FYMM/MMP IIIb 2020 Solutions to Problem Set 3

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1. Question 1. We have

$$f : S^1 \to S^2$$

   (a) From page 45 of the lecture notes we have

$$X = X^\mu \frac{\partial}{\partial x^\mu}$$

   And from page 51 of the elcture notes we have

$$f_* X = X^\mu \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial}{\partial y^\alpha}$$

   Following the example of page 51 we have

$$x^1 = \phi; y^1 = \theta, y^2 = \varphi$$

$$V = \dot{\phi}(t) \frac{\partial}{\partial \phi}$$

$$X = V, X^\mu = \dot{\phi}(t), \frac{\partial y^\alpha}{\partial x^\mu} = \frac{\partial \theta}{\partial \phi} = \frac{1}{2}$$

$$\frac{\partial}{\partial y^\alpha} = \frac{\partial \varphi}{\partial \phi} = 1$$

   So

$$f_* V = \dot{\phi}(\frac{1}{2} \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \varphi})$$

   Overall, for $c(t) = \phi(t) = at$

$$\dot{\phi} = a$$

$$V = a \frac{\partial}{\partial \phi}$$

   So

$$f_* V = a(\frac{1}{2} \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \varphi})$$

   (b) $c(t) = \phi(t) = 2\pi \sin(t)$ we have

$$\dot{\phi} = 2\pi \cos(t)$$

$$V = 2\pi \cos(t) \frac{\partial}{\partial \phi}$$

$$f_* V = 2\pi \cos(t)(\frac{1}{2} \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \varphi})$$

2. Following pg 53, Proving that XY is not a vector field. If we have two smooth vector fields X and Y and we apply XY to the smooth function f acting on M we would have

$$XYf = (X^\mu \partial_\mu Y^\mu \partial_\mu) f = X^\mu \partial_\mu \Big[Y^\nu \partial_\nu f \Big]$$
$$= X^\mu \partial_\mu Y^\mu \partial_\mu f + X^\mu Y^\mu (\partial_\mu \partial_\mu f)$$
$$= X^\mu (\partial_\mu Y^{nu}) \partial_\nu f + X^\mu Y^\nu \partial_\mu \partial_\nu f$$

The first term is a vector field but the second is not as we cannot write XY as a function of Z i.e $XYf = Z^\mu \partial_\mu f$.
Showing that $[X,Y]$ is a smooth vector field, we apply the same method as before

$$[X,Y]f = XYf - YXf$$
$$= X^\mu (\partial_\mu Y^\nu) \partial_\nu f + X^\mu Y^\nu (\partial_\mu \partial_\nu f) - (Y^\mu (\partial_\mu X^\nu) \partial_\nu f + Y^\mu X^\nu (\partial_\mu \partial_\nu f))$$
$$= X^\mu (\partial_\mu Y^\nu) \partial_\nu f - Y^\mu (\partial_\mu X^\nu) \partial_{nu} f$$

Both terms are vector fields and satisfy

$$[X,Y]f = Z^\mu \partial_\mu f$$

So $[X,Y]$ is a smooth vector field.
To prove the first identity we have

$$[X, fY] = X(fY) - (fY)X$$
$$= XfY + fXY - fYX$$
$$= (Xf)Y + f[X,Y]$$

To prove the second identity:

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] =$$
$$= XYZ - XZY - YZX + ZYX + YZX - YXZ - ZXY + XZY + ZXY - ZYX - XYZ + YXZ$$
$$= 0$$

3. Hamilton's equations as an example of a flow generated by a vector field.

$$H : M \to \mathbb{R}$$

   (a) Vector field X_H gives integral curves

$$x_H(t) = \text{ the basis } \mu \rightleftarrows i$$
$$\dot{x}^\mu_H = X^\mu_H$$

   For $\mu = 1 \ldots N$. So we have
$$\frac{\partial q_i}{\partial t} = -\frac{\partial H}{\partial p_i}$$
   Holds for $\mu = N+1 \ldots 2N$. Thus the integral curves for $x_H(t)$ are equivalent to the Hamiltonian equations of motion.

(b) We have $M = \mathbb{R}^2 \to N = 1, H = \frac{1}{2}(p^2 + q^2)$. This means that

$$X_H = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}$$

$$= \frac{\partial}{\partial p}(\frac{1}{2}(p^2 + q^2))\frac{\partial}{\partial q} - \frac{\partial}{\partial q}(\frac{1}{2}(p^2 + q^2))\frac{\partial}{\partial p}$$

$$= p\frac{\partial}{\partial q} - q\frac{\partial}{\partial p}$$

So we need to solve

$$\dot{q} = p, \dot{p} = -q$$

With $q(0) = 1, p(0) = 0$. Using wolffram alpha we get

$$p(t) = -\sin(t)$$

$$q(t) = \cos(t)$$

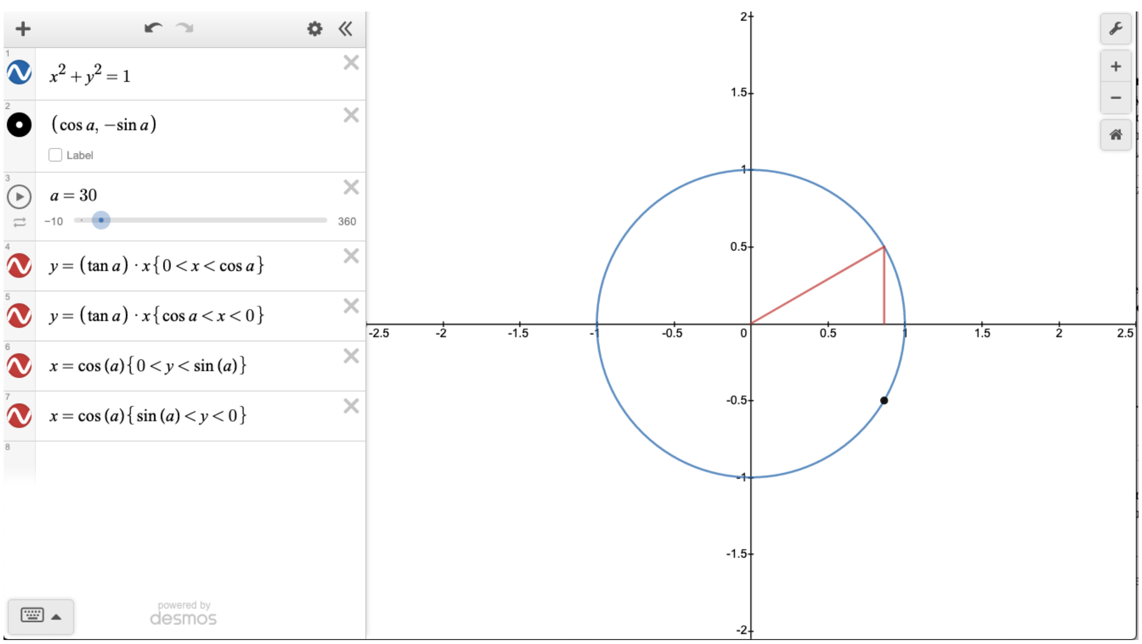

Figure 1: The differential equations describe a unit circle but with clockwise rotation. This figure shows the desmos representation of this.

(c) Now with $H = \frac{1}{2}(p^2 - q^2)$ and $x_0 = (1, 1)$ So we have

$$X_H = p\frac{\partial}{\partial q} + q\frac{\partial}{\partial p}$$

The differential equations are then

$$\dot{q} = p, \dot{p} = q$$

With

$$q(0) = 1, p(0) = 1$$

Solved using wolffram alpha

$$p(t) = e^t$$

$$q(t) = e^t$$

Which is a straight line.

(d) $M = T^2$, T^2 is the 2-torus. $H = \cos(p)$.

$$X_H = \frac{\partial}{\partial p}(\cos(p))\frac{\partial}{\partial q} - \frac{\partial}{\partial q}(\cos(p))\frac{\partial}{\partial p}$$

$$= -\sin(p)\frac{\partial}{\partial q} - 0$$

Solved using wolffram alhpa gives

$$\dot{q} = -\sin(p), \dot{p} = 0$$

$$p(t) = c_1$$

Where c_1 is a constnat or would equal the initial conditions.

$$q(t) = c_2 - t\sin(c_2)$$

I don't know how this would look exactly but $q(t)$ is a striahgt line around q which is the case of a 2-torus, q would be the angle and $q(t)$ would be a straight line on T^2.

4. The vector field

$$X = X^\mu(x)\frac{\partial}{\partial x^\mu}$$

$$g = g_{\mu\nu}(x)dx^\mu \otimes dx^\nu$$

The lie derivative $L_X g$ would therefore be (following Lecture notes)

$$L_X g = (L_x g_{\mu\nu})dx^\mu \otimes dx^\nu + g_{\mu\nu}(L_X dx^\mu) \otimes dx^\nu + g_{\mu\nu}dx^\mu \otimes (L_X dx^\nu)$$

$$= (X^\alpha \partial_\alpha g_{\mu\nu})dx^\mu \otimes dx^\nu + g_{\mu\nu}(\partial_\alpha X^\mu)dx^\alpha \otimes dx^\nu + g_{\mu\nu}dx^\mu \otimes \partial_\alpha X^\alpha dx^\alpha$$