Quantum Information B Fall 2020 Exam Solutions

Jake Muff
Student number: 015361763
21/12/2020

1 Exercise 8.15

A projective measurement is perfomed on a single qubit in the basis $|+\rangle, |-\rangle$, where

$$|\pm\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}$$

$$\rho \to \mathcal{E}(\rho) = |+\rangle\langle+|\rho|+\rangle\langle+| + |-\rangle\langle-|\rho|-\rangle\langle-|$$

For a pure state (assumption) we would have

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

ρ is a projective measurement so

$$\rho = \frac{1}{2}(1+\cos(\theta))|0\rangle\langle 0| + \frac{1}{2}(1-\cos(\theta))|1\rangle\langle 1|$$

$$+\frac{1}{2}\sin(\theta)(\cos(\theta) - i\sin(\theta))|0\rangle\langle 1| + \frac{1}{2}\sin(\theta)(\cos(\theta)+i\sin(\theta))|1\rangle\langle 0|$$

This evolves like the equation above, with

$$\langle+|\rho|+\rangle = \frac{1}{2}(\langle 0| + \langle 1|)\rho(|0\rangle + |1\rangle)$$

$$= \frac{1}{2}(1+\sin(\theta)\cos(\theta))$$

And

$$\langle-|\rho|-\rangle = \frac{1}{2}(\langle 0| - \langle 1|)\rho(|0\rangle + |1\rangle)$$

$$= \frac{1}{2}(1-\sin(\theta)\cos(\theta))$$

$$\Rightarrow \mathcal{E}(\rho) = \frac{1}{4}(1+\sin(\theta)\cos(\theta))(|0\rangle + |1\rangle)(\langle 0| + \langle 1|)$$

$$+\frac{1}{4}(1-\sin(\theta)\cos(\theta))(|0\rangle - |1\rangle)(\langle 0| - \langle 1|)$$

$$= \frac{1}{2}\underbrace{(|0\rangle\langle 0| + |1\rangle\langle 1|)}_{=I} + \frac{1}{2}\sin(\theta)\cos(\theta)(|0\rangle\langle 1| + |1\rangle\langle 0|)$$

$$= \frac{1}{2}(I + \sin(\theta)\cos(\theta)(|0\rangle\langle 1| + |1\rangle\langle 0|))$$

Which in the form of the geometric picture (eq 8.87) we would have

$$\vec{r} \cdot \vec{\sigma} = \sin(\theta)\cos(\theta)(|0\rangle\langle 1| + |1\rangle\langle 0|)$$

Which has the corresponding map

$$(r_x, r_y, r_z) \to (r_x, 0, 0) = (\sin(\theta), \cos(\theta), 0, 0)$$

Therefore, it is projected onto the x axis of the bloch sphere. Illustrated this would be like fig 8.9 but stretched on the x axis as both y and z components of the bloch vector are lost.

2 Exercise 10.9

The 3 qubit phase flip code, where P_i, Q_i are projectors onto the $|0\rangle, |1\rangle$ states of the ith qubit. The 3 qubit phase flip code is given by

$$|0_L\rangle = |+++\rangle$$

$$|1_L\rangle = |---\rangle$$

The projectors are

$$P_i = |0\rangle_i\langle 0|_i$$

$$Q_i = |1\rangle_i\langle 1|_i$$

From Ex 10.8 P must be in total

$$P = |0_L\rangle\langle 0_L| + |1_L\rangle\langle 1_L|$$

$$P = |+++\rangle\langle +++| + |---\rangle\langle ---|$$

Clearly

$$PI^2P = P$$

For $P_1 = |0\rangle_1\langle 0|_1$, acting on the 1st qubit

$$PP_1^2P = \frac{1}{2}(|+++\rangle\langle 0++| + |---\rangle\langle 0--|) \times (|0++\rangle\langle +++| + |0--\rangle\langle ---|)$$

$$= \frac{1}{2}P$$

This is also the same result for P_2, P_3. For P_1P_2 we have

$$PP_1P_2P = \frac{1}{2}(|+++\rangle\langle 0++| + |---\rangle\langle 0--|) \times (|+0+\rangle\langle +++| + |-0-\rangle\langle ---|)$$

$$= \frac{1}{4}P$$

Also, $PP_1P_3 = \frac{1}{4}P$, $PP_2P_3 = \frac{1}{4}P$. So we can easily see that for $i \neq j$, $PP_iP_jP = \frac{1}{4}P$. For the Q projectors we have

$$PQ_1^2P = \frac{1}{2}(|+++\rangle\langle 1++| - |---\rangle\langle 1--|) \times (|1++\rangle\langle +++| - |1--\rangle\langle ---|)$$

$$= \frac{1}{2}P$$

$$PQ_1Q_2P = \frac{1}{4}P$$

$$PQ_1Q_3P = \frac{1}{4}P$$

So we see that for $i \neq j$ $PQ_iQ_jP = \frac{1}{4}P$ and for $i = j$ $PQ_iQ_jP = \frac{1}{2}P$. We also see that the cross prjector terms

$$PQ_iP_jP = 0$$

$$PP_iQ_jP = 0$$

Summarising all this information we can see that the phase flip code protects against the error set as the Quantum error correction conditions hold for a hermitian α_{ij}. The components of α are

$$I^2 = 1, P_i^2 = \frac{1}{2}, Q_i^2 = \frac{1}{2}, IP_i = \frac{1}{2}, IQ_i = \frac{1}{2}$$

$$P_iP_j = \frac{1}{4}, Q_iQ_j = \frac{1}{4}, P_iQ_j = 0, Q_iP_j = 0$$