Open Quantum Systems Fall 2020 Answers to Exercise Set 7

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20/11/2020

1 Exercise 1

1. Because A is invertible and square the determinant $detA \neq 0$ and because the determinant is the product of eigenvalues, the eigenvalues must be positive which means that $A \geq 0$, thus $A^\dagger A$ is positive.
   Since $A^\dagger A$ is positive definite, it must be invertible as it does not have any eigenvalues equal to 0.

2. If we set
   $$(A^\dagger A)^{1/2} = \sqrt{A^\dagger A} = P$$
   This is useful later on. Since A is invertible and $A^\dagger A$ is positive definite, meaning all the eigenvalues of $A^\dagger A$ are positive, therefore all the eigenvalues of P must be positive so P is positive definite.

3. $$\frac{A}{\sqrt{A^\dagger A}} = \frac{A}{P} = AP^{-1} = U$$
   To prove this is unitary we can use spectral decomposition.
   $$A(A^\dagger A)^{-1/2} = AVD^{-1/2}V^\dagger$$
   Where V is unitary and thus $V^\dagger$ is unitary. If we use SVD we can show that
   $$A = WD^{1/2}V^\dagger$$
   Such that
   $$AVD^{-1/2} = WD^{1/2}V^\dagger VD^{-1/2} = W$$
   Therefore U is unitary.

4. Using the definitions given for U and P we can write out UP and show that this is equal to A
   $$UP = A(A^\dagger A)^{-1/2}(A^\dagger A)^{1/2} = A$$
   $$A = UP$$

2 Exercise 2: Invertibility of a Quantum Channel

$$\Gamma(\rho) = \sum_i M_i \rho M_i^\dagger$$

Where ρ and M_i are square and $\sum_i M_i^\dagger M_i = \mathbb{I}$.

1. Assuming that the Quantum channel is completely positive and trace preserving (CPTP), then the left side would be a sum of positive terms and each must be proportional to $\psi\psi^\dagger$.

   $$\Gamma'(\Gamma(\psi\psi^\dagger)) = \sum_{i,j} N_j M_i \psi\psi^\dagger M_i^\dagger N_j^\dagger$$

   Where $\Gamma'(\rho) = \sum_j N_j \rho N_j^\dagger$ and that this also obeys completeness. So for each i and j we would have

   $$N_j M_i = \lambda_j \mathbb{I}$$

   This is also true if we follow Nielsen and Chaung *Theorem 8.3: Unitary freedom in the operator sum representation*, which states that there must be complex numbers λ_{ji} that satisfy the answer.

2. Using the completeness relation as well as results from (a) we would have

   $$M_b^\dagger M_a = M_b^\dagger \Big(\sum_j N_j^\dagger N_j \Big) M_a$$

   $$= \sum_j \lambda_{jb}^* \lambda_{ja} \mathbb{I} = \beta_{ba} \mathbb{I}$$

   Where we have substituted β_{ba} for $\sum_j \lambda_{jb}^* \lambda_{ja}$ and used the fact that $\sum_j N_j^\dagger N_j = \mathbb{I}$

3. Since Γ is a linear map where M_i are $d \times d$ matrcies, a system would have dimension d and as such we can use the results from Exercise 1 to see that a polar decomposition of M_a s.t $A^\dagger A \equiv M_a^\dagger M_a$ would give

   $$M_a = \sqrt{M_a^\dagger M_a} U_a = \sqrt{\beta_{aa}} U_a$$

4. From the previous 2 results we can say that

   $$M_b^\dagger M_a = \sqrt{\beta_{aa}\beta_{bb}} U_b^\dagger U_a = \beta_{aa} \mathbb{I}$$

   Which, rearranged gives

   $$U_a = \frac{\beta_{ba}}{\sqrt{\beta_{aa}\beta_{bb}}} U_b$$

5. These results show that each M_a us proportional to a unitary matrix U_a and $\Gamma(\rho)$ is a unitary map meaning that it can be written as

   $$\Gamma(\rho) = U \rho U^\dagger$$

3 Exercise 3

$$H = \sum_{j=1}^{N} \hbar\omega_j b_j^\dagger b_j$$

$$H|i_1, i_2 \dots i_N\rangle = \Big(\sum_{j=1}^{N} i_j \hbar\omega_j\Big)|i_1, i_2 \dots i_N\rangle \tag{1}$$

1. Using equation (1) for the thermal state $\rho_{th} = e^{-\beta H}$ i.e sub $H = e^{-\beta H}$ gives

$$e^{-\beta H}|i_1, i_2 \dots i_N\rangle = e^{\sum_j -\beta i_j \hbar\omega_j}|i_1, i_2 \dots i_N\rangle$$

From the orthonormal basis of H (eq 2 in Ex) we can write the elements of $e^{-\beta H}$ in that basis such that

$$e^{-\beta H} = \langle k_1, k_2, \dots k_N|e^{-\beta H}|i_1, i_2, \dots, i_N\rangle$$

The basis is orthonormal so it can be written in terms of kronecker delta

$$e^{-\beta H} = e^{\sum_j -\beta i_j \hbar\omega_j}\delta_{k_1} \dots \delta_{k_N}, \delta_{i_1} \dots \delta_{i_N}$$

So the thermal state can be given by

$$\sum_{i_1, i_2, \dots i_N}^{+\infty} = |i_1, i_2 \dots i_N\rangle\langle i_1, i_2 \dots i_N|e^{\sum_j -\beta i_j \hbar\omega_j}$$

2. Find a purification of the thermal state. Because we have $\mathbb{H} \otimes \mathbb{H}$, ψ will be of the form

$$H|i_1, i_2, \dots i_N\rangle \otimes H|i'_1, i'_2, \dots i'_N\rangle$$

Where $|i'\rangle$ denotes the orthonormal eigenbasis of the second hilbert space. ρ_{th} can be diagonalized and written as $\rho = \sum_{i=1}^{N} p_i|i\rangle\langle i|$ for the basis $|i\rangle$. Because we have another copy of the hilbert space $\mathbb{H}$, denoted by $\mathbb{H}_D$ which has an orthonormal eigenbasis as discussed in the previous question then $|\psi\rangle$ can be defined by $\mathbb{H} \otimes \mathbb{H}$ as in the question

$$|\psi\rangle = \sum_i \sqrt{p_i}|i\rangle \otimes |i'\rangle$$

In terms of the question this would give

$$|\psi\rangle = \sum_i \sqrt{e^{\sum_j -\beta i_j \hbar\omega_j}}|i_1, i_2, \dots i_N\rangle \otimes |i'_1, i'_2, \dots i'_N\rangle$$

This is verified by solving the trace

$$\mathrm{Tr}_2(|\psi\rangle\langle\psi|) = \mathrm{Tr}_2(\psi\psi^\dagger)$$

$$= \text{Tr}_2\Big[\Big(\sum_i \sqrt{e^{\sum_j -\beta i_j \hbar\omega_j}}|i_1, i_2, \dots i_N\rangle \otimes |i_1', i_2', \dots i_N'\rangle\Big)$$

$$\Big(\sum_k \sqrt{e^{\sum_j -\beta k_j \hbar\omega_j}}\langle k_1, k_2, \dots k_N| \otimes \langle k_1', k_2', \dots k_N'|\Big)\Big]$$

$$= \text{Tr}_2\Big(\sum_{i,k} \sqrt{e^{\sum_j -\beta i_j \hbar\omega_j} e^{\sum_j -\beta k_j \hbar\omega_j}}|i_1, i_2, \dots i_N\rangle\langle k_1, k_2, \dots k_N| \otimes |i_1', i_2', \dots i_N'\rangle\langle k_1', k_2', \dots k_N'|\Big)$$

$$= \delta_{ik}\sqrt{e^{\sum_j -\beta i_j \hbar\omega_j} e^{\sum_j -\beta k_j \hbar\omega_j}}|i_1, i_2, \dots i_N\rangle\langle k_1, k_2, \dots k_N|$$

$$= \sum_i \sqrt{\left(e^{\sum_j -\beta i_j \hbar\omega_j}\right)^2}|i_1, i_2, \dots i_N\rangle\langle i_1, i_2, \dots i_N|$$

$$= e^{\sum_j -\beta i_j \hbar\omega_j} = \rho_{th}$$

4 Exercise 4

1. Because $b_j^\dagger$ and b_j are ladder operators I can apply ladder operator properties derived in many resources. For this question I particularly referenced Chapter 7 of Nielsen and Chaung: QIQC.

   $$a^\dagger a|n\rangle = n|n\rangle$$

   From this I applied the right part of the equation to get

   $$b_j^\dagger b_j' \psi = b_j^\dagger b_j' \sum_i \sqrt{e^{\sum_j -\beta i_j \hbar\omega_j}}|i_1, i_2, \dots i_N\rangle \otimes |i_1', i_2', \dots i_N'\rangle$$

   And the left side

   $$\psi^\dagger b_j^\dagger b_j' \psi = \sum_j \sum_i \sqrt{e^{\sum_j -\beta i_j \hbar\omega_j}}\sqrt{e^{\sum_j -\beta j_k \hbar\omega_j}}|i_1, i_2, \dots i_N\rangle\langle j_1, j_2, \dots j_N| \otimes |i_1', i_2', \dots i_N'\rangle\langle j_1', j_2', \dots j_N'|\Big)$$

   I am not sure how this comes out at 0. I'm sure that the ladder operators play a larger part in this question but I'm not quite sure how. As such, I did not answer the second part of this question.