FYMM/MMP IIIb 2020 Solutions to Problem Set 2

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1. If we have
$$\mathbb{R} \setminus \{p_1\} \text{ and } \mathbb{R}^2 \setminus \{p_2\}$$
Which are not homeomorphic as they are connect connected so
$$\mathbb{R} \setminus \{p\}$$
is not connected but
$$\mathbb{R}^2 \setminus \{p\}$$
is connected. So, from the theorem $\mathbb{R}$ must not be homeomorphic to $\mathbb{R}^2$.

2. $\pi_2(M)$ is a two-loop. Need to find the fundamental groups.
   (a) Not answered.
   (b) $M = T^2$, $T^2 = S^1 \times S^1$. The fundamental group of S^1 is $\{e\}$ so we have
   $$\pi_2\{T^2\} = \{e\} \times \{e\} = \{e\}$$
   So $\pi_2(M) = \{e\}$

3. S^1 in cartesian coordinates is
$$x^2 + y^2 = 1$$
Such that (x, y) is a point in $\mathbb{R}^2$. So S^1 is a subset of $\mathbb{R}^2$: $S^1 = \{x \in \mathbb{R}^2 | \sum_i^2 (x^i)^2 = 1\}$. So we have 2 sets of coordinate neighbourhoods
$$U_{1+} \equiv \{(x, y) \in S^1 | x > 0\}$$
$$U_{1-} \equiv \{(x, y) \in S^1 | x < 0\}$$
$$U_{2+} \equiv \{(x, y) \in S^1 | y > 0\}$$
$$U_{2-} \equiv \{(x, y) \in S^1 | y < 0\}$$
With coordinate functions
$$V_{1\pm}(x, y) = y$$
Where $V_{1\pm}$ maps $U_{1\pm} \to (-1, 1)$. And
$$V_{2\pm}(x, y) = x$$
Where $V_{2\pm}$ maps $U_{2\pm} \to (-1, 1)$.
These are coordinate neighbourhoods and functions if they are open and a homeomorphism. These can be shown to be homeomorphisms by construction onverses from the transistion functions: $\psi_{U\pm}$ maps $(-1, 1) \to S^1$
$$\psi_{V_{1\pm}}(x, y) = (\pm\sqrt{1 - x^2}, x)$$

$$= (\pm\sqrt{1-y^2}, y)$$
$$\psi_{V2\pm}(x,y) = (x, \pm\sqrt{1-x^2})$$
$$= (y, \pm\sqrt{1-y^2})$$

4. Express the vector field
$$V = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$
in *paraboloidal* coordinates (u, v, φ), the coordinate transformation is
$$x = uv\cos\varphi \ , \ y = uv\sin\varphi \ , \ z = \frac{1}{2}(u^2 - v^2).$$
So I need to find functions of u, v and φ. If we divide y by x we get a function of φ
$$\frac{y}{x} = \frac{uv\sin(\varphi)}{uv\cos(\varphi)} = \tan(\varphi)$$
And that
$$x^2 + y^2 = (uv\cos(\varphi))^2 + (uv\sin(\varphi))^2$$
$$= u^2v^2\cos^2(\varphi) + u^2v^2\sin^2(\varphi)$$
Because $\sin^2 + \cos^2 = 1$ this can be written as
$$= u^2v^2$$
Notice that
$$\sqrt{u^2v^2 + (\frac{1}{2}(u^2 - v^2))^2} = \frac{u^2 + v^2}{2}$$
So adding z and taking another square root gives u and adding $-z$ and another square root gives v (derived from wolffram alpha)
$$u = \sqrt{\sqrt{u^2v^2 + (\frac{1}{2}(u^2 - v^2))^2} + (\frac{1}{2}u^2 - v^2)}$$
$$= \sqrt{\sqrt{x^2 + y^2 + z^2} + z}$$
And
$$v = \sqrt{\sqrt{x^2 + y^2 + z^2} - z}$$
From the lecture notes, a general transformation of coordinates is given by
$$X = X^\mu \frac{\partial}{\partial x^\mu} = Y^\mu \frac{\partial}{\partial y\mu}$$

Where
$$X^\mu \frac{\partial}{\partial x^\mu} = X^v \frac{\partial y^\mu}{\partial x^v} \frac{\partial}{\partial y^\mu}$$
From the chain rule. The version we need is
$$\frac{\partial y^\mu}{\partial x^v} \Rightarrow y^\mu = (u, v, \varphi)$$
So that we have 3 partial fractions for u, 3 for v and 3 for φ for x, y, z
$$\frac{\partial u}{\partial x} = \frac{x}{2\sqrt{\sqrt{x^2+y^2+z^2}+z} \cdot \sqrt{x^2+y^2+z^2}}$$
Subsituting u and v
$$= \frac{uv\cos(\varphi)}{\sqrt{\sqrt{u^2v^2+(\frac{1}{2}(u^2-v^2))^2}+\frac{1}{2}(u^2-v^2)}\sqrt{u^2v^2+(\frac{1}{2}(u^2-v^2))^2}}$$
$$= \frac{v\cos(\varphi)}{u^2+v^2}$$
Doing this for the rest then gives
$$\frac{\partial u}{\partial y} = \frac{v\sin(\varphi)}{u^2+v^2}$$
$$\frac{\partial u}{\partial z} = \frac{u}{u^2+v^2}$$
$$\frac{\partial v}{\partial x} = \frac{u\cos(\varphi)}{u^2+v^2}$$
$$\frac{\partial v}{\partial y} = \frac{u\sin(\varphi)}{u^2+v^2}$$
$$\frac{\partial v}{\partial z} = -\frac{v}{u^2+v^2}$$
$$\frac{\partial \varphi}{\partial x} = -\frac{\sin(\varphi)}{uv}$$
$$\frac{\partial \varphi}{\partial y} = -\frac{\cos(\varphi)}{uv}$$
$$\frac{\partial \varphi}{\partial z} = 0$$
All of which were evaluated using wolfram alpha. So that the vector field in paraboloidal coordinates is
$$X = \frac{u+v(\sin(\varphi)+\cos(\varphi))}{u^2+v^2}\frac{\partial}{\partial u} + \frac{-v+u(\sin(\varphi)+\cos(\varphi))}{u^2+v^2}\frac{\partial}{\partial v} + \frac{\cos(\varphi)-\sin(\varphi)}{uv}\frac{\partial}{\partial \varphi}$$