Open Quantum Systems Fall 2020 Answers to Exercise Set 5

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1 Random Phases

$$\psi = a\phi_1 + b\phi_2$$

a and b have the condition that they must $|a|^2 = |b|^2 = 1$.

$$\psi(t) = ae^{i\theta_1}\phi_1 + be^{i\theta_2}\phi_2$$

With probability

$$P(\theta_1, \theta_2) = \frac{1}{\sqrt{2\pi\lambda_1 t}}\frac{1}{\sqrt{2\pi\lambda_2 t}}e^{-\frac{\theta_1^2}{2\lambda_1 t}}e^{-\frac{\theta_2^2}{2\lambda_2 t}}$$

The density matrix can be written as an integral from a state vector $\psi(t)$ from the statistical description of the density matrix

$$\rho(t) = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

$$\rho(t) = \sum_i p_i \psi(t)\psi^\dagger(t)$$

$$= \int_{-\infty}^{\infty} P(\theta_1, \theta_2)\psi(t)\psi^\dagger(t)$$

So we have

$$\psi(t)\psi^\dagger(t) = (ae^{i\theta_1}\phi_1 + be^{i\theta_2}\phi_2)\cdot(a^*e^{-i\theta_1}\phi_1^\dagger + b^*e^{-i\theta_2}\phi_2^\dagger)$$
$$= |a|^2\phi_1\phi_1^\dagger + ab^*e^{i\theta_1 - i\theta_2}\phi_1\phi_2^\dagger + ba^*e^{i\theta_2 - i\theta_1}\phi_2\phi_1^\dagger + |b|^2\phi_2\phi_2^\dagger$$

1. The density matrix at time t is

$$\rho(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(\theta_1, \theta_2)\psi(t)\psi^\dagger(t)d\theta_1 d\theta_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\lambda_1 t}}\frac{1}{\sqrt{2\pi\lambda_2 t}}e^{-\frac{\theta_1^2}{2\lambda_1 t}}e^{-\frac{\theta_2^2}{2\lambda_2 t}}\dots$$

$$\cdot\Big[|a|^2\phi_1\phi_1^\dagger + ab^*e^{i\theta_1 - i\theta_2}\phi_1\phi_2^\dagger + ba^*e^{i\theta_2 - i\theta_1}\phi_2\phi_1^\dagger + |b|^2\phi_2\phi_2^\dagger\Big]d\theta_1 d\theta_2$$

To make this look easier I introduce the substitutions $c = 2\lambda_1 t$ and $d = 2\lambda_2 t$ so that the equation looks like

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi c}}\frac{1}{\sqrt{\pi d}} e^{-\frac{\theta_1^2}{c}} e^{-\frac{\theta_2^2}{d}} \psi(t)\psi^\dagger(t) d\theta_1 d\theta_2$$

The gaussian part of the integral is easily seen now such that the first part of the integral evaluates at 1. Also notice that the parts of $\psi(t)\psi^\dagger(t)$ which contribute to θ_1, θ_2 can be split up as such

$$ab^*(e^{i\theta_1 - i\theta_2}) = ab^*(e^{i\theta_1}e^{-i\theta_2})$$

$$ba^*(e^{i\theta_2 - i\theta_1}) = ba^*(e^{i\theta_2}e^{-i\theta_1})$$

And that split up and evaluated through the integral also $= 1$. So the solution is

$$\rho(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(\theta_1, \theta_2)\psi(t)\psi^\dagger(t) d\theta_1 d\theta_2$$

$$= |a|^2\phi_1\phi_1^\dagger + |b|^2\phi_2\phi_2^\dagger \ldots$$

N.B Not sure how the value of $e^{-\frac{1}{2}t(\lambda_1+\lambda_2)}$ comes into the solution to the integral.

2. Show that $\rho(t)$ satisfies the master equation

$$\rho(t) = |a|^2\phi_1\phi_1^\dagger + |b|^2\phi_2\phi_2^\dagger + e^{-\frac{1}{2}t(\lambda_1+\lambda_2)}(ab^*\phi_1\phi_2^\dagger + ba^*\phi_2\phi_1^\dagger)$$

Taking the derivative of this w.r.t t

$$\frac{d}{dt}\rho(t) = -\frac{(\lambda_1+\lambda_2)(ba^*\phi_2\phi_1^\dagger + ab^*\phi_1\phi_2^\dagger)e^{-\frac{1}{2}t(\lambda_1+\lambda_2)}}{2}$$

$$= -\frac{1}{2}(\lambda_1+\lambda_2)\Big[\rho(t) - (\phi_1\phi_1^\dagger\rho(t)\phi_1\phi_1^\dagger + \phi_2\phi_2^\dagger\rho(t)\phi_2\phi_2^\dagger)\Big]$$

$$= -\frac{1}{2}(\lambda_1+\lambda_2)\Big[\rho(t) - \sum_i \phi_i\phi_i^\dagger\rho(t)\phi_i\phi_i^\dagger\Big]$$

Expanding and splitting $-\frac{1}{2}(\lambda_1+\lambda_2)$ so that

$$\Rightarrow \sqrt{\frac{1}{2}(\lambda_1+\lambda_2)}\phi_i\phi_i^\dagger \cdot -\sqrt{\frac{1}{2}(\lambda_1+\lambda_2)}\phi_i\phi_i^\dagger \equiv -\frac{1}{2}(\lambda_1+\lambda_2)$$

So the first term can be written as

$$\sqrt{\frac{1}{2}(\lambda_1+\lambda_2)}\phi_i\phi_i^\dagger \ \rho(t) \ (\sqrt{\frac{1}{2}(\lambda_1+\lambda_2)}\phi_i\phi_i^\dagger)^\dagger$$

Where

$$(\sqrt{\frac{1}{2}(\lambda_1+\lambda_2)}\phi_i\phi_i^\dagger)^\dagger = -\sqrt{\frac{1}{2}(\lambda_1+\lambda_2)}\phi_i^\dagger\phi_i$$

The second term can also be written but it has 4 terms in (from $i = 1$ to 2) so we have

$$\frac{1}{2}(\lambda_1+\lambda_2)\rho(t) + \rho(t)\frac{1}{2}(\lambda_1+\lambda_2)$$

Which is the anti commutation relation with an extra value of 2 included from which the prefactor $\frac{1}{2}$ outside of the commutation brackets to be able to use this simplification. So we have

$$\frac{d}{dt}\rho(t) = \sqrt{\frac{1}{2}(\lambda_1+\lambda_2)}\phi_i\phi_i^\dagger\rho(t)\cdot -\sqrt{\frac{1}{2}(\lambda_1+\lambda_2)}\phi_i^\dagger\phi_i$$

$$-\frac{1}{2}(\frac{1}{2}(\lambda_1+\lambda_2)\rho(t) + \rho(t)\frac{1}{2}(\lambda_1+\lambda_2))$$

$$= \sum_i \Big[\sqrt{\frac{1}{2}(\lambda_1+\lambda_2)}\phi_i\phi_i^\dagger\rho(t)\cdot -\sqrt{\frac{1}{2}(\lambda_1+\lambda_2)}\phi_i^\dagger\phi_i$$

$$-\frac{1}{2}\{-\sqrt{\frac{1}{2}(\lambda_1+\lambda_2)}\phi_i^\dagger\phi_i\sqrt{\frac{1}{2}(\lambda_1+\lambda_2)}\phi_i\phi_i^\dagger, \rho(t)\}\Big]$$

Substituting

$$L_i = \sqrt{\frac{1}{2}(\lambda_1+\lambda_2)}\phi_i\phi_i^\dagger$$

So

$$L_i^\dagger = -\sqrt{\frac{1}{2}(\lambda_1+\lambda_2)}\phi_i^\dagger\phi_i$$

Gives

$$\frac{d}{dt}\rho(t) = \sum_i \Big[L_i\rho(t)L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho(t)\}\Big]$$

Note. $\sum_i^2 \phi_i^\dagger\phi_i\phi_i\phi_i^\dagger = 1$

2 Unitary Jump

1. Density operator at $t+dt$. $\psi(t)$ has probability $P = \lambda dt$ to jump to $e^{-iG}\psi(t)$. So it has $1-\lambda dt$ of staying $\psi(t)$. For normal time evolution we have

$$\rho(t) = Pe^{-iHt}\rho(0)e^{iHt}$$

Where $\rho(0)$ is the initial density matrix. For this the initial density matrix is $\rho(t)$ and we need to add on the probability that the state is unchanged to satisfy the total probability.

$\rho(t+dt) =$ Probability to remain unchanged$\cdot\rho(t)+$Probability to change$\cdot e^{-iG}\rho(0=t)e^{iG}$

$$\rho(t+dt) = (1-\lambda dt)\rho(t) + \lambda dt e^{-iG}\rho(t)e^{iG}$$

2. Show that $\rho(t)$ satisfies the differential equation

$$\begin{aligned}\frac{d}{dt}\rho(t) &= \frac{d}{dt}\Big[(1-\lambda dt)\rho(t) + \lambda t e^{-iG}\rho(t)e^{iG}\Big] \\ &= -\lambda\rho(t) + \lambda e^{-iG}\rho(t)e^{iG} \\ &= -\lambda\Big[\rho(t) - e^{-iG}\rho(t)e^{iG}\Big]\end{aligned}$$

To solve the time evolution we use the equation for $\rho(t+dt)$ and set $dt = 0$.

3. Finding L in the master equation form given.

$$\frac{d}{dt}\rho(t) = L\rho(t)L^\dagger - \frac{1}{2}\{L^\dagger L, \rho(t)\}$$

Setting this equal to the equation in the previous equation

$$L\rho(t)L^\dagger - \frac{1}{2}\{L^\dagger L, \rho(t)\} = -\lambda\Big[\rho(t) - e^{-iG}\rho(t)e^{iG}\Big]$$

By directly comparing :

$$L\rho(t)L^\dagger \to \lambda e^{-iG}\rho(t)e^{iG}$$

So that

$$L = \sqrt{2\lambda}\cdot e^{-iG}$$

Because

$$L^\dagger = (\sqrt{2\lambda}\cdot e^{-iG})^\dagger = \sqrt{2\lambda}e^{iG}$$

With the 2 being there so that when expanded you get 2λ which multiplied by $\frac{1}{2}$ gives just λ which we were looking for.

4. Showing that the off diagonal elements satisfy the differential equation. We have

$$E_1 = g_1\ ,\ E_2 = g_2$$

Where E_i are eigenvalues so $|E_i\rangle$ are eigenvectors.

$$|E_1\rangle = |g_1\rangle\ ,\ |E_2\rangle = |g_2\rangle$$

If the diagonals are g_1 and g_2 so we have

$$\rho_{ii}(t) = |g_i\rangle^\dagger \rho(t)|g_i\rangle$$

$$\frac{d}{dt}\rho_{ii}(t) = 0$$

This is because $|g_i\rangle^\dagger \rho(t)|g_i\rangle$ will always equal a constant and the differential of a constant is 0. For $\rho_{12}(t)$

$$\rho_{12}(t) = |g_1\rangle^\dagger \rho(t)|g_2\rangle$$

From part (b)

$$\frac{d}{dt}\rho_{12}(t) = -\lambda[\rho_{12}(t) - e^{-ig_1}\rho_{12}(t)e^{ig_2}]$$

$$= -\lambda\rho_{12}(t) + \lambda\rho_{12}(t)e^{i(g_2-g_1)}$$

$$= -\lambda\rho_{12}(t)[1 - e^{i(g2-g1)}]$$

3 Random Unitary transformation

In a time dt

$$\psi(t+dt) = e^{-iG\theta}\psi(t)$$

With probability

$$P(\theta) = \frac{1}{\sqrt{2\pi\lambda dt}}e^{-\frac{\theta^2}{2\lambda dt}}$$

1. Find density matrix at time $t+dt$. First need to show that order θ^3 and higher can be neglected. We can Taylor expand

   $$e^{-iG\theta}\psi(t)$$

   Around θ. So we get

   $$e^{-iG\theta} - iG\theta e^{-iG\theta} - \frac{1}{2}G^2\theta^2 e^{-iG\theta} + \frac{1}{6}iG^3\theta^3 e^{-iG\theta}\dots$$

   Lets also expand $e^{iG\theta}$ around θ

   $$e^{iG\theta} + iG\theta e^{iG\theta} - \frac{1}{2}G^2\theta^2 e^{iG\theta} - \frac{1}{6}iG^3\theta^3 e^{iG\theta}\dots$$

   So $e^{-iG\theta}\rho(t)e^{iG\theta}$ expanded in θ will have terms of order θ^3 and higher cancel out as the signs will be different. Can also say that that for t we have $e^{-iG\theta}$ and for dt we have $e^{-iG\theta}\rho(t)e^{iG\theta}$ so naturally dt will always have 1 order higher of θ past θ^3.
   For $\rho(t+dt)$ it is then a simpler version of Part 1.

   $$\rho(t+dt) = \int_{-\infty}^{\infty} P(\theta)e^{-iG\theta}\rho(t)e^{-iG\theta}$$

Using my expansions I used mathematica to evaluate $e^{-iG\theta}\rho(t)e^{-iG\theta}$

$$e^{-iG\theta}\rho(t)e^{-iG\theta} = \frac{4\rho(t) + G^4\theta^4\rho(t) - 8G^2\theta^2\rho(t)}{4} + iG^3\theta^3\rho(t) - 2iG\theta\rho(t)$$

Obviously this ignored the commuatative effect so it needed to be split up as

$$G\rho(t)G \neq G^2\rho(t)$$

$$\frac{1}{2}G^2\theta^2\rho(t) \neq \rho(t)\frac{1}{2}G^2\theta^2$$

And ignoring terms θ^3 or higher

$$= \rho(t) - \frac{1}{2}G^2\theta^2\rho(t) - \frac{1}{2}G^2\theta^2 + G\rho(t)G\theta^2$$

So we have

$$\rho(t+dt) = \int_{-\infty}^{\infty} P(\theta)\Big[\rho(t) - \frac{1}{2}G^2\theta^2\rho(t) - \rho(t)\frac{1}{2}G^2\theta^2 + G\rho(t)G\theta^2]$$

Like in question 1 we have a guassian integral from $P(\theta)$ so the solution is

$$\rho(t+dt) = \rho(t) - \frac{\theta^3}{2}\Big[G^2\rho(t) + \rho(t)G^2 - 2G\rho(t)G\Big]$$

And substituting $\theta^3 = \lambda dt$

$$\rho(t+dt) = \rho(t) - \frac{\lambda dt}{2}\Big[G^2\rho(t) + \rho(t)G^2 - 2G\rho(t)G\Big]$$

2. Finding L so that the derivative satisfies the master equation. Evaluating the derivative like in question 2 we can find the derivative from the equation above with $dt = t$

$$\begin{aligned}\frac{d}{dt} &= \frac{d}{dt}\Big[\rho(0) - \frac{\lambda t}{2}\Big[G^2\rho(t) + \rho(t)G^2 - 2G\rho(t)G\Big]\Big]\\ &= 0 - \frac{\lambda}{2}\Big[G^2\rho(t) + \rho(t)G^2 - 2G\rho(t)G\Big]\end{aligned}$$

Which can be simplified to

$$= -\frac{\lambda}{2}G\{G,\rho(t)\} + \lambda G\rho(t)G$$

As we can taker out a factor of G outside the brackets then inside the brackets is the commutation relation + an extra factor to bring it back to the original. Therefore

$$L = \sqrt{\lambda}G$$

3. Showing that the components of density operator in basis of eigenvectors of G satify the differential equation. Lie before in question 2d, for the off diagonal elements they have values of g_1 and g_2 (g_i, g_j). Substituting g_i and g_j into the differential equation above (g_i is left side multiplier of G and g_j is right side multiplier of G)

$$\frac{d}{dt}\rho_{ij}(t) = -\frac{\lambda}{2}\Big[g_i^2\rho_{ij}(t) + \rho_{ij}(t)g_j^2 - 2g_i\rho_{ij}(t)g_j\Big]$$
$$= -\frac{\lambda}{2}\Big[g_i^2 + g_j^2 - 2g_ig_j\Big]\rho ij(t)$$
$$= -\frac{\lambda}{2}(g_i - g_j)^2\rho ij(t)$$

4 State Exchange

Two orthonormal vectors such that

$$\psi(t) = a(t)\phi_1 + b(t)\phi_2$$

In time dt with probability λdt, $\psi(t)$ undergoes

$$\psi(t) \to a(t)\phi_2 b(t)\phi_1$$

1. Showing that the state operator satisfies the master equation in the canonical pauli x basis. To begin with (from Unitary Jumpy question) we have

   $$\rho(t+dt) = (1-\lambda dt)\rho(t) + \lambda dt\sigma_x\rho(t)\sigma_x$$

   So the derivative of this is

   $$\frac{d}{dt}\rho(t) = -\lambda\Big[\rho(t) - \sigma_x\rho(t)\sigma_x\Big]$$
   $$= -\lambda\rho(t) + \lambda\sigma_x\rho(t)\sigma_x$$

   However, this implies that $L = \sqrt{2\lambda}\sigma_x$, so for $L = \sqrt{\lambda}\sigma_x$ the differential equation would be

   $$\frac{d}{dt}\rho(t) = -\frac{\lambda}{2}\Big[\rho(t) - \sigma_x\rho(t)\sigma_x\Big]$$

   Whiich satisfies the master equation as the master equation expanded gives

   $$\sqrt{\lambda}\sigma_x\rho(t)(\sqrt{\lambda}\sigma_x)^\dagger - \frac{1}{2}(\sqrt{\lambda}\sigma_x(\sqrt{\lambda}\sigma_x)^\dagger\rho(t) + \rho(t)\sqrt{\lambda}\sigma_x(\sqrt{\lambda}\sigma_x)^\dagger)$$

   And σ_x is in the basis $\phi_{1,2}$ such that it is in the same form as the previous lindblad equations.

2. The Pauli matrices have eigenvalues of $+1$ and -1 which implies for that opposite diagonals have different signs.

5 The Lindblad Equation

1. Showing that the Lindblad equation is trace preserving.

$$\frac{d}{dt}\mathrm{Tr}\rho(t) = -i\mathrm{Tr}([H,\rho(t)]) + \sum_i \Big[\mathrm{Tr}(L_i\rho(t)L_i^\dagger) - \frac{1}{2}\mathrm{Tr}(\{L_i^\dagger L_i, \rho(t)\})\Big]$$

$$= -i\mathrm{Tr}([H,\rho(t)]) + \sum_i \Big[\mathrm{Tr}(L_i\rho(t)L_i^\dagger) - \frac{1}{2}\mathrm{Tr}(L_i^\dagger L_i\rho(t)) - \frac{1}{2}\mathrm{Tr}(\rho(t)L_i^\dagger L_i)\Big]$$

Using the cyclic property of traces where

$$\mathrm{Tr}(ABC) = \mathrm{Tr}(ACB) = \mathrm{Tr}(BAC)$$

The second part of the equation becomes

$$\ldots + \sum_i \Big[\mathrm{Tr}(ABC) - \frac{1}{2}\mathrm{Tr}(ACB) - \frac{1}{2}\mathrm{Tr}(BAC)\Big]$$

$$\Rightarrow 0$$

And because $H = 0$

$$-i\mathrm{Tr}([H,\rho(t)]) = -i\mathrm{Tr}(H,\rho(t),\rho(t)H) = -i\mathrm{Tr}(0) = 0$$

And we have

$$\frac{d}{dt}\mathrm{Tr}\rho(t) = 0$$

2. Show that the state operator is valid at all times.
$\rho(t)$ is hermitian (self adjoint) due to

$$\rho(t)^\dagger = \Big(\sum_i M_i(t)\rho_0 M_i^\dagger(t)\Big)^\dagger$$

$$= \sum_i M_i\rho_0^\dagger M_i^\dagger = \rho(t)$$

For a longer (in my opinion more rigorous method) prove the hermicity from the full Lindblad equation

$$\rho(t+dt)^\dagger = \rho(t)^\dagger + dt(\frac{d}{dt}\rho(t))$$

$$= \rho(t) + dt\Big[-i[H,\rho(t)] + \sum_i \Big[L_i\rho(t)L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho(t)\}\Big]\Big]^\dagger$$

$$= \rho(t) + dt\Big[-i[\rho(t),H] + \sum_i \Big[(L_i\rho(t)L_i^\dagger)^\dagger - \frac{1}{2}(L_i^\dagger L_i\rho(t))^\dagger - \frac{1}{2}(\rho(t)L_i^\dagger L_i)^\dagger\Big]$$

$$= \rho(t) + dt\Big[- i[H, \rho(t)] + \sum_i \Big[L_i \rho(t) L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho(t)\}\Big]\Big]^\dagger$$

$$= \rho(t + dt)$$

For trace of 1 we have

$$\mathrm{Tr}\rho(t) = \mathrm{Tr}\Big(\sum_i M_i \rho_0 M_i^\dagger \Big)$$

$$= \mathrm{Tr}\Big(\rho_0 \sum_i M_i M_i^\dagger\Big) = \mathrm{Tr}(\rho_0) = 1$$

For semi-positive definite

$$\rho(t) = \sum_i M_i(t) \rho_0 M_i^\dagger(t)$$

$$= \sum_i \rho_0 \langle \psi | M_i \rangle \langle M_i | \psi \rangle$$

$$= \sum_i \rho_0 |\langle \psi | M_i \rangle|^2 \geq 0$$

For an arbitrary vector $|\psi\rangle$ in the state space, using the fact that $\mathrm{Tr}(\rho_0) = 1$