Quantum Information A
Problem Set 1, Solutions

Problem 1

The matrix representation of A w.r.t. an input basis $|v_i\rangle$ and an output basis $|w_i\rangle$ is defined by the elements A_{ij} in

$$A|v_j\rangle = \sum_i A_{ij}|w_i\rangle. \tag{1}$$

With orthonormal $|w_i\rangle$, we can write this as

$$A_{ij} = \langle w_i|A|v_j\rangle. \tag{2}$$

Here, in addition to being orthonormal, our input and output bases are the same, namely,

$$|v_1\rangle = |w_1\rangle = |0\rangle \qquad\qquad |v_2\rangle = |w_2\rangle = |1\rangle, \tag{3}$$

so our matrix elements are

$$A_{11} = \langle 0|A|0\rangle = \langle 0|1\rangle = 0 \qquad\qquad A_{12} = \langle 0|A|1\rangle = \langle 0|0\rangle = 1 \tag{4}$$

$$A_{21} = \langle 1|A|0\rangle = \langle 1|1\rangle = 1 \qquad\qquad A_{22} = \langle 1|A|1\rangle = \langle 1|0\rangle = 0. \tag{5}$$

That is,

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ (= X) \tag{6}$$

so A is the X gate (or quantum NOT gate if you prefer) as we could have expected.

We can instead use the $|\pm\rangle$ states as our basis, i.e.,

$$|v_1\rangle = |w_1\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad\qquad |v_2\rangle = |w_2\rangle = |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle). \tag{7}$$

These states are orthonormal since

$$\langle \pm|\pm\rangle = \frac{1}{2}(\langle 0|0\rangle + \langle 1|1\rangle) = 1 \qquad\qquad \langle +|-\rangle = \frac{1}{2}(\langle 0|0\rangle - \langle 1|1\rangle) = 0. \tag{8}$$

The action of A on these states is

$$A|+\rangle = \frac{1}{\sqrt{2}}(A|0\rangle + A|1\rangle) = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle) = |+\rangle \tag{9}$$

$$A|-\rangle = \frac{1}{\sqrt{2}}(A|0\rangle - A|1\rangle) = \frac{1}{\sqrt{2}}(|1\rangle - |0\rangle) = -|-\rangle \tag{10}$$

and our matrix elements become

$$A_{11} = \langle +|A|+\rangle = \langle +|+\rangle = 1 \qquad\qquad A_{12} = \langle +|A|-\rangle = -\langle +|-\rangle = 0 \tag{11}$$

$$A_{21} = \langle -|A|+\rangle = \langle -|+\rangle = 0 \qquad\qquad A_{22} = \langle -|A|-\rangle = -\langle -|-\rangle = -1. \tag{12}$$

Therefore

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{13}$$

A is diagonal in this basis because the $|\pm\rangle$ states are eigenstates of A $(= X)$; a fact which you will undoubtedly come across many times during this course.

Problem 2

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{14}$$

$$|01\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{15}$$

$$|10\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{16}$$

$$|11\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{17}$$

Problem 3
This is just straightforward matrix multiplication

$$U_{CN}|00\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |00\rangle \tag{18}$$

$$U_{CN}|01\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |01\rangle \tag{19}$$

$$U_{CN}|10\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |11\rangle \tag{20}$$

$$U_{CN}|11\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = |10\rangle \tag{21}$$

Problem 4
This is again a very straightforward problem just checking that you understand the concept of bitwise XOR. The answers are

(a) 11010

(b) 11110

(c) $11000 \oplus 11100 = 00100$

Problem 5

The straightforward method of doing this exercise is to just use the given formulae:

$$H|+\rangle = H\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) = \frac{1}{2}(|0\rangle + |1\rangle + |0\rangle - |1\rangle) = |0\rangle \tag{22}$$

$$H|-\rangle = H\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) = \frac{1}{2}(|0\rangle + |1\rangle - |0\rangle + |1\rangle) = |1\rangle. \tag{23}$$

Other possible methods would be, e.g., to show that $H^2 = I$ from which it follows that $H|+\rangle = H^2|0\rangle = |0\rangle$, or to use an outer product representation for H:

$$H = |+\rangle\langle 0| + |-\rangle\langle 1| \tag{24}$$

and since clearly $H = H^\dagger$, i.e., H is Hermitian, then also

$$H = H^\dagger = |0\rangle\langle +| + |1\rangle\langle -| \tag{25}$$