

Supplement: Generators and Relations for Real Stabilizer Operators

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This supplement establishes that the reduced relations of **Definition 5.6** imply the typed relations of **Definition 5.1** (as listed in the 9 figures, from Figure 4 to Figure 12 in **Appendix B**).

1 Proof of Proposition 5.7

We show that the typed relations, viewed as relations between un-typed circuits, follow from the reduced relations. That is, in this appendix, we *forget* the typing.

Once typing is forgotten, the relations in **Figure 8** and **Figure 9** coincide so that it is sufficient to derive those in **Figure 8**. Similarly, the first 12 rules of **Figure 6** are identical to the next 12 rules of **Figure 6** as un-typed rules so we only derive the first 12 of them. Lastly this is also true in the last 8 rules that appear in **Figure 5**, so we will only prove the leftmost column.

We start by proving a few useful lemmas before deriving the typed relations. For each one of the figures from **Figure 4** to **Figure 12**, we number the relations left-to-right and row-by-row in order to conveniently refer to them.

1.1 Lemmas

Lemma 1.1. *The following is a consequence of R2 and R3.*

$$\boxed{H} \bullet \boxed{H} \boxed{H} \bullet \boxed{H} = \text{———} \quad (\text{L1})$$

Lemma 1.2. *The following rules are a consequence of R1 – R18*

$$\begin{array}{c} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \\ \text{---} \bullet \text{---} \text{---} \bullet \text{---} \end{array} = \text{---} \text{---} \quad (\text{L2})$$

$$\begin{array}{c} \text{---} \bullet \text{---} \text{---} \bullet \text{---} \\ \boxed{H} \text{---} \bullet \text{---} \boxed{H} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \end{array} = \text{---} \text{---} \quad (\text{L3})$$

$$\begin{array}{c} \text{---} \bullet \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \\ \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \end{array} = \begin{array}{c} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \\ \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \text{---} \end{array} \quad (\text{L4})$$

$$\begin{array}{c} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \text{---} \bullet \text{---} \boxed{H} \end{array} = \begin{array}{c} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \text{---} \\ \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \end{array} \quad (\text{L5})$$

$$\begin{array}{c} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \text{---} \\ \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \end{array} = \begin{array}{c} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \\ \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \text{---} \end{array} \quad (\text{L6})$$

$$\begin{array}{c} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \text{---} \\ \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \end{array} = \begin{array}{c} \bullet \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \text{---} \\ \text{---} \bullet \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \end{array} \quad (\text{L7})$$

$$\begin{array}{c} \bullet \text{---} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \text{---} \\ \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \end{array} = \begin{array}{c} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \text{---} \\ \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \text{---} \end{array} \quad (\text{L8})$$

Lemma 1.3. *The following rules are a consequence of R1 – R18*

$$\begin{array}{c} \bullet \\ | \\ \boxed{H} \text{---} \bullet \text{---} \boxed{H} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \\ | \\ \bullet \end{array} = \begin{array}{c} \bullet \\ | \\ \boxed{H} \text{---} \bullet \text{---} \boxed{H} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \\ | \\ \bullet \end{array} \quad (\text{L9})$$

$$\begin{array}{c}
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\end{array}
\quad (L10)$$

[illegible]

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\end{array}
\quad (L12)$$

(L13)

[illegible]

$$\begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} = \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \quad (L15)$$

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\text{---}
\end{array}
\quad (L16)$$

[illegible]

[illegible]

$$\begin{array}{c}
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\end{array}
\begin{array}{c}
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\text{---}
\end{array}
=
\begin{array}{c}
\text{---} \\
\text{---} \\
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\end{array}
\begin{array}{c}
\bullet \\
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\end{array}
\begin{array}{c}
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\text{---}
\end{array}
\quad (L19)$$

1.2 Proofs of Lemmas

L1:

$$\begin{array}{c}
 \boxed{H} \text{---} \bullet \text{---} \boxed{H} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \xrightarrow{R3} \boxed{H} \text{---} \bullet \text{---} \bullet \text{---} \boxed{H} \\
 \xrightarrow{R2} \boxed{H} \boxed{H} \text{---} \\
 \xrightarrow{R3} \text{---}
 \end{array}$$

L2:

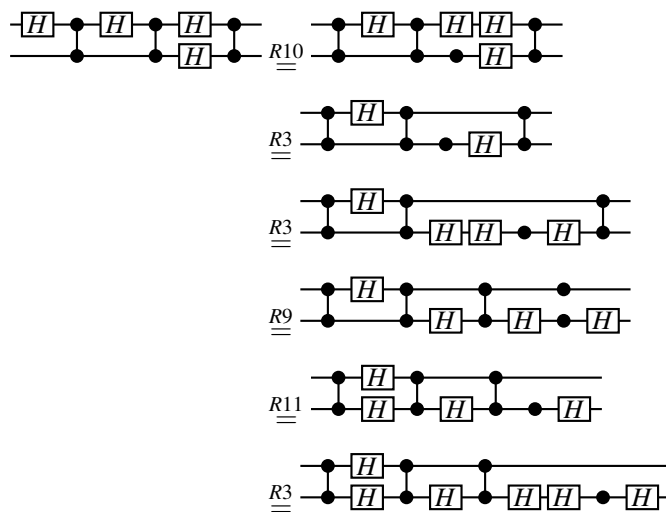
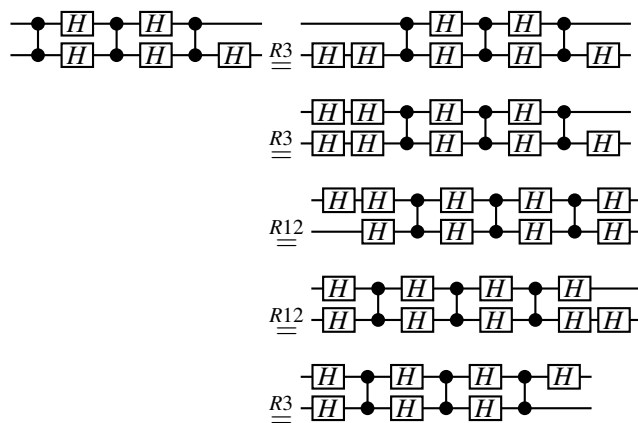
$$\begin{array}{c}
 \boxed{H} \text{---} \bullet \text{---} \boxed{H} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \xrightarrow{R3} \boxed{H} \text{---} \bullet \text{---} \bullet \text{---} \boxed{H} \\
 \xrightarrow{R5} \boxed{H} \boxed{H} \text{---} \\
 \xrightarrow{R3} \text{---}
 \end{array}$$

L3

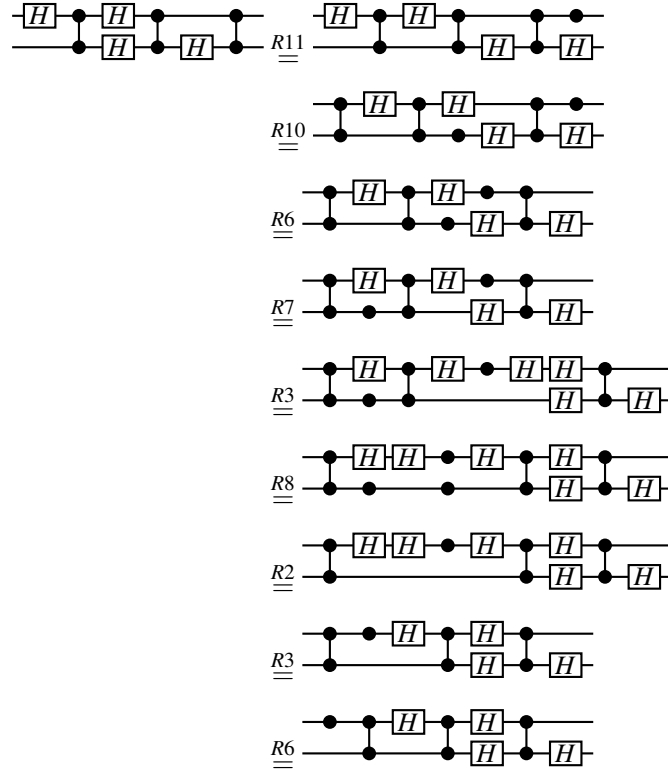
$$\begin{array}{c}
 \boxed{H} \text{---} \bullet \text{---} \boxed{H} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \xrightarrow{R3} \boxed{H} \text{---} \bullet \text{---} \bullet \text{---} \boxed{H} \\
 \xrightarrow{R5} \boxed{H} \boxed{H} \text{---} \\
 \xrightarrow{R3} \text{---}
 \end{array}$$

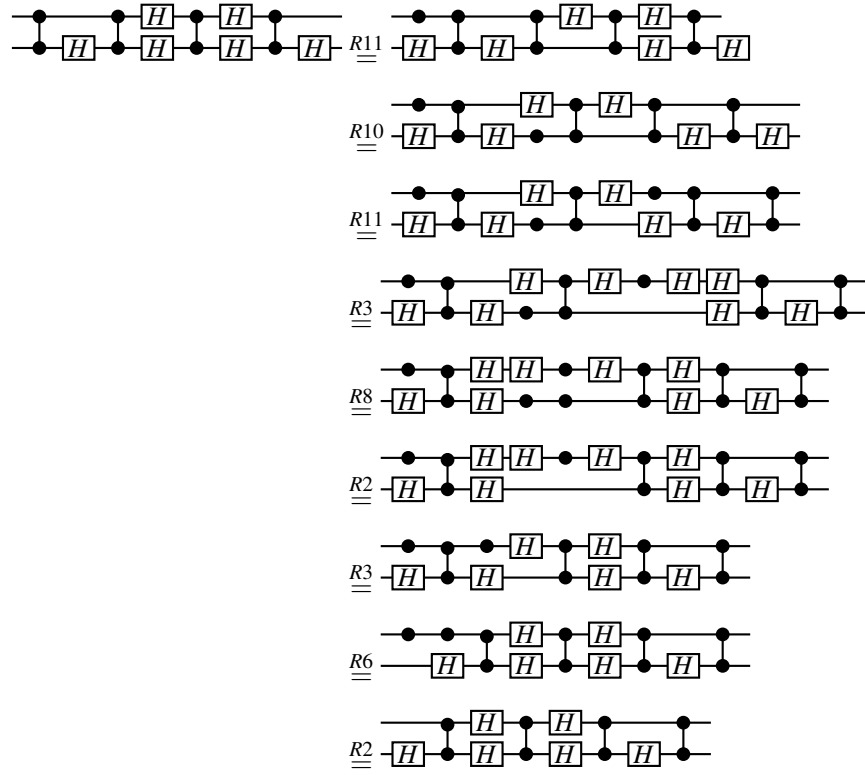
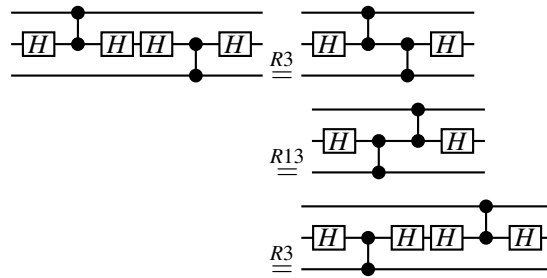
L4:

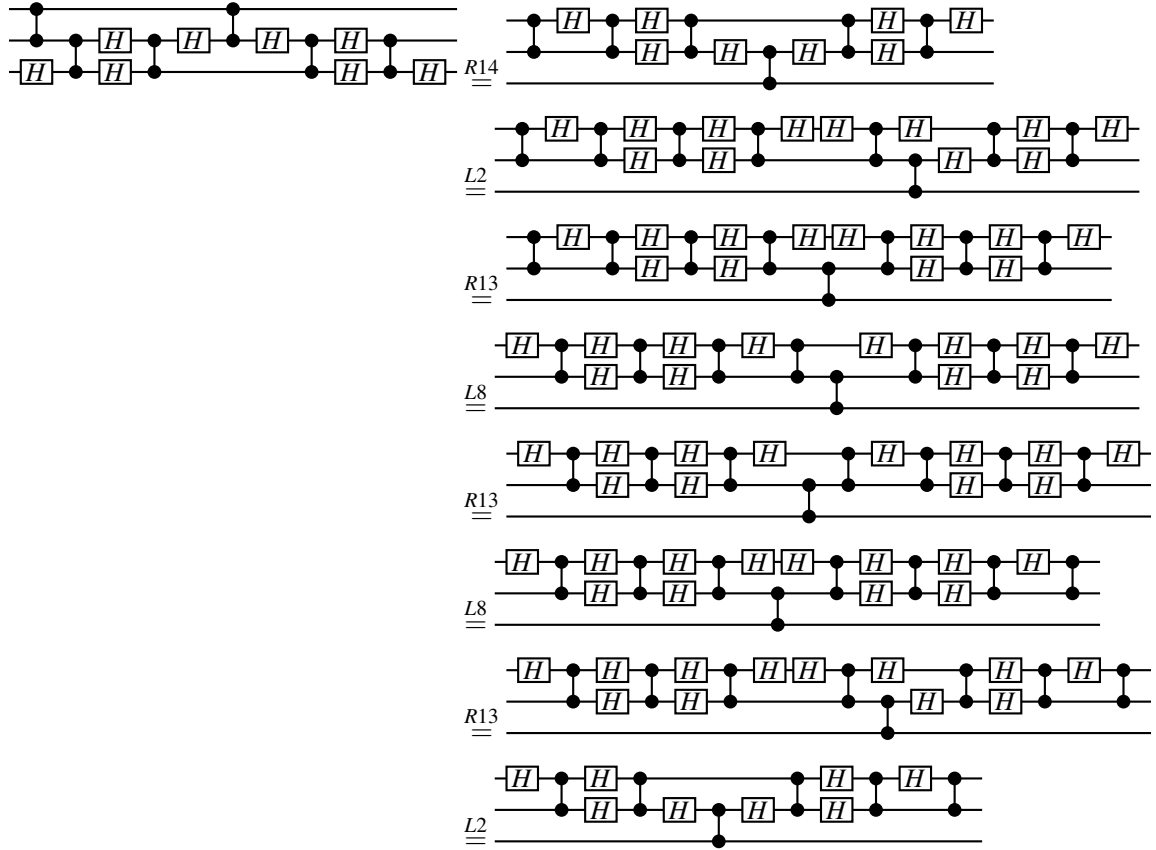
$$\begin{array}{c}
 \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \xrightarrow{R11} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \\
 \xrightarrow{R3} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \\
 \xrightarrow{R3} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \\
 \xrightarrow{R8} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \\
 \xrightarrow{R10} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \\
 \xrightarrow{R3} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H}
 \end{array}$$

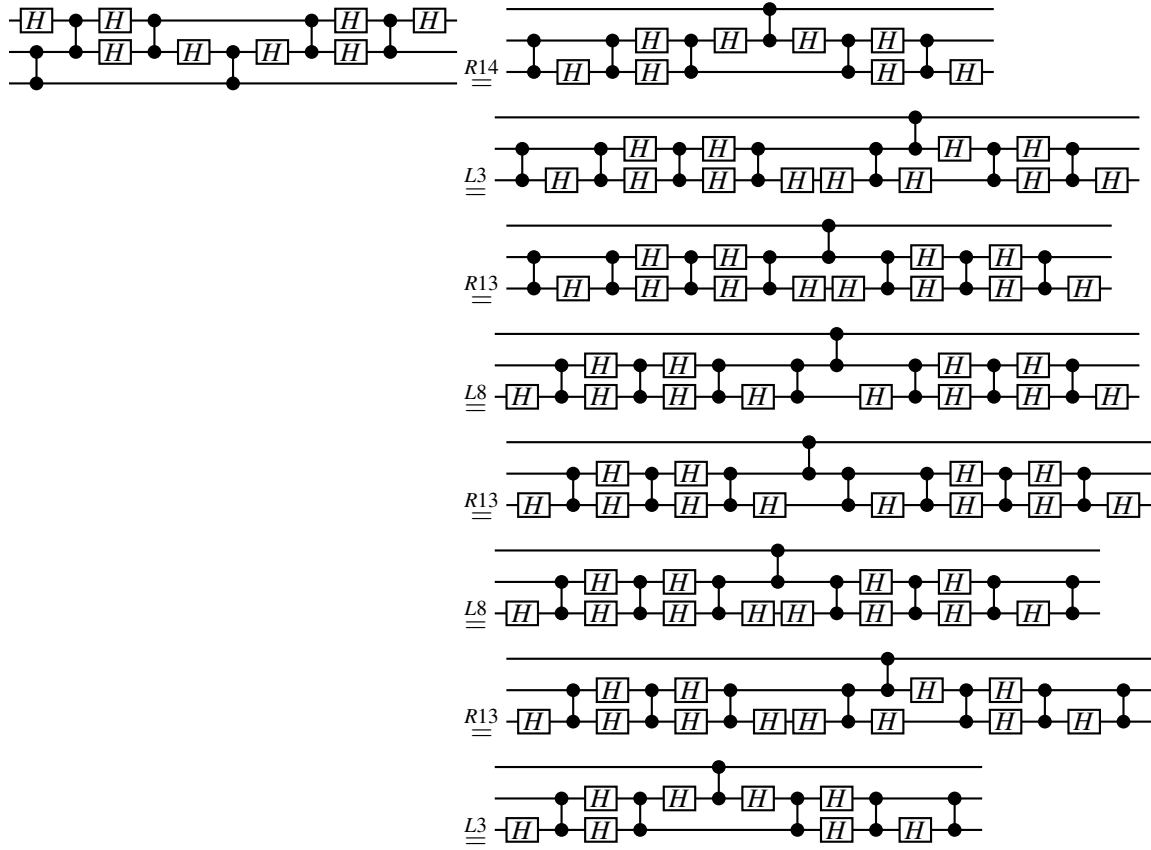
L5:**L6:**

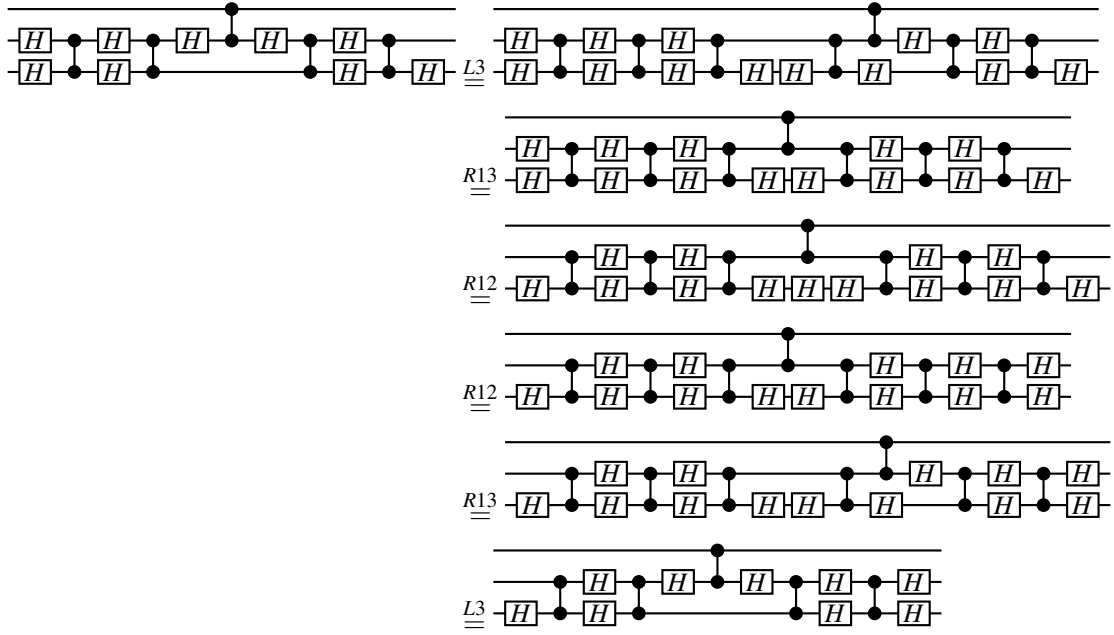
L7:

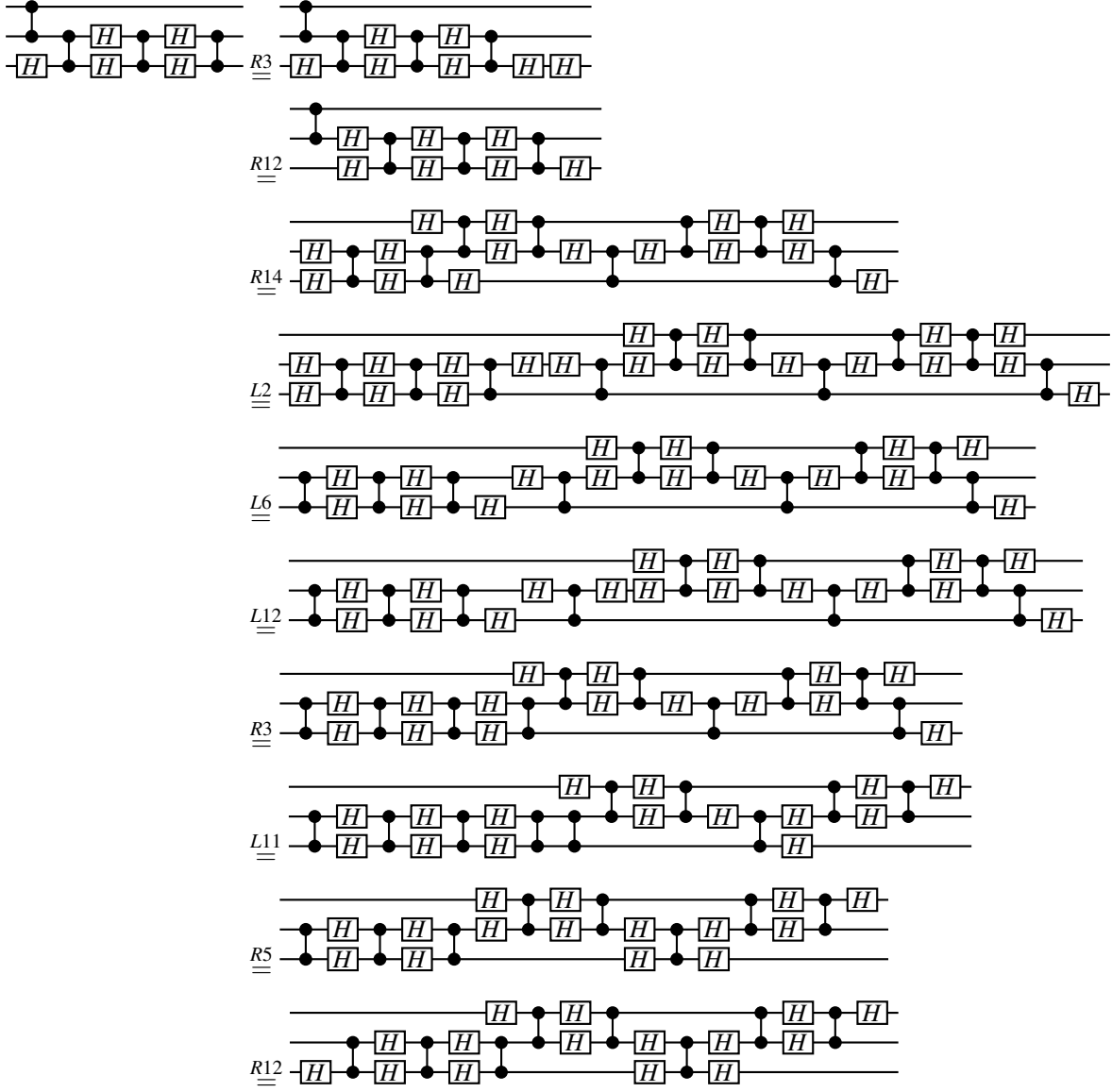


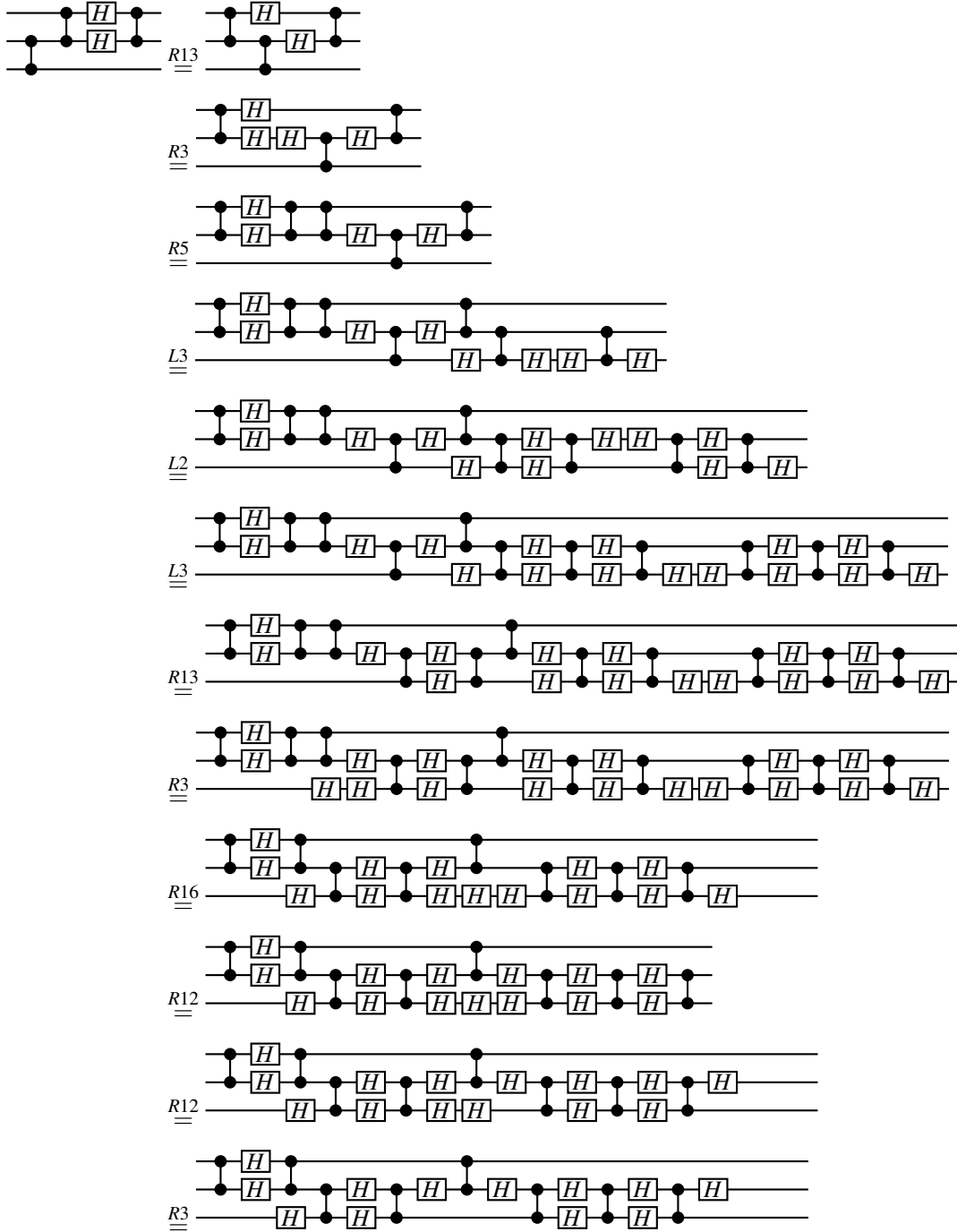
L8:**L9:**

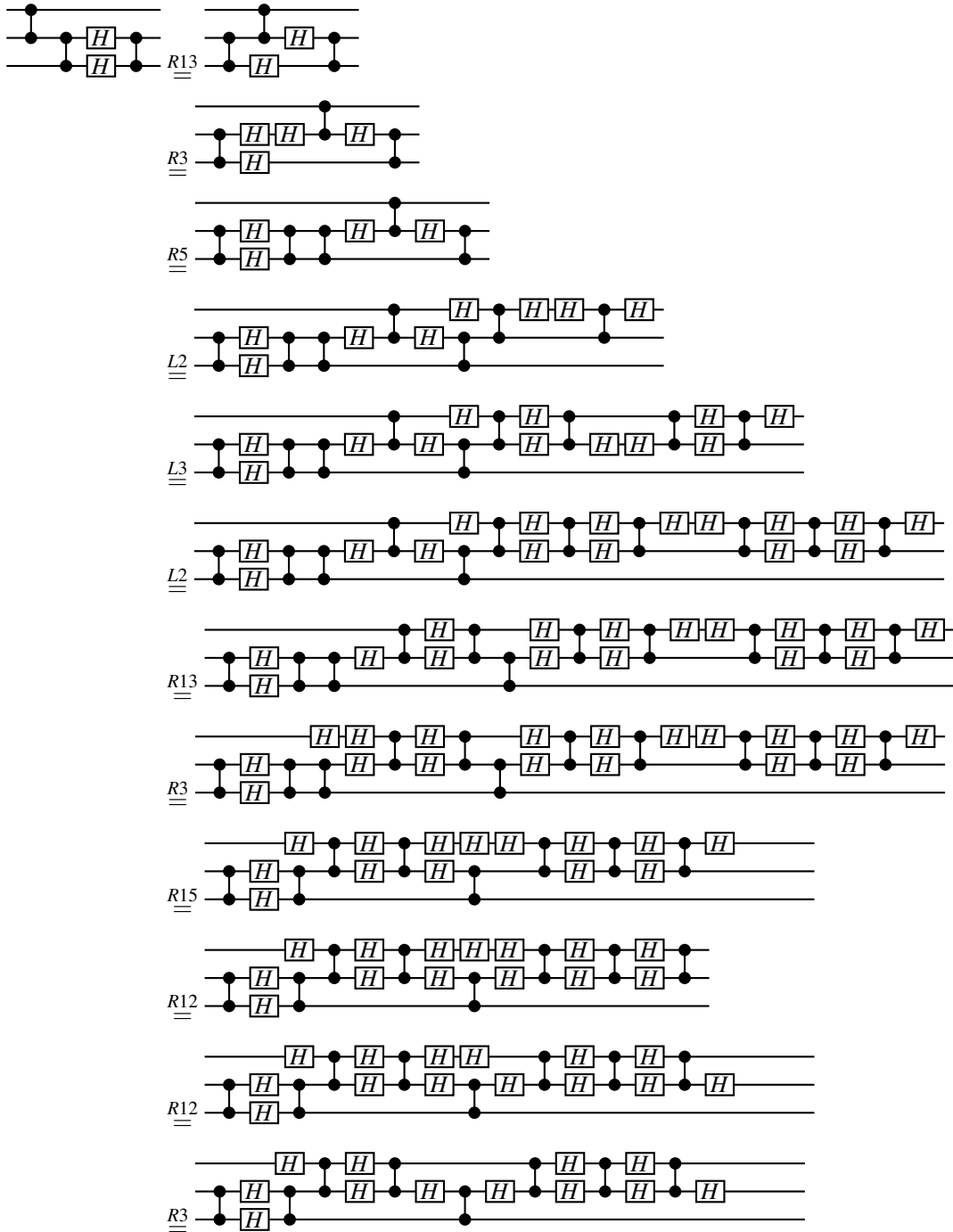
L10:

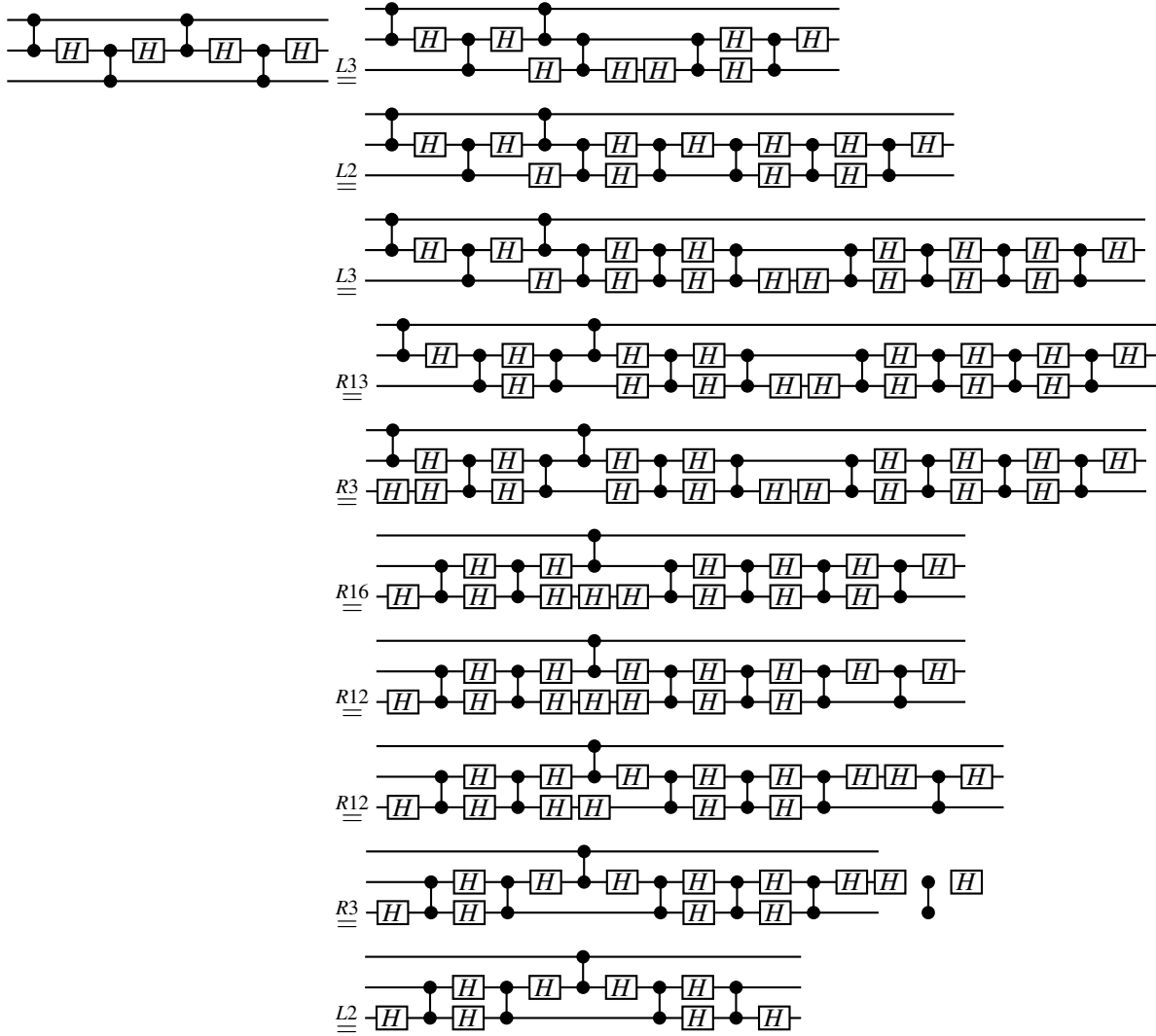
L11:

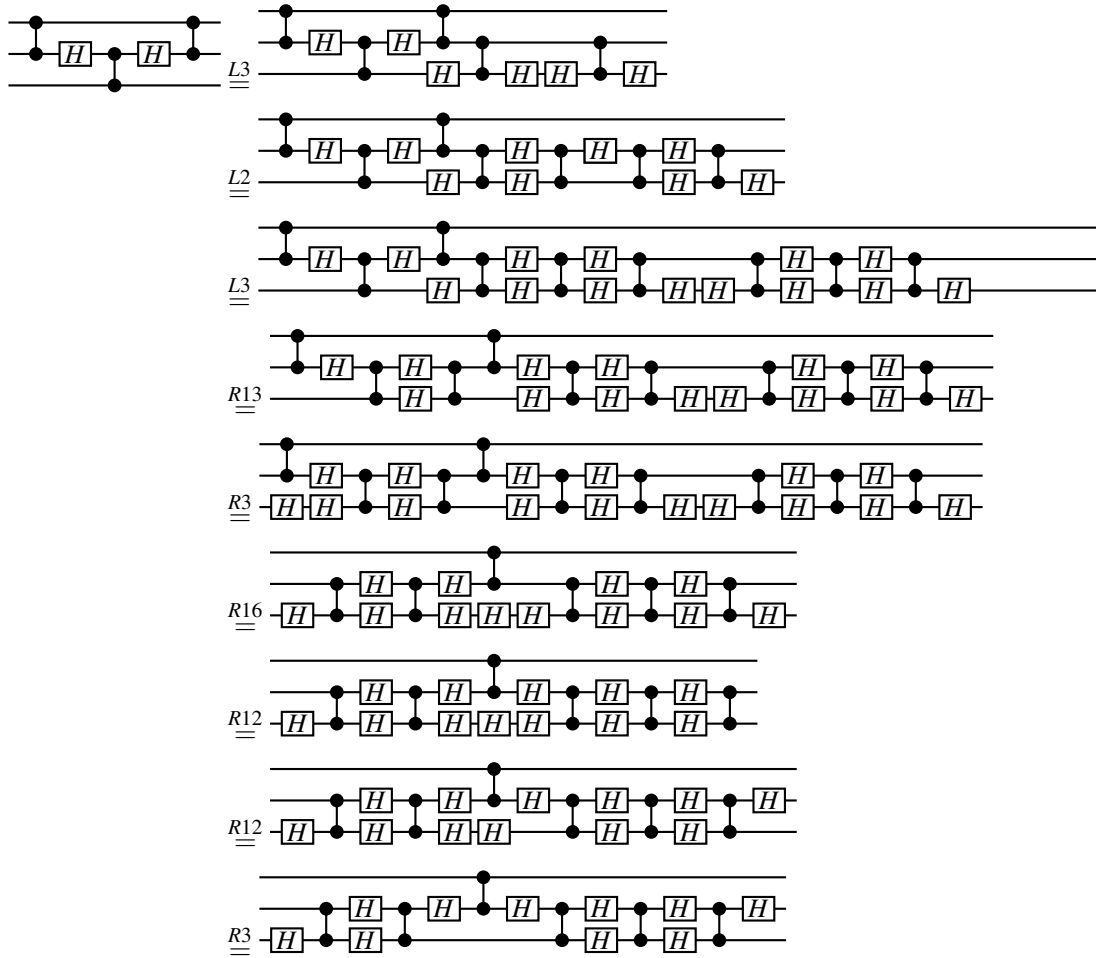
L12

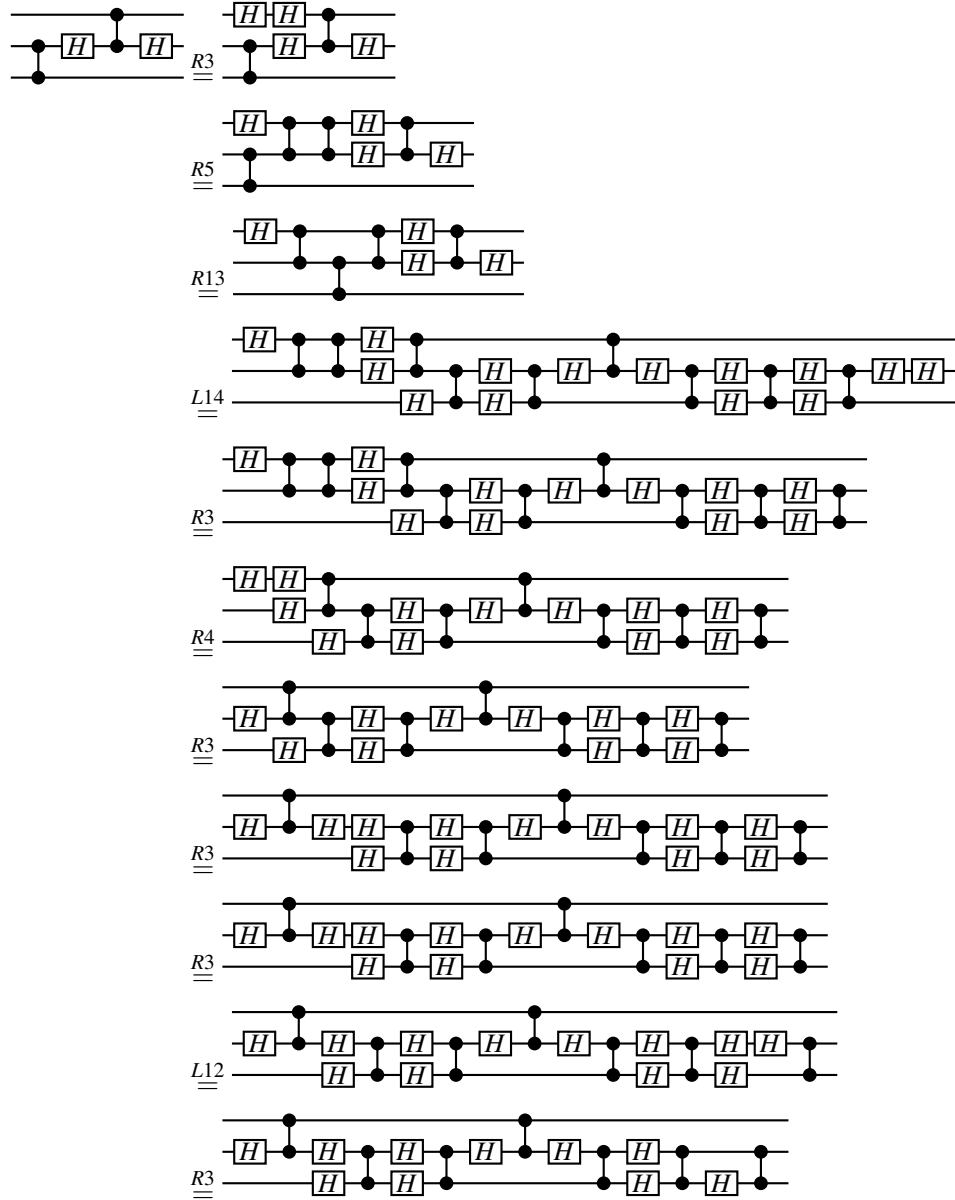
L13:

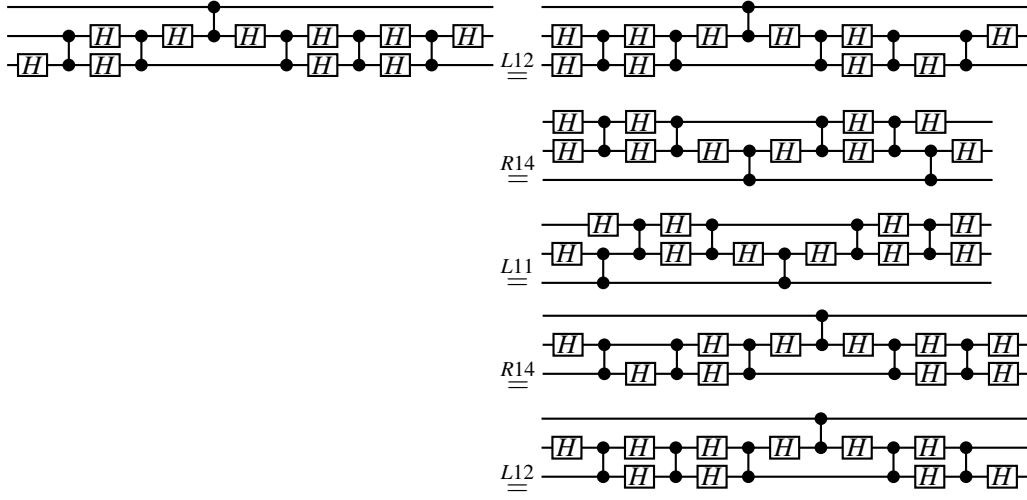
L14:

L15:

L16:

L17:

L18:

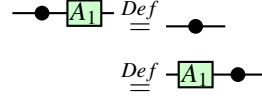
L19:

1.3 Proofs of Rewrite Rules

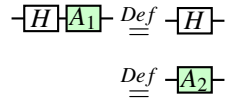
Proofs

Figure 4:

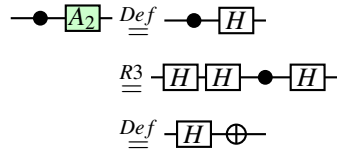
4.1



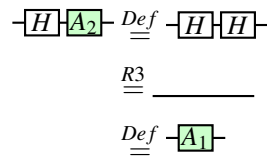
4.2



4.3



4.4



4.5

$$\begin{array}{c}
 \text{---} \bullet \boxed{A_3} \text{---} \stackrel{Def}{=} \text{---} \bullet \text{---} \\
 \stackrel{Def}{=} \text{---} \boxed{A_3} \bullet \text{---}
 \end{array}$$

4.6

$$\begin{array}{c}
 \text{---} \boxed{H} \boxed{A_3} \text{---} \stackrel{Def}{=} \text{---} \boxed{H} \text{---} \\
 \stackrel{Def}{=} \text{---} \boxed{A_3} \boxed{H} \text{---}
 \end{array}$$

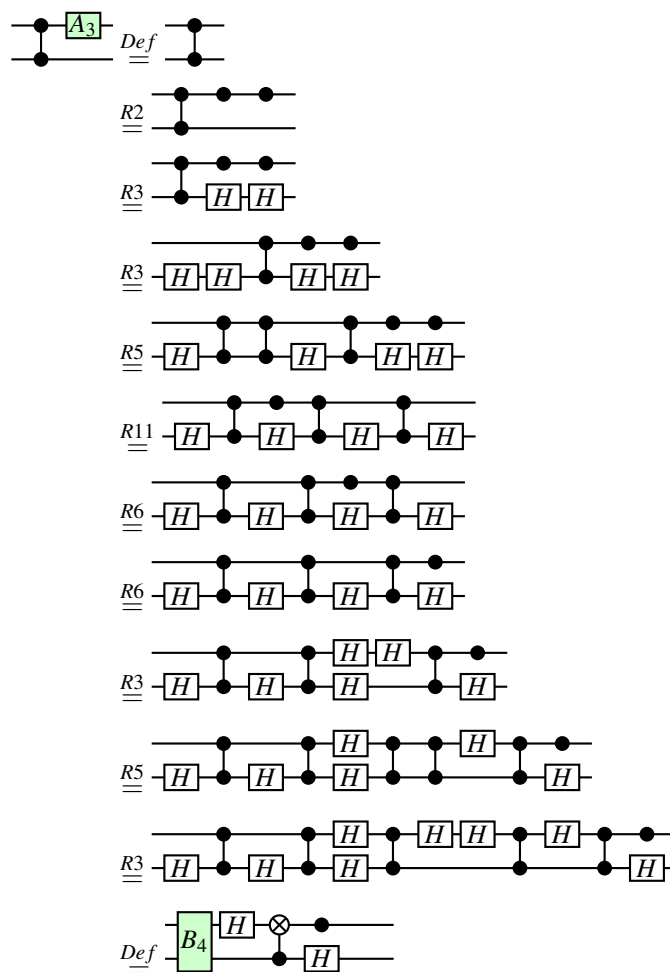
4.7

$$\begin{array}{c}
 \begin{array}{c} \text{---} \bullet \boxed{A_1} \text{---} \\ \text{---} \bullet \text{---} \end{array} \stackrel{Def}{=} \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} \\
 \stackrel{Def}{=} \begin{array}{c} \text{---} \boxed{A_1} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array}
 \end{array}$$

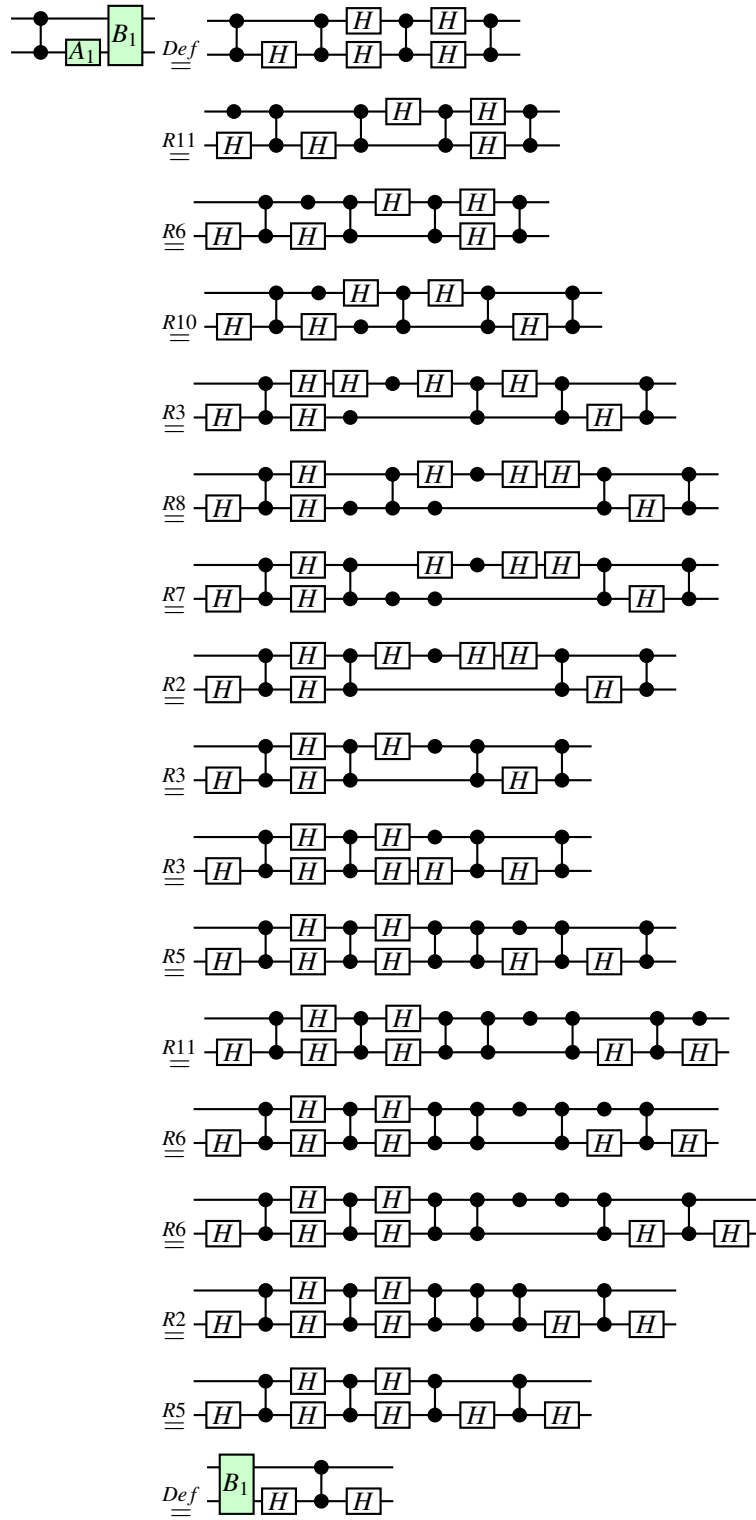
4.8

$$\begin{array}{c}
 \begin{array}{c} \text{---} \bullet \boxed{A_2} \text{---} \\ \text{---} \bullet \text{---} \end{array} \stackrel{Def}{=} \begin{array}{c} \text{---} \bullet \boxed{H} \text{---} \\ \text{---} \bullet \text{---} \end{array} \\
 \stackrel{R3}{=} \begin{array}{c} \text{---} \bullet \boxed{H} \text{---} \\ \text{---} \bullet \boxed{H} \boxed{H} \text{---} \end{array} \\
 \stackrel{R5}{=} \begin{array}{c} \text{---} \bullet \boxed{H} \bullet \bullet \text{---} \\ \text{---} \bullet \boxed{H} \bullet \bullet \boxed{H} \text{---} \end{array} \\
 \stackrel{Def}{=} \begin{array}{c} \text{---} \boxed{B_2} \bullet \text{---} \\ \text{---} \bullet \boxed{H} \text{---} \end{array}
 \end{array}$$

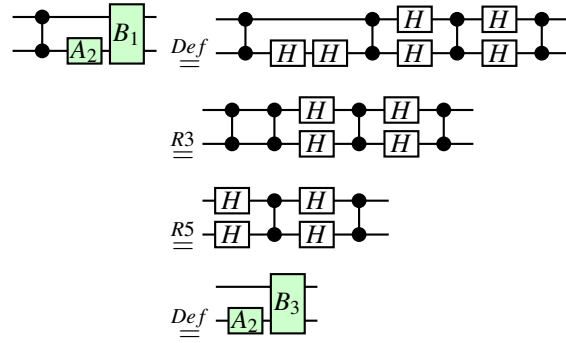
4.9



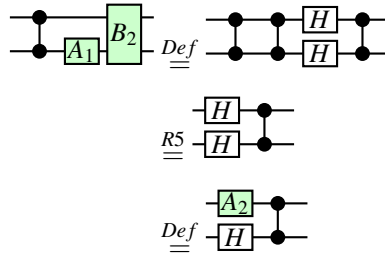
4.10



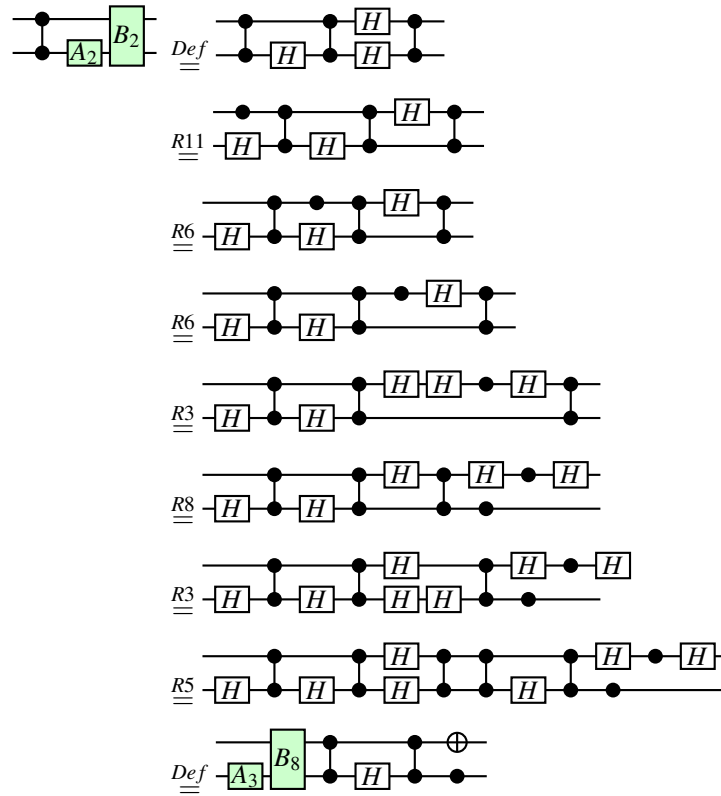
4.11



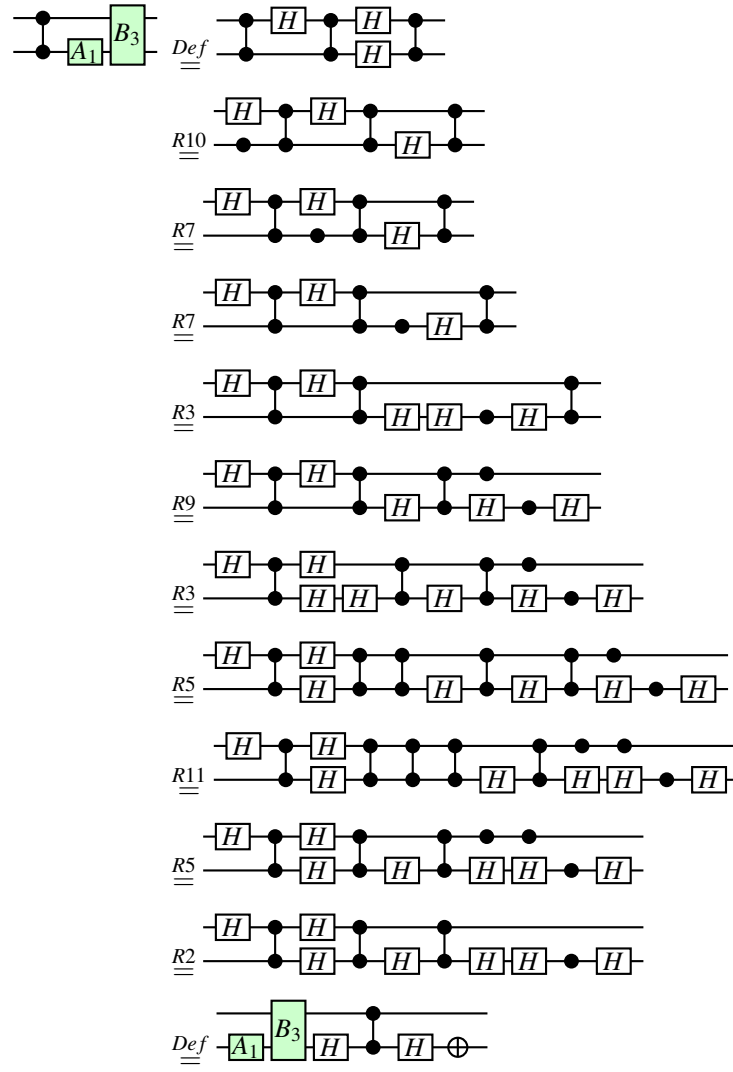
4.12



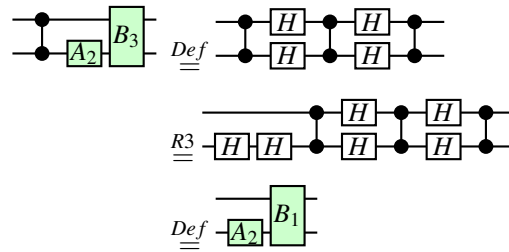
4.13



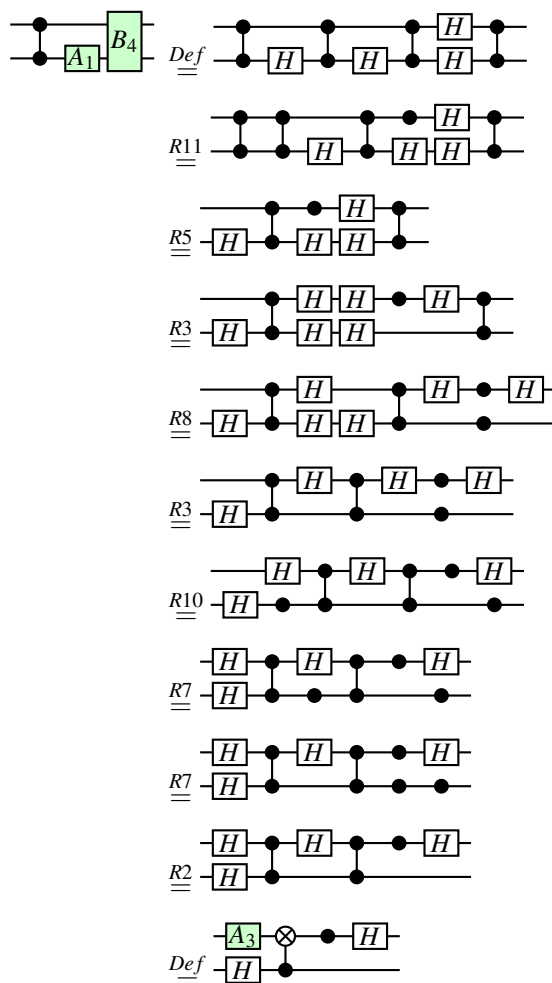
4.14



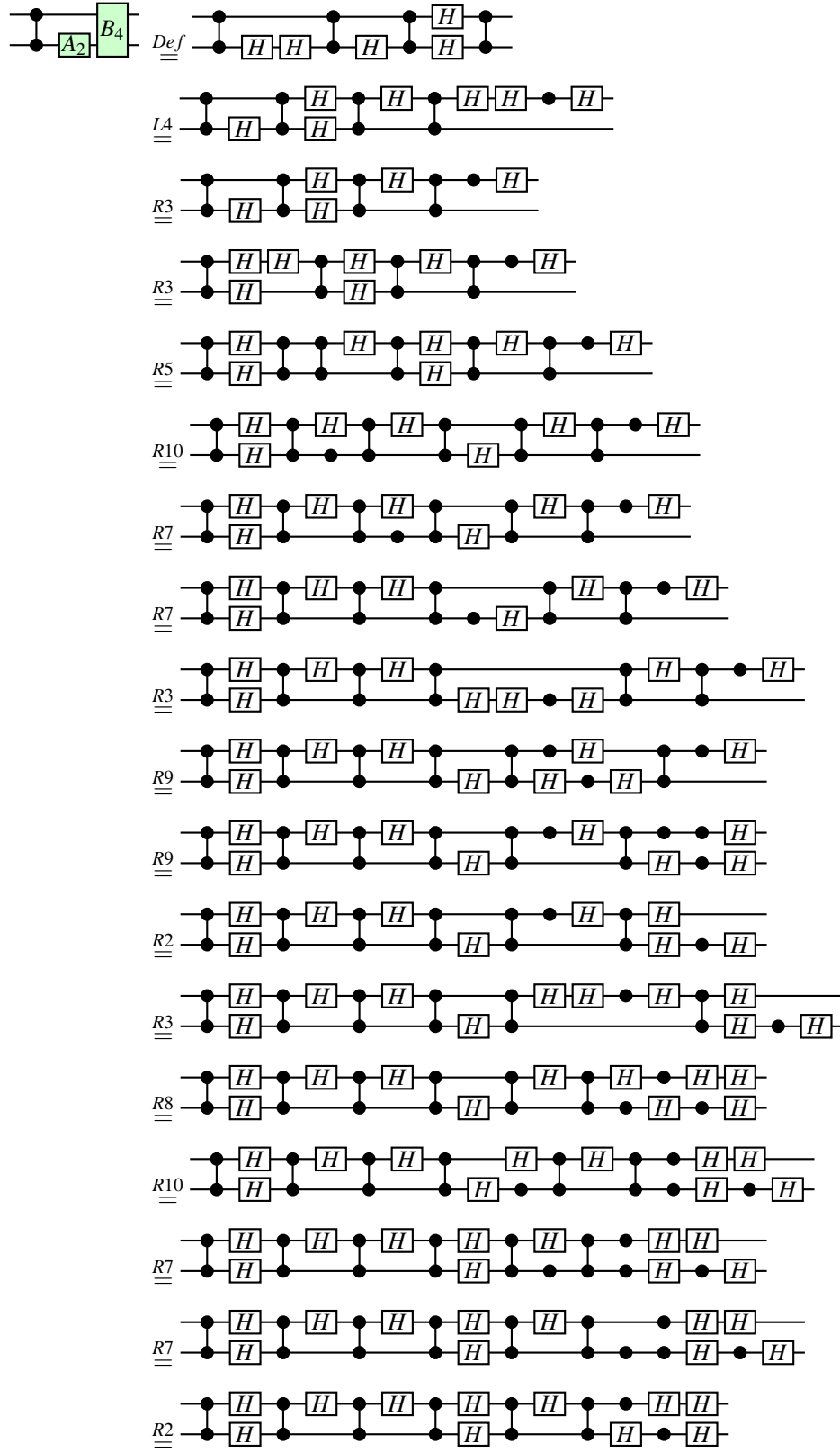
4.15

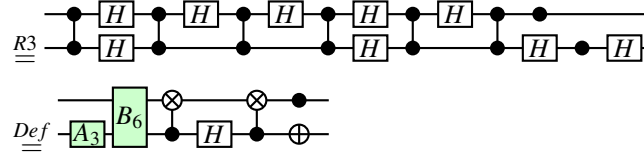


4.16

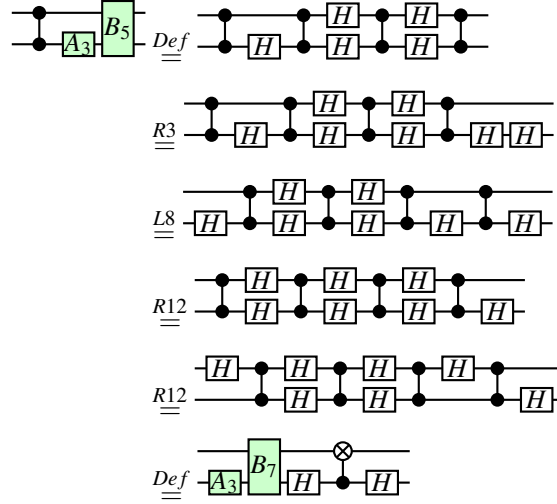


4.17

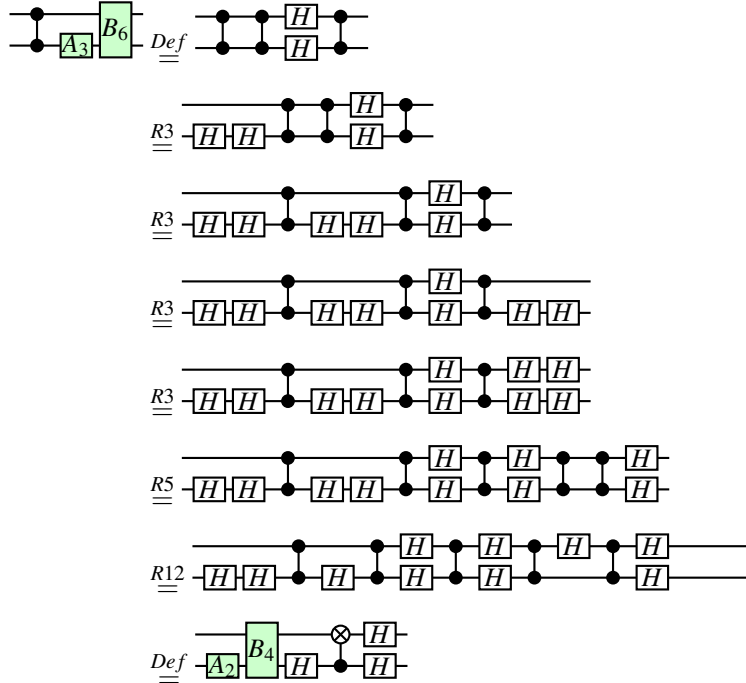


**Figure 5:**

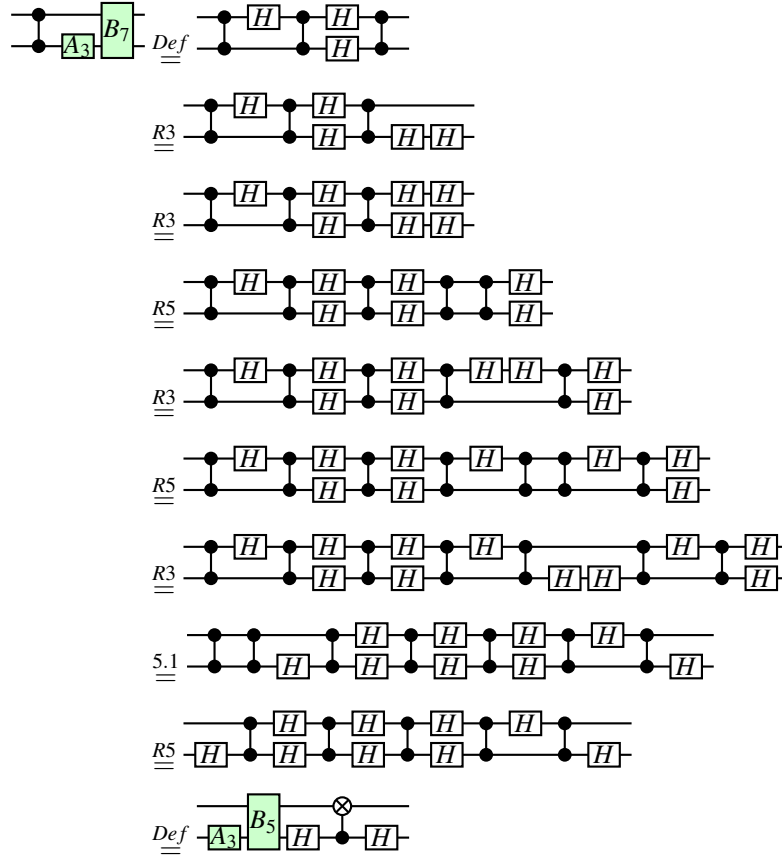
5.1



5.2



5.3



5.4

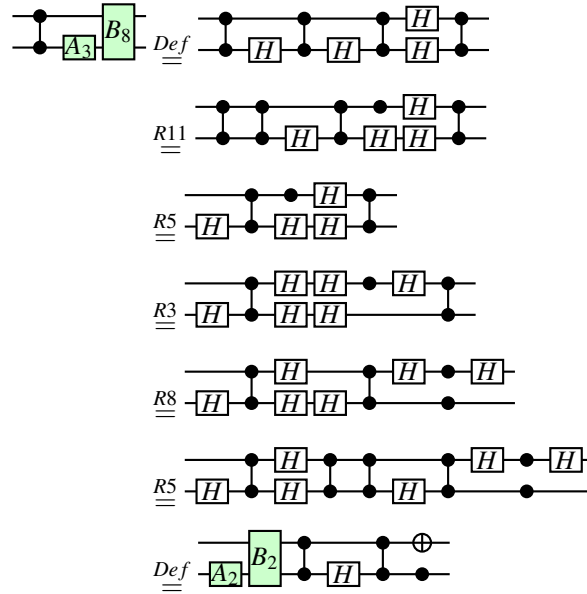
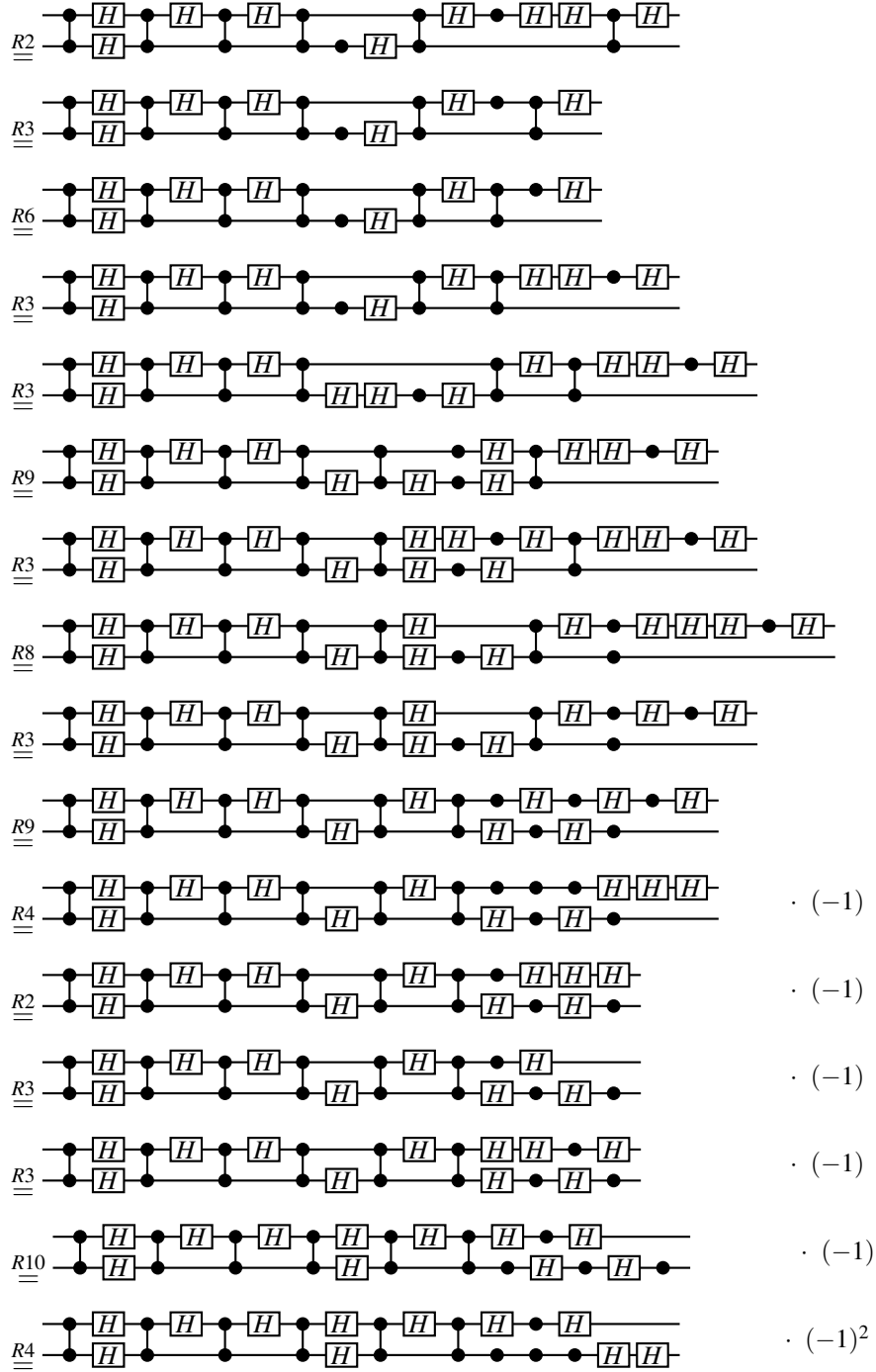
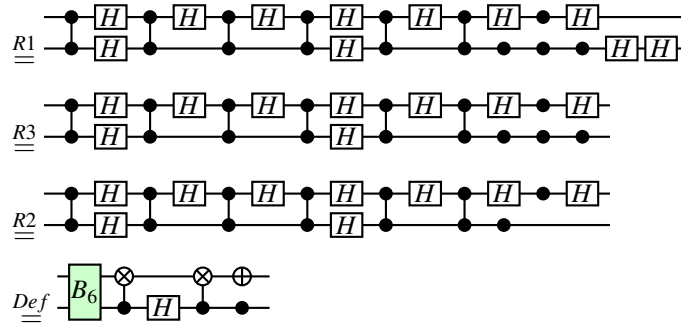
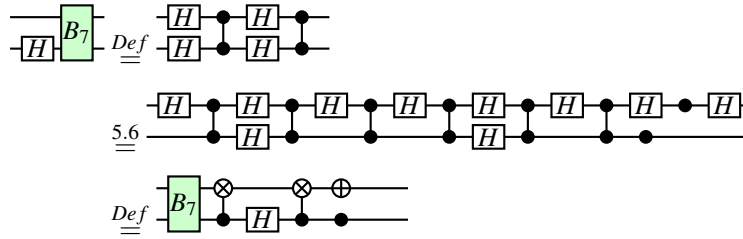


Figure 1 illustrates the decomposition of a 6-qubit gate B_6 into a sequence of 2-qubit gates. The top diagram shows B_6 as a green box. The subsequent diagrams, labeled L_2 , L_1 , L_4 , R_{11} , R_3 , R_8 , L_1 , R_7 , R_5 , L_4 , R_8 , and R_8 , show the step-by-step construction of the circuit using H gates and CNOTs. The bottom diagram shows the final circuit with 12 gates.

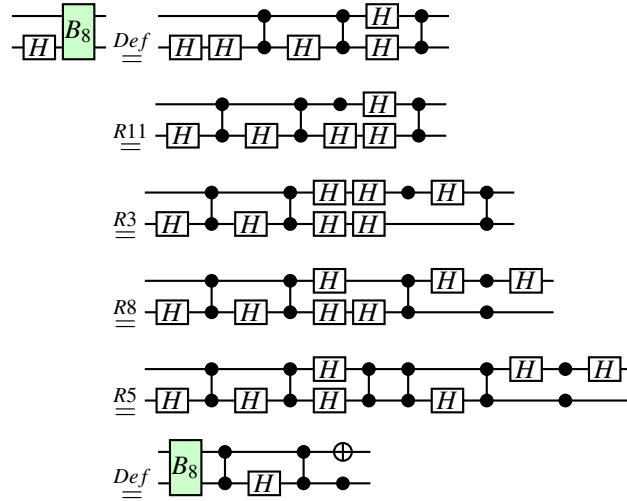




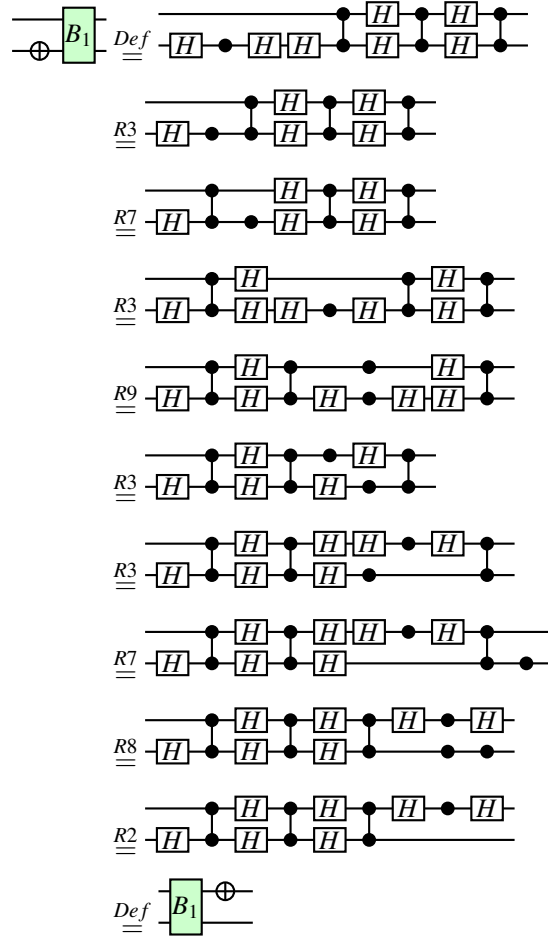
5.7



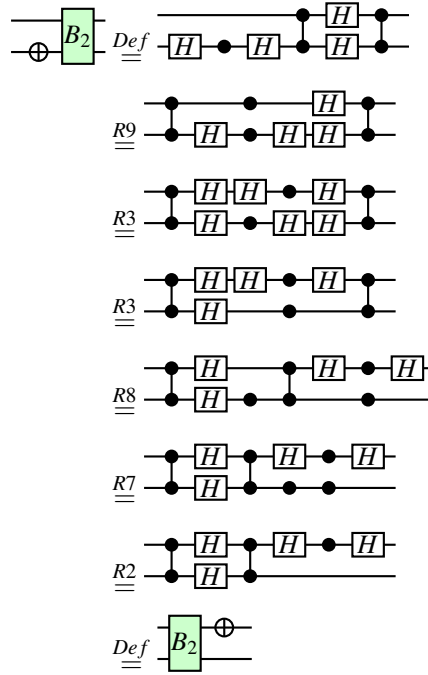
5.8



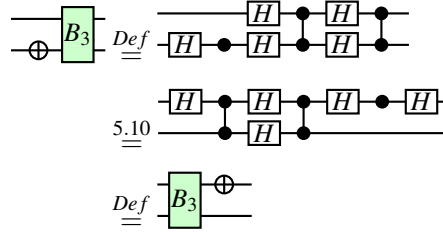
5.9



5.10



5.11



5.12

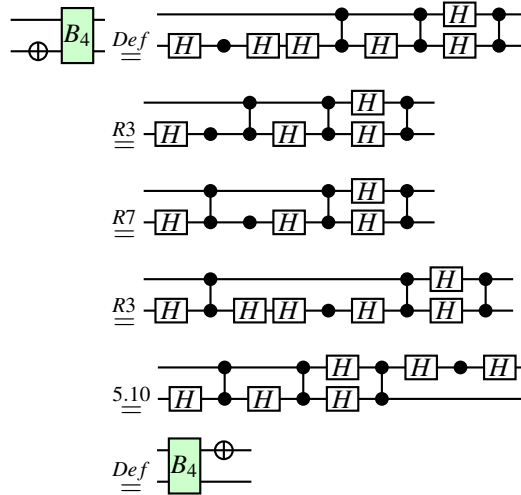
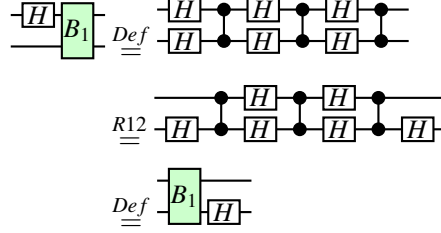
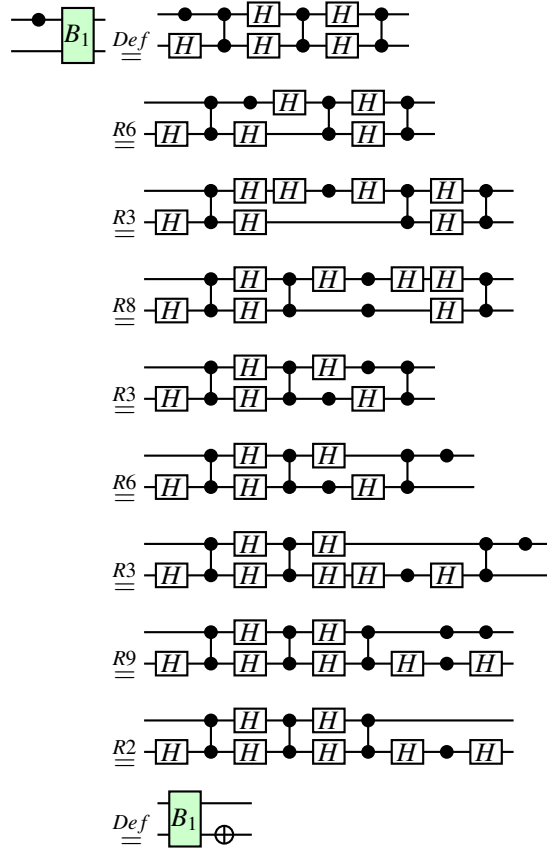


Figure 6:

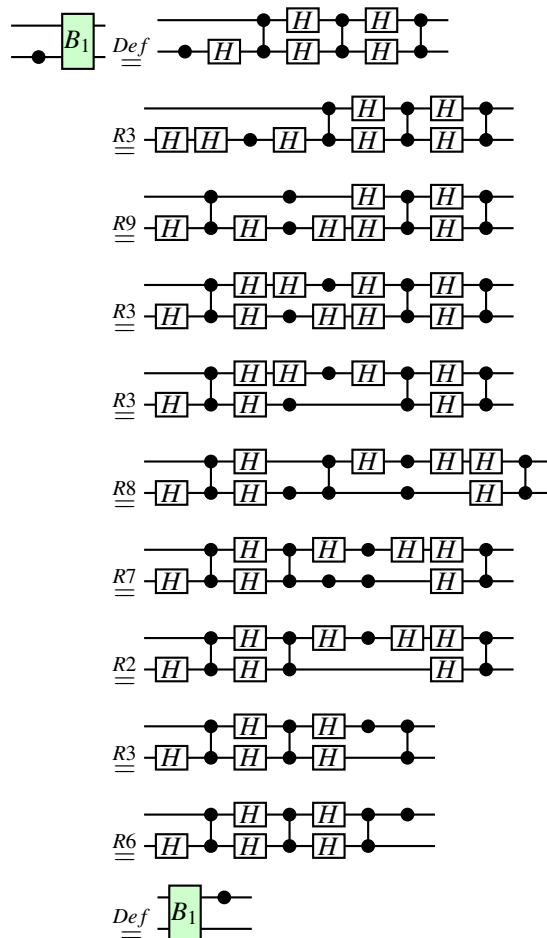
6.1



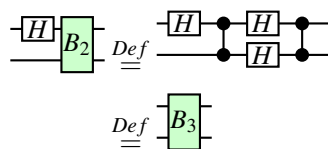
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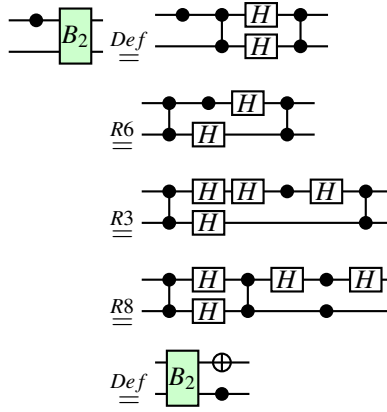
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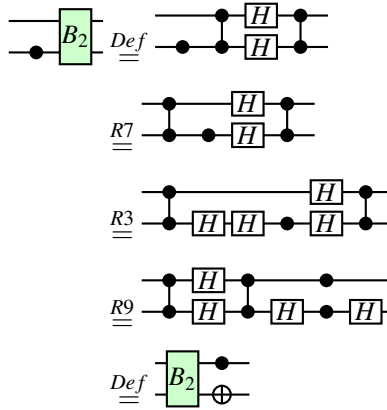
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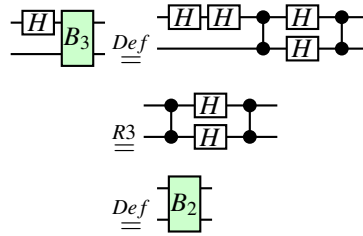
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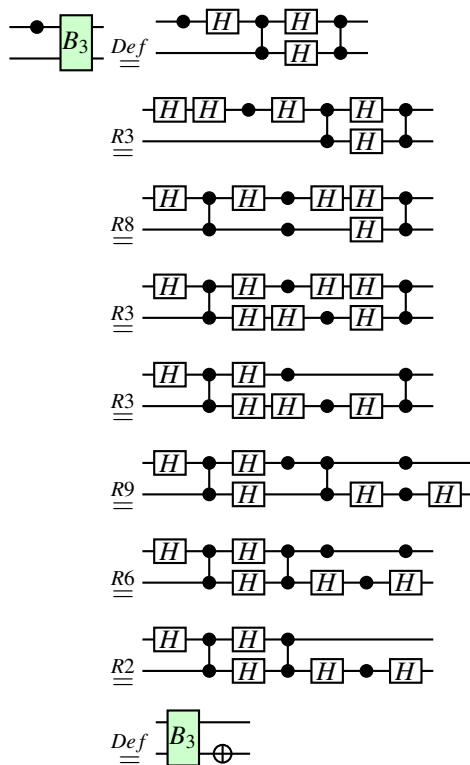
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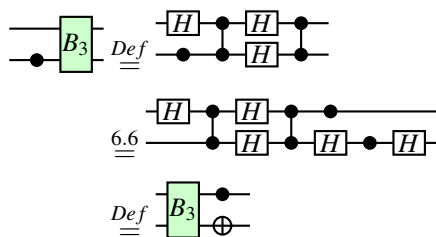
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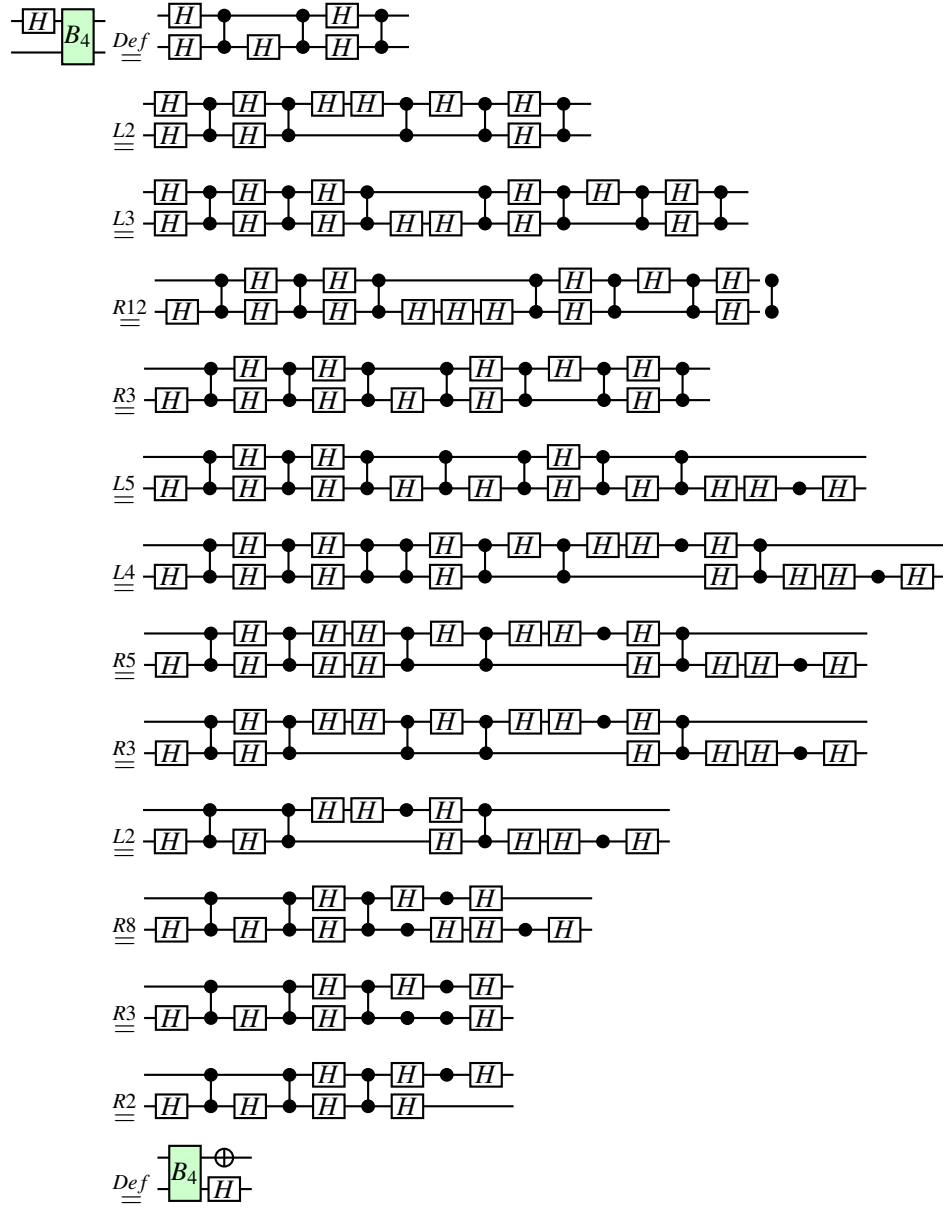
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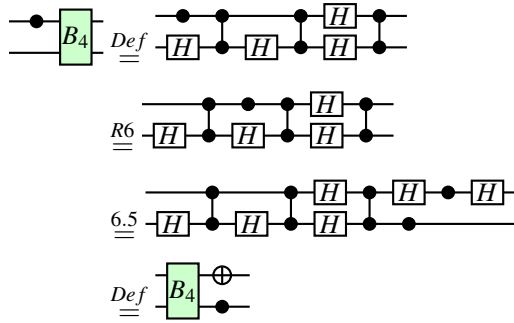
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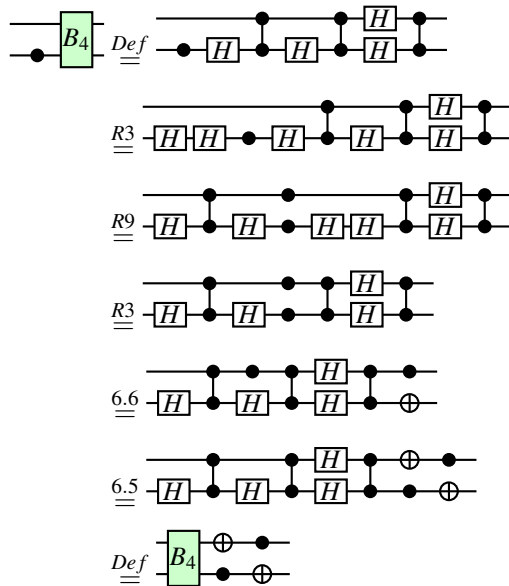
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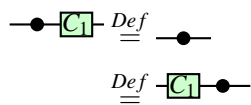
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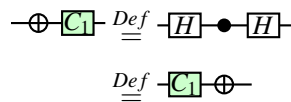
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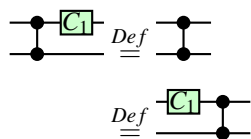
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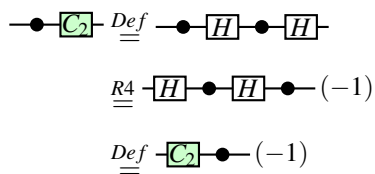
6.14



6.15

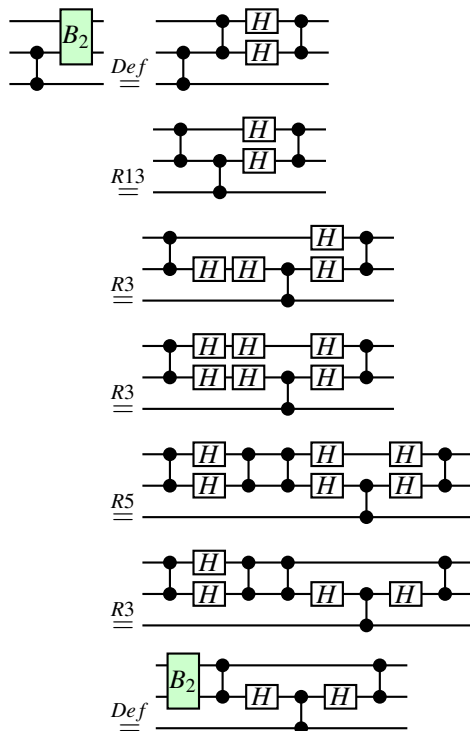


6.16

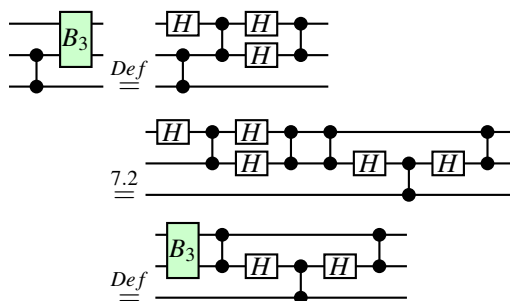


The figure displays a sequence of quantum circuit diagrams. The top circuit, labeled *Def*, consists of three horizontal qubit lines. The top line has a green box labeled B_1 followed by a CNOT gate with the middle line as target. The middle line has a CNOT gate with the bottom line as target. The bottom line has a CNOT gate with the middle line as target. This is followed by a sequence of circuits labeled R_{12} , R_{13} , R_{14} , R_{12} , and L_{16} , each with a double underline. The bottom circuit, also labeled *Def*, shows a different arrangement of gates and qubit connections, including a green box labeled B_1 on the top line.

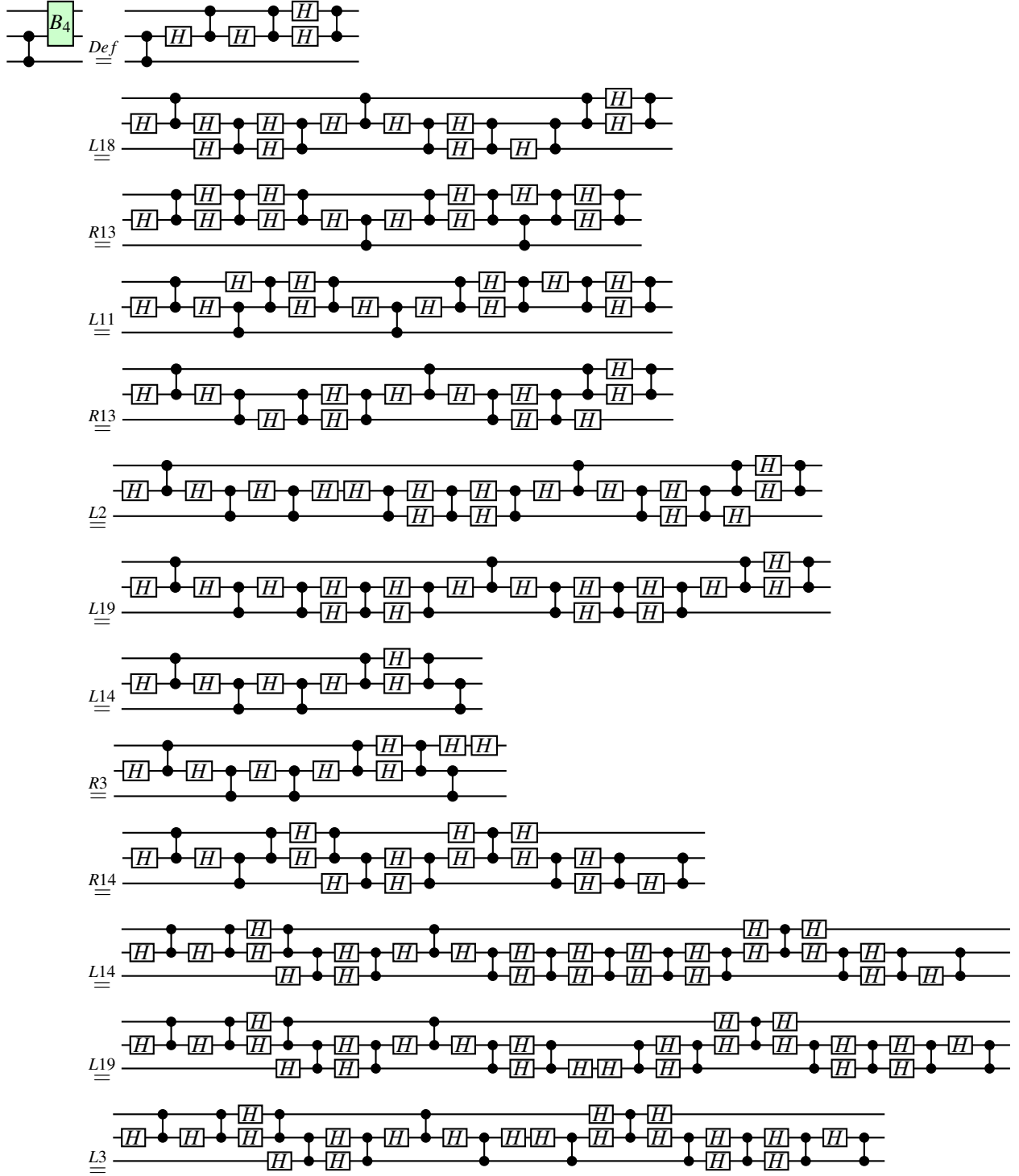
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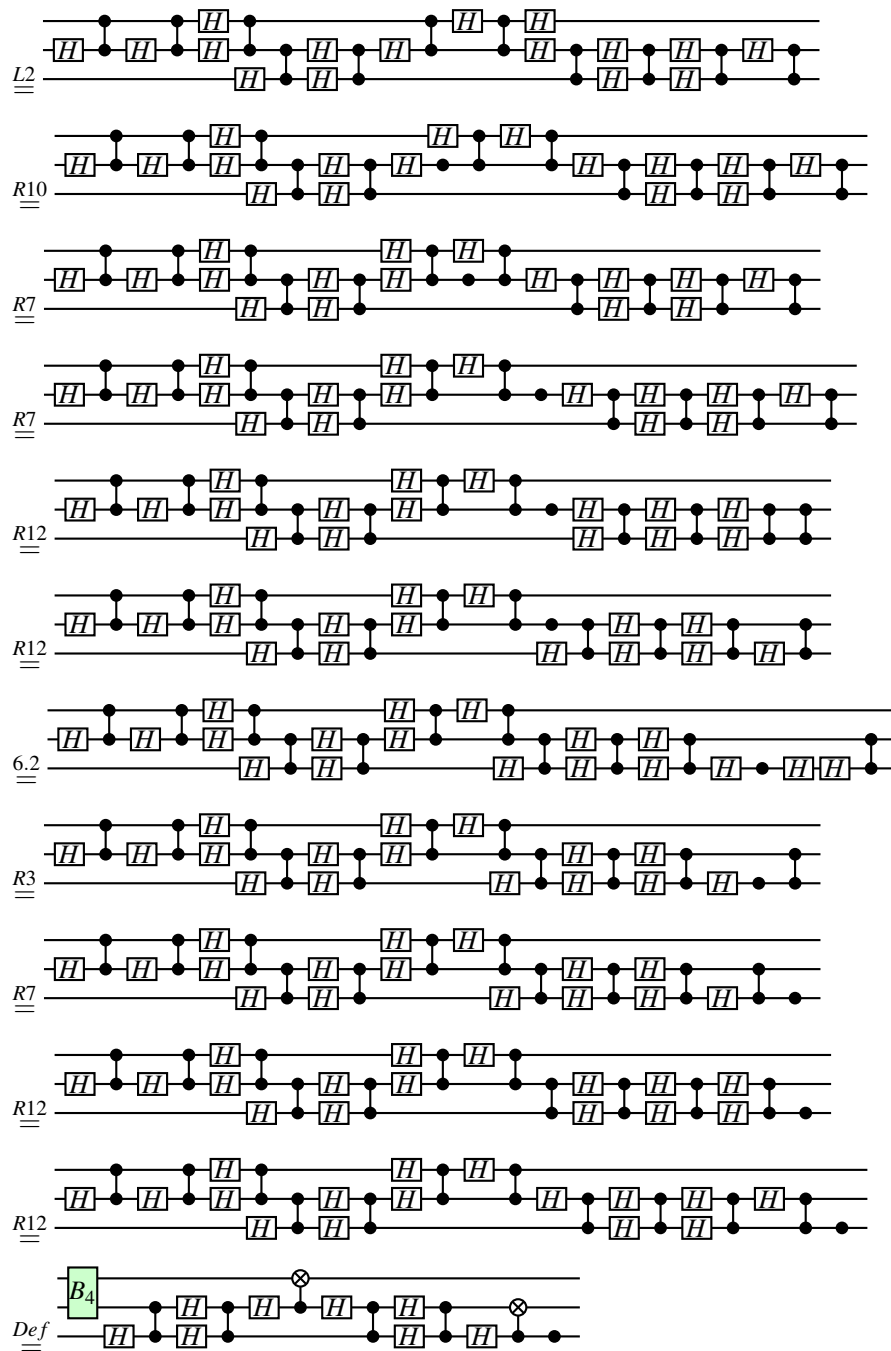


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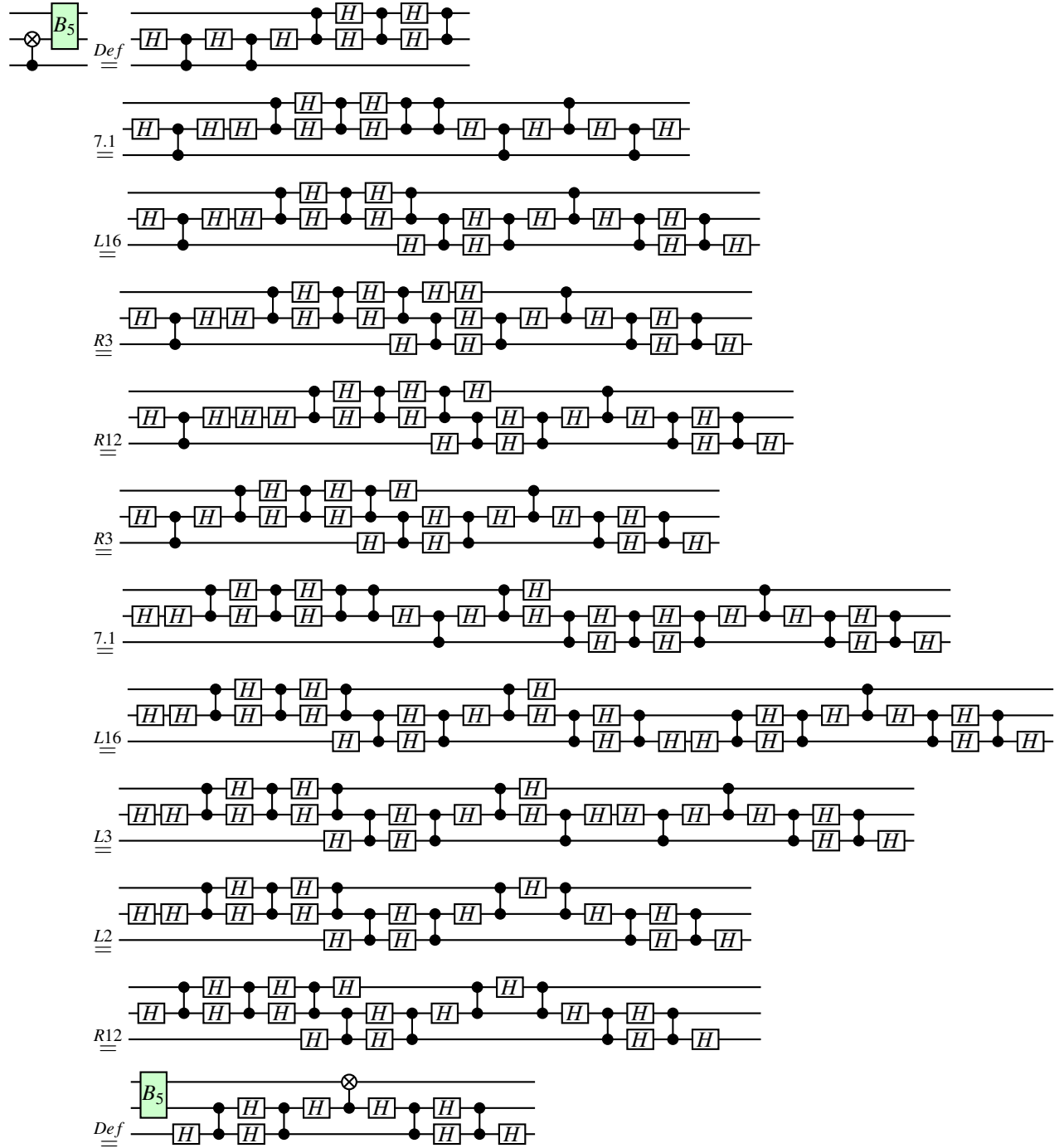


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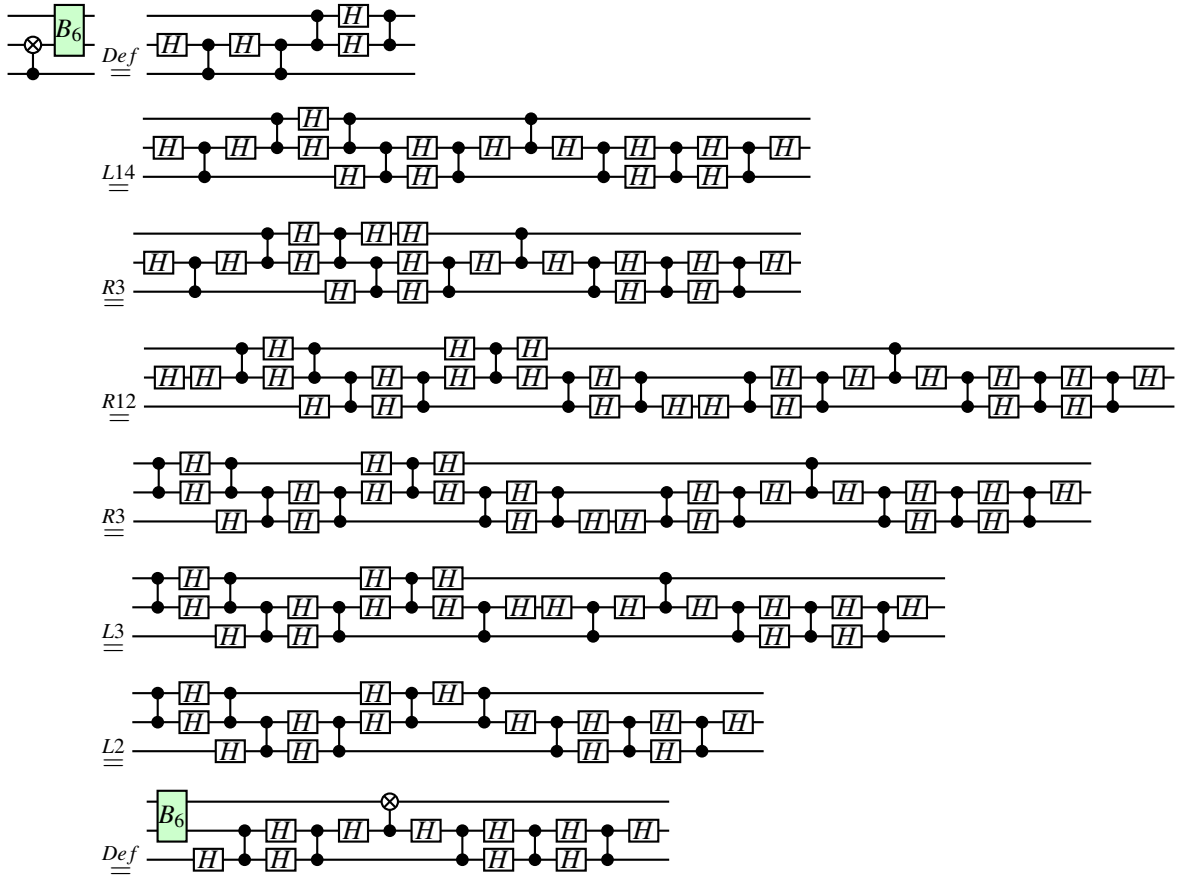




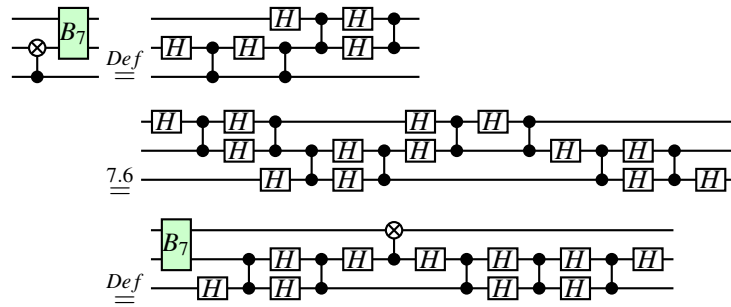
7.5



7.6



7.7



7.8

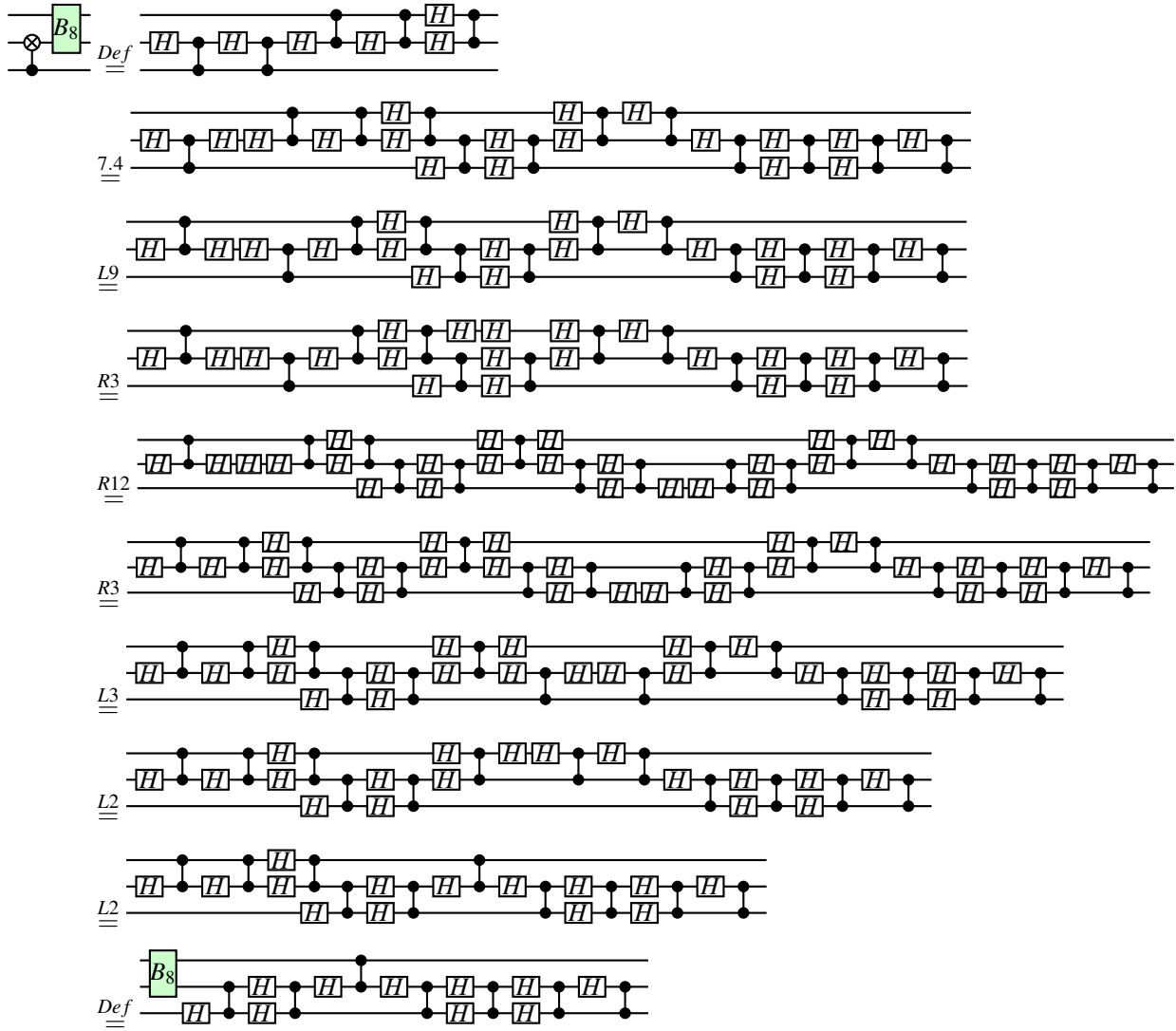
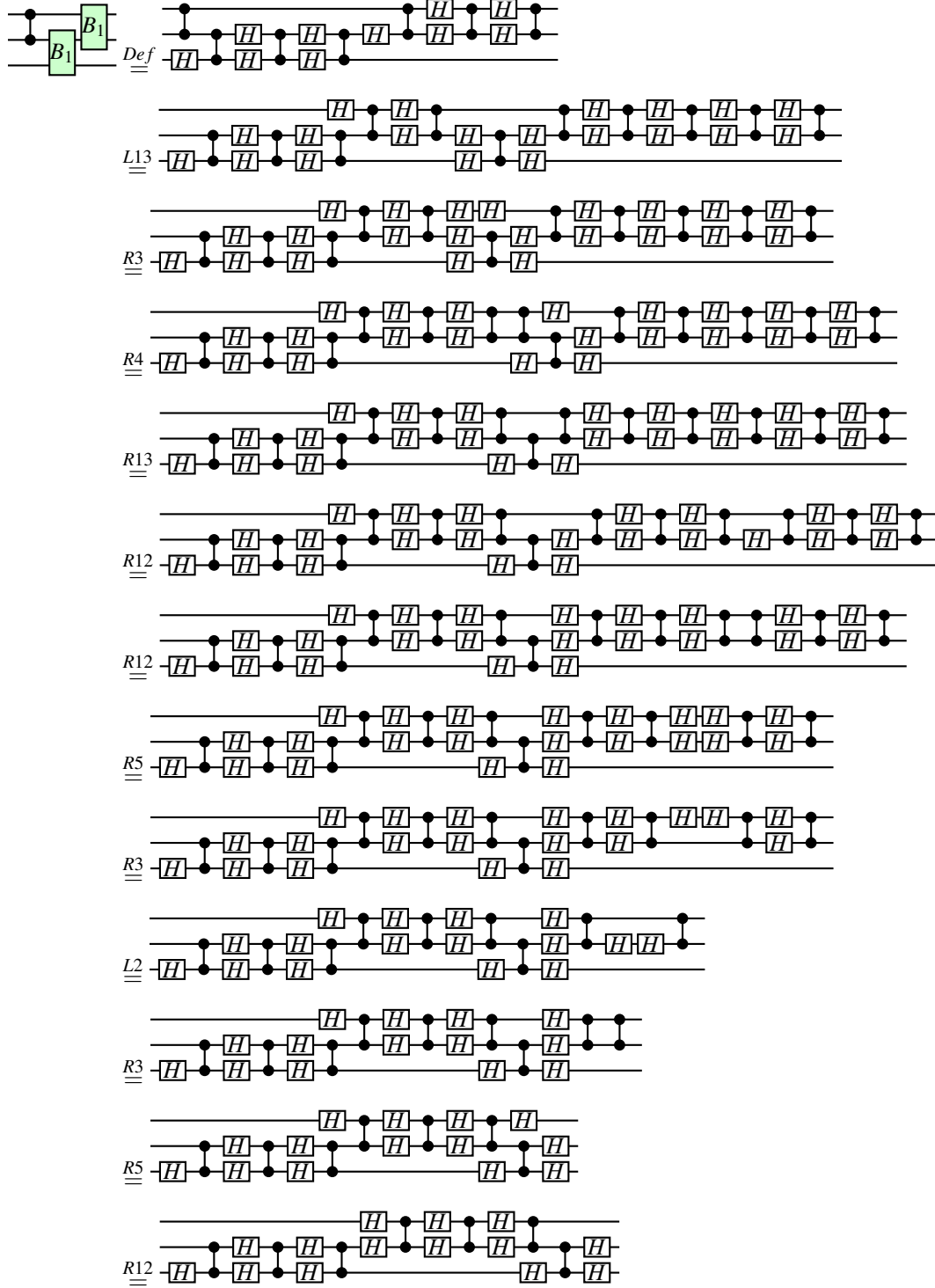
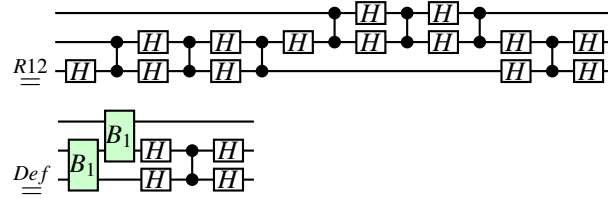
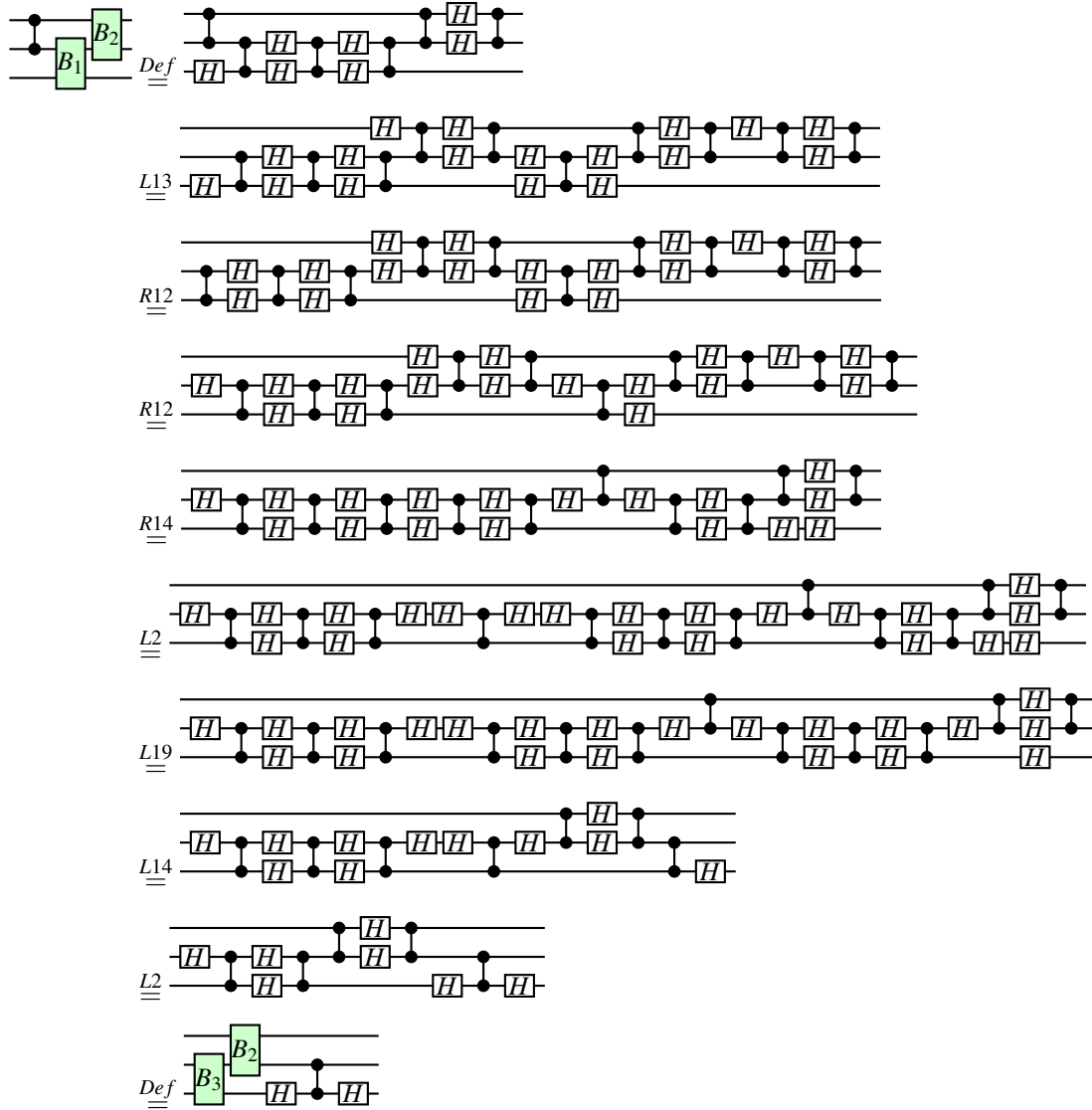


Figure 8:
8.1

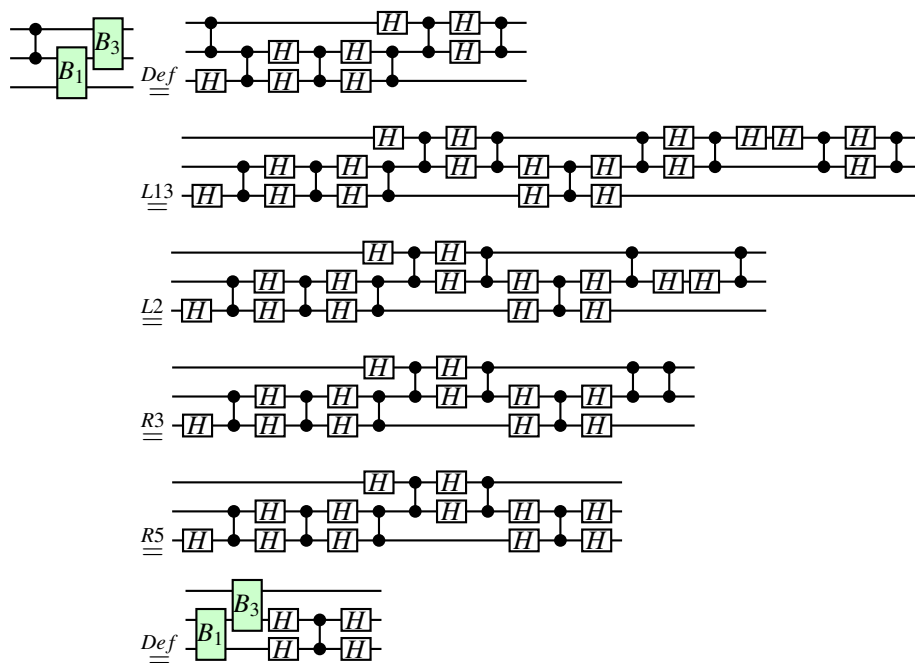




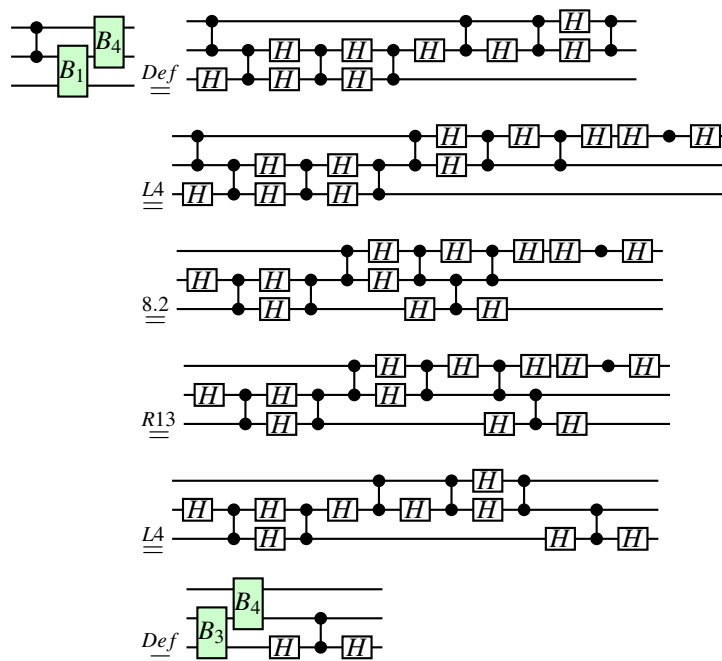
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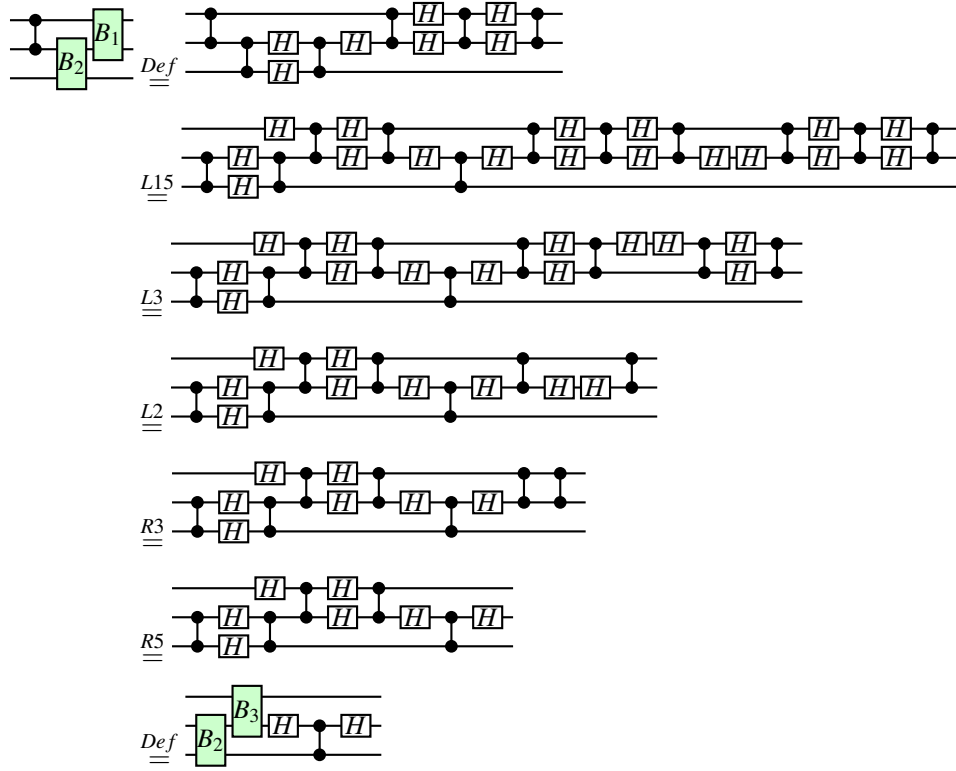
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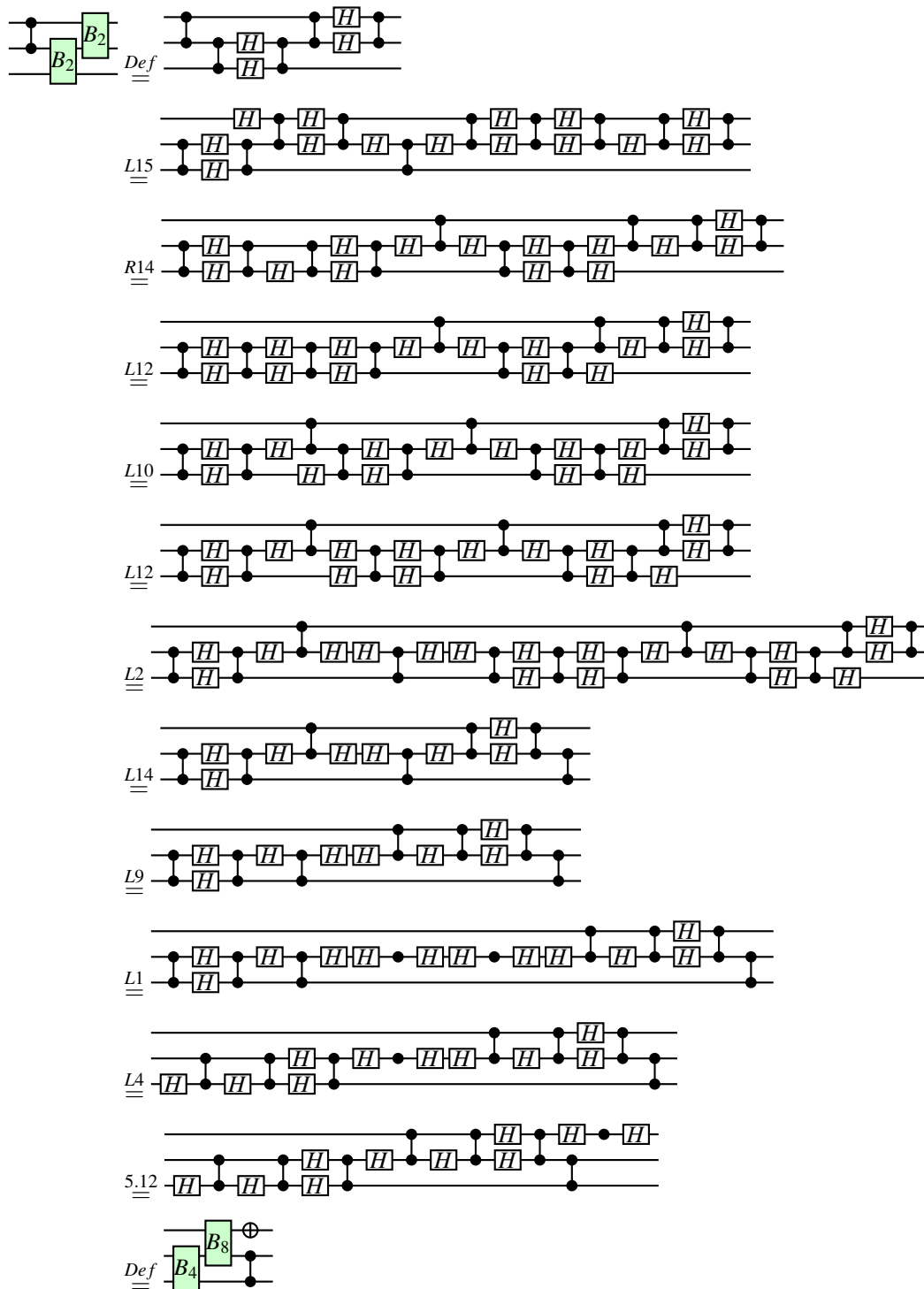
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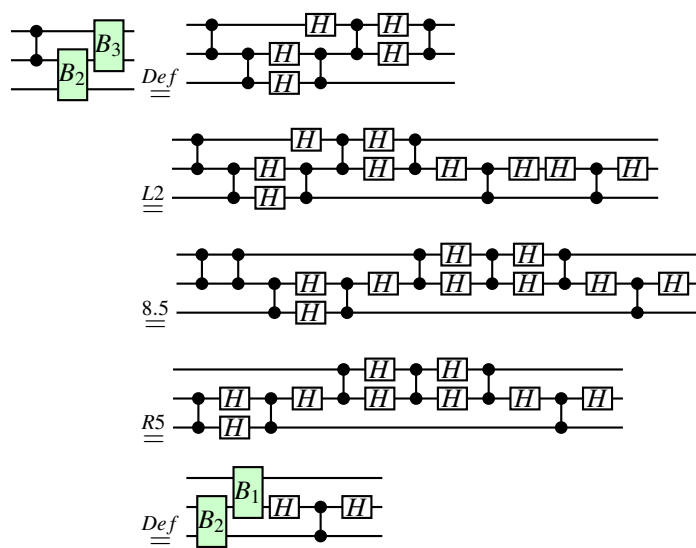
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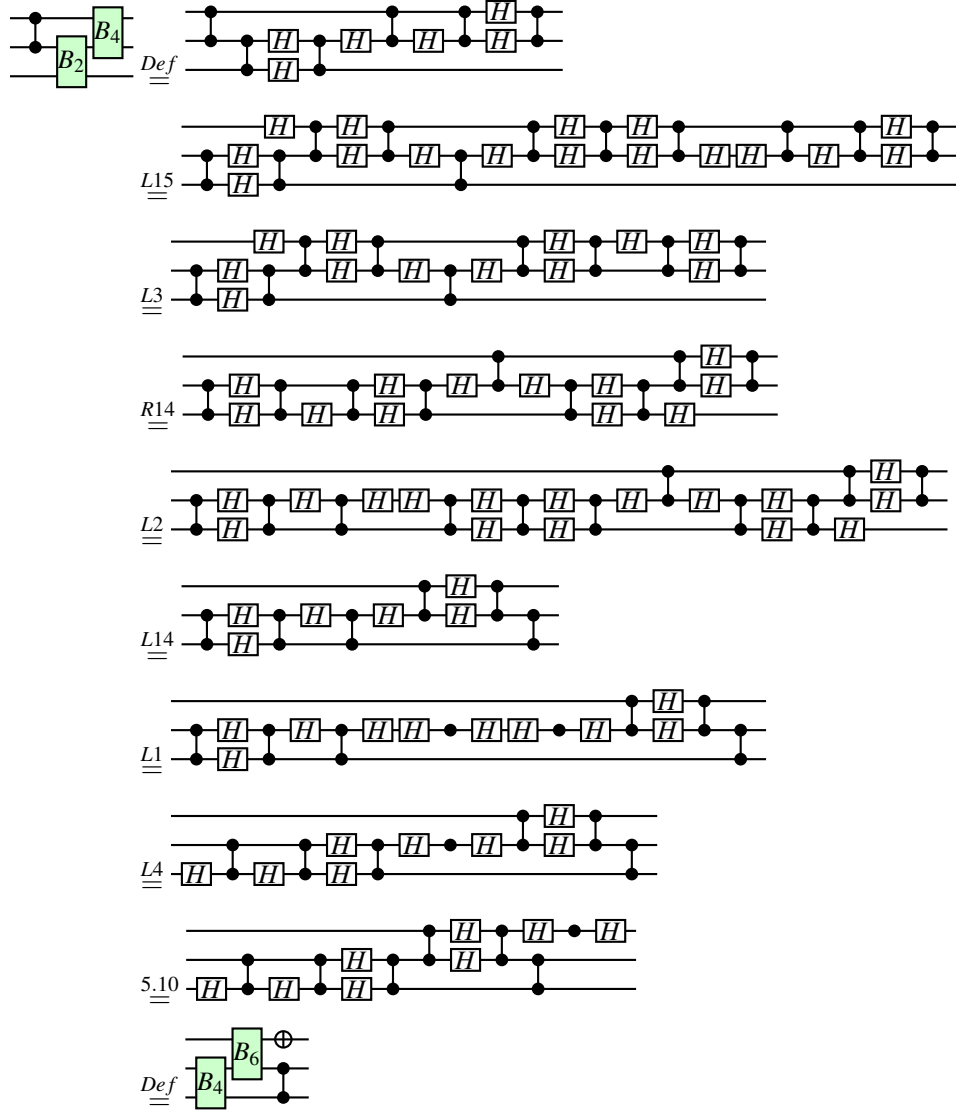
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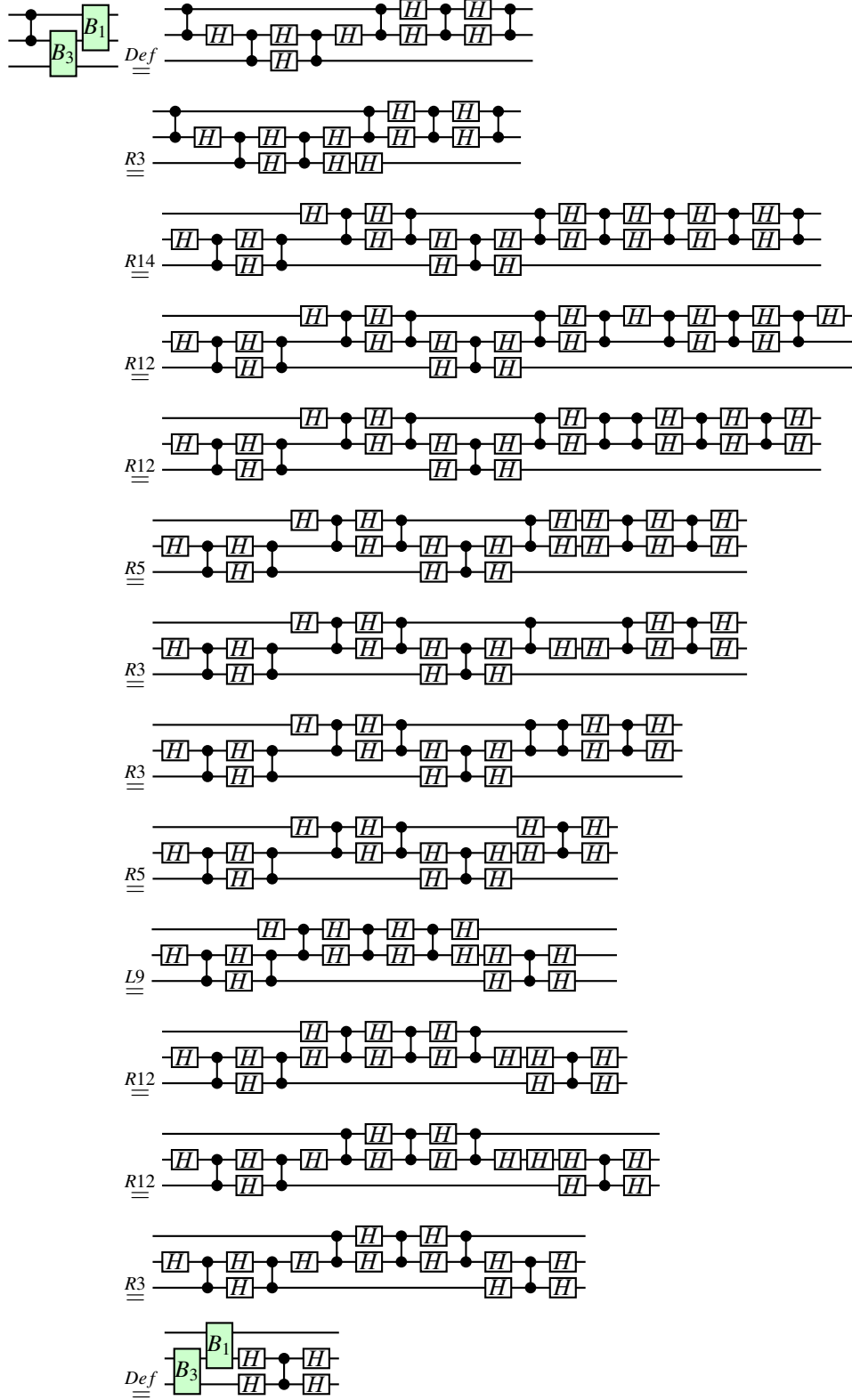
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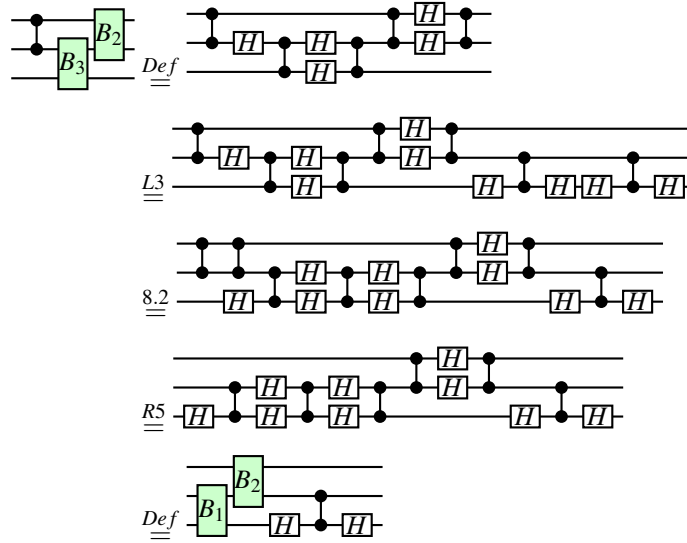
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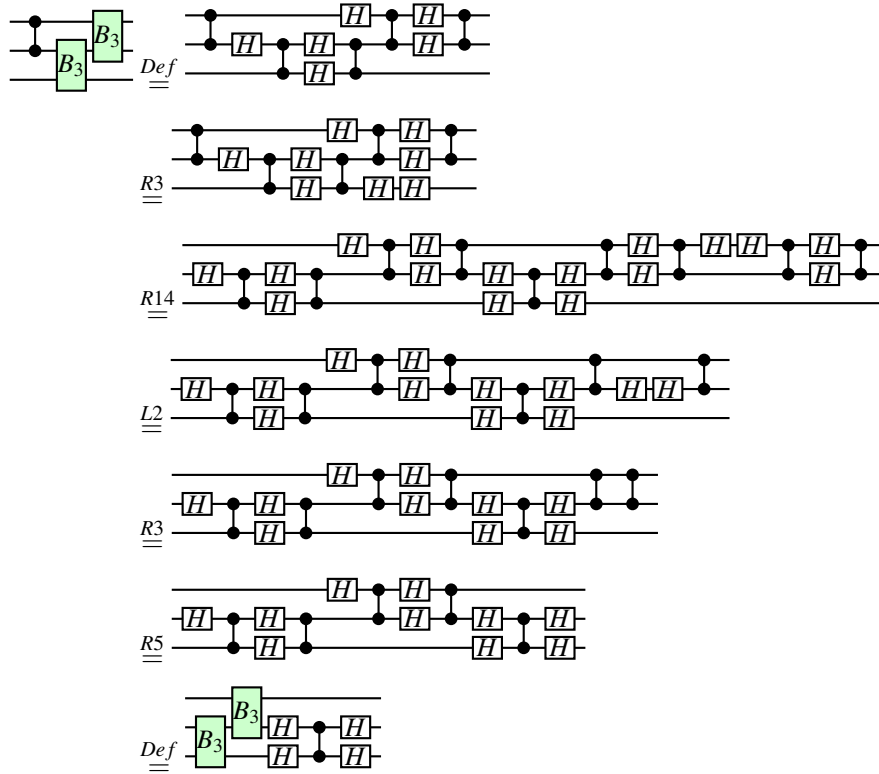
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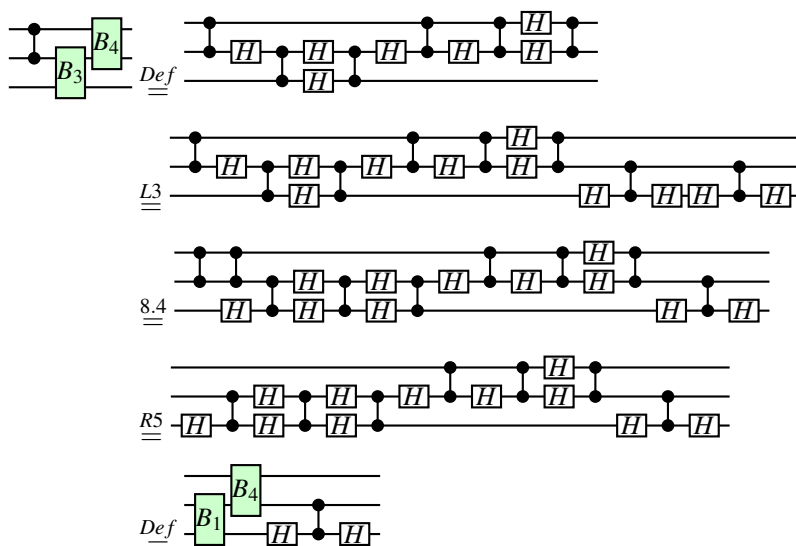
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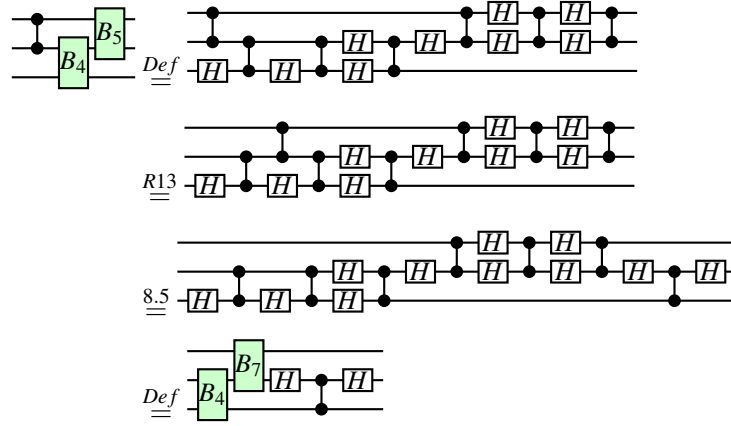
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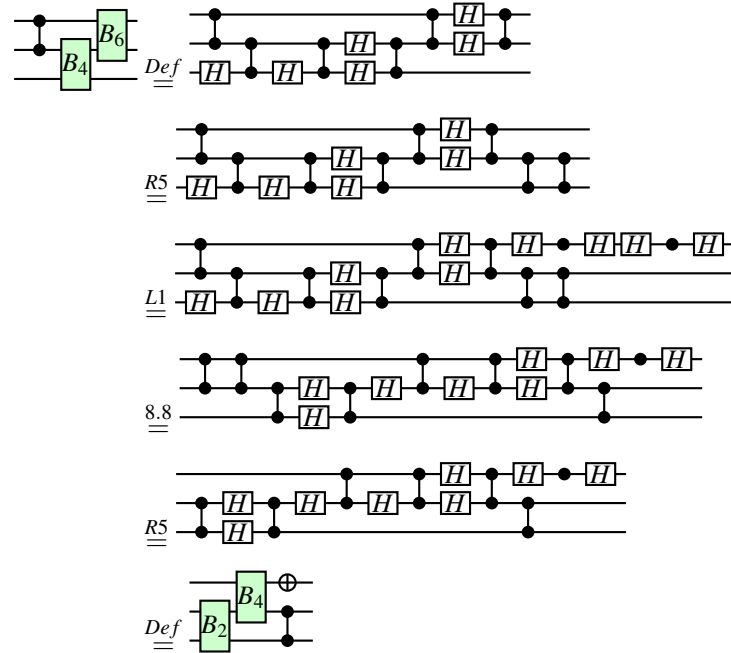
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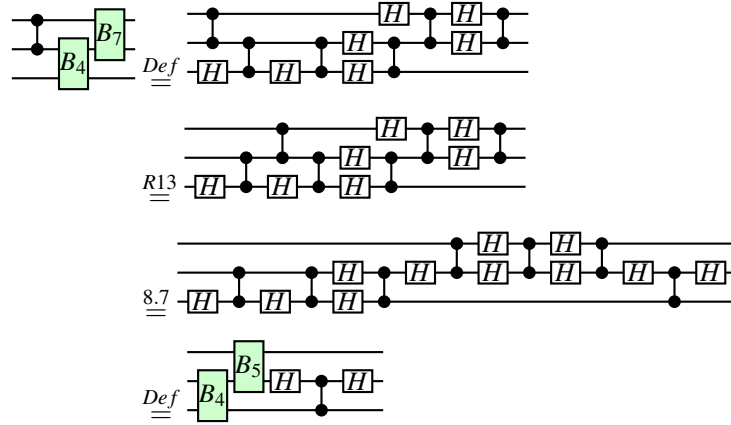
8.13



8.14



8.15



8.16

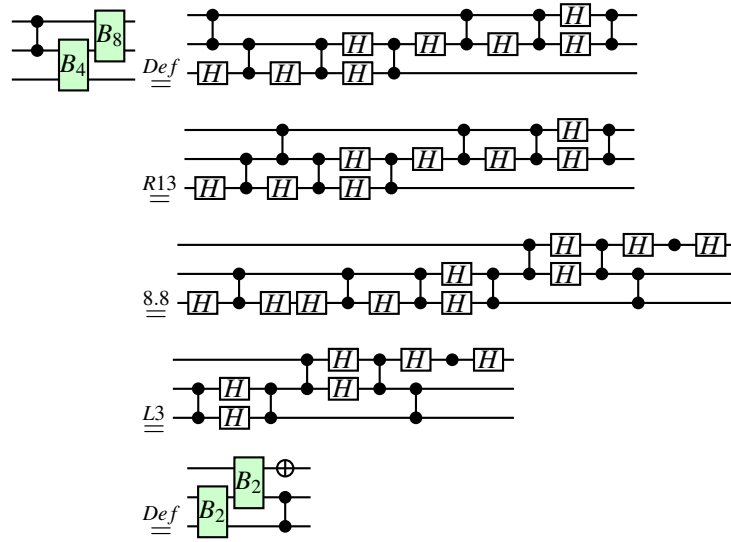
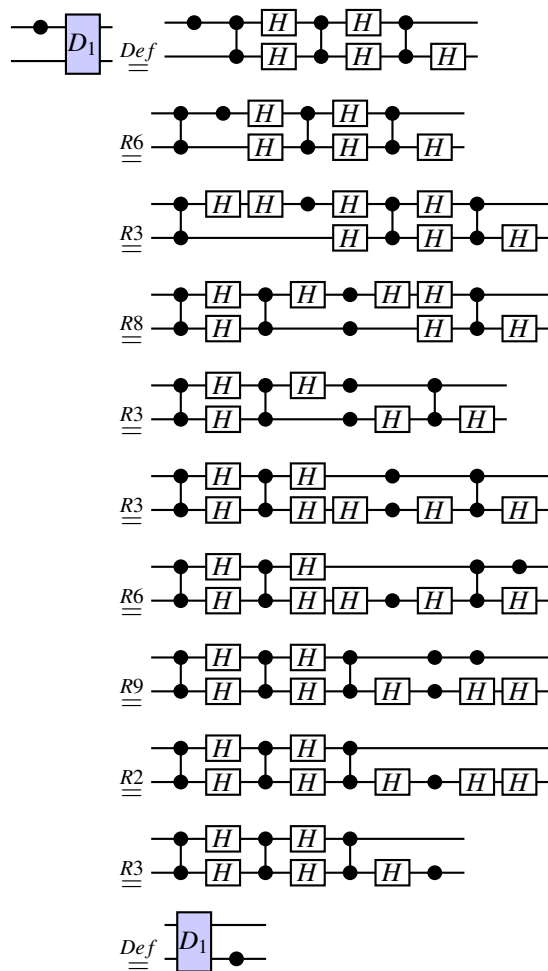
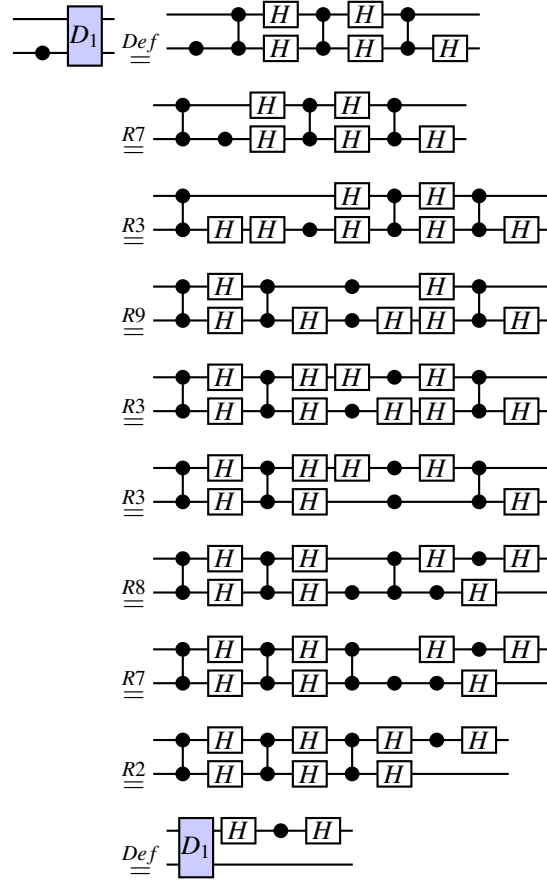


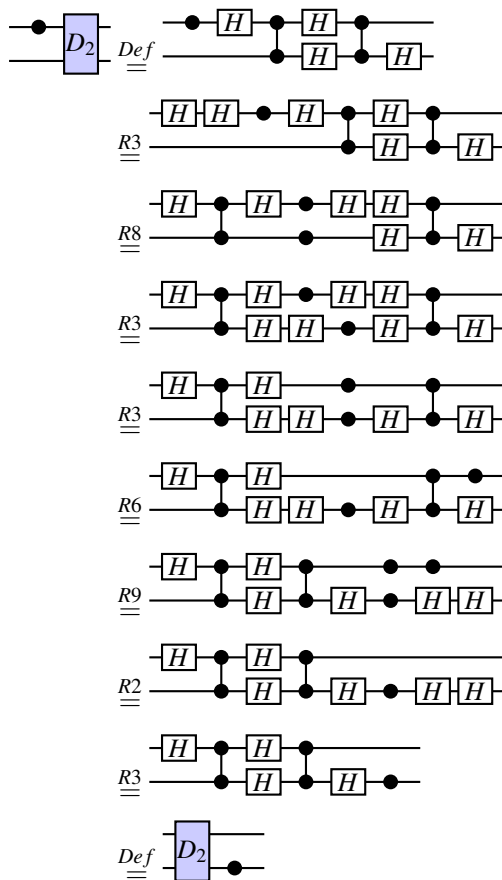
Figure 10:
10.1



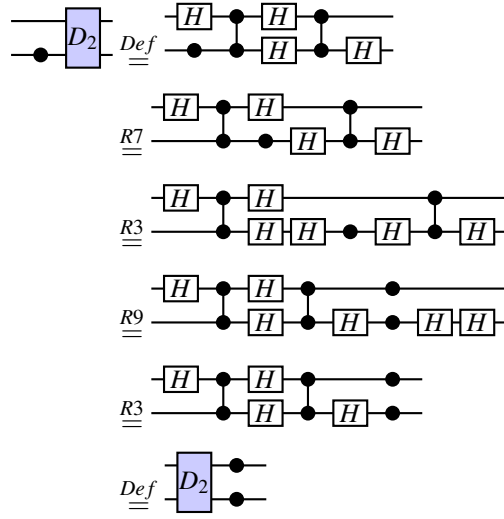
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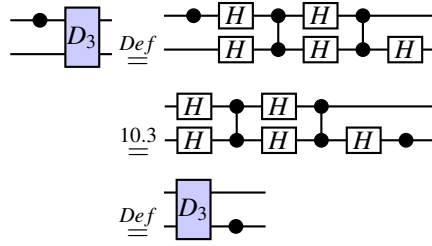
10.3



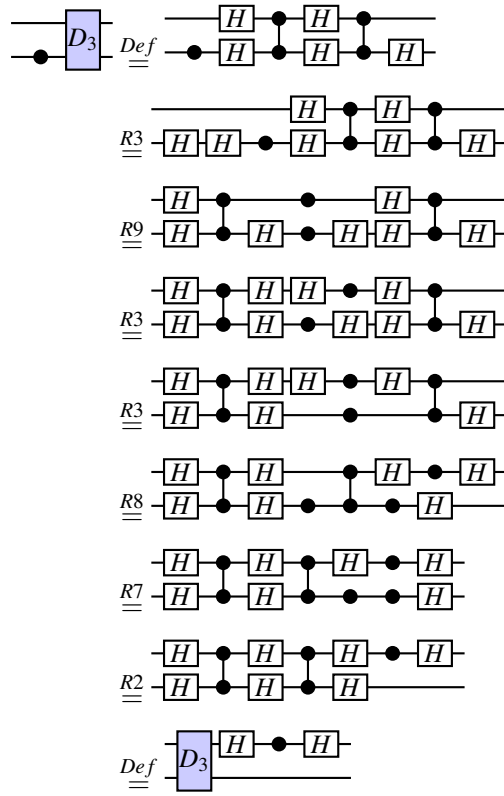
10.4



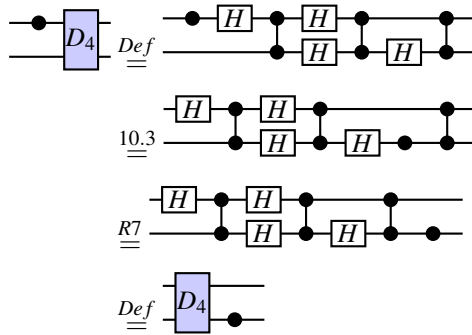
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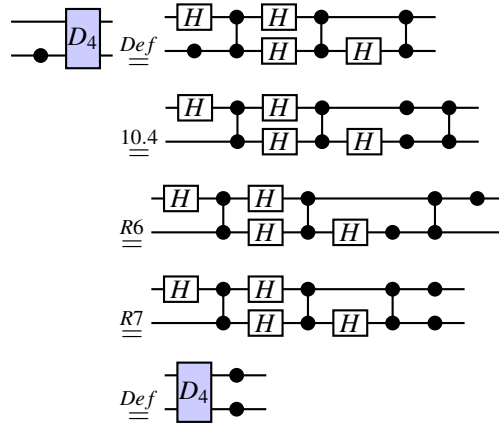
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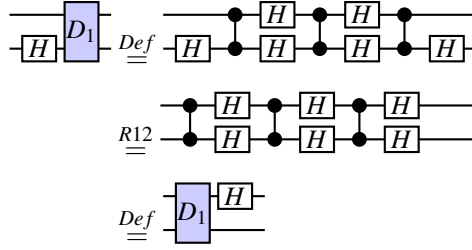
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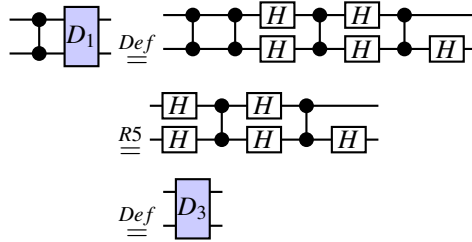
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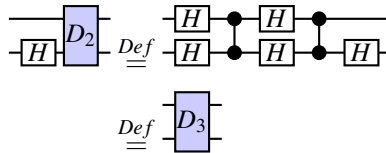
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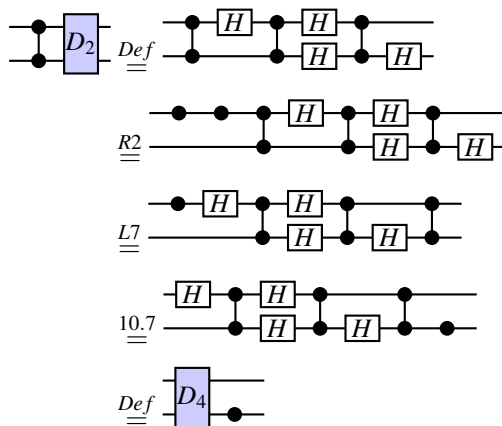
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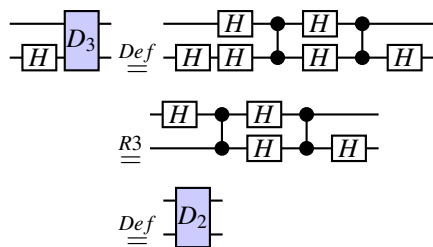
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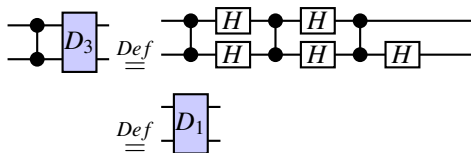
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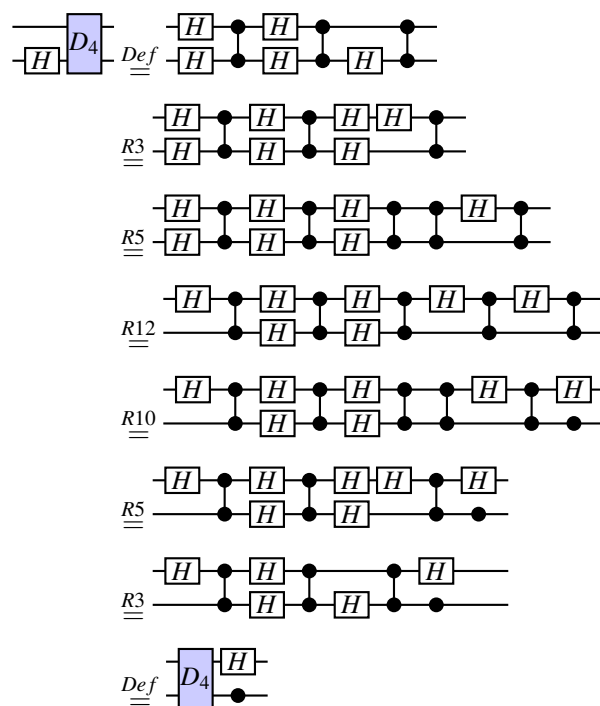
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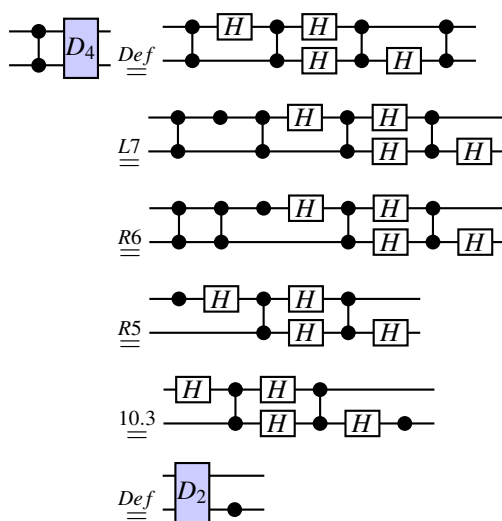
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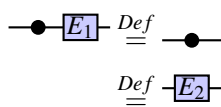
10.15



10.16



10.17



10.18

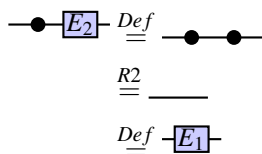
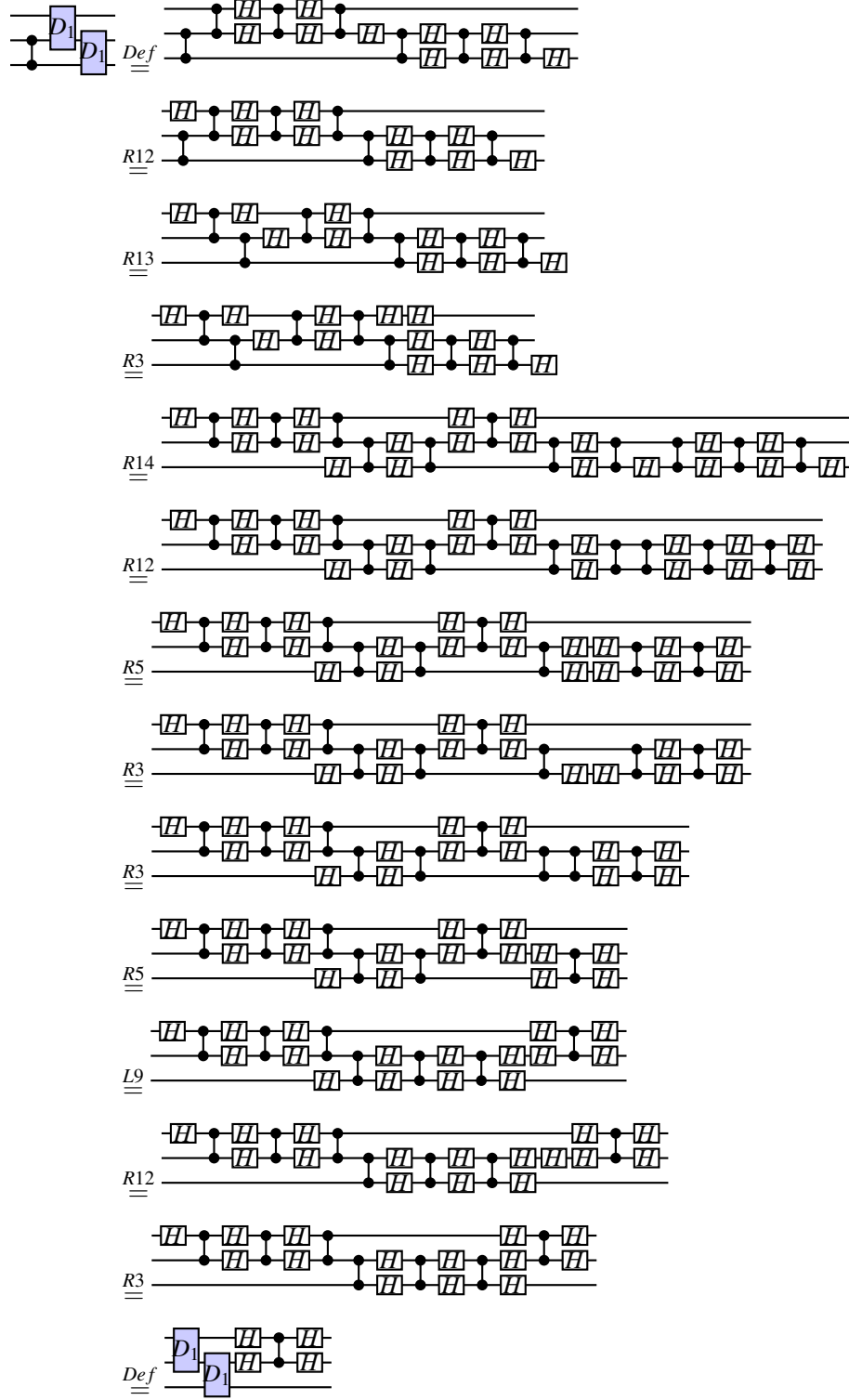
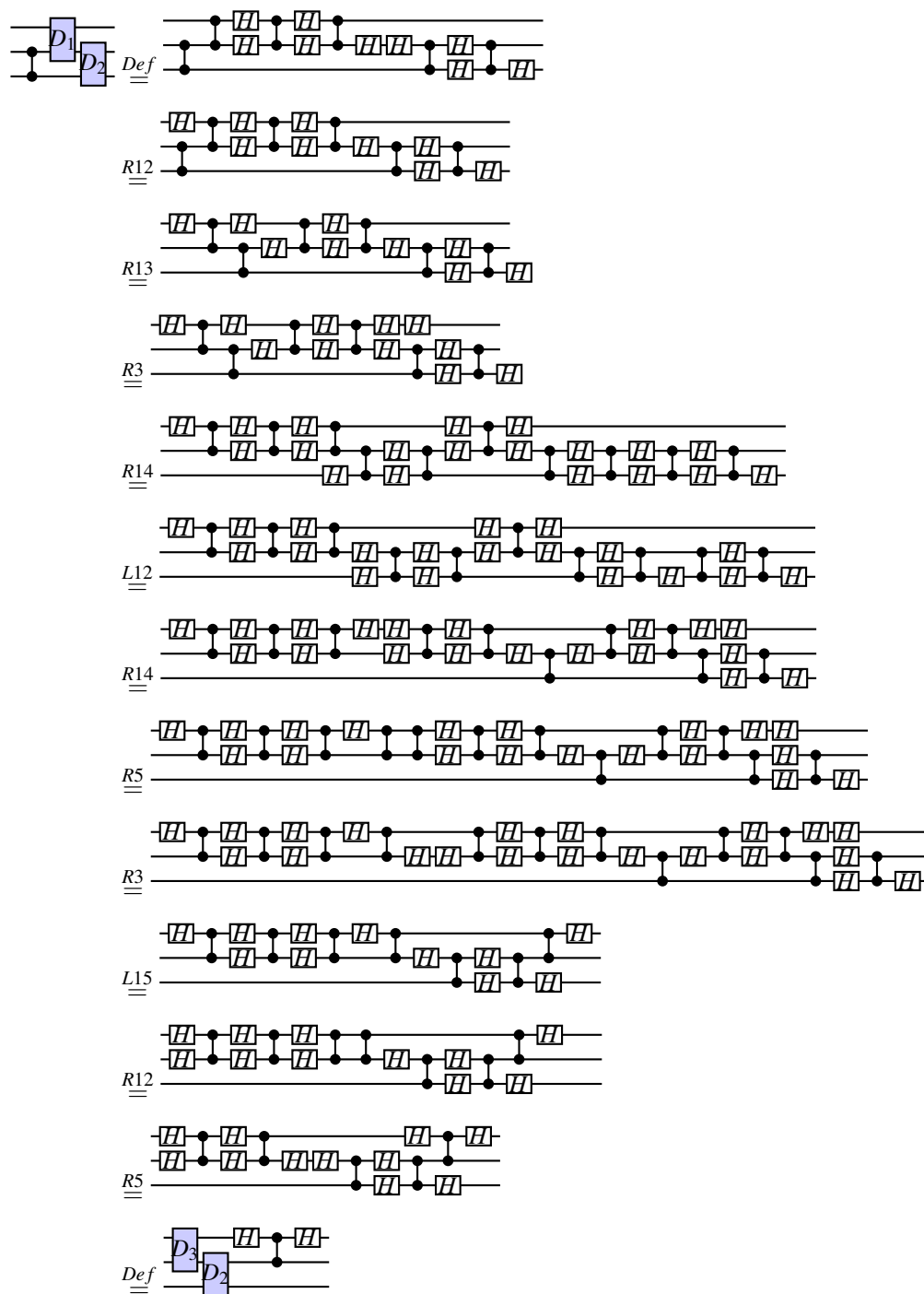


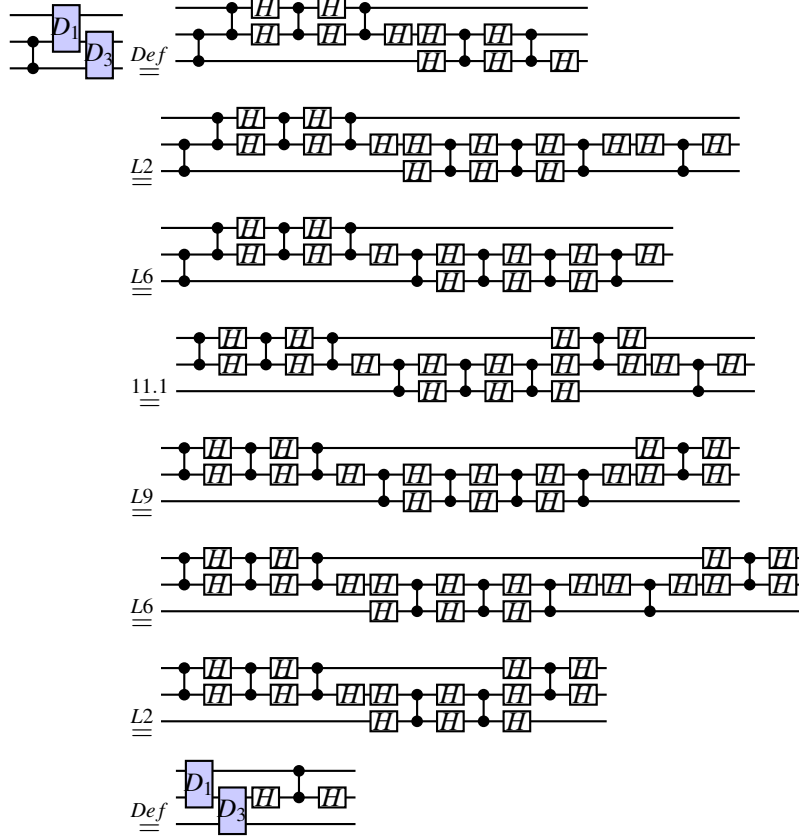
Figure 11:
11.1



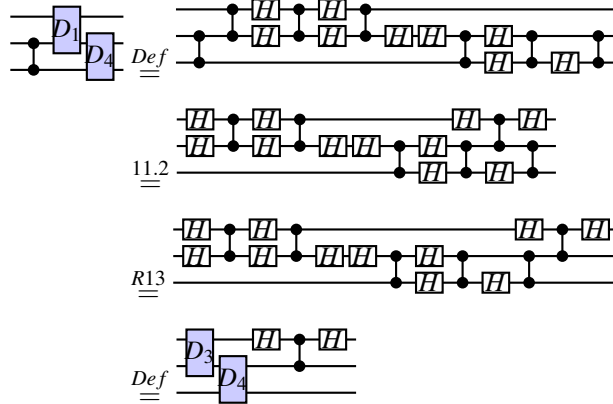
11.2



11.3



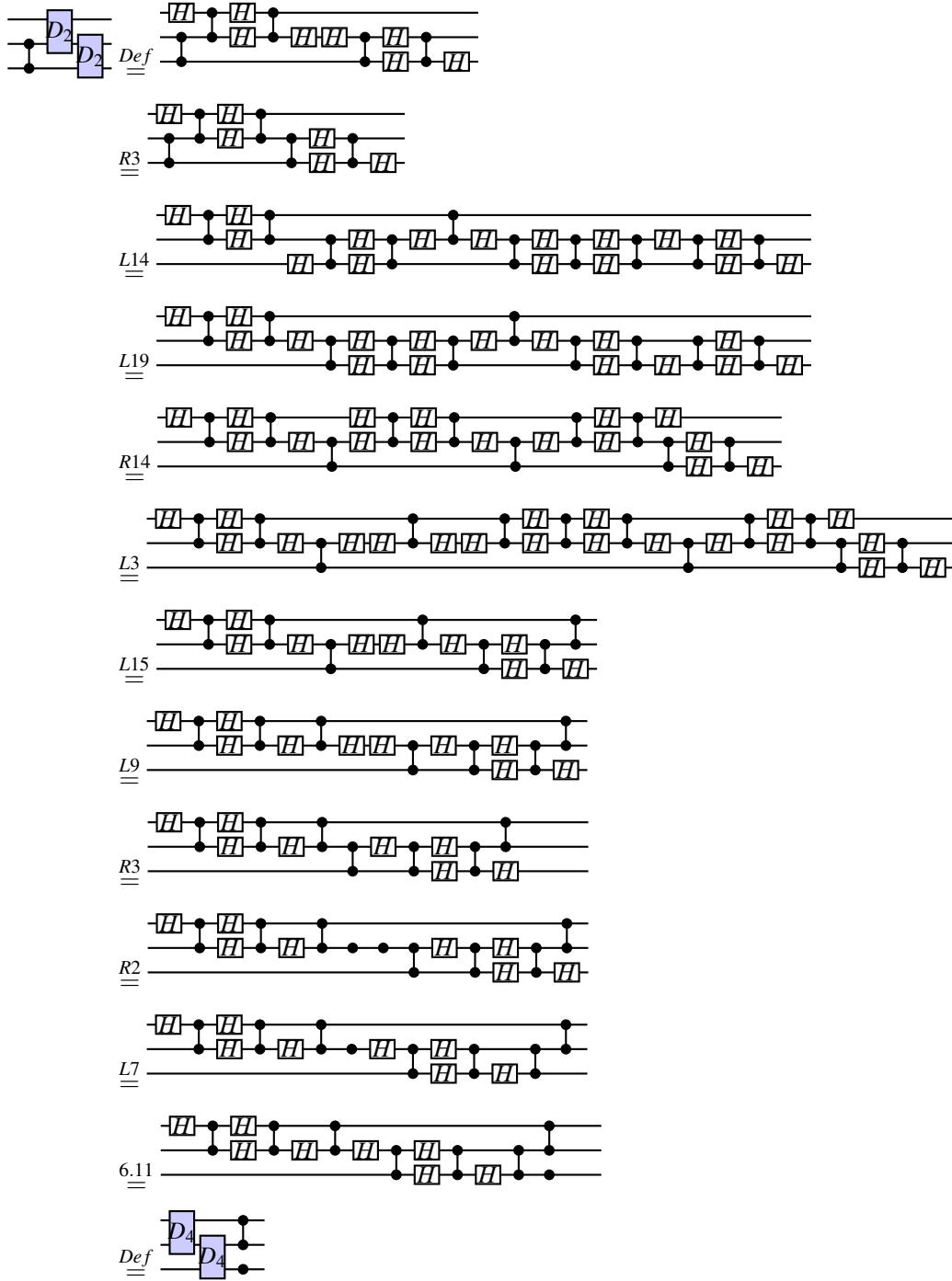
11.4



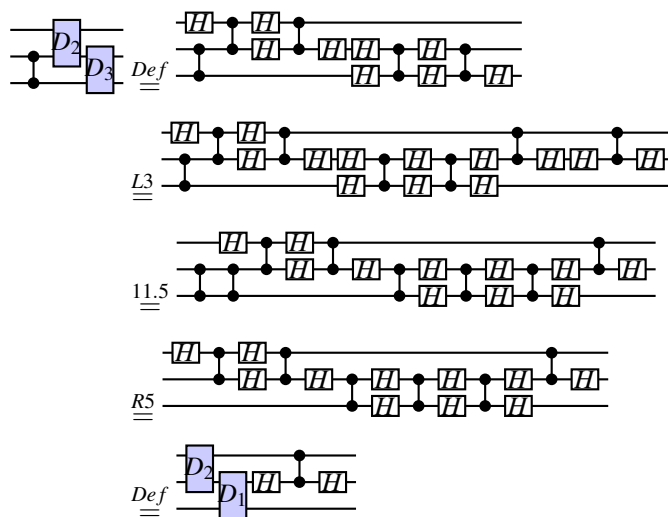
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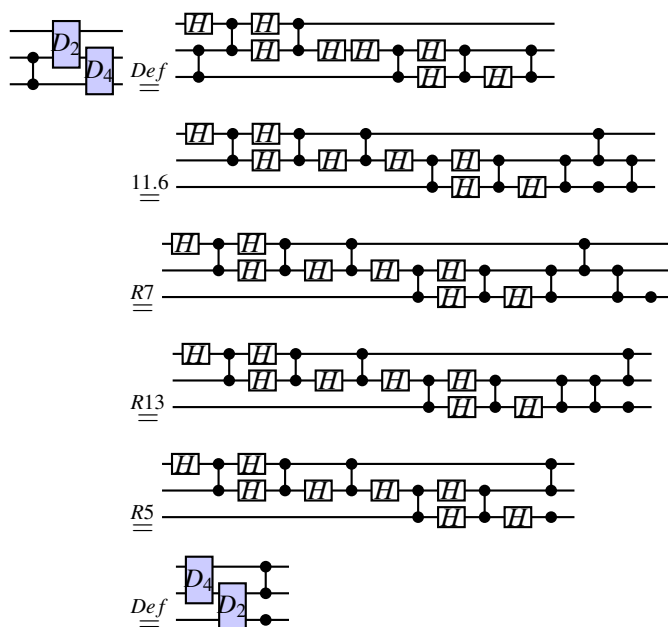
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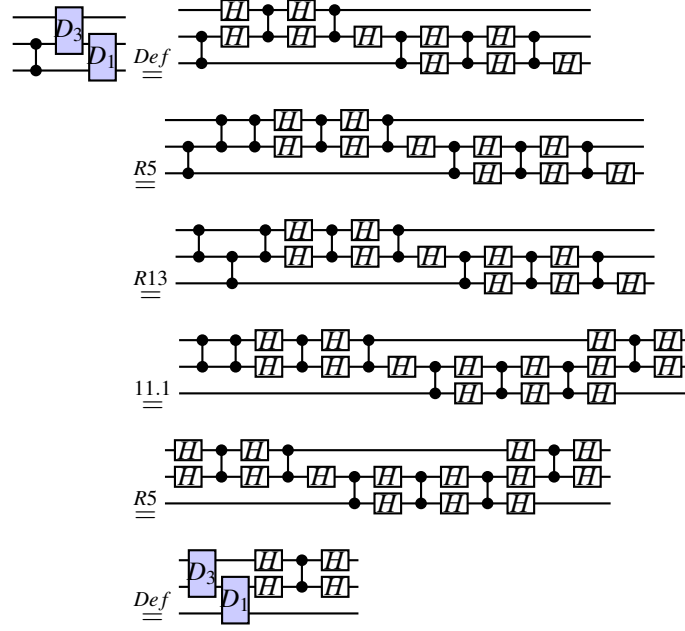
11.7



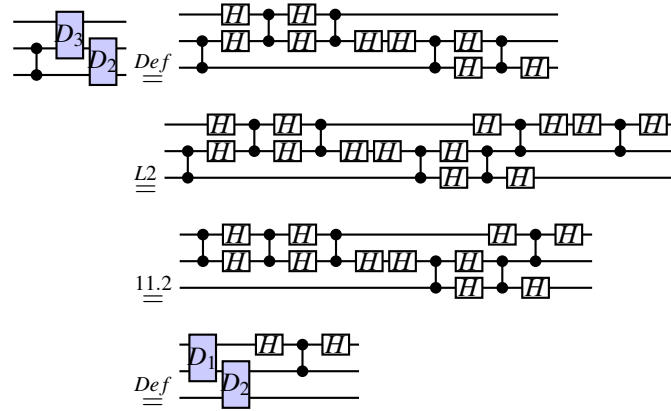
11.8



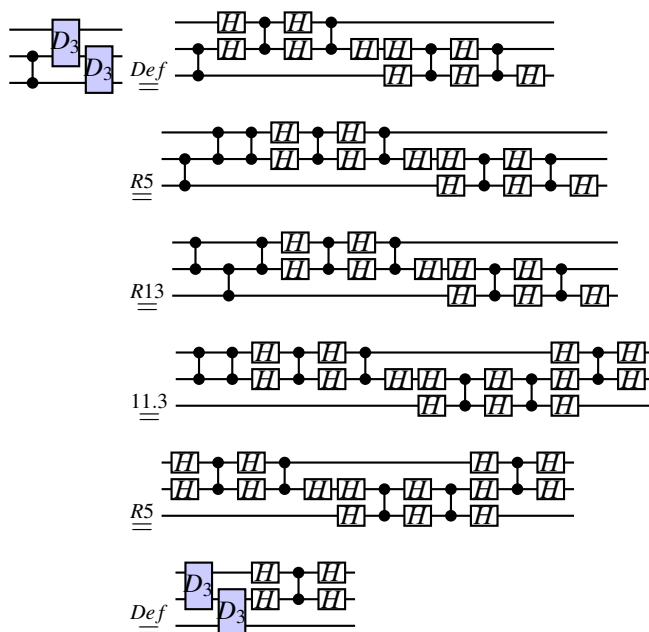
11.9



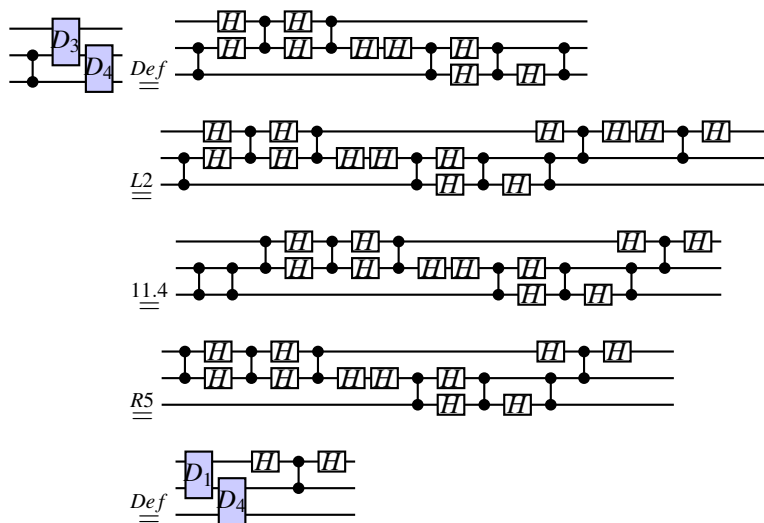
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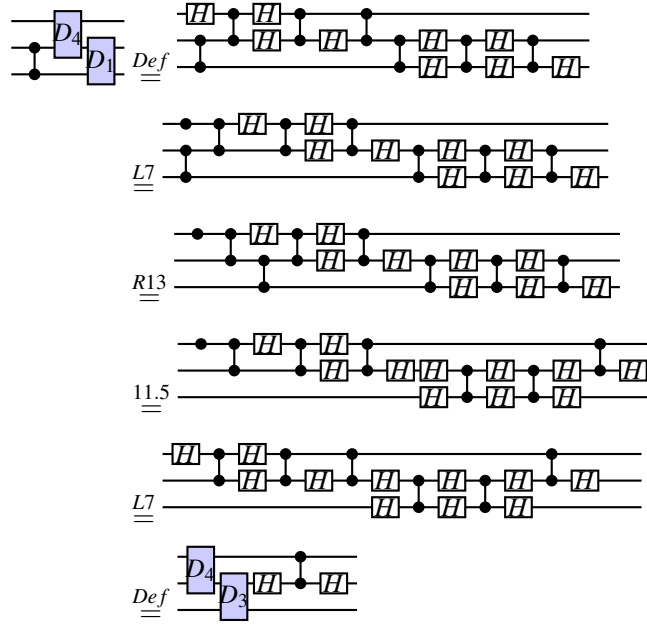
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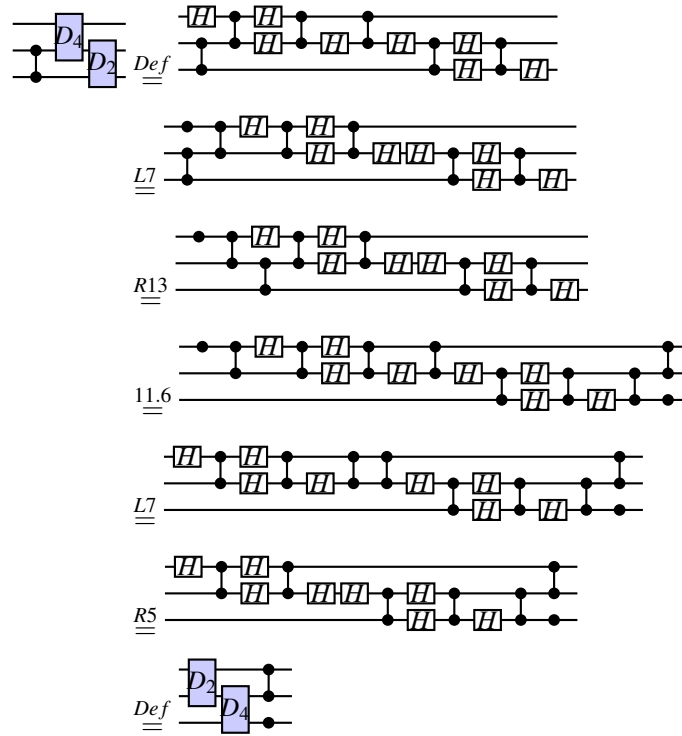
11.12



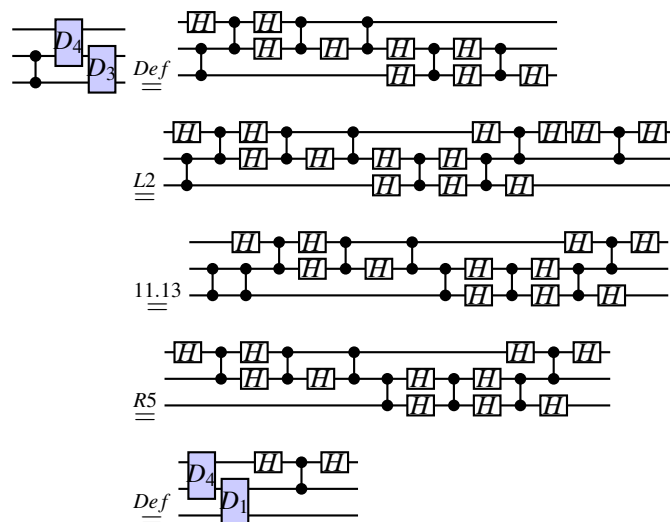
11.13



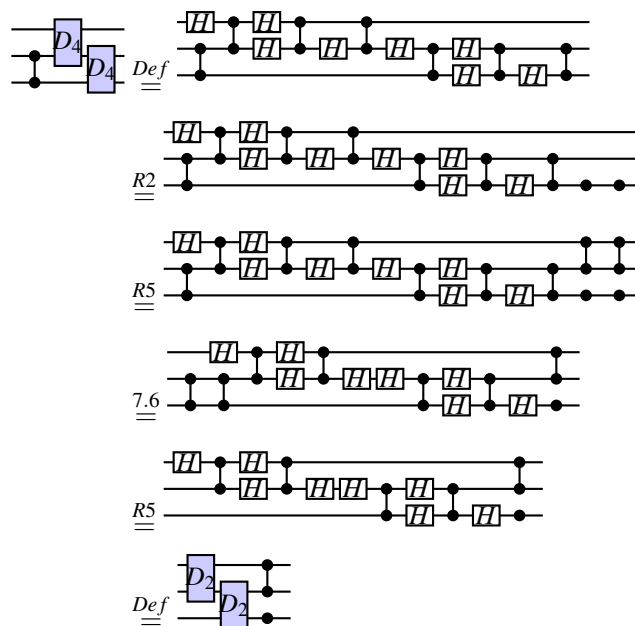
11.14



11.15



11.16

**Figure 12:**

12.1

$$-A_1-C_1-E_1- \stackrel{Def}{=} \underline{\hspace{1cm}}$$

The figure displays a sequence of quantum circuit diagrams. The top diagram, labeled 'Def' and 'L3', shows a circuit with two CNOT gates and four Hadamard gates. The subsequent diagrams, labeled 'L2', 'R3', and 'R5', show intermediate steps in the decomposition. The bottom diagram shows the final decomposition into three blocks: B_1 (green), C_1 (green), and D_1 (blue), labeled 'Def'.