These questions come from Robert Alicki Quantum Dynamical systems page 86/146

Open Quantum Systems: Solutions 6

Paolo Muratore-Ginanneschi and Brecht Donvil

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Problem Set 6 2018

Exercise 1

Show that a 2×2 matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{1}$$

is positive iff

$$a_{11} \geq 0,\ a_{21} = \bar{a}_{12} \text{ and } |a_{12}|^2 \leq a_{11}a_{22} \tag{2}$$

2pts

Exercise 2

Consider the map on $M_2(\mathbb{C})$

$$\Gamma : \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \to \begin{pmatrix} \delta a_{11} + (1-\delta)a_{22} & \mu a_{12} \\ \bar{\mu} a_{21} & \delta' a_{22} + (1-\delta')a_{11} \end{pmatrix} \tag{3}$$

a. Clearly Γ is unity preserving, show that Γ is positive, i.e. it maps positive matrices into positive matrices, iff

$$0 \leq \delta \leq 1,\ 0 \leq \delta' \leq 1 \text{ and } |\mu| \leq \sqrt{\delta\delta'} + \sqrt{(1-\delta)(1-\delta')} \tag{4}$$

2pts

b. The Choi matrix for a map Γ is

$$C = \sum_{i,j=1}^{2} \Gamma(e_{ij}) \otimes e_{ij}. \tag{5}$$

A linear mapping is completely positive if and only if it's Choi matrix is positive semi-definite ($C \geq 0$). Find the conditions for μ, δ and δ' such that $C \geq 0$. Note that they are more restrictive than (4).
2pts

Exercise 3

Consider the completely postive map from the last exercise. In this exercise we will find a Kraus decomposition for it:

$$\Gamma(A) = \sum_{j}^{4} W_j A W_j^\dagger, \tag{6}$$

$$W_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}. \tag{7}$$

a. Write both sides of (6) for the elementary matrices e_{ij}.
2pts

b. Note that the equations obtained for the matrix elements of W_j in a. are underdetermined, i.e. multiple solutions are possible. Let us chose W_1 and W_2 to be diagonal matrices and W_3 and W_4 to be off-diagonal. What equations do you obtain for a_j, b_j, c_j and d_j?
2pts

c. Solve the equations obtained in b. (Hint: since $|\mu| \leq \sqrt{\delta\delta'}$ it is possible to write $\mu = \sqrt{\delta\delta'}\sin(\theta)e^{i\phi}$ for $0 \leq \theta \leq \pi/2$ and $0 \leq \phi \leq 2\pi$)
2pts