***Networks, Crowds, and Markets*, Chapter 15: Sponsored Search Markets**
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Motivation: person types in a query into Google, several available ad slots (1,2,3), how do we sell them to advertisers for this specific query?

Key issue: we do not really know advertisers' values when it comes to different queries. If we knew true valuations, we would simply match them to the available queries in a profit-maximizing way. We do not, so we need an auction. Generally, we analyze how auction allows us to maximize advertisers' total valuation for the slots, which is not necessarily what search engine is interested in (sum of prices in can charge).

Two available procedures:

- VCG (Vickrey-Clarke-Groves) – generalization of the second-price auction to the multi-item case (several slots for one query). Not used in practice.
- Generalized Second Price auction – what is actually used in the industry – can result in untruthful bidding and socially sub-optimal solutions.

Basic setup:

- Number of potential slots per search query
- Assumption 1 – Advertisers know click-through rate (CTR) per slot (not problematic)
- Assumption 2 - CTR depends on the slot, but not the ad in that slot (can be easily relaxed)
- Assumption 3 – CTR does not depend on ads is other slots (not well understood even now)
- Assumption 4 – Advertiser has revenue per click (intrinsic to advertiser, does not depend on what was being shown when the user clicked on the ad)

Matching market construction:

- Set of buyers (advertisers) and set of sellers (slots)
- Each buyer j has valuation for the item offered by seller i: (v_{ij} - value an advertiser j obtains from acquiring slot i)
- Goal is to match up buyers with sellers so that no buyer purchases two different items, no item is sold to two different buyers.
- If r_i is a CTR, v_j is revenue per click, then $v_{ij} = r_i v_j$
- We will assume for simplicity equal number of buyers and sellers for the matching (advertisers and slots) – but it is not necessary
    - Can add fictitious slots with CTR=0, if advertisers > slots
    - Can add fictitious advertisers with v_j=0, if advertisers < slots

Now we can get market-clearing prices (prices at which each advertiser prefers a different slot).

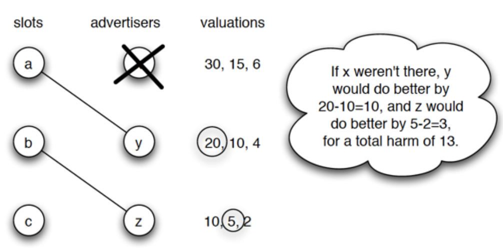
- Each seller i announces price p_i for his item (slot)
- Each buyer evaluates potential payoff: $v_{ij} - p_i$

- We build preferred-seller-graph – each buyer linked to his preferred seller(s) from which he gets highest payoff
- The prices are market-clearing if the graph has a perfect matching – we can assign each buyer an item that maximizes his payoff
- These market-clearing prices exist for each market, there is a procedure to construct them, and they maximize total buyer valuation (Ch. 10)

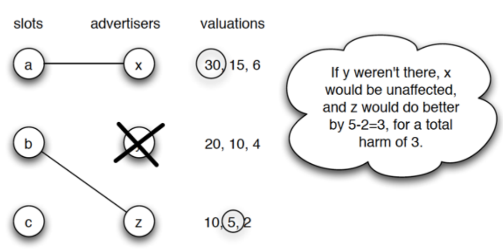
Vickrey-Clarke-Groves procedure – encouraging truthful bidding

- Under first-price auctions, there is no incentive to reveal the truth, rather, one is incentivized to shave the price, leading to very turbulent market, everyone trying to change prices all the time.
- So we want to generalize the idea of truth-revealing second-price auction to the multi-item situation, but how? To do it, we need to change the way we think about the second-price auction.
- Under second price auction, *auction winner is charged the amount of harm he causes others*.
    - That is, if winner's valuation is v_1, if he weren't there, the second largest bidder would have gotten value $v_2 < v_1$, but he does not. Also, all subsequent bidders would still get nothing. So v_2 is the total harm caused by the top bidder.
- This *Vickrey-Clarke-Groves (VCG) principle* encourages truth telling in more general situations. *Alternative formulations:*
    - Each individual is charged the harm they cause the rest of the world
    - Each individual is charged a price equal to the total amount better off everyone else would be if the individual weren't there.
- Additionally, VCG, like a second-price auction, produces allocation that maximizes social welfare – the bidder who values the item the most gets it.

Applying VCG



(a) *Determining how much better off y and z would be if x were not present*



(b) *Determining how much better off x and z would be if y were not present*

- Each buyer only knows his own valuation, and only cares what slot he gets, not what others get.
- VCG procedure:
    - Ask buyers for their valuations (need not be truthful)
    - Assign items to buyers so as to maximize valuation
    - Then, VCG says price buyer j should pay for item from seller i is the harm he causes to other buyers through acquisition of the item
    - See Fig. 15.4 (to the right) for an example
- More formally
    - S is the number of sellers, S-i – seller i removed
    - B is the set of buyers, B-j – buyer j removed
    - V_B^S is max. total valuation over all possible perfect matchings of sellers and buyers
    - V_{B-j}^{S-i} is value of optimal matching if buyer j gets item i and both get removed from sets
    - V_{B-j}^{S} is best value if buyer gets removed
    - So **VCG price** that reflects harm is: $p_{ij} = V_{B-j}^{S} - V_{B-j}^{S-i}$
    - Note, VCG prices are personalized, in contrast to announced prices in general market-matching

- And it turns out this mechanism is truth-revealing – *if items are assigned and prices are computed as described, dominant strategy for every buyer will be to reveal true valuation, and the resulting assignment maximizes total valuation of perfect matching by design*.
- Why is the mechanism *truth-revealing*?
    - True valuation payoff: $v_{ij} - p_{ij}$
    - If valuation is fake,
        - if it changes assigned item to h, payoff is $v_{hj} - p_{hj}$
        - else, payoff is the same as true one – because prices are based on assignments
    - We want to show: $v_{hj} - p_{hj} \leq v_{ij} - p_{ij}$
    - This is true simply because the right side is obtaining by optimally matching all sellers and buyers, but the left side is obtained by first matching buyer j with some arbitrary item h, and then optimally matching the rest of buyers and sellers.

$$v_{hj} - \left[V_{B-j}^{S} - V_{B-j}^{S-h}\right] \leq v_{ij} - \left[V_{B-j}^{S} - V_{B-j}^{S-i}\right]$$

$$v_{hj} + V_{B-j}^{S-h} \leq v_{ij} + V_{B-j}^{S-i} = V_{B}^{S}$$

    - VCG pricing scheme penalizes untruthful value declarations that change assignment.

ADVANCED MATERIAL

- *VCG prices are always market-clearing.*
- There are multiple possible market-clearing prices, but *VCG are the smallest possible market clearing prices, moreover the minimum is unique*.
- The proof is in the advanced section. It proceeds in two steps.
- First, the proof shows that when we zero out valuation of j (remove j), and look at the item i that j formerly got in the original market, before he was zeroed out, it turns out we can then shift the assignment to all other buyers using the edges of the alternating path, which will give us a perfect matching in the preferred-seller graph of the zeroed-out market *at the same prices*. This shows that the prices that are market-clearing in the full market are also market-clearing for the zeroed-out market. Such zeroing out reduces payout of the excluded buyer to zero.
- Second, the proof uses algebraic manipulation to show that VCG price equation is exactly implied by the equation of total payoff at market-clearing prices upon zeroing out a buyer, which are the same prices if we did not zero out a buyer.

The Generalized Second Price Auction (search industry practice)

- Each advertiser j announces a single bid b_j
- GSP procedure assigns each slot i to the i^{th} highest bidder at price $(i+1)^{th}$ highest bid
- The seller gets $\sum r_i \, b_{i+1}$

When there is a single slot, GSP=VCG=Second-price sealed auction

Problems:

- Truth-telling may not be an equilibrium.
- There may be multiple and socially non-optimal equilibria.
- The revenue of GSP may be more or less than VCG depending on the equilibrium.

There is always one Nash equilibrium that maximize total advertiser valuation.
How to construct a socially optimal equilibrium:

1. Determine market clearing prices for the matching market problem, $p_j, j = 1, \dots, n$
2. Derive prices per click, $p_j^* = p_j / r_j$
3. Find bids that result in these prices per click, $b_j = p_{j-1}^*, j = 2, \dots, n$; $b_1 > p_1^*$
4. Use the market-clearing property to verify that these bids form a Nash equilibrium.

Ad Quality

- Ad quality affects the clickthrough rates, thus the search engine revenues.
- The search engine wants high quality ads occupying high slots.
- Google uses a quality factor, q_j, as a fudge factor on the clickthrough rate and assigns slots in descending order of $b_j q_j$
- The exact computation of ad quality is unknown to the advertisers.

Further issues

- Market behavior when the rules of slot allocation are not explicit due to ad quality
- Extrapolation from existing bids to implied bids on more complex queries