Quantum Mechanics IIa 2021
Solutions to Problem Set 3

Jake Muff 6/02/21

Problem 1

Periodically Driven Harmonic Oscillator where $t < 0$ in the ground state and for $t > 0$ we have perturbing potential

$$V(x,t) = F_0 x \cos(\omega t)$$

With Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 x^2$$

In the interaction picture we have

$$\langle x \rangle = \langle \psi | x | \psi \rangle$$

$$= \langle \psi | e^{iH_0 t} x e^{-iH_0 t} | \psi \rangle$$

Where

$$|\psi\rangle = \sum_n c_n(t)|n\rangle \tag{1}$$

Starting at $t = 0$ we have $c_n^{(0)}(t) = \delta_{n0}$

$$c_n^{(0)} = c_0^{(0)} = c_0(t) - 1$$

$$c_n^{(1)}(t) = \frac{-i}{\hbar}\int_{t_0}^{t} \langle n|V_I(t')|i\rangle dt'$$

$$= \frac{-i}{\hbar}\int_{t_0}^{t} e^{i\omega_{ni}t'} V_{ni}(t') dt'$$

$$= \frac{-i}{\hbar}\int_{0}^{t} V_{n0}(t') e^{in\omega_0 t'}$$

$$= \frac{-i}{\hbar}\int_{0}^{t} e^{i(E_n - E_0)t'/\hbar} \langle n|F_0 x \cos(\omega t')|0\rangle dt'$$

Using the hint

$$\langle n'|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\sqrt{n+1}\delta_{n',n+1} + \sqrt{n}\delta_{n',n-1})$$

So

$$\langle n|F_0 x \cos(\omega t')|0\rangle \equiv F_0 \langle n|x|0\rangle \cos(\omega t')$$

$$= \sqrt{\frac{\hbar}{2m\omega}}(\delta_{n,1})$$

Thus,

$$\begin{aligned}
c_n^{(1)}(t) &= \frac{-i}{\hbar}\int_0^t e^{i\omega_0 t'} F_0 \cos(\omega t')\sqrt{\frac{\hbar}{2m\omega_0}}\delta_{n1} dt' \\
&= \frac{-i}{\hbar}\sqrt{\frac{\hbar}{2m\omega_0}} F_0 \delta_{n1}\int_0^t e^{i\omega_0 t'}\cos(\omega t') dt' \\
&= \frac{-i}{\hbar}\sqrt{\frac{\hbar}{2m\omega_0}} F_0 \int_0^t e^{i\omega_o t'}\Big(\frac{e^{i\omega t'} + e^{-i\omega t'}}{2}\Big) dt' \\
&= \frac{-i}{\hbar}\sqrt{\frac{\hbar}{2m\omega_0}} F_0 \cdot \text{integral}
\end{aligned}$$

The integral is evaluated as

$$\begin{aligned}
\int_0^t e^{i\omega_o t'}\Big(\frac{e^{i\omega t'} + e^{-i\omega t'}}{2}\Big) &= \Big[\frac{ie^{-it'(\omega-\omega_0)}}{\omega-\omega_0} - \frac{ie^{it'(\omega+\omega_0)}}{\omega+\omega_0}\Big]_0^t \\
&= -i\Big(\frac{1-e^{-it(\omega-\omega_0)}}{\omega-\omega_0} + \frac{e^{it(\omega+\omega_0)}-1}{\omega+\omega_0}\Big)
\end{aligned}$$

For $n > 1$, $c_n^{(1)} = 0$ clearly. Here I changed into the schrodinger picture, because it was easier to calculate (and understand what was going on). I still use the calculations above in the final answer.

$$\begin{aligned}
|\psi\rangle_I &= \sum_n c_n(t)|n\rangle \\
&= 1|0\rangle + c_1(t)|1\rangle
\end{aligned}$$

Therefore

$$|\psi\rangle_S = e^{-iH_0 t/\hbar}|\psi\rangle_I$$

For a simple harmonic Oscillator we have

$$H_0|0\rangle = \frac{1}{2}\hbar\omega_0|0\rangle$$

$$H_0|1\rangle = \frac{3}{2}\hbar\omega_0|1\rangle$$

Thus

$$|\psi\rangle_S = e^{-i\omega_0 t/2}|0\rangle + c_1(t)e^{-3i\omega_0 t/2}|1\rangle$$

$$\langle x\rangle =_S \langle\psi|x|\psi\rangle_S$$

$$= (e^{i\omega_0 t/2}\langle 0| + c_1^\dagger(t)e^{3i\omega_0 t/2}\langle 1|)\cdot x\cdot(e^{-i\omega_0 t/2}|0\rangle + c_1(t)e^{-3i\omega_0 t/2}|1\rangle) \tag{2}$$

x here can be represented in ladder operator formalism with

$$x = \sqrt{\frac{\hbar}{2m\omega_0}}(a + a^\dagger)$$

Where

$$a = \sqrt{\frac{m\omega_0}{2}}(x + \frac{i}{m}\hat{p})$$

$$a^\dagger = \sqrt{\frac{m\omega_0}{2}}(x - \frac{i}{m}\hat{p})$$

Using this equation (2) can be split into two

$$c_1^\dagger e^{i\omega_0 t}\langle 1|x|0\rangle = c_1^\dagger e^{i\omega_0 t}\sqrt{\frac{\hbar}{2m\omega_0}}$$

$$c_1 e^{-i\omega_0 t}\langle 0|x|1\rangle = c_1 e^{-i\omega_0 t}\sqrt{\frac{\hbar}{2m\omega_0}}$$

So that

$$\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega_0}}(c_1 e^{-i\omega_0 t} + c_1^\dagger e^{i\omega_0 t})$$

We have from before:

$$c_1(t) = \frac{-i}{\hbar}\sqrt{\frac{\hbar}{2m\omega_0}}F_0\Big(-i\Big(\frac{1 - e^{-it(\omega-\omega_0)}}{\omega - \omega_0} + \frac{e^{it(\omega+\omega_0)} - 1}{\omega + \omega_0}\Big)\Big)$$

Substituting this in

$$\langle x\rangle = \frac{1}{\hbar}\frac{\hbar}{2m\omega_0}F_0\Big(e^{-i\omega_0 t}\Big(\frac{1 - e^{i(\omega+\omega_0)t}}{\omega + \omega_0}\Big) + e^{-i\omega_0 t}\Big(\frac{1 - e^{i(\omega-\omega_0)t}}{\omega - \omega_0}\Big)\Big)$$

$$= \frac{1}{\hbar}\frac{\hbar}{2m\omega_0}F_0\Big(\Big(\frac{e^{-i\omega_0 t} - e^{i\omega t}}{\omega + \omega_0}\Big) + \Big(\frac{e^{-i\omega_0 t} - e^{-i\omega t}}{\omega - \omega_0}\Big)\Big)$$

$$= \frac{1}{\hbar}\frac{\hbar}{2m\omega_0}F_0\frac{\cos(\omega_0 t) - \cos(\omega t)}{\omega_0^2 - \omega^2}\Big((\omega - \omega_0) + (\omega + \omega_0)\Big)$$

$$= \frac{1}{\hbar}\frac{\hbar}{2m\omega_0}F_0 2\omega_0\Big(\frac{\cos(\omega_0 t) - \cos(\omega t)}{\omega_0^2 - \omega^2}\Big)$$

$$= \frac{F_0}{m}\frac{\cos(\omega_0 t) - \cos(\omega t)}{\omega_0^2 - \omega^2}$$

Is this valid for $\omega = \omega_0$ (Needs answering)

Problem 2

Simple Harmonic Oscillator with

$$V(x,t) = Ax^2 e^{\frac{-t}{\tau}}$$

Probability that after $t >> \tau$ system transitions to a higher excited state. Transistion probability for $|i\rangle \to |n\rangle$ with $n \neq i$ is

$$P(i \to n) = |c_n^{(1)}(t) + c_n^{(2)}(t) + \dots|^2$$

For this we need

$$\langle n'|x^2|n\rangle = \sqrt{\frac{\hbar}{2m\omega_0}}\Big(\sqrt{n}\langle n'|x|n-1\rangle + \sqrt{n-1}\langle n'|x|n+1\rangle\Big)$$

$$= \frac{\hbar}{2m\omega_0}\Big(\sqrt{n(n-1)}\delta_{n-2,n'} + (2n+1)\delta_{nn'} + \sqrt{(n+1)(n+2)}\delta_{n+2,n'}\Big)$$

So

$$\langle n'|x^2|0\rangle = \frac{\hbar}{2m\omega_0}(\delta_{0n'} + \sqrt{2}\delta_{2n'})$$

Ignoring $\delta_{0n'}$.

$$c_n^{(0)} = \delta_{n0}$$

$$c_n^{(1)} = \frac{-i}{\hbar}\int_0^t e^{i(E_n - E_0)t'/\hbar}\langle n'|Ax^2 e^{-t/\tau}|0\rangle dt'$$

$$= \frac{-i}{\hbar}A\int_0^t e^{i\omega_0 t'} e^{-t/\tau}\langle n'|x^2|0\rangle dt'$$

$$= \frac{-i}{\hbar}A\frac{\hbar}{2m\omega_0}\sqrt{2}\delta_{n2}\int_0^t e^{i\omega_0 t'} e^{-t/\tau} dt'$$

$$= \frac{-i}{\hbar}A\frac{\hbar}{2m\omega_0}\sqrt{2}\delta_{n2}\Big[\frac{e^{i\omega_0 t'} - \frac{t'}{\tau}}{i\omega_0 - \frac{1}{\tau}}\Big]_0^t$$

Problem 3