Quantum Information A Fall 2020
Solutions to Problem Set 3

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1. What is the Polar Decomposition of a positive matrix P? Of a Unitary Matrix U? Of a Hermitian Matrix H?

   (a) A positive matrix, P, is diagonalizable, meaning

   $$P = \sum_i \lambda_i |i\rangle\langle i|$$

   Where $\lambda_i \geq 0$. A matrix J can be viewed as

   $$J = \sqrt{P^\dagger P} = \sqrt{PP} = \sqrt{P^2}$$

   $$= \sum_i \sqrt{\lambda_i^2} |i\rangle\langle i| = \sum_i \lambda_i |i\rangle\langle i| = P$$

   So the polar decomposition is therefore

   $$P = UP; \ \forall P$$

   Since U is a unitary matrix i.e $U^\dagger U = I$ and $U = I$ then $P = P$

   (b) For a unitary matrix U
   Lets say that U is decomposed such that $U = KJ$ where K is unitary and J is a positive operator such that $J = \sqrt{U^\dagger U} = \sqrt{I} = I$.
   Unitary matrices/operators are invertible such that $K = UJ^{-1} = UI^{-1} = UI = U$.
   The polar decomposition of U is then $U = U$

   (c) For a Hermitian matrix H such that $H = UJ$ where $J = \sqrt{H^\dagger H} = \sqrt{HH} = \sqrt{H^2}$, which can be re-written as

   $$H = U\sqrt{H^2}$$

   From spectral decomposition we know that $H \neq \sqrt{H^2}$, the proof is as follows

   $$H = \sum_i \lambda_i |i\rangle\langle i|$$

   Where $\lambda \in \mathbb{R}$:

   $$\sqrt{H^2} = \sqrt{\sum_i \lambda_i^2 |i\rangle\langle i|} = \sum_i \sqrt{\lambda_i^2} |i\rangle\langle i| = \sum_i |\lambda_i| |i\rangle\langle i| \neq H$$

2. Find the left and right Polar Decompositions of the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$A^\dagger A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

The left polar decomposition is

$$A = UJ$$

And the right polar decomposition

$$A = KU$$

Where J and K are positive operators.

To find the matrices for K and J we use spectral decomposition and notice that $U = AJ^{-1}$

Lets find the eigenvalues for $A^\dagger A$:

$$det(A^\dagger A - \lambda I) = | \begin{pmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = 0$$

$$= (2-\lambda)(1-\lambda) - 1$$

$$= 2 - 2\lambda - \lambda + \lambda^2 - 1$$

$$1 - 3\lambda + \lambda^2 = 0$$

So

$$\lambda_\pm = \frac{3 \pm \sqrt{5}}{2}$$

With associated eigenvectors

$$|\lambda_+\rangle = \begin{pmatrix} \frac{\sqrt{5}+1}{2} \\ 1 \end{pmatrix}$$

$$|\lambda_-\rangle = \begin{pmatrix} -\frac{\sqrt{5}+1}{2} \\ 1 \end{pmatrix}$$

Now the spectral decomposition of $A^\dagger A$ is

$$A^\dagger A = \lambda_+ |\lambda_+\rangle\langle\lambda_+| + \lambda_- |\lambda_-\rangle\langle\lambda_-|$$

So when $J = \sqrt{A^\dagger A}$ the spectral decomposition is

$$\sqrt{A^\dagger A} = \sqrt{\lambda_+} |\lambda_+\rangle\langle\lambda_+| + \sqrt{\lambda_-} |\lambda_-\rangle\langle\lambda_-|$$

Which works out to be

$$J = \begin{pmatrix} 5.854 & 1.618 \\ 1.618 & 2.236 \end{pmatrix}$$

Now $U = AJ^{-1}$, where

$$J^{-1} = \begin{pmatrix} 0.214 & -0.155 \\ -0.155 & 0.559 \end{pmatrix}$$

So that

$$U = \begin{pmatrix} 0.214 & -0.155 \\ 0.059 & 0.405 \end{pmatrix}$$

K is calculated from the spectral decomposition of $\sqrt{AA^\dagger}$ such that

$$K = \begin{pmatrix} 4.227 & 1.618 \\ 3.611 & 3.851 \end{pmatrix}$$

The left polar decomposition is then

$$A = UJ = \begin{pmatrix} 0.214 & -0.155 \\ 0.059 & 0.405 \end{pmatrix} \begin{pmatrix} 5.854 & 1.618 \\ 1.618 & 2.236 \end{pmatrix}$$

And the right polar decomposition is

$$A = KU = \begin{pmatrix} 4.227 & 1.618 \\ 3.611 & 3.851 \end{pmatrix} \begin{pmatrix} 0.214 & -0.155 \\ 0.059 & 0.405 \end{pmatrix}$$

These values may be a little off as the calculator had some rounding error when it came to the surds and couldn't represent some decimals as fractions or surds. To keep thinks consistent I kept to 3 decimal places.

3. Show that $\vec{v} \cdot \vec{\sigma}$ has eigenvalues ± 1 and that the projectors onto the corresponding eigenspaces are given by $P_\pm = \frac{I \pm \vec{v} \cdot \vec{\sigma}}{2}$
This question is similar to exercise 2.35 from the book.
We know $\vec{v} \cdot \vec{\sigma}$ is hermitian and that $(\vec{v} \cdot \vec{\sigma})^2 = I$. So we can say

$$(\vec{v} \cdot \vec{\sigma})^2 |\lambda\rangle = I|\lambda\rangle = \lambda$$

Where $|\lambda\rangle$ is the eigenvector with λ eigenvalue. Therefore we have

$$\lambda^2 |\lambda\rangle = |\lambda\rangle$$

Thus $\lambda^2 = 1;\ \lambda = \pm 1$ Or using the determinant.

$$\vec{v} \cdot \vec{\sigma} = \sum_i^3 v_i \sigma_i$$

$$= \begin{pmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{pmatrix}$$

Computing the determinant

$$|\begin{pmatrix} v_3 - \lambda & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 - \lambda \end{pmatrix}|$$

$$= \lambda^2 - 1 = 0$$

Note that $|\vec{v}| = 1$.
So $\lambda = \pm 1$.
Now we know from eq 2.35 in the book that the projector is equal to $P = \sum_i |i\rangle\langle i|$. To answer the second part of the question we can show that the outer products of the eigenvectors will be what we're looking for. However, I will also provide a proof the projector is related to this outer product, thus tying them together to show that $P_\pm = \frac{I \pm \vec{v}\cdot\vec{\sigma}}{2}$.
For $\lambda = 1$ the outer product is

$$|\lambda_1\rangle\langle\lambda_1| = \frac{1+v_3}{2} \begin{pmatrix} 1 \\ \frac{1-v_3}{v_1-iv_2} \end{pmatrix} \begin{pmatrix} 1 & \frac{1-v_3}{v_1-iv_2} \end{pmatrix}$$

$$= \frac{1}{2}\Big(I + \begin{pmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{pmatrix}\Big)$$

$$= \frac{I + \vec{v}\cdot\vec{\sigma}}{2}$$

For $\lambda = -1$

$$|\lambda_{-1}\rangle\langle\lambda_{-1}| = \frac{1-v_3}{2} \begin{pmatrix} 1 \\ -\frac{1+v_3}{v_1-iv_2} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1+v_3}{v_1-iv_2} \end{pmatrix}$$

$$= \frac{1}{2}\Big(I - \begin{pmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{pmatrix}\Big)$$

$$= \frac{I - \vec{v}\cdot\vec{\sigma}}{2}$$

Now we need to prove that

$$P_\pm = |\lambda_{\pm 1}\rangle\langle\lambda_{\pm 1}|$$

And thus

$$P_\pm = \frac{I \pm \vec{v}\cdot\vec{\sigma}}{2}$$

Suppose that $|\phi\rangle \in \mathbb{C}^2$ such that

$$\langle\phi|(P_\pm - |\lambda_{\pm 1}\rangle\langle\lambda_{\pm 1}|)|\phi\rangle = 0$$

As $\vec{v}\cdot\vec{\sigma}$ is hermitian (proven previously) thus the eigenvectors are orthonormal, $|\phi\rangle$ can be written as a linear combination

$$|\phi\rangle = A|\lambda_{\pm 1}\rangle + B|\lambda_{\pm 1}\rangle$$

Where $|A|^2 + |B|^2 = 1$ and theyre both in $\mathbb{C}$ (Seen this before) we can write

$$\langle\phi|(P_{\pm} - |\lambda_{\pm}\rangle\langle\lambda_{\pm}|)|\phi\rangle$$

$$= \langle\phi|P_{\pm}|\phi\rangle - \langle\phi|\lambda_{\pm}\rangle\langle\lambda_{\pm}|\phi\rangle$$

So

$$\langle\phi|P_{\pm}|\phi\rangle = \langle\phi|\frac{1}{2}(I \pm \vec{v}\cdot\vec{\sigma})|\phi\rangle$$

$$= \frac{1}{2} \pm \frac{1}{2}\langle\phi|\vec{v}\cdot\vec{\sigma}|\phi\rangle$$

$$= \frac{1}{2} \pm \frac{1}{2}(|A|^2 - |B|^2) = \frac{1}{2} \pm \frac{1}{2}(2|A|^2 - 1)$$

Where subsituting the previous values $\langle\phi|\lambda_1\rangle\langle\lambda_1|\phi\rangle = |A|^2$ and $\langle\phi|\lambda_{-1}\rangle\langle\lambda_{-1}|\phi\rangle = |B|^2$
So we have that

$$\langle\phi|P_{\pm} - |\lambda_{\pm 1}\rangle\langle\lambda_{\pm 1}||\phi\rangle = 0$$

And

$$P_{\pm} = |\lambda_{\pm 1}\rangle\langle\lambda_{\pm 1}|.$$

So that the projector is equal to the outer product of the eigenvectors.

4. Calculate the probability of obtaining the result $+1$ for a measurement of $\vec{v}\cdot\vec{\sigma}$, given that the state prior to measurement is $|0\rangle$. What is the state of the system after measurement if $+1$ is obtained?
Probability of $+1$ state after measurement

$$P(+1||0\rangle)$$

$+1$ state corresponds to λ_1 therefore we have

$$P(\langle\lambda_1|0\rangle) = \langle\lambda_1|0\rangle\langle 0|\lambda_1\rangle$$

$$= \langle 0|\lambda_1\rangle\langle\lambda_1|0\rangle$$

$$= \langle 0|\frac{1}{2}(I + \vec{v}\cdot\vec{\sigma})|0\rangle$$

$$= \frac{1}{2}(1 + v_3)$$

Given that outcome m occured, the state after measurement is

$$\frac{P_m|\psi\rangle}{\sqrt{P(m)}}$$

So we get

$$\begin{aligned}
&\frac{|\lambda_1\rangle\langle\lambda_1|0\rangle}{\sqrt{\langle 0|\lambda_1\rangle\langle\lambda_1|0\rangle}} \\
&= \frac{1}{\sqrt{\frac{1}{2}(1+v_3)}} \cdot \frac{1}{2}\begin{pmatrix} 1+v_3 \\ v_1+iv_2 \end{pmatrix} \\
&= \sqrt{\frac{1+v_3}{2}}\begin{pmatrix} 1 \\ \frac{1-v_3}{v_1-iv_2} \end{pmatrix} \\
&= |\lambda_1\rangle
\end{aligned}$$

From the previous exercise.

5. If you have an orthonormal basis $e_1, \ldots, e_n$ of a vector space V chosen so that the first $1 \leq k < n$ vectors are a basis of a k-dimensional subspace W, a projection operator P that projects to W is simply

$$P = \sum_{i=1}^{k} e_i e_i^\dagger \ .$$

(In ket notation with $e_i = |i\rangle$, $P = \sum_{i=1}^{k} |i\rangle\langle i|$.) What if you have a basis which is not even orthogonal? Consider the vectors

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \ ; \ u_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

spanning a two-dimensional subspace W (the xy-plane) of $\mathbf{R}^3$. Note that u_1, u_2 are not orthogonal. In this case one can construct a projection operator P which projects to W as follows. Construct a 3×2 matrix

$$A = [u_1 u_2] \ ,$$

the notation means that the two vectors u_1, u_2 are the columns of A. Then

$$P = A(A^T A)^{-1} A^T$$

is a projection operator to W. Verify this: show that in general the above $P^2 = P$, and by using the given u_1, u_2 calculate the matrix P explicitly and verify that it projects to W by showing that the vector Pv with an arbitrary

$$v = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

is in W. Next, show that the matrix $G \equiv A^T A$ is in fact a Gram matrix with $G_{ij} = u_i \cdot u_j$.
Answer:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$P = A(A^T A)^{-1} A^T$$

In general (for any P), P satisfies

$$P^2 = P$$

Because

$$P = \sum_i |i\rangle\langle i|$$

$$\begin{aligned} P^2 &= (\sum_i |i\rangle\langle i|)(\sum_j |j\rangle\langle j|) \\ &= \sum_{i,j} |i\rangle\langle i||j\rangle\langle j| \\ &= \sum_{i,j} |i\rangle\langle j|\delta_{i,j} \\ &= \sum_i |i\rangle\langle i| = P \end{aligned}$$

Calculating the matrix P explicitly

$$\begin{aligned} P &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \left[\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \right]^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Vector Pv with

$$v = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

Such that

$$Pv = \begin{pmatrix} v_x \\ v_y \\ 0 \end{pmatrix}$$

v_2 spans the 2D subspace W of $\mathbb{R}^3$ so Pv is in W

$$G = A^T A$$

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$G_{ij} = u_i \cdot u_j$$

$$G_{ij} = u_i^\dagger u_j$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}^\dagger \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

With inner product $\langle u_i, u_j \rangle$ so that $G_{12} = 1$ verifies it.

6. Suppose Bob is given a quantum state chosen from a set $|\psi_1\rangle \dots |\psi_m\rangle$ of linearly independent states. Construct a POVM $\{E_1, E_2, \dots, E_{m+1}\}$ such that if outcome E_i occurs, $1 \le i \le m$, then Bob knows with certainty that he was given the state $|\psi_i\rangle$. The POVM must be such that $\langle \psi_i | E_i | \psi_i \rangle > 0$ for each i.

   To construct the POVM we want that $\langle \psi_i | E_j | \psi_i \rangle = 0$ for every $i \neq j$ and we have a condition that $\langle \psi_i | E_i | \psi_i \rangle > 0$. We're looking to find $\langle u_j | \phi_i \rangle = 0$ $i \neq j$ for such $E_j = a|u_j\rangle\langle u_j|$, which as we know can be rewritten in the form of a projector as $E_j = aP_j$ where P_j is the projector onto an orthogonal complement U_j of the set of linearly independent states. For all j from $j = 1, \dots, m$ we know that the sum $\sum_j E_j = I$, which is called the completeness condition. Pulling these two facts together we can construct E_{m+1}

$$E_{m+1} = I - a \sum_{j=1}^{m} P_j = I - a \sum_{j=1}^{m} |j\rangle\langle j| = I - \sum_{j=1}^{m} E_j$$

As the states are linearly independent the trace must be greater than 0, $Tr(P_j\psi_j) > 0$ and that E_{m+1} is positive.
Our POVM elements are then

$$E_j = aP_j = a|j\rangle\langle j| = \frac{1}{m}P_j$$

$$E_{m+1} = I - a\sum_{j=1}|j\rangle\langle j| = I - \sum_{j=1}^{m} E_j$$

If we take $a = 1/m$ we can prove that this satisifies the condition that $\langle\psi_i|E_i|\psi_i\rangle > 0$

$$\langle\psi_i|E_{m+1}|\psi_i\rangle = \langle\psi_i|(I - a\sum_{j=1}^{m}|u_j\rangle\langle u_j|)|\psi_i\rangle$$

$$= \langle\psi_i|\psi_i\rangle - \frac{1}{m}\sum_{j=1}^{m}\langle\psi_i|u_j\rangle\langle u_j|\psi_i\rangle$$

$$\geq 1 - \frac{1}{m}\sum_{j=1}^{m} 1$$

$$\geq 1 - \frac{1}{m}(m \times 1) = 0$$

Therefore $E_{m+1} \geq 0$