Summary Table of Contents

Table of Contents

Annex F

This annex contains detailed proofs of some of the results utilized in the Methods section.

F-I- Additivity of Values at Current Prices

Note: The variable definitions and notations from Subsection A-III, "Treatment of Data Category 1: GDP and Final Expenditure Components," are maintained in this subsection (see A-III-2.2, 'Variable Definitions and Notations').

To show that values at current prices are additive, let's prove that the annual/quarterly value at current prices of aggregate A is equal to the sum of the annual/quarterly values at current prices of its disjoint subcomponents $\{i\}_{1 \le i \le m}$.

The values at current prices of component i are given by,

$$\forall t, \qquad C_i^t = \sum_{(i,j) \in R_i} \sum_{p=1}^{Tr_{i,j}^t} P_{i,j,p}^t * U_{i,j,p}^t$$

We multiply and divide the terms of the outer summation by the corresponding $U_{i,j}^t = \sum_{p=1}^{Tr_{i,j}^t} U_{i,j,p}^t$, and thus,

$$\forall t, \qquad C_i^t = \sum_{(i,j) \in R_i} \bar{P}_{i,j}^t * U_{i,j}^t$$

Where we defined,

$$\bar{P}_{i,j}^t = \frac{\sum_{p=1}^{Tr_{i,j}^t} P_{i,j,p}^t * U_{i,j,p}^t}{\sum_{p=1}^{Tr_{i,j}^t} U_{i,j,p}^t}$$

For aggregate A, we have that, $R_A = \bigcup_{i=1}^{m} R_i$. Therefore,

$$\forall t, \qquad C_A^t = \sum_{(A,j)\in \bigcup_{i=1}^{m} R_i} \bar{P}_{A,j}^t * U_{A,j}^t$$

Since the $\{R_i\}_{1\le i\le m}$ are disjoint,

$$\forall t, \qquad C_A^t = \sum_{i=1}^{m} \sum_{(i,j)\in R_i} \bar{P}_{i,j}^t * U_{i,j}^t$$

Hence,

$$\forall t, \qquad C_A^t = \sum_{i=1}^{m} C_i^t$$

Consequently, we showed that values at current prices are additive.

F-II- Aggregation of Laspeyres Volume Indices from Previous Year

Notes:

- The variable definitions and notations from Subsection A-III, "Treatment of Data Category 1: GDP and Final Expenditure Components," are maintained in this subsection (see A-III-2.2, 'Variable Definitions and Notations').
- By "additivity" we refer to cross-item additivity.
- All series are supposed to have positive elements.

F-II-1. Annual Series

We want to calculate the annual Laspeyres volume indices from previous year for A, the aggregate of the disjoint subcomponents $\{i\}_{1\le i\le m}$, from year 1 to year y_{end}, $\left(Q_A^{y-1\to y}\right)_{1\le y\le y_{end}}$.

The Laspeyres volume index from previous year for component i in year y is given by,

$$Q_i^{y-1\to y} = \frac{K_i^{y-1\to y}}{C_i^{y-1}}$$

Where $K_i^{y-1\to y}$ is the volume measure at previous year prices for component i in year y. It is defined by,

$$K_i^{y-1\to y} = \sum_{(i,j)\in R_i} \bar{P}_{i,j}^{y-1} * U_{i,j}^{y}$$

Where,

$$\bar{P}_{i,j}^{y-1} = \frac{\sum_{p=1}^{Tr_{i,j}^{y-1}} P_{i,j,p}^{y-1} * U_{i,j,p}^{y-1}}{\sum_{p=1}^{Tr_{i,j}^{y-1}} U_{i,j,p}^{y-1}}$$

a) Additivity of Volume Measures at Previous Year Prices

To show that annual volume measures at previous year prices are additive, let's prove that the annual volume measure at previous year prices of aggregate A is equal to the sum of the annual volume measures at previous year prices of its disjoint subcomponents $\{i\}_{1\le i\le m}$.

We have that $R_A = \bigcup_{i=1}^{m} R_i$. Therefore, the annual volume measure at previous year prices of aggregate A is given by,

$$\forall y, \qquad K_A^{y-1\to y} = \sum_{(A,j)\in \bigcup_{i=1}^{m} R_i} \bar{P}_{A,j}^{y-1} * U_{A,j}^{y}$$

Since the $\{R_i\}_{1\le i\le m}$ are disjoint,

$$\forall y, \qquad K_A^{y-1\to y} = \sum_{i=1}^{m} \sum_{(i,j)\in R_i} \bar{P}_{i,j}^{y-1} * U_{i,j}^{y}$$

Therefore,

$$\forall y, \qquad K_A^{y-1\to y} = \sum_{i=1}^{m} K_i^{y-1\to y}$$

Consequently, we showed that annual volume measures at previous year prices are additive.

b) Aggregation of Laspeyres Volume Indices from Previous Year

Let's derive the aggregation equation for annual Laspeyres volume indices from previous year.

For aggregate A, we have that,

$$\forall y, \qquad Q_A^{y-1\to y} = \frac{K_A^{y-1\to y}}{C_A^{y-1}}$$

Using the additivity property of volume measures at previous year prices, which we proved in the precedent Item a,

$$\forall y, \qquad Q_A^{y-1\to y} = \sum_{i=1}^{m} \frac{K_i^{y-1\to y}}{C_A^{y-1}}$$

We multiply and divide each term of the summation by the corresponding C_i^{y-1}, and thus,

$$\forall y, \qquad Q_A^{y-1\to y} = \sum_{i=1}^{m} \frac{C_i^{y-1}}{C_A^{y-1}} * \frac{K_i^{y-1\to y}}{C_i^{y-1}}$$

Consequently, we obtain the aggregation equation for annual Laspeyres volume indices from previous year,

$$\forall y, \qquad Q_A^{y-1\to y} = \sum_{i=1}^{m} \frac{C_i^{y-1}}{C_A^{y-1}} * Q_i^{y-1\to y}$$

F-II-2. Quarterly Series

We want to calculate the quarterly Laspeyres volume indices from previous year for A, the aggregate of the disjoint subcomponents $\{i\}_{1\le i\le m}$, for the period between year 1 and year y_{end}, $\left(Q_A^{y-1\to(s,y)}\right)_{1\le y\le y_{end}, 1\le s\le 4}$.

The quarterly Laspeyres volume index from previous year for component i in quarter s of year y is given by,

$$Q_i^{y-1\to(s,y)} = \frac{K_i^{y-1\to(s,y)}}{\frac{1}{4} * \sum_{s=1}^{4} C_i^{(s,y-1)}}$$

$K_i^{y-1\to(s,y)}$ is the volume measure at previous year prices for component i in quarter s of year y. It is defined by,

$$K_i^{y-1\to(s,y)} = \sum_{(i,j)\in R_i} \bar{P}_{i,j}^{y-1} * U_{i,j}^{(s,y)}$$

Where,

$$\bar{P}_{i,j}^{y-1} = \frac{\sum_{s=1}^{4}\sum_{p=1}^{Tr_{i,j}^{(s,y-1)}} P_{i,j,p}^{(s,y-1)} * U_{i,j,p}^{(s,y-1)}}{\sum_{s=1}^{4}\sum_{p=1}^{Tr_{i,j}^{(s,y-1)}} U_{i,j,p}^{(s,y-1)}}$$

a) Additivity of Volume Measures at Previous Year Prices

To show that quarterly volume measures at previous year prices are additive, let's prove that the quarterly volume measure at previous year prices of aggregate A is equal to the sum of the quarterly volume measures at previous year prices of its disjoint subcomponents $\{i\}_{1\leq i\leq m}$.

We have that $R_A = \bigcup_{i=1}^{m} R_i$. Therefore, the quarterly volume measure at previous year prices of aggregate A is given by,

$$\forall y, \forall 1 \leq s \leq 4, \qquad K_A^{y-1\to(s,y)} = \sum_{(A,j)\in\bigcup_{i=1}^{m} R_i} \bar{P}_{A,j}^{y-1} * U_{A,j}^{(s,y)}$$

Since the $\{R_i\}_{1\leq i\leq m}$ are disjoint,

$$\forall y, \forall 1 \leq s \leq 4, \qquad K_A^{y-1\to(s,y)} = \sum_{i=1}^{m}\sum_{(i,j)\in R_i} \bar{P}_{i,j}^{y-1} * U_{i,j}^{(s,y)}$$

Therefore,

$$\forall y, \forall 1 \leq s \leq 4, \qquad K_A^{y-1\to(s,y)} = \sum_{i=1}^{m} K_i^{y-1\to(s,y)}$$

Consequently, we showed that quarterly volume measures at previous year prices are additive.

b) Aggregation of Laspeyres Volume Indices from Previous Year

Let's derive the aggregation equation for quarterly Laspeyres volume indices from previous year.

For aggregate A, we have that,

$$\forall y, \forall 1 \leq s \leq 4, \qquad Q_A^{y-1\to(s,y)} = \frac{K_A^{y-1\to(s,y)}}{\frac{1}{4} * \sum_{s=1}^{4} C_A^{(s,y-1)}}$$

Using the additivity of volume measures at previous year prices,

$$\forall y, \forall 1 \leq s \leq 4, \qquad Q_A^{y-1\to(s,y)} = \sum_{i=1}^{m} \frac{K_i^{y-1\to(s,y)}}{\frac{1}{4} * \sum_{s=1}^{4} C_A^{(s,y-1)}}$$

We multiply and divide each term of the summation by the corresponding $\frac{1}{4} * \sum_{s=1}^{4} C_i^{(s,y-1)}$, and thus,

$$\forall y, \forall 1 \leq s \leq 4, \qquad Q_A^{y-1\to(s,y)} = \sum_{i=1}^{m} \frac{\sum_{s=1}^{4} C_i^{(s,y-1)}}{\sum_{s=1}^{4} C_A^{(s,y-1)}} * \frac{K_i^{y-1\to(s,y)}}{\frac{1}{4} * \sum_{s=1}^{4} C_i^{(s,y-1)}}$$

Consequently, we obtain the aggregation equation for quarterly Laspeyres volume indices from previous year,

$$\forall y, \forall 1 \leq s \leq 4, \qquad Q_A^{y-1\to(s,y)} = \sum_{i=1}^{m} \frac{\sum_{s=1}^{4} C_i^{(s,y-1)}}{\sum_{s=1}^{4} C_A^{(s,y-1)}} * Q_i^{y-1\to(s,y)}$$

F-III- Annual Overlap Technique: Annual Links

Notes:

- The variable definitions and notations from Subsection A-III, “Treatment of Data Category 1: GDP and Final Expenditure Components,” are maintained in this subsection (see A-III-2.2, ‘Variable Definitions and Notations’).
- All series are supposed to have positive elements.

a) Relation between Annual Links and Quarterly Laspeyres Volume Indices from Previous Year:

Let's show that,

$$\forall k, \qquad G_i^{k-1\to k} = \frac{1}{4} * \sum_{s=1}^{4} Q_i^{k-1\to(s,k)}$$

The annual links employed in chain-linking quarterly Laspeyres volume indices with the annual overlap technique are given by, (see Item c of A-III-3.1.2)

$$\forall k, \qquad G_i^{k-1\to k} = \frac{\sum_{s=1}^{4}\sum_{j=1}^{n_i}\bar{P}_{i,j}^{k-1} * U_{i,j}^{(s,k)}}{\sum_{s=1}^{4}\sum_{j=1}^{n_i}\bar{P}_{i,j}^{(s,k-1)} * U_{i,j}^{(s,k-1)}}$$

We rearrange the outer summation in the numerator, and we multiply and divide each of its terms by $\frac{1}{4}$,

$$\forall k, \qquad G_i^{k-1\to k} = \frac{1}{4} * \sum_{s=1}^{4} \frac{\sum_{j=1}^{n_i}\bar{P}_{i,j}^{k-1} * U_{i,j}^{(s,k)}}{\frac{1}{4} * \sum_{s=1}^{4}\sum_{j=1}^{n_i}\bar{P}_{i,j}^{(s,k-1)} * U_{i,j}^{(s,k-1)}} = \frac{1}{4} * \sum_{s=1}^{4} \frac{K_i^{k-1\to(s,k)}}{\frac{1}{4} * \sum_{s=1}^{4} C_i^{(s,k-1)}}$$

Therefore,

$$\forall k, \qquad G_i^{k-1\to k} = \frac{1}{4} * \sum_{s=1}^{4} Q_i^{k-1\to(s,k)}$$

b) Relation between Chain-Linked Annual Links and Quarterly Chain-Linked Laspeyres Volume Indices

We consider years starting from 0 and suppose that we have the required data from year -1 to calculate indices for year 0. Let $0 \le b$.

Let's show that,

$$\forall y \ge 0, \qquad G_i^{b\to y} = \frac{1}{4} * \sum_{s=1}^{4} Q_i^{b\to(s,y)}$$

Let $y \ge 0$,

The chain-linked annual link in year y with reference year -1 is given by,

$$G_i^{-1\to y} = \prod_{k=0}^{y} G_i^{k-1\to k}$$

If $y = 0$, we already showed in the previous item, a, that,

$$G_i^{-1\to 0} = \frac{1}{4} * \sum_{s=1}^{4} Q_i^{-1\to(s,0)}$$

If $y > 0$, we isolate $G_i^{y-1\to y}$ from the product $\prod_{k=0}^{y} G_i^{k-1\to k}$ and substitute its expression obtained in previous Item a,

$$G_i^{-1\to y} = \left(\prod_{k=0}^{y-1} G_i^{k-1\to k}\right) * \frac{1}{4} * \sum_{s=1}^{4} Q_i^{y-1\to(s,y)}$$

Hence,

$$G_i^{-1\to y} = \frac{1}{4} * \sum_{s=1}^{4} \left(\prod_{k=0}^{y-1} G_i^{k-1\to k}\right) * Q_i^{y-1\to(s,y)}$$

Based on the chain-linking formula (18) adapted for the reference year -1,

$$G_i^{-1\to y} = \frac{1}{4} * \sum_{s=1}^{4} Q_i^{-1\to(s,y)}$$

Therefore, we showed that,

$$\forall y \geq 0, \qquad G_i^{-1\to y} = \frac{1}{4} * \sum_{s=1}^{4} Q_i^{-1\to(s,y)}$$

Dividing both sides of the above equation by $G_i^{-1\to b}$ yields the final result,

$$\forall y \geq 0, \qquad G_i^{b\to y} = \frac{1}{4} * \sum_{s=1}^{4} Q_i^{b\to(s,y)}$$

F-IV- Temporal Consistency of Quarterly and Annual Series

Notes:

- The variable definitions and notations from Subsection A-III, “Treatment of Data Category 1: GDP and Final Expenditure Components,” are maintained in this subsection (see A-III-2.2, ‘Variable Definitions and Notations’).
- All series are supposed to have positive elements.
- By “temporal consistency” we refer to the temporal consistency of quarterly series with respect to annual series.

a) Implications for the Annual Links in the Annual Overlap Technique

We suppose that values at current prices, respectively volume measures at previous year prices, are temporally consistent over the years $\{k \geq 0\}$, respectively $\{k \geq 1\}$. Let's deduce the implications for the annual links used in the annual overlap technique.

The annual links employed in chain-linking quarterly Laspeyres volume indices with the annual overlap technique are given by, (see Item c of A-III-3.1.2)

$$\forall k \geq 1, \qquad G_i^{k-1\to k} = \frac{\sum_{s=1}^{4}\sum_{j=1}^{n_i} \bar{P}_{i,j}^{k-1} * U_{i,j}^{(s,k)}}{\sum_{s=1}^{4}\sum_{j=1}^{n_i} \bar{P}_{i,j}^{(s,k-1)} * U_{i,j}^{(s,k-1)}} = \frac{\sum_{s=1}^{4} K_i^{k-1\to(s,k)}}{\sum_{s=1}^{4} C_i^{(s,k-1)}}$$

Under the assumption of temporally consistent values at current prices and volume measures at previous year prices,

$$\forall k \geq 1, \qquad G_i^{k-1\to k} = \frac{\sum_{s=1}^{4} K_i^{k-1\to(s,k)}}{\sum_{s=1}^{4} C_i^{(s,k-1)}} = \frac{K_i^{k-1\to k}}{C_i^{k-1}}$$

Therefore,

$$\forall k \geq 1, \qquad G_i^{k-1\to k} = Q_i^{k-1\to k}$$

Hence,

$$\forall y \geq 1, \qquad \prod_{k=1}^{y} G_i^{k-1\to k} = \prod_{k=1}^{y} Q_i^{k-1\to k}$$

That is, with the fact that $G_i^{0\to 0} = 1 = Q_i^{0\to 0}$,

$$\forall y \geq 0, \qquad G_i^{0\to y} = Q_i^{0\to y}$$

For $b \geq 0$, we divide the above equation by $G_i^{0\to b} = Q_i^{0\to b}$, and thus,

$$\forall y \geq 0, \qquad G_i^{b\to y} = Q_i^{b\to y}$$

Consequently, under the assumption of temporally consistent values at current prices, respectively volume measures at previous year prices, over the years $\{y \geq 0\}$, respectively $\{y \geq 1\}$, the chain-linked annual links with reference year b, respectively annual links, match the annual chain-linked Laspeyres volume indices with reference year b, respectively annual Laspeyres volume indices from previous year, over the years $\{y \geq 0\}$, respectively $\{y \geq 1\}$.

b) Implications for the Laspeyres Volume Indices

We suppose that values at current prices and volume measures at previous year prices are temporally consistent over the years $\{y \geq 0\}$ and $\{y \geq 1\}$, respectively. Let's uncover the implications for the Laspeyres volume indices when the quarterly series are chain-linked using the annual overlap technique.

First let's show that the Laspeyres volume indices from previous year are also temporally consistent over years $\{y \geq 1\}$. We know from F-III-a that,

$$\forall y \geq 1, \qquad G_i^{y-1\to y} = \frac{1}{4} * \sum_{s=1}^{4} Q_i^{y-1\to(s,y)}$$

Therefore, using the result from previous Item a, which is valid in the case of the temporal consistency property assumed above, $\forall y \geq 1,\ G_i^{y-1\to y} = Q_i^{y-1\to y}$, we get,

$$\forall y \geq 1, \qquad Q_i^{y-1\to y} = \frac{1}{4} * \sum_{s=1}^{4} Q_i^{y-1\to(s,y)}$$

Consequently, we showed that under the temporal consistency property assumed above, the Laspeyres volume indices from previous year are temporally consistent over years $\{y \geq 1\}$.

Now, let's prove that the temporal consistency of the Laspeyres volume indices from previous year over the years $\{y \geq 1\}$ is equivalent to the temporal consistency of the chain-linked Laspeyres volume indices over the years $\{y \geq 0\}$.

The temporal consistency of the Laspeyres volume indices from previous year over the years $\{y \geq 1\}$ can be written as follows,

$$\forall y \geq 1, \qquad Q_i^{y-1\to y} = \frac{1}{4} * \sum_{s=1}^{4} Q_i^{y-1\to(s,y)}$$

Based on the result from F-III-a that $\forall y \geq 1,\ G_i^{y-1\to y} = \frac{1}{4} * \sum_{s=1}^{4} Q_i^{y-1\to(s,y)}$, the above equation is equivalent to,

$$\forall y \geq 1, \qquad Q_i^{y-1\to y} = G_i^{y-1\to y}$$

We showed at the end of previous Item a that the above equation implies that,

$$\forall y \geq 0, \qquad G_i^{b\to y} = Q_i^{b\to y}$$

Where $b \geq 0$.

The reverse implication comes from transforming the above equation into the below system and dividing the terms of its first equation by the terms of its second equation.

$$\forall y \geq 1, \qquad \begin{cases} G_i^{b\to y} = Q_i^{b\to y} \\ G_i^{b\to y-1} = Q_i^{b\to y-1} \end{cases}$$

Therefore, we have that, $\forall y \geq 1,\ Q_i^{y-1\to y} = \frac{1}{4} * \sum_{s=1}^{4} Q_i^{y-1\to(s,y)}$ is equivalent to $\forall y \geq 0$, $G_i^{b\to y} = Q_i^{b\to y}$.

We also have from F-III-b,

$$\forall y \geq 0, \qquad G_i^{b\to y} = \frac{1}{4} * \sum_{s=1}^{4} Q_i^{b\to(s,y)}$$

Hence, $\forall y \geq 1,\ Q_i^{y-1\to y} = \frac{1}{4} * \sum_{s=1}^{4} Q_i^{y-1\to(s,y)}$ is equivalent to,

$$\forall y \geq 0, \qquad \frac{1}{4} * \sum_{s=1}^{4} Q_i^{b\to(s,y)} = Q_i^{b\to y}$$

The temporal consistency of the Laspeyres volume indices from previous year over the years $\{y \geq 1\}$ is equivalent to the temporal consistency of the chain-linked Laspeyres volume indices over the years $\{y \geq 0\}$ when quarterly series are chain-linked using the annual overlap technique.

c) Transferability of Temporal Consistency from Subcomponents to Aggregates

Let's suppose that a set of disjoint subcomponents $\{i\}_{1\leq i\leq m}$ have temporally consistent values at current prices and chain-linked Laspeyres volume indices (quarterly series are chain-linked with annual overlap) over the years $\{y \geq 0\}$ and show that their aggregate A has the same properties.

The temporal consistency assumption for a subcomponent i can be written as follows,

$$\forall y \geq 0, \qquad \begin{cases} \displaystyle\sum_{s=1}^{4} C_i^{(s,y)} = C_i^{y} \\ \displaystyle\frac{1}{4} * \sum_{s=1}^{4} Q_i^{b\to(s,y)} = Q_i^{b\to y} \end{cases}$$

According to the result from previous Item b the second equation is equivalent to,

$$\forall y \geq 1, \qquad \frac{1}{4} * \sum_{s=1}^{4} Q_i^{y-1\to(s,y)} = Q_i^{y-1\to y}$$

We substitute the expressions of the volume indices from previous year, and we use the temporal consistency of values at current prices in the left-hand side term's denominator. Hence,

$$\forall y \geq 1, \qquad \frac{1}{4} * \sum_{s=1}^{4} \frac{K_i^{y-1\to(s,y)}}{\frac{1}{4} * C_i^{y-1}} = \frac{K_i^{y-1\to y}}{C_i^{y-1}}$$

Simplifying yields,

$$\forall y \geq 1, \qquad \sum_{s=1}^{4} K_i^{y-1\to(s,y)} = K_i^{y-1\to y}$$

Now, for aggregate A, let's begin by proving the temporal consistency of values at current prices.

Based on the cross-item additivity of values at current prices (see Annex F-I), we express the annual sums of the quarterly values at current prices of aggregate A in terms of its subcomponents' values at current prices as follows,

$$\forall y \geq 0, \qquad \sum_{s=1}^{4} C_A^{(s,y)} = \sum_{s=1}^{4}\sum_{i=1}^{m} C_i^{(s,y)}$$

We invert the order of the summations in the right-hand side and use the temporal consistency of the subcomponents' values at current prices,

$$\forall y \geq 0, \qquad \sum_{s=1}^{4} C_A^{(s,y)} = \sum_{i=1}^{m} C_i^y$$

Using the cross-item additivity of values at current prices we get,

$$\forall y \geq 0, \qquad \sum_{s=1}^{4} C_A^{(s,y)} = C_A^y$$

Therefore, the values at current prices for aggregate A are also temporally consistent over years $\{y \geq 0\}$.

Now, let's prove the temporal consistency of chain-linked Laspeyres volume indices for aggregate A.

Based on the expression of the Laspeyres volume index from previous year for aggregate A, and using the temporal consistency of values at current prices in the expression's denominator, we write the annual averages of the quarterly Laspeyres volume indices from previous year for aggregate A as follows,

$$\forall y \geq 1, \qquad \frac{1}{4} * \sum_{s=1}^{4} Q_A^{y-1 \to (s,y)} = \sum_{s=1}^{4} \frac{K_A^{y-1 \to (s,y)}}{C_A^{y-1}}$$

Using the cross-item additivity of volume measures at previous year prices (see Annex F-II-2.a),

$$\forall y \geq 1, \qquad \frac{1}{4} * \sum_{s=1}^{4} Q_A^{y-1 \to (s,y)} = \frac{1}{C_A^{y-1}} \sum_{s=1}^{4} \sum_{i=1}^{m} K_i^{y-1 \to (s,y)}$$

We invert the order of the summations in the right-hand side and use the temporal consistency of the subcomponents' volume measures at previous year prices,

$$\forall y \geq 1, \qquad \frac{1}{4} * \sum_{s=1}^{4} Q_A^{y-1 \to (s,y)} = \frac{1}{C_A^{y-1}} \sum_{i=1}^{m} K_i^{y-1 \to y}$$

Using the cross-item additivity of volume measures at previous year prices,

$$\forall y \geq 1, \qquad \frac{1}{4} * \sum_{s=1}^{4} Q_A^{y-1\to(s,y)} = \frac{K_A^{y-1\to y}}{C_A^{y-1}} = Q_A^{y-1\to y}$$

Therefore, aggregate A's Laspeyres volume indices from previous year are temporally consistent over years $\{y \geq 1\}$, and so are its chain-linked Laspeyres volume indices over years $\{y \geq 0\}$, by equivalence (see previous Item b). Consequently, we showed that the temporal consistency of values at current prices and chain-linked Laspeyres volume indices (quarterly series are chain-linked with annual overlap) over the years $\{y \geq 0\}$ transfers from subcomponents to their aggregate.

F-V- Implicit Price Deflators

Notes:

- The variable definitions and notations from Subsection A-III, "Treatment of Data Category 1: GDP and Final Expenditure Components," are maintained in this subsection (see A-III-2.2, 'Variable Definitions and Notations').
- All series are supposed to have positive elements.

F-V-1. Definitions

F-V-1.1. Paasche Price Indices

a) Annual Series

The annual Paasche price index from previous year $PP_i^{y-1\to y}$ for component i in year y is defined in the equation below.

$$PP_i^{y-1\to y} = \frac{C_i^y}{K_i^{y-1\to y}} = \frac{\sum_{j=1}^{n_i} \bar{P}_{i,j}^y * U_{i,j}^y}{\sum_{j=1}^{n_i} \bar{P}_{i,j}^{y-1} * U_{i,j}^y}$$

Where,

$$\bar{P}_{i,j}^y = \frac{\sum_{p=1}^{Tr_{i,j}^y} P_{i,j,p}^y * U_{i,j,p}^y}{\sum_{p=1}^{Tr_{i,j}^y} U_{i,j,p}^y}$$

The annual Paasche price indices from previous year $\left(PP_i^{y-1\to y}\right)_{1\le y}$ can be chain-linked to the reference year 0 using the below formula.

$$PP_i^{0\to y} = \begin{cases} \prod_{k=1}^{y} PP_i^{k-1\to k} & for\ y \ge 1 \\ 1 & for\ y = 0 \end{cases}$$

The resulting annual chain-linked Paasche price indices with reference year 0 can be re-referenced to year $b \ge 0$ using the below equation.

$$PP_i^{b\to y} = \frac{PP_i^{0\to y}}{PP_i^{0\to b}}$$

b) Quarterly Series

The quarterly Paasche price index from previous year $PP_i^{y-1\to(s,y)}$ for component i in quarter s of year y is defined in the equation below.

$$PP_i^{y-1\to(s,y)} = \frac{C_i^{(s,y)}}{K_i^{y-1\to(s,y)}} = \frac{\sum_{j=1}^{n_i} \bar{P}_{i,j}^{(s,y)} * U_{i,j}^{(s,y)}}{\sum_{j=1}^{n_i} \bar{P}_{i,j}^{y-1} * U_{i,j}^{(s,y)}}$$

Where,

$$\begin{cases} \bar{P}_{i,j}^{(s,y)} = \dfrac{\sum_{p=1}^{Tr_{i,j}^{(s,y)}} P_{i,j,p}^{(s,y)} * U_{i,j,p}^{(s,y)}}{\sum_{p=1}^{Tr_{i,j}^{(s,y)}} U_{i,j,p}^{(s,y)}} \\ \bar{P}_{i,j}^{y-1} = \dfrac{\sum_{s=1}^{4} \bar{P}_{i,j}^{(s,y-1)} * U_{i,j}^{(s,y-1)}}{\sum_{s=1}^{4} U_{i,j}^{(s,y-1)}} \end{cases}$$

The quarterly Paasche price indices from previous year $\left(PP_i^{y-1\to(s,y)}\right)_{0\le y, 1\le s\le 4}$ can be chain-linked, with annual overlap, to the reference year -1 using the below formula.

$$PP_i^{-1\to(s,y)} = \begin{cases} \left(\prod_{k=0}^{y-1} PG_i^{k-1\to k}\right) * PP_i^{y-1\to(s,y)} & for\ y \ge 1 \\ PP_i^{y-1\to(s,y)} & for\ y = 0 \end{cases}$$

Where $\left(PG_i^{k-1\to k}\right)$ are the annual links,

$$PG_i^{k-1\to k} = \frac{\sum_{s=1}^{4}\sum_{j=1}^{n_i} \bar{P}_{i,j}^{(s,k)} * U_{i,j}^{(s,k)}}{\sum_{s=1}^{4}\sum_{j=1}^{n_i} \bar{P}_{i,j}^{k-1} * U_{i,j}^{(s,k)}} = \frac{\sum_{s=1}^{4} C_i^{(s,k)}}{\sum_{s=1}^{4} K_i^{k-1\to(s,k)}}$$

Where $\bar{P}_{i,j}^{(s,k)}$ and $\bar{P}_{i,j}^{k-1}$ are as defined in the system above.

The annual links are annual Paasche-type price indices from previous year based on temporally aggregated quarterly values. When the quarterly values at current prices and volume measures at previous year prices are temporally consistent with their annual counterparts, then $PG_i^{k-1\to k} = PP_i^{k-1\to k}$.

The quarterly chain-linked Paasche price indices with reference year -1 can be re-referenced to year $b \geq 0$ using the below equation.

$$PP_i^{b\to(s,y)} = \frac{PP_i^{-1\to(s,y)}}{PG_i^{-1\to b}} = PG_i^{b\to y-1} * PP_i^{y-1\to(s,y)}$$

F-V-1.2. Definition of Paasche Prices Indices Using Laspeyres Volume Measures

a) Annual Series

Let's show that,

$$\forall b, y \geq 0, \qquad PP_i^{b\to y} = \frac{C_i^y}{C_i^b} * \frac{1}{Q_i^{b\to y}} = \frac{C_i^y}{K_i^{b\to y}}$$

First, let's work on the bellow expression where $y \geq 0$,

$$\frac{C_i^y}{C_i^0} * \frac{1}{Q_i^{0\to y}}$$

If $y = 0$, then,

$$\frac{C_i^y}{C_i^0} * \frac{1}{Q_i^{0\to y}} = 1$$

If $y \neq 0$, we substitute $Q_i^{0\to y} = \prod_{k=1}^{y} \frac{K_i^{k-1\to k}}{C_i^{k-1}}$ into the expression,

$$\frac{C_i^y}{C_i^0} * \frac{1}{Q_i^{0\to y}} = \frac{C_i^y}{C_i^0} * \prod_{k=1}^{y} \frac{C_i^{k-1}}{K_i^{k-1\to k}}$$

We break down the product $\prod_{k=1}^{y} \frac{C_i^{k-1}}{K_i^{k-1\to k}}$ as follows,

$$\frac{C_i^y}{C_i^0} * \frac{1}{Q_i^{0\to y}} = \frac{C_i^y}{C_i^0} * \prod_{k=1}^{y} C_i^{k-1} * \prod_{k=1}^{y} \frac{1}{K_i^{k-1\to k}}$$

We integrate $\frac{C_i^y}{C_i^0}$ into the product $\prod_{k=1}^{y} C_i^{k-1}$,

$$\frac{C_i^y}{C_i^0} * \frac{1}{Q_i^{0\to y}} = \prod_{k=2}^{y+1} C_i^{k-1} * \prod_{k=1}^{y} \frac{1}{K_i^{k-1\to k}}$$

By performing a change of variable where $k' = k - 1$ in the product $\prod_{k=2}^{y+1} C_i^{k-1}$,

$$\frac{C_i^y}{C_i^0} * \frac{1}{Q_i^{0\to y}} = \prod_{k=1}^{y} C_i^{k} * \prod_{k=1}^{y} \frac{1}{K_i^{k-1\to k}} = \prod_{k=1}^{y} \frac{C_i^k}{K_i^{k-1\to k}}$$

We showed that

$$\frac{C_i^y}{C_i^0} * \frac{1}{Q_i^{0\to y}} = \begin{cases} \prod_{k=1}^{y} \frac{C_i^k}{K_i^{k-1\to k}} & if\ y \geq 1 \\ 1 & if\ y = 0 \end{cases}$$

Therefore, $\frac{C_i^y}{C_i^0} * \frac{1}{Q_i^{0\to y}}$ is equal to the annual chain-linked Paasche price index with reference year 0 for component i in year y, $PP_i^{0\to y}$ (see Item a of F-V-1-1). We will refer to this result as Result (X).

Now, let's work on the bellow expression where $y, b \geq 0$,

$$\frac{C_i^y}{C_i^b} * \frac{1}{Q_i^{b\to y}}$$

We know that $Q_i^{b\to y} = \frac{Q_i^{0\to y}}{Q_i^{0\to b}}$, and thus,

$$\frac{C_i^y}{C_i^b} * \frac{1}{Q_i^{b\to y}} = \frac{C_i^y}{C_i^b} * \frac{Q_i^{0\to b}}{Q_i^{0\to y}}$$

By multiplying both the numerator and denominator of the right-hand side by C_i^0 we get,

$$\frac{C_i^y}{C_i^b} * \frac{1}{Q_i^{b\to y}} = \frac{C_i^y}{C_i^0} * \frac{1}{Q_i^{0\to y}} * \frac{1}{\frac{C_i^b}{C_i^0} * \frac{1}{Q_i^{0\to b}}}$$

Using Result (X),

$$\frac{C_i^y}{C_i^b} * \frac{1}{Q_i^{b\to y}} = \frac{PP_i^{0\to y}}{PP_i^{0\to b}} = PP_i^{b\to y}$$

Consequently, we showed that the annual implicit price deflator $P_i^{b\to y} = \frac{C_i^y}{C_i^b} * \frac{1}{Q_i^{b\to y}}$ is equal to the annual chain-linked Paasche price index with reference year b, $PP_i^{b\to y}$.

Furthermore, using the fact that $C_i^b * Q_i^{b\to y} = K_i^{b\to y}$, we get,

$$PP_i^{b\to y} = \frac{C_i^y}{C_i^b} * \frac{1}{Q_i^{b\to y}} = \frac{C_i^y}{K_i^{b\to y}}$$

b) Quarterly Series

Let's show that,

$$\forall b, y \geq 0, \forall 1 \leq s \leq 4, \qquad PP_i^{b\to(s,y)} = \frac{C_i^{(s,y)}}{\frac{1}{4} * \sum_{s=1}^{4} C_i^{(s,b)}} * \frac{1}{Q_i^{b\to(s,y)}} = \frac{C_i^{(s,y)}}{K_i^{b\to(s,y)}}$$

let's work on the bellow expression where $b, y \geq 0$ and $1 \leq s \leq 4$, (we suppose that we have the required data from year -1 to calculate indices for year 0)

$$\frac{C_i^{(s,y)}}{\frac{1}{4} * \sum_{s=1}^{4} C_i^{(s,b)}} * \frac{1}{Q_i^{b\to(s,y)}}$$

We know that $Q_i^{b\to(s,y)} = G_i^{b\to y-1} * Q_i^{y-1\to(s,y)}$, and thus,

$$\frac{C_i^{(s,y)}}{\frac{1}{4} * \sum_{s=1}^{4} C_i^{(s,b)}} * \frac{1}{Q_i^{b\to(s,y)}} = \frac{1}{\sum_{s=1}^{4} C_i^{(s,b)}} * \frac{1}{G_i^{b\to y-1}} * \frac{C_i^{(s,y)}}{\frac{1}{4} * Q_i^{y-1\to(s,y)}}$$

We multiply both the numerator and denominator of the right-hand side by $\sum_{s=1}^{4} C_i^{(s,y-1)}$,

$$\frac{C_i^{(s,y)}}{\frac{1}{4} * \sum_{s=1}^{4} C_i^{(s,b)}} * \frac{1}{Q_i^{b\to(s,y)}} = \frac{\sum_{s=1}^{4} C_i^{(s,y-1)}}{\sum_{s=1}^{4} C_i^{(s,b)}} * \frac{1}{G_i^{b\to y-1}} * \frac{C_i^{(s,y)}}{\frac{1}{4} * \left(\sum_{s=1}^{4} C_i^{(s,y-1)}\right) * Q_i^{y-1\to(s,y)}}$$

$G_i^{b\to y-1}$ is an annual chain-linked Laspeyres-type volume index based on temporally aggregated quarterly values. Moreover, $\sum_{s=1}^{4} C_i^{(s,y-1)}$ and $\sum_{s=1}^{4} C_i^{(s,b)}$ are the corresponding annual values at current prices in years $y-1$ and b, respectively, obtained by temporally aggregating the quarterly values at current prices. Therefore, we can apply the result from the previous Item, a, (with -1 as a starting year instead of 0) to $\frac{\sum_{s=1}^{4} C_i^{(s,y-1)}}{\sum_{s=1}^{4} C_i^{(s,b)}} * \frac{1}{G_i^{b\to y-1}}$. This yields the annual chain-linked Paasche-type price index with reference year b, which is based on temporally aggregated quarterly values. It is the chain-linked annual link used to chain-link quarterly Paasche price indices from previous year with annual overlap, $PG_i^{b\to y-1}$ (see Item b of F-V-1-1).

We also substitute $\frac{1}{4} * \sum_{s=1}^{4} C_i^{(s,y-1)} * Q_i^{y-1\to(s,y)} = K_i^{y-1\to(s,y)}$ into the equation. Hence,

$$\frac{C_i^{(s,y)}}{\frac{1}{4} * \sum_{s=1}^{4} C_i^{(s,b)}} * \frac{1}{Q_i^{b\to(s,y)}} = PG_i^{b\to y-1} * \frac{C_i^{(s,y)}}{K_i^{y-1\to(s,y)}}$$

Consequently, with $PP_i^{y-1\to(s,y)} = \frac{C_i^{(s,y)}}{K_i^{y-1\to(s,y)}}$ (see Item b of F-V-1-1),

$$\frac{C_i^{(s,y)}}{\frac{1}{4} * \sum_{s=1}^{4} C_i^{(s,b)}} * \frac{1}{Q_i^{b\to(s,y)}} = PG_i^{b\to y-1} * PP_i^{y-1\to(s,y)} = PP_i^{b\to(s,y)}$$

We showed that the quarterly implicit price deflator $P_i^{b\to(s,y)} = \frac{C_i^{(s,y)}}{\frac{1}{4}*\sum_{s=1}^{4} C_i^{(s,b)}} * \frac{1}{Q_i^{b\to(s,y)}}$ is equal to the quarterly chain-linked, with annual overlap, Paasche price index with reference year b, $PP_i^{b\to(s,y)}$.

Furthermore, using the fact that $\frac{1}{4} * \sum_{s=1}^{4} C_i^{(s,b)} * Q_i^{b\to(s,y)} = K_i^{b\to(s,y)}$, we get,

$$PP_i^{b\to(s,y)} = \frac{C_i^{(s,y)}}{\frac{1}{4} * \sum_{s=1}^{4} C_i^{(s,b)}} * \frac{1}{Q_i^{b\to(s,y)}} = \frac{C_i^{(s,y)}}{K_i^{b\to(s,y)}}$$

F-V-2. Re-referencing Volume Measures Using Implicit Price Deflators

a) Annual Series

Let $b, d, y \geq 0$.

Let's work on the below expression.

$$\frac{K_i^{b\to y}}{K_i^{d\to y}}$$

We multiply both numerator and denominator by C_i^y and rearrange the expression as follows,

$$\frac{K_i^{b\to y}}{K_i^{d\to y}} = \frac{\frac{C_i^y}{K_i^{d\to y}}}{\frac{C_i^y}{K_i^{b\to y}}}$$

Using the result from Item a of F-V-1.2, $\forall b, y \geq 0,\ PP_i^{b\to y} = \frac{C_i^y}{K_i^{b\to y}}$,

$$\frac{K_i^{b\to y}}{K_i^{d\to y}} = \frac{PP_i^{d\to y}}{PP_i^{b\to y}}$$

Using the definition, $\forall b, y \geq 0,\ PP_i^{b\to y} = \frac{PP_i^{0\to y}}{PP_i^{0\to b}}$,

$$\frac{K_i^{b\to y}}{K_i^{d\to y}} = \frac{PP_i^{0\to b}}{PP_i^{0\to d}}$$

Therefore,

$$\frac{K_i^{b\to y}}{K_i^{d\to y}} = \frac{1}{PP_i^{b\to d}}$$

Consequently, we showed that,

$$\forall b, d, y \geq 0, \qquad K_i^{b\to y} * PP_i^{b\to d} = K_i^{d\to y}$$

For instance, to move from reference year b to the previous year $y - 1$, which is equivalent to unchaining the volume measure,

$$K_i^{y-1\to y} = K_i^{b\to y} * PP_i^{b\to y-1}$$

b) Quarterly Series

Let $b, d, y \geq 0$ and $1 \leq s \leq 4$.

Let's work on the below expression.

$$\frac{K_i^{b\to(s,y)}}{K_i^{d\to(s,y)}}$$

We multiply both numerator and denominator by $C_i^{(s,y)}$ and rearrange the expression as follows,

$$\frac{K_i^{b\to(s,y)}}{K_i^{d\to(s,y)}} = \frac{\frac{C_i^{(s,y)}}{K_i^{d\to(s,y)}}}{\frac{C_i^{(s,y)}}{K_i^{b\to(s,y)}}}$$

Hence, using the result from Item b of F-V-1.2, $\forall b, y \geq 0, \forall\ 1 \leq s \leq 4,\ PP_i^{b\to(s,y)} = \frac{C_i^{(s,y)}}{K_i^{b\to(s,y)}}$,

$$\frac{K_i^{b\to(s,y)}}{K_i^{d\to(s,y)}} = \frac{PP_i^{d\to(s,y)}}{PP_i^{b\to(s,y)}}$$

Using the definition $\forall b, y \geq 0, \forall 1 \leq s \leq 4, PP_i^{b\to(s,y)} = \frac{PG_i^{-1\to y-1}}{PG_i^{-1\to b}} * PP_i^{y-1\to(s,y)}$, where $PG_i^{-1\to y}$ is the chain-linked annual link used to chain-link, with annual overlap, the Paasche price indices from previous year to reference year -1, we get,

$$\frac{K_i^{b\to(s,y)}}{K_i^{d\to(s,y)}} = \frac{PG_i^{-1\to b}}{PG_i^{-1\to d}} = \frac{1}{PG_i^{b\to d}}$$

Consequently, we showed that,

$$\forall b, d, y \geq 0, \forall 1 \leq s \leq 4, \qquad K_i^{b\to(s,y)} * PG_i^{b\to d} = K_i^{d\to(s,y)}$$

For instance, to move from reference year b to the previous year $y - 1$, which is equivalent to unchaining the volume measure,

$$K_i^{y-1\to(s,y)} = K_i^{b\to(s,y)} * PG_i^{b\to y-1}$$

If the quarterly and annual values at current prices and chain-linked Laspeyres volume indices are temporally consistent (the temporal consistency of chain-linked Laspeyres volume indices implies that the chain-linked annual links $G_i^{b\to y}$ match the annual chain-linked Laspeyres volume indices $Q_i^{b\to y}$, see System (24), in Item f of A-III-3.1.2), then we know from the relation $PG_i^{b\to y} = \frac{\sum_{s=1}^{4} C_i^{(s,y)}}{\sum_{s=1}^{4} C_i^{(s,b)}} * \frac{1}{G_i^{b\to y}}$ (this equality is proved implicitly in Item b of F-V-1.2) that $PG_i^{b\to d} = PP_i^{b\to d}$, and thus,

$$\forall b, d, y \geq 0, \forall 1 \leq s \leq 4, \qquad K_i^{b\to(s,y)} * PP_i^{b\to d} = K_i^{d\to(s,y)}$$

F-VI- Additive Contributions to Growth Rates of Chain-linked Volume Measures

Notes:

- The variable definitions and notations from Subsection A-III, "Treatment of Data Category 1: GDP and Final Expenditure Components," are maintained in this subsection (see A-III-2.2, 'Variable Definitions and Notations').
- By "additivity" we refer to cross-item additivity.
- The assumption in the beginning of Sub-subsubsection A-III-6.1, "Theoretical Framework" (System (27)), is maintained in this subsection.

F-VI-1. Annual Series

We want to decompose the annual growth rate of aggregate A's chain-linked Laspeyres volume measure in monetary terms (referred to as chain-linked Laspeyres volume measure in this subsection) into contributions from the components $\{i\}_{1\leq i\leq m}$.

The annual growth rate of aggregate A's chain-linked Laspeyres volume measure with reference year b in year y is given by, (we assume that the elements of aggregate A's series are positive)

$$g_{K_A^b}^{y} = \frac{K_A^{b\to y} - K_A^{b\to y-1}}{K_A^{b\to y-1}}$$

We transform the volume measures in the numerator into volume measure at previous year prices and value at current prices using the result from Item a of F-V-2, which gives that,

$$\begin{cases} K_A^{b\to y} = \dfrac{K_A^{y-1\to y}}{PP_A^{b\to y-1}} \\ K_A^{b\to y-1} = \dfrac{C_A^{y-1}}{PP_A^{b\to y-1}} \end{cases}$$

Hence,

$$g_{K_A^b}^{y} = \frac{1}{K_A^{b\to y-1} * PP_A^{b\to y-1}} * \left(K_A^{y-1\to y} - C_A^{y-1}\right)$$

Using our assumption that,

$$\forall y, \qquad \begin{cases} C_A^{y} = \sum_{i=1}^{m} (-1)^{f(i)} * C_i^{y} \\ K_A^{y-1\to y} = \sum_{i=1}^{m} (-1)^{f(i)} * K_i^{y-1\to y} \end{cases}$$

We get,

$$g_{K_A^b}^{y} = \frac{1}{K_A^{b\to y-1} * PP_A^{b\to y-1}} * \sum_{i=1}^{m} (-1)^{f(i)}\left(K_i^{y-1\to y} - C_i^{y-1}\right)$$

We use the result from Item a of F-V-2, which gives that,

$$\begin{cases} K_i^{y-1\to y} = K_i^{b\to y} * PP_i^{b\to y-1} \\ C_i^{y-1} = K_i^{b\to y-1} * PP_i^{b\to y-1} \end{cases}$$

Hence,

$$g_{K_A^b}^{y} = \frac{1}{K_A^{b\to y-1} * PP_A^{b\to y-1}} * \sum_{i=1}^{m} (-1)^{f(i)} * \left(K_i^{b\to y} - K_i^{b\to y-1}\right) * PP_i^{b\to y-1}$$

Therefore,

$$g_{K_A^b}^y = \sum_{i=1}^{m} (-1)^{f(i)} * \frac{K_i^{b \to y} - K_i^{b \to y-1}}{K_A^{b \to y-1}} * \frac{PP_i^{b \to y-1}}{PP_A^{b \to y-1}}$$

We decomposed the annual growth rate of aggregate A's chain-linked Laspeyres volume measure with reference year b into additive contributions from components $\{i\}_{1 \le i \le m}$. The contribution of component i is given by the below expression.

$$(-1)^{f(i)} * \frac{K_i^{b \to y} - K_i^{b \to y-1}}{K_A^{b \to y-1}} * \frac{PP_i^{b \to y-1}}{PP_A^{b \to y-1}}$$

F-VI-2. Quarterly Series

We want to decompose the quarterly growth rates of aggregate A's chain-linked Laspeyres volume measure in monetary terms (referred to as chain-linked Laspeyres volume measure in this subsection) into contributions from the components $\{i\}_{1 \le i \le m}$.

F-VI-2.1. Quarter-on-Quarter Growth Rate

The Quarter-on-Quarter growth rate of aggregate A's chain-linked Laspeyres volume measure with reference year b in quarter s of year y is given by,

$$g_{K_A^b}^{(s,y)} = \begin{cases} \dfrac{K_A^{b \to (s,y)} - K_A^{b \to (s-1,y)}}{K_A^{b \to (s-1,y)}} & if\ s = 2,3,4 \\ \dfrac{K_A^{b \to (1,y)} - K_A^{b \to (4,y-1)}}{K_A^{b \to (4,y-1)}} & if\ s = 1 \end{cases}$$

Therefore, we distinguish between two cases depending on the quarter s.

a) For $s = 2,3,4$

$$g_{K_A^b}^{(s,y)} = \frac{K_A^{b \to (s,y)} - K_A^{b \to (s-1,y)}}{K_A^{b \to (s-1,y)}}$$

We transform the volume measures in the numerator into volume measures at previous year prices using the result from Item b of F-V-2, which gives that,

$$\begin{cases} K_A^{b\to(s,y)} = \dfrac{K_A^{y-1\to(s,y)}}{PG_A^{b\to y-1}} \\ K_A^{b\to(s-1,y)} = \dfrac{K_A^{y-1\to(s-1,y)}}{PG_A^{b\to y-1}} \end{cases}$$

Hence,

$$g_{K_A^b}^{(s,y)} = \frac{1}{K_A^{b\to(s-1,y)} * PG_A^{b\to y-1}} * \left(K_A^{y-1\to(s,y)} - K_A^{y-1\to(s-1,y)}\right)$$

Using our assumption that,

$$\forall(s.y), \qquad K_A^{y-1\to(s,y)} = \sum_{i=1}^{m}(-1)^{f(i)} * K_i^{y-1\to(s,y)}$$

We get,

$$g_{K_A^b}^{(s,y)} = \frac{1}{K_A^{b\to(s-1,y)} * PG_A^{b\to y-1}} * \sum_{i=1}^{m}(-1)^{f(i)} * \left(K_i^{y-1\to(s,y)} - K_i^{y-1\to(s-1,y)}\right)$$

We use the result from Item b of F-V-2, which gives that,

$$\begin{cases} K_i^{y-1\to(s,y)} = K_i^{b\to(s,y)} * PG_i^{b\to y-1} \\ K_i^{y-1\to(s-1,y)} = K_i^{b\to(s-1,y)} * PG_i^{b\to y-1} \end{cases}$$

Therefore,

$$g_{K_A^b}^{(s,y)} = \sum_{i=1}^{m}(-1)^{f(i)} * \frac{K_i^{b\to(s,y)} - K_i^{b\to(s-1,y)}}{K_A^{b\to(s-1,y)}} * \frac{PG_i^{b\to y-1}}{PG_A^{b\to y-1}}$$

We decomposed the Quarter-on-Quarter growth rate of aggregate A's chain-linked Laspeyres volume measure with reference year b for $s = 2,3,4$ into additive contributions from components $\{i\}_{1\leq i\leq m}$. The contribution of component i is given by the below expression.

$$(-1)^{f(i)} * \frac{K_i^{b\to(s,y)} - K_i^{b\to(s-1,y)}}{K_A^{b\to(s-1,y)}} * \frac{PG_i^{b\to y-1}}{PG_A^{b\to y-1}}$$

b) For $s = 1$

$$g_{K_A^b}^{(1,y)} = \frac{K_A^{b\to(1,y)} - K_A^{b\to(4,y-1)}}{K_A^{b\to(4,y-1)}}$$

We transform the volume measures in the numerator into volume measures at previous year prices using the result from Item b of F-V-2, which gives that,

$$\begin{cases} K_A^{b\to(1,y)} = \dfrac{K_A^{y-1\to(1,y)}}{PG_A^{b\to y-1}} \\ K_A^{b\to(4,y-1)} = \dfrac{K_A^{y-2\to(4,y-1)}}{PG_A^{b\to y-2}} \end{cases}$$

Hence,

$$g_{K_A^b}^{(1,y)} = \frac{1}{K_A^{b\to(4,y-1)}} * \left(\frac{K_A^{y-1\to(1,y)}}{PG_A^{b\to y-1}} - \frac{K_A^{y-2\to(4,y-1)}}{PG_A^{b\to y-2}} \right)$$

Using our assumption that,

$$\forall(s.y), \qquad K_A^{y-1\to(s,y)} = \sum_{i=1}^{m} (-1)^{f(i)} * K_i^{y-1\to(s,y)}$$

We get,

$$g_{K_A^b}^{(1,y)} = \frac{1}{K_A^{b\to(4,y-1)}} * \sum_{i=1}^{m} (-1)^{f(i)} * \left(\frac{K_i^{y-1\to(1,y)}}{PG_A^{b\to y-1}} - \frac{K_i^{y-2\to(4,y-1)}}{PG_A^{b\to y-2}} \right)$$

We use the result from Item b of F-V-2, which gives that,

$$\begin{cases} K_i^{y-1\to(1,y)} = K_i^{b\to(1,y)} * PG_i^{b\to y-1} \\ K_i^{y-2\to(4,y-1)} = K_i^{b\to(4,y-1)} * PG_i^{b\to y-2} \end{cases}$$

Hence,

$$g_{K_A^b}^{(1,y)} = \frac{1}{K_A^{b\to(4,y-1)}} * \sum_{i=1}^{m} (-1)^{f(i)} * \left(K_i^{b\to(1,y)} * \frac{PG_i^{b\to y-1}}{PG_A^{b\to y-1}} - K_i^{b\to(4,y-1)} * \frac{PG_i^{b\to y-2}}{PG_A^{b\to y-2}} \right)$$

Within each term of the summation, we add and subtract $(-1)^{f(i)} * K_i^{b\to(4,y-1)} * \frac{PG_i^{b\to y-1}}{PG_A^{b\to y-1}}$, and thus,

$$g_{K_A^b}^{(1,y)} = \sum_{i=1}^{m} (-1)^{f(i)} * \left(\frac{K_i^{b\to(1,y)} - K_i^{b\to(4,y-1)}}{K_A^{b\to(4,y-1)}} * \frac{PG_i^{b\to y-1}}{PG_A^{b\to y-1}} + \frac{K_i^{b\to(4,y-1)}}{K_A^{b\to(4,y-1)}} * \left(\frac{PG_i^{b\to y-1}}{PG_A^{b\to y-1}} - \frac{PG_i^{b\to y-2}}{PG_A^{b\to y-2}} \right) \right)$$

At this point, we can define the additive contribution of component i to the Quarter-on-Quarter growth of aggregate A's chain-linked Laspeyres volume measure with reference year b in quarter 1 of year y as given by,

$$(-1)^{f(i)} * \left(\frac{K_i^{b\to(1,y)} - K_i^{b\to(4,y-1)}}{K_A^{b\to(4,y-1)}} * \frac{PG_i^{b\to y-1}}{PG_A^{b\to y-1}} + \frac{K_i^{b\to(4,y-1)}}{K_A^{b\to(4,y-1)}} * \left(\frac{PG_i^{b\to y-1}}{PG_A^{b\to y-1}} - \frac{PG_i^{b\to y-2}}{PG_A^{b\to y-2}} \right) \right)$$

However, An additional term $- (-1)^{f(i)} * \frac{\sum_{s=1}^{4} K_i^{b\to(s,y-1)}}{\sum_{s=1}^{4} K_A^{b\to(s,y-1)}} * \left(\frac{PG_i^{b\to y-1}}{PG_A^{b\to y-1}} - \frac{PG_i^{b\to y-2}}{PG_A^{b\to y-2}} \right)$ is added to attenuate the effect of the term $\frac{K_i^{b\to(4,y-1)}}{K_A^{b\to(4,y-1)}} * \left(\frac{PP_i^{b\to y-1}}{PP_A^{b\to y-1}} - \frac{PP_i^{b\to y-2}}{PP_A^{b\to y-2}} \right)$ (Arnaud, Boussard, Poissonnier, and Soual 2014). Therefore, the contribution of component i is given by,

$$(-1)^{f(i)} * \left(\frac{K_i^{b\to(1,y)} - K_i^{b\to(4,y-1)}}{K_A^{b\to(4,y-1)}} * \frac{PG_i^{b\to y-1}}{PG_A^{b\to y-1}} + \left(\frac{K_i^{b\to(4,y-1)}}{K_A^{b\to(4,y-1)}} - \frac{\sum_{s=1}^{4} K_i^{b\to(s,y-1)}}{\sum_{s=1}^{4} K_A^{b\to(s,y-1)}} \right) * \left(\frac{PG_i^{b\to y-1}}{PG_A^{b\to y-1}} - \frac{PG_i^{b\to y-2}}{PG_A^{b\to y-2}} \right) \right)$$

The contributions are still additive. In fact, we show below that the sum across components $\{i\}_{1\le i\le m}$ of the additional term is null. Furthermore, the additional term follows a similar additive relationship to the one assumed in System (27). Therefore, the aggregate contributions of subsets of $\{i\}_{1\le i\le m}$ are consistent.

$$\sum_{i=1}^{m} -(-1)^{f(i)} * \frac{\sum_{s=1}^{4} K_i^{b\to(s,y-1)}}{\sum_{s=1}^{4} K_A^{b\to(s,y-1)}} * \left(\frac{PG_i^{b\to y-1}}{PG_A^{b\to y-1}} - \frac{PG_i^{b\to y-2}}{PG_A^{b\to y-2}} \right)$$

$$= \sum_{i=1}^{m} -(-1)^{f(i)} * \frac{\sum_{s=1}^{4} K_i^{b\to(s,y-1)}}{\sum_{s=1}^{4} K_A^{b\to(s,y-1)}} * \frac{PG_i^{b\to y-1}}{PG_A^{b\to y-1}} + (-1)^{f(i)} * \frac{\sum_{s=1}^{4} K_i^{b\to(s,y-1)}}{\sum_{s=1}^{4} K_A^{b\to(s,y-1)}} * \frac{PG_i^{b\to y-2}}{PG_A^{b\to y-2}}$$

$$= \sum_{i=1}^{m} -(-1)^{f(i)} * \frac{\sum_{s=1}^{4} C_i^{(s,y-1)}}{\sum_{s=1}^{4} C_A^{(s,y-1)}} + (-1)^{f(i)} * \frac{\sum_{s=1}^{4} K_i^{y-2\to(s,y-1)}}{\sum_{s=1}^{4} K_A^{y-2\to(s,y-1)}}$$

$$= \sum_{s=1}^{4} - \frac{\sum_{i=1}^{m} (-1)^{f(i)} * C_i^{(s,y-1)}}{\sum_{s=1}^{4} C_A^{(s,y-1)}} + \frac{\sum_{i=1}^{m} (-1)^{f(i)} * K_i^{y-2\to(s,y-1)}}{\sum_{s=1}^{4} K_A^{y-2\to(s,y-1)}}$$

$$= \sum_{s=1}^{4} -\frac{C_A^{(s,y-1)}}{\sum_{s=1}^{4} C_A^{(s,y-1)}} + \frac{K_A^{y-2\to(s,y-1)}}{\sum_{s=1}^{4} K_A^{y-2\to(s,y-1)}}$$

$$= -1 + 1$$

$$= 0$$

Where in the first equality, we expanded the expression. In the second equality, we used the definition of the annual chain-linked Paasche price index as an implicit price deflator (for annual temporally aggregated quarterly values), $PG_i^{b\to y-1} = \frac{\sum_{s=1}^{4} C_i^{(s,y-1)}}{\sum_{s=1}^{4} K_i^{b\to(s,y-1)}}$, and the result from item b of F-V-2, $K_i^{b\to(s,y-1)} * PG_i^{b\to y-2} = K_i^{y-2\to(s,y-1)}$, in both the numerators and denominators. In the third equality, we swapped the order of the summations and used our assumed additive relationships between aggregate A and components $\{i\}_{1\le i\le m}$'s values at current prices and volume measures at previous year prices to obtain the fourth equality. From the third equality, we can see that the additional term follows a similar additive relationship to the one assumed in System (27).

F-VI-2.2. Year-over-Year Growth Rate

The Year-over-Year growth rate of aggregate A's chain-linked Laspeyres volume measure with reference year b in quarter s of year y is given by,

$$g_{K_A^b,YoY}^{(s,y)} = \frac{K_A^{b\to(s,y)} - K_A^{b\to(s,y-1)}}{K_A^{b\to(s,y-1)}}$$

The decomposition of the above Year-over-Year growth rate is very similar to that of the Quarter-on-Quarter growth rate when $s = 1$ (see Item b of F-VI-2.1) and yields the below additive contribution of component i.

$$(-1)^{f(i)} * \left(\frac{K_i^{b\to(s,y)} - K_i^{b\to(s,y-1)}}{K_A^{b\to(s,y-1)}} * \frac{PG_i^{b\to y-1}}{PG_A^{b\to y-1}} + \left(\frac{K_i^{b\to(s,y-1)}}{K_A^{b\to(s,y-1)}} - \frac{\sum_{s=1}^{4} K_i^{b\to(s,y-1)}}{\sum_{s=1}^{4} K_A^{b\to(s,y-1)}} \right) \right.$$
$$\left. * \left(\frac{PG_i^{b\to y-1}}{PG_A^{b\to y-1}} - \frac{PG_i^{b\to y-2}}{PG_A^{b\to y-2}} \right) \right)$$

F-VII- Volume and Price Components for GDP per Capita

Notes:

- The variable definitions and notations from Subsection A-III, "Treatment of Data Category 1: GDP and Final Expenditure Components," are maintained in this subsection (see A-III-2.2, 'Variable Definitions and Notations').

F-VII-1. The Chain-Linked Laspeyres Volume Index for GDP per Capita

We assume that the values at current prices and the volume measures at previous year prices for GDP per capita are linked to those of GDP by the bellow equations.

$$\forall k, \qquad \begin{cases} C_{GPC}^{k} = \dfrac{C_{GDP}^{k}}{POP^{k}} \\ K_{GPC}^{k-1\to k} = \dfrac{K_{GDP}^{k-1\to k}}{POP^{k}} \end{cases}$$

Where the subscript GPC refers to GDP per capita and POP^{y} is the population size.

Therefore, the Laspeyres volume index from previous year for GDP per capita is given by:

$$Q_{GPC}^{k-1\to k} = \frac{K_{GPC}^{k-1\to k}}{C_{GPC}^{k-1}} = \frac{\dfrac{K_{GDP}^{k-1\to k}}{POP^{k}}}{\dfrac{C_{GDP}^{k-1}}{POP^{k-1}}} = \frac{POP^{k-1}}{POP^{k}} * Q_{GDP}^{k-1\to k}$$

Chain-linking to year 0 yields,

$$\forall y \geq 0, \qquad Q_{GPC}^{0\to y} = \prod_{k=1}^{y} Q_{GPC}^{k-1\to k} = \prod_{k=1}^{y} \frac{POP^{k-1}}{POP^{k}} * \prod_{k=1}^{y} Q_{GDP}^{k-1\to k}$$

Hence,

$$\forall y \geq 0, \qquad Q_{GPC}^{0\to y} = \frac{POP^0}{POP^y} * Q_{GDP}^{0\to y}$$

Re-referencing to year b gives the chain-linked Laspeyres volume index with reference year b for GDP per capita,

$$\forall y \geq 0, \qquad Q_{GPC}^{b\to y} = \frac{Q_{GPC}^{0\to y}}{Q_{GPC}^{0\to b}} = \frac{POP^b}{POP^y} * Q_{GDP}^{b\to y}$$

Furthermore, the chain-linked Laspeyres volume measure in monetary terms with reference year b for GDP per capita is given by,

$$\forall y \geq 0, \qquad K_{GPC}^{b\to y} = C_{GPC}^{b} * Q_{GPC}^{b\to y} = \frac{C_{GDP}^{b}}{POP^b} * \frac{POP^b}{POP^y} * Q_{GDP}^{b\to y}$$

Consequently,

$$\forall y \geq 0, \qquad K_{GPC}^{b\to y} = \frac{K_{GDP}^{b\to y}}{POP^y}$$

F-VII-2. The Chain-Linked Paasche Price Index for GDP per Capita

We maintain the assumption from F-VII-1 and use the resulting relation between $K_{GPC}^{b\to y}$ and $K_{GDP}^{b\to y}$.

$$\forall y \geq 0, \qquad K_{GPC}^{b\to y} = \frac{K_{GDP}^{b\to y}}{POP^y}$$

Based on the definition of the chain-linked Paasche price index (see Item a of F-V-1.2),

$$\forall y \geq 0, \qquad PP_{GPC}^{b\to y} = \frac{C_{GPC}^{y}}{K_{GPC}^{b\to y}}$$

Substituting the expressions of C_{GPC}^{b} and $K_{GPC}^{b\to y}$ yields,

$$\forall y \geq 0, \qquad PP_{GPC}^{b\to y} = \frac{\frac{C_{GDP}^{y}}{POP^y}}{\frac{K_{GDP}^{b\to y}}{POP^y}} = PP_{GDP}^{b\to y}$$