FYMM/MMP IIIb 2020 Problem Set 4

Please submit your solutions for grading by **Monday 23.11.** in Moodle. (This week's problems might need more time than the previous ones so it could be useful to start early enough and prepare questions for the tutorial.)

1. Consider the compact manifold S^1, and recall that it can be given an atlas with two charts (U_i, φ_i), $i = 1, 2$, such that for $i = 1, 2$ we set

$$U_i = \{\mathrm{e}^{\mathrm{i}\theta} | \theta \in I_i\}\,, \quad \varphi_i^{-1} : I_i \to U_i\,, \quad \varphi_i^{-1}(\theta) = \mathrm{e}^{\mathrm{i}\theta}\,,$$

using the parameter domains $I_1 = (-\pi, \pi)$ and $I_2 = (0, 2\pi)$. Define for every $p = \mathrm{e}^{\mathrm{i}\theta} \in U_1$ the vectors

$$X_p = \sin^2(\theta)\, \partial_\theta\,, \qquad Y_p = \theta\, \partial_\theta\,,$$

in T_pS^1, using the coordinate presentation on the chart U_1. Show that there is a unique choice of X_p for $p = \mathrm{e}^{\mathrm{i}\pi}$ which makes X into a (smooth) vector field X on S^1. In contrast, show that it is not possible to extend Y into a smooth vector field on S^1. (These results should become obvious once you tranform the vectors for $p \in U_1 \cap U_2$ into the coordinate system of U_2). Compute the (complete) flow map $\sigma : \mathbb{R} \times S^1 \to S^1$ generated by X. Compute also all *fixed points* of the flow, i.e., points $p \in S^1$ for which $\sigma(t, p) = p$ for all t.

2. Show that under a coordinate transformation $(x_1, \ldots, x_n) \mapsto (y_1(\vec{x}), \ldots, y_n(\vec{x}))$ in $\mathbb{R}^n$ the n-form

$$dy^1 \wedge dy^2 \wedge \cdots \wedge dy^n = J(\vec{y}, \vec{x}) dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n,$$

where J is the Jacobian determinant

$$J(\vec{y}, \vec{x}) = \det\left(\frac{\partial y^i}{\partial x^j}\right).$$

Hint: you may need the following determinant formula

$$\det\left(\frac{\partial y^i}{\partial x^j}\right) = \varepsilon^{j_1 \cdots j_n} \frac{\partial y^1}{\partial x^{j_1}} \cdots \frac{\partial y^n}{\partial x^{j_n}} \tag{1}$$

which uses the completely antisymmetric ε-symbol. This is the general determinant formula

$$\det M = \det(M_{ij}) = \varepsilon^{j_1 \cdots j_n} M_{1j_1} \cdots M_{nj_n} \tag{2}$$

applied to the Jacobian matrix. You can find the formula for the determinant of an $n \times n$ matrix e.g. in Wikipedia.

3. Show that the exterior product satisfies the two properties below. For our purpose you may use the basis expansions, and direct computations. Let ω_q be a q-form

$$\omega_q = \frac{1}{q!}\omega_{\mu_1\mu_2\cdots\mu_q} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_q} \ ,$$

and η_r be a r-form. Show that

(a) $\omega_q \wedge \eta_r = (-1)^{qr}\eta_r \wedge \omega_q$

(b) $\omega_q \wedge \omega_q = 0$, when q is odd

Hint: (a) implies (b).

4. Consider the following differential forms in $\mathbb{R}^3$:

$$\alpha = xdx + ydy + zdz \ ; \ \beta = zdx + xdy + ydz \ ; \ \gamma = xydz \ .$$

(a) Is α closed or exact? Is γ closed or exact?

(b) Calculate $\alpha \wedge \beta$ and $(\alpha + \gamma) \wedge (\alpha + \gamma)$.
(*Hint:* the previous problem may be helpful.)

5. (Nakahara 6.2) Let $M = \mathbb{R}^3$, $\omega = \omega_x dx + \omega_y dy + \omega_z dz$. Show that Stokes' theorem implies

$$\int_S (\nabla \times \vec{\omega}) \cdot d\vec{S} = \oint_C \vec{\omega} \cdot d\vec{s} \ , \tag{3}$$

where $\vec{\omega} = (\omega_x, \omega_y, \omega_z)$ and C is the boundary of a surface S. In a similar vein, for $\psi = \frac{1}{2}\psi_{\mu\nu} dx^\mu \wedge dx^\nu$, show

$$\int_V \nabla \cdot \vec{\psi} \ dV = \oint_S \vec{\psi} \cdot d\vec{S} \ , \tag{4}$$

where the components of the vector $\vec{\psi}$ are $\psi^\lambda = \varepsilon^{\lambda\mu\nu}\psi_{\mu\nu}$ and S is the boundary of a volume V.

(*N.B.* This problem is meant to be non-trivial. By using the definition of integral of an n-form, motivate how the integrals are reduced to the usual integrals in $\mathbb{R}^3$. No need to consider several charts. Wikipedia might be useful.)