FYMM/MMP IIIa 2020 Solutions to Problem Set 6

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1. **Mattress flipping.**
   In this question we have 4 different operations:

   (a) I $\rightarrow$ do nothing, the identity operation
   (b) R $\rightarrow$ flip 180 degrees along the long side
   (c) P $\rightarrow$ flip 180 degrees along the short side
   (d) Y $\rightarrow$ rotation of 180 degrees around the center

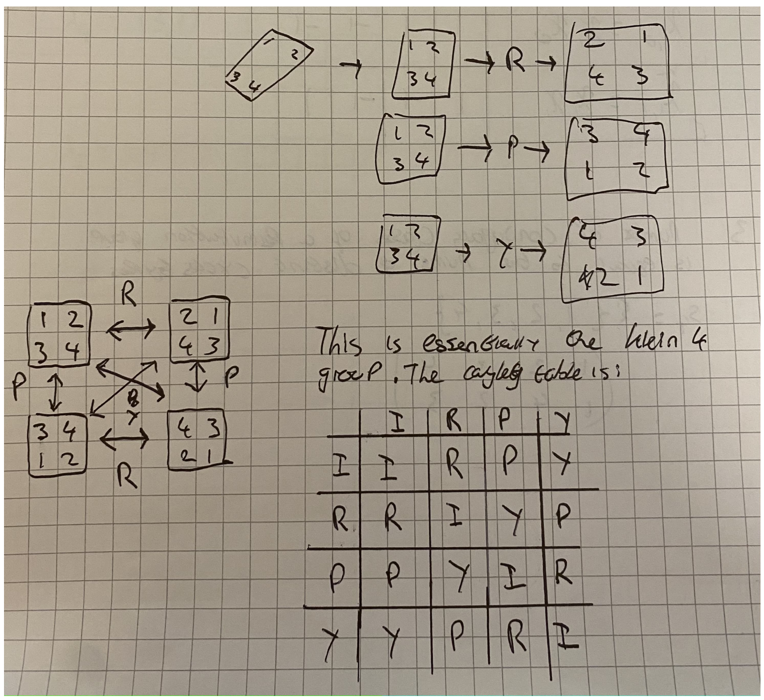

Figure 1: Picture of Mattress flipping diagrams. Don't know why they have rendered badly. Will attach also.

   This is essentially the klein 4 group

   This is the klein four group also known as the viergruppe $\mathbb{Z}_2 \times \mathbb{Z}_2$ group with a matching cayley table

2. Construct the character table of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

	I	R	P	Y
I	I	R	P	Y
R	R	I	Y	P
P	P	Y	I	R
Y	Y	P	R	I

Table 1: Cayley table for operations in the mattress flipping

	(0,0)	(0,1)	(1,0)	(1,1)
$\chi_{0,1} = \chi_0\chi_0$	1	1	1	1
$\chi_{0,1} = \chi_0\chi_1$	1	-1	1	-1
$\chi_{1,0} = \chi_1\chi_0$	1	1	-1	-1
$\chi_{1,1} = \chi_1\chi_1$	1	-1	-1	1

Table 2: Character table for $\mathbb{Z}_2 \times \mathbb{Z}_2$

3. Proving that the number of conjugacy classes of a permutation group is equal to the number of disjoint cycle types.

$$S_4 = \{1, 2, 3, 4\}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$$

The S_4 has cycle types of

$$(12), (13), (14), (23), (24), (34) \rightarrow 2 - cycles$$

$$(12)(34), (13)(24), (14)(23) \rightarrow Products of 2 - cycles$$

$$(123), (124), (132), (134), (142), (143), (234), (243) \rightarrow 3 - cycles$$

$$(1234), (1234), (1324), (1342), (1423), (1432) \rightarrow 4 - cycles$$

Along the the cycle type () i.e the empty cycles which sometimes denoted e, we have 5 cycle types. The character table for S_4 shows: Which has 5 conjugacy classes.

	()	(1,2)(3,4)	(1,2)	(1,2,3,4)	(1,23)
Trivial representation	1	1	1	1	1
Sign representation	1	1	-1	-1	1
Irreducible representation	2	2	0	0	-1
Standard representation	3	-1	1	-1	0
Product of standard and sign representation	3	-1	-1	1	0

Table 3: Character table for S_4

This shows that for S_4 the number of classes is equal to the number of cycle types.

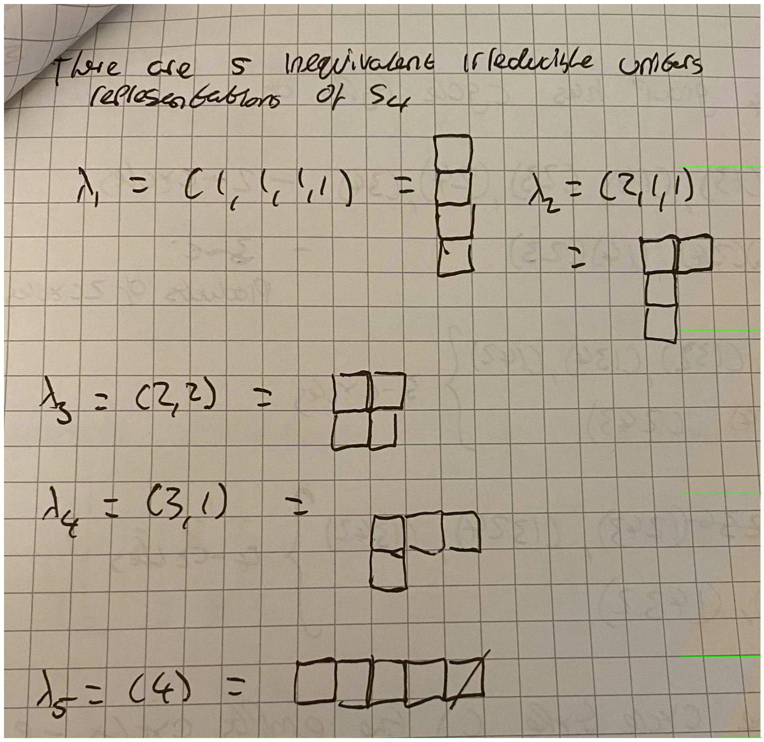

Figure 2: Young diagrams for S_4 also showing 5 inequivalent irreducible unitary representations

4. Given a vector space V, prove that every $\omega \in (V^*)^*$ can be uniquely associated with a vector $\vec{v} \in V$ such that $\omega(f) = \langle f, \vec{v} \rangle$. We know that $\omega \in (V^*)^*$ is a dual of a dual such that $\omega : V^* \to \mathbb{C}$. A vector can be written as

$$\vec{v} = v^j \vec{e}_j$$

The dual basis is

$$e^{*j}(\vec{e}_j) = \delta^i_j \to \ dimV = dimV^* = n$$

Let f be a linear function with $v \in \mathbb{C}$

$$f(\vec{v}) = f_i e^{*i}(v^j \vec{e}_j) = f_i v^j e^{*i}(\vec{e}_j)$$
$$= f_i v^i \delta^i_j = f_i v^i = \langle f, \vec{v} \rangle$$

So for $\omega(f)$

$$\omega(f) = v_i e^{**i}(f) = v_i e^{**i}(f^j \vec{e}_j) = v_i f^j e^{**i}(\vec{e}^*_j)$$
$$= v_i f^j \delta^i_j = v_i f^i = f_i v^i = \langle f, \vec{v} \rangle$$

5. Let the (1,0)-tensor R have the components

$$R^1 = a \; ; R^2 = a^2 \; ; R^3 = a^4$$

and the (0,1)-tensor S have the components

$$S_1 = -b \; ; S_2 = c \; ; S_3 = -d \; .$$

Calculate all the components T^μ_ν of the (1,1)-tensor $T = R \otimes S$.

$$T^1_1 = R^1 S_1 = -ab \;\; T^2_1 = R^2 S_1 = -a^2 b$$
$$T^1_2 = R^1 S_2 = ac \;\; T^2_2 = R^2 S_2 = a^2 c$$
$$T^1_3 = R^1 S_3 = -ad \;\; T^2_3 = R^2 S_3 = -a^2 d$$
$$T^3_1 = R^3 S_1 = -a^4 b \;\; T^3_2 = R^3 S_2 = a^4 c$$
$$T^3_3 = R^3 S_3 = -a^4 d$$