# Open Quantum Systems: Exercises 4

Paolo Muratore-Ginanneschi and Brecht Donvil

October 9, 2020

## Exercise 1: Schwartz's inequality and uncertainty relation

(Nielsen and Chuang, exercise 2.57)
Suppose $\{L_l\}$ and $\{M_m\}$ are two sets of measurement operators.

a. Show that a measurement defined by the set of measurement operators $\{M_m\}$ followed by a measurement defined by the set $\{L_l\}$ is physically equivalent to a single measurement defined by the measurement operators $\{N_{ml}\}$ with $N_{ml} \equiv L_l M_m$. More precisely show that the probability of consecutive measurements is equal to the probability of the measurement $N_{ml}$ and that the resulting states are equal. *2pts*

b. If the first measurement goes unrecorded, the state $\rho_1$ after the first measurement is

$$\rho_1 = \sum_m M_m \rho M_m^\dagger.$$

What is the state after the second measurement is performed? *1pt*

c. Imagine that b. also the last measurement is unrecorded. Compare the final state to the state resulting from an unrecorded measurement of $\{N_{ml}\}$. *1pt*

## Exercise 2

The spin of a spin 1/2 particle has two possible values in any direction: $\pm\hbar/2$. It is a two level system, a qubit, which is described by the Hilbert space $\mathbb{C}^2$. It's equation of motion in a time dependent magnetic field $B(t)$ is given by

$$i\hbar\frac{\mathrm{d}\psi}{\mathrm{d}t}(t) = \vec{\sigma}\cdot B(t)\psi(t),$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the Pauli vector. Let us now consider the magnetic field

$$B(t) = \frac{\hbar}{2}(\omega_1 \cos\omega t, \omega_1 \sin\omega t, \omega_0).$$

(a) Give the equation of motion for the components $\psi_\pm(t)$ of $\psi(t) = (\psi_+(t), \psi_-(t))$ *2pts*

(b) In order to solve the equations of motion obtained in (a), it is convenient to define new variables

$$\psi_\pm(t) = e^{\mp i\omega_0 t/2}\phi_\pm(t).$$

Find the equations of motion of $\phi_\pm(t)$ *1pt*

(c) Now define

$$\xi_-(t) = e^{i(\omega_0-\omega)t}\phi_-(t).$$

Find the equation of motion for $\xi_-(t)$. Use this to find expressions for $\psi_\pm(t)$. *1pt*

(d) If at time 0, the $z$-spin component is $\hbar/2$ find the probability $P(t)$ that it is $-\hbar/2$ at a later time $t$. At which times does $P(t)$ reach a local maximum? *1pt*

## Exercise 3

Photons can have either a linear or circular polarisation. In the case of linear polarisation, the polarisation can be described in terms of two orthogonal directions. Let us call them the horizontal and vertical direction. The polarisation of the photon is thus a two level system, we choose the orthogonal basis $|h\rangle = (0,1)$ and $|v\rangle = (1,0)$ corresponding to the horizontal and vertical directions. The polarisation can be rotated by an angle $\theta$ with a unitary transformation

$$U(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{1}$$

(a) Show that rotating $n$ times with angle $\theta$ is the same as rotating with an angle $n\theta$, more precisely, show that

$$U^n(\theta) = U(n\theta) \tag{1}$$

*2pts*
(b) The projector on the horizontally polarised photon state is $P_h = |h\rangle\langle h|$, what is the projector on a photon state with polarisation rotated $\pi/4$ from the horizontal direction? *1pt*
(c) What is the probability of finding the photon in the horizontal state $|h\rangle$ after rotating it from the horizontal state by an angle $\theta$. *1pt*

## Exercise 4

We are still considering the polarised photons from exercise 3.
(a) Imagine that we apply $n$ rotations angle $\theta$ to the a photon with horizontal polarisation. After every rotation we project the polarisation back on the horizontal direction. Concretely, we apply to the state $|h\rangle$ the transformation

$$M_n = \underbrace{P_hU(\theta)P_h...U(\theta)P_hU(\theta)}_{n \text{ times}}$$

Calculate the vector

$$|\phi\rangle = M_n|h\rangle$$

*2pts*
(b) What is the probability of that the probability for the photon to be have horizontal polarisation after applying $M_n$ to $|h\rangle$ (calculate $|\langle h|\phi\rangle|^2$). *1pt*
(c) What happens to the probability for small $\theta$? This effect is called the quantum Zeno effect. *1pt*

## Exercise 5

Suppose that $A$, $B \in M_n\mathbb{C}$ commute, show that

$$e^A e^B = e^{A+B}.$$

Find a counterexample to show that this is not true for non-commuting matrices. (Hint: Pauli matrices) *3pts*

Ex 2.41 N&C
Problem 11 Chap 3 Problems&Solutions in QC&QI