Open Quantum Systems: Exercise Session 3

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Ex 2.71 NC

Exercise 1: Purity Test

Remember that a pure state is of the form $\rho = \psi\psi^\dagger$, where ψ is a state vector. Show that ρ is a pure state if and only if

$$\mathrm{tr}(\rho^2) = 1 \tag{1}$$

Exercise 2: Qubit State Operator

In physics a system described by the Hilbert space [1] $\mathbb{C}^2$ is called a qubit. Show that the density matrix of a qubit ρ can always be written in the form

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$$\rho = \frac{1}{2}(\mathbb{I} + v \cdot \sigma)$$

where $v \in \mathbb{R}^3$, $\|v\| \leq 1$ and $v \cdot \sigma = v_1\sigma_x + v_2\sigma_y + v_3\sigma_z$. Show that $\|v\| = 1$ if and only if the qubit is in a pure state.

Exercise 3

From the last exercise we know that the pure states of a qubit represent points on the unit sphere in $\mathbb{R}^3$, which in this context is called the Bloch sphere.

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(a) Show that every unit vector of the qubit can be written as

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle,$$

where $|0\rangle$, $|1\rangle$ is a fixed orthonormal basis.
The two angles $(\phi, \theta) \in [0, \pi[\times[0, 2\pi[$ are the coordinates of the pure states on the Bloch sphere in the given basis.
(b) Show that orthonormal bases correspond to antipodal points on the Bloch sphere.
(c) Express the transition probability between two states of the qubit as a geometric property of the corresponding points on the Bloch sphere.

[1] with the inner product $\langle\phi|\psi\rangle = \phi_1^*\psi_1 + \phi_2^*\psi_2$

Exercise 4

(Ballentine [2], exercise 2.5)
Which of the following are state operators? Are they pure states? If so decompose them in pure unit vectors.

$$\rho_1 = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{3}{4} \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{pmatrix}$$

$$\rho_3 = \frac{1}{3}|u\rangle\langle u| + \frac{2}{3}|v\rangle\langle v| + \frac{\sqrt{2}}{3}|v\rangle\langle u| + \frac{\sqrt{2}}{3}|u\rangle\langle v|,$$

where $\langle u|u\rangle = \langle v|v\rangle = 1$ and $\langle u|v\rangle = 0$,

$$\rho_4 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & 0 \end{pmatrix}, \quad \rho_5 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix}.$$

Exercise 5

Consider a two level system with Hamiltonian

$$H = \frac{\hbar\omega}{2}\sigma_z + \frac{\Omega}{2}\sigma_x$$

(a) Find the eigenvalues and eigenvectors of the Hamiltonian
(b) Use (a) the to solve the Schrödinger equation

$$i\hbar\frac{\mathrm{d}\psi}{\mathrm{d}t}(t) = H\psi(t)$$

for an initial state $\psi_0 \in \mathbb{C}^2$.
(c) Write the von Neumann equation for the qubit density matrix ρ. From Exercise 2 we know that the density matrix can written as $\rho = \frac{1}{2}(\mathbb{I} + v_1(t)\sigma_x + v_2(t)\sigma_y + v_3(t)\sigma_z)$. Use the von Neumann equation to find differential equations for $v_1(t)$, $v_2(t)$ and $v_3(t)$.

[2] Leslie E. Ballentine, Quantum Mechanics: A Modern Development