Quantum Mechanics IIa 2021 Problem Set 1 (v2)

Solutions are due in 2 pm on Wednesday Jan 27, as a pdf file into a return box that will be in the course Moodle page.

5.1 1. A simple harmonic oscillator (in one dimension) is subjected to a perturbation

$$\lambda V = bx$$

where b is a real constant. Let us denote the unperturbed energy eigenstates of the harmonic oscillator as $|u_n\rangle$.

(a) Calculate the energy shift of the ground state to *lowest non-vaninshing order*.

(b) Solve this problem *exactly* and compare with your result obtained above.

Hint: you may assume without proof that

$$\langle u_{n'}|x|u_n\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\sqrt{n+1}\delta_{n',n+1} + \sqrt{n}\delta_{n',n-1}) \ .$$

Griffths 2. Show by an explicit calculation using the Hydrogen energy eigenstate wavefunctions $\psi_{nlm}(\boldsymbol{x}) = \langle \boldsymbol{x}|nlm\rangle$ that the matrix elements

$$\langle 100|\hat{z}|nlm\rangle$$

vanish if the quantum number $l \neq 1$. Notation: $\boldsymbol{x}$ denotes a vector $\vec{x}$, I will sometimes use this convention in the problem sets. (Hint for the solution: just the angular part of the wavefunction is relevant.)

3. In the linear Stark effect, we calculated (see my lecture notes) the shifts in energy $\Delta_\pm^{(1)}$ associated with the eigenstates

$$|2,\pm\rangle = \frac{1}{\sqrt{2}}(|200\rangle \pm |210\rangle)$$

to be

$$\Delta_\pm^{(1)} = \pm 3ea_0|\vec{E}|$$

where $\vec{E}$ is the external electric field. Let us interpret the energy shift to correspond to a dipole moment $\vec{d}_\pm$ which the atom develops as it is in the state $|2,\pm\rangle$. The interaction energy between an electric dipole and the electric field is

$$H' = -\vec{d}\cdot\vec{E} \ .$$

What are the dipole moments $\vec{d}_\pm$: what is their magnitude $|\vec{d}_\pm|$ and how are they oriented with respect to the external electric field $\vec{E}$? (Hint: this problem is almost trivial.)

5.3 4. Consider a particle in a two-dimensional potential

$$V_0 = \begin{cases} 0 \ , & for\ 0 \le x \le L, 0 \le y \le L \ , \\ \infty \ , & otherwise\ . \end{cases}$$

(a) Write the energy eigenfunctions for the ground state and for the first excited state.

(b) We now add a perturbation of the form

$$\lambda V_1 = \begin{cases} \lambda xy \ , & for\ 0 \le x \le L, 0 \le y \le L \ , \\ 0 \ , & otherwise\ . \end{cases}$$

Obtain the zeroth-order energy eigenfunctions and the first-order energy shifts for the ground state and the first excited state.

5.10 5. The two-dimensional harmonic oscillator has the Hamiltonian

$$\begin{aligned} H_0 &= \tfrac{p_x^2+p_y^2}{2m} + \tfrac{1}{2}m\omega_0^2(x^2+y^2) \\ &= \hbar\omega_0(a^\dagger a + b^\dagger b + 1) \ , \end{aligned}$$

where

$$\begin{aligned} x &= \tfrac{1}{\beta\sqrt{2}}(a+a^\dagger) \\ y &= \tfrac{1}{\beta\sqrt{2}}(b+b^\dagger) \ . \end{aligned} \tag{1}$$

where $\beta \equiv m\omega_0/\hbar$.

(a) In the level number representation $|nm\rangle$, (n and m are the level numbers associated with a, b), what are the energy levels and their degeneracies? What are the eigenstates $|nm\rangle$ of the first excited state (the energy level above the ground state)?

(b) Let us turn on a perturbation

$$\lambda V = \lambda xy \ .$$

What happens to the first excited energy level? Calculate the energy shifts to first order in perturbation theory and find the "diagonal" energy eigenstates.

Hint: use the annihilation/creation operator representation (1) for x, y.