Quantum Information A Fall 2020 Solutions to Problem Set 6

Jake Muff

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1 Answers

1. Exercise 4.24 from Nielsen and Chaung. Verify that figure 4.9 implements the Toffoli gate.

$$T = e^{i\pi/8} e^{i\pi Z/8}$$

Where Z is the Pauli Z gate. Circuits read left to right but matrix operators read right to left. Couple of useful identities

$$TT^\dagger = ST^\dagger T^\dagger = \mathbf{I}$$

$$XT^\dagger X = e^{i\pi/4} T$$

As $XZX = \mathbf{I}$. The first qubit is set to 0 so $T|0\rangle = |0\rangle$, so nothing happens to the first qubit. For the other two we have Which effectively does nothing as the $CNOT$

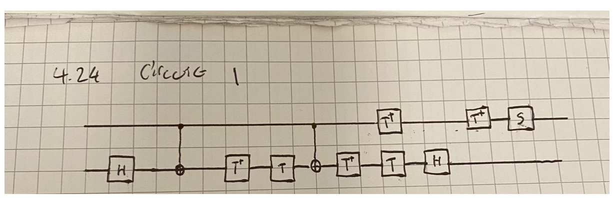


Figure 1: Circuit for Ex 4.24

gates cancel out.
If the first of these qubits is set to 0, the second qubit is

$$HTXT^\dagger TXT^\dagger H = HTXXT^\dagger H = HTT^\dagger H = HH = I$$

And for the first qubit

$$e^{i\pi/4} SXT^\dagger XT^\dagger |0\rangle = e^{i\pi/4} SXT^\dagger X|0\rangle$$

$$= e^{i\pi/4} SXT^\dagger |1\rangle$$

$$= e^{i\pi/4} SXe^{i\pi/4} |1\rangle$$

$$= S|0\rangle = |0\rangle$$

First of the two remaining qubits is set to 1. The first qubit is

$$e^{i\pi/4} SXT^\dagger XT^\dagger |1\rangle = SXT^\dagger X|1\rangle = SXT^\dagger |0\rangle$$

$$= SX|0\rangle = S|1\rangle = i|1\rangle$$

The second qubit is

$$HTXT^\dagger XTXT^\dagger XH$$

As we know

$$XT^\dagger X = e^{i\pi/4}$$

So we have

$$HTXT^\dagger XTXT^\dagger XH = e^{i\pi/2} HTTTTH = e^{i\pi/2} HZH = -iX$$

The phase factor here cancels with the other qubits phase factor.
Alternate method:
We can also verify that it by considering the circuit gate by gate labelling the operations $1 \rightarrow 14$ denoted $O_1 \rightarrow O_{14}$. In this case the initial process is

$$O_1 = |x, y, z\rangle$$

Where x, y, z denotes the 3 qubits as inputs. Then, using the identities listed before,

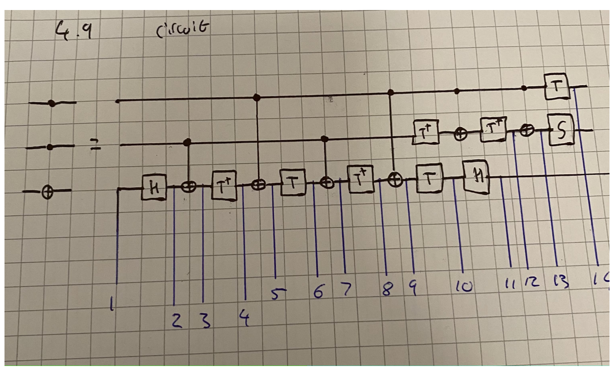


Figure 2: Circuit for 4.9

we can skip to effectively the 9th gate in the figure 2. Reading from right to left we have

$$O_9 = |x, y\rangle \otimes X^x T^\dagger X^y T X^x T^\dagger X^y H|z\rangle$$

$$O_{10} = |x\rangle \otimes T^\dagger |y\rangle \otimes TX^x T^\dagger X^y T X^x T^\dagger X^y H|z\rangle$$

$$O_{11} = |x\rangle \otimes X^x T^\dagger |y\rangle \otimes HTX^x T^\dagger X^y T X^x T^\dagger X^y H|z\rangle$$

$$O_{12} = |x\rangle \otimes T^\dagger X^x T^\dagger |y\rangle \otimes HTX^x T^\dagger X^y T X^x T^\dagger X^y H|z\rangle$$

$$O_{13} = |x\rangle \otimes X^x T^\dagger X^x T^\dagger |y\rangle \otimes HTX^xT^\dagger X^y TX^xT^\dagger X^y H|z\rangle$$
$$O_{14} = e^{i\pi/4}|x\rangle \otimes SX^x T^\dagger X^x T^\dagger |y\rangle \otimes HTX^xT^\dagger X^y TX^xT^\dagger X^y H|z\rangle$$

So we have $T|x\rangle = e^{i\pi/4}|x\rangle$.
If we set $x = 0$ i.e the first qubit to 0, then the outcome will be

$$|0\rangle \otimes |y\rangle \otimes |z\rangle = TOFFOLI|0, y, z\rangle$$

If $x = 1$ then,

$$e^{i\pi x/4} SX^x T^\dagger X^x T^\dagger |S\rangle = e^{i\pi/4} SXT^\dagger XT^\dagger |y\rangle$$
$$= S|y\rangle = (i)^y|y\rangle$$

So when $y = 0$ the outcome is

$$|1\rangle \otimes |0\rangle \otimes HTXT^\dagger TXT^\dagger H|z\rangle = |1, 0, z\rangle$$

If both x and $y = 1$ the outcome will be

$$|1, 1\rangle \otimes iHTXT^\dagger XTXT^\dagger XH|z\rangle$$

$$= TOFFOLI|1, 0, z\rangle$$

with

$$(TXT^\dagger X)^2 = \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}^2 = -iZ$$

Which leads us to

$$|1, 1\rangle \otimes HZH|z\rangle = |1, 1\rangle \otimes X|z\rangle = TOFFOLI|1, 1, z\rangle$$

2. Exercise 4.27 from Nielsen-Chaung.
Need to implement the permutation

$$U = (1234567)$$

Using cycle notation we can be broken down into products of 2-cycles

$$U = (12)(23)(34)(45)(56)(67)$$

We can write the actions of the $CNOT$ and $TOFFOLI$ gate as permutations shown by the table

So we see as an example we have

$$C_{12}T_2 = (46)(57)(57) = (46)$$

Gate	Control Qubit	Target Qubit	Permutations
$C_{1,2}$	1	2	(46)(57)
$C_{1,2}$	2	1	(26)(37)
$C_{1,2}$	1	3	(45)(67)
$C_{1,2}$	3	1	(15)(37)
$C_{1,2}$	2	3	(23)(67)
$C_{1,2}$	3	2	(13)(57)
T_1	2,3	1	(37)
T_2	1,3	2	(57)
T_3	1,2	3	(67)

Table 1: Table showing how the $CNOT$ and $TOFFOLI$ gates relate to permutations

So the transformation would be

$$|000\rangle \to |000\rangle$$
$$|001\rangle \to |111\rangle$$
$$|010\rangle \to |001\rangle$$
$$|011\rangle \to |010\rangle$$
$$|100\rangle \to |011\rangle$$
$$|101\rangle \to |100\rangle$$
$$|110\rangle \to |101\rangle$$
$$|111\rangle \to |110\rangle$$

3. Exercise 4.31 from Nielsen-Chaung. Prove (4.32), (4.33), (4.34)
   Doing this for computational basis states (each $\to$ is an operation i.e for CX_1C there are 3 arrows for $C \to X_1 \to C$):

   (a) (4.32), CX_1C in computation basis states is

   $$|0\rangle|0\rangle \to |0\rangle|0\rangle \to |1\rangle|0\rangle \to |1\rangle|1\rangle$$
   $$|0\rangle|1\rangle \to |0\rangle|1\rangle \to |1\rangle|0\rangle \to |1\rangle|0\rangle$$
   $$|1\rangle|0\rangle \to |1\rangle|1\rangle \to |0\rangle|1\rangle \to |0\rangle|1\rangle$$
   $$|1\rangle|1\rangle \to |1\rangle|0\rangle \to |0\rangle|0\rangle \to |0\rangle|0\rangle$$

   For X_1X_2 we have:

   $$|0\rangle|0\rangle \to |1\rangle|0\rangle \to |1\rangle|1\rangle$$
   $$|0\rangle|1\rangle \to |1\rangle|1\rangle \to |1\rangle|0\rangle$$

$$|1\rangle|0\rangle \rightarrow |0\rangle|0\rangle \rightarrow |0\rangle|1\rangle$$
$$|1\rangle|1\rangle \rightarrow |0\rangle|1\rangle \rightarrow |0\rangle|0\rangle$$

So we see that in computation basis they are the same and $CX_1C = X_1X_2$

(b) (4.33), $CY_1C = Y_1X_2$. For CY_1C we have:

$$|0\rangle|0\rangle \rightarrow |0\rangle|0\rangle \rightarrow i|1\rangle|0\rangle \rightarrow i|1\rangle|1\rangle$$
$$|0\rangle|1\rangle \rightarrow |0\rangle|0\rangle \rightarrow i|1\rangle|1\rangle \rightarrow i|1\rangle|0\rangle$$
$$|1\rangle|0\rangle \rightarrow |1\rangle|1\rangle \rightarrow -i|0\rangle|1\rangle \rightarrow -i|0\rangle|1\rangle$$
$$|1\rangle|1\rangle \rightarrow |1\rangle|0\rangle \rightarrow -i|0\rangle|0\rangle \rightarrow -i|0\rangle|0\rangle$$

And for Y_1Y_2

$$|0\rangle|0\rangle \rightarrow i|1\rangle|0\rangle \rightarrow i|1\rangle|1\rangle$$
$$|0\rangle|1\rangle \rightarrow i|1\rangle|1\rangle \rightarrow i|1\rangle|0\rangle$$
$$|1\rangle|0\rangle \rightarrow -i|0\rangle|0\rangle \rightarrow -i|0\rangle|1\rangle$$
$$|1\rangle|1\rangle \rightarrow -i|0\rangle|1\rangle \rightarrow -i|0\rangle|0\rangle$$

And the LHS = RHS so $CY_1C = Y_1Y_2$

(c) (4.34), $CZ_1C = Z_1$:
For CZ_1C

$$|0\rangle|0\rangle \rightarrow |0\rangle|0\rangle \rightarrow |0\rangle|0\rangle \rightarrow |0\rangle|0\rangle$$
$$|0\rangle|1\rangle \rightarrow |0\rangle|0\rangle \rightarrow |0\rangle|1\rangle \rightarrow |0\rangle|1\rangle$$
$$|1\rangle|0\rangle \rightarrow |1\rangle|1\rangle \rightarrow -|1\rangle|1\rangle \rightarrow -|1\rangle|0\rangle$$
$$|1\rangle|1\rangle \rightarrow |1\rangle|0\rangle \rightarrow -|1\rangle|0\rangle \rightarrow -|1\rangle|1\rangle$$

And for Z_1

$$|0\rangle|0\rangle \rightarrow |0\rangle|0\rangle$$
$$|0\rangle|1\rangle \rightarrow |0\rangle|1\rangle$$
$$|1\rangle|0\rangle \rightarrow -|1\rangle|0\rangle$$
$$|1\rangle|1\rangle \rightarrow -|1\rangle|1\rangle$$

Therefore, $CZ_1C = Z_1$

4. Exercise 4.32 from Nielsen-Chaung.
Suppose we have

$$P_0 = |0\rangle\langle 0| \ ; \ P_1 = |1\rangle\langle 1|$$

which are projectors onto $|0\rangle$ and $|1\rangle$, and we denote the density matrix before measurement as ρ and after measurement as ρ'. From earlier in N&C we know that given an outcome i, the state of the quantum system is

$$\frac{P_i \rho P_i}{p(i)}$$

Such that, in a computational basis there is probability $p(0)$ that outcome $i = 0 \to |0\rangle$ occurs and $p(1)$ that outcome $i = 1 \to |1\rangle$ occurs so

$$\rho' = p(0)\frac{P_0 \rho P_0}{p(0)} + p(1)\frac{P_1 \rho P_1}{p(1)}$$

$$\rho' = P_0 \rho P_0 + P_1 \rho P_1$$

For the second part of the question where we find the reduced density matrix when the first qubit is not affected by the measurement, i.e $tr_2(\rho) = tr_2(\rho')$
We know that

$$\rho = \sum_i \underline{|\psi_1\rangle|\psi_2\rangle}\langle\psi_1|\langle\psi_2|$$

$$Tr_2(\rho) = \sum_i p_i Tr_2(|\psi_1\rangle|\psi_2\rangle\langle\psi_1|\langle\psi_2|)$$

$$= \sum_i p_1 Tr_2(|\psi_1\rangle\langle\psi_1| \otimes |\psi_2\rangle\langle\psi_2|)$$

$$= \sum_i p_i(|\psi_1\rangle\langle\psi_1|)\langle\psi_2|\psi_2\rangle$$

Post measurement trace but not after observation, so we use the first part:

$$\rho' = \sum_i p_i(P_0|\psi_1\rangle|\psi_2\rangle\langle\psi_1|\langle\psi_2|P_0 + P_1|\psi_1\rangle|\psi_2\rangle\langle\psi_1|\langle\psi_2|P_1)$$

$$Tr_2(\rho') = \sum_i p_i Tr_2(|\psi_1\rangle\langle\psi_1| \otimes P_0|\psi_2\rangle\langle\psi_2|P_0) + Tr_2(|\psi_1\rangle\langle\psi_1| \otimes P_1|\psi_2\rangle\langle\psi_2|P_1)$$

$$Tr_2(\rho') = \sum_i p_i(|\psi_1\rangle\langle\psi_1|)(\langle\psi_2|P_0|\psi_2\rangle + \langle\psi_2|P_1|\psi_2\rangle$$

We also have that $P_0 + P_1 = I$, therefore,

$$Tr_2(\rho') = Tr_2(\rho)$$

5. Exercise 4.37 from Nielsen-Chaung. Decomposition of

$$U = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

into a product of 2-level unitary matrices. For this question I followed the above example (4.45 - 4.50) and extrapolated for the larger matrix. Using equation (4.45) in N&C

$$U_1 = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Using equation (4.48)

$$U_2 = \begin{pmatrix} \frac{\sqrt{6}}{3} & 0 & \frac{\sqrt{3}}{3} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{6}}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Using equation 4.50

$$U_3 = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

Now U_4 follows from (4.50) due to U being an extra dimension.

$$U_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{4}(1-i) & \frac{3}{4} - \frac{i}{4} & 0 \\ 0 & \frac{3}{4} + \frac{i}{4} & -\frac{\sqrt{3}}{4}(1+i) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

And U_5 and U_6 come from 4.51 repeating $U = V_1 \dots V_k$ where k is defined in the text so we have an extra two matrices

$$U_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{6}}{5} & 0 & -\frac{\sqrt{3}}{3}i \\ 0 & 0 & 1 & 0 \\ 0 & \frac{\sqrt{3}}{3}i & 0 & -\frac{\sqrt{6}}{3} \end{pmatrix}$$

$$U_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}i \\ 0 & 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}i \end{pmatrix}$$

So we have

$$U = U_1 U_2 U_3 U_4 U_5 U_6$$

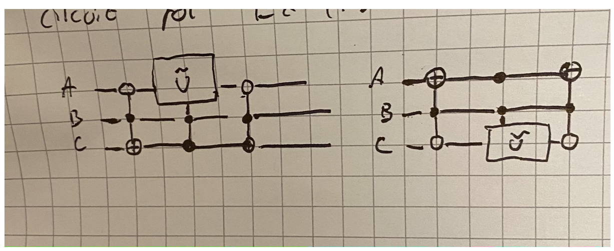
6. Exercise 4.39 from Nielsen-Chuang. Find a quantum circuit using single qubit operators and $CNOTS$ to implement

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & d \end{pmatrix}$$

Much like in the example in 4.58, however the a, b, c, d complex numbers are in different positions. Working from the example we have gray codes of

A	B	C
0	1	0
1	1	0
1	1	1

Which, again using figure 4.16 and comparing to equation 4.59 and 4.58 should get



2 Comments

1. The second method I provided I found to be a better example of my thought process which is why it is so messy. I think I managed to answer the question I just found it to be a very lengthy process. I know Esko showed an example way of answering the question in the lecture which I tried to replicate but struggled to understand his thought process. The second method also shows in the notation what each bit is doing at at one point in the circuit, which I found incredibly useful.

2. Question 2 was good as it was good to draw knowledge from MMPIIIa

3. Question 3 was lengthy, perhaps there is a better, shorter way of proving the equations. I am unsure. Was not difficult though.

4. Question 4 was interesting as it helped my understand of measurement/observation process a lot better. Not sure my proof is that rigorous though, could perhaps be lengthened.

5. Question 5 was extremely lengthy and seemed somewhat pointless to use such a large matrix. The question was incredibly useful in understand the content but the most difficult part of the question was its length which was due to the size of the matrix. I just hope I am right in my extrapolation for a large matrix, because apart from that you could follow the previous example.

6. Question 6 was similar to question 5 in sense that there was an example to follow which was slightly different. I enjoyed this question though because it was a different way of constructing quantum circuits compared to Question 1.

7. Overall I think this Problem Set was easily the most length/difficult. For the sake of brevity I excluded a lot of my notes I took whilst doing this Problem Set, particularly for Question 5 and 6 in which the notes got a bit messy.
   I did try to use a LaTeXQuantum circuit package, however it was such a lengthy process for the quantum circuits I had drawn I found it easier just to attach photos of them.