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CS682 Computer Vision

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Q4 Write Up

1) Prove that there exist a homography H that satisfies

$x_1 \equiv H \cdot x_2$

Solution:- 3D rotations can be defined as

$$R\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Think of as change of basis where $r_i = r(i,:)$ are orthogonal basis vectors.

Lots of parameters try to capture 3DOFs. Some of the helpful ones are orthogonal matrix, axis angle, exponential maps.

Perspective projection can be given by

$$\lambda \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$ where (x,y) and λ

can be computed by,

$\lambda x = f X$

$\lambda = Z$

$$x = \frac{\lambda x}{\lambda} = \frac{fX}{Z}$$

We can get 3D affine transformation by,

3D translations

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + T = \begin{bmatrix} X + t_x \\ Y + t_y \\ Z + t_z \end{bmatrix} \quad \text{— ①}$$

3D rotation or any invertible matrix R

$$R \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad \text{— ②}$$

From ① & ② we get,

$$R \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + T = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Now to get camera projection,

$$\lambda \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\downarrow} \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \end{bmatrix}}_{\downarrow} \underbrace{\begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}}_{\downarrow}$$

camera intrinsic matrix K which can include skew and non-square pixel size

camera extrinsics which is the rotation and translation

3D point in world coordinates.

We will now place world coordinate frame on object plane.

$$\lambda \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \end{bmatrix} \begin{bmatrix} X \\ Y \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \end{bmatrix} \begin{bmatrix} X \\ Y \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ r_{31} & r_{32} & t_z \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} fr_{11} & fr_{12} & ft_x \\ fr_{21} & fr_{22} & ft_y \\ r_{31} & r_{32} & t_z \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} \quad \text{— ③}$$

This holds true for any intrinsic matrix K and $\begin{bmatrix} fr_{11} & fr_{12} & ft_x \\ fr_{21} & fr_{22} & ft_y \\ r_{31} & r_{32} & t_z \end{bmatrix}$ is a 3×3 H matrix.

Now let us consider two points x_1 and x_2 in the images of a plane Π, then using ③ we get

for x_1,

$$\lambda_1 \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} = H_1 \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} \quad \text{— ④}$$

$$\lambda_2 \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} = H_2 \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} \quad \text{— ⑤}$$

We know that multiplication of a matrix A and its inverse A^{-1} results in a Identity matrix I.

consider ④,

Multiply with inverse H_1^{-1} on both the sides,

$$H_1^{-1} \lambda_1 \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} = \underbrace{H_1^{-1} H_1}_{I} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

$$H_1^{-1} \lambda_1 \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} \quad — ⑥$$

Substitute ⑥ in ⑤,

$$\lambda_2 \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} = H_2 H_1^{-1} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$$

Generalizing the above equation,

$$\lambda \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} = H_2 H_1^{-1} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$$

Both the LHS and RHS are related by a scale factor

$\therefore$ If the scene is planar, then images from any two cameras are related by a homography.

Estimating homographies for above scenario.

If (x_1, y_1) and H are given then (x_2, y_2) can be found by

$$\lambda \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$$

$$x_2 = \frac{\lambda x_2}{\lambda} = \frac{a x_1 + b y_1 + c}{g x_1 + h y_1 + i}$$

We can also estimate H, if the two points are given,

$$x_2 (gx_1 + hy_1 + i) = ax_1 + by_1 + c$$

$$\vdots$$

$$AH(:) = \begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix} \rightarrow \text{homogeneous linear system}$$

$$\min_{\|H(:)\|^2 = 1} \|AH(:)\|^2 \quad \text{Minimum right singular A vector}$$

$$(\text{eigenvector of } A^T A)$$

2.) Now we need to show that there exists a Homography H that satisfies the proof explained in Qn 1 given two cameras are separated by a pure rotation.

<u>Solution:</u> Given $x_1 = K_1 [I\,0] X$

$x_2 = K_2 [R\,0] X$ where I is 3×3 Identity matrix.

matrix and R is 3×3 Rotation matrix and

The K_1 and K_2 are the intrinsic matrices described in Q1.

Rewriting the given equations like in Q1.

$$\lambda_1 \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} = K_1 [I\,0] \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} \quad \text{where } K_1 = \begin{bmatrix} f_1 & 0 & 0 \\ 0 & f_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\lambda_2 \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} = K_2 [R\,0] \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} \quad K_2 = \begin{bmatrix} f_2 & 0 & 0 \\ 0 & f_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

K_1 and K_2 are distinct in our case.

$$\lambda_1 \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} = K_1 \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} \quad — ① \quad \text{as } [I\,0] \text{ is Identity matrix.}$$

$$\lambda_2 \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} = K_2 R \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} \quad — ②$$

Taking K_1^{-1} on both sides of ①

$$K_1^{-1} \lambda_1 \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} = K_1^{-1} K \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} \qquad \text{We know } A^{-1}A = I$$

$$K_1^{-1} \lambda_1 \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} \quad — ③$$

using ③ in ②

$$\lambda_2 \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} = K_2 R K_1^{-1} \lambda_1 \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$$

Generalizing,

$$\lambda \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} = K_2 R K_1^{-1} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$$

$\therefore$ If camera rotates about the center, then the images are related by a homography irrespective of scene depth.

3.) Show that H^2 is the homography corresponding to a rotation of 2θ.

We already know that there exist a homography between two points x_1 and x_2 given the cameras are separated by pure rotation.

From Q2,

$$\lambda \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} = K_2 R K_1^{-1} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} \quad — ①$$

Its is given that the intrinsic parameters K are constant. Therefore ① becomes

$$\lambda \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} = R \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$$

where $K_2 = K_1$

and $K_2 K_1^{-1} = I$

R is the 3×3 rotation matrix which is the Homography matrix H for our scenario. Let the angle of rotation be θ.

Now if we multiply H matrix with itself then

$H^2 = H \times H \Rightarrow R^2$

Taking the square inside the matrix R results in the conversion of θ in Sin and Cos elements to 2θ as $\cos^2\theta = \frac{1+\cos 2\theta}{2}$; $\sin^2\theta = \frac{1-\cos 2\theta}{2}$