# Quantum Information B Fall 2020 Solutions to Problem Set 2

Jake Muff

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# 1 Answers

1. Exercise 8.3.
System AB in state $\rho_{AB}$ brought into contact with system CD in state $|0\rangle$. Two systems interact with $U$. After interactions discard A and D so we have a state $\rho'$ of system BC. So intially we have

$$\rho_{AB} \otimes |0\rangle_{CD}\langle 0|_{CD}$$

Then after

$$\rho'_{BC} = \sum_{ij} \langle i_A|\langle j_D|(U\rho_{AB}|0\rangle\langle 0|U^\dagger)|j_D\rangle|i_A\rangle$$

$$= \sum_{i,j} \langle i_A|\langle j_D|(U\rho_{AB} \otimes |0\rangle_C \otimes |0\rangle_D \langle 0|_C \otimes \langle 0|_D U^\dagger)|j_D\rangle|i_A\rangle$$

$$= \sum_{i,j} \langle i_A| \otimes \langle j_D|U|0\rangle_C \otimes |0\rangle_D \cdot \rho_{AB} \cdot \langle 0|_C \otimes \langle 0|_D U^\dagger|j_D\rangle|i_A\rangle$$

The first term in the above equation is $E_{ij}$ because of the general equation $E_k \equiv \langle e_k|U|e_0\rangle$ from the book. So

$$E_{ij} = \sum_{i,j} \langle i_A| \otimes \langle j_D|U|0\rangle_C \otimes |0\rangle_D$$

The second term is simply just $\rho$ and the third term is the hermitian of $E_{ij}$. If we collect $i, j$ into $k$ so we have

$$\rho'_{BC} = \sum_k E_k \rho_{AB} E_k^\dagger$$

For the second part we have

$$\sum_k E_k^\dagger E_k = \sum_{i,j} \langle 0|_C \otimes \langle 0|_D U^\dagger |i_A\rangle \otimes |j_D\rangle \cdot \langle i_A| \otimes \langle j_D|U|0\rangle_C \otimes |0\rangle_D$$

$$= \langle 0|_C \langle 0|_D U^\dagger U|0\rangle_C|0\rangle_D = I = I_{AB}$$

2. Exercise 8.4.

$$U = P_0 \otimes I + P_1 \otimes X$$

With $P_0 \equiv |0\rangle\langle 0|, P_1 \equiv |1\rangle\langle 1|$. So we have

$$\varepsilon(\rho) = \mathrm{Tr}(U(\rho \otimes |0\rangle\langle 0|)U^\dagger)$$

$$= \sum_k \langle k|U(\rho \otimes |0\rangle\langle 0|)U^\dagger|k\rangle$$

$$\sum_k \langle k|(P_0 \otimes I + P_1 \otimes X)\rho \otimes |0\rangle\langle 0|P_0 \otimes I + P_1 \otimes X|k\rangle$$

$U^\dagger U$ can be shown to be unitary through

$$U = P_0 \otimes I + P_1 \otimes X$$

$$U^\dagger U = (P_0 \otimes I + P_1 \otimes X)^\dagger (P_0 \otimes I + P_1 \otimes X)$$
$$= P_0 \otimes I + P_1 \otimes X^2$$
$$(P_0 + P_1) \otimes I$$
$$= I$$

Making use of $X^2 = I$, $P_0 P_1 = |0\rangle\langle 0| \otimes |1\rangle\langle 1| = 0$ and $P_1 P_0 = |1\rangle\langle 1| \otimes |0\rangle\langle 0| = 0$ So we can write

$$\varepsilon(\rho) = \sum_k P_0 \rho P_0 \otimes \langle k|I|0\rangle\langle 0|I|k\rangle$$
$$+ P_0 \rho P_1 \otimes \langle k|I|0\rangle\langle 0|X|k\rangle$$
$$+ P_1 \rho P_0 \otimes \langle k|X|0\rangle\langle 0|I|k\rangle$$
$$+ P_1 \rho P_1 \otimes \langle k|X|0\rangle\langle 0|X|k\rangle$$
$$= P_0 \rho P_0 + P_1 \rho P_1$$

3. Exercise 8.9. We have a set of quantum operations $\{\varepsilon_m\}$ where

$$U|\psi\rangle|e_0\rangle = \sum_{mk} E_{mk}|\psi\rangle|m, k\rangle$$

With projector

$$P_m \equiv \sum_k |m, k\rangle\langle m, k|$$

Performing $U$ on $\rho \otimes |e_0\rangle\langle e_0|$ then measuring $P_m$ gives $m$ with probability $\mathrm{Tr}(\varepsilon_m(\rho))$ with post measurement state

$$\varepsilon_m(\rho)/\mathrm{Tr}(\varepsilon_m(\rho))$$

So we have

$$\rho = \sum_i p_i |\psi\rangle\langle\psi|$$

And from the bottom of p365 we have

$$\rho'_\psi = \frac{1}{p(m)} \mathrm{Tr}(P_m U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger)$$

$$= \frac{1}{p(m)} \mathrm{Tr}(\sum_k |m,k\rangle\langle m,k| U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger)$$

$$= \frac{1}{p(m)} \sum_{k,i} p_i \langle m,k|U(|\psi\rangle \otimes |e_0\rangle)(\langle\psi| \otimes \langle e_0|)U^\dagger|m,k\rangle$$

$$= \frac{1}{p(m)} \sum_{k,i} p_i E_{m,k}|\psi\rangle\langle\psi|E^\dagger_{m,k}$$

$$= \frac{1}{p(m)} \sum_k E_{mk}\rho E^\dagger_{mk} = \frac{\varepsilon_m(\rho)}{p(m)}$$

$$\rho'_\psi = \frac{\varepsilon_m(\rho)}{p(m)}$$

Where $p(m)$ is

$$p(m) = \mathrm{Tr}(P_m U(P \otimes |e_0\rangle\langle e_0|)U^\dagger)$$

$$= \sum_{i,m,k} p_i \langle\psi_i|E^\dagger_{mk}E_{mk}|\psi_i\rangle \otimes \langle m,k||m,k\rangle$$

$$= \sum_{i,m,k} p_i \langle\psi_i|E^\dagger_{mk}E_{mk}|\psi_i\rangle$$

$$= \mathrm{Tr}(\sum_{i,m,k} E_{mk}|\psi_i\rangle p_i \langle\psi_i|E^\dagger_{mk})$$

$$= \mathrm{Tr}(\sum_k E_{mk}\rho E^\dagger_{mk})$$

$$= \mathrm{Tr}(\varepsilon_m(\rho))$$

So probability to get $m$ when measuring $P_m$ is

$$p(m) = \mathrm{Tr}(\varepsilon_m(\rho))$$

And the post measurement state is

$$\rho'_\psi = \frac{\varepsilon_m(\rho)}{p(m)} = \frac{\varepsilon_m(\rho)}{\mathrm{Tr}(\varepsilon_m(\rho))}$$

4. Exercise 8.17. Verfying

$$\frac{I}{2} = \frac{\rho + X\rho X + Y\rho Y + Z\rho Z}{4}$$

Through

$$\varepsilon(A) \equiv \frac{A + XAX + YAY + ZAZ}{4}$$
$$= \frac{1}{4}(A + \sum_{i}^{3} \sigma_i A \sigma_i)$$

We know that pauli matrices have the property that $\sigma_i^2 = I$ so

$$\varepsilon(I) = \frac{1}{4}(I + XIX + YIY + ZIZ)$$

$$= \frac{1}{4}(I + 3I) = I$$

And for $\varepsilon(\sigma_i)$

$$\varepsilon(\sigma_i) = \sum_i \frac{1}{4}(\sigma_i + \sum_{i \neq j} \sigma_j \sigma_i \sigma_j + \sigma_i)$$
$$= \frac{1}{4}(2\sigma_i - 2\sigma_i) = 0$$

In the bloch sphere representation this is

$$\rho = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma})$$

$$\varepsilon(\rho) = \frac{1}{4}(\rho + \sum_i \sigma_i \rho \sigma_i)$$
$$= \frac{1}{2} I = \frac{\rho + X\rho X + Y\rho Y + Z\rho Z}{4}$$

5. Exercise 8.21. This exercise relies heavily on chapter 7 which we skipped in lectures and I am reading for the first time for this exercise so my answer to this question may miss some bits out.

$$H = \chi(a^\dagger b + b^\dagger a)$$

We have a system that interacts with the environment, so the intitial state of the whole system would be ( I am answering this like I have answered a similar question on another course)

$$\rho(0) = \rho_S(0) \otimes \rho_E(0)$$

Where $\rho_S$ is the density operator for the system and $\rho_E$ is for the environment. This evolves with time evolution according to

$$\rho(t) = U(t)\rho_S(0) \otimes \rho_E(0)U^\dagger$$

Where $U(t) \equiv \exp(-iHt/\hbar) = \exp(-iHt)$ where $\hbar = 1$. The partial trace over the environment gives us the reduced density operator

$$\rho_S(t) = \mathrm{Tr}_E\Big(U(t)\rho_S(0) \otimes \rho_E(0)U^\dagger(t)\Big)$$

$$= \sum_{k_i} \langle k_i|U(t)\rho_S(0) \otimes \rho_E(0)U^\dagger(t)|k_i\rangle$$

Where $|k_i\rangle$ satisfies $\sum_{i=1}^{\infty} k_i = k$ so this can be written in the operator notation as

$$E_k = \sum_{k_i}^{k} \langle k_i|U(t)|0_i\rangle$$

$$= \langle k_b|U|0_b\rangle$$

That last part is in the context of the question. This can be wirtten in its Krauss representation as

$$\rho_S(t) = \sum_{k=0}^{\infty} E_k \rho_S(0) E_K^\dagger$$

Which satisfies completeness i.e $\sum_k E_k^\dagger E_k = I$. From previous exercises we can write

$$E_k = \sum_{m,n} E_{m,n}^k |m\rangle\langle n|$$

Where $|n\rangle$ is an eigenstate of $a^\dagger a$ and as such is an orthonormal basis of the system.

$$E_{m,n}^k = \sum_{k_i}^{k} \langle m|\langle k_i|U|0_i\rangle|n\rangle$$

$$= \langle m|\langle k_b|U|0_b\rangle|n\rangle$$

Now using Exercise 7.4 from the book

$$U|0_b\rangle|n\rangle$$

Can be written as

$$\frac{U(a^\dagger)^n}{\sqrt{n!}}|0_b\rangle|0\rangle$$

$$= \frac{(a^\dagger(-t))^n}{\sqrt{n!}}|0_b\rangle|0\rangle$$

Using this we can define

$$E^k_{m,n}(E^k_{m,n})^\dagger = \binom{n}{k}(\cos^2(\chi\Delta t)^{(n-k)})(1-\cos^2(\chi\Delta t))^k\delta_{m,n-k}$$

This comes from expanding out the Hamiltonian in $U$ and using the commutation relation. ( I admit I worked a little backwards here as I knew I needed it to be of the form where it includes the binomial expansion)

$$[a, a^\dagger] = 1$$

We can divide by $(E^k_{m,n})^\dagger$ noting that if we impose the condition that the state $|n\rangle$ satisfies $n \geq k$ then elements of $E^k_{m,n}$ must be real so we get

$$E_k = \sum_n \sqrt{\binom{n}{k}}\sqrt{(1-\gamma)^{n-k}\gamma^k}|n-k\rangle\langle n|$$

Where $\gamma = 1 - \cos^2(\chi\Delta t)$