Open Quantum Systems Fall 2020 Answers to Exercise Set 4

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1 Answers

1. **Exercise 1**

   (a) Nielsen and Chaung Exercise 2.57

   $$\{L_l\} \to \{M_m\} = \{N_{lm}\}$$

   Suppose we have a state $|\phi\rangle$, the sttate of the system after measurement is given by

   $$|\phi\rangle = \frac{L_l|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}}$$

   We then have another set of measurements such that

   $$\langle\phi|M_m^\dagger M_m|\phi\rangle = \frac{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}{\langle\psi|L_l^\dagger L_l|\psi\rangle}$$

   So totally we have

   $$\begin{aligned}\frac{M_m|\phi\rangle}{\sqrt{\langle\phi|M_m^\dagger M_m|\phi\rangle}} &= \frac{M_m L_l|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}} \cdot \frac{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}}{\sqrt{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}} \\ &= \frac{M_m L_l|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}} \\ &= \frac{N_{lm}|\psi\rangle}{\langle\psi|N_{lm}^\dagger N_{lm}|\psi\rangle}\end{aligned}$$

   (b) State after the second measurement is performed

   (c) Last measurement unrecorded

2. **Exercise 2**

   (a) Equations of motion for the components $\psi_\pm(t)$

(b) Equations of motion for $\xi_-(t)$

(c) Probability $P(t)$ that it is $-\hbar/2$. When is $P(t)$ a maximum?

3. **Exercise 3**

(a) Rotating n times with angle θ is the same as rotating with an angle $n\theta$

$$U(\theta)^n = \begin{pmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{pmatrix}^n$$

We can trivially show that for $n = 2$ we have

$$U(\theta)^2 = \begin{pmatrix} cos^2(\theta) - sin^2(\theta) & 2sin(\theta)cos(\theta) \\ -2sin(\theta)cos(\theta) & cos^2(\theta) - sin^2(\theta) \end{pmatrix}$$

Which, using trig identities simplifies down to

$$U(\theta)^2 = \begin{pmatrix} cos(2\theta) & sin(2\theta) \\ -sin(2\theta) & cos(2\theta) \end{pmatrix}$$

And this applies for integer values of n. This follows from De Moivre's theorem such that

$$(cos(x) + isin(x))^n = cos(nx) + isin(nx)$$

Which is related to matrix representation in Group Theory

$$z = x + iy \in \mathbb{C} \rightarrow \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \in GL(2)$$

And $(cos(x)+isin(x))^n$ is an isomorphism from $\mathbb{C}$ to $GL(2)$. Notice how the $y's$ is the above matrix are swapped. In this case the unitary transformation representation a clockwise rotation or a $-\theta$ rotation as opposed to the convention. Rotations will have this same property rotating clockwise or anticlockwise.
In summary this property $U(\theta)^n = U(n\theta)$ is a direct consequence of the fact that for matrices of this type are isomorphic to the space of complex number $\mathbb{C}$ which I why De Moivre's theorem can be used here.

(b) Projector on a photon state with polarisation rotated $\pi/4$ from the horizontal? The horizontal polarisation is $|h\rangle$ and the vertical polarisation is $|v\rangle$. The projector onto $|h\rangle$ and $|v\rangle$ is

$$P_h = |h\rangle\langle h|$$

$$P_v = |v\rangle\langle v|$$

When the polarisation is rotated by $\pi/4$ which si the same as 45 degrees the projector onto the photon state will be

$$P_{\frac{\pi}{4}} = \frac{|h+v\rangle\langle h+v|}{2}$$

Additionally the state after going through polarisation is

$$\frac{|h + v\rangle}{\sqrt{2}}$$

With probability equal to $\frac{1}{2}$

(c) Probability of finding the photon in the horizontal state after rotation from horizontal by θ
After a rotation of θ from the horizontal, the projector will be

$$P_\theta = (cos(\theta)|h\rangle + sin(\theta)|v\rangle)(cos(\theta)\langle h| + sin(\theta)\langle v|)$$

With a probability

$$p(\theta) = |(cos(\theta)\langle h| + sin(\theta)\langle v|)|h\rangle|^2$$

$$p(\theta) = cos^2(\theta)$$

4. **Exercise 4**

(a) Calculate $|\phi\rangle = M_n|h\rangle$
After n rotations the photon is rotated

$$U(\theta)^n|h\rangle = U(n\theta)|h\rangle = cos(n\theta)|h\rangle + sin(n\theta)|v\rangle$$

Such that

$$|\phi\rangle_n = cos(n\theta)|h\rangle + sin(n\theta)|v\rangle$$

(b) Probability of the probability for the photon to have horizontal polarisation after applying M_n to $|h\rangle$
Probability to get horizontal polarisation is simply

$$p_n = cos^{2n}(\theta)$$

(c) What happens to probability for small θ?
For small θ the probability tends to 1 with state $|h\rangle$

5. **Exercise 4**

(a) Show that

$$e^A e^B = e^{A+B}$$

For commuting matrices
From definition

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

So we have

$$e^A e^B = \Big(\sum_{k=0}^{\infty} \frac{A^k}{k!}\Big)\Big(\sum_{k=0}^{\infty} \frac{B^k}{k!}\Big)$$

$$= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{A^k B^j}{k!j!} = \sum_{l=0}^{\infty}\sum_{k=0}^{l} \frac{A^k B^{l-k}}{k!(l-k)!}$$

$$= \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{k=0}^{l} \frac{l!}{k!(l-k)!} A^k B^{l-k}$$

$$= \sum_{l=0}^{\infty} \frac{(A+B)^l}{l!} = e^{A+B}$$

Which only works if we can use the fact that $AB = BA$ and they commute.

(b) As a counter example for non-commuting matrices we have the Pauli matrices which have the property

$$[\sigma_j, \sigma_k] = 2i \sum_{l=1}^{3} \epsilon_{jkl}\sigma_l$$

Lets use the σ_1 and σ_2 pauli matrices and calculate $e^A e^B$ and e^{A+B}

$$e^{\sigma_1} e^{\sigma_2} = \Big(\sum_{k=0}^{\infty} \frac{1}{k!}\sigma_1^k\Big)\Big(\sum_{k=0}^{\infty} \frac{1}{k!}\sigma_2^k\Big)$$

$$= \begin{pmatrix} \frac{e^2+1}{2e} & \frac{e^2-1}{2e} \\ \frac{e^2-1}{2e} & \frac{e^2+1}{2e} \end{pmatrix} \cdot \begin{pmatrix} e & 0 \\ 0 & \frac{1}{e} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{e^2+1}{2} & \frac{e^2-1}{2e^2} \\ \frac{e^2-1}{2} & \frac{e^2+1}{2e^2} \end{pmatrix}$$

On the other side we have

$$e^{\sigma_1+\sigma_2} = \exp\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{e^{2\sqrt{2}}+\sqrt{2}e^{2\sqrt{2}}-1+\sqrt{2}}{2\sqrt{2}e^{\sqrt{2}}} & \frac{e^{2\sqrt{2}}-1}{2\sqrt{2}e^{\sqrt{2}}} \\ \frac{e^{2\sqrt{2}}-1}{2\sqrt{2}e^{\sqrt{2}}} & \frac{-e^{2\sqrt{2}}(-\sqrt{2}+1)+1+\sqrt{2}}{2\sqrt{2}e^{\sqrt{2}}} \end{pmatrix}$$

Clearly

$$e^{\sigma_1} e^{\sigma_2} \neq e^{\sigma_1+\sigma_2}$$

2 Appendix

1. I didn't fully understand what the notation with M_n meant in exercise 4 as I computed this and my answer didn't make sense. The answer i've given makes sense conceptually to me so thats why I gave it. Here is what I got using the notation in the exercise

   We know that

$$P_h = |h\rangle\langle h| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

   Such that

$$P_h U^n(\theta) P_h = P_h U(n\theta) P_h$$
$$= \begin{pmatrix} 0 & 0 \\ sin(n\theta) & 0 \end{pmatrix}$$

   And

$$U^n(\theta) P_h U^n(\theta) = \begin{pmatrix} sin(n\theta)cos(n\theta) & sin^2(n\theta) \\ cos^2(n\theta) & cos(n\theta)sin(n\theta) \end{pmatrix}$$

   So that

$$M_n = \begin{pmatrix} 0 & 0 \\ sin^2(n\theta)cos(n\theta) & sin^3(n\theta) \end{pmatrix}$$

   And

$$|\phi\rangle = \begin{pmatrix} 0 & 0 \\ sin^3(n\theta) & 0 \end{pmatrix}$$

   This didn't make sense as the probably given using this answer wouldn't tend to 1 as I know it should as I've studied the Quantum Zeno effect previously.