Quantum Information A Fall 2020 Problem Set 2

Solutions are due in 4 pm on Tuesday Sep 8.

1. Consider the vectors $w_1 = (1, 2, 2)$, $w_2 = (-1, 0, 2)$, $w_3 = (0, 0, 1)$. Compute the Gram matrix G with entries G_{ij} given by the scalar products of pairs of vectors

$$G_{ij} = w_i \cdot w_j \ ,$$

   you can simplify the computation by noticing that G is a symmetric matrix. Show that the vectors w_i are linearly independent, by showing that the Gram determinant $\det G \neq 0$. The vectors then form a basis of $\mathbf{R}^3$. Show that the basis is not orthogonal. Then, apply the Gram-Schmidt process to contruct an orthonormal basis v_1, v_2, v_3.

2. A matrix A is *normal*, if $A^\dagger A = AA^\dagger$. Show that the matrix

$$M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

   is not normal. Show that a *Hermitian* matrix A (satisfying $A^\dagger = A$) is normal.

3. Show by using the Cauchy-Schwarz inequality that the norm given by the inner product $||x|| = \sqrt{\langle x|x\rangle}$ satisfies the triangle inequality: every pair of vectors x, y satisfies

$$||x + y|| \leq ||x|| + ||y|| \ .$$

4. (Exercise 2.11 of the book): Find the eigenvectors, eigenvalues, and diagonal representations of the Pauli matrices X, Y, Z.

5. Exercise 2.18 of the book: Show that all eigenvalues λ of a unitary matrix satisfy $|\lambda| = 1$, that is, they can be written in the form $e^{i\theta}$ for some real θ. (In other words, they belong to the origin-centered unit circle in the complex plane).

6. (Exercise 2.22 of the book): Show that the eigenvectors of a Hermitian operator with different eigenvalues are necessarily orthogonal.