Open Quantum Systems: Exercises 5

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Quantum master equations I: Random Phases

Consider a state vector written in the basis ϕ_1, ϕ_2 whose phase factors undergo a random walk. Given an initial state vector

$$\psi = a\phi_1 + b\phi_2$$

and suppose that at time t it has evolved to

$$\psi(t) = ae^{i\theta_1}\phi_1 + be^{i\theta_2}\phi_2$$

with probability

$$P(\theta_1, \theta_2) = \frac{1}{\sqrt{2\pi\lambda_1 t}} \frac{1}{\sqrt{2\pi\lambda_2 t}} e^{-\frac{\theta_1^2}{2\lambda_1 t}} e^{-\frac{\theta_2^2}{2\lambda_2 t}}.$$

a. Show that the density matrix at time t equals

$$\begin{aligned}\rho(t) &= \int_{-\infty}^{\infty} d\theta_1 \int_{-\infty}^{\infty} d\theta_2 P(\theta_1, \theta_2)\psi(t)\psi^\dagger(t) \\ &= |a|^2\phi_1\phi_1^\dagger + |b|^2\phi_2\phi_2^\dagger + e^{-\frac{1}{2}t(\lambda_1+\lambda_2)}(ab^*\phi_1\phi_2^\dagger + ba^*\phi_2\phi_1^\dagger)\end{aligned}$$

2pts

b. Show that $\rho(t)$ satisfies the master equation

$$\frac{d}{dt}\rho(t) = \sum_i \left[L_i\rho(t)L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i\rho(t)\}\right]$$

with $\{a, b\} = ab + ba$ and $L_i = \sqrt{\frac{1}{2}(\lambda_1 + \lambda_2)}\phi_i\phi_i^\dagger$.
2pts

Quantum master equations II: Unitary Jump

Suppose that in a time dt, a state vector $\psi(t)$ has probability λdt to jumping to $e^{-iG}\psi(t)$, where G is some hermitian operator.

a. Show that the density operator at a time $t + dt$ is given by

$$\rho(t + dt) = (1 - \lambda dt)\rho(t) + \lambda dte^{-iG}\rho(t)e^{iG}$$

1pt

b. Show that $\rho(t)$ satisfies the differential equation

$$\frac{d}{dt}\rho(t) = -\lambda[\rho(t) - e^{-iG}\rho(t)e^{iG}]$$

1pt

c. Find the operator L such that

$$\frac{d}{dt}\rho(t) = L\rho(t)L^\dagger - \frac{1}{2}\{L^\dagger L, \rho(t)\}$$

1pt

d. Let G have eigenvalues g_1, g_2 and eigenvectors $\mathbf{g_1}$ and $\mathbf{g_2}$. Show that the diagonal elements $\rho_{ii}(t) = \mathbf{g_i}^\dagger\rho(t)\mathbf{g_i}$ are constant in time and the off-diagonal element $\rho_{12}(t) = \mathbf{g_1}^\dagger\rho(t)\mathbf{g_2}$ satisfies the differential equation

$$\frac{d}{dt}\rho_{12}(t) = -\lambda\rho_{12}(t)(1 - e^{i(g_2-g_1)}).$$

2pts

Quantum master equations III: Random Unitary transformation

Suppose that in a time dt a state vector $\psi(t)$ undergoes a transformation to

$$\psi(t+dt) = e^{-iG\theta}\psi(t)$$

for a self-adjoint matrix G, with probability

$$P(\theta, t) = \frac{1}{\sqrt{4\pi\lambda dt}}e^{-\frac{\theta^2}{4\lambda dt}}$$

a. Show that the density operator at time $t + dt$ equals (up to first order in dt)

$$\rho(t+dt) = \rho(t) - \frac{\lambda dt}{2}\left[G^2\rho(t) + \rho(t)G^2 - 2G\rho(t)G\right]$$

Hint: expand $e^{-iG\theta}\rho(t)e^{iG\theta}$ in θ. Show that terms of order θ^3 or higher will be of higher order in dt and can thus be neglected.
2pts

b. Find an operator L such that the density operator satisfies the master equation

$$\frac{d}{dt}\rho(t) = L\rho(t)L^\dagger - \frac{1}{2}\{L^\dagger L, \rho(t)\}$$

2pts

c. Let G have eigenvalues g_1 and g_2. Let $\rho_{ij}(t)$ be the components of the density operator in the basis of eigenvectors of G, show that they satisfy the differential equation

$$\frac{d}{dt}\rho_{ij} = -\frac{\lambda}{2}(g_i - g_j)^2\rho_{ij}(t)$$

1pt

Quantum master equations IV: State exchange

Let ϕ_1, ϕ_2 be two orthonormal vectors and define the state $\psi(t) = a(t)\phi_1 + b(t)\phi_2$. Suppose that in a time dt, with probability λdt $\psi(t)$ interchanges its basis states, i.e.

$$\psi(t) \to a(t)\phi_2 + b(t)\phi_1$$

a. Let σ_x be the canonical Pauli matrix in the basis $\phi_{1,2}$ (meaning σ_x is 0 on the diagonal and 1 on the off-diagonal). Show that the state operator satisfies the master equation

$$\frac{d}{dt}\rho(t) = L\rho(t)L^\dagger - \frac{1}{2}\{L^\dagger L, \rho(t)\}$$

with $L = \sqrt{\lambda}\sigma_x$.
2pts

b. Show that the matrix elements satisfy

$$\frac{d}{dt}\rho_{11}(t) = -\frac{d}{dt}\rho_{22}(t) = -\lambda[\rho_{11} - \rho_{22}],$$

$$\frac{d}{dt}\rho_{12}(t) = -\frac{d}{dt}\rho_{21}(t) = -\lambda[\rho_{12} - \rho_{21}].$$

1pt

Quantum master equations V: Lindblad equation

In the last exercises we have derived master equations for density operators arising from various situations. All of these master equations were of the form

$$\frac{d}{dt}\rho(t) = -i[H, \rho(t)] + \sum_i \left[L_i\rho(t)L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho(t)\} \right],$$

where H is a hermitian operator (in the above examples $H = 0$). An equation of this form is often referred to as the Lindblad equation. This equation is central to open quantum systems and will be discussed in detail in the coming lectures.

a. Show that that the Lindblad equation is trace preserving, concretely show that

$$\frac{d}{dt}\operatorname{Tr}\rho(t) = 0$$

1pt

b. The solution of the Lindblad equation will be of the form

$$\rho(t) = \sum_i M_i(t)\rho_0 M_i^\dagger(t)$$

where ρ_0 is the initial state and $\sum_i M_i^\dagger(t)M_i(t) = \mathbb{I}$. Show that $\rho(t)$ is a valid state operator at all times (show that $\rho(t)$ has trace 1 given that ρ_0 has trace 1, is self-adjoint and semi-positive definite)
3pts