FYMM/MMP IIIb 2020 Solutions to Problem Set 6

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1. Calculating the Riemann tensor, the Ricci tensor and the scalar curvature for unit sphere S^2

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2$$

$$g = d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi$$

The metric from the previous exercise is

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

With inverse

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix}$$

The connection coefficients are calculated from

$$\Gamma^\lambda_{\alpha\beta} = g^{\lambda\mu}\Gamma_{\mu\beta\alpha}$$

Where

$$\Gamma_{\mu\beta\alpha} = \frac{1}{2}(\partial_\alpha g_{\beta\mu} + \partial_\beta g_{\mu\alpha} - \partial_\mu g_{\alpha\beta})$$

Therefore we have

$$\Gamma_{\phi\phi\theta} = \Gamma_{\phi\theta\phi} = \frac{1}{2}(\partial_\theta g_{\phi\phi} + \partial_\phi g_{\phi\theta} - \partial_\phi g_{\phi\theta})$$

$$= \frac{1}{2}\partial_\theta g_{\phi\phi}$$

$$\sin\theta\cos\theta$$

$$\Gamma_{\theta\phi\phi} = \frac{1}{2}(\partial_\phi g_{\theta\phi} + \partial_\phi g_{\theta\phi} - \partial_\theta g_{\phi\phi})$$

$$= -\sin\theta\cos\theta$$

I have neglected to show 0 terms. And

$$\Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = g^{\phi\phi}\Gamma_{\phi\phi\theta}$$

$$= \frac{1}{\sin^2\theta}\sin\theta\cos\theta$$

$$= \frac{\cos\theta}{\sin\theta}$$

$$\Gamma^{\theta}_{\phi\phi} = g^{\theta\theta}\Gamma_{\theta\phi\phi}$$
$$= -\sin\theta\cos\theta$$

The Riemann tensor is given by

$$R^{\kappa}_{\lambda\mu\nu} = \partial_{\mu}\Gamma^{\kappa}_{\nu\lambda} - \partial_{\nu}\Gamma^{\kappa}_{\mu\lambda} + \Gamma^{\eta}_{\mu\eta}\Gamma^{\kappa}_{\mu\eta} - \Gamma^{\eta}_{\mu\lambda}\Gamma^{\kappa}_{\nu\eta}$$

In the case of this question this is

$$R^{\theta}_{\phi\theta\phi} = \partial_{\theta}\Gamma^{\theta}_{\phi\phi} - \partial_{\phi}\Gamma^{\theta}_{\phi\theta} + \Gamma^{\eta}_{\phi\phi}\Gamma^{\theta}_{\eta\theta} - \Gamma^{\eta}_{\phi\theta}\Gamma^{\theta}_{\eta\phi}$$
$$= \partial_{\theta}(-\sin\theta\cos\theta) - 0 + 0\cdot\Gamma^{\eta}_{\phi\phi} - \Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\phi\theta}$$
$$= (-\cos^2\theta + \sin^2\theta) - (-\sin\theta\cos\theta)\left(\frac{\cos\theta}{\sin\theta}\right)$$
$$= \sin^2\theta$$

The Ricci Tensor

$$(\mathrm{Ric})_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}$$
$$g^{ab}R_{a\mu b\nu}$$

So we have

$$R_{\theta\theta} = g^{ab}R_{a\theta b\theta} = g^{\theta\theta}R_{\theta\theta\theta\theta} + g^{\phi\phi}R_{\phi\theta\phi\theta}$$
$$= (1\cdot 0) + \left(\frac{1}{\sin^2\theta}\sin^2\theta\right)$$
$$= 1$$

And

$$R_{\phi\phi} = g^{ab}R_{a\phi b\phi} = g^{\theta\theta}R_{\theta\phi\theta\phi} + g^{\phi\phi}R_{\phi\phi\phi\phi}$$
$$= (1\cdot\sin^2\theta) + \left(\frac{1}{\sin^2\theta}\cdot 0\right)$$
$$= \sin^2\theta$$
$$R_{\theta\phi} = g^{ab}R_{a\theta b\phi} = g^{\theta\theta}R_{\theta\theta\theta\phi} + g^{\phi\phi}R_{\phi\theta\phi\phi}$$
$$= (1\cdot 0) + \left(\frac{1}{\sin^2\theta}\cdot 0\right)$$
$$= 0$$

This is also the same for $R_{\phi\theta} = 0$. The Scalar curvature denoted by R_S is

$$R_S = g^{\mu\nu}(\mathrm{Ric})_{\mu\nu} = g^{\mu\nu}R_{\mu\nu}$$
$$R_S = g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi}$$
$$= (1\cdot 1) + \left(\frac{1}{\sin^2\theta}\cdot\sin^2\theta\right)$$
$$= 2$$

2. Symmetry of a sphere

$$g = d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi$$

Killing vector fields given by

$$X^\xi \partial_\xi g_{\mu\nu} + \partial_\mu X^\alpha g_{\alpha\nu} + \partial_\nu X^\beta g_{\mu\beta} = 0$$

The metric is diagonal so we can write

$$X^\xi \partial_\xi g_{\mu\nu} + \partial_\mu X^\nu g_{\nu\nu} + \partial_\nu X^\mu g_{\mu\mu} = 0$$

The L_i killing vectors can replace X^j in the above equation for different μ, ν as either θ or ϕ. So we have Beginning with $\mu, \nu = \theta, \theta$

$$L_1^\xi \partial_\xi g_{\theta\theta} + \partial_\theta L_1^\theta g_{\theta\theta} + \partial L_1^\theta g_{\theta\theta} = 0$$
$$L_1^\xi \cdot 0 + 0 \cdot g_{\theta\theta} = 0$$

$$L_2^\xi \partial_\xi g_{\theta\theta} + \partial_\theta L_2^\theta g_{\theta\theta} + \partial_\theta L_2^\theta g_{\theta\theta} = 0$$
$$L_2^\xi \cdot 0 + 0 \cdot g_{\theta\theta} = 0$$

$$L_3^\xi \partial_\xi g_{\theta\theta} + \partial_\theta L_3^\theta g_{\theta\theta} + \partial_\theta L_3^\theta g_{\theta\theta} = 0$$
$$L_3^\xi \cdot 0 + 0 \cdot g_{\theta\theta} = 0$$

Now with $\mu, \nu = \phi, \phi$

$$L_1^\xi \partial_\xi g_{\phi\phi} + \partial_\phi L_1^\phi g_{\phi\phi} + \partial_\phi L_1^\phi g_{\phi\phi} = 0$$
$$-\cos\phi \partial_\theta \sin^2\theta + (\partial_\phi \sin\phi \cot\theta) \sin^2\theta + (\partial_\phi \sin\phi \cot\theta) \sin^2\theta = 0$$
$$-2\cos\phi \cos\theta \sin\theta + 2\cos\phi \cot\theta \sin^2\theta = 0$$

$$L_2^\xi \partial_\xi g_{\phi\phi} + \partial_\phi L_2^\phi g_{\phi\phi} + \partial_\phi L_2^\phi g_{\phi\phi} = 0$$
$$\sin\phi \partial_\theta \sin^2\theta + (\partial_\phi \cos\phi \cot\theta) \sin^2\theta + (\partial_\phi \cos\phi \cot\theta) \sin^2\theta = 0$$
$$2\sin\phi \cos\theta \sin\theta - 2\sin\phi \cot\theta \sin^2\theta = 0$$

$$L_3^\xi \partial_\xi g_{\phi\phi} + \partial_\phi L_3^\phi g_{\phi\phi} + \partial_\phi L_3^\phi g_{\phi\phi} = 0$$
$$0 \cdot \partial_\theta g_{\phi\phi} + 0 \cdot g_{\phi\phi} = 0$$

Now with $\mu, \nu = \phi\theta$

$$\underbrace{L_1^\xi \partial_\xi g_{\phi\theta}}_{=0} + \partial_\theta L_1^\phi g_{\phi\phi} + \partial_\phi L_1^\theta g_{\theta\theta} = 0$$

$$\frac{-\sin\phi g_{\phi\phi}}{\sin^2\theta} + \cos\phi = 0$$

$$\underbrace{L_2^\xi \partial_\xi g_{\phi\theta}}_{=0} + \partial_\theta L_2^\phi g_{\phi\phi} + \partial_\phi L_2^\theta g_{\theta\theta} = 0$$

$$-\frac{\cos\phi g_{\phi\phi}}{\sin^2\theta} + \sin\phi = 0$$

$$\underbrace{L_3^\xi \partial_\xi g_{\phi\theta}}_{=0} + \partial_\theta L_3^\phi g_{\phi\phi} + \partial_\phi L_3^\theta g_{\theta\theta} = 0$$

$$(0 \cdot 0) + (0 \cdot 0) = 0$$

And so, $L_{1,2,3}$ are the killing vectors of g. Now to calculate the commutators $[L_a, L_b]$

$$[L_1, L_2] = [-\cos\phi\partial_\theta + \cot\theta\sin\phi\partial_\phi\ ,\ \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi]$$

$$= \cos\phi\partial_\theta\Big(-\sin\phi\partial_\theta - \cot\theta\cos\phi\partial_\phi\Big) - \cot\theta\sin\phi\partial_\phi\Big(-\sin\phi\partial_\theta - \cot\theta\cos\phi\partial_\phi\Big)$$

$$+\sin\phi\partial_\theta\Big(\cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi\Big) + \cot\theta\cos\phi\partial_\phi\Big(\cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi\Big)$$

$$= -\cos\phi\cos\phi\partial_\theta^2 - \cos^2\phi(\partial_\theta\cot\theta)\partial_\phi - \cot\theta\cos^2\phi\partial_\theta\partial_\phi + \cot\theta\sin\phi\cos\phi\partial_\theta + \cot\theta\sin^2\phi\partial_\phi\partial_\theta$$

$$-\cot^2\theta\sin^2\phi\partial_\phi + \cot^2\theta\sin\phi\cos\phi\partial_\phi^2 + \sin\phi\cos\phi\partial_\theta^2 - \sin^2\phi(\partial_\theta\cot\theta)\partial_\phi - \cot\theta\sin^2\phi\partial_\theta\partial_\phi$$

$$-\cot\theta\sin\phi\cos\phi\partial_\theta + \cot\theta\cos^2\phi\partial_\phi\partial_\theta - \cos^2\theta\cos^2\phi\partial_\phi - \cot^2\theta\sin\phi\cos\phi\partial_\phi^2$$

$$= -\cot^2\theta\partial_\phi - (\partial_\theta\cot\theta)\partial_\phi$$

$$= \partial_\phi = \frac{\partial}{\partial\phi} = L_3$$

This is then repeated for $[L_2, L_3]$ and $[L_3, L_1]$ so we have

$$[L_1, L_2] = L_3$$

$$[L_2, L_3] = L_1$$

$$[L_3, L_1] = L_2$$

I think this represents the angular momentum operator.

3. T_a are $N \times N$ matrices

$$[T_a, T_b] = i f_{abc} T_c$$

$$\chi_a = \sum_{i,j=1}^{N} (T_a)_{ij} b_i^\dagger b_j$$

$$\chi_b = \sum_{k,l=1}^{N} (T_b)_{kl} b_k^\dagger b_l$$

$$[\chi_a, \chi_b] = \Big[\sum_{i,j=1}^{N} (T_a)_{ij} b_i^\dagger b_j\Big]\Big[\sum_{k,l=1}^{N} (T_b)_{kl} b_k^\dagger b_l\Big]$$

$$-\Big[\sum_{k,l=1}^{N} (T_b)_{kl} b_k^\dagger b_l\Big]\Big[\sum_{i,j=1}^{N} (T_a)_{ij} b_i^\dagger b_j]$$

$$= (T_a)_{ij}(T_b)_{kl}[b_i^\dagger b_j, b_k^\dagger b_l]$$

$$= (T_a)_{ij}(T_b)_{kl}(b_i^\dagger b_j b_k^\dagger b_l - b_k^\dagger b_l b_i^\dagger b_j)$$

$$= (T_a)_{ij}(T_b)_{kl}(b_i^\dagger [b_j, b_k^\dagger b_l] + [b_i^\dagger, b_k^\dagger b_l] b_j)$$

$$= (T_a)_{ij}(T_b)_{kl}(b_i^\dagger [b_j, b_k^\dagger] b_l + b_k^\dagger [b_i^\dagger, b_l] b_j)$$

$$(T_a)_{ij}(T_b)_{kl}(b_I^\dagger \delta_{jk} b_l - b_k^\dagger \delta_{il} b_j)$$

$$(T_a T_b)_{il} b_i^\dagger b_l - (T_a T_b)_{kj} b_k^\dagger b_j$$

$$[T_a, T_b]_{ij} b_i^\dagger b_j = i f_{abc} \chi_c$$

4. (a) The Gell-Mann matrices are

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The generators of SU(3) are

$$T_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad T_2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad T_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad T_4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$T_5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad T_6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad T_7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad T_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The generators T_1, T_2, T_3 form a SU(2) subgroup of SU(3) and since they are orthogonal for $a, b = \{1, 2, 3\}$, $c = \{4, 5, 6, 7\}$, $f_{abc} = 0$ and the only other values are

$$f_{147} = \frac{1}{2}$$

and

$$f_{458} = \frac{\sqrt{3}}{2}$$

Note that this also applies for other structure constants which aren't 0. The structure constants can also be calculated by the commutators of the generators as shown

$$[T_1, T_4] = \frac{1}{2}T_7 \ , \ \mathrm{Tr}([T_1, T_4]T_c) = \delta^{c7} \to f_{147} = \frac{1}{2}$$

(b) As with page 81 of the lecture notes we just need to show that $[J_a, J_b] = i\epsilon_{abc}J_c$, so we have

$$[\lambda_2, \lambda_5] = i\lambda_7$$

$$[\lambda_5, \lambda_2] = i\lambda_2$$

$$[\lambda_7, \lambda_2] = i\lambda_5$$

Thus, $\lambda_2 = J_1, \lambda_5 = J_2, \lambda_7 = J_3$ and clearly

$$[\lambda_a, \lambda_b] = i\epsilon_{abc}\lambda_c$$