Statistical Methods Exercises 8 Autumn 2020

Return your solutions by 12.00 Finnish time on Thursday 12.11.2020 to Moodle course page: `https://moodle.helsinki.fi/course/view.php?id=30207`

1. **LS estimates.** The specific heat capacity (C) of a substance is the amount of energy (E) required to change its temperature a given amount per unit of mass m, $C = E/m\Delta T$, where ΔT is change in temperature. See e.g. `http://en.wikipedia.org/wiki/Specific_heat_capacity`.

   A way to determine C for a liquid is to heat it in a thermally isolated container with an electric heater while measuring the temperature of the system at different times. The heater produces as much heat energy in the time t as it consumes electrical energy E; the energy produced is $E = U \cdot I \cdot t$ (U is the voltage and I the current). A perfect transfer of the heat from the heater to the substance can be assumed.

   A chemist student uses an electric heater at $U = 12\,\mathrm{V}$ and $I = 10\,\mathrm{A}$ to heat $m = 1000$ g of an unknown liquid. He/she obtains the following results for the change in temperature at different times:

t (s)	352 ± 5	701 ± 9	1048 ± 9	1398 ± 9
ΔT (°C)	10.0	19.7	30.2	40.4
t (s)	1751 ± 9	2099 ± 15	2446 ± 15	2805 ± 15
ΔT (°C)	49.9	60.5	70.4	80.0

   (i) Determine the specific heat C of the liquid and its uncertainty using a Least Squares (LS) estimate. Assume there is no uncertainty on the temperature measurements and that C is constant over the measured temperature range. What are the χ^2 and P-values of your LS estimate (hint: get P-value from functions in statistics/mathematics packages or web applets e.g. google for "Chi2 calculator" or "Chi2 applet")?

   (ii) Check if the C variance obtained in (i) is equal to RCF-bound ($= (1 + (\partial b/\partial\alpha))^2 / E\left[-(\partial^2 \ln L/\partial\alpha^2)\right]$, where $b =$ bias). Hint: $\ln L = -\chi^2/2$.

   (iii) Based on the results, do you suspect that the student altered his/her measurements (or overestimated the uncertainties) to improve the result? Motivate. *Exercise gives max 8 points instead of usual 6.*

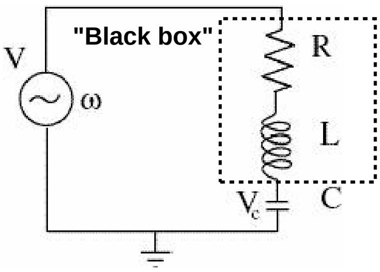
2. **LS fit with penalty functions** An unknown electronics circuit ("Black box", see figure below) induces a signal measured at V_c. If the capacity C is known, one can determine the internal resistance R and inductance L of the "Black box" by measuring θ as function of the frequency ω:

$$\cot\theta = (L/R)\omega - (1/RC)(1/\omega) \quad \Rightarrow \quad y = \alpha_1 x - \alpha_2/x\,,$$

   when $y \equiv \cot\theta$, $\alpha_1 \equiv \omega_0 L/R$, $\alpha_2 \equiv 1/(\omega_0 RC)$, $x \equiv \omega/\omega_0$ and $\omega_0 =$

1 rad/s. A measurement at five frequencies ω by connecting a known capacitor, $C = 0.02\ \mu$F, to the circuit, gave the following results:

$y^{meas} \pm \sigma_{y^{meas}}$	$x^{meas} \pm \sigma_{x^{meas}}$
-4.02 $\pm$ 0.50	22000 $\pm$ 440
-2.74 $\pm$ 0.25	22930 $\pm$ 470
-1.15 $\pm$ 0.08	23880 $\pm$ 500
1.49 $\pm$ 0.09	25130 $\pm$ 530
6.87 $\pm$ 1.90	26390 $\pm$ 540



(i) Determine the α_1 and α_2 values (and their errors) for the "Black box" using the Least Squares (LS) method neglecting the uncertainties in x^{meas}. What is the χ^2_{min} and P-value of your solution? Calculate L and R (and their errors) from α_1 and α_2 applying error propagation.

(ii) Determine the α_1 and α_2 values (and their errors) for the "Black box" taking now into account both uncertainties in x^{meas} and y^{meas}. Hint: Introduce the uncertainties in x as additional terms in the χ^2 sum and allow the actual value used for x_i in the $\cot\theta$ formula to vary around x_i^{meas} using $\sigma_{x_i{}^{meas}}$ as a "penalty" function i.e. the further one goes away from the x_i^{meas} in terms of $\sigma_{x_i{}^{meas}}$, the more additional contribution one gets to the χ^2 sum:

$$\chi^2 = \sum_{i=1}^{N}\left[\left(y_i^{meas} - y_i(x_i)\right)^2/\sigma^2_{y_i{}^{meas}}\right] + \sum_{i=1}^{N}\left[\left(x_i - x_i^{meas}\right)^2/\sigma^2_{x_i{}^{meas}}\right],$$

What is χ^2_{min} and P-value now? Calculate again L and R (with errors) from α_1 and α_2. Any significant changes? *Exercise gives max 12 points.*