Quantum Information A Fall 2020 Solutions to Problem Set 2

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1. Question 1.

$$w1 = (1, 2, 2) \ w_2 = (-1, 0, 2) \ w_3 = (0, 0, 1)$$

Gram Matrix given by $G_{ij} = w_i \cdot w_j$:

$$G = w^\dagger w$$

$$= \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 0 \\ 2 & 2 & 1 \end{pmatrix}^\dagger \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 0 \\ 2 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 2 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 0 \\ 2 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 3 & 2 \\ 3 & 5 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

This resulting matrix is of the form $G^T G$ and therefore symmetric giving the symmetric Gram matrix.
The determinant of $det(G) = 4$ as shown:

$$|G| = | \begin{pmatrix} 9 & 3 & 2 \\ 3 & 5 & 2 \\ 2 & 2 & 1 \end{pmatrix} |$$

$$= 9| \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} | - 3| \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} | + 2| \begin{pmatrix} 3 & 5 \\ 2 & 2 \end{pmatrix} |$$

$$= 9(5 - 4) - 3(3 - 4) + 2(6 - 10)$$

$$= 4 \neq 0$$

Therefore the vectors w_i are linearly independent.
The bases are orthogonal if every pair has an inner product 0 i.e $\langle w_i || w_j \rangle = 0$, therefore proving that every pair doesn't have an inner product equal to 0 shows that the basis is not orthogonal.

$$\langle w_1, w_2 \rangle = 3$$

$$\langle w_1, w_3 \rangle = 2$$

$$\langle w_2, w_3 \rangle = 2$$

Using the Gram-Schmidt to construct an orthonormal basis v_1, v_2, v_3 :

$$|v_1\rangle = \frac{|w_1\rangle}{|||w_1\rangle||}$$

$$= \frac{|w_1\rangle}{\sqrt{\langle w_1|w_1\rangle}}$$

$$\langle w_1|w_1\rangle = \begin{pmatrix} 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 9$$

$$\therefore \sqrt{\langle w_1|w_1\rangle} = 3$$

$$|v_1\rangle = \frac{|w_1\rangle}{3} = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

For $|v_2\rangle$ using $|v_1\rangle$:

$$|v_2\rangle = \frac{|w_2\rangle - \sum_i^k \langle v_i|w_2\rangle|v_i\rangle}{|||w_2\rangle - \sum_i^k \langle v_i|w_2\rangle|v_i\rangle||}$$

$$= \frac{|w_2\rangle - \langle v_1|w_2\rangle|v_1\rangle}{|||w_2\rangle - \langle v_1|w_2\rangle|v_1\rangle||}$$

$$\langle v_1|w_2\rangle = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = 1$$

$$|v_2\rangle = \frac{|w_2\rangle - |v_1\rangle}{|||w_2\rangle - |v_1\rangle||}$$

$$|||w_2\rangle - |v_1\rangle|| = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix} = \begin{pmatrix} -4/3 \\ -2/3 \\ 4/3 \end{pmatrix} = |x\rangle$$

$$\langle x|x\rangle = \begin{pmatrix} -\frac{4}{3} & -\frac{2}{3} & \frac{4}{3} \end{pmatrix} \begin{pmatrix} -4/3 \\ -2/3 \\ 4/3 \end{pmatrix} = 4$$

$$\therefore |||w_2\rangle - |v_1\rangle|| = 2$$

So we have

$$|v_2\rangle = \frac{|w_2\rangle - |v_1\rangle}{2}$$

$$= \frac{\begin{pmatrix} -4/3 \\ -2/3 \\ 4/3 \end{pmatrix}}{2}$$

$$= \begin{pmatrix} -2/3 \\ -1/3 \\ 2/3 \end{pmatrix}$$

$$|v_2\rangle = \frac{1}{3}\begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$$

For $|v_3\rangle$ we compute the same thing but now we have multiple terms in our sums ($\sum_i^{k=2}$):

$$|v_3\rangle = \frac{|w_3\rangle - \sum_i^{k=2}\langle v_i|w_3\rangle|v_i\rangle}{|||w_3\rangle - \sum_i^{k=2}\langle v_i|w_3\rangle|v_i\rangle||}$$

$$\sum_i^{k=2}\langle v_i|w_3\rangle|v_i\rangle = \langle v_1|w_3\rangle|v_1\rangle + \langle v_2|w_3\rangle|v_2\rangle$$

$$\langle v_1|w_3\rangle = \frac{2}{3}$$

$$\langle v_2|w_3\rangle = \frac{2}{3}$$

So we have :

$$\langle v_1|w_3\rangle|v_1\rangle + \langle v_2|w_3\rangle|v_2\rangle = \frac{2}{3}|v_1\rangle + \frac{2}{3}|v_2\rangle$$

$$= \begin{pmatrix} -2/9 \\ 2/9 \\ 8/9 \end{pmatrix}$$

Putting this into our Gram-Schmidt fraction:

$$|v_3\rangle = \frac{|w_3\rangle - \begin{pmatrix} -2/9 \\ 2/9 \\ 8/9 \end{pmatrix}}{|||w_3\rangle - \begin{pmatrix} -2/9 \\ 2/9 \\ 8/9 \end{pmatrix}||} = \frac{\begin{pmatrix} 2/9 \\ -2/9 \\ 1/9 \end{pmatrix}}{1/3}$$

$$|v_3\rangle = \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

These vectors are orthongonal to each other and it is trivial to find out that they all have length $= 1$ and are therefore unit vectors, therefore they form an orthonormal basis.

2. Question 2

$$A^\dagger A = AA^\dagger$$
$$M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$
$$M^\dagger M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$
$$MM^\dagger = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$
$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

And M is not *normal*
Showing that a hermitian matrix A is normal:
A hermitian matrix has the property

$$A^\dagger = A$$

Therefore

$$(A^\dagger)A = A \cdot A = A(A^\dagger)$$

Hence A is normal.

3. Question 3
The Cauchy-Schwarz inequality is:

$$|\langle x|y\rangle|^2 \leq \langle x|x\rangle\langle y|y\rangle$$

Starting with

$$||x+y|| = \sqrt{\langle x+y|x+y\rangle}$$

And squaring both sides

$$||x+y||^2 = \langle x+y|x+y\rangle$$
$$= ||x||^2 + 2\langle x|y\rangle + ||y||^2$$

Now using the Cauchy-Schwarz inequality

$$||x+y||^2 \leq ||x||^2 + 2\langle x|y\rangle + ||y||^2$$
$$\leq ||x||^2 + 2||x||||y|| + ||y||^2$$

The RHS of which equals (factorising)

$$||x||^2 + 2||x||||y|| + ||y||^2 = (||x|| + ||y||)^2$$

So we have

$$||x + y||^2 \leq (||x|| + ||y||)^2$$

And the square root

$$||x + y|| \leq ||x|| + ||y||$$

4. Starting with the X matrix:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$det(X - \lambda I) = det(\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}) = 0$$

$$\lambda = \pm 1$$

For $\lambda_1 = -1$:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvector

$$|\lambda_1\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Normalized eigenvector:

$$|\lambda_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For $\lambda_2 = 1$:

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvector

$$|\lambda_2\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Normalized eigenvector:

$$|\lambda_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For the diagonal representation X must satisfy

$$X = \sum_i \lambda_i |i\rangle\langle i|$$

$$\lambda_1 \cdot |\lambda_1\rangle\langle\lambda_1| + \lambda_2 \cdot |\lambda_2\rangle\langle\lambda_2|$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

For the Pauli Y matrix:

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$det(Y - \lambda I) = det(\begin{pmatrix} -\lambda & -i \\ -i & -\lambda \end{pmatrix}) = 0$$

$$\lambda = \pm 1$$

For $\lambda_1 = -1$:

$$\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvector

$$|\lambda_1\rangle = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Normalized eigenvector:

$$|\lambda_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

For $\lambda_2 = 1$:

$$\begin{pmatrix} -1 & -i \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvector

$$|\lambda_2\rangle = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Normalized eigenvector:

$$|\lambda_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

The diagonal representation uses the hermitian transpose † therefore we can write

$$-1 \cdot \lambda_1 \lambda_1^\dagger + 1 \cdot \lambda_2 \lambda_2^\dagger$$

So Y has diagonal representation of

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

For the Pauli Z matrix:

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$det(Z - \lambda I) = det(\begin{pmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix}) = 0$$

$$\lambda = \pm 1$$

For $\lambda_1 = 1$:

$$\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvector

$$|\lambda_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

For $\lambda_2 = -1$:

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvector

$$|\lambda_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Y has diagonal representation of

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

5. Question 5
   Let us say that $|v\rangle$ is an eigenvector with eigenvalue λ_v such that

$$U|v\rangle = \lambda|v\rangle$$

   So that

$$\langle v|v\rangle = 1$$
$$= \langle v|I|v\rangle$$
$$= \langle|U^\dagger U|v\rangle$$
$$= \lambda_v \lambda_v^* \langle v|v\rangle$$
$$= ||\lambda_v||^2 = 1$$
$$\therefore \lambda = e^{i\theta}$$

6. (Exercise 2.22 of the book): Show that the eigenvectors of a Hermitian operator with different eigenvalues are necessarily orthogonal.
   Let us state a hermitian operator A which has $|v_i\rangle$ eigenvectors and λ_i eigenvalues i.e

$$A|v_i\rangle = \lambda_i|v_i\rangle$$

Or

$$A|v_j\rangle = \lambda_j|v_j\rangle$$

So we have

$$\langle v_i|A|v_j\rangle = \lambda_j\langle v_i|v_j\rangle$$

Or

$$\langle v_i|A|v_j\rangle = \lambda_i\langle v_i|v_j\rangle$$

We then have

$$\langle v_i|A|v_j\rangle - \langle v_i|A|v_j\rangle = (\lambda_j - \lambda_i)\langle v_i|v_j\rangle = 0$$

So either $\lambda_j = \lambda_i$ or $\langle v_i|v_j\rangle = 0$
Meaning that if $\lambda_j \neq \lambda_i$ they are orthogonal to each other.
We can also say that because A is hermitian $A = A^\dagger$ so

$$\langle v_i|A|v_j\rangle = \langle v_i|A^\dagger|v_j\rangle = (\langle v_j|A|v_i\rangle)^*$$

$$= \lambda_i^*\langle v_j|v_i\rangle^* = \lambda_i^*\langle v_i|v_j\rangle = \lambda_i\langle v_i|v_j\rangle$$

Therefore

$$(\lambda_i - \lambda_j)\langle v_i|v_j\rangle = 0$$

and the same outcome as above.