Quantum Information A Fall 2020 Problem Set 1

Solutions are due in 4 pm on Tuesday Sep 8.

1. Suppose V is a vector space with basis vectors $|0\rangle$ and $|1\rangle$, and A is a linear operator from V to V such that
$$A|0\rangle = |1\rangle \ ; \ A|1\rangle = |0\rangle \ .$$
Give a matrix representation for A, with respect to the input and output basis $|0\rangle, |1\rangle$. Find input and output bases which give rise to a different matrix representation of A. (For example, you can use the states $|\pm\rangle$ appearing in problem 5.)

2. Let the basis of V be $\{|0\rangle, |1\rangle\} \equiv \{e_1, e_2\}$. The tensor product vector space $V \otimes V$ has the basis
$$\begin{aligned} &\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\} = \{|0\rangle|0\rangle, |0\rangle|1\rangle, |1\rangle|0\rangle, |1\rangle|1\rangle\} \\ &\equiv \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\} \ . \end{aligned} \tag{1}$$
Let us use the two-component notation
$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ ; \ |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ . \tag{2}$$
In the component notation, the tensor product of vectors becomes the Kronecker product,
$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a\begin{pmatrix} c \\ d \end{pmatrix} \\ b\begin{pmatrix} c \\ d \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix} \ . \tag{3}$$
Show that
$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \ ; \ |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \ ; \ |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \ ; \ |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \ . \tag{4}$$

3. Using the above 4-component notation, show that the CNOT operator
$$U_{CN} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{5}$$
acts on the (computational) basis states as
$$U_{CN}|00\rangle = |00\rangle \ ; \ U_{CN}|01\rangle = |01\rangle \ ; \ U_{CN}|10\rangle = |11\rangle \ ; \ U_{CN}|11\rangle = |10\rangle \ . \tag{6}$$

4. Recall that (from Mathematical Methods of Physics IIIa or equivalent) the abelian group $\mathbf{Z}_2$ can be realized as the set $\{0, 1\}$ with addition modulo two as the product. The latter is the same as the XOR operation of bits, and we denote it with the symbol $\oplus$:

$$0 \oplus 0 = 1 \oplus 1 = 0 \ ; \ 0 \oplus 1 = 1 \oplus 0 = 1 \ . \tag{7}$$

Note that $x \oplus x = 0$. Recall also that we can define a product group

$$\mathbf{Z}_2^n = \underbrace{\mathbf{Z}_2 \times \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2}_{n \ times} = \{0, 1\}^n \tag{8}$$

with elements $x = (x_1, x_2, \ldots, x_n)$ with each $x_i \in \mathbf{Z}_2$, with the product

$$x \oplus y = (x_1 \oplus y_1, x_2 \oplus y_2, \ldots, x_n \oplus y_n) \ . \tag{9}$$

Next we shorten the notation and write

$$x = x_1 x_2 \cdots x_n = (x_1, x_2, \ldots, x_n) \ . \tag{10}$$

You notice that x is a binary number with n bits, e.g. $x = 011011$ for $n = 6$. The above rule then extends the XOR operation for n-bit binary numbers. Converting the binary numbers to decimal numbers and back gives fun results, e.g. $3 \oplus 5 = 011 \oplus 101 = 110 = 6$. Practise this by calculating the results (in binary form) of

(a) $01011 \oplus 10001$

(b) $10001 \oplus 01111$

(c) $(01101 \oplus 10101) \oplus 11100$

5. The Hadamard gate is represented by

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \ . \tag{11}$$

Define the states

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = H|0\rangle \ ; \ |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = H|1\rangle \ . \tag{12}$$

Let H act on them, what are the resulting states $H|+\rangle, H|-\rangle$?