FYMM/MMP IIIb 2020 Solutions to Problem Set 4

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1. Compact manifold S^1 on the chart U_1, this has coordinate presentation

$$(x, y) = (\cos(\theta), \sin(\theta))$$

Where

$$U_1 = \{e^{i\theta} | \theta \in I_i\}$$
$$I_1 = (-\pi, \pi)$$
$$I_2 = (0, 2\pi)$$

Like in the lecture notes we need to partition the unity, so for $x = \cos(\theta), \theta = \cos^{-1}(x)$ and we have

$$\frac{\partial}{\partial\theta} = \frac{\partial\theta}{\partial x}\frac{\partial}{\partial x} = \frac{\partial}{\partial x}\cos^{-1}(x)\frac{\partial}{\partial x}$$
$$= -\frac{1}{\sqrt{1-x^2}}\frac{\partial}{\partial x}$$

This comes from the trig identitity $\sin^2 + \cos^2 = 1$ which gives

$$\cos(\theta) = \pm\sqrt{1 - \sin^2(\theta)}$$

$$x = \pm\sqrt{1-y^2} \to (\sqrt{1-y^2}, y)$$

And

$$\sin(\theta) = \pm\sqrt{1 - \cos^2(\theta)}$$
$$y = \pm\sqrt{1-x^2} \to (x, \pm\sqrt{1-x^2})$$

So for $y = \sin(\theta) \to \theta = \sin^{-1}(y)$

$$\frac{\partial}{\partial\theta} = \frac{\partial\theta}{\partial y}\frac{\partial}{\partial y} = \frac{\partial}{\partial y}\sin^{-1}(y)\frac{\partial}{\partial y}$$

$$= \frac{1}{\sqrt{1-y^2}}\frac{\partial}{\partial y}$$

For $X_p = \sin^2(\theta)\partial_\theta$

$$X_p = \sin^2(\sin^{-1}(y))\frac{1}{\sqrt{1-y^2}}\frac{\partial}{\partial y}$$

$$= \frac{y^2}{\sqrt{1-y^2}}\frac{\partial}{\partial y}$$

For the missing point we take the right and left limits s.t $p = e^{i\theta} \to p(-\pi) = p(\pi) = (-1, 0)$

$$\lim_{y \to 0^+} X_p = 0$$

$$\lim_{y \to 0^-} X_p = 0$$

In order for X_p to be continuous $X_p(\pi)$ has to be equal to 0. So there is only 1 unique choice for X_p which makes it into a smooth vector field.
For Y we have the limits as $\theta \to \pi$ or $\theta \to -\pi$

$$\lim_{\theta \to \pi} \theta \partial_\theta = \pi \partial_\theta$$

$$\lim_{\theta \to -\pi} \theta \partial_\theta = -\pi \partial_\theta$$

For both of these they correspond to the same point as $\cos(\pi), \sin(\pi) = \cos(-\pi), \sin(-\pi)$.
For the vector field to be smooth on S^1 we need to have

$$\frac{\partial a}{\partial \theta}|_{\theta = \pi} \neq 0$$

Where a is a smooth function. So we have

$$\lim_{\theta \to \pi} Y_p a = \pi \partial_\theta a$$

$$\lim_{\theta \to -\pi} Y_p a = -\pi \partial_\theta a$$

These two are not equal to each other so there is no value of Y_p for which $\theta = \pi$, thus Y_p cannot be extended into a smooth vector field on S^1

2. From the lecture notes we have

$$dy^\nu = \frac{\partial y^\nu}{\partial x^\mu} dx^\mu = \partial_\mu y^\nu dx^\mu$$

So

$$dy^1 \wedge dy^2 \wedge \ldots \wedge dy^n = \partial_{\mu_1} y^1 dx^{\mu_1} \wedge \partial_{\mu_2} y^2 dx^{\mu_2} \wedge \ldots \wedge \partial_{\mu_n} y^n dx^{\mu_n}$$

The indicies of wedge products are totally anti-symmetric so

$$\varepsilon^{1 \ldots n} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots dx^{\mu_n} = \varepsilon^{\mu_1 \ldots \mu_n} dx^1 \wedge dx^2 \wedge dx^n$$

Therefore the equation above simplifies to

$$\varepsilon^{1 \ldots n} dy^1 \wedge dy^2 \wedge \ldots \wedge dy^n = \varepsilon^{\mu_1 \ldots \mu_n} \partial_{\mu_1} y^1 dx^1 \wedge \partial_{\mu_2} y^2 dx^2 \wedge \ldots \wedge \partial_{\mu_n} y^n dx^n$$

$$= \varepsilon^{\mu_1 \ldots \mu_n} \frac{\partial y^1}{\partial x^{\mu_1}} dx^1 \wedge \frac{\partial y^2}{\partial x^{\mu_2}} dx^2 \wedge \ldots \wedge \frac{\partial y^n}{\partial x^{\mu_n}} dx^n$$

Now using the definition of $J(\vec{y}, \vec{x})$ down using the determinant equation gives

$$\varepsilon^{1 \ldots n} dy^1 \wedge dy^2 \wedge \ldots \wedge dy^n = J(\vec{y}, \vec{x}) dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$$

Where $\varepsilon^{1 \ldots n} = 1$

3. Let ω_q be a q- form

$$\omega_q = \frac{1}{q!}\omega_{\mu_1\mu_2\cdots\mu_q} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_q} \ ,$$

   (a) Showing that $\omega_q \wedge \eta_r = (-1)^{qr}\eta_r \wedge \omega_q$. Suppose, for example we have

   $$\omega_q = \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_q$$

   Where $\omega_n = dx^{\mu_n}$ for example. η_r will be an r-form version

   $$\eta_r = \eta_1 \wedge \eta_2 \wedge \ldots \wedge \eta_r$$

   Where $\eta_n = dx^{\alpha_n}$ for example. This means that

   $$\omega_q \wedge \eta_r = \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_q \wedge \eta_1 \wedge \eta_1 \wedge \ldots \wedge \eta_r$$

   To now get $\eta_r \wedge \omega_q$ we swap each of the ω's and η's by moving each η_n over q times. By doing each swap each time we have a negative sign which totally gives $(-1)^{qr}$, the r part comes from us having r, η's so that it takes qr swaps. Therefore we have

   $$\omega_q \wedge \eta_r = (-1)^{qr}\eta_r \wedge \omega_q$$

   (b) $\omega_q \wedge \eta_r = -\omega_q \wedge \eta_r$ implies that $\omega_q \wedge \omega_q = -\omega_q \wedge \omega_q$ meaning that

   $$\omega_q \wedge \omega_q = 0$$

4. Following differential forms in $\mathbb{R}^3$:

$$\alpha = xdx + ydy + zdz$$

$$\beta = zdx + xdy + ydz$$

$$\gamma = dydz$$

   (a) α is closed because we can define a function which has gradient equal to α such that

   $$\nabla \times \alpha = 0$$

   For α as it is, this is very simple

   $$\frac{1}{2}(x^2 + y^2 + z^2)\nabla \times \alpha = 0$$

   $$= \frac{1}{2}(2xdx + 2ydy + 2zdz) \times \alpha = (xdx + ydy + zdz) \times \alpha = 0$$

   For γ at (1,1,1), $\nabla \times \vec{F} =< x, -y, 0 >$ where $F_1 = 0, F_2 = 0, F_3 = xy \rightarrow \vec{F} =< F_1, F_2, F_3 >$. So γ is not closed as

   $$\nabla \times \vec{F} \neq 0$$

   N.B To calculate curls and gradients I used an online calculator.

(b) $\alpha \wedge \beta$, so from question 3 we have

$$\alpha \wedge \beta = (xdx + ydy + zdz) \wedge (zdz + xdy + ydz)$$

$$= (x^2 - yz)dx \wedge dy + (y^2 - zx)dy \wedge dz + (z^2 - xy)dz \wedge dx$$

And for $(\alpha + \gamma) \wedge (\alpha + \gamma)$ this would simply be

$$(\alpha + \gamma) \wedge (\alpha + \gamma) = 0$$

5. $M = \mathbb{R}^3, \omega = \omega_x dx + \omega_y dy + \omega_z dz$

$$\int_S (\nabla \times \vec{\omega}) \cdot d\vec{S} = \oint_C \vec{\omega} \cdot d\vec{s}$$

Where $\vec{\omega} = (\omega_x, \omega_y, \omega_z)$. Working from the LHS to the RHS by expanding out the curl on $\vec{\omega}$. As ω is a 2-form it can be written as

$$\omega = \omega_x dx \wedge dy + \omega_y dy \wedge dz + \omega)zdz \wedge dx$$

Such that $d\omega$ can be written as

$$d\omega = \Big(\frac{\partial \omega_y}{\partial x} - \frac{\partial \omega_z}{\partial y}\Big) dx \wedge dy$$

$$+\Big(\frac{\partial \omega_z}{\partial y} - \frac{\partial \omega_y}{\partial z}\Big) dy \wedge dz$$

$$+\Big(\frac{\partial \omega_z}{\partial x} - \frac{\partial \omega_x}{\partial z}\Big) dx \wedge dz$$

Therefore, the curl is

$$\nabla \times \vec{\omega} = \frac{\partial \omega_y}{\partial x} dx \wedge dy - \frac{\partial \omega_y}{\partial z} dy \wedge dz$$

To compute this is much easier if we have the same wedge products. Rewriting

$$\nabla \times \vec{\omega} = \frac{\partial \omega_y}{\partial x} dx \wedge dy + \frac{\partial \omega_y}{\partial z} dz \wedge dy$$

So that the LHS integral is

$$\int_S \nabla \times \vec{\omega} d\vec{s} = \int_S \Big(\frac{\partial \omega_y}{\partial x} dx \wedge dy + \frac{\partial \omega_y}{\partial z} \wedge dy\Big) d\vec{s}$$

This can now be put into the form needed on the RHS of eq (3) from PS by noting that

$$A = \omega_y ds$$

Such that

$$dA = \frac{\partial \omega_y}{\partial x} dx \wedge dy + \frac{\partial \omega_y}{\partial z} dz \wedge dy$$

So the above integral is

$$\int_S dA \cdot d\vec{s}$$

Which from the lecture notes and assuming that we have a single chart, this is equivalent to the closed curve integral

$$\int_S dA \cdot d\vec{s} = \oint_C A \cdot d\vec{s}$$

$$= \oint_C \omega_y dy \cdot d\vec{s}$$

$$= \oint_C \vec{\omega} \cdot d\vec{s}$$

Second part of this question not answered.