Quantum Mechanics IIa 2021   Problem Set 2

Solutions are due in 2 pm on Wednesday Feb 3, as a pdf file into the return box in the course Moodle page.

Constant

1. Calculate the first order correction to the third eigenenergy $E_3^{(0)}$ (the ground state is $E_1^{(0)}$) for a particle in a one-dimensional box with infinite walls at $x = 0, x = L$, due to the following perturbations:

   (a) $V = 10^{-3} E_1 x/L$ ← const

   (b) $V = 10^{-3} E_1 (x/L)^2$

   (c) $V = 10^{-3} E_1 \sin(x/L)$.

   E_n

2. Consider a particle of mass m in a potential well

$$V(x) = \begin{cases} \infty\ , & x \leq -L/2,\ x \geq L/2 \\ 0\ , & -L/2 < x < -a/2,\ a/2 < x < L/2 \\ V_0\ , & -a/2 \leq x \leq a/2 \end{cases}$$

   where $L > a$ and $V_0 > 0$. This is an infinite potential well with a bump in the middle. Assume that the bump at the bottom of the well can be considered a small perturbation.

   (a) Calculate the corrected second eigenenergy and eigenfunction to first order in the perturbation.

   (b) What dimensionless ratio must be small compared to 1 in order for your approximation to be valid?

   (c) First-order corrected energies corresponding to even eigenstates are greater than energies corresponding to odd eigenstates. Why? (Hint: Note the behavior of eigenstates at the origin.)

3. **Periodically driven two-state system**. Consider the time-dependent Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H(t)|\psi(t)\rangle \tag{1}$$

   with a time dependent Hamiltonian of the form

$$H(t) = H_0 + V(t)\ .$$

   Let $|n\rangle$ denote the eigenstates of H_0 with eigenvalues E_n. Expand the state (in this problem it is sufficient to work in the Schrödinger picture)

$$|\psi(t)\rangle = \sum_n c_n(t) e^{-iE_n t/\hbar} |n\rangle$$

(a) Show that the coefficients $c_m(t)$ satisfy the equation

$$i\hbar \frac{dc_m(t)}{dt} = \sum_n e^{i\omega_{mn}t} V_{mn}(t) c_n(t) \tag{2}$$

with

$$\omega_{mn} \equiv \frac{E_m - E_n}{\hbar} \; ; \; V_{mn}(t) = \langle m|V(t)|n\rangle \; .$$

(b) Assume that the potential contains a small parameter so that we can treat $V(t)$ perturbatively. Use an initial condition $c_m^{(0)}(t_0) = \delta_{mi}$ and show that to first order in $V(t)$,

$$c_m(t) = \delta_{mi} - \frac{i}{\hbar} \int_{t_0}^{t} dt' e^{i\omega_{mi}t'} V_{mi}(t') \; . \tag{3}$$

Next, consider a two-state system with

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \; ; \; |2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

with the unperturbed Hamiltonian

$$H_0 = \begin{pmatrix} \hbar\omega_0/2 & 0 \\ 0 & -\hbar\omega_0/2 \end{pmatrix}$$

and with the interaction potential

$$V(t) = \begin{pmatrix} 0 & -2\hbar\lambda\cos\omega t \\ -2\hbar\lambda\cos\omega t & 0 \end{pmatrix} \; .$$

This is a simplified model for the interaction of an atom with a time-dependent laser field, or for the interaction of a spin half particle with an oscillating magnetic field.

(c) Let $t_0 = 0$. Assume that initially the system is at ground state, $c_m^{(0)}(0) = \delta_{m1}$. Solve (3) to get

$$c_1(t) = 1 \; ; \; c_2(t) = \lambda \left[\frac{e^{i(\omega+\omega_0)t}}{\omega+\omega_0} - \frac{e^{-i(\omega-\omega_0)t}}{\omega-\omega_0} \right] \; . \tag{4}$$

Typo: wrong solution, not normalized, will get eztra term

Next, we find a more accurate solution of this driven two-level system.

(d) First, substitute the energy eigenvalues and the interaction potential into (2) to obtain two coupled differential equations. Then, invoking the so-called rotating-wave approximation, drop the rapidly oscillating terms $e^{\pm i(\omega+\omega_0)t}$ to derive the simplified equations

$$\frac{dc_1(t)}{dt} = i\lambda e^{i\delta t} c_2(t) \; ; \; \frac{dc_2(t)}{dt} = i\lambda e^{-i\delta t} c_1(t) \tag{5}$$

where $\delta \equiv \omega - \omega_0$ is the *detuning parameter*. **Note:** We assume $\delta = \omega - \omega_0$ is small and ω, ω_0 are large.

typo: signs should be flipped

(e) Then, show that

$$
\begin{aligned}
c_1(t) &= e^{i\delta t/2}\left[\cos(\Omega t/2) - i\frac{\delta}{\Omega}\sin(\Omega t/2)\right] \\
c_2(t) &= e^{-i\delta t/2}\frac{2i\lambda}{\Omega}\sin(\Omega t/2)
\end{aligned}
$$

where $\Omega = \sqrt{\delta^2 + 4\lambda^2}$ is the *Rabi frequency*, are solutions of (5) satisfying the same initial condition $c_m^{(0)}(0) = \delta_{m1}$ and the normalization condition $|c_1|^2 + |c_2|^2 = 1$.

Hint: Transform (5) into second order differential equations for c_1 and c_2. Then substitute $c_1(t) = e^{i\delta t/2}x(t)$, $c_2(t) = e^{-i\delta t/2}y(t)$ to obtain differential equations of a simple harmonic oscillator for x, y.

(f) Finally, show that for short times $\delta t, \Omega t \ll 1$ the above result and the perturbative result agree, with

+ something

$$
|c_2(t)|^2 \approx (\lambda t)^2 \ .
$$

Q3 worth double points

Hint: Apply the rotating-wave approximation to (4).

Lesson: the validity of the perturbative result does not only depend on having a small expansion parameter, but also on the range of time.

4. (Optional problem, no points given. However, you may find it interesting, as NMR spectroscopy is a useful tool, and a basis for the magnetic resonance imaging in medicine.) Read about the nuclear magnetic resonance (NMR) from Kimmo Tuominen's lecture notes: section 3.1.2 starting at p. 60, to understand it as an application of the periodically driven two-state system. Prepare to discuss it in the next week's exercise session. You don't have to write anything, but if you like, you can write a short summary. The lecture notes can be found in the QMII Moodle.