Chapter 4: Foundations for inference

Introduction to Statistics

Young, Underemployed and Optimistic

Coming of Age, Slowly, in a Tough Economy

Young adults hit hard by the recession. A plurality of the public (41%) believes young adults, rather than middle-aged or older adults, are having the toughest time in today's economy. An analysis of government economic data suggests that this perception is correct. The recent indicators on the nation's labor market show a decline in the

Tough economic times altering young adults' daily lives, long-term plans. While negative trends in the labor market have been felt most acutely by the youngest workers, many adults in their late 20s and early 30s have also felt the impact of the weak economy. Among all 18- to 34-year-olds, fully half (49%) say they have taken a job they didn't want just to pay the bills, with 24% saying they have taken an unpaid job to gain work experience. And more than one-third (35%) say that, as a result of the poor economy, they have gone back to school. Their personal lives have also been affected: 31% have postponed either getting married or having a baby (22% say they have postponed having a baby and 20% have put off getting married). One-in-four (24%) say they have moved back in with their parents after living on their own.

http://pewresearch.org/pubs/2191/young-adults-workers-labor-market-pay-careers-advancement-recession

Margin of error

The general public survey is based on telephone interviews conducted Dec. 6-19, 2011, with a nationally representative sample of 2,048 adults ages 18 and older living in the continental United States, including an oversample of 346 adults ages 18 to 34. A total of 769 interviews were completed with respondents contacted by landline telephone and 1,279 with those contacted on their cellular phone. Data are weighted to produce a final sample that is representative of the general population of adults in the continental United States. Survey interviews were conducted under the direction of Princeton Survey Research Associates International, in English and Spanish. Margin of sampling error is plus or minus 2.9 percentage points for results based on the total sample and 4.4 percentage points for adults ages 18-34 at the 95% confidence level.

- 41% ± 2.9%: We are 95% confident that 38.1% to 43.9% of the public believe young adults, rather than middle-aged or older adults, are having the toughest time in today's economy.
- 49% ± 4.4%: We are 95% confident that 44.6% to 53.4% of 18-34 years olds have taken a job they didn't want just to pay the bills.

Parameter estimation

- We are often interested in *population parameters*.
- Since complete populations are difficult (or impossible) to collect data on, we use *sample statistics* as *point estimates* for the unknown population parameters of interest.
- Sample statistics vary from sample to sample.
- Quantifying how sample statistics vary provides a way to estimate the *margin of error* associated with our point estimate.
- But before we get to quantifying the variability among samples, let's try to understand how and why point estimates vary from sample to sample.

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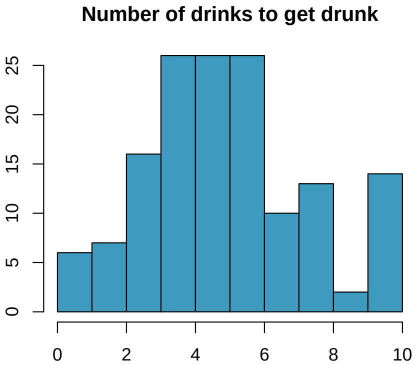
Suppose we randomly sample 1,000 adults from each state in the US. Would you expect the sample means of their heights to be the same, somewhat different, or very different?

Not the same, but only somewhat different.

The following histogram shows the distribution of number of drinks it takes a group of college students to get drunk. We will assume that this is our population of interest. If we randomly select observations from this data set, which values are most likely to be selected, which are least likely?



Suppose that you don't have access to the population data. In order to estimate the average number of drinks it takes these college students to get drunk, you might sample from the population and use your sample mean as the best guess for the unknown population mean.

- Sample, with replacement, ten students from the population, and record the number of drinks it takes them to get drunk.
- Find the sample mean.
- Plot the distribution of the sample averages obtained by members of the class.

1	7	16	3	31	5	46	4	61	10	76	6	91	4	106	6	121	6	136	6
2	5	17	10	32	9	47	3	62	7	77	6	92	0.5	107	2	122	5	137	7
3	4	18	8	33	7	48	3	63	4	78	5	93	3	108	5	123	3	138	3
4	4	19	5	34	5	49	6	64	5	79	4	94	3	109	1	124	2	139	10
5	6	20	10	35	5	50	8	65	6	80	5	95	5	110	5	125	2	140	4
6	2	21	6	36	7	51	8	66	6	81	6	96	6	111	5	126	5	141	4
7	3	22	2	37	4	52	8	67	6	82	5	97	4	112	4	127	10	142	6
8	5	23	6	38	0	53	2	68	7	83	6	98	4	113	4	128	4	143	6
9	5	24	7	39	4	54	4	69	7	84	8	99	2	114	9	129	1	144	4
10	6	25	3	40	3	55	8	70	5	85	4	100	5	115	4	130	4	145	5
11	1	26	6	41	6	56	3	71	10	86	10	101	4	116	3	131	10	146	5
12	10	27	5	42	10	57	5	72	3	87	5	102	7	117	3	132	8		
13	4	28	8	43	3	58	5	73	5.5	88	10	103	6	118	4	133	10		
14	4	29	0	44	6	59	8	74	7	89	8	104	8	119	4	134	6		
15	6	30	8	45	10	60	4	75	10	90	5	105	3	120	8	135	6		

Example:

List of random numbers: 59, 121, 88, 46, 58, 72, 82, 81, 5, 10

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8	5	23	6	38	0	53	2	68	7	83	6	98	4	113	4	128	4	143	6
9	5	24	7	39	4	54	4	69	7	84	8	99	2	114	9	129	1	144	4
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Sample mean: (8+6+10+4+5+3+5+6+6+6) / 10 = 5.9

Sampling distribution

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Approximately 5.39, the true population mean.

Central limit theorem

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The distribution of the sample mean is well approximated by a normal model:

$$\bar{x} \sim N\left(mean = \mu, SE = \frac{\sigma}{\sqrt{n}}\right),$$

where SE is represents *standard error*, which is defined as the standard deviation of the sampling distribution. If σ is unknown, use s.

- It wasn't a coincidence that the sampling distribution we saw earlier was symmetric, and centered at the true population mean.
- We won't go through a detailed proof of why $SE = \frac{\sigma}{\sqrt{n}}$, but note that as n increases SE decreases.
  - As the sample size increases we would expect samples to yield more consistent sample means, hence the variability among the sample means would be lower.

CLT - conditions

Certain conditions must be met for the CLT to apply:

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2. *Sample size/skew:* Either the population distribution is normal, or if the population distribution is skewed, the sample size is large.
   - the more skewed the population distribution, the larger sample size we need for the CLT to apply
   - for moderately skewed distributions $n > 30$ is a widely used rule of thumb

   This is also difficult to verify for the population, but we can check it using the sample data, and assume that the sample mirrors the population.

Confidence intervals

- A plausible range of values for the population parameter is called a *confidence interval*.
- Using only a sample statistic to estimate a parameter is like fishing in a murky lake with a spear, and using a confidence interval is like fishing with a net.

We can throw a spear where we saw a fish but we will probably miss. If we toss a net in that area, we have a good chance of catching the fish.

- If we report a point estimate, we probably won't hit the exact population parameter. If we report a range of plausible values we have a good shot at capturing the parameter.

Photos by Mark Fischer (http://www.flickr.com/photos/fischerfotos/7439791462) and Chris Penny (http://www.flickr.com/photos/clearlydived/7029109617) on Flickr.

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$$\begin{aligned} \bar{x} \pm 2 \times SE &= 3.2 \pm 2 \times 0.25 \\ &= (3.2 - 0.5, 3.2 + 0.5) \\ &= (2.7, 3.7) \end{aligned}$$

Which of the following is the correct interpretation of this confidence interval?

We are 95% confident that

(a) the average number of exclusive relationships college students in this sample have been in is between 2.7 and 3.7.

(b) college students on average have been in between 2.7 and 3.7 exclusive relationships.

(c) a randomly chosen college student has been in 2.7 to 3.7 exclusive relationships.

(d) 95% of college students have been in 2.7 to 3.7 exclusive relationships.

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Confidence interval, a general formula

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Conditions when the point estimate = $\bar{x}$:

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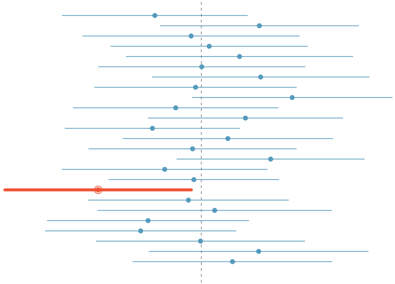
Note: We will discuss working with samples where $n < 30$ in the next chapter.

What does 95% confident mean?

- Suppose we took many samples and built a confidence interval from each sample using the equation $point\ estimate \pm 2 \times SE$.
- Then about 95% of those intervals would contain the true population mean (μ).

- The figure shows this process with 25 samples, where 24 of the resulting confidence intervals contain the true average number of exclusive relationships, and one does not.



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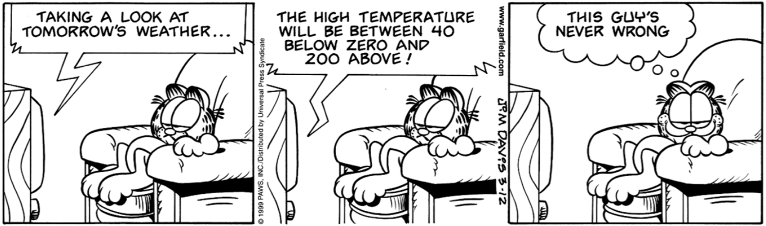
Can you see any drawbacks to using a wider interval?



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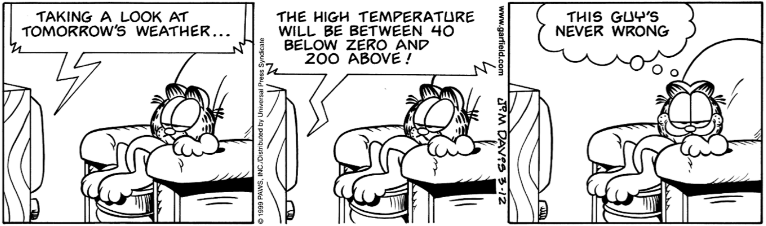

If the interval is too wide it may not be very informative.

Image source: http://web.as.uky.edu/statistics/users/earo227/misc/garfield_weather.gif

Changing the confidence level

$$point\ estimate \pm z^{\star} \times SE$$

- In a confidence interval, $z^{\star} \times SE$ is called the *margin of error*, and for a given sample, the margin of error changes as the confidence level changes.
- In order to change the confidence level we need to adjust $z^{\star}$ in the above formula.
- Commonly used confidence levels in practice are 90%, 95%, 98%, and 99%.
- For a 95% confidence interval, $z^{\star} = 1.96$.
- However, using the standard normal (z) distribution, it is possible to find the appropriate $z^{\star}$ for any confidence level.

Which of the below Z scores is the appropriate $z^{\star}$ when calculating a 98% confidence interval?

(a) $Z = 2.05$

(b) $Z = 1.96$

(c) $Z = 2.33$

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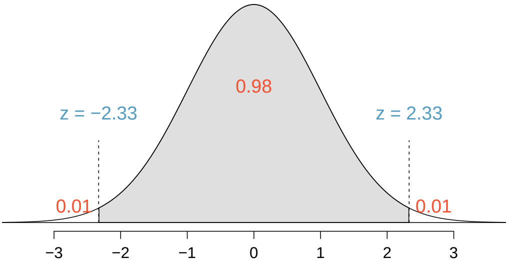

Remember when...

Gender discrimination experiment:

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		Promoted	Not Promoted	Total
Gender	Male	21	3	24
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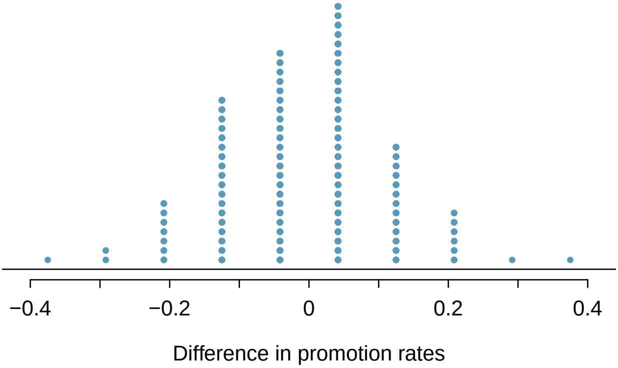
Possible explanations:

- Promotion and gender are *independent*, no gender discrimination, observed difference in proportions is simply due to chance. $\rightarrow$ *null* - (nothing is going on)
- Promotion and gender are *dependent*, there is gender discrimination, observed difference in proportions is not due to chance. $\rightarrow$ *alternative* - (something is going on)

Result



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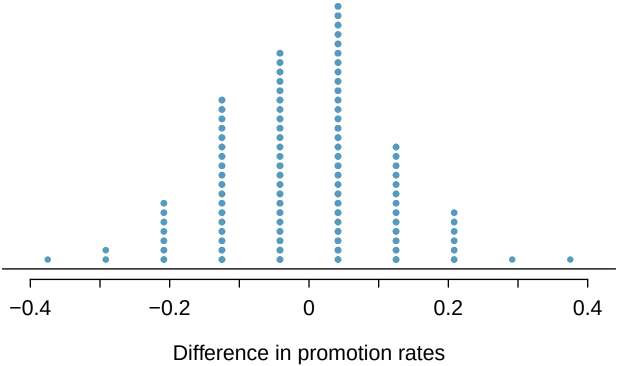

Since it was quite unlikely to obtain results like the actual data or something more extreme in the simulations (male promotions being 30% or more higher than female promotions), we decided to reject the null hypothesis in favor of the alternative.

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We'll formally introduce the hypothesis testing framework using an example on testing a claim about a population mean.

Testing hypotheses using confidence intervals

Earlier we calculated a 95% confidence interval for the average number of exclusive relationships college students have been in to be (2.7, 3.7). Based on this confidence interval, do these data support the hypothesis that college students on average have been in more than 3 exclusive relationships.

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- Since the null value is included in the interval, we do not reject the null hypothesis in favor of the alternative.
- This is a quick-and-dirty approach for hypothesis testing. However it doesn't tell us the likelihood of certain outcomes under the null hypothesis, i.e. the p-value, based on which we can make a decision on the hypotheses.

Number of college applications

A similar survey asked how many colleges students applied to, and 206 students responded to this question. This sample yielded an average of 9.7 college applications with a standard deviation of 7. College Board website states that counselors recommend students apply to roughly 8 colleges. Do these data provide convincing evidence that the average number of colleges all Duke students apply to is higher than recommended?

http://www.collegeboard.com/student/apply/the-application/151680.html

Setting the hypotheses

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- We test the claim that the average number of colleges Duke students apply to is greater than 8

$$H_A : \mu > 8$$

Number of college applications - conditions

Which of the following is not a condition that needs to be met to proceed with this hypothesis test?

(a) Students in the sample should be independent of each other with respect to how many colleges they applied to.

(b) Sampling should have been done randomly.

(c) The sample size should be less than 10% of the population of all Duke students.

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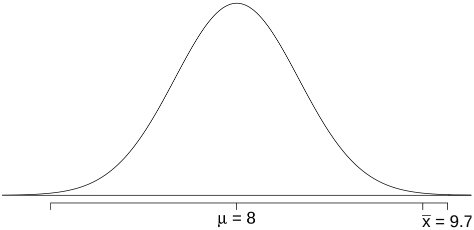

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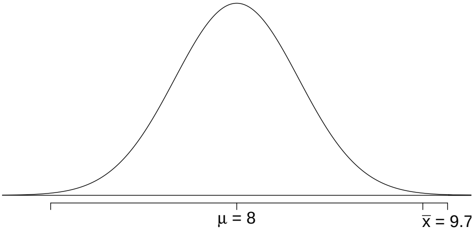

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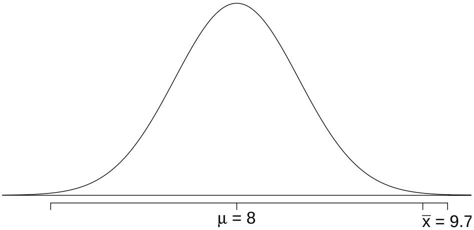

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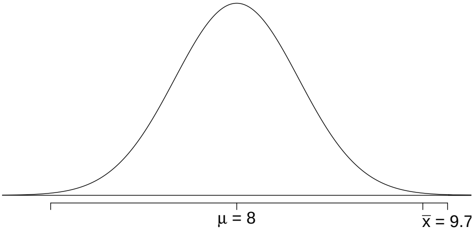

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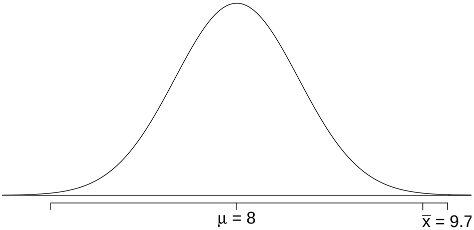

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Yes, and we can quantify how unusual it is using a p-value.

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- We then use this test statistic to calculate the *p-value*, the probability of observing data at least as favorable to the alternative hypothesis as our current data set, if the null hypothesis were true.

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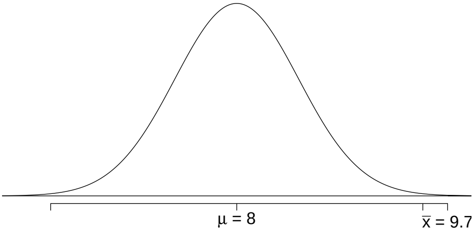
Number of college applications - p-value

p-value: probability of observing data at least as favorable to H_A as our current data set (a sample mean greater than 9.7), if in fact H_0 were true (the true population mean was 8).



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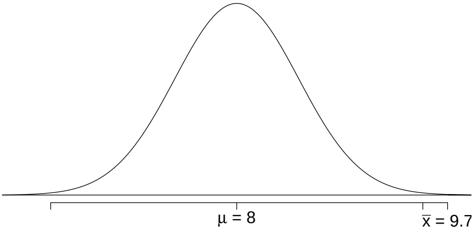

$$P(\bar{x} > 9.7 \mid \mu = 8) = P(Z > 3.4) = 0.0003$$

Number of college applications - Making a decision

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- Since p-value is *low* (lower than 5%) we *reject* H_0.
- The data provide convincing evidence that Duke students apply to more than 8 schools on average.
- The difference between the null value of 8 schools and observed sample mean of 9.7 schools is *not due to chance* or sampling variability.

A poll by the National Sleep Foundation found that college students average about 7 hours of sleep per night. A sample of 169 college students taking an introductory statistics class yielded an average of 6.88 hours, with a standard deviation of 0.94 hours. Assuming that this is a random sample representative of all college students *(bit of a leap of faith?)*, a hypothesis test was conducted to evaluate if college students on average sleep less than 7 hours per night. The p-value for this hypothesis test is 0.0485. Which of the following is correct?

(a) Fail to reject H_0, the data provide convincing evidence that college students sleep less than 7 hours on average.
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(c) Reject H_0, the data prove that college students sleep more than 7 hours on average.
(d) Fail to reject H_0, the data do not provide convincing evidence that college students sleep less than 7 hours on average.
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Two-sided hypothesis testing with p-values

- If the research question was "Do the data provide convincing evidence that the average amount of sleep college students get per night is *different* than the national average?", the alternative hypothesis would be different.

$$H_0 : \mu = 7$$
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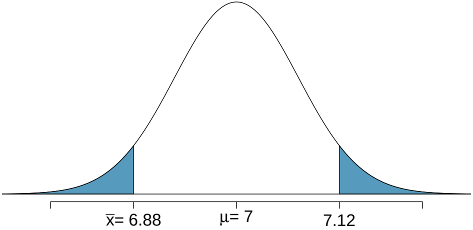
- Hence the p-value would change as well:



p-value
$= 0.0485 \times 2$
$= 0.097$

Decision errors

- Hypothesis tests are not flawless.
- In the court system innocent people are sometimes wrongly convicted and the guilty sometimes walk free.
- Similarly, we can make a wrong decision in statistical hypothesis tests as well.
- The difference is that we have the tools necessary to quantify how often we make errors in statistics.

Decision errors (cont.)

There are two competing hypotheses: the null and the alternative. In a hypothesis test, we make a decision about which might be true, but our choice might be incorrect.

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- A *Type 1 Error* is rejecting the null hypothesis when H_0 is true.
- A *Type 2 Error* is failing to reject the null hypothesis when H_A is true.
- We (almost) never know if H_0 or H_A is true, but we need to consider all possibilities.

Hypothesis Test as a trial

If we again think of a hypothesis test as a criminal trial then it makes sense to frame the verdict in terms of the null and alternative hypotheses:

$$H_0: \text{ Defendant is innocent}$$
$$H_A: \text{ Defendant is guilty}$$

Which type of error is being committed in the following cirumstances?

- Declaring the defendant innocent when they are actually guilty
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Which error do you think is the worse error to make?

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Which error do you think is the worse error to make?

"better that ten guilty persons escape than that one innocent suffer"

– William Blackstone

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$$P(\text{Type 1 error}) = \alpha$$

- This is why we prefer small values of α – increasing α increases the Type 1 error rate.

Choosing a significance level

- Choosing a significance level for a test is important in many contexts, and the traditional level is 0.05. However, it is often helpful to adjust the significance level based on the application.
- We may select a level that is smaller or larger than 0.05 depending on the consequences of any conclusions reached from the test.
- If making a Type 1 Error is dangerous or especially costly, we should choose a small significance level (e.g. 0.01). Under this scenario we want to be very cautious about rejecting the null hypothesis, so we demand very strong evidence favoring H_A before we would reject H_0.
- If a Type 2 Error is relatively more dangerous or much more costly than a Type 1 Error, then we should choose a higher significance level (e.g. 0.10). Here we want to be cautious about failing to reject H_0 when the null is actually false.

the next two slides are provided as a brief summary of hypothesis testing...

Recap: Hypothesis testing framework

1. Set the hypotheses.
2. Check assumptions and conditions.
3. Calculate a *test statistic* and a p-value.
4. Make a decision, and interpret it in context of the research question.

Recap: Hypothesis testing for a population mean

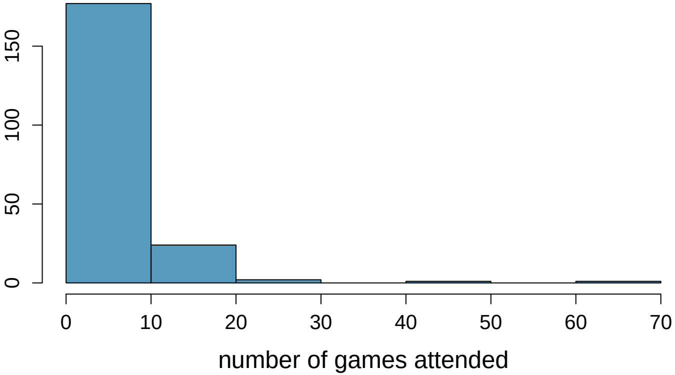
1. Set the hypotheses
   - $H_0 : \mu = null\ value$
   - $H_A : \mu <$ or $>$ or $\neq null\ value$
2. Calculate the point estimate
3. Check assumptions and conditions
   - Independence: random sample/assignment, 10% condition when sampling without replacement
   - Normality: nearly normal population or $n \geq 30$, no extreme skew – or use the t distribution
4. Calculate a *test statistic* and a p-value (draw a picture!)

$$Z = \frac{\bar{x} - \mu}{SE},\ where\ SE = \frac{s}{\sqrt{n}}$$

5. Make a decision, and interpret it in context
   - If p-value $< \alpha$, reject H_0, data provide evidence for H_A
   - If p-value $> \alpha$, do not reject H_0, data do not provide evidence for H_A

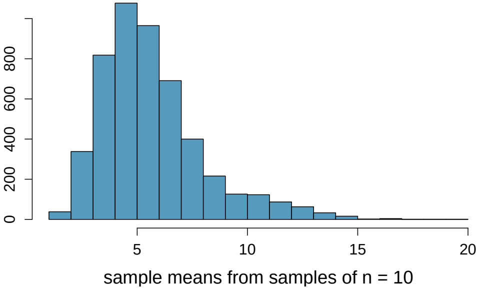
Average number of basketball games attended

Next let's look at the population data for the number of basketball games attended:



Average number of basketball games attended (cont.)

Sampling distribution, n = 10:



What does each observation in this distribution represent?

Is the variability of the sampling distribution smaller or larger than the variability of the population distribution? Why?

Average number of basketball games attended (cont.)

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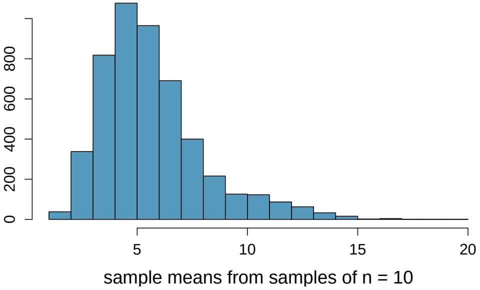

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Sample mean ($\bar{x}$) of samples of size $n = 10$.

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Average number of basketball games attended (cont.)

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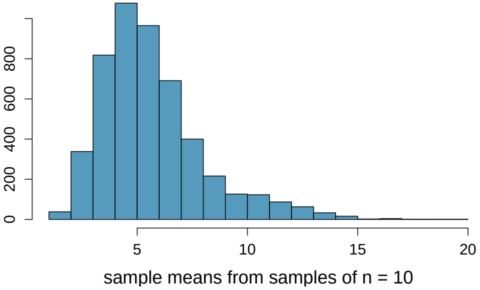


What does each observation in this distribution represent?

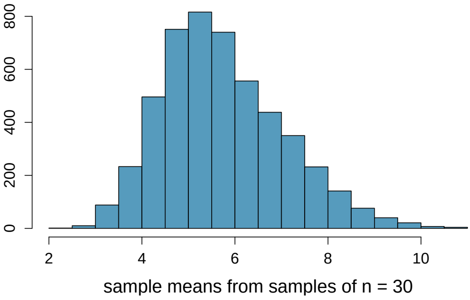
Sample mean ($\bar{x}$) of samples of size $n = 10$.

Is the variability of the sampling distribution smaller or larger than the variability of the population distribution? Why?

Smaller, sample means will vary less than individual observations.

Average number of basketball games attended (cont.)

Sampling distribution, n = 30:



How did the shape, center, and spread of the sampling distribution change going from $n = 10$ to $n = 30$?

Average number of basketball games attended (cont.)

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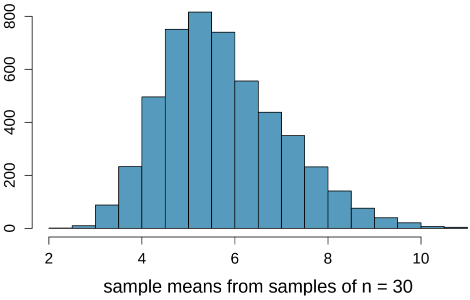
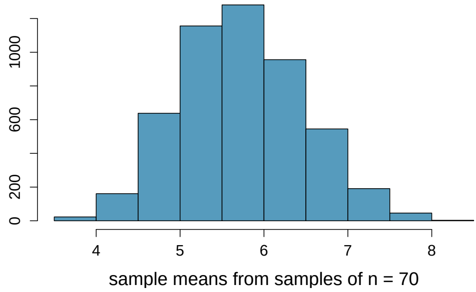


How did the shape, center, and spread of the sampling distribution change going from $n = 10$ to $n = 30$?

Shape is more symmetric, center is about the same, spread is smaller.

Average number of basketball games attended (cont.)

Sampling distribution, n = 70:



Average number of basketball games attended (cont.)

The mean of the sampling distribution is 5.75, and the standard deviation of the sampling distribution (also called the *standard error*) is 0.75. Which of the following is the most reasonable guess for the 95% confidence interval for the true average number of basketball games attended by students?

(a) 5.75 ± 0.75

(b) $5.75 \pm 2 \times 0.75$

(c) $5.75 \pm 3 \times 0.75$

(d) cannot tell from the information given

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(b) $5.75 \pm 2 \times 0.75 \ \rightarrow (4.25, 7.25)$

(c) $5.75 \pm 3 \times 0.75$

(d) cannot tell from the information given

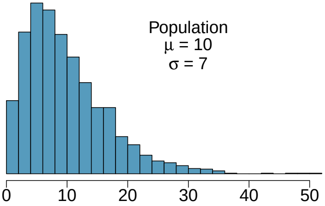
Four plots: Determine which plot (A, B, or C) is which.

(1) At top: distribution for a population ($\mu = 10, \sigma = 7$),

(2) a single random sample of 100 observations from this population,

(3) a distribution of 100 sample means from random samples with size 7, and

(4) a distribution of 100 sample means from random samples with size 49.

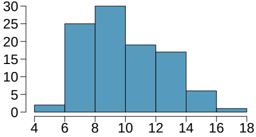


(a) A - (3); B - (2); C - (4)

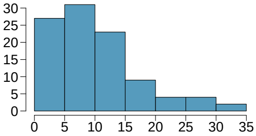
(b) A - (2); B - (3); C - (4)

(c) A - (3); B - (4); C - (2)

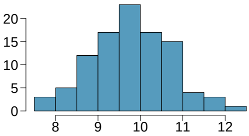
(d) A - (4); B - (2); C - (3)



Plot A



Plot B



Plot C

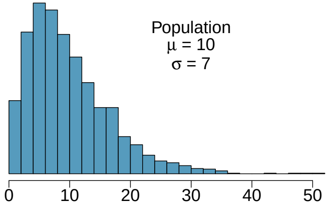
Four plots: Determine which plot (A, B, or C) is which.

(1) At top: distribution for a population ($\mu = 10, \sigma = 7$),

(2) a single random sample of 100 observations from this population,

(3) a distribution of 100 sample means from random samples with size 7, and

(4) a distribution of 100 sample means from random samples with size 49.



(a) *A - (3); B - (2); C - (4)*

(b) A - (2); B - (3); C - (4)

(c) A - (3); B - (4); C - (2)

(d) A - (4); B - (2); C - (3)