Quantum Information A Fall 2020 Solutions to Problem Set 4

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27/09/20

1 Answers

1. Excercise 2.67 from Nielsen and Chaung.

$$V = W \oplus W_{\perp}$$

Prove that there exists a unitary operator $U' : V \to V$ which extends U.

$$U'|w\rangle = U|w\rangle \forall w \in W$$

Suppose we have 3 orthonormal basis for $W, W_{\perp}$ and the image of $U_{\perp}$ is

$$U_{\perp} = (Image(U))_{\perp}$$

And the basis is

$$|w_i\rangle, |w_j'\rangle, |u_j'\rangle$$

So

$$U' : V \to V$$

$$U' = \sum_i |u_i\rangle\langle w_i| + \sum_j |u_j'\rangle\langle w_j'|$$

Where $|u_i\rangle = U|w_i\rangle$. We now need to prove that U' is an extension of U

For all $|w\rangle \in W$

$$U'|w\rangle = \Big(\sum_i |u_i\rangle\langle w_i| + \sum_j |u_j'\rangle\langle w_j'|\Big)|w\rangle$$

$$= \sum_i |u_i\rangle\langle w_i|w\rangle + \sum_j |u_j'\rangle\langle w_j'|w\rangle$$

$$= \sum_i |u_i\rangle\langle w_i|w\rangle$$

Therefore $|w_j'\rangle \perp |w\rangle$ and

$$= \sum_i U|w_i\rangle\langle w_i|w\rangle$$

$$= U|w\rangle$$

And U' is an extension of U

2. Excercise 2.72 from Nielsen and Chuang. Bloch sphere for mixed states. From Theorem 2.5 in the book where an operator ρ is the density operator associated to some ensemble $\{p_i, |\psi_i\rangle\}$ iff it satisfies

   (a) $Tr(\rho) = 1$

   (b) ρ is a positive operator

   (a) (1)
   ρ can be represented in matrix form as

$$\begin{pmatrix} a & b \\ b^* & d \end{pmatrix}$$

   Where $a, d \in \mathbb{R}$ and $b \in \mathbb{C}$. From theorem 2.5 then $Tr(\rho) = a + d = 1$. Looking at section 1.2 of Nielsen and Chaung and the Pauli exercises done previously we can show that

$$a = \frac{1+r_3}{2} \; ; \; d = \frac{1-r_3}{2}$$

$$b = \frac{r_1 - ir_2}{2}$$

   Where $r_i \in \mathbb{R}^3$.We then have

$$\rho = \begin{pmatrix} a & b \\ b^* & d \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+r_3 & r_1 - ir_2 \\ r_1 + ir_2 & 1 - r_3 \end{pmatrix}$$

$$= \frac{1}{2}(\mathbb{I} + \vec{r} \cdot \vec{\sigma})$$

   where $\vec{\sigma}$ are the pauli matrices. For a 'mixed' state qubit we have to prove that ρ is positive (e.g Ex 2.71). If ρ is positive then eigenvalues will be non negative. Lets find the eigenvalues

$$det(\rho - \lambda\mathbb{I}) = \begin{pmatrix} a - \lambda & b \\ b^* & d - \lambda \end{pmatrix}$$

$$= (a - \lambda)(d - \lambda) - |b|^2$$

$$= \lambda^2 - (a + d)\lambda + ad - |b|^2 = 0$$

   So the eigenvalues are (from quadratic formula)

$$\lambda_\pm = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad - |b|^2)}}{2}$$

$$= \frac{1 \pm \sqrt{1 - 4(\frac{1-r_3^2}{4} - \frac{r_1^2+r_2^2}{4})}}{2}$$

$$= \frac{1 \pm \sqrt{|\vec{r}|^2}}{2}$$

$$= \frac{1 \pm |\vec{r}|}{2}$$

Since we assume that ρ is positive then $\frac{1-|\vec{r}|}{2} \geq 0$ which means that $|\vec{r}|$ must be less than or equal to 1. So

$$\rho = \frac{\mathbb{I} + \vec{r} \cdot \vec{\sigma}}{2}$$

(b) (2)
If $\rho = \frac{\mathbb{I}}{2}$ then $\vec{r} = 0$ clearly. Make sense as this is the origin spoint for the bloch sphere.

(c) (3)
From page 100 and Ex 2.71 a pure state has $Tr(\rho^2) = 1$.

$$\rho^2 = \frac{1}{2}(\mathbb{I} + \vec{r} \cdot \vec{\sigma}) \cdot \frac{1}{2}(\mathbb{I} + \vec{r} \cdot \vec{\sigma})$$

$$= \frac{1}{4}(\mathbb{I} + 2\vec{r} \cdot \vec{\sigma} + |\vec{r}|^2\mathbb{I})$$

Now

$$Tr(\rho^2) = Tr\Big(\frac{1}{4}(\mathbb{I} + 2\vec{r} \cdot \vec{\sigma} + |\vec{r}|^2\mathbb{I})\Big) = 1$$

Recognising that $Tr(\mathbb{I}) = 2$ and $Tr(\vec{\sigma}) = 0$ we get

$$Tr(\rho^2) = \frac{1}{4}(2 + 2|\vec{r}|^2) = 1$$

And solved gives

$$|\vec{r}| = 1$$

(d) (4) Showing that for pure states the descriptions of the Bloch vector we have given coincides with section 1.2 i.e

$$|\psi\rangle = cos(\frac{\theta}{2})|0\rangle + e^{i\phi}sin(\frac{\theta}{2})|1\rangle$$

$$P = |\psi\rangle\langle\psi|$$

$$P = \begin{pmatrix} cos^2(\theta/2) & e^{-i\phi}cos(\theta/2)sin(\theta/2) \\ e^{i\phi}cos(\theta/2)sin(\theta/2) & sin^2(\theta/2) \end{pmatrix}$$

$$= \begin{pmatrix} cos^2(\theta/2) & cos(\phi)cos(\theta/2)sin(\theta/2) - isin(\phi)cos(\theta/2)sin(\theta/2) \\ cos(\phi)cos(\theta/2)sin(\theta/2) + isin(\phi)cos(\theta/2)sin(\theta/2) & 1 - cos^2(\theta/2) \end{pmatrix}$$

Like in (1) place into the same form so that

$$1 + r_3 = 2cos^2(\theta/2) \ ; \ r_1 = 2cos(\phi)cos(\theta/2)sin(\theta/2)$$

$$r_3 = 2cos^2(\theta/2) \ ; \ r_2 = 2sin(\phi)cos(\theta/2)cos(\theta/2)sin(\theta/2)$$

So

$$|\vec{r}|^2 = 4cos(\theta/2)(cos^2(\theta/2) - cos^2(\theta/2)) + 1 = 1$$

This is not really necessary as it pretty much works the same as (1), however it is good to know this it works.

3. Exercise 2.73. Let ρ be a density operator. A minimal ensemble for ρ is an ensemble $\{p_i, |\psi_i\rangle\}$ containing a number of elements equal to the rank of ρ. Let $|\psi\rangle$ be any state in the support of ρ. Show that there is a minimal ensemble for ρ that contains $|\psi\rangle$ and that in any such ensemble $|\psi\rangle$ must appear with probability

$$p_i = \frac{1}{\langle\psi_i|\rho^{-1}|\psi_i\rangle}$$

where ρ^{-1} is the inverse of ρ.
From theorem 2.6 in Nielsen and Chaung (pg 102) and the density formalism for postulate 2 of QM we can transform this eigen decomposition. Where the eigen-decomposition is

$$\rho = \sum_k^N \lambda_k |k\rangle\langle k|$$

Where N is the dimension of the hilbert space. Suppose we have a variable p such that $p_k > 0$ for $k = 1 \ldots l$ where $l = rank(\rho)$ and $p_k = 0$ for $k = l + 1 \ldots N$. So we have

$$\rho = \sum_k^N \lambda_k |k\rangle\langle k|$$

$$= \sum_{k=1}^l p_k |k\rangle\langle k|$$

$$= \sum_{k=1}^l |\tilde{k}\rangle\langle\tilde{k}|$$

Where $|\tilde{k}\rangle = \sqrt{\lambda_k}|k\rangle$ and $\langle\tilde{k}| = \sqrt{\lambda_k}\langle k|$
Suppose that $|\psi_i\rangle$ is in support of ρ, meaning that

$$|\psi_i\rangle = \sum_{k=1}^l a_{ik}|k\rangle$$

Where $\sum_k |a_{ik}|^2 = 1$. So we have the probability as

$$p_i = \frac{1}{\sum_k \frac{|a_{ik}|^2}{\lambda_k}}$$

We also define a new variable as

$$b_{ik} = \frac{\sqrt{p_i} a_{ik}}{\sqrt{\lambda_k}}$$

Such that

$$\sum_k |b_{ik}|^2 = \sum_k \frac{p_i |a_{ik}|^2}{\lambda_k}$$
$$= p_i \sum_k \frac{|a_{ik}|^2}{\lambda_k} = 1$$

Now we can use the Gram-schmidt procedure to construct an orthonormal basis $\{u_i\}$ such that a unitary operator U has this basis

$$U = [u_{i1} \dots u_{ik} \dots u_{il}]$$

Another ensemble can then be defined by

$$\left[|\tilde{\psi_1}\rangle \dots |\tilde{\psi_i}\rangle \dots |\tilde{\psi_l}\rangle\right]$$
$$= \left[|\tilde{k_1}\rangle \dots |\tilde{k_l}\rangle\right] U^T$$

Noticing that we have subsituted $|\tilde{\psi_i}\rangle = \sqrt{p_i}|\psi_i\rangle$. Using the theorem above (2.6) we can find ρ in terms of this

$$\rho = \sum_k |\tilde{k}\rangle\langle\tilde{k}| = \sum_k |\tilde{\psi_k}\rangle\langle\tilde{\psi_k}|$$

And the inverse

$$\rho^{-1} = \sum_k \frac{1}{\lambda_k} |k\rangle\langle k|$$

So $\langle\psi_i|\rho^{-1}|\psi_i\rangle$ is

$$\langle\psi_i|\rho^{-1}|\psi_i\rangle = \sum_k \frac{1}{\lambda_k} \langle\psi_i|k\rangle\langle k|\psi_i\rangle$$
$$= \sum_k \frac{|a_{ik}|^2}{\lambda_k} = \frac{1}{p_i}$$

Take the inverse of this will simply give p_i so

$$p_i = \frac{1}{\langle\psi_i|\rho^{-1}|\psi_i\rangle} = p_i$$

This question was very difficult but the previous 3 pages in Nielsen Chaung help a lot.

4. Exercise 2.75 from Nielsen and Chaung. For each of the four Bell states find the reduced density operator for each qubit.
   The Bell states are

$$00 : |\Phi_+\rangle \to \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$01 : |\Phi_-\rangle \to \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$10 : |\psi_+\rangle \to \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

$$11 : |\psi_-\rangle \to \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

   To calculate the reduced density operator

$$|\Phi_+\rangle\langle\Phi_+| = \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)$$

$$|\Phi_-\rangle\langle\Phi_-| = \frac{1}{2}(|00\rangle\langle 00| - |00\rangle\langle 11| - |11\rangle\langle 00| - |11\rangle\langle 11|)$$

$$|\psi_+\rangle\langle\psi_+| = \frac{1}{2}(|01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + |10\rangle\langle 10|)$$

$$|\psi_-\rangle\langle\psi_-| = \frac{1}{2}(|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| - |10\rangle\langle 10|)$$

   Computing the traces

$$Tr(|\Phi_\pm\rangle\langle\Phi_\pm|) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \mathbb{I}/2$$

$$Tr(|\psi_\pm\rangle\langle\psi_\pm|) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \mathbb{I}/2$$

5. Exercise 2.79 from Nielsen and Chaung. Finding the Schmidt decompositions of the states

$$\frac{|00\rangle + \langle 11|}{\sqrt{2}} \; ; \; \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}$$

   and

$$\frac{|00\rangle + |01\rangle + |10\rangle}{\sqrt{3}}$$

   The Schmidt decomposition is

$$|\psi\rangle = \sum_i \sqrt{\lambda_i} |i_A\rangle |i_B\rangle$$

With

$$\rho^A = \sum_i \lambda_i^2 |i_A\rangle\langle i_A|$$

$$\rho^B = \sum_i \lambda_i^2 |i_B\rangle\langle i_B|$$

(a) $\frac{|00\rangle + \langle 11|}{\sqrt{2}}$
We see that it is already decomposed as

$$\frac{|00\rangle + \langle 11|}{\sqrt{2}} = \sum_{i=1}^{2} \frac{1}{\sqrt{2}} |i\rangle\langle i| = |\psi\rangle$$

(b) $\frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}$ can be broken down into

$$\Big(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\Big) \otimes \Big(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\Big)$$

$$= |\psi\rangle|\psi\rangle$$

(c) $|\psi\rangle = \frac{|00\rangle + |01\rangle + |10\rangle}{\sqrt{3}}$

$$\rho^A = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)\langle\psi|$$

$$= \frac{1}{3}(2|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|)$$

$$= \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Calculating the eigenvalues gives

$$\lambda_{\pm} = \frac{3 \pm \sqrt{5}}{6}$$

Using the notation that

$$\lambda_+ = \lambda_0 \; ; \; \lambda_- \; \lambda_1$$

$$\lambda_0 = \frac{3 + \sqrt{5}}{6}$$

with eigenvector

$$|\lambda_0\rangle = \sqrt{\frac{2}{5 + \sqrt{5}}} \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}$$

And

$$\lambda_1 = \frac{3 - \sqrt{5}}{6}$$

with eigenvector

$$|\lambda_1\rangle = \sqrt{\frac{2}{5-\sqrt{5}}}\begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}$$

So

$$\rho^A = \lambda_0|\lambda_0\rangle\langle\lambda_0| + \lambda_1|\lambda_1\rangle\langle\lambda_1|$$

And

$$|\psi\rangle = \sum_{i=0}^{1}\sqrt{\lambda_i}|\lambda_i\rangle|\lambda_i\rangle$$

6. Exercise 2.82 from Nielsen and Chaung. Suppose $\{p_i, |\psi_i\rangle\}$ is an ensemble of states with $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ for a Quantum System A. A system R with orthonormal basis $|i\rangle$

   (a) Show that $\sum_i \sqrt{p_i}|\psi_i\rangle|i\rangle$ is a purification of ρ. Let $|\psi\rangle = \sum_i \sqrt{p_i}|\psi_i\rangle|i\rangle$. The trace is system R is

$$Tr_R(|\psi\rangle\langle\psi|) = \sum_{i,j}\sqrt{p_i}\sqrt{p_j}|\psi_i\rangle\langle\psi_j|Tr_R(|i\rangle\langle j|)$$

$$= \sum_{i,j}\sqrt{p_i}\sqrt{p_j}|\psi_i\rangle\langle\psi_j|\delta_{ij}$$

$$= \sum_i p_i|\psi_i\rangle\langle\psi_i| = \rho$$

   And thus $|\psi\rangle$ is a purification of ρ

   (b) Measure R in basis $|i\rangle$, what is the probability to get i and the corresponding state?

   A projector P is defined by $P = \mathbb{I}\otimes|i\rangle\langle i|$ so that the probability to get i is equal to

$$Tr\Big[P|\psi\rangle\langle\psi|\Big] = \langle\psi|P|\psi\rangle$$

$$= \langle\psi|\mathbb{I}\otimes|i\rangle\langle i||\psi\rangle$$

$$= p_i\langle\psi_i|\psi_i\rangle = p_i$$

   After measuring, the state will be

$$\frac{P|\psi\rangle}{\sqrt{p_i}} = \frac{(\mathbb{I}\otimes|i\rangle\langle i|)|\psi\rangle}{\sqrt{p_i}}$$

$$= \frac{\sqrt{p_i}|\psi_i\rangle|i\rangle}{\sqrt{p_i}}$$

$$= |\psi_i\rangle|i\rangle$$

   System A is the trace of the above state i.e

$$Tr(|\psi_i\rangle|i\rangle) = |\psi_i\rangle$$

(c) (3) $|AR\rangle$ be any purification of ρ to a system AR. Show that $|i\rangle$ in R can be measured such that the correspodning post measurement state for a system A is $|\psi_i\rangle$ with probability p_i.
Schmidt decomposition of $|AR\rangle$ is

$$|AR\rangle = \sum_i \sqrt{\lambda_i}|\phi_i^A\rangle|\phi_i^R\rangle$$

As $|AR\rangle$ is a purification of ρ we can write

$$Tr_R(|AR\rangle\langle AR|) = \sum_i \lambda_i|\phi_i^A\rangle\langle\phi_i^A|$$

$$= \sum_i p_i|\psi_i\rangle\langle\psi_i|$$

Using theorem 2.6 in the book and the proof that follows, where

$$\sqrt{\lambda_i}|\phi_i^A\rangle = \sum_j u_{ij}\sqrt{p_j}|\psi_j\rangle$$

We have

$$|AR\rangle = \sum_i \Big(\sum_j u_{ij}\sqrt{p_j}|\psi_j\rangle\Big)|\phi_i^R\rangle$$

$$= \sum_j \sqrt{p_j}|\psi_j\rangle \otimes \Big(\sum_i u_{ij}|\phi_i^R\rangle\Big)$$

$$= \sum_j \sqrt{p_j}|\psi_j\rangle|j\rangle$$

$$= \sum_i \sqrt{p_i}|\psi_i\rangle|i\rangle$$

Such that $|i\rangle = \sum_k u_{ki}|\phi_k^R\rangle$. As u_{ij} is unitary $|j\rangle$ is implied to be an orthonormal basis for the system R. So if we measure R with respect to $|j\rangle$ we get j with $P(p_j)$ (probability) and the state after measurement is $|\psi_j\rangle$. So for any purification of $|AR\rangle$ there is an orthonormal basis $|i\rangle$.

7. Voluntary problem: Exercise 3.2 from the book. Do not hand in a solution, but just think about it. You may wish to google for "Turing number" or "Description number + Turing machine" for hints. I am not sure how useful the hint in the book is...?

2 Appendix

8. For Question 1, proving that U' is a unitary operator which is assumed in the answering of the question.

$$(U'^{\dagger})U' = \Big(\sum_{i=1}^{dimW} |w_i\rangle\langle u_i| + \sum_{j=1}^{dimW_{\perp}} |w_j\rangle\langle u_j|\Big) \cdot \Big(\sum_i |u_i\rangle\langle w_i| + \sum_j |u'_j\rangle\langle w'_j|\Big)$$

$$= \sum_i |w_i\rangle\langle w_i| + \sum_j |w'_j\rangle\langle w'_j| = \mathbb{I}$$

We also calculate $U'(U')^{\dagger}$

$$U'(U')^{\dagger} = \Big(\sum_i |u_i\rangle\langle w_i| + \sum_j |u'_j\rangle\langle w'_j|\Big) \cdot \Big(\sum_i |w_i\rangle\langle u_i| + \sum_j |w'_j\rangle\langle u'_j|\Big)$$

$$= \sum_i |u_i\rangle\langle u_i| + \sum_j |u'_j\rangle\langle u'_j| = \mathbb{I}$$

Which proves that U' is a unitary operator.