FYMM/MMP IIIb 2020 Solutions to Problem Set 1

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1. Check whether the following (X, τ) is a topological space or not: $X = \{0, 1, 2\}$ and $\tau = \{\{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$?
   Answer.
   T1: $\emptyset \in \tau, X \in \tau$
   X is in τ but $\emptyset$ is not in τ (as $\emptyset$ is the empty set it is $\emptyset = \{\}$)so this is not a topological space and we stop here ignoring **T2, T3**

2. $X_1 = (\mathbb{R}, \tau_{disc}), X_2 = (\mathbb{R}, \tau_{triv})$. Show that the identity map

   $$id : X_1 \to X_2, x \mapsto x$$

   Is not a homeomorphism.
   $f : X \to y$ is a homeomorphism if f is continuous and has inverse $f : y \to X$. If we have a subset of $\mathbb{R}$ e.g $a \subseteq \mathbb{R}$ and $a \neq \emptyset$ then a is not closed (i.e open) in the discrete topology so

   $$X_1 \to X_2$$

   Is continuous. But in the trivial topology a is closed because $\neq \emptyset$ so the map

   $$X_2 \to X_1$$

   Is not continuous. $X_2 \to X_1$ is the inverse map, which is not continuous so the identity map id is not a homeomorphism.

3. Show that $\mathbb{R}^n$ with the usual topology is Hausdorff
   All spaces X with metric topology are Hausdorff. From the definition of metric suppose we have

   $$x, y \in \mathbb{R}^n, x \neq y$$

   Following from **M2**:

   $$a = d(x, y) \geq 0$$

   Which represents an open ball with radius $a/2$ centered around x. We have 2 disjoint neighbourhoods

   $$N_x(x, a/2)$$

   Which is a neighbour consisting of x. And

   $$N_y(y, a/2)$$

   Which is a neighbourhood consisting of y. These are clearly disjoint (I couldn't come up with a clear rigorous mathematical way to show this). The neighbour being disjoint means that $\mathbb{R}^n$ is hausdorff.
   N.B: I think a better proof lies in proving $\mathbb{R}^n$ has metric topology equivalent to the usual topology but I understood this method better.

4. Well defined meaning: $gH' = g'H \to g = g'h, h \in H$. In this question for well definedness the map looks similar to a homotopy group with γ as a loop. Following the lecture notes we need to find an equivalence relation such that

$$f \circ \gamma \sim f \circ \gamma'$$

In N.

$$\gamma \sim \gamma'$$

In M.
So, if we have F as a homotopy between γ and γ' then $f \circ F$ needs to be continuous. From the lecture notes there are 3 to check

$$f \circ F(s, 0) = f \circ \gamma(s)$$

$$f \circ F(0, t) = f \circ F(1, t) = f(x_0)$$

$$f \circ F(s, 1) = f \cdot \gamma'(s)$$

As f is continuous then $f \circ \gamma$ must be as well. With $f \circ \gamma$ being continuous and a homotopy then it is well defined.
To be an isomorphism f_* needs to be a homomorphism and bijective. For homomorphism we need

$$f_*(\gamma\gamma') = f_*(\gamma)f_*(\gamma')$$

$$f_*(\gamma\gamma') = [f \circ (\gamma\gamma')]$$

$$f_*(\gamma)f_*(\gamma') = (f \circ \gamma)(f \circ \gamma')$$

$$f \circ (\gamma\gamma') = (f \circ \gamma)(f \circ \gamma')$$

And there is a homomorphism.
For a bijection we need to show injectiveness and surjectiveness. For injectiveness: We have

$$f \circ \gamma = f \circ \gamma'$$

Which is a homotopy between two loops. As f is a homeomorphism and continuous then f^{-1} is continuous as well. As I showed before there is an equivalence relation between the loops so we can apply

$$f^{-1} \circ F$$

Where F is the homotopy between γ and γ'. $f^{-1} \circ F$ is also a homotopy between γ and γ' so

$$(\gamma) = (\gamma')$$

For surjectiveness (pulling from 4.2.3 in Lecture notes): If we have a loop in the topological invariant such that

$$\alpha \in \pi_1(M, f(x_0))$$

Then, since f^{-1} is a homeomorphism we can take $f^{-1} \circ \alpha$ like before where we defined well definedness and see that in itself $f^{-1} \circ \alpha$ is a loop in N. Following previous work we would also have

$$f_*(f^{-1} \circ \alpha) = \alpha$$

I think this proves that the function maps onto. So f_* is a bijection and thus, an isomorphism.

5. **Examples of Homotopy groups**.

   (a) Suppose $M = \mathbb{R}^3 \setminus \{\text{point}\}$. Identify $\pi_1(M)$.
   M is a 3D space with a point cut out such that any loop in M can be deformed. So, the fundamental group must be $\{e\}$

   $$\pi_1(M) = \{e\}$$

   (b) Didn't answer

   (c) Didn't answer