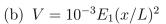
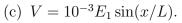
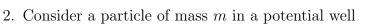
## Quantum Mechanics IIa 2021 Problem Set 2

Solutions are due in 2 pm on Wednesday Feb 3, as a pdf file into the return box in the course Moodle page. Constant

- 1. Calculate the first order correction to the third eigenenergy  $E_3^{(0)}$  (the ground state is  $E_1^{(0)}$ ) for a particle in a one-dimensional box with infinite walls at x=0, x=L, due to the following perturbations:









$$V(x) = \begin{cases} \infty , & x \le -L/2, \ x \ge L/2 \\ 0 , & -L/2 < x < -a/2, \ a/2 < x < L/2 \\ V_0 , & -a/2 \le x \le a/2 \end{cases}$$

where L > a and  $V_0 > 0$ . This is an infinite potential well with a bump in the middle. Assume that the bump at the bottom of the well can be considered a small perturbation.

- (a) Calculate the corrected second eigenenergy and eigenfunction to first order in the perturbation.
- (b) What dimensionless ratio must be small compared to 1 in order for your approximation to be valid?
- (c) First-order corrected energies corresponding to even eigenstates are greater than energies corresponding to odd eigenstates. Why? (Hint: Note the behavior of eigenstates at the origin.)
- 3. **Periodically driven two-state system**. Consider the time-dependent Schrödinger equation

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle$$
 (1)

with a time dependent Hamiltonian of the form

$$H(t) = H_0 + V(t) .$$

Let  $|n\rangle$  denote the eigenstates of  $H_0$  with eigenvalues  $E_n$ . Expand the state (in this problem it is sufficient to work in the Schrödinger picture)

$$|\psi(t)\rangle = \sum_{n} c_n(t)e^{-iE_nt/\hbar}|n\rangle$$

(a) Show that the coefficients  $c_m(t)$  satisfy the equation

$$i\hbar \frac{dc_m(t)}{dt} = \sum_n e^{i\omega_{mn}t} V_{mn}(t) c_n(t)$$
 (2)

with

$$\omega_{mn} \equiv \frac{E_m - E_n}{\hbar} \; ; \; V_{mn}(t) = \langle m|V(t)|n\rangle \; .$$

(b) Assume that the potential contains a small parameter so that we can treat V(t) perturbatively. Use an initial condition  $c_m^{(0)}(t_0) = \delta_{mi}$  and show that to first order in V(t),

$$c_m(t) = \delta_{mi} - \frac{i}{\hbar} \int_{t_0}^t dt' e^{i\omega_{mi}t'} V_{mi}(t') . \qquad (3)$$

Next, consider a two-state system with

$$|1\rangle = \begin{pmatrix} 0\\1 \end{pmatrix} \; ; \; |2\rangle = \begin{pmatrix} 1\\0 \end{pmatrix}$$

with the unperturbed Hamiltonian

$$H_0 = \left( \begin{array}{cc} \hbar \omega_0 / 2 & 0\\ 0 & -\hbar \omega_0 / 2 \end{array} \right)$$

and with the interaction potential

$$V(t) = \begin{pmatrix} 0 & -2\hbar\lambda\cos\omega t \\ -2\hbar\lambda\cos\omega t & 0 \end{pmatrix} .$$

This is a simplified model for the interaction of an atom with a time-dependent laser field, or for the interaction of a spin half particle with an oscillating magnetic field.

(c) Let  $t_0 = 0$ . Assume that initially the system is at ground state,  $c_m^{(0)}(0) = \delta_{m1}$ . Solve (3) to get

$$c_1(t) = 1$$
;  $c_2(t) = \lambda \left[ \frac{e^{i(\omega + \omega_0)t}}{\omega + \omega_0} - \frac{e^{-i(\omega - \omega_0)t}}{\omega - \omega_0} \right]$ .

Next, we find a more accurate solution of this driven two-level system.

(d) First, substitute the energy eigenvalues and the interaction potential into (2) to obtain two coupled differential equations. Then, invoking the so-called rotating-wave approximation, drop the rapidly oscillating terms  $e^{\pm i(\omega+\omega_0)t}$  to derive the simplified equations

$$\frac{dc_1(t)}{dt} = i\lambda e^{i\delta t}c_2(t) \; ; \; \frac{dc_2(t)}{dt} = i\lambda e^{-i\delta t}c_1(t)$$
 (5)

where  $\delta \equiv \omega - \omega_0$  is the detuning parameter. Note: We assume  $\delta = \omega - \omega_0$  is small and  $\omega, \omega_0$  are large.

typo: signs should be flipped

solution, not normalized, will

get eztra term

(e) Then, show that

$$c_1(t) = e^{i\delta t/2} \left[ \cos(\Omega t/2) - i \frac{\delta}{\Omega} \sin(\Omega t/2) \right]$$
$$c_2(t) = e^{-i\delta t/2} \frac{2i\lambda}{\Omega} \sin(\Omega t/2)$$

where  $\Omega = \sqrt{\delta^2 + 4\lambda^2}$  is the *Rabi frequency*, are solutions of (5) satisfying the same initial condition  $c_m^{(0)}(0) = \delta_{m1}$  and the normalization condition  $|c_1|^2 +$  $|c_2|^2 = 1.$ 

**Hint:** Transform (5) into second order differential equations for  $c_1$  and  $c_2$ . Then substitute  $c_1(t) = e^{i\delta t/2}x(t)$ ,  $c_2(t) = e^{-i\delta t/2}y(t)$  to obtain differential equations of a simple harmonic oscillator for x, y.

(f) Finally, show that for short times  $\delta t$ ,  $\Omega t \ll 1$  the above result and the perturbative result agree, with + DOMOCHINS

 $|c_2(t)|^2 \approx (\lambda t)^2$ .

Q3 worth double points

**Hint:** Apply the rotating-wave approximation to (4).

Lesson: the validity of the perturbative result does not only depend on having a small expansion parameter, but also on the range of time.

4. (Optional problem, no points given. However, you may find it interesting, as NMR spectroscopy is a useful tool, and a basis for the magnetic resonance imaging in medicine.) Read about the nuclear magnetic resonance (NMR) from Kimmo Tuominen's lecture notes: section 3.1.2 starting at p. 60, to understand it as an application of the periodically driven two-state system. Prepare to discuss it in the next week's exercise session. You don't have to write anything, but if you like, you can write a short summary. The lecture notes can be found in the QMII Moodle.