FYMM/MMP IIIa 2020 Solutions to Problem Set 4

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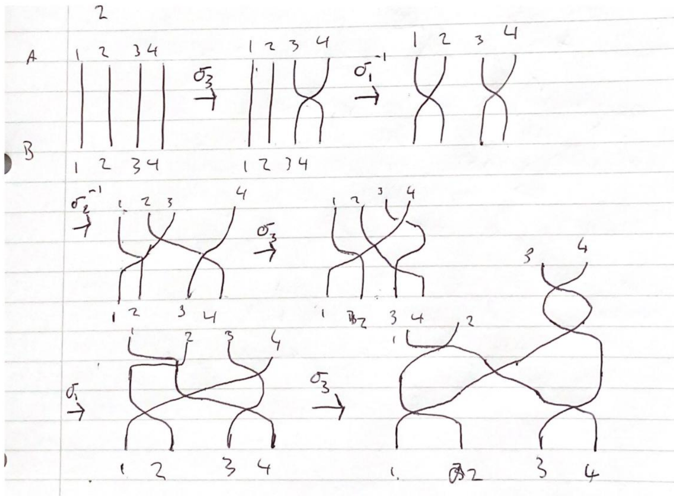
1. Question 1

$$D_4 = \langle r, f | r^4, f^2, rfrf \rangle$$

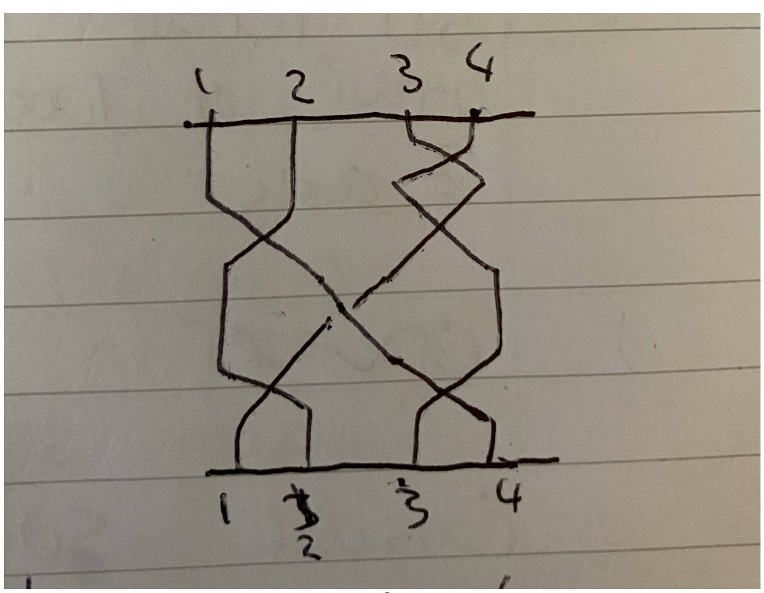
   Where r = 90 degree rotation and f = reflection. Need to find and motivate the presentation for D_n. Clearly $r^4 = e$ as this is equivalent to a 360 degree rotation and clearly $f^2 = e$ due to it being reflected twice. $rfrf = e$ as well.
   For a polygon with n sides it can be rotated by 360 degree as the operation r and reflected by f. The n sided polygon will always naturally have 2 reflections, one after the other to return it to its original state due to a natural property of reflections so for D_n $f^2 = e$ always.
   The n sided polygon rotated n times through through rotation r will also similarly return it to its original state as a nature of the rotation around a point, therefore $r^n = e$.
   Also a reflection (f) then a rotation (r) then a reflection (f) then a rotation (r) where r is 90 degrees will always be the identity for n sided polygons. So

$$D_n = \langle r, f | r^n, f^2, rfrf \rangle$$

2. Draw a picture of the braid (of 4 strands) $\sigma_3 \sigma_1^{-1} \sigma_2^{-1} \sigma_3 \sigma_1 \sigma_3$.
   σ_i^{-1} moves the $i+1$ strand over the ith strand. σ_3 moves the 3rd strand over the 4th strand. σ_1^{-1} moves the 2nd strand over the first strand. σ_2^{-1} moves the 3rd strand over the 2nd strand. σ_1 moves the 1st strand over the 2nd strand and σ_3 moves the 3rd strand over the 4th strand. When drawing we take the equation from left to right and draw from the bottom up.



Kuva 1: Process of me creating the braid going through each step



Kuva 2: Drawing of the braid in final form

3. Question 3
   SO(3) left action on the sphere $S^2 \in \Re^3$. SO(3) is the 3D rotation group i.e the group of rotations of the xy plane about the z axis.
   Didn't really understand how to use the vector x to parametrize this isotropy group.

4. Consider the set of *Möbius transformations*

$$\text{Mob} = \left\{ f_A : \mathbb{C} \to \mathbb{C} | f_A(z) = \frac{az+b}{cz+d}; \ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}) \right\} \tag{1}$$

   (a) Show that Mob is a group, with composition of mappings as the product.
   To show this go through the usual steps to prove a group, starting with closure and associativity (G0 & G1).
   Let $f_A, f_B \in$ Mob with SL matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}; \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

   Such that

$$(f_B \circ f_A)(z) = \frac{a'a + b'c)z + (b'b + b'd)z}{(c'a + cd')z + (cd' + dd')z}$$
$$= f_{BA}(z)$$

   With BA the product of B and A, $BA \in SL$ and $f_{BA} \in$ Mob. Proving closure with associativity inherited.
   For G2, the exitence of the unit element. The unit element of Mobis $f_{\mathbb{1}}$ which is the complex identity $f_{\mathbb{1}} = id_c$ so we have

$$f_A \circ f_{\mathbb{1}} = f_{A\mathbb{1}} = f_A = f_{\mathbb{1}} \circ f_A$$

   Now we know trivially that $\mathbb{1} \in SL$ amd $f_{\mathbb{1}} \in$ Mob.
   Existence of the inverse.

$$A^{-1} \in SL$$
$$f_A \circ f_{A^{-1}} = f_{AA^{-1}} = f_{\mathbb{1}} = f_{A^{-1}} \circ f_A$$

   So

$$f_{A^{-1}} = f_A^{-1}$$

   And it is a group.

   (b) Show that the mapping

$$f : SL(2,\mathbb{C}) \to \text{Mob}; \ f(A) = f_A \tag{2}$$

   is a homomorphism.
   For it to be a homomorphism it needs to have $\forall g_1, g_2 \in G, f(g_1 g_2) = f(g_1)f(g_2)$.
   So from closure we have

$$f(AB) = f_A B = f_A \circ f_B = f(A) \circ f(B)$$

   So there exists a group homomorphism.

(c) Find a subgroup H of SL$(2,\mathbb{C})$ such that the quotient group SL$(2,\mathbb{C})/H$ is isomorphic to Mob. Give reasons why.
Taken inspiration from example at the top of page 22 of the lecture notes. Let G and H be groups such that

$$\phi : G \to H \quad \text{Be a homomorphism}$$

Then we have

i. Kernel of ϕ is a normal subgroup of G
ii. Image of ϕ is a subgroup of H
iii. Image of ϕ is isomorphic to the quotient group $G/Ker(\phi)$

The kernel is

$$ker(f) = \{A \in SL(2,\mathbb{C}) | f_A = e = f_{\mathbb{1}} = id_c\}$$

If we expand the condition that

$$f_A(z) = z \ \forall z$$

Then we see that $az + b$ is equal to $cz^2 + dz$ where, due to linear independence, $c = b$ and $b = 0$ and $a = d$. Now we also have a property that $A \in SL(2,\mathbb{C}) \to det(A) = 1$ Therefore, $a = d = \pm 1$ and the kernel is

$$ker(f) = \{A \in SL(2,\mathbb{C}) | \mathbb{1}, -\mathbb{1}\}$$

From the theorem above, part iii, and using part (b) of this question we have

$$f : SL(2,\mathbb{C}) \to \text{Mob} \ ; \ f(A) = f_A$$

is a group homomorphism so

$$\text{Mob} \cong SL(2,\mathbb{C})/\mathbb{Z}_2$$

5. Let V_1, V_2 be vector spaces, $L : V_1 \to V_2$ a linear map. Show that ImL and $KerL$ are vector subspaces of V_1 and V_2.
V_1, V_2 are vector spaces over a field denoted F
The kernel and image of L is

$$ker(L) = \{\vec{v_1} \in V_1 | L(\vec{v_1}) = 0\}$$

$$Im(L) = \{\vec{v_2} \in V_2 | \exists \vec{v_2} \in V_1 s.t L(\vec{v_1}) = \vec{v_2}\}$$

To be a vector subspace we need to have that

$$u, v \in V \text{ and } u + v \in V$$

And

$$v \in V, u \in V, vu \in V$$

where u, v are vectors and V a vector space and I've replaced the arrow notation. So, for the image $Im(L)$ $u, v \in Im(L)$ there exists a

$$w, y \in V_1 \to L(w) = u \ ; \ L(y) = v$$

$$u + v = L(w) + L(y) = L(w + y)$$

Hence

$$u + v \in Im(L)$$

And for our second condition $u \in Im(L)$ so that $u = L(w)$ for some $w \in V_1$ and a constant c in the field F so that

$$cw \in V_1$$

For all $c \in F$. Then

$$cv = cL(w) = L(cw) \in Im(L)$$

So Im(L) is a vector subspace.
For the kernel we have $u, v \in Ker(L)$ so $L(u) = 0$ and $L(v) = 0$. Using the same process as before

$$L(u + v) = L(u) + L(v) = 0 + 0 = 0$$

So, $u + v \in Ker(L)$. For the second condition we have $u \in ker(L)$ and $c \in F$ so that

$$L(cu) = cL(u) = c \cdot 0 = 0$$

So $cu \in Ker(L)$
And $Ker(L)$ is a vector subspace.