Open Quantum Systems: Exercise session 1

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Exercise 1

(a) Prove that the trace of an operator A, $\mathrm{Tr}\, A = \sum_n u_n^\dagger A u_n$ is independent of the orthonormal basis $\{u_n\}$ chosen for its evaluation. *1pt*
(b) Show that $\mathrm{Tr}(Avv^\dagger) = v^\dagger Av$. *1pt*

Exercise 2

(Ballentine, exercise 1.3)
Consider the vector space $M_2(\mathbb{C})$, show that the Pauli matrices form a basis for this vector space:

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

3pts

Exercise 3: Hilbert-Schmidt product

(Ballentine, exercise 1.4)
(a) If A, B and C are matrices of the same shape, show that $(A, B) = \mathrm{Tr}(A^\dagger B)$ has all the properties of an inner product. I.e. show that

1. $(A, B) = \overline{(B, A)}$, where $\bar{\cdot}$ denotes the complex conjugate. *0.5pts*
2. $(A + C, B) = (A, B) + (C, B)$ and for $c \in \mathbb{C}$, $(A, cB) = c(A, B)$. *0.5pts*
3. $(A, A) > 0$ for A not the zero-matrix. *0.5pts*

This inner product is called the HIlbert-Schmidt product.
(b) Show that the Pauli matrices (see exercise 3) are orthogonal with respect to the inner product (A, B). *1.5pts*

Exercise 4

Prove that for $X \in M_n(\mathbb{C})$, the $n \times n$-dimensional complex matrices, the following statements are equivalent

a. X is of the form $B^\dagger B$, with $B \in M_n(\mathbb{C})$.

b. $\phi^\dagger X \phi \geq 0 \; \forall \phi \in \mathbb{C}^n$.

c. $X = X^\dagger$ and every eigenvalue is non-negative.

(show for example $a \Rightarrow b$, $b \Rightarrow c$, $c \Rightarrow a$) *3pts*

Exercise 5

(a) Show that for A, $B \in M_n(\mathbb{C})$ the trace is cyclic i.e. $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. *1pt*
(b) Consider X, $Y \in M_n(\mathbb{C})$ and $X, Y \geq 0$, show that $\mathrm{Tr}(XY) \geq 0$. *1pt*
(c) Is XY a positive matrix? *1pt*
hint: for (b) use (a) and Exercise 4

Exercise 6

The exponential of a matrix A is defined as

$$\exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

Show that

a. Let A be fully diagonalisable, i.e. $A = \sum_i a_i v_i v_i^\dagger$ for $a_i \in \mathbb{C}$ and $\{v_i\}$ and orthonormal basis. Then $\exp(A) = \sum_i \exp(a_i) v_i v_i^\dagger$. *1pt*

b. Let σ_i be one of the Pauli matrices, show that $\exp(i\sigma_i t) = \cos(t)\mathbb{I} + i\sin(t)\sigma_i$ for $t \in \mathbb{R}$ and $\mathbb{I}$ the identity matrix. *1pt*

c. Show that $\frac{d}{dt}\exp(At) = A\exp(At)$. *1pt*

Exercise 7: BBGKY hierarchy

Consider an system of N identical particles with the probability density functional $f_N(\mathbf{q}_1 \ldots \mathbf{q}_N, \mathbf{p}_1 \ldots \mathbf{p}_N)$, where $\mathbf{q_i}$ and $\mathbf{p}_i$ are 3-dimensional vectors for the position and momentum coordinates respectively. The probability density function satisfies the Liouville equation

$$\frac{\partial f_N}{\partial_t} + \sum_{i=1}^{N} \frac{\mathbf{p}_i}{m} \cdot \frac{\partial f_N}{\partial \mathbf{q}_i} + \sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial f_N}{\partial \mathbf{p}_i} = 0$$

with the net force acting on the i-th particle

$$\mathbf{F}_i = - \sum_{j=1, j\neq i}^{N} \frac{\partial \Phi_{ij}}{\partial \mathbf{q}_i} - \frac{\partial \Phi^{ext}}{\partial \mathbf{q}_i}$$

where Φ_{ij} is the pair interaction between particles and Φ^{ext} an external potential. We define the n-particle density functional

$$f_n(\mathbf{q}_1 \ldots \mathbf{q}_n, \mathbf{p}_1 \ldots \mathbf{p}_n) = \int f_N(\mathbf{q}_1 \ldots \mathbf{q}_N, \mathbf{p}_1 \ldots \mathbf{p}_N) d\mathbf{q}_{n+1} \ldots d\mathbf{q}_N d\mathbf{p}_{n+1} \ldots d\mathbf{p}_N$$

1. Show that the one-particle density function satisfies

$$\frac{\partial f_1}{\partial_t} + \frac{\mathbf{p}_1}{m} \cdot \frac{\partial f_1}{\partial \mathbf{q}_1} - \frac{\partial \Phi^{ext}}{\partial \mathbf{q}_1} = (N-1) \int \frac{\partial \Phi_{12}}{\partial \mathbf{q}_1} \frac{\partial f_2}{\partial \mathbf{p}_1} d\mathbf{q}_2 d\mathbf{p}_2$$

   2pts

2. Find the corresponding equation for the n-particle density function. *2pts*