FYMM/MMP IIIb HW 7 (exam) Dec 16 2020

Please remember to include your *name and student id number* on the first sheet of every file you submit. You are allowed to use notes and other references but *do not to discuss or write answers with other people*; exception: you are welcome to discuss and clarify the questions during the exercise sessions on Wed Dec 9, 13:15–14:00, and Thu Dec 10, 13:15–14:00.

Please answer all of the questions below and return them in Moodle by Wednesday, Dec 16. Each problem will be graded, with the maximum points from each item shown below.

Exercise 1 (max. 6 points total)

Derive the fundamental groups of the following topological spaces (endowed with the standard topologies inherited as subsets of $\mathbb{R}^2$ or $\mathbb{R}^3$):

$$M_1 = \mathbb{R}^2 \setminus \{(0,0)\}\,, \quad M_2 = \mathbb{R}^2 \setminus \{(0,0),(1,0),(0,1)\}\,,$$
$$M_3 = S^2\,, \quad M_4 = S^2 \setminus \{(1,0,0)\}\,, \quad M_5 = S^2 \setminus \{(0,0,-1),(0,0,1)\}\,.$$

You do not need to give precise mathematical details, but you should mention if you rely on results given in the lecture notes, and explain any possible geometric arguments for instance using pictures. (*Hint:* Two of the spaces can be deformation retracted to a circle.) Which, if any, of the manifolds are simply connected?

Exercise 2 (6 points)

Consider a differentiable manifold M and a vector field X defined on it. Let d denote the corresponding exterior derivative, and suppose ∇ is an affine connection on M. If f is any function, the following notations considered in the course denote certain (q,r)-tensor fields on M:

$$T = \nabla_X(df)\,, \quad U = d(\nabla_X f)\,, \quad V = d(\mathcal{L}_X(df))\,, \quad W = \mathcal{L}_X(d(df))\,.$$

As in the notes, here $\mathcal{L}_X$ denotes the Lie derivative associated with the flow of the vector field X.

(a) For each of the tensors T, U, V, and W, give the appropriate q and r for which it is a (q,r)-tensor field. (For example, the field X itself is a $(1,0)$-tensor field, and thus has $q = 1$ and $r = 0$.) (*2 points*)

(b) Let us next focus on one chart and denote the associated components of the vector field by X^μ and the associated connection coefficients by $\Gamma^\lambda{}_{\mu\nu}$. Using these, write down and simplify, as much as possible in this generality, the components of each of the four tensor fields. (*4 points*)

Exercise 3 (6 points)

In spherical coordinates (θ, ϕ), $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$, consider the vector field

$$X_p\big|_{p=(\theta,\phi)} = \frac{1}{\sin\theta}\partial_\phi \,,$$

and the manifold $M = S^2 \setminus \{(0,0,-1),(0,0,1)\}$ (i.e., the sphere where north and south pole have been removed).

(a) Explain why X is a smooth vector field on M but it cannot be extended into a smooth vector field on S^2. (*1 point*)

(b) Compute the maximal flow generated by X. Is it complete? (*2 points*)

(c) Consider the following map defined from M to $\mathbb{R}^2$,

$$f(\theta, \phi) = \left(\cot\frac{\theta}{2}\cos\phi \,, \cot\frac{\theta}{2}\sin\phi\right) \,.$$

Compute $U = f(M)$ explicitly. It is an open subset of $\mathbb{R}^2$, and thus a differentiable manifold with one chart. Check that $f : M \to U$ is a differentiable map. (*1 point*)

(d) Compute the pushed vector field $f_* X$ explicitly on U. Describe the resulting flow in qualitative terms (e.g. with a schematic picture). (*2 points*)

Exercise 4 (6 points)

(**Inspired by Kruskal–Szekeres coordinates for black holes**)

Consider the following differentiable submanifold of $\mathbb{R}^2$

$$M = \{(u,v)\,|\,uv < 1\} \,,$$

and the following metric tensor defined on it

$$ds^2 = -\frac{1}{(1-uv)^2}\,(du \otimes dv + dv \otimes du) \,.$$

Show that this is a metric which makes M into a pseudo-Riemannian manifold. Compute the associated Levi–Civita connection coefficients. Which of the following curves are geodesics? (All of the curves are defined for $t \in \mathbb{R}$.)

$$c(t) = (0, t) \,, \qquad c(t) = (t, -t) \,, \qquad c(t) = (\mathrm{e}^t, -\mathrm{e}^{-t}) \,.$$

If the curve is a geodesic, is it spacelike, timelike or null?

Exercise 5 (6 points)

After reading Sections 6.1–6.4 from the lecture notes, describe the classification of all irreducible Hermitian (also called "unitary") finite-dimensional representations of the simple Lie algebra $\mathfrak{su}(3)$ of the (simply connected compact simple) Lie group SU(3). Given one such representation of the algebra, how can you build a unitary representation of the Lie group SU(3) from it?

(If this looks too much, you will get 50% of the maximum for completing the Exercise for the Lie algebra $\mathfrak{su}(2)$ of the Lie group SU(2).)