Review of “EnFK-based Dynamic Estimation of Seperated Flows with a Low-Order Vortex Model”

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For this project, I reviewed [1], where the authors used an Ensemble Kalman Filter to estimate the circulating, unsteady aerodynamics over a plate with a set of pressure measurements. The focus of this final project is to present, assess, and critique their estimation application. In this report, I first introduce vortex modeling as it is applied in [1] and some key terms to understand the work. Next, I present the author’s estimation technique and justify or critique their choices. To support their claims and aid my analysis, I reproduce some figures of relevant simulation examples from their work and connect the findings back to their estimation technique. To conclude, I discuss some future research directions based on their work and summarize changes or critiques I believe would improve the author’s argument and formulation.

I. Nomenclature

t	=	nondimensional time
α	=	angle of attack
Re	=	Reynolds number
(x, y)	=	two-dimensional blob position
Γ	=	blob vortex strength
n	=	number of blobs approximating the flow
$\boldsymbol{x}$	=	state vector
$\boldsymbol{m}$	=	estimated measurement vector, $\boldsymbol{m} = \boldsymbol{h}(\boldsymbol{x})$
m	=	number of pressure measurements along the plate
$LESP^c$	=	critical leading edge suction pressure
$\boldsymbol{f}$	=	dynamic model vector
$\boldsymbol{h}$	=	measurement model vector
$\boldsymbol{X}$	=	augmented state vector, $\boldsymbol{X} = [\boldsymbol{x}, \boldsymbol{m}]^T$
H	=	measurement Jacobian of the augmented state vector
N	=	number of ensembles in the filter
M_k	=	measurement covariance among the ensembles
C_k	=	state-measurement cross covariance among the ensembles
P^a	=	augmented state covariance
$\boldsymbol{z}$	=	measurement vector
V_k	=	theoretical measurement covariance
K_k	=	Kalman gain matrix
σ	=	covariance of an individual state among the ensembles
LE	=	leading edge or the edge facing incoming flow
TE	=	trailing edge or the edge downstream of the flow

Superscripts, subscripts, and styling

k	=	subscript, time index
j	=	superscript or subscript, ensemble element
BOLD	=	bolded variables indicates a vector
$\sim$	=	tilde indicates a filtered estimate

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- = superscript, indicates a pre-filtered estimate rather than the ground truth

∧ = hat, the mean quantity of the ensembles

II. Background and Scope

This section will present a necessary background in aerodynamic modeling to help understand the paper [1] and the estimation results. After, I will define the scope of this project and expectations for the report.

A. Vortex Modeling Background

Vortex elements represent discontinuities or circulations in a flow. In unsteady aerodynamic modeling, these elements make approximations of the flow and are often used as low-order models. The primary benefit of these models are their computational efficiency - tracking vortices (i.e. blobs) can give a quick approximation of the flow, while "true" flows require high-fidelity CFD simulations. These vortex models that approximate the flow can vary from low fidelity-quick computation to high-fidelity-slow computation models. In this project, the authors use [2] as the basis of their vortex model. In brief, these models determine blob position and strength to coarsely approximate the discontinuities and the resulting fluid pressure and velocity distribution. The reader is referred to [3] for approximation and modeling details.

While the nuances of low-order vortex modeling are left to the reader, I will present a quick analysis of the Leading Edge Suction Pressure (LESP). This term characterizes how low the relative pressure is at the LE of a plate and is a time-continuous property of the flow. If the LESP is sufficnetly high (i.e. the relative pressure is sufficiently low) and above a critical LESP value ($LESP^c$), new vortices are shed from the LE. While the critical LESP value is usually a property of the flow characteristics, it will vary with changes to the plate geometry or position, and therefore, is quite difficult to exactly estimate. More recent literature on calculating and modeling the LESP is found in [4, 5]. The author's vortex model in [2] provides the dynamics to calculate the critical LESP. Finally, Fig. 1 presents a graphical representation of the role of this parameter.

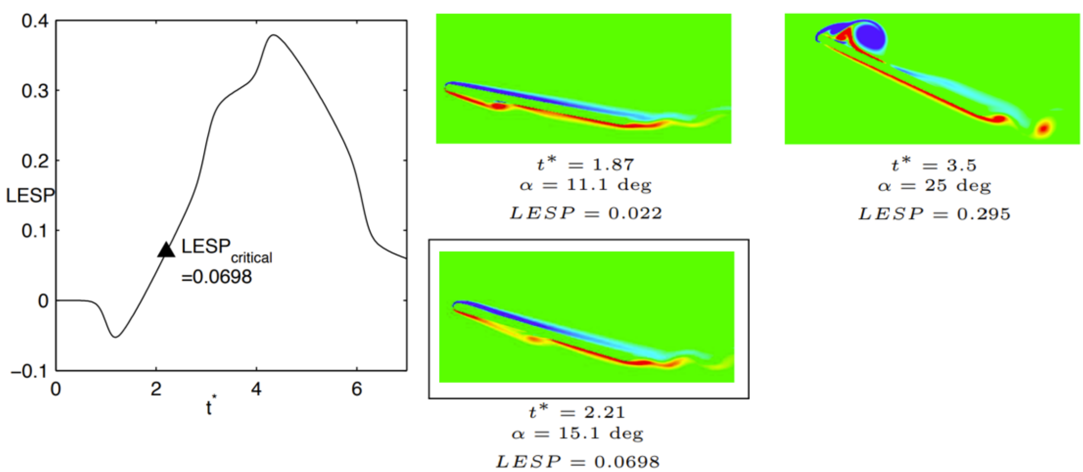

Fig. 1 In this paper's simulations, they increase a plate's angle of attack relative to the flow. As α increases, the LESP follows, eventually reaching the critical value, where the plate begins to shed vortices. The figures on the right show CFD simulations of the experiment, with the rightmost showing the shedding vortex after the critical LESP is exceeded. Figures are reproduced from [4].

B. Motivation

The control of highly-agile aircraft relies on the quick and real-time estimation of the unsteady aerodynamics over an airfoil. Therefore, the goal of using a low-order vortex model is that it may be run online and efficiently computed. The high-level motivation of the author's work is to develop a filter that can estimate the unsteady aerodynamics *in* *real time* using this lower-order model and a set of pressure measurements. This work may find application in high-α research or for controlling agile aircraft.

C. Project Scope

The goal of this final project is to explain how they approach this complex filtering problem. What makes this paper interesting and unique is the high dimension of the system and the highly-nonlinear nature of the dynamics. In section III, I will first present the dynamic and measurement model used. In section IV, I will present their filtering technique exactly as found in the work. I will comment on and critique their filter in this section as well. In section V, I will summarize their simulation results and will tie some of their findings back to their filter derivation, extending the analysis beyond what was found in the paper. Finally, in section VI, I will detail my revision to the work and potential extensions to help clarify or build upon their work.

III. Dynamic and Measurement Models

In this section, I will present a general overview of the dynamic and measurement model, but leave the theoretical details to the reader to review. One key paper detailing the author's vortex models is [2]; this would present the necessary details to begin reproducing their results.

To start, the author's dynamic model [1] is a nonlinear state transition function $\boldsymbol{f_k}$ that propagates the state one time step by:

1) computing the bound vortex sheet strength on the plate using the vortex blob positions and strengths
2) computing the velocities of the vortex blobs
3) advecting the plate and the vortex blobs
4) applying the vortex aggregation algorithm, reducing the strength of any aggregated blobs to zero instead of completely removing the blob
5) releasing a new vortex blob from each edge of the plate with strengths based on the Kutta condition at the trailing edge and the current estimate of the critical LESP.

Details of the vortex aggregation algorithm (step 4) are found in [6] and pages 4-5 of [1]. To quickly summarize for the relevancy of the project, a pairwise velocity correction is made between *each* blob in the simulation; details found in [7]. Then, the blobs are convected one time step and the corresponding impulse is calculated. The error between the pairwise velocity correction and the corresponding impulse is calculated. This error is characterizing how much of an effect each blob has on each other in the simulation, so those with low error are then accumulated to reduce the number of blobs.

The author's measurement model $\boldsymbol{h_k}$ is calculated using the change in impulse of the blob on the plate over a time step to compute the force. With a known force along the plate, pressure may be found, forming the basis of their measurement model [2].

IV. Methodology and Application

This section will detail the methods deployed in [1] to estimate the unsteady aerodynamics. The author's derivation is first presented from the paper, followed by my critique and discussion.

A. Filter Application

The authors use an EnKF with a vortex model to estimate the unsteady flow over a plate [1, 2]. In the EnKF, we must first define a state vector that defines the system, a dynamic model, and a measurement model. In [1], the vortex model is composed of a set of *n* distinct blobs, each with a two dimensional position and strength. These blobs, along with the estimated critical LESP, will define the state vector

$$\boldsymbol{x_k} = [x_k^{(1)}, y_k^{(1)}, \Gamma_k^{(1)}, \ldots, x_k^{(n)}, y_k^{(n)}, \Gamma_k^{(n)}, LESP_k^c]^T \in \mathbb{R}^{3n+1} \tag{1}$$

The dynamic and measurement model are then defined by the vortex-blob and vortex-plate interactions. Here, the authors treat the vortex propagation and accumulation algorithm as the dynamic model and the vortex-surface pressure calculation as the measurement model, both highly nonlinear functions. A description of these are found in the previous section, but they may take the general form below:

$$\boldsymbol{x}_{\boldsymbol{k+1}}^{(\boldsymbol{j})} = \boldsymbol{f}(\boldsymbol{x}_{\boldsymbol{k}}^{(\boldsymbol{j})}) \tag{2}$$

$$\boldsymbol{m}_{\boldsymbol{j},\boldsymbol{k}} = \boldsymbol{h}_{\boldsymbol{k}}(\tilde{\boldsymbol{x}}_{\boldsymbol{k}}^{(\boldsymbol{j})}) \tag{3}$$

It is worth noting that the dynamics do not have a noise term, i.e. there is no process noise in the model, but there is measurement noise. The authors do not, unfortunately, tell us how *much* measurement noise or what type or noise it is. The authors then convert the problem into a classical Kalman formulation. One issue, however, is that the measurement function is highly nonlinear, which much be linearized (note: they *claim* that H must be linearized, but as I will show later, other derivations exist where this is not necessary). The authors first augment the state vector to include the measurements.

$$\boldsymbol{X}_{\boldsymbol{j},\boldsymbol{k}} = \begin{bmatrix} \tilde{\boldsymbol{x}}_{\boldsymbol{j},\boldsymbol{k}}^{-} \\ \boldsymbol{m}_{\boldsymbol{j},\boldsymbol{k}} \end{bmatrix} \in \mathbb{R}^{3n+1+M} \tag{4}$$

$$H = \begin{bmatrix} 0^{(3n+1)\times(3n+1)} & 0^{(3n+1)\times M} \\ 0^{M\times(3n+1)} & I^{M\times M} \end{bmatrix} \forall k \tag{5}$$

Then, when the Jacobian of the augmented state vector is taken with respect to $\boldsymbol{m}_{\boldsymbol{k}}$, linearizing H becomes tractable. Next, the authors define the cross covariance between the measurement and non-augmented states, the measurement covariance, and the augmented state covariance

$$M_k = \frac{1}{N-1}\sum_{j=1}^{N}(\boldsymbol{m}_{\boldsymbol{j},\boldsymbol{k}} - \bar{\boldsymbol{m}}_{\boldsymbol{k}})(\boldsymbol{m}_{\boldsymbol{j},\boldsymbol{k}} - \bar{\boldsymbol{m}}_{\boldsymbol{k}})^T \tag{6}$$

$$C_k = \frac{1}{N-1}\sum_{j=1}^{N}(\boldsymbol{m}_{\boldsymbol{j},\boldsymbol{k}} - \bar{\boldsymbol{m}}_{\boldsymbol{k}})(\tilde{\boldsymbol{x}}_{\boldsymbol{j},\boldsymbol{k}}^{-} - \bar{\boldsymbol{x}}_{\boldsymbol{k}})^T \tag{7}$$

$$P_k^a = \frac{1}{N-1}\sum_{j=1}^{N}(\boldsymbol{X}_{\boldsymbol{j},\boldsymbol{k}} - \bar{\boldsymbol{X}}_{\boldsymbol{k}})(\boldsymbol{X}_{\boldsymbol{j},\boldsymbol{k}} - \bar{\boldsymbol{X}}_{\boldsymbol{k}})^T = \begin{bmatrix} P_k^- & C_k \\ C_k^T & M_k \end{bmatrix} \tag{8}$$

Note that P_k^a is composed of the cross and measurement covariances as a block matrix. Then, when the Kalman gain of the augmented system is computed, the zero matrices from the linearized H eliminate the state covariance, yielding

$$K_k^a = P_k^a H^T (V_k + H P_k^a H^T)^{-1} = \begin{bmatrix} C_k (V_k + M_k)^{-1} \\ M_k (V_k + M_k)^{-1} \end{bmatrix} \tag{9}$$

The Kalman update of each ensemble is

$$\tilde{\boldsymbol{x}}_{\boldsymbol{j},\boldsymbol{k}} = \tilde{\boldsymbol{x}}_{\boldsymbol{j},\boldsymbol{k}}^{-} + C_k (V_k + M_k)^{-1}(\boldsymbol{z}_{\boldsymbol{k}} + \boldsymbol{v}_{\boldsymbol{k}}^{(\boldsymbol{j})} - \boldsymbol{h}(\tilde{\boldsymbol{x}}_{\boldsymbol{j},\boldsymbol{k}}^{-})) \tag{10}$$

$$\boldsymbol{v}^{(\boldsymbol{j})} = \mathcal{N}(0, V_k) \tag{11}$$

B. Covariance Inflation

In [1], the authors claim that the covariance ensemble, P_k will become ever-decreasing. This seems to agree with other literature [7–9], but is generally more applicable for very large state systems (e.g. meteorological data), which may not be realistic here. Regardless, the authors employ covariance inflation as part of their filter, where the ensemble state is updated after computing the sample mean, but before computing the sample covariance:

$$\tilde{\boldsymbol{x}}_{\boldsymbol{j},\boldsymbol{k}}^{-} \leftarrow \hat{\boldsymbol{x}}_{\boldsymbol{k}}^{-} + \beta(\tilde{\boldsymbol{x}}_{\boldsymbol{j},\boldsymbol{k}}^{-} - \hat{\boldsymbol{x}}_{\boldsymbol{k}}^{-}) + \boldsymbol{\alpha}_{\boldsymbol{j},\boldsymbol{k}} \tag{12}$$

where

- $\beta = 1\%$ is the multiplicate inflation
- α is the additive inflation which adds zero-mean Gaussian noise depending on the state.

- vortex positions undergo additive perturbations drawn from $\mathcal{N}(0, 10^{-5}c)$,
- vortex strengths undergo additive perturbations drawn from $\mathcal{N}(0, 10^{-3}/\Delta t)$, and
- the critical LESP undergoes additive perturbations drawn from $\mathcal{N}(0, 5 \times 10^{-5})$

The authors do not discuss how the parameters were tuned, only that they found a "suitable set." As shown in the next section, this added covariance will become critical for the filter's success. I will later discuss how this covariance inflation appears to compensate for the author's derivation of the Kalman gain.

C. Discussion of Approach

I agree that the EnKF is an appropriate choice of filter given that the dimension of the state is quite large and continues to grow with time. The author's algebra appears correct and reproducible. In a strict sense, their augmentation of the state vector and linearization does match the classical Kalman filter gain, but I am unsure as to why they took this approach - this is my first critique of their work. I first compared their derivation to Crassidis' in [10] and found a striking difference. First, in Crassidis' derivation, he does not linearize $\boldsymbol{h}$. By not taking this step, Crassidis' Kalman gain is (in his notation):

$$K_k = P^{e_x e_y}(P^{e_y e_y})^{-1} \tag{13}$$

where $P^{e_x e_y} = C_k$ and $P^{e_y e_y} = M_k$. I first note that the augmentation and linearization steps in the previous section are unnecessary if $boldsymbol h$ does not need to be linearized, which would make the problem much simpler. Regardless, I next compare the update steps between Crassidis' implementation and the author's.

In comparison, the author's gain included the addition of an inverted term for measurement variance, V_k, which is **likely underestimating the Kalman gain compared to Crassidis' derivation**. Interestingly, the authors noted the need for covariance inflation, where they add zero-mean Gaussian noise to the states between time steps, to achieve reasonable results with their filter. This approach is noted throughout the literature [8], but in this formulation, it seems to be compensating for the inverse measurement variance. The authors should have elaborated on this tension between adding variance but also reducing it in the Kalman gain. Finally, when I looked through the citations on their EnKF approach, they only list one from 1994 [11]. I think more relevant citations to back up their approach is needed given this difference and the necessity of covariance inflation. Overall, their approach is *much more complicated than Crassidis'* and warrants further investigation into why they performed it this way.

V. Simulation and Results

The authors then simulated their filter. In their simulation, they initialized $N = 50$ ensembles and $M = 50$ pressure measurement points along a thin plate. They simulated $Re = 500$ flow over the plate angled at $\alpha = 20^o$ to induce unsteady aerodynamics. Their simulations included three cases: a pulse-free case, a single pulse case at $t = 3$ nondimensional units, and a double-pulse case at both $t = 3$ and $t = 5$. Here, the pulse represents a "gust" that instantaneously perturbs the plate. In each case, three filtering approaches were taken: no covariance inflation is applied, multiplicative covariance inflation only is applied, and both multiplicative and additive covariance inflation is applied. This section will be composed of a subsection presenting their results and a subsection of my discussion

A. Pulse-Free Case

For the no-pulse case, the simulation was run and blobs were released according to the estimated critical LESP. Two selected time slices are shown in Figs. 2-3; the reader is referred to Figs. 10-13 in [1] for complete plots. By looking closely at each figure, the multiplicative-only and no inflation cases show blobs that "teleport" and appear to ignore the natural advection shown in the ground-truth CFD. Note that the author's are reporting this teleporting as they conduct their research, but this is harder to see in static figures. It is slightly clearer in the original paper than in this report. The multiplicative and additive inflation, however, show blobs that appear to more naturally convect according to the ground-truth CFD.

Next, the authors show a figure comparing the estimated pressure distribution along the plate through time, reproduced here in Fig. 4. As with the time-series plots from Figs. 2-3, the no inflation and multiplicative inflation demonstrate poor estimates of the flow and show strong cross-crossing of the pressure distribution. While these filters appears to capture nominal trends in the ground truth, they makes a poor spatial estimation. The multiplicative and additive inflation, however, shows a close approximation to the ground truth and appears estimate the unsteady aerodynamics well.

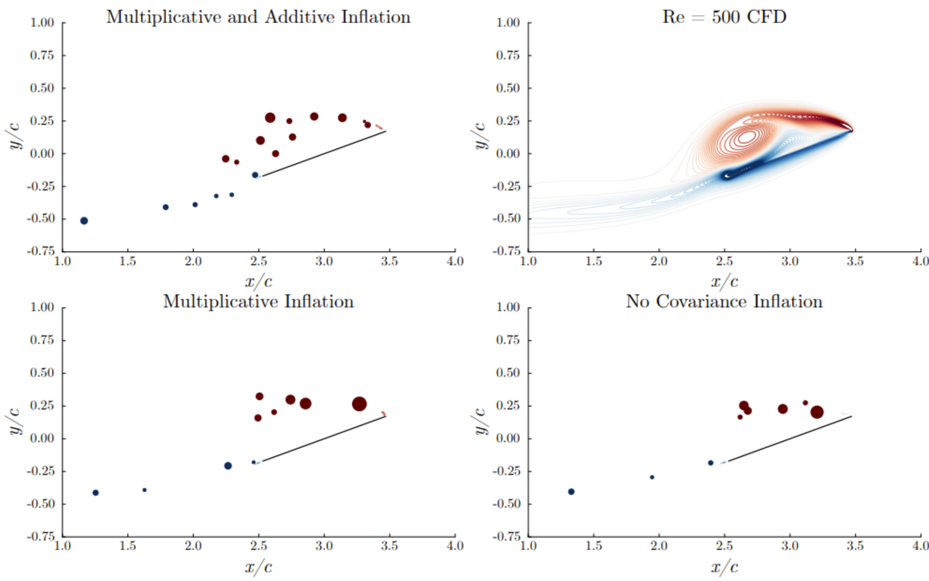

Fig. 2 Estimates of the no-pulse flow using plate pressure measurements and the EnKF at t = 3. The three filtering cases are compared to the ground-truth CFD simulation. Reproduced from [1].

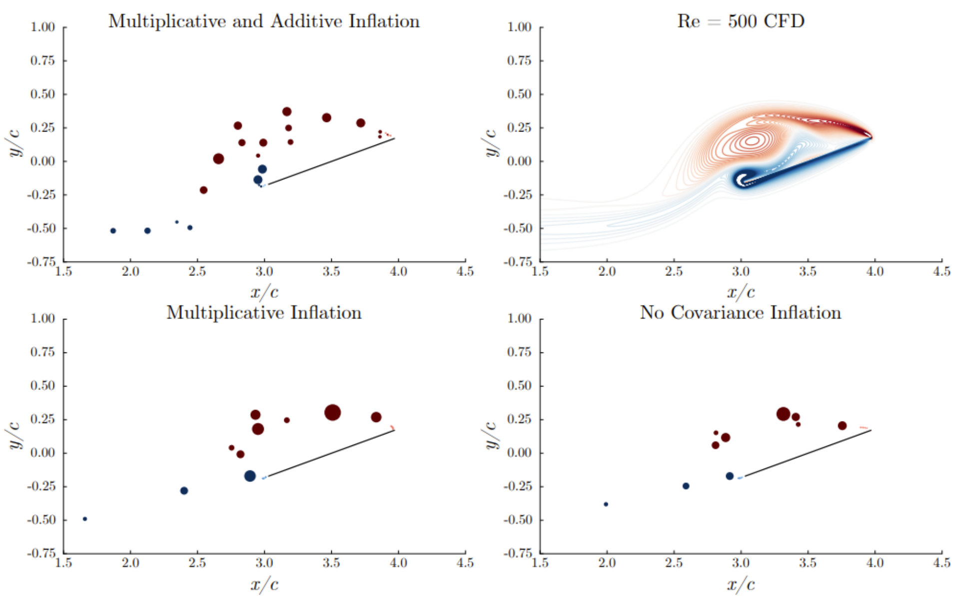

Fig. 3 Estimates of the no-pulse flow using plate pressure measurements and the EnKF at t = 4. The three filtering cases are compared to the ground-truth CFD simulation. Reproduced from [1].

Finally, the authors present some evidence to explain the results they see. First, the authors plotted the ensemble covariance of the position, strength, and critical LESP of the three filtering cases, see Fig. 5 in this text. For full details, the reader is referred to Figs. 5-7 in [1]. Under no and multiplicative inflation, the critical LESP covariance collapses to machine precision, while in the multiplicative and additive inflation case, the critical LESP uncertainty keeps a non-trivial value. This collapse indicates that the ensembles are all converging to the same, likely erroneous, value, while the filtering case with added inflation keeps enough variance among the ensembles to make a good estimate.

From the results presented thus far, the authors conclude that the covariance collapse produces poor results for two reasons. First, the gain, a product of the ensemble covariance, goes to zero, and the estimator is relying entirely on the

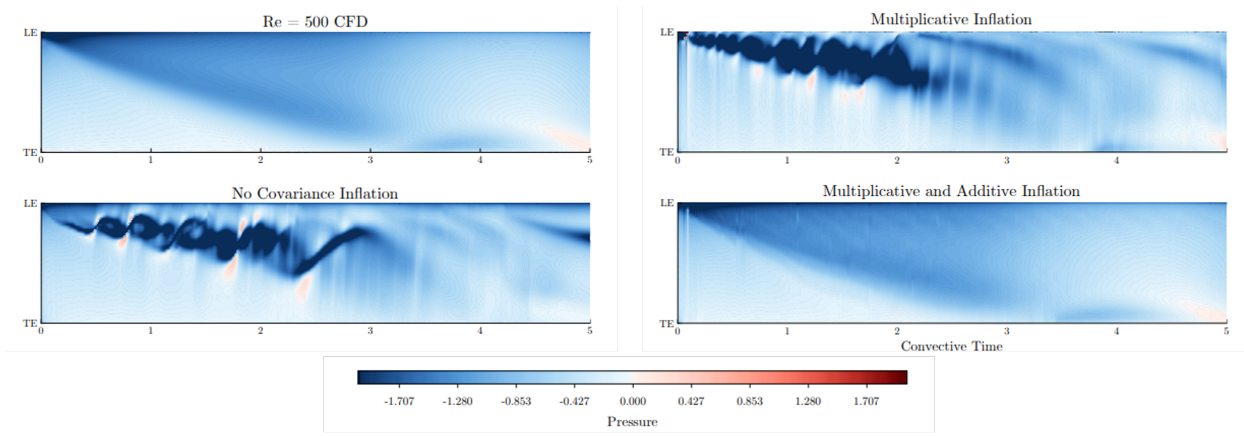

Fig. 4 The estimated pressure across the plate through time (x-axis) for the three filtering cases as compared to the ground-truth CFD with no pulse. Reproduced from [1].

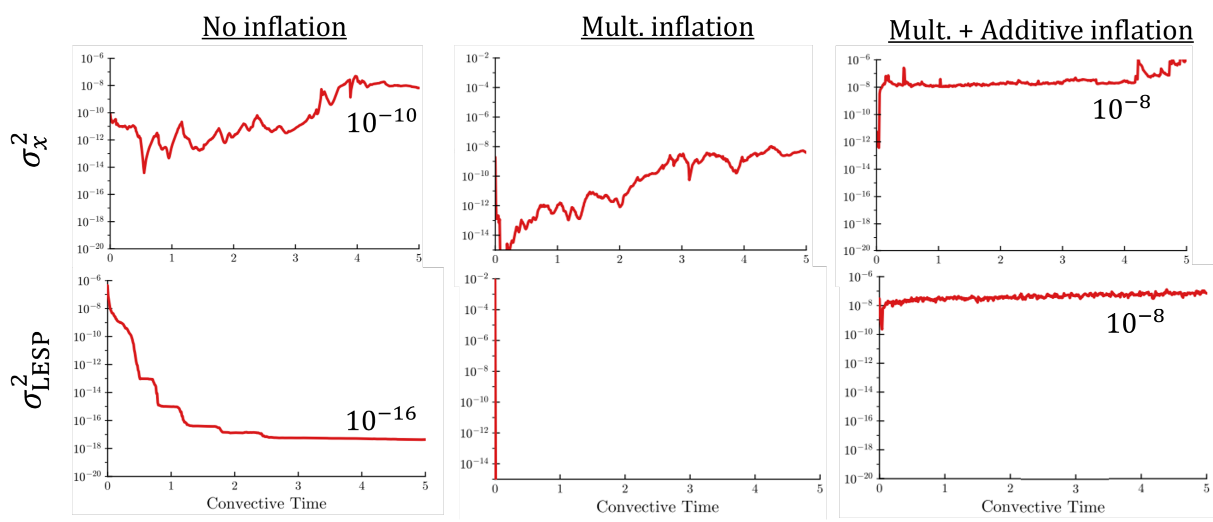

Fig. 5 The ensemble covariance of the blob position (x-only) and the critical LESP. Summarized from [1] with key values called out. ***Note the scales on these figures!***

model, rather than the measurements, to make the estimate. Because the blob model is, by construction for a real-time application, low fidelity, the filter is producing poor results. Second, the critical LESP is significantly mismodeled when the covariance collapses. Because the LESP is incorrect, vortex blobs may not be released from the LE and the filter can only move existing blob positions to try and match the pressure distribution. This helps explain the "teleporting" of blobs they noted.

B. Pulsed Cases

Many of the same results shown in the pulse-free case return under the pulsed cases. Subsequently, this section of the author's work seems almost like a validation section, and I will present just the necessary results. In the single and double pulse case, the estimated pressure distribution through time are found in Figs. 6-7. Full details are found in [1] and include: Fig. 14 - normal force approximation of a single pulse, Fig. 18-19 - ensemble covariances for the different filtering types of a single pulse, Fig. 20-23 - time series plots of the blobs under a single pulse; Fig. 25 - normal force approximation of two pulses, Fig. 28-29 - ensemble covariances for the different filtering types of two pulses, Fig. 30-33 - time series plots of the blobs under two pulses.

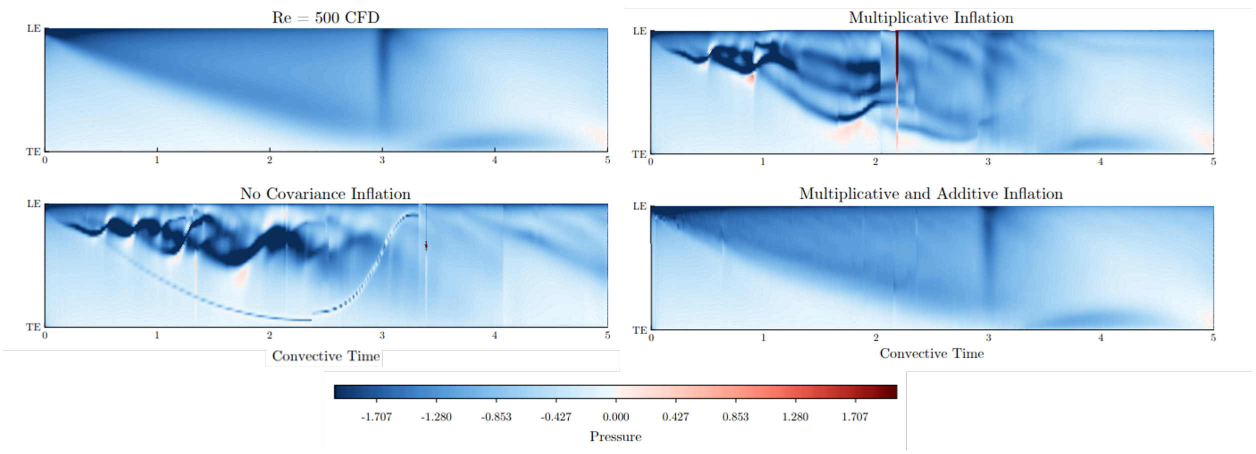

Fig. 6 The estimated pressure across the plate through time (x-axis) for the three filtering cases as compared to the ground-truth CFD with a single pulse at t = 3. Reproduced from [1].

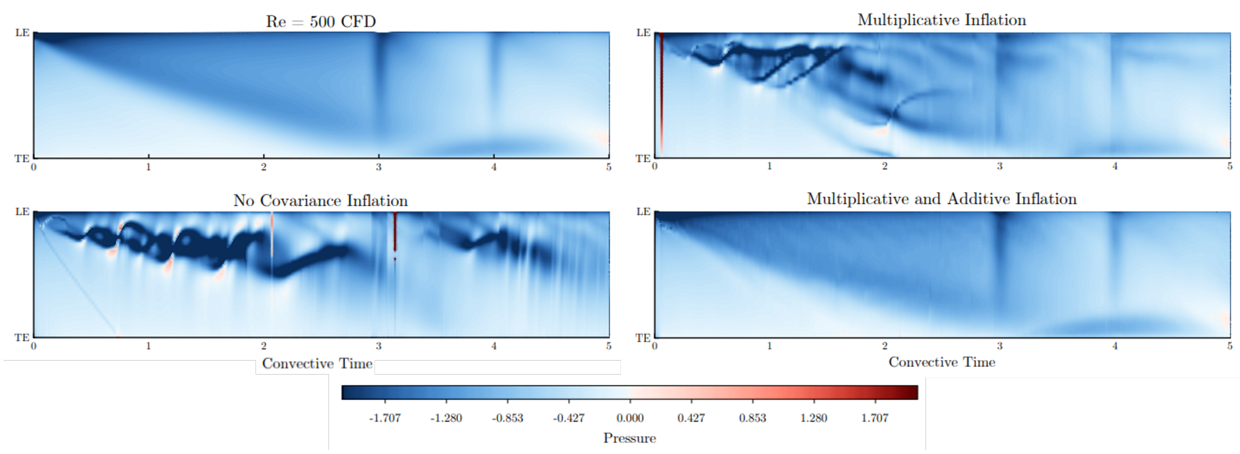

Fig. 7 The estimated pressure across the plate through time (x-axis) for the three filtering cases as compared to the ground-truth CFD with a single pulse at t = 5. Reproduced from [1].

C. Discussion of Results

To start, I agree with the author's conclusions. The evidence suggests that a collapse in covariance inflation causes the filter to rely too strongly on the low fidelity model. Their evidence of the ensemble covariances and the "teleporting" blobs supports this well. To critique their work however, I believe the necessity of covariance inflation may arise from their approach. From my discussion in the filter derivation section, these results seem to be symptomatic of the effective gain reduction under their derivation. While the authors get consistent results using covariance inflation, I am curious if the covariance would be necessary under a better researched or differing implementation. The authors do discuss how the covariance inflation is necessary for consistent results, but they do not do so in context with their derivation. This lack of connection or analysis between their derivation and the results is, again, my major critique of the work. In addition, there are some analyses that, if included, would strengthen their argument or clarify the paper:

- The scaling of the covariance figures is inconsistent and makes comparison between different filtering cases difficult or impossible. For example, the authors note covariance collapse and see the critical LESP ensemble covariance going to machine precision in one figure but do not scale another figure where it is likely occurring such that this collapse could be seen Fig. 5.
- Given a ground truth, the authors do not present a quantitative error analysis of their filter. "Good agreement" is left to the eye of the reader, but the authors had quantitative data to characterize this. It seems odd why they didn't make this quantitative comparison. In addition, the authors should consider presenting consistency results, like calculating autocorrelation or NES, to demonstrate the performance of their filter.

- A major argument for their work is the ability to work online or in real-time, but the authors do not present evidence of a computation count, computation time, or FLOPS that could indicate whether their implementation would work in real time.
- The authors claim that the covariance is collapsing and not incorporating measurement information, but only discuss this rather than demonstrate it. They can quantify this by measuring the Kalman gain through time (via a matrix norm or the likes), which would strengthen their argument.

VI. Extension or Revision of their Approach

A. Revisions

To summarize my main critiques of the work thus far:

- The author's derivation appears to be underestimating the Kalman gain when compared to other derivations. This appears to impact their results, but the authors do not connect their derivation to the filter's performance.
- The authors do not present a quantitative analysis of the error and rely on qualitative comparisons between high-fidelity CFD results and the filter to make their claim.
- The work does not perform an computational cost analysis, which would be pertinent to online estimation.

These critiques form the basis of how I would revise the work. There are two main revisions that I would perform if building on or reviewing their work. First, I would perform a quantitative analysis of the error using pressure, the ground-truth vortex model with no measurement or modeling error, or some other relevant aerodynamic quantity (such as normal force, coefficient of lift and drag, or other pertinent quantities). Second I would, at a minimum, provide timing details on how long the filter takes to run under a defined machine. Ideally, I would instead extend the analysis to count floating-point operations, but this may be unnecessary given simple timing of the filter.

B. Extensions

After conducting a close reading of the paper in context with the class material, there are some logical extensions that would extend or elaborate on their hypothesis. I would first implement Crassidis' description of the EnKF and compare how the results differ with and without linearizing $\boldsymbol{h}$. This would help determine what role covariance inflation is playing in the estimation process and confirm if the covariance collapse is fundamental to the filter or a facet of their implementation.

The author's conclude that inaccurate estimation of the critical LESP results in qualitatively poor results and justify their reasoning by tracking how to filter moves the blobs. Therefore, they determine that correct estimation of the critical LESP is key to yielding accurate results. As an extension, I would consider using a more sophisticated model of the critical LESP in the dynamics to try and improve the results of the filter.

Existing works that deploy the EnKF have states much larger than the one here [8]. The primary advantage of the EnKF in high-dimensional spaces is computational savings. In this problem, however, the dimension relatively small, and the author's dynamic model requires significant computational loads to aggregate and convect blobs. Similarly, the measurement model must numerically solve $2n + 1$ integrals, which is costly. As an extension, I would try estimating the unsteady flow with a mesh-based estimation scheme using the same pressure measurements. An example of this approach is in [12]. While the state would be much higher (on the order of $10^4 - 10^5$), the computational load of the dynamics would be much smaller by not performing the costly computations in the blob-based dynamic and measurement model. Therefore, direct estimation of the flow might be computationally comparable to the author's vortex-based filtering scheme and yield potentially more accurate results.

Finally, I would consider using a Kalman filter itself. The EnKF is usually deployed for systems with massive states, but here, the state dimension is, at most, around 100. To first compare, the Kalman filter must make an inverse operation, $O(n^3)$, which would be one million computations. The EnKF takes a $O(nMN)$ operation, which is 250,000 computations. By calculating the covariance directly, rather than from a sample estimate as in the EnKF, the standard Kalman filter might show improved results and this modest 4x increase in computation cost might be worth it.

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