MATHEMATICAL MODEL IN MARKETING: QUESTION 3

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1 Task 1

Define A as Adjacency Matrix where A_{ij} denotes the pair of Follower i and Leader j. $\mathbb{1}$ is One Vector so that each element of $\mathbb{1}$ is 1. Follower Vector represents the number of accounts that each user follows, which can be expressed as

$$F = A\mathbb{1}. \tag{1}$$

Similarly, Leader Vector L represents the number of accounts that each user leads , which can be expressed as

$$L = A^T\mathbb{1}. \tag{2}$$

The average number of followers is

$$F_1 = \frac{\mathbb{1}^T A\mathbb{1}}{\mathbb{1}^T\mathbb{1}} \tag{3}$$

The average number of leaders is

$$L_1 = \frac{\mathbb{1}^T A^T\mathbb{1}}{\mathbb{1}^T\mathbb{1}} \tag{4}$$

Given that both $\mathbb{1}^T A\mathbb{1}$ and $\mathbb{1}^T A^T\mathbb{1}$ are scalars, matrix transpose implies that

$$\{\mathbb{1}^T A\mathbb{1}\}^T = \mathbb{1}^T A^T\mathbb{1} \tag{5}$$

Thus,

$$F_1 = L_1 \tag{6}$$

Intuitively, there is one follower and one leader within each pair. Summing up all the pairs, the total number of followers equals to the total number of leaders. Dividing by the number of users, the average number of leaders equals to the average number of followers.

2 Task 2

Mathematical Explanations for the phenomenon that the average of followers is smaller than the average of followers of leaders.

The average number of followers of leaders is

$$F_2 = \frac{\mathbb{1}^T A^T A \mathbb{1}}{\mathbb{1}^T A \mathbb{1}} \tag{7}$$

From the Cauchy–Schwarz inequality,

$$\frac{\mathbb{1}^T A \mathbb{1}}{\mathbb{1}^T \mathbb{1}} \leq \frac{\mathbb{1}^T A^T A \mathbb{1}}{\mathbb{1}^T A \mathbb{1}} \tag{8}$$

Moreover, the two sides are equal if and only if $A\mathbb{1}$ and $\mathbb{1}$ are linearly dependent. This is consistent with the observation that if Adjacency Matrix A is fully connected (every user leads and follows each other without any exception), then $F_1 = F_2$. Otherwise, the average of followers is smaller than the average of followers of leaders.

Intuitively, we can recall the process of calculating the average of followers of leaders – first find the leaders for each user and then find the followers of each leader. On top of this, consider the fact that some celebrities, like Katy Perry, Justin Beiber and Barack Obama, have huge number of followers each on Twitter. Compared with ordinary users, these celebrities has two major differences: (1) From the perspective of being leaders, these celebrities have more followers; (2) From the perspective of followers, these celebrities are followed by more users. Thus, when calculating the average of followers of leaders, these two effects were added. This is consistent with the fact that, given $F = A\mathbb{1}$,

$$\mathbb{1}^T A^T A \mathbb{1} = \sum_{i=1} F_i^2 \tag{9}$$

$$F_2 = \frac{\sum_{i=1} F_i^2}{\sum_{i=1} F_i} \tag{10}$$

Take the example of Barack Obama, F_{Obama} is much larger than average, F_1, which makes the value of F_2 outnumber F_1.

3 Task 3

Mathematical Explanations for the phenomenon that the average number of leaders of followers is much higher than the average number of leaders.

The average number of leaders of followers is

$$L_2 = \frac{\mathbb{1}^T A A^T \mathbb{1}}{\mathbb{1}^T A^T \mathbb{1}} \tag{11}$$

From the Cauchy–Schwarz inequality,

$$\frac{\mathbb{1}^T A^T \mathbb{1}}{\mathbb{1}^T \mathbb{1}} \leq \frac{\mathbb{1}^T A A^T \mathbb{1}}{\mathbb{1}^T A^T \mathbb{1}} \tag{12}$$

Moreover, the two sides are equal if and only if $A^T\mathbb{1}$ and $\mathbb{1}$ are linearly dependent. This is consistent with the observation that if Adjacency Matrix A is fully connected (every user leads and follows each other without any exception), then $L_1 = L_2$. Otherwise, the average of followers is smaller than the average of followers of leaders.

Intuitively, we can think of the process of calculating the average of leaders of followers – first find the followers for each user and then find the leaders of each follower. On top of this, consider the fact that some people, like MarQuis Trill and Megamix Champion, follow millions of people on Twitter. Compared with ordinary users, these people has two major differences: (1) From the perspective of being followers, these people have more leaders; (2) From the perspective of leaders, these people are led by more users. Thus, when calculating the average of leaders of followers, these two effects were added. This is consistent with the fact that, given $L = A^T \mathbb{1}$,

$$\mathbb{1}^T A^T A \mathbb{1} = \sum_{i=1} L_i^2 \tag{13}$$

$$L_2 = \frac{\sum_{i=1} L_i^2}{\sum_{i=1} L_i} \tag{14}$$

Take the example of MarQuis Trill, L_{Trill} is much larger than average, L_1, which makes the value of F_2 outnumber F_1.

4 Task 4

Given the fact that $F_1 = L_1$ and $F_2 < L_2$, we have

$$\mathbb{1}^T A^T A \mathbb{1} > \mathbb{1} A A^T \mathbb{1} \tag{15}$$

The inequality above implied that

$$\sum_{i=1} L_i^2 > \sum_{i=1} F_i^2 \tag{16}$$

The inequality can be rewritten as

$$\sum_{i=1} \{L_i^2 - L_1\}^2 > \sum_{i=1} \{F_i - F_1\}^2 \tag{17}$$

The large discrepancy in the two averages ($F_2 << L_2$) arises from the fact that the variance of the number of leaders is larger than the variance of the number of followers. This is consistent with the observation that superstars and celebrities are at the center of viral marketing. It is common that people will follow superstars and celebrities while it is hard for ordinary people to get much attention, which accounts for the fact that Leader Vector (L) has larger variance compared with Follower Vector (F). Twitter created networks for its users and celebrities/superstars are widely connected within the Twitter network. This implies the potential of Tweeting as a Marketing Tool especially by influential tweets and retweets from celebrities and superstars.