Open Quantum Systems: Exercise session 7

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Intro: a qubit interacting with a bath

In this exercise session we will derive the Lindblad equation for a concrete example. We put $\hbar = 1$.

We suppose that the bath is described by a free electromagnetic field. The Hamiltonian for the bath is therefore a series ranging over the field modes

$$H_E = \sum_k \omega_k b_k^\dagger b_k \tag{1}$$

The integer k codes all of the information specifying each mode: its frequency, direction, transverse structure and polarization. The annihilation and creation operators for each mode are independent and they obey the Bosonic commutation relations

$$[b_k \,, b_l^\dagger] = \delta_{kl} \tag{2}$$

The Hamiltonian of the qubit (two-level "atom") is

$$H_S = \frac{\omega_a}{2}\sigma_z \tag{3}$$

Here ω_a is the energy difference between the ground $|g\rangle$ and excited $|e\rangle$ states, and

$$\sigma_z = |e\rangle\langle e| - |g\rangle\langle g| = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4}$$

is the "inversion operator" for the qubit. Recall that $\hat{\sigma}_z$ together with

$$\sigma_x \equiv \sigma_1 = |e\rangle\langle g| + |g\rangle\langle e| = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \& \qquad \sigma_y \equiv \hat{\sigma}_2 = -\imath|e\rangle\langle g| + \imath|g\rangle\langle e| = \begin{bmatrix} 0 & -\imath \\ \imath & 0 \end{bmatrix} \tag{5}$$

are Pauli matrices satisfying

$$\sigma_j \hat{\sigma}_k = \delta_{jk}\mathsf{I} + \imath\, \epsilon_{jkl}\, \sigma_l \tag{6}$$

for I the identity matrix.

The coupling of the electromagnetic field to a qubit can be described by the so-called dipole-coupling Hamiltonian

$$H_I = \sigma_x B = \sigma_x \sum_k g_k (b_k + b_k^\dagger) \tag{7}$$

The *real* coefficient g_k is proportional to the dipole matrix element for the transition.

The total Hamiltonian operator is therefore

$$\hat{H} = H_E + H_S + H_I \tag{8}$$

Question 1

1. Prove that in the interaction picture

$$\begin{aligned} H_I(t) &= \sigma_x(t)B(t) \\ &= \sum_k g_k \left(\frac{\sigma_x - i\,\sigma_y}{2} e^{-i\,\omega_a t} + \frac{\sigma_x + i\,\sigma_y}{2} e^{i\,\omega_a t} \right) \left(b_k e^{-i\,\omega_k\,t} + b_k^\dagger e^{i\,\omega_k\,t} \right). \end{aligned} \tag{9}$$

   Here $\sigma_x(t)$ and $B(t)$ are the interaction picture representations of σ_x and B:

$$\sigma_x(t) = e^{iH_S t} \sigma_x e^{-iH_S t} \tag{10}$$

$$B(t) = e^{iH_E t} B e^{-iH_E t} \tag{11}$$

2. Show that for $\sigma_+ = \frac{\sigma_x + i\,\sigma_y}{2}$, $\sigma_- = \frac{\sigma_x - i\,\sigma_y}{2}$

$$[H_S, \sigma_\pm] = \pm\omega_a \sigma_\pm \tag{12}$$

3. What is the evolution equation for the state operator in the interaction picture?

Question 2

We assume that at time $t = 0$

$$\rho(0) = \rho_S(0) \otimes \rho_B. \tag{13}$$

a. By integrating and interating the evolution equation for the interaction picture denstiy matrix $\rho(t)$, show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -i[H_I, \rho(0)] - \int_0^t \mathrm{d}s\, [H_I(t), [H_I(s), \rho(s)]]. \tag{14}$$

b. Now take the partial trace of the bath degrees of freedom under the assumption that

$$\mathrm{Tr}_B[H_I(t), \rho(0)] = 0. \tag{15}$$

   and that the Born approximation holds. This approximation assumes that since the coupling between system and bath is week, the system will barely affect the environment. Therefore

$$\rho(t) \approx \rho_S(t) \otimes \rho_B. \tag{16}$$

   The final result should be

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = -\int_0^t \mathrm{d}s\ \mathrm{Tr}_B[H_I(t), [H_I(s), \rho_S(s) \otimes \rho_B]]. \tag{17}$$

c. Now we perform the Markov approximation: replace $\rho_S(s)$ by $\rho_S(t)$ in the integral. Additionally, make the a change of variables in the integral $s \to t - s$. We introduce two time scales: τ_R is the timescale over which the qubit varies due to the interaction with the environment. τ_B is the timescale over which bath correlation functions decay (Note that the integral contains bath correlation functions: $\mathrm{Tr}(...\rho_E)$). Assume that $\tau_R \gg \tau_B$, which means that you can let the upper limit of the integral go to infinity.

Question 3

a. Show that the integral you obtained in Exercise 2 c. can be written as

$$\begin{aligned}\dot{\rho}_S(t) = &- \{\Gamma(\omega_a)\,[\sigma_+\,\sigma_-\,\rho_S(t) - \sigma_-\,\rho_S(t)\,\sigma_+] + \text{Hermitian conjugate}\} \\ &- \{\Gamma(-\omega_a)\,[\sigma_-\,\sigma_+\,\rho_S(t) - \sigma_+\,\rho_S(t)\,\sigma_-] + \text{Hermitian conjugate}\}\end{aligned} \tag{18}$$

where

$$\Gamma(\omega) = \int_0^{+\infty} \mathrm{d}s\, e^{i\omega s}\, \mathrm{Tr}_B(B^\dagger(t)B(t-s)\rho_B) \tag{19}$$

and you perform the secular approximation, which means that terms proportional to $e^{\pm 2\omega_a}$ can be neglected.

b. Assuming that $[H_E, \rho_B] = 0$, show that

$$\Gamma(\omega) = \int_0^{+\infty} \mathrm{d}s\, e^{i\omega s}\, \mathrm{Tr}_B(B^\dagger(s)B\rho_B). \tag{20}$$

Write

$$\Gamma(\omega) = \frac{1}{2}\gamma(\omega) + iS(\omega) \tag{21}$$

Where γ and S are real-valued function, and show that

$$\gamma(\omega) = \Gamma(\omega) + \Gamma^*(\omega) = \int_{-\infty}^{+\infty} \mathrm{d}s\, e^{i\omega s}\, \mathrm{Tr}_B(B^\dagger(s)B\rho_B). \tag{22}$$

c. Show with (**??**) that equation (**??**) becomes

$$\begin{aligned}\dot{\rho}_S(t) = &- i[S(\omega_a)\sigma_+\sigma_- + S(-\omega_a)\sigma_-\sigma_+, \rho_S(t)] \\ &+ \gamma(\omega_a)\big(\sigma_-\,\rho_S(t)\,\sigma_+ - \frac{1}{2}\{\sigma_+\,\sigma_-\,, \rho_S(t)\}\big) \\ &+ \gamma(-\omega_a)\big(\sigma_+\,\rho_S(t)\,\sigma_- - \frac{1}{2}\{\sigma_-\,\sigma_+\,, \rho_S(t)\}\big)\end{aligned} \tag{23}$$

d. Let us now assume that we have a continuous spectrum in the bath:

$$H_E = \int_0^{\infty} \mathrm{d}\omega\, \omega\, b^\dagger(\omega)b(\omega) \tag{24}$$

and

$$B = \int_0^{\infty} \mathrm{d}\omega\, g(\omega)(b^\dagger(\omega) + b(\omega)) \tag{25}$$

Using that

$$\int_{-\infty}^{+\infty} \mathrm{d}s\, e^{i\omega s} = 2\pi\delta(\omega) \tag{26}$$

show that

$$\gamma(-\omega_a) = 2\pi g^2(\omega_a)\, \mathrm{Tr}_B(b^\dagger(\omega_a) \int \mathrm{d}\omega\, g(\omega)(b^\dagger(\omega) + b(\omega))\rho_B) \tag{27}$$

and

$$\gamma(\omega_a) = 2\pi g^2(\omega_a)\, \mathrm{Tr}_B(b(\omega_a) \int \mathrm{d}\omega\, g(\omega)(b^\dagger(\omega) + b(\omega))\rho_B) \tag{28}$$