MATHEMATICAL MODEL IN MARKETING: QUESTION 1

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1 Task 1

Denote the Story Set S = $\{S_1, S_2, S_3, ..., S_n\}$. Denote the candidate ranking $r_1 \underset{R}{\succ} r_2 \underset{R}{\succ} r_3 ... \underset{R}{\succ} r_n$ as R = $(r_1, r_2, r_3, ..., r_n)$ where $r_i \in S \quad \forall i = 1, 2, 3, .., n$ and $r_i \neq r_j \quad \forall i \neq j$. Denote the ranking by Algorithm t ($t = 1, 2, 3, .., m$) as $A_t = (A_{t1}, A_{t2}, A_{t3}, ..., A_{tn})$ where $A_{t1} \underset{A_t}{\succ} A_{t2} \underset{A_t}{\succ} A_{t3} ... \underset{A_t}{\succ} A_{tn}$ and $A_{ti} \in S \quad \forall i = 1, 2, 3, .., n$ and $A_{ti} \neq A_{tj} \quad \forall i \neq j$.

Define $O_i(x, y)$ as the ordering of the pair (x,y) based on Ranking i where

$$O_i(x, y) = \begin{cases} 1, & \text{if } x \underset{i}{\succ} y \\ 0, & \text{otherwise} \end{cases}$$

Using these notations, the Linear Ordering Problem can be formulated as

$$\underset{R=(r_1,r_2,r_3,...,r_n)}{max} \sum_{i=1}^{m} \sum_{(x,y)\subset R} O_R(x, y)$$

s.$t.\forall t \in [2, n] \quad 1 \leq O_i(x_1, x_2) + O_i(x_2, x_3) + ... + O_i(x_{t-1}, x_t) + O_i(x_t, x_1) \leq t-1$

Given the notation for combinations ${}^nC_k = \frac{n!}{k!(n-k)!}$, for each algorithm with n unique stories, when $t = 2$,

$$\forall x \neq y, \quad O_i(x, y) + O_i(y, x) = 1$$

imposes ${}^nC_2$ constraints and when $t = 3$,

$$\forall x \neq y \neq z, \quad 1 \leq O_i(x, y) + O_i(y, z) + O(z, x) \leq 2$$

imposes ${}^nC_3$ constraints. Following this logic, the total number of constraints with respects to m and n is

$$\sum_{t=2}^{n} m({}^nC_t) = m({}^nC_2 + {}^nC_3 + ... + {}^nC_n) = m(2^n - n - 1)$$

2 Task 2

The term NP-hard refers to any problem that is at least as hard as any problem in NP. To date, no one has found a polynomial-time algorithm for any problem in NP. This implies that LOP cannot be solved with a polynomial-time algorithm.

3 Task 3

In order to obtain a reversal, we assume that the real preference ordering for the pair (S_1, S_2),is $S_1 \succ S_2$. Then the reversal is obtained when an algorithm ranks S_2 ahead of S_1.

$$\Pr(Reversal) = Pr(u_2 > u_1) = Pr(v_2 + \epsilon_2 > v_1 + \epsilon_1) = Pr(\epsilon_2 > v_1 + \epsilon_1 - v_2)$$

Given that each ϵ_k has an independent, extreme value distribution,

$$\Pr(Reversal) = Pr(\epsilon_2 > \epsilon_1 + v_1 - v_2) = \frac{e^{v_1 - v_2}}{e^{v_1 - v_2} + e^{v_2 - v_1}}$$

$$\Pr(Non-Reversal) = 1 - Pr(Reversal) = \frac{e^{v_2 - v_1}}{e^{v_1 - v_2} + e^{v_2 - v_1}}$$

Define Non-Reversal: $NR_{i,j}(x, y)$ as the ordering of the pair (x,y) based on Ranking i and Ranking j where

$$NR_{i,j}(x,y) = \begin{cases} 1, & \text{if } O_i(x,y) = O_j(x,y) = 0 \quad or \quad O_i(x,y) = O_j(x,y) = 1 \\ 0, & \text{otherwise} \end{cases}$$

We assume that each unique pair of stories from S is independent of each other and that rankings given by different algorithms are independent of each other, $\forall t$, $\forall i \neq j$, the probability of a non-reversal for the pair (S_i, S_j) is

$$\Pr\{NR_{R,A_t}(S_i, S_j)\}$$

$$= \frac{e^{v_j - v_i}}{e^{v_i - v_j} + e^{v_j - v_i}} \Pr\{O_R(S_i, S_j) = 1\} + \frac{e^{v_i - v_j}}{e^{v_i - v_j} + e^{v_j - v_i}} \Pr\{O_R(S_i, S_j) = 0\}$$

If we further assume that the candidate ranking is based on the deterministic part v_k such that $(r_1, r_2, ..., r_n)$ implies that $v_1 > v_2 > ... > v_n$, then $\forall t$, $\forall i \neq j$ and $v_i > v_j$, the probability of a non-reversal for the pair (S_i, S_j) is

$$\Pr\{NR_{R,A_t}(S_i, S_j)\} = \frac{e^{v_j - v_i}}{e^{v_i - v_j} + e^{v_j - v_i}}$$

On top of the probability of non-reversals, the expected number of non-reversals across all newsfeed algorithms is

$$\sum_{t=1}^{m} \sum_{(S_i,S_j) \subset S} Pr\{NR_{R,A_t}(S_i, S_j)\} = m \sum_{(S_i,S_j) \subset S} \frac{e^{v_j - v_i}}{e^{v_i - v_j} + e^{v_j - v_i}}$$

4 Task 4

Define Reversal: $R_{i,j}(x, y)$ as the ordering of the pair (x,y) based on Ranking i and Ranking j where

$$R_{i,j}(x,y) = \begin{cases} 1, & \text{if } O_i(x,y) \neq O_j(x,y) \\ 0, & \text{otherwise} \end{cases}$$

Given that reversal and non-reversal are mutually exclusive, $\forall (x,y) \subset S$,

$$R_{R,A_j}(x,y) + NR_{R,A_j}(x,y) = m$$

For the candidate ranking, R $= (r_1, r_2, ..., r_n)$, $\forall (x,y) \subset R$, whenever $O_i(x,y) = 1$, we have $NR_{R,A_j}(x,y) = 1$. Given that $x \underset{R}{\succ} y$, both $O_i(x,y) = 1$ and $NR_{R,A_i}(x,y) = 1$ imply that $x \underset{A_i}{\succ} y$. As a consequence, we have the same objective functions from maximizing the expected number of reversals and optimizing the discrete linear ordering problem.

The simplest example is $n = 2$. If among m algorithms, more than $\frac{m}{2}$ algorithms prefer S_1 to S_2, the optimal solution to the discrete linear ordering problem is S_1; and vice verse. From the perspective of non-reversals, when the candidate ranking agrees with the algorithm, we will have a non-reversal instead of a reversal. To maximize the number of non-reversals, we also need to find the candidate ranking that agrees with more than $\frac{m}{2}$ algorithms.

Admittedly, when $n > 2$, social preferences cannot be generated by pair-wise majority viting. However, the intuition holds that $\forall i \forall (x,y) \in R$, the events leading to $O_i(x,y) = 1$ are exactly the same as those correpsonding to $NR_{R,A_i}(x,y) = 1$. The conclusion is stated as follows.

Assuming that $(r_1^*, r_2^*, r_3^*, ..., r_n^*)$ is the solution to the problem of maximizing the expected number of non-reversals, $(r_1^*, r_2^*, r_3^*, ..., r_n^*)$ will also maximize

$$\sum_{i=1}^{m} \sum_{(x,y) \subset R} O_i(x,y)$$

Assuming that $(r_1^*, r_2^*, r_3^*, ..., r_n^*)$ is the optimal solution to the discrete linear ordering problem, R $= (r_1^*, r_2^*, r_3^*, ..., r_n^*)$ will also maximize

$$\sum_{t=1}^{m} \sum_{(S_i,S_j) \subset S} Pr\{NR_{R,A_t}(S_i,S_j)\}$$

5 Task 5

Assuming that each unique pair of stories from S are independent of each other, given the fact that the probability of a non-reversal for the pair (S_i, S_j)

$$Pr\{NR_{R,A_t}(S_i,S_j)\}$$

the probability of non-reversal across all pairs between R and A_j is

$$\mathrm{P}r\{NP_{R,A_t}\} = \prod_{(S_i,S_j) \subset S} Pr\{NR_{R,A_t}(S_i,S_j)\}$$

Assuming that rankings given by different algorithms are independent of each other, the likelihood of non-reversal across all pairs between R and all algorithms is

$$\mathrm{L} = \prod_{t=1}^{m} Pr\{NP_{R,A_t}\} = \prod_{t=1}^{m} \prod_{(S_i,S_j) \subset S} Pr\{NR_{R,A_t}(S_i,S_j)\}$$

Maximizing lnL with respects to $(r_1, r_2, ..., r_n)$ is equivalent to Maximizing log-likelihood function with respects to $(r_1, r_2, ..., r_n)$.

$$\max_{R=(r_1,r_2,r_3,...,r_n)} lnL = \max_{R=(r_1,r_2,r_3,...,r_n)} \sum_{t=1}^{m} \ln\{Pr[NP_{R,A_t}]\}$$

$$= \max_{R=(r_1,r_2,r_3,...,r_n)} \sum_{t=1}^{m} \sum_{(S_i,S_j)\subset S} \ln\{Pr[NR_{R,A_t}(S_i, S_j)]\}$$

This formulation is equivalent to multinomial logit model with Maximum Likelihood Estimation method. McFadden(1974) shows that the log-likelihood function for MNL is globally concave for linear-in-parameter utility. Thus, maximizing the likelihood function is a convex optimization problem.