10/2/20 CS682 Mithilaesh Jayakumar
G01206238

1) Suppose that a linear system $Ax = b$ is inconsistent.
This is often when the number of equation exceeds number of unknowns (Heterogeneous system).
In geometric terms, inconsistency means that b is not in image of A. So, we look for x such that $y = Ax$ is as close to b as possible.

$$\|Ax - b\| \text{ is minimum}$$

In other words, we are interested in a vector x^* such that

$$Ax^* = \text{proj}_{\text{im}A}\, b$$

Such a vector x^* is the solution for least squares

$$\|Ax - b\|^2 = \sum_k \left((A_x)_k - b_k\right)^2$$

Consider,

$$Ax^* = \text{proj}_{\text{im}A}\, b$$

$$b - Ax^* \perp \text{im}A \qquad (b - Ax^* \text{ is normal to im } A)$$

$$b - Ax^* \text{ is in } \ker A^T$$

$$A^T(b - Ax^*) = 0$$

$$A^TAx^* = A^Tb$$

$A^T A$ is a symmetric square matrix, if $A^T A$ is invertible then A has trivial kernel, therefore least squares solution is

$$x^* = (A^T A)^{-1} A^T b$$

(or) $$A_{x}^* = (A^T A)^{-1} A^T b$$

where $(A^T A)^{-1} A^T$ is a standard matrix of orthogonal projection onto the image A.

Disadvantage:-

If $A^T A$ is not invertible, then there are infinitely many least square solutionsbecause of which it results in poor fitting.

2) First, the equation $ax_i + by_i + c = 0$ represents a straight line in xy plane. If the points (x_i, y_i) for $i = 1, \dots N$ lie exactly on the line, then

$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots \\ x_N & y_N & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

i.e a homogeneous systen $Ax = 0$ with A $(n \times 3)$ matrix

The solution of a homogeneous system is required to determine the fundamental matrix from the given coordinates of the corresponding points in two images.

The solution of any homogeneous system $Ax = 0$ has two special features

(1) There is always a trivial solution $x = 0$

(2) If $x \neq 0$ is a solution, then kx, where k is a arbitary scalar.

These lead to the homogeneous least squares problem, where we need to find a least squares solution $\hat{x}$ such that the below conditions are fulfilled,

$$\|A\hat{x}\|^2 = \min$$

$$\|B\hat{x}\| = 1$$

The solution to the homogeneous least squares problem can be found by the computation of generalized eigen value decomposition.

By using Lagrange's method,

$$L(x,\lambda) = \|Ax\|^2 + \lambda(1 - \|Bx\|^2)$$

where λ is the Lagrange multiplier. Setting the partial derivatives of L with respect to x and λ to zero, we get,

$$A^T A x = \lambda B^T B x \quad \text{and} \quad \|Bx\|^2 = 1$$

so, $L = \|Ax\|^2 + \lambda(1 - \|Bx\|^2) = x^T A^T A x + \lambda(1 - x^T B^T B x) = \lambda$

The solution that minimizes L is, in particular, the generalized eigenvector e_{min} related to smallest generalized eigen value λ_{min}, ie $\hat{x} = e_{min}$. In order to fulfill the condition $\|B\hat{x}\| = 1$, a simple scaling is required

$$\boxed{\hat{x} = (1/\|Be_{min}\|)\, e_{min}}$$

This approach is best because,

→ No singular cases

→ No need of approximate values and iterations

→ Orthogonal residuals are minimized.

3.) Based on the explanations given for the homogeneous and heterogeneous least squares solution, we can conclude that homogeneous least square problem solution makes more sense geometrically.

1)

Let us represent a general conic as,

$$F(a, x) = a \cdot x = ax^2 + bxy + cy^2 + dx + ey + f = 0$$

where $a = [a\ b\ c\ d\ e\ f]^T$ and $x = [x^2\ xy\ y^2\ x\ y\ 1]^T$

The fitting of a general conic may be done by minimizing the sum of squared algebric distances:

$$D_A(a) = \sum_{i=1}^{N} F(x_i)^2$$

In order to avoid any trivial solution $a = 0$ and recognize any multiple of a solution a representing the same conic, we add a constraint

$a^T C a = 1$ where C is 6×6 constraint matrix

Introducing a Lagrange multiplier λ and differentiating we get,

$\| D a \|^2$, subject to $a^T C a = 1$ becomes

$$\boxed{\begin{gathered} 2D^T Da - 2\lambda Ca = 0 \\ a^T Ca = 1 \end{gathered}}$$

rewritten as,

$$Sa = \lambda Ca$$

$$a^T Ca = 1$$

where S is $D^T D$

2) The minimum value of N that leads to a unique solution would be $\boxed{N=2}$

3) A conic in a plane is represented as,

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

This can be converted to homogeneous coordinates by replacing $x \to x_1/x_3$ and $y \to x_2/x_3$

Now the homogeneous conic equation is,

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

This can be written in matrix form as,

$$x^T C x = 0$$

where C is the conic coefficient matrix,

$$C = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

The matrix C is a homogeneous representation of a conic that has 5 degrees of freedom, 6 elements of the symmetric matrix minus one for scale.

4) Recall that for a point correspondence $x \leftrightarrow x'$, the points are related by the homography matrix H such that $x' = Hx$. Since H is invertible, this can also be written as $x = H^{-1}x'$. Substituting this into the conic equation we get,

$$x'^T H^{-T} C H^{-1} x' = 0$$

This gives the transformation rule for a conic,

$$C' = H^{-T} C H^{-1}.$$

Since H has 8 degrees of freedom and a conic provides 5 degrees of freedom, at least 2 conics would be required to solve for H.