FYMM/MMP III Problem Set 1

Please submit your solutions for grading by Monday 7.9. in Moodle (there is a link where you can do this after the exercise sheet).

1. Consider the following constructions; check each one whether it is a semigroup, monoid, group or none of them. Why?
   - The set of real numbers $\mathbb{R}$, with raising to power as multiplication: $x \cdot y \equiv x^y$, $x, y \in \mathbb{R}$.
   - The set of positive natural numbers $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$ with the greatest common divisor of $m, n \in \mathbb{N}_+$ as their product: $m \cdot n \equiv \gcd(m, n)$.
   - The set of nonzero rational numbers $\mathbb{Q} \setminus \{0\}$, with the usual product as multiplication: $(m/n) \cdot (p/q) = (mp/nq)$.
2. Show that $|S_N| = N!$.
3. Consider the group $G = \{e, x_1, x_2, x_3, x_4, x_5\}$, where

$$e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; ; \; x_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$x_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \; ; \; x_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$x_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \; ; \; x_5 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} ,$$

   and the law of composition is the matrix multiplication. Show that G is isomorphic to a known group, give an explicit construction of the isomorphism.
4. An equilateral triangle is symmetric under reflections, with the line passing through the center and one of the vertices as the reflection axis; and symmetric under 120 degree counterclockwise rotations (with the center as the fixed point). Let e be the identity map (do nothing), a a rotation by 120 degrees, and b the above mentioned reflection. Consider the group generated by e, a ja b with composition of symmetry operations as the multiplication rule. What is the order of the group? (Hint: greater than three.) Construct the multiplication table (Cayley table) of the group.