Time Series Analysis & Recurrent Neural Networks

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Exercise 6

To be uploaded before the exercise group on January 13th, 2021

Task 1. Poisson latent variable models.
In 'ex6file.mat' (or according *.xls files), you will find variables $\mathbf{U} = \{\mathbf{u}_t\}$ (named 'u'), $t = 1...T$, as well as parameters $\mathbf{A}$, $\mathbf{B}$, Σ, $\boldsymbol{\eta}_0$, and Γ obtained from the following model (see also Ch7 eqns. 7.85):

$$
\begin{aligned}
&c_t^{(i)}|\mathbf{z}_t \sim Poisson[\lambda_t^{(i)}\Delta t] \text{ with } \lambda_t^{(i)} = \exp(\log[\eta_i^{(0)}] + \boldsymbol{\eta}_i^{(1)}\mathbf{z}_t),\\
&\mathbf{z}_t = \mathbf{A}\mathbf{z}_{t-1} + \mathbf{B}\mathbf{u}_t + \boldsymbol{\epsilon}_t, \text{ with } \boldsymbol{\epsilon}_t \sim N(\mathbf{0}, \Sigma),\\
&\mathbf{z}_1 \sim N(\boldsymbol{\mu}_0, \Sigma).
\end{aligned}
$$

Note that matrix Γ collects the $i = 1...N$ vectors $\boldsymbol{\eta}_i^{(1)}$ in its rows, i.e. $\Gamma_{i,:} = \boldsymbol{\eta}_i^{(1)}$, and $\boldsymbol{\eta}_0$ is a vector collecting the η_{0i}.

1. Create time series of $M \times T$ ($M = 2$, $T = 100$) dimensional latent states $\mathbf{Z} = \{\mathbf{z}_t\}$ (named 'z'), and $N \times T$ dimensional observations $\mathbf{C} = \{\mathbf{c}_t\}$ (named 'c') from these variables and parameter settings, and plot them.
2. What is the joint data log-likelihood $\log p(\{\mathbf{c}_t\mathbf{z}_t\}|\theta)$ of your generated time series?

Task 2. Fixed points, stability, and bifurcations.
Consider the univariate nonlinear map

$$x_{t+1} = f(x_t, w, \theta) = w \cdot \sigma(x_t) + \theta, \text{ with } \sigma(x) = \frac{1}{1+e^{-x}}.$$

1. For $w = 7$ and $\theta = -2.5$, find the fixed points of the system. Visualize these in a graph. Are they stable?
2. For $w = 7$, plot the bifurcation graph as a function of $\theta \in [-10\ 0]$. Include both stable and unstable objects. How does the system change its dynamical properties as θ is varied within this range?

Task 3. Nonlinear systems, oscillations, and chaos.
Consider the 'Ricker map', with parameter $r \in \mathbb{R}$, and variable $x_t \in \mathbb{R}$:

$$x_{t+1} = r x_t e^{-x_t},$$

1. What are the fixed point(s) of this map? How many are there?
2. Explore the behavior of the map for a few values $r \in [\exp(1), ..., \exp(4)]$ (covering the extremes of this interval), and comment on the dynamics.