Question 2

December 2, 2020

1 Question 2

1.1 Part 1

$$L(s,b) = \frac{(s+b)^n e^{-(s+b)}}{n!}$$

To calculate the p value for the background only i.e $s = 0$, with $n = 15, b = 2.8$ we have to use the following formula with the cumulative χ^2 distribution function.

$$p = 1 - \sum_{n=0}^{m} (n; v) = 1 - [1 - F_{\chi^2}(2\nu; n_{dof})]$$

In the case of this question $m = 14$ because we are going from $n = 15 \to \infty$ to our sum is then $n = 0 \to m = 14$. We also have $\nu = s + b = b$ and the degrees of freedom $n_{dof} = 2(m+1) = 30$. This can also be explictly verified using a chi2 applet for $F_{\chi^2}(\nu = 5.6, n_{dof} = 30)$ where $2\nu = 2b = 5.6$

$$\text{p-value} = 2.9 \times 10^{-7}$$

1.2 Part 2

For the second part we have to calculate how many standard deviations (=std) from a standard gaussian. To calculate this we use

$$\sigma = \sqrt{1-p}$$

It is easy to see that this is just the square root of the poisson function or the square root of 1-p-value. The standard normal distribution has $\mu = 0, \sigma = 1$ so we can use the ppf (percent points function or Quartile function) to return the significance. The signifiance is greater than 5 so it is a new discovery.

$$Z = 5.132042$$

1.3 Part 3

For part 3 we introduce a new function in the code which is the Likelihood ratio, simply likelihood for s over likelihood for $\hat{s}$. To get $\hat{s}$ we have to maximise $L(s,b)$ i.e $\frac{\partial L}{\partial b} = 0$. Analytically this is

$$\frac{\partial L}{\partial b} = \frac{ne^{-(s+b)}(s+b)^{n-1} - e^{-(s+b)}(s+b)^n}{n!}$$

We know that $b > 0, s > 0$ so

$$(s+b)^{n-1}(n - (s+1)) = 0$$

So either $(s+b) = 0$ or $n - (s+b) = 0$ and $\hat{s} = n - b$. This is verified graphically (desmos.com) and introduced as a function for which the likelihood ratio depends on. In the code below I plotted $-2\ln\lambda(\hat{b}, n)$ vs s like in the hint and used *scipy* to find the confidence intervals matching the roots of the plot for when $-2\ln\lambda = 1$.

1.4 Part 4

Part 4 was particularly tricky but the hint and reference helped. I now introduced a new function for the likelihood ratio using a b as a uniformly distributed random number between $[b - \sigma_b]$, $[b + \sigma_b]$. Note that a 1 σ confidence interval is 68.3% for a standard gaussian distributed. So to find the CI I find the ratio of the likelihood function which is less than or equal to 1 by introducing another function which finds the amount under 1 and then finding fraction of them. This fraction was plotted and shows that it has very similar roots to part 3 with very similar confidence interval. I am not sure if I avoided undercoverage or not with this method.

1.5 Code

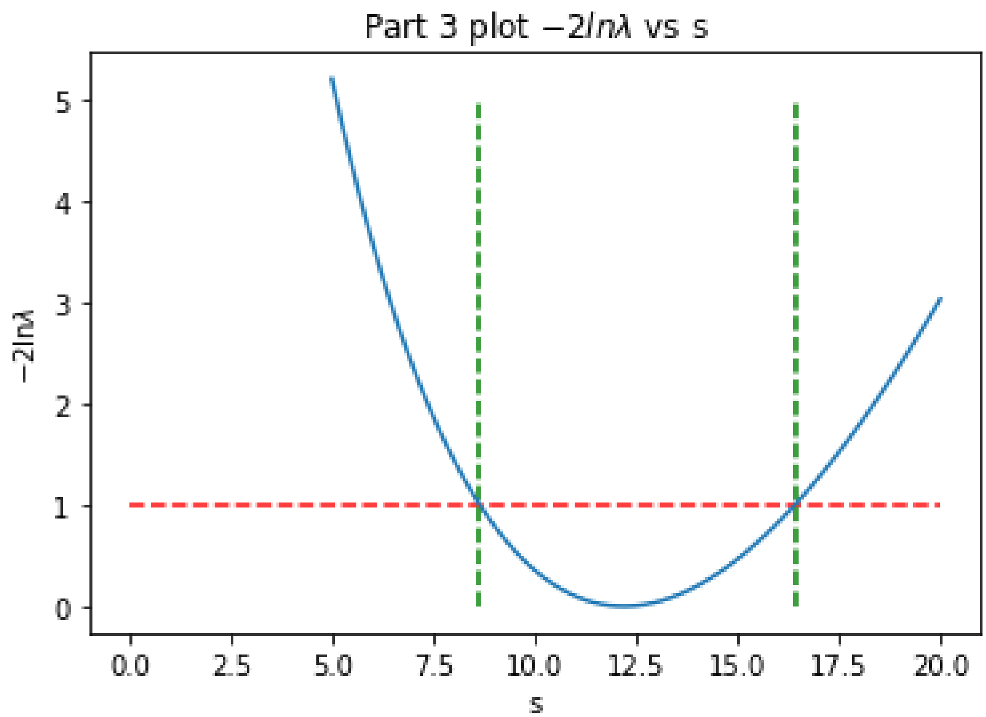
[64]:
```
import matplotlib.pyplot as plt
import numpy as np
import scipy.optimize
import scipy.stats

#functions
def likelihood(s, b, n):
    return ((s+b)**n * np.exp(-(s+b))) / np.math.factorial(n)


def poisson_func(m, v):
    n_dof = 2*(m+1)
    return 1 - scipy.stats.chi2.cdf(2*v, n_dof)


def s_hat(b, n):
    return n - b


def likelihood_ratio(s, b, n):
    return likelihood(s, b, n) / likelihood(s_hat(b, n), b, n)


def likelihood_ratio_b_uniform(s, b, n, b_hat):
    return likelihood(s, b_hat, n) / likelihood(s_hat(b, n), b_hat, n)


def likelihood_ratio_under_limit(s, b, n,sigma_b): #n_b = 10000, limit=1
    b_arr = np.random.uniform(b-sigma_b,b+sigma_b,10000)
    frac_sat_con = np.sum(-2*np.log(likelihood_ratio_b_uniform(s, b, n, b_arr))
↪<= 1) / 10000
```

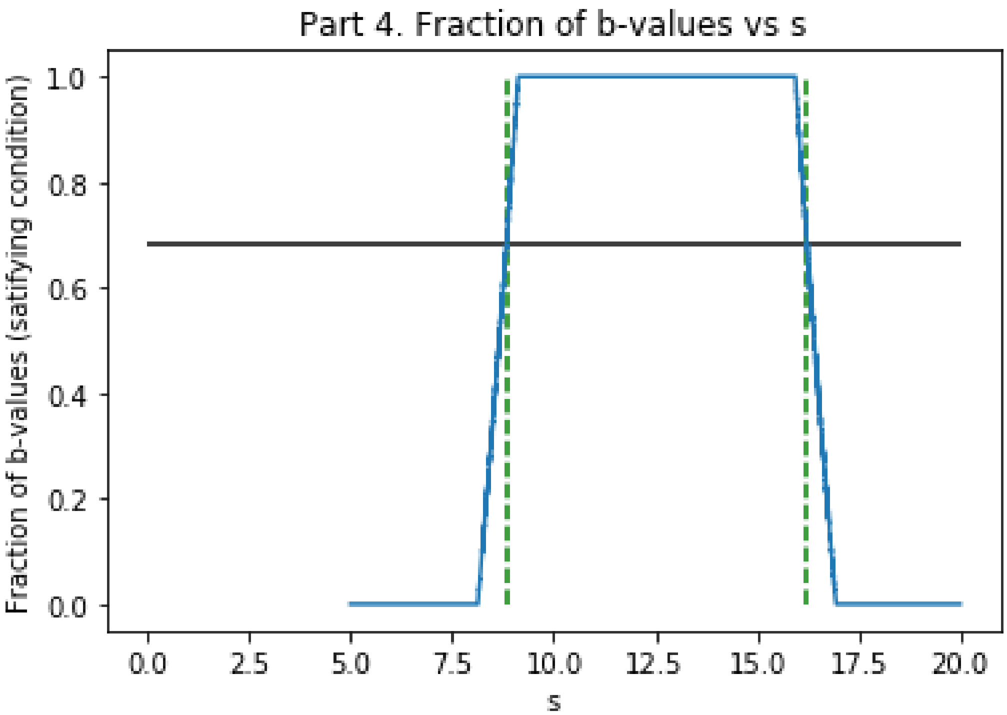
```
    return frac_sat_con


#init
b=2.8
n_obs = 15
m = n_obs-1

s=0
v = s+b
Poi= poisson_func(m, v)
p=1-Poi
print("p-value:", p)

std = np.sqrt(Poi)
n_sigma = scipy.stats.norm.ppf(std)
print("P-value corresponds to %f standard deviations" %n_sigma)


plt.figure(1)

s_arr = np.linspace(5,20, 10000)
plt.plot(s_arr, -2*np.log(likelihood_ratio(s_arr, b, n_obs)))
plt.title("Part 3 plot $-2ln \lambda$ vs s")
plt.xlabel("s")
plt.ylabel(r"$-2\ln \lambda$")


sol1_min = scipy.optimize.minimize_scalar(lambda s: -2*np.
 ↪log(likelihood_ratio(s, b, n_obs)), bracket=show_range)
print("sol1 x = %.5f" %sol1_min.x)
#print(sol1_min)

sol1_lower = scipy.optimize.root_scalar(lambda s: -2*np.log(likelihood_ratio(s,␣
 ↪b, n_obs)) - 1, bracket=[show_range[0], sol_min.x])
sol1_upper = scipy.optimize.root_scalar(lambda s: -2*np.log(likelihood_ratio(s,␣
 ↪b, n_obs)) - 1, bracket=[sol_min.x, show_range[1]])
#print(sol1_lower)
#print(sol1_upper)

plt.hlines(1,0,20,  colors='Red', linestyle='dashed')
plt.vlines(sol_lower.root, 0, 5, colors='Green', linestyle='dashed')
plt.vlines(sol_upper.root, 0, 5, colors='Green', linestyle='dashed')
print("-------------------------------------------------")

print("s for part 3 with +- uncertainties from scipy")
```

```
print("s = %.3f + %.3f, - %.3f" %(s_hat(b, n_obs),sol_upper.root - s_hat(b,␣
 ↪n_obs),s_hat(b, n_obs) - sol_lower.root))



sigma_b = 0.5



frac_arr = np.zeros_like(s_arr)
for i in range(fracs.size):
    frac_arr[i] = likelihood_ratio_under_limit(s_arr[i], b, n_obs, sigma_b)
plt.figure(2)
plt.plot(s_arr, frac_arr)
plt.title("Part 4. Fraction of b-values vs s")
plt.xlabel("s")
plt. ylabel("Fraction of b-values (satifying condition)")

frac_limit = scipy.stats.norm.cdf(1) - scipy.stats.norm.cdf(-1)
plt.hlines(frac_limit, 0,20)

sol2_lower = scipy.optimize.root_scalar(
    lambda s: -2*np.log(likelihood_ratio_under_limit(s, b, n_obs, sigma_b)) -␣
 ↪frac_limit,bracket=[show_range[0], sol_min.x])
sol2_upper = scipy.optimize.root_scalar(
    lambda s: -2*np.log(likelihood_ratio_under_limit(s, b, n_obs, sigma_b)) -␣
 ↪frac_limit,bracket=[sol_min.x, show_range[1]])
#print(sol2_lower)
#print(sol2_upper)

plt.vlines(sol2_lower.root,0,1, colors='Green', linestyle='dashed')
plt.vlines(sol2_upper.root,0,1, colors='Green', linestyle='dashed')
print("--------------------------------------------------")
print("s for part 4 with +- uncertainties from scipy")

print("s = %.3f + %.3f, - %.3f" %(s_hat(b, n_obs),sol2_upper.root - s_hat(b,␣
 ↪n_obs),s_hat(b, n_obs) - sol2_lower.root))
```

```
p-value: 2.866153162583984e-07
P-value corresponds to 5.132042 standard deviations
sol1 x = 12.20000
-------------------------------------------------
s for part 3 with +- uncertainties from scipy
s = 12.200 + 4.213, - 3.547
-------------------------------------------------
s for part 4 with +- uncertainties from scipy
s = 12.200 + 4.010, - 3.343
```

```
/usr/local/anaconda3/lib/python3.7/site-packages/ipykernel_launcher.py:95:
RuntimeWarning: divide by zero encountered in log
/usr/local/anaconda3/lib/python3.7/site-packages/ipykernel_launcher.py:97:
RuntimeWarning: divide by zero encountered in log
```




Part 4. Fraction of b-values vs s
Fraction of b-values (satifying condition)
1.0
0.8
0.6
0.4
0.2
0.0
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
s