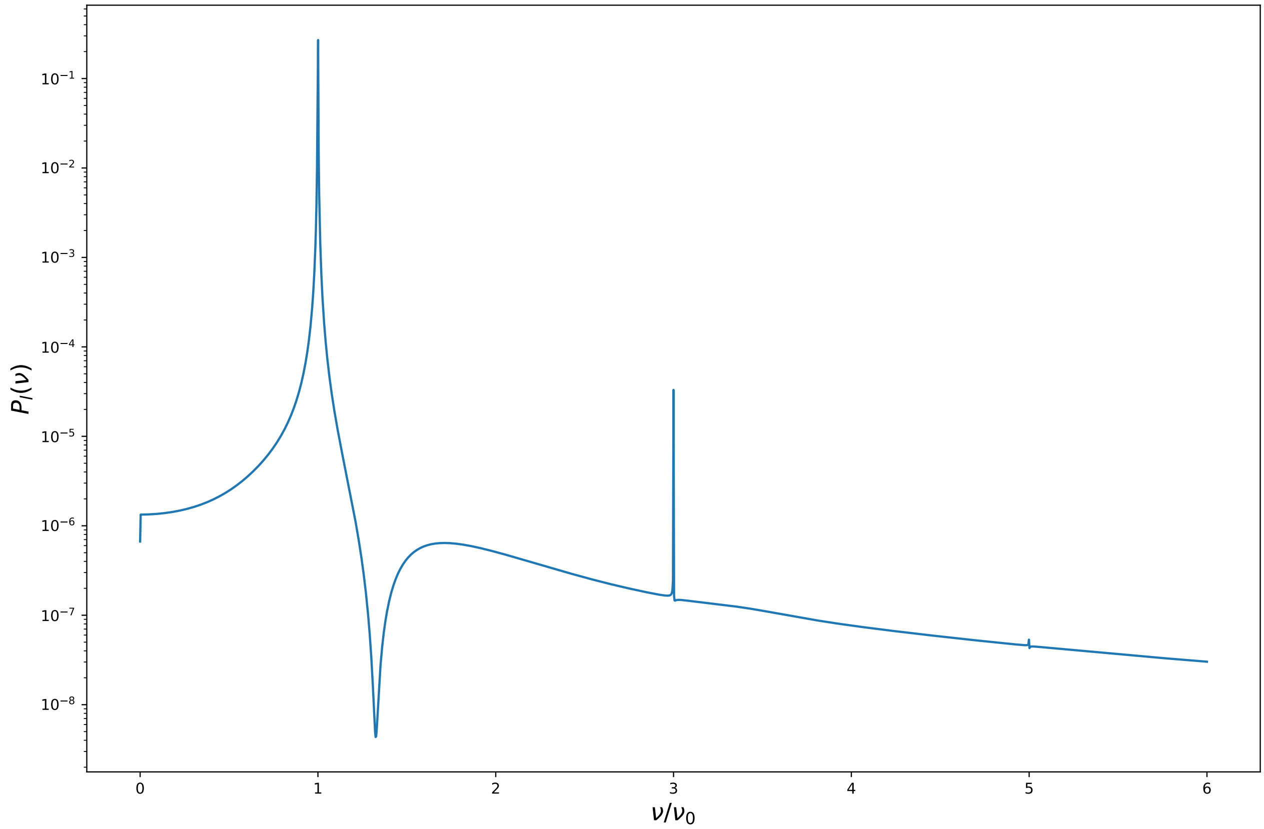

Figure 5: Periodogram of the driven pendulum from problem 2 for $Q = 0.5$.

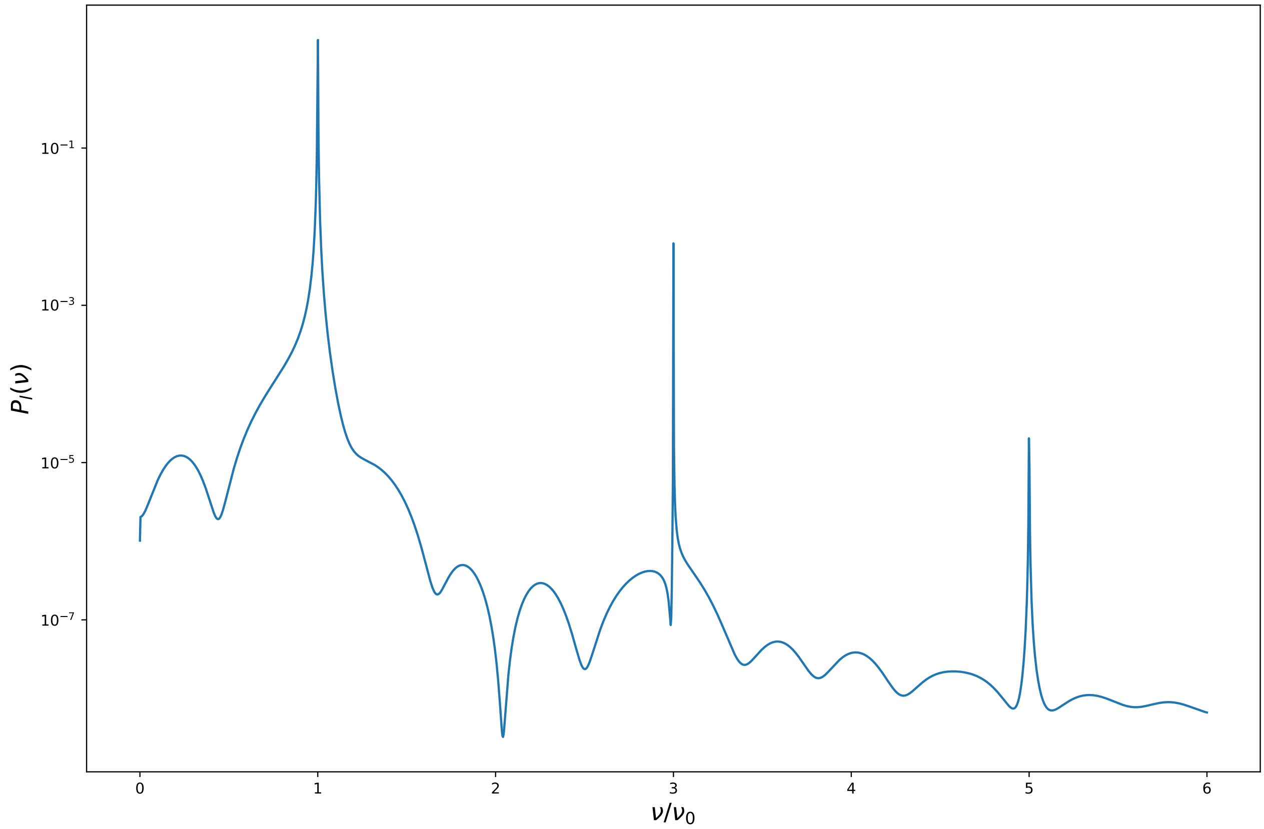

Figure 6: Periodogram of the driven pendulum from problem 2 for $Q = 0.9$.

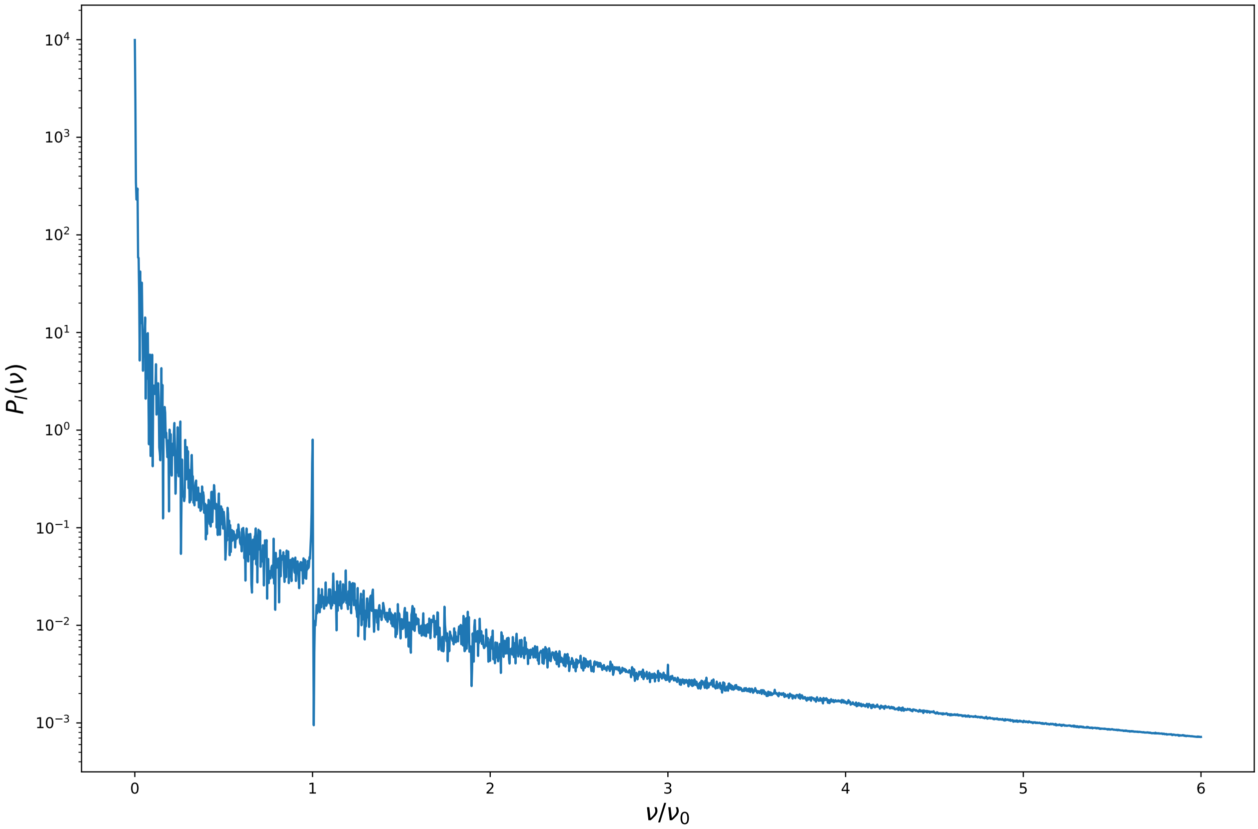

Figure 7: Periodogram of the driven pendulum from problem 2 for $Q = 1.2$.

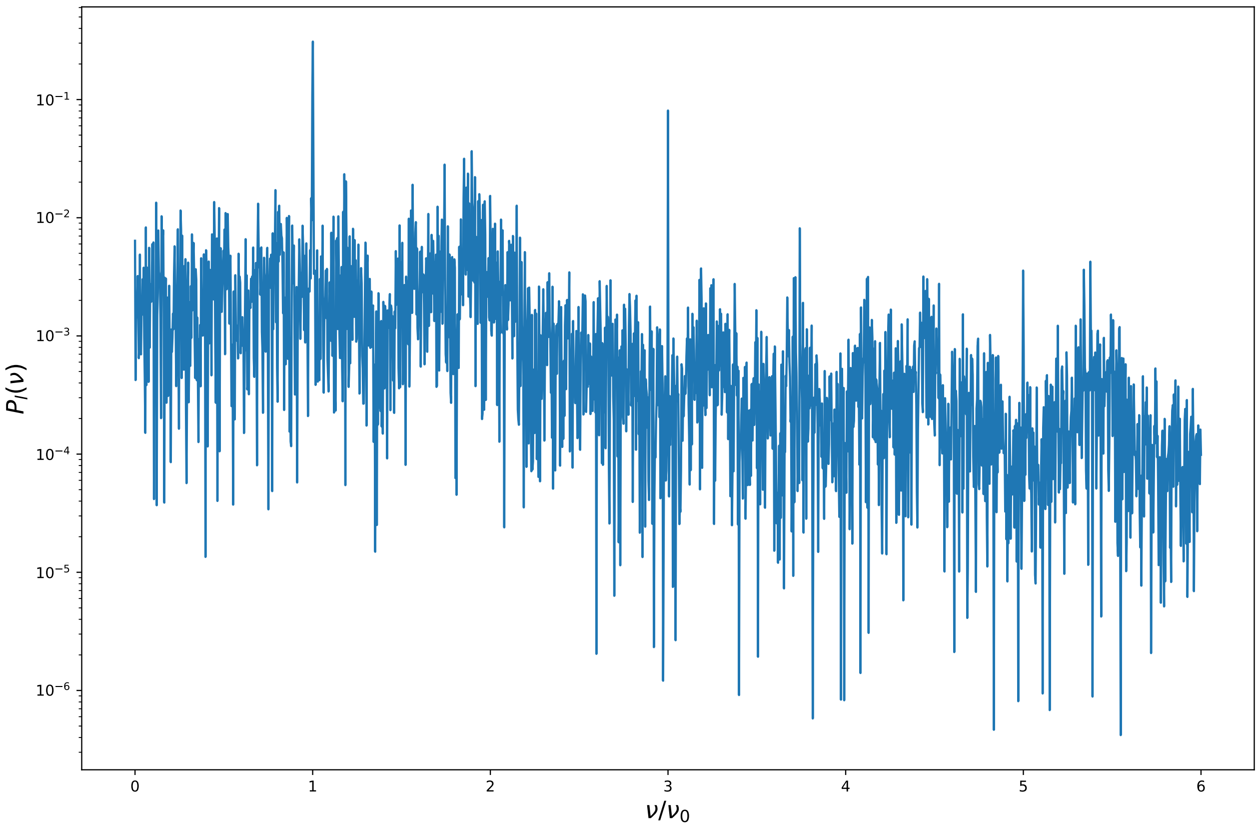

Figure 8: Periodogram of the driven pendulum from problem 2 for $Q = 1.2$. Here, the trajectory was wrapped into $[-\pi, \pi)$.