Taller 2 p(2.1)

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Dado el polinomio de lagrange para los puntos $[x_i, x_{i+1}, x_{1+2}, x_{i+1}]$

$$
\begin{aligned}
L(x) = &\frac{(x-x_{i+1})(x-x_{i+2})(x-x_{i+3})}{(x_i-x_{i+1})(x_i-x_{i+2})(x_i-x_{i+3})} \cdot f(x_i) \\
&+ \frac{(x-x_i)(x-x_{i+2})(x-x_{i+3})}{(x_{i+1}-x_i)(x_{i+1}-x_{i+2})(x_{i+1}-x_{i+3})} \cdot f(x_{i+1}) \\
&+ \frac{(x-x_i)(x-x_{i+1})(x-x_{i+3})}{(x_{i+2}-x_i)(x_{i+2}-x_{i+1})(x_{i+2}-x_{i+3})} \cdot f(x_{i+2}) \\
&+ \frac{(x-x_i)(x-x_{i+1})(x-x_{i+2})}{(x_{i+3}-x_i)(x_{i+3}-x_{i+1})(x_{i+3}-x_{i+2})} \cdot f(x_{i+3})
\end{aligned}
$$

Realizamos la integracion del polinomio de la siguiente manera:

$$
\begin{aligned}
\int_{x_i}^{x_{i+3}} L(x)\,dx = &\frac{1}{(x_i-x_{i+1})(x_i-x_{i+2})(x_i-x_{i+3})} f(x_i) \int_{x_i}^{x_{i+3}} (x-x_{i+1})(x-x_{i+2})(x-x_{i+3})\,dx \\
&+ \frac{1}{(x_{i+1}-x_i)(x_{i+1}-x_{i+2})(x_{i+1}-x_{i+3})} f(x_{i+1}) \int_{x_i}^{x_{i+3}} (x-x_i)(x-x_{i+2})(x-x_{i+3})\,dx \\
&+ \frac{1}{(x_{i+2}-x_i)(x_{i+2}-x_{i+1})(x_{i+2}-x_{i+3})} f(x_{i+2}) \int_{x_i}^{x_{i+3}} (x-x_i)(x-x_{i+1})(x-x_{i+3})\,dx \\
&+ \frac{1}{(x_{i+3}-x_i)(x_{i+3}-x_{i+1})(x_{i+3}-x_{i+2})} f(x_{i+3}) \int_{x_i}^{x_{i+3}} (x-x_i)(x-x_{i+1})(x-x_{i+2})\,dx
\end{aligned}
$$

Realizamos las integrales individualmente asumiendo que $h = \frac{a-b}{3}$donde $a = x_i$ y $b = x_{i+3}$:

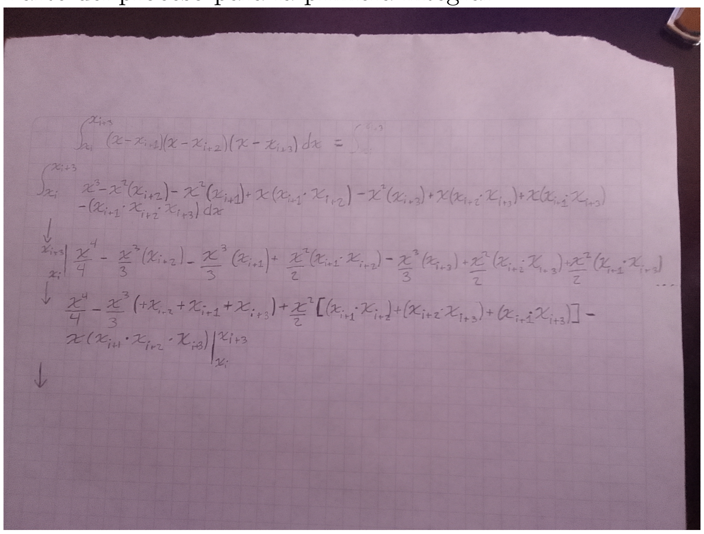
1. $\int_{x_i}^{x_{i+3}} (x-x_{i+1})(x-x_{i+2})(x-x_{i+3})\,dx = \frac{h}{8}$

2. $\int_{x_i}^{x_{i+3}} (x-x_i)(x-x_{i+2})(x-x_{i+3})\,dx = \frac{h}{8}$

3. $\int_{x_i}^{x_{i+3}} (x - x_i)(x - x_{i+1})(x - x_{i+3})\, dx = -\frac{h}{8}$

4. $\int_{x_i}^{x_{i+3}} (x - x_i)(x - x_{i+1})(x - x_{i+2})\, dx = \frac{h}{8}$

Parte del proceso para la primera integral



$$\int_{x_i}^{x_{i+3}} (x-x_{i+1})(x-x_{i+2})(x-x_{i+3})\, dx =$$

$$\int_{x_i}^{x_{i+3}} x^3 - x^2(x_{i+2}) - x^2(x_{i+1}) + x(x_{i+1}\cdot x_{i+2}) - x^2(x_{i+3}) + x(x_{i+2}\cdot x_{i+3}) + x(x_{i+1}\cdot x_{i+3}) - (x_{i+1}\cdot x_{i+2}\cdot x_{i+3})\, dx$$

$$\downarrow$$

$$\left. \frac{x^4}{4} - \frac{x^3}{3}(x_{i+2}) - \frac{x^3}{3}(x_{i+1}) + \frac{x^2}{2}(x_{i+1}\cdot x_{i+2}) - \frac{x^3}{3}(x_{i+3}) + \frac{x^2}{2}(x_{i+2}\cdot x_{i+3}) + \frac{x^2}{2}(x_{i+1}\cdot x_{i+3}) \right|_{x_i}^{x_{i+3}} \ldots$$

$$\downarrow$$

$$\left. \frac{x^4}{4} - \frac{x^3}{3}(+x_{i+2} + x_{i+1} + x_{i+3}) + \frac{x^2}{2}[(x_{i+1}\cdot x_{i+2}) + (x_{i+2}\cdot x_{i+3}) + (x_{i+1}\cdot x_{i+3})] - x(x_{i+1}\cdot x_{i+2}\cdot x_{i+3}) \right|_{x_i}^{x_{i+3}}$$

$$\downarrow$$

Ahora la integral se ve asi:

$$\int_{x_i}^{x_{i+3}} L(x)\, dx = \frac{h}{8}\Bigg[\frac{1}{(x_i - x_{i+1})(x_i - x_{i+2})(x_i - x_{i+3})} f(x_i) + \frac{1}{(x_{i+1} - x_i)(x_{i+1} - x_{i+2})(x_{i+1} - x_{i+3})} f(x_{i+1})$$
$$- \frac{1}{(x_{i+2} - x_i)(x_{i+2} - x_{i+1})(x_{i+2} - x_{i+3})} f(x_{i+2}) + \frac{1}{(x_{i+3} - x_i)(x_{i+3} - x_{i+1})(x_{i+3} - x_{i+2})} f(x_{i+3})\Bigg]$$

Comparando con los pesos para la regla de Simpson 3/8

$$\frac{1}{(x_i - x_{i+1})(x_i - x_{i+2})(x_i - x_{i+3})} = \frac{1}{-h^3}$$
$$\frac{1}{(x_{i+1} - x_i)(x_{i+1} - x_{i+2})(x_{i+1} - x_{i+3})} = \frac{-1}{h^3}$$
$$\frac{1}{(x_{i+2} - x_i)(x_{i+2} - x_{i+1})(x_{i+2} - x_{i+3})} = \frac{1}{h^3}$$
$$\frac{1}{(x_{i+3} - x_i)(x_{i+3} - x_{i+1})(x_{i+3} - x_{i+2})} = \frac{1}{h^3}$$

Obtenemos que la integral:

$$\int_{x_i}^{x_{i+3}} L(x)\,dx = \frac{h}{8}\left[\frac{1}{-h^3}f(x_i) - \frac{1}{h^3}f(x_{i+1}) + \frac{1}{h^3}f(x_{i+2}) + \frac{1}{h^3}f(x_{i+3})\right]$$

Simplifacando:

$$\begin{aligned}
\int_{x_i}^{x_{i+3}} L(x)\,dx &= \frac{h}{8}\left[\frac{1}{-h^3}f(x_i) - \frac{1}{h^3}f(x_{i+1}) + \frac{1}{h^3}f(x_{i+2}) + \frac{1}{h^3}f(x_{i+3})\right] \\
&= \frac{h}{8}\left[\frac{1}{-h^3}f(x_i) - \frac{1}{h^3}f(x_{i+1}) + \frac{1}{h^3}f(x_{i+2}) + \frac{1}{h^3}f(x_{i+3})\right] \\
&= \frac{1}{8}\left[-f(x_i) + 3f(x_{i+1}) + 3f(x_{i+2}) + f(x_{i+3})\right]
\end{aligned}$$

Obtenemos al final:

$$\int_{x_i}^{x_{i+3}} f(x)\,dx \approx \frac{3h}{8}\left[f(x_i) + 3f(x_{i+1}) + 3f(x_{i+2}) + f(x_{i+3})\right]$$