MATHEMATICAL MODEL IN MARKETING: QUESTION 2

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1 Task 1

1.1 Devise a polynomial-time algorithm that produces a stable assignment of the n houses to the n people.

Recursively implement the following two steps until all the n people is excluded.

Step 1: Denote $\boldsymbol{N}$ as the group of n people. Let every member declare their most desirable house. Consider the set $\boldsymbol{M} = \{$ j: Person j declares that House j is exactly his/her favourate house$\}$. Assume $m = |\boldsymbol{M}|$, then $m \leq 0$ and $m \geq n$. $\forall j \in \boldsymbol{M}$, the assignment of House j to Person j is coalition proof because Person j has already achieved the highest utility. Simultaneously, Person j is not willing to exchange House j with any other house. Update $\boldsymbol{N}$ to be $\boldsymbol{N} - \boldsymbol{M}$.

Step 2: Starting from any member N_1 of $\boldsymbol{N}$, N_1 declare the best house for him/her, which currently belongs to N_2. Create the set $\boldsymbol{Pool} = \{N_1\}$. Then N_2 declare the best house for him/her, which currently belongs to N_3. If N_3 is exactly N_1, then we can find a coalition $\boldsymbol{\{N_1, N_2\}}$ to exchange houses and then $\boldsymbol{N}$ should be updated as $\boldsymbol{N} - \boldsymbol{\{N_1, N_2\}}$. Every member in the new $\boldsymbol{N}$ will resay their favorite house. If $N_3 \neq N_1$, we update $\boldsymbol{Pool} = \{N_1, N_2\}$. Then N_3 declare the best house for him/her, which currently belongs to N_4. If $N_4 \in \boldsymbol{Pool}$, then we can find a coalition so that we can update $\boldsymbol{N}$ by excluding this coalition from $\boldsymbol{N}$. If $N_4 \notin \boldsymbol{Pool}$, we update $\boldsymbol{Pool} = \{N_1, N_2, N_3\}$. Proceed this procedure until we find a coalition.

1.2 Prove that the algorithm terminates and obtains a stable assignment.

This algorithm will always find a coalition regardless of the size of $\boldsymbol{N}$.

$\forall k \in \boldsymbol{N}$, when $N_k - 1$ declares N_k, N_k's favorite house is either in $\boldsymbol{Pool}$ or not. If N_k's favorite house is in $\boldsymbol{Pool}$, then this algorithm finds a coalition. If all of the first n-1 persons cannot find their favorite house in $\boldsymbol{Pool}$, N_n's favorite house must be in $\boldsymbol{Pool}$ given the fact that Person n is not perfectly satisfied with House n.

On top of this, the new group $\boldsymbol{N}$ updates their preferences every time we find a coalition and exclude every member in the coalition from $\boldsymbol{N}$. Thus, this algorithm terminates until no element remains in $\boldsymbol{N}$. The time complexity for this algorithm is polynomial because at most n steps are needed for each round.

This algorithm can obtain a stable assignment because every time we find a coalition and reassign houses, every one in this coalition will live in the house that he likes most. As a consequence, no one has any incentive to move after being reassigned to their favorite house (as declared).

2 Task 2

It is a dominant strategy for each person to truthfully reveal his or preferences.

Whenever we find a coalition, every one in this coalition will move to the house that he declares as his favorite one regardless of others' declarations. If he/she does not truthfully reveal his/her preferences, he/she will not be better off under the following two circumstances.

Case 1: he/she is included in the coalition in this round if he truthfully declares his preferences. In this case, he/she will get the house that is declared. Thus, the outcome from any untruthful declaration must be inferior to truthful declarations.

Case 2: he/she is not included in the coalition during this round. In this case, truthful declaration is still not inferior to untruthful declaration. For example, assume $\boldsymbol{N} = \{A, B, C, D\}$. Assume that Person A's preference ordering is $A \preceq D \preceq C \preceq B$. However, House B is not achievable for Person A because Person B likes House D most and Person D likes House B most. It is possible that Person A can achieve his second best by declaring that House C is his favorite. However, after excluding $\{B, D\}$ from $\boldsymbol{N}$, House C naturally becomes his favorite house when Person A updates his preferences in the next round.

To conclude, it is a dominant strategy to for each person to truthfully reveal his/her preferences.