Open Quantum Systems: Solutions 7

Paolo Muratore-Ginanneschi and Brecht Donvil

November 12, 2020

Exercise 1: Polar Decomposition for Invertible Matrices

Let A be an invertible square matrix

a. Show that $A^\dagger A$ is positive definite ($A > 0$) and thus invertible
1pt

b. Show that $(A^\dagger A)^{1/2}$ is also positive definite *1pt*

c. Show that $A(A^\dagger A)^{-1/2}$ is unitary *1pt*

d. Using the above, show that A can be written as the product of a unitary matrix U and a positive definite matrix P *1pt*

$$A = UP$$

The above expression is called the **Polar Decomposition** of a matrix. In general, any square matrix can be written as the product of a unitary and a semi-positive definite matrix.

Exercise 2: Invertibility of a Quantum Channel

A quantum channel is a linear map

$$\Gamma(\rho) = \sum_i M_i \rho M_i^\dagger,$$

where ρ and the M_i are $d \times d$ matrices and $\sum_i M_i^\dagger M_i = \mathbb{I}$. In this exercise we will show that Γ is only invertible if all the M_i are proportional to a single unitary matrix, meaning that Γ is a unitary map.

a. If Γ is invertible there exists another map $\Gamma'(\rho) = \sum_j N_j \rho N_j^\dagger$ such that

$$\Gamma'(\Gamma(\psi\psi^\dagger)) = \sum_{i,j} N_j M_i \psi\psi^\dagger M_i^\dagger N_j^\dagger = \psi\psi^\dagger.$$

Conclude from the above equation that $N_j M_i = \lambda_{ji}\mathbb{I}$, with λ_{ji} some complex number.
1pt

b. Show that

$$M_b^\dagger M_a = \sum_j \lambda_{jb}^* \lambda_{ja} \mathbb{I} = \beta_{ba}\mathbb{I}$$

2pts

c. Use the result of last exercise that

$$M_a = \sqrt{\beta_{aa}} U_a$$

where U_a is a unitary matrix.
1pt

d. Show that

$$U_a = \frac{\beta_{ba}}{\sqrt{\beta_{aa}\beta_{bb}}} U_b$$

2pts

e. Conclude that $\Gamma(\rho) = U\rho U^\dagger$, where U is a unitary matrix.
1pt

Exercise 3

Exercise 3 Solutions 8-2018

Let us consider a system of N harmonic oscillators, with Hamiltonian

$$H = \sum_{j=1}^{N} \hbar\omega_j b_j^\dagger b_j. \tag{1}$$

The orthonormal eigenbasis of H is given by

$$\left\{ |i_1, i_2 \ldots, i_N\rangle = \frac{(b_1^\dagger)^{i_1}}{\sqrt{i_1!}} \frac{(b_2^\dagger)^{i_2}}{\sqrt{i_2!}} \ldots |0\rangle \right\} \text{ for } i_1,\, i_2,\, \ldots \in \mathbb{N}, \tag{2}$$

the states have energy

$$H|i_1, i_2 \ldots, i_N\rangle = \big(\sum_{j=1}^{N} i_j \hbar\omega_j\big)|i_1, i_2 \ldots, i_N\rangle. \tag{3}$$

a. Show that the thermal state $\rho_{th} = e^{-\beta H}$ can be written in terms of the basis (2) as

$$\sum_{i_1, i_2, \ldots, i_N = 0}^{+\infty} |i_1, i_2 \ldots, i_N\rangle\langle i_1, i_2 \ldots, i_N| e^{-\beta \sum_j i_j \hbar\omega_j} \tag{4}$$

2pts

b. Find a purification of the thermal state. Concretely, double the original Hilbert space $\mathbb{H} \to \mathbb{H}_D = \mathbb{H} \otimes \mathbb{H}$, and find a vector Ψ in $\mathbb{H}_D$, such that

$$\rho_{th} = \mathrm{Tr}_2(\Psi\Psi^\dagger), \tag{5}$$

where Tr_2 traces over the second Hilbert space.
2pts

Exercise 4

Continuation of Exercise 3.

a. Show that the expectation value

$$\Psi^\dagger b_j^\dagger b_{j'} \Psi = 0 \tag{6}$$

for all $j \neq j'$ and where both operators act on the first Hilbert space of the product $\mathbb{H} \to \mathbb{H}_D = \mathbb{H} \otimes \mathbb{H}$.
2pts

b. Calculate

$$\Psi^\dagger b_j^\dagger b_j \Psi \tag{7}$$

where both operators act on the first Hilbert space of the product $\mathbb{H} \to \mathbb{H}_D = \mathbb{H} \otimes \mathbb{H}$.
2pts