Quantum Information B Fall 2020 Solutions to Problem Set 3

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1 Answers

1. Exercise 8.26: Circuit model for Phase damping. Suppose the qubit is in a state

$$|\psi\rangle = a|0\rangle b|1\rangle$$

So that initially we have

$$|\psi_0\rangle = (a|0\rangle + b|1\rangle) \otimes |0\rangle$$

The circuit does the operation so that

$$|\psi\rangle_{out} = a|0\rangle \otimes |0\rangle + b|1\rangle \otimes (\cos(\frac{\theta}{2})|0\rangle + \sin(\frac{\theta}{2})|1\rangle)$$

The rotation R_y is

$$R_y(\theta) = \cos(\frac{\theta}{2})I - i\sin(\frac{\theta}{2})Y$$

So we have

$$|\psi\rangle_{out} = (a|0\rangle + b\cos(\frac{\theta}{2})|1\rangle) \otimes |0\rangle + (b\sin(\frac{\theta}{2})|1\rangle) \otimes |1\rangle$$

Tracing over the environment gives operation elements $E_k = \langle k_b|U|0_b\rangle = \langle k_E|U|0_E\rangle$

$$\mathrm{Tr}(\rho \otimes |\psi_0\rangle\langle\psi|)$$

$$= (a|0\rangle + b\cos(\frac{\theta}{2})|1\rangle) \otimes |0\rangle(a^*\langle 0| + b^*\cos(\frac{\theta}{2})\langle 1|) \otimes \langle 0|$$

$$+(b\sin(\frac{\theta}{2})|1\rangle) \otimes |1\rangle(b\sin(\frac{\theta}{2})\langle 1|) \otimes \langle 1|$$

$$= |a|^2 + ab^*\cos(\frac{\theta}{2})|0\rangle\langle 0| + ba^*\cos(\frac{\theta}{2})|1\rangle\langle 0| + |b|^2$$

$$= \begin{pmatrix} |a|^2 & ba^*\cos(\frac{\theta}{2}) \\ ab^*\cos(\frac{\theta}{2}) & |b|^2 \end{pmatrix}$$

For amplitude damping in the book it has E_0 and E_1 and then applied equation 8.107

$$\varepsilon_{AD}(\rho) = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger$$

But for Phase Damping we have a new set of operation elements $\tilde{E}_0$ and $\tilde{E}_1$

$$\varepsilon(\rho) = \tilde{E}_0\rho\tilde{E}_0^\dagger + \tilde{E}_1\rho\tilde{E}_1^\dagger$$

Where

$$\tilde{E}_0 = \sqrt{\alpha}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tilde{E}_1 = \sqrt{1-\alpha}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\varepsilon(\rho) = \sqrt{\alpha}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} |a|^2 & ba^* \\ ab^* & |b|^2 \end{pmatrix}\sqrt{\alpha}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$+\sqrt{1-\alpha}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} |a|^2 & ba^* \\ ab^* & |b|^2 \end{pmatrix}\sqrt{1-\alpha}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} |a|^2 & 2ba^*\alpha - ba^* \\ 2ab^*\alpha - ab^* & |b|^2 \end{pmatrix}$$

Comparing we have

$$2ba^*\alpha - ba^* = ba^*\cos(\frac{\theta}{2})$$

$$ba^*(2\alpha - 1) = ba^*\cos(\frac{\theta}{2})$$

$$2\alpha - 1 = \cos(\frac{\theta}{2})$$

$$\alpha = \frac{\cos(\frac{\theta}{2}) + 1}{2}$$

Doing the same for $2ab^*\alpha - ab^* = ab^*\cos(\frac{\theta}{2})$ yields the same result. This matches the previous Phase Damping quantum operation so the circuit can be used to model it given that θ is chosen appropriately.

2. Ex 8.29.

$$\varepsilon(\rho)\sum_\alpha M_\alpha\rho M_\alpha^\dagger$$

$$\varepsilon(I) = \sum_\alpha M_\alpha I M_\alpha^\dagger$$

which is equivalent to

$$\sum_\alpha M_\alpha M_\alpha^\dagger = I$$

For the de polarizing channel:

$$\varepsilon(\rho) = \frac{pI}{2} + (1-p)\rho$$

From Exercise 8.17 which we did before

$$\varepsilon(I) = \frac{I + \sum_{i=1}^{3} \sigma_i I \sigma_i}{4} = \frac{4I}{4} = I$$

For Phase Damping:

$$\varepsilon(I) = (1 - \frac{1}{2}p)I + \frac{1}{2}p \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= I$$

Alternatively

$$\varepsilon(I) = \tilde{E}_0 I \tilde{E}_0^\dagger + \tilde{E}_1 I \tilde{E}_1^\dagger$$

$$= \sqrt{\alpha} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sqrt{\alpha} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$+ \sqrt{1-\alpha} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sqrt{1-\alpha} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= I$$

For Amplitude Damping:

$$\varepsilon(I) = E_0 I E_0^\dagger + E_1 I E_1^\dagger$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1+\gamma & 0 \\ 0 & 1-\gamma \end{pmatrix} \neq I$$

3. Exercise 8.31. This descrives the interaction between 2 harmonic oscillators where $a^\dagger a$ is the system and $bb^\dagger$ is the environment.

$$H = \chi a^\dagger a (b + b^\dagger)$$

$$\rho_{nm} = \langle n|\rho|m\rangle$$

$$U = \exp(-i\chi a^\dagger a (b + b^\dagger)\Delta t)$$

So we need to apple U to $|\psi\rangle$ where

$$|\psi\rangle = C|n\rangle + D|m\rangle \otimes |0\rangle$$

So that we get

$$U|\psi\rangle = C \cdot U|n\rangle + D \cdot U|m\rangle \otimes |0\rangle$$

Then trace over the environment i.e

$$\mathrm{Tr}(U|\psi\rangle\langle\psi|U^\dagger)$$
$$= |C|^2 U|n\rangle U^\dagger\langle n| + |D|^2 U|m\rangle U^\dagger\langle m| + CD^* U|n\rangle U^\dagger\langle m| + C^* DU|m\rangle U^\dagger\langle n|$$

N.B At this point I did not know how to get $U, U^\dagger$ into a state where the trace gives an exponential with a $(n-m)^2$ factor in.

4. Problem 8.1. Solve

$$\dot{\rho} = -\frac{\lambda}{2}(\sigma_+\sigma_-\rho + \rho\sigma_+\sigma_- - 2\sigma_-\rho\sigma_+)$$

In the notation from the book (which I found a confusing approach to master equations and didn't fully understand. Also "from bloch vector representation for $\tilde{\rho}$" where is this in the book?). I express the solution to the differential equation as

$$\rho(t) = \varepsilon(\rho(0)) = E_0\rho(0)E_0^\dagger + E_1\rho(0)E_1^\dagger$$

With the $\gamma = -\frac{\lambda}{2}$ variable in the book such that

$$\gamma' = 1 - e^{\lambda t}$$

$$\rho(t) = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma'} \end{pmatrix} \rho(0) \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma'} \end{pmatrix} + \begin{pmatrix} 0 & \sqrt{\gamma'} \\ 0 & 0 \end{pmatrix} \rho(0) \begin{pmatrix} 0 & 0 \\ \sqrt{\gamma'} & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \rho(0) + \rho(0)\gamma' & 0 \\ 0 & \rho(0)(1-\gamma') \end{pmatrix}$$
$$= \begin{pmatrix} \rho(0)(2 - e^{\lambda t}) & 0 \\ 0 & \rho(0)e^{\lambda t} \end{pmatrix}$$

5. Exercise 9.1. Trace distance between probability distribution (1,0) and probability distribution $(\frac{1}{2}, \frac{1}{2})$

(a)

$$D((1,0), (\frac{1}{2}, \frac{1}{2}))$$
$$= \frac{1}{2}(|1 - \frac{1}{2}| + |0 - \frac{1}{2}|)$$
$$= \frac{1}{2}$$

(b) $(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$ and $(\frac{3}{4}, \frac{1}{8}, \frac{1}{8})$

$$D((\frac{1}{2}, \frac{1}{3}, \frac{1}{6}), (\frac{3}{4}, \frac{1}{8}, \frac{1}{8}))$$
$$= \frac{1}{2}(|\frac{1}{2} - \frac{3}{4}| + |\frac{1}{3} - \frac{1}{8}| + |\frac{1}{6} - \frac{1}{8}|)$$
$$= \frac{1}{4}$$

6. Exercise 9.2. Trace distance between $(p, 1-p)$ and $(q, 1-q)$

$$
\begin{aligned}
&D((p, 1-p), (q, 1-q)) \\
&= \frac{1}{2}(|p-q| + |(1-p) - (1-q)|) \\
&= \frac{1}{2}(|p-q| + |-p+q|) \\
&= \frac{1}{2}(2|p-q|) \\
&= |p-q|
\end{aligned}
$$