Problem 5 (11.3)

Find the fixed probability vector w for each of the following regular matrices:

a) $P = \begin{pmatrix} .75 & .25 \\ .5 & .5 \end{pmatrix}$

We know that $w = wP$

$$\begin{cases} w_1 + w_2 = 1 \\ (w_1 \; w_2)\begin{pmatrix} .75 & .25 \\ .5 & .5 \end{pmatrix} = (w_1 \; w_2) \end{cases}$$

$$\begin{cases} w_1 + w_2 = 1 \\ \frac{3}{4} w_1 + \frac{1}{2} w_2 = w_1 \\ \frac{1}{4} w_1 + \frac{1}{2} w_2 = w_2 \end{cases}$$

$\Rightarrow w_1 = 1 - w_2$ (1)

$\Rightarrow w_1\left(\frac{3}{4} - 1\right) + \frac{1}{2} w_2 = 0$ (2)

(1) into (2)

$$(1 - w_2)\left(-\frac{1}{4}\right) + \frac{1}{2} w_2 = 0$$

$$-\frac{1}{4} + \frac{1}{4} w_2 + \frac{1}{2} w_2 = 0$$

$$-\frac{1}{4} + \frac{3}{4} w_2 = 0$$

$$w_2 = \frac{1}{4}(4) = \frac{1}{3}$$

$w_1 = 1 - \frac{1}{3}$

$w_1 = \frac{2}{3}$

$$w = \left(\frac{2}{3}, \frac{1}{3}\right)$$

b) $P = \begin{pmatrix} .9 & .1 \\ .1 & .9 \end{pmatrix}$

We know that $\omega = \omega P$

$\omega_1 + \omega_2 = 1$

$(\omega_1 \ \omega_2)\begin{pmatrix} .9 & .1 \\ .1 & .9 \end{pmatrix} = (\omega_1 \ \omega_2)$

$\begin{cases} \omega_1 + \omega_2 = 1 \\ \frac{9}{10}\omega_1 + \frac{1}{10}\omega_2 = \omega_1 \\ \frac{1}{10}\omega_1 + \frac{9}{10}\omega_2 = \omega_2 \end{cases}$

$\Rightarrow \omega_1 = 1 - \omega_2$ (1)

$\Rightarrow -\frac{1}{10}\omega_1 + \frac{1}{10}\omega_2 = 0$ (2)

(1) in (2)

$-\frac{1}{10}(1-\omega_2) + \frac{1}{10}\omega_2 = 0$

$-\frac{1}{10} + \frac{\omega_2}{10} + \frac{\omega_2}{10} = 0$

$-\frac{1}{10} + \frac{1}{5}\omega_2 = 0$

$\begin{cases} \omega_2 = \left(\frac{1}{10}\right)(5) = \frac{1}{2} \\ \omega_1 = \frac{1}{2} \end{cases}$

$\omega = \left(\frac{1}{2}, \frac{1}{2}\right)$

c) $P = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 0 & 2/3 & 1/3 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$

We know that $\omega = \omega P$

$\omega_1 + \omega_2 + \omega_3 = 1$

$(\omega_1 \quad \omega_2 \quad \omega_3) \begin{pmatrix} 3/4 & 1/4 & 0 \\ 0 & 2/3 & 1/3 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} = (\omega_1 \quad \omega_2 \quad \omega_3)$

$\begin{cases} \omega_1 + \omega_2 + \omega_3 = 1 \\ \frac{3}{4}\omega_1 + \frac{1}{4}\omega_2 = \omega_1 \\ \frac{1}{4}\omega_1 + \frac{2}{3}\omega_2 + \frac{1}{4}\omega_3 = \omega_2 \\ \frac{1}{3}\omega_2 + \frac{1}{2}\omega_3 = \omega_3 \end{cases}$

$\Rightarrow \omega_3 = 1 - \omega_2 - \omega_1$ (1)

$\Rightarrow -\frac{1}{4}\omega_1 + \frac{1}{4}\omega_3 = 0 \Rightarrow \omega_1 = \omega_3$ (2)

$\Rightarrow \frac{1}{4}\omega_1 - \frac{1}{3}\omega_2 + \frac{1}{4}\omega_3 = 0$ (4)

$\Rightarrow \frac{1}{3}\omega_2 - \frac{1}{2}\omega_3 = 0$ (3)

(1) in (3) $\frac{1}{3}\omega_2 - \frac{1}{2}(1 - \omega_2 - \omega_1) = 0$

$\frac{1}{3}\omega_2 - \frac{1}{2} + \frac{1}{2}\omega_2 + \frac{1}{2}\omega_1 = 0$

$-\frac{1}{2} + \frac{1}{2}\omega_1 + \frac{5}{6}\omega_2 = 0$

$\omega_2 = \frac{6}{5}\left(\frac{1}{2} - \frac{1}{2}\omega_1\right)$ (5)

(4) $\frac{1}{4}\omega_1 - \frac{1}{3}\omega_2 + \frac{1}{4}\omega_3 = 0$

(2) $\omega_1 = \omega_3$

$\hookrightarrow \frac{1}{2}\omega_1 - \frac{1}{3}\omega_2 = 0$

(5) $\frac{1}{2}\omega_1 - \frac{1}{3}\cdot\frac{6}{5}\left(\frac{1}{2} - \frac{1}{2}\omega_1\right) = 0$

$\frac{1}{2}\omega_1 - \frac{1}{5} + \frac{1}{5}\omega_1 = 0$

$\frac{5\omega_1}{10} + \frac{2}{10}\omega_1 = \frac{1}{5}$

$\frac{7}{10}\omega_1 = \frac{1}{5}$

$\omega_1 = \frac{1}{5}\cdot\frac{10}{7} = \frac{2}{7}$

$\omega_1 = \omega_3 = \frac{2}{7}$

$\omega_2 = 1 - \frac{2}{7} - \frac{2}{7} = \frac{3}{7}$

$\omega = \left(\frac{2}{7}, \frac{3}{7}, \frac{2}{7}\right)$