Open Quantum Systems   Fall 2020   Answers to Exercise Set 3

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1. ## Exercise 1

$$\rho = \psi\psi^\dagger$$

Let ρ be a density operator. From spectral decomposition of ρ we know:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

For $p_i \geq 0$ with $\sum_i p_i = 1$. So

$$\rho^2 = \sum_{i,j} p_i p_j |i\rangle\langle i|j\rangle\langle j|$$

$$= \sum_{i,j} p_i p_j |i\rangle\langle j|\delta_{ij}$$

$$= \sum_i p_i^2 |i\rangle\langle i|$$

So

$$Tr(\rho^2) = Tr\left(\sum_i p_i^2 |i\rangle\langle i|\right)$$

$$= \sum_i p_i^2 Tr(|i\rangle\langle i|)$$

$$= \sum_i p_i^2 \langle i|i\rangle$$

$$= \sum_i p_i^2 = 1$$

And we have

$$\sum_i p_i^2 \leq \sum_i p_i$$

So, therefore,

$$p_i^2 \leq p_i \qquad \boxed{1.1}$$

Now if ρ is assumed to be pure then $\rho = |\psi\rangle\langle\psi|$ and we have

$$Tr(\rho^2) = Tr(|\psi\rangle\langle\psi|\psi\rangle\langle\psi|)$$

$$= Tr(|\psi\rangle\langle\psi|) = \langle\psi|\psi|\rangle = 1$$ ✓

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$\boxed{1.2}$

2. **Exercise 2: Qubit State Operator**

Theorem 1 *An operator ρ is the density operator associated to some ensemble $\{p_i, |\psi_i\rangle\}$ if and only if it satisfies*

(a) $Tr(\rho) = 1$

(b) ρ *is a positive operator*

ρ can be represented in matrix form as

$$\begin{pmatrix} a & b \\ b^* & d \end{pmatrix}$$

Where $a, d \in \mathbb{R}$ and $b \in \mathbb{C}$. From theorem 1 then $Tr(\rho) = a + d = 1$.

$$a = \frac{1 + v_3}{2} \; ; \; d = \frac{1 - v_3}{2} \checkmark$$

$$b = \frac{v_1 - iv_2}{2} \checkmark$$

Where $v_i \in \mathbb{R}^3$. We then have

$$\rho = \begin{pmatrix} a & b \\ b^* & d \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 + v_3 & v_1 - iv_2 \\ v_1 + iv_2 & 1 - v_3 \end{pmatrix}$$

$$= \frac{1}{2}(\mathbb{I} + \vec{v} \cdot \vec{\sigma}) \checkmark$$

where $\vec{\sigma}$ are the pauli matrices.

A pure state has $Tr(\rho^2) = 1$.

$$\rho^2 = \frac{1}{2}(\mathbb{I} + \vec{v} \cdot \vec{\sigma}) \cdot \frac{1}{2}(\mathbb{I} + \vec{v} \cdot \vec{\sigma})$$

$$= \frac{1}{4}(\mathbb{I} + 2\vec{v} \cdot \vec{\sigma} + |\vec{v}|^2 \mathbb{I})$$

Now

$$Tr(\rho^2) = Tr\left(\frac{1}{4}(\mathbb{I} + 2\vec{v} \cdot \vec{\sigma} + |\vec{v}|^2 \mathbb{I})\right) = 1$$

Recognising that $Tr(\mathbb{I}) = 2$ and $Tr(\vec{\sigma}) = 0$ we get

$$Tr(\rho^2) = \frac{1}{4}(2 + 2|\vec{v}|^2) = 1$$

And solved gives

$$|\vec{v}| = 1 \checkmark$$

Conversely, if $|\vec{v}| = 1$ then

$$Tr(\rho^2) = \frac{1}{4}(2 + 2|\vec{v}|^2) = 1 \checkmark$$

2.1

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3. **Exercise 3**

(a) The qubit has 2 basic states spanning $\mathbb{C}^2$ which are $|0\rangle$ and $|1\rangle$. Every unit vector in the space $\mathbb{C}^2$ is a state vector so we have states that are linear combinations of $|0\rangle$ and $|1\rangle$ of the form

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Where α, β are complex numbers with the property

$$|\alpha|^2 + |\beta|^2 = 1$$

For the bloch sphere we can express $|\psi\rangle$ as

$$|\psi\rangle = r_\alpha e^{i\phi_\alpha}|0\rangle + r_\beta e^{i\phi_\beta}|1\rangle$$

Where $r_\alpha, r_\beta, \phi_\alpha, \phi_\beta$ are real. We can multiply this by a global phase factor $e^{i\gamma}$ which has no observable consequences as

$$|e^{i\gamma}\alpha|^2 = (e^{i\gamma}\alpha)^*(e^{i\gamma}\alpha) = (e^{-i\gamma}\alpha)(e^{i\gamma}\alpha) = \alpha^*\alpha = |\alpha|^2$$

So we have

$$|\psi'\rangle = r_\alpha|0\rangle + r_\beta e^{i(\phi_\beta - \phi_\alpha)}|1\rangle$$
$$= r_\alpha|0\rangle + r_\beta e^{i\phi}|1\rangle$$

Which in cartesian coordinates is

$$|\psi'\rangle = r_\alpha|0\rangle + (x + iy)|1\rangle$$

This is constrained by $\langle\psi'|\psi'\rangle = 1$. So

$$|r_\alpha|^2 + |x + iy|^2 = r_\alpha^2 + x^2 + y^2 = 1$$

Which is the equation for a unit sphere in 3D.
In spherical polar coordinates we have

$$x = r\sin(\theta)\cos(\phi)$$
$$y = r\sin(\theta)\sin(\phi)$$
$$z = r\cos(\theta)$$

Writing r_α to z and $r = 1$

$$|\psi'\rangle = z|0\rangle + (x + iy)|1\rangle$$
$$= \cos(\theta)|0\rangle + \sin(\theta)\Big(\cos(\phi) + i\sin(\phi)\Big)|1\rangle$$

$$= \cos(\theta)|0\rangle + e^{i\phi}\sin(\theta)|1\rangle$$

Using half angle relations and splitting the bloch sphere is ha

$$|\psi\rangle = \cos(\frac{\theta}{2})|0\rangle + e^{i\phi}\sin(\frac{\theta}{2})|1\rangle$$

The reasons why the half angle relation can be used here is ii

(b) To show that orthonormal bases correspond to antipodal poi
sphere, consider $|\chi\rangle$ correspodning to the opposite point on tl

$$|\chi\rangle = \cos(\frac{\pi - \theta}{2})|0\rangle + e^{i(\phi+\pi)}\sin(\frac{\pi - \theta}{2})|1\rangle$$

$$= \cos(\frac{\pi -}{2})|0\rangle - e^{i\phi}\sin(\frac{\pi - \theta}{)}|1\rangle$$

So

$$\langle\chi|\psi\rangle = \cos(\frac{\theta}{2})\cos(\frac{\pi - \theta}{2}) - \sin(\frac{\theta}{2})\sin(\frac{\pi - \theta}{2})$$

Using the relation $cos(a + b) = cos(a)cos(b) - sin(a)sin(b)$ w

$$\langle\chi|\psi\rangle = \cos(\frac{\pi}{2}) = 0$$

And opposite points correspond to orthgonal qubit states w
bases.

(c) In the bloch sphere representation, in terms of probability we
states occupying the edge or surface of the bloch sphere w.
occupying between the centre and the edge of the sphere.
A quantum state where ρ is in a mixed state is a statistical ei
where $k \in \mathbb{Z}^+$. Each pure state $|\psi_k\rangle$ occured with probability
probabilities p_k and the pure states are the eigenvalues and
respectively and can be written as

$$\rho = \sum_k p_k|\psi_k\rangle\langle\psi_k|$$

So $\rho_1^T = \rho_1$ And the trace $Tr(\rho_1^2) = \frac{7}{4} \neq 4$. ρ_1 is a hermitian matrix and normalizable but not positive definite as it has a negative eigenvalue

$$\lambda_1 = \frac{-\sqrt{10} + 2}{4}$$

$Tr(\rho_1^2)$ also $\neq 1$ so it is not an acceptable operator. For ρ_2

$$\rho_2 = \begin{pmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{pmatrix}$$

$$\rho_2^2 = \begin{pmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{pmatrix}$$

So $\rho_2^T = \rho_2$ and $\rho_2 = \rho_2^2$. The eigenvalues are $\lambda_1 = 0, \lambda_2 = 1$ so this is an acceptable operator and a pure state as it is hermitian, normalizable and positive definite. Also $Tr(\rho_2^2) = 1$ so it is pure (and $\rho_2 = \rho_2^2$). Because we have

$$|\alpha|^2 + |\beta|^2 = 1$$

The state vector is

$$|\psi\rangle = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}$$

For ρ_3

$$\rho_3 = \frac{1}{3}|u\rangle\langle u| + \frac{2}{3}|v\rangle\langle v| + \frac{\sqrt{2}}{3}|v\rangle\langle u| + \frac{\sqrt{2}}{3}|u\rangle\langle v|$$

The outer product of two orthonormal vectors is the delta function so

$$\langle u|u\rangle = \langle v|v\rangle = 1, \langle u|v\rangle = 0$$

This is normalizable, hermitian and is postitive definite. It also has $Tr(\rho_3^2) = 1$ as

$$(\frac{1}{3})^2 + (\frac{2}{3})^+ (\frac{\sqrt{2}}{3})^2 + (\frac{\sqrt{2}}{3})^2 = 1$$

So it is a pure state. The state vector is

$$|\psi\rangle = \frac{1}{\sqrt{3}}(|u\rangle + \sqrt{2}|v\rangle)$$

ρ_4 has negative eigenvalues so it is not positive definite. It is also not pure as $Tr(\rho_4^2) = \frac{5}{8} \neq 1$

ρ_5 is normalizable, hermitian and positive definite. The eigenvalues are

$$\lambda_1 = \frac{1}{4}, \lambda_2 = \frac{-\sqrt{5} + 3}{8} > 0, \lambda_3 = \frac{\sqrt{5} + 3}{8}$$

$$\rho_5^2 = \begin{pmatrix} \frac{5}{16} & 0 & \frac{3}{16} \\ 0 & \frac{1}{16} & 0 \\ \frac{3}{16} & 0 & \frac{1}{8} \end{pmatrix}$$

With

$$Tr(\rho_5^2) = \frac{1}{2} \neq 1 \checkmark \qquad \boxed{6.1}$$

So it is not pure.

5. Exercise 5

$$H = \frac{\hbar w}{2}\sigma_z + \frac{\Omega}{2}\sigma_x$$

(a) Find the eigenvalues and eigenvectors
   The hamiltonian can be written in matrix form as

$$H = \begin{pmatrix} \frac{\hbar w}{2} & \frac{\Omega}{2} \\ \frac{\Omega}{2} & -\frac{\hbar w}{2} \end{pmatrix}$$

The eigenvalues of this equation can be determined from $det(H - EI)$ where E is the discrete energies or eigenvalues of the hamiltonian. For this system they are

$$E_\pm = \pm\sqrt{\left(\frac{\hbar w}{2}\right)^2 + \left(\frac{\Omega}{2}\right)^2} \checkmark$$

The eigenvectros are then

$$|E_+\rangle = \begin{pmatrix} \hbar w + E \\ \frac{\Omega}{2} \end{pmatrix} \checkmark$$

$$|E_-\rangle = \begin{pmatrix} \hbar w - E \\ \frac{\Omega}{2} \end{pmatrix}$$

Normalizing these to get the normalized eigenvectors

$$|E_+\rangle = \frac{1}{\left(\frac{\hbar w}{2}\right)^2 + E\hbar w + \left(\frac{\Omega}{2}\right)^2} \begin{pmatrix} \hbar w + E \\ \frac{\Omega}{2} \end{pmatrix}$$

$$|E_-\rangle = \frac{1}{\left(\frac{\hbar w}{2}\right)^2 - E\hbar w + \left(\frac{\Omega}{2}\right)^2} \begin{pmatrix} \hbar w - E \\ \frac{\Omega}{2} \end{pmatrix}$$

(b) To solve the schrodinger equation

$$i\hbar\frac{d\psi}{dt}(t) = H\psi(t)$$

The solution is given by

$$|\psi(t)\rangle = \exp(-i\hat{H}t/\hbar)|\psi(t=0)\rangle$$

Where in our case the initial state is $\psi_0 \in \mathbb{C}^2$. For a qubit state an example initial state could be $\psi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, however this is ignored as it is not explicitly stated in the question.

We need to find $\exp(-i\hat{H}t/\hbar)$ which we can do using the *Cayley-Hamilton* theorem, since $E_+ \neq E_-$ where $E_+ = E$ and $E_- = -E$, so we solve the system of equations

$$e^{-iEt/\hbar} = c_0 + c_1 E$$
$$e^{iEt/\hbar} = c_0 - c_1 E$$

Then

$$e^{-i\hat{H}t/\hbar} = c_0 + c_1\hat{H}$$

$$= \begin{pmatrix} c_0 + c_1\frac{\hbar w}{2} & c_1\frac{\Omega}{2} \\ c_1\frac{\Omega}{2} & c_0 - c_1\frac{\hbar w}{2} \end{pmatrix}$$

So

$$c_0 = \cos(Et/\hbar) \ ; \ c_1 = \frac{-i\sin(Et/\hbar)}{E}$$

And

$$e^{-i\hat{H}t/\hbar} = \begin{pmatrix} \cos(Et/\hbar) - \frac{i\sin(Et/\hbar)}{E}\frac{\hbar w}{2} & \frac{-i\sin(Et/\hbar)}{E}\frac{\Omega}{2} \\ \frac{-i\sin(Et/\hbar)}{E}\frac{\Omega}{2} & \cos(Et/\hbar) + \frac{i\sin(Et/\hbar)}{E}\frac{\hbar w}{2} \end{pmatrix}$$

This is then placed in

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi_0\rangle$$

For the example state above $\psi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ this would correspond to

$$|\psi(t)\rangle = \begin{pmatrix} \cos(Et/\hbar) - \frac{i\sin(Et/\hbar)}{E}\frac{\hbar w}{2} \\ \frac{-i\sin(Et/\hbar)}{E}\frac{\Omega}{2} \end{pmatrix} ✅$$

(c) The von neumann equation for a qubit density matrix is

$$i\hbar\frac{\partial\rho}{\partial t} = \left[H, \rho\right]$$

Where the RHS denotes the commutation relation. With

$$\rho(t) = e^{-i\hat{H}t}\rho(0)e^{i\hat{H}t}$$

At this point we can see that the commutation relation between H and ρ can be broken down into individual $\vec{v} \cdot \vec{\sigma}$ to find the differential equations for each. For $v_1(t)$ with σ_x

$$i\hbar \frac{\partial v_1(t)\sigma_x}{\partial t} = \left[H, v_1(t)\sigma_x \right]$$

The RHS or the commuation relation is

$$\left[H, v_1(t)\sigma_x \right] = \begin{pmatrix} \frac{\hbar w}{2} & \frac{\Omega}{2} \\ \frac{\Omega}{2} & -\frac{\hbar w}{2} \end{pmatrix} \begin{pmatrix} 0 & v_1(t) \\ v_1(t) & 0 \end{pmatrix} - \begin{pmatrix} 0 & v_1(t) \\ v_1(t) & 0 \end{pmatrix} \begin{pmatrix} \frac{\hbar w}{2} & \frac{\Omega}{2} \\ \frac{\Omega}{2} & -\frac{\hbar w}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \hbar w v_1(t) \\ -\hbar w v_1(t) & 0 \end{pmatrix} = \hbar w \begin{pmatrix} 0 & v_1(t) \\ -v_1(t) & 0 \end{pmatrix}$$

$$= i\hbar w v_1(t)\sigma_y$$

So the von neumann equation is

$$i\hbar \frac{\partial v_1(t)\sigma_x}{\partial t} = i\hbar w v_1(t)\sigma_y$$

$$\frac{\partial v_1(t)\sigma_x}{\partial t} = w v_1(t)\sigma_y$$

$$\frac{\partial v_1(t)}{\partial t} = -iw v_1(t)\sigma_z$$

For $v_2(t)$ with σ_y

$$i\hbar \frac{\partial v_2(t)\sigma_y}{\partial t} = \left[H, v_2(t)\sigma_y \right]$$

The RHS or the commuation relation is

$$\left[H, v_2(t)\sigma_y \right] = \begin{pmatrix} \frac{\hbar w}{2} & \frac{\Omega}{2} \\ \frac{\Omega}{2} & -\frac{\hbar w}{2} \end{pmatrix} \begin{pmatrix} 0 & -iv_2(t) \\ iv_2(t) & 0 \end{pmatrix} - \begin{pmatrix} 0 & -iv_2(t) \\ iv_2(t) & 0 \end{pmatrix} \begin{pmatrix} \frac{\hbar w}{2} & \frac{\Omega}{2} \\ \frac{\Omega}{2} & -\frac{\hbar w}{2} \end{pmatrix}$$

$$= \begin{pmatrix} i\Omega v_2(t) & -i\hbar w v_2(t) \\ -i\hbar w v_2(t) & -i\Omega v_2(t) \end{pmatrix} = i\Omega v_2(t)\sigma_z - i\hbar w v_2(t)\sigma_x$$

So the von neumann equation is

$$i\hbar \frac{\partial v_2(t)\sigma_y}{\partial t} = i\Omega v_2(t)\sigma_z - i\hbar w v_2(t)\sigma_x$$

$$\frac{\partial v_2(t)\sigma_y}{\partial t} = v_2(t)\left(\frac{\Omega}{\hbar}\sigma_z - w\sigma_x \right)$$

$$\frac{\partial v_2(t)}{\partial t} = \frac{-i}{\hbar}\Omega v_2(t)\sigma_x - iw v_2(t)\sigma_z$$

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For $v_3(t)$ with σ_z

$$i\hbar \frac{\partial v_3(t)\sigma_z}{\partial t} = \left[H, v_3(t)\sigma_z \right]$$

The RHS or the commuation relation is

$$\left[H, v_3(t)\sigma_z \right] = \begin{pmatrix} \frac{\hbar w}{2} & \frac{\Omega}{2} \\ \frac{\Omega}{2} & -\frac{\hbar w}{2} \end{pmatrix} \begin{pmatrix} v_3(t) & 0 \\ 0 & -v_3(t) \end{pmatrix} - \begin{pmatrix} v_3(t) & 0 \\ 0 & -v_3(t) \end{pmatrix} \begin{pmatrix} \frac{\hbar w}{2} & \frac{\Omega}{2} \\ \frac{\Omega}{2} & -\frac{\hbar w}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\Omega v_3(t) \\ \Omega v_3(t) & 0 \end{pmatrix} = -i\Omega v_3(t)\sigma_y$$

So the von neumann equation is

$$i\hbar \frac{\partial v_3(t)\sigma_z}{\partial t} = -i\Omega v_3(t)\sigma_y$$

$$\frac{\partial v_3(t)\sigma_z}{\partial t} = \frac{-\Omega}{\hbar} v_3(t)\sigma_y$$

$$\frac{\partial v_1(t)}{\partial t} = \frac{-i}{\hbar}\Omega v_3(t)\sigma_x$$

In summary we have

$$\frac{\partial v_1(t)}{\partial t} = \boxed{9.1} iwv_1(t)\sigma_z$$

$$\frac{\partial v_2(t)}{\partial t} = \frac{-i}{\hbar}\Omega v_2(t)\sigma_x - iwv_2(t)\sigma_z$$

$$\frac{\partial v_1(t)}{\partial t} = \frac{-i}{\hbar}\Omega v_3(t)\sigma_x$$

If we had an initial state given or initial conditions, these differential equations could be solved. ✓ $\boxed{9.2}$

1 Appendix

1. Alternate to Question 3: Showing that for pure states the descriptions of the Bloch vector we have given coincides with

$$|\psi\rangle = cos(\frac{\theta}{2})|0\rangle + e^{i\phi}sin(\frac{\theta}{2})|1\rangle$$

$$\rho = |\psi\rangle\langle\psi|$$

$$\rho = \begin{pmatrix} cos^2(\theta/2) & e^{-i\phi}cos(\theta/2)sin(\theta/2) \\ e^{i\phi}cos(\theta/2)sin(\theta/2) & sin^2(\theta/2) \end{pmatrix}$$

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$$= \begin{pmatrix} cos^2(\theta/2) & cos(\phi)cos(\theta/2)sin(\theta/2) - isin(\phi)cos(\theta/2)sin(\theta/2) \\ cos(\phi)cos(\theta/2)sin(\theta/2) + isin(\phi)cos(\theta/2)sin(\theta/2) & 1 - cos^2(\theta/2) \end{pmatrix}$$

Like in Exercise 2 place into the same form so that

$$1 + v_3 = 2cos^2(\theta/2) \; ; \; v_1 = 2cos(\phi)cos(\theta/2)sin(\theta/2)$$

$$v_3 = 2cos^2(\theta/2) \; ; \; v_2 = 2sin(\phi)cos(\theta/2)cos(\theta/2)sin(\theta/2)$$

So

$$|\vec{v}|^2 = 4cos(\theta/2)(cos^2(\theta/2) - cos^2(\theta/2)) + 1 = 1$$

This is not really necessary as it pretty much works the same as in exercise 2, however it is good to know this it works.

2. **Half angles on the Bloch Sphere**

$$|\psi\rangle = cos(\theta')|0\rangle + e^{i\phi}sin(\theta')|1\rangle$$

For $\theta' = 0, |\psi\rangle = |0\rangle$ and if $\theta' = \frac{\pi}{2}$ then $ket\psi = e^{i\phi}|1\rangle$ so that $0 \leq \theta' \leq \frac{pi}{2}$. This allows us to generate all the points within the bloch sphere.

Now we consider a state vector $|\psi'\rangle$ which corresponds to the opposite point on the bloch sphere with polar coordinates $(1, \pi - \theta', \phi + \pi)$ such that

$$|\psi'\rangle = cos(\pi - \theta')|0\rangle + e^{i(\phi+\pi)}sin(\pi - \theta')|1\rangle$$

$$= -cos(\theta')|0\rangle + e^{i\phi}e^{i\pi}sin(\theta')|1\rangle$$

So we can see that

$$|\psi'\rangle = -|\psi\rangle$$

Therefore, we only have to consider the upper hemisphere of the bloch sphere from $0 \leq \theta' \leq \frac{\pi}{2}$ as antipodal points in the lower hemisphere only different by a factor or -1.

We can map the point on the upper hemisphere onto points on a sphere by stating the definition

$$\theta = 2\theta' \; s.t \; \theta' = \frac{\theta}{2}$$

Which gives us the equation we were aiming for. This is for $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$

3. **Note**:
Many of these questions were questions I have come across before and I have referred to course notes done at University of Helsinki and previous teaching. Question 5 was particularly hard I found. 10.1

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