Quantum Mechanics IIa 2021
Solutions to Problem Set 3

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Problem 1

Periodically Driven Harmonic Oscillator where $t < 0$ in the ground state and for $t > 0$ we have perturbing potential

$$V(x,t) = F_0 x \cos(\omega t)$$

With Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 x^2$$

In the interaction picture we have

$$\langle x \rangle = \langle \psi | x | \psi \rangle$$

$$= \langle \psi | e^{iH_0 t} x e^{-iH_0 t} | \psi \rangle$$

Where

$$|\psi\rangle = \sum_n c_n(t) |n\rangle \tag{1}$$

Starting at $t = 0$ we have $c_n^{(0)}(t) = \delta_{n0}$

$$c_n^{(0)} = c_0^{(0)} = c_0(t) - 1$$

$$c_n^{(1)}(t) = \frac{-i}{\hbar} \int_{t_0}^{t} \langle n | V_I(t') | i \rangle dt'$$

$$= \frac{-i}{\hbar} \int_{t_0}^{t} e^{i\omega_{ni} t'} V_{ni}(t') dt'$$

$$= \frac{-i}{\hbar} \int_{0}^{t} V_{n0}(t') e^{in\omega_0 t'}$$

$$= \frac{-i}{\hbar} \int_{0}^{t} e^{i(E_n - E_0)t'/\hbar} \langle n | F_0 x \cos(\omega t') | 0 \rangle dt'$$

Using the hint

$$\langle n' | x | n \rangle = \sqrt{\frac{\hbar}{2m\omega}} (\sqrt{n+1}\delta_{n',n+1} + \sqrt{n}\delta_{n',n-1})$$

So

$$\langle n | F_0 x \cos(\omega t') | 0 \rangle \equiv F_0 \langle n | x | 0 \rangle \cos(\omega t')$$

$$= \sqrt{\frac{\hbar}{2m\omega}} (\delta_{n,1})$$

Thus,

$$\begin{aligned}
c_n^{(1)}(t) &= \frac{-i}{\hbar}\int_0^t e^{i\omega_0 t'} F_0 \cos(\omega t')\sqrt{\frac{\hbar}{2m\omega_0}}\delta_{n1} dt' \\
&= \frac{-i}{\hbar}\sqrt{\frac{\hbar}{2m\omega_0}} F_0 \delta_{n1}\int_0^t e^{i\omega_0 t'}\cos(\omega t')dt' \\
&= \frac{-i}{\hbar}\sqrt{\frac{\hbar}{2m\omega_0}} F_0 \int_0^t e^{i\omega_o t'}\Big(\frac{e^{i\omega t'}+e^{-i\omega t'}}{2}\Big)dt' \\
&= \frac{-i}{\hbar}\sqrt{\frac{\hbar}{2m\omega_0}} F_0 \cdot \text{integral}
\end{aligned}$$

The integral is evaluated as

$$\begin{aligned}
\int_0^t e^{i\omega_o t'}\Big(\frac{e^{i\omega t'}+e^{-i\omega t'}}{2}\Big) &= \Big[\frac{ie^{-it'(\omega-\omega_0)}}{\omega-\omega_0} - \frac{ie^{it'(\omega+\omega_0)}}{\omega+\omega_0}\Big]_0^t \\
&= -i\Big(\frac{1-e^{-it(\omega-\omega_0)}}{\omega-\omega_0} + \frac{e^{it(\omega+\omega_0)}-1}{\omega+\omega_0}\Big)
\end{aligned}$$

For $n > 1$, $c_n^{(1)} = 0$ clearly. Here I changed into the schrodinger picture, because it was easier to calculate (and understand what was going on). I still use the calculations above in the final answer.

$$\begin{aligned}
|\psi\rangle_I &= \sum_n c_n(t)|n\rangle \\
&= 1|0\rangle + c_1(t)|1\rangle
\end{aligned}$$

Therefore

$$|\psi\rangle_S = e^{-iH_0 t/\hbar}|\psi\rangle_I$$

For a simple harmonic Oscillator we have

$$H_0|0\rangle = \frac{1}{2}\hbar\omega_0|0\rangle$$

$$H_0|1\rangle = \frac{3}{2}\hbar\omega_0|1\rangle$$

Thus

$$|\psi\rangle_S = e^{-i\omega_0 t/2}|0\rangle + c_1(t)e^{-3i\omega_0 t/2}|1\rangle$$

$$\langle x\rangle =_S \langle\psi|x|\psi\rangle_S$$

$$= (e^{i\omega_0 t/2}\langle 0| + c_1^\dagger(t)e^{3i\omega_0 t/2}\langle 1|)\cdot x\cdot(e^{-i\omega_0 t/2}|0\rangle + c_1(t)e^{-3i\omega_0 t/2}|1\rangle) \tag{2}$$

x here can be represented in ladder operator formalism with

$$x = \sqrt{\frac{\hbar}{2m\omega_0}}(a+a^\dagger)$$

Where

$$a = \sqrt{\frac{m\omega_0}{2}}(x + \frac{i}{m}\hat{p})$$

$$a^\dagger = \sqrt{\frac{m\omega_0}{2}}(x - \frac{i}{m}\hat{p})$$

Using this equation (2) can be split into two

$$c_1^\dagger e^{i\omega_0 t}\langle 1|x|0\rangle = c_1^\dagger e^{i\omega_0 t}\sqrt{\frac{\hbar}{2m\omega_0}}$$

$$c_1 e^{-i\omega_0 t}\langle 0|x|1\rangle = c_1 e^{-i\omega_0 t}\sqrt{\frac{\hbar}{2m\omega_0}}$$

So that

$$\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega_0}}(c_1 e^{-i\omega_0 t} + c_1^\dagger e^{i\omega_0 t})$$

We have from before:

$$c_1(t) = \frac{-i}{\hbar}\sqrt{\frac{\hbar}{2m\omega_0}}F_0\Big(-i\Big(\frac{1-e^{-it(\omega-\omega_0)}}{\omega-\omega_0} + \frac{e^{it(\omega+\omega_0)}-1}{\omega+\omega_0}\Big)\Big)$$

Substituting this in

$$\langle x\rangle = \frac{1}{\hbar}\frac{\hbar}{2m\omega_0}F_0\Big(e^{-i\omega_0 t}\Big(\frac{1-e^{i(\omega+\omega_0)t}}{\omega+\omega_0}\Big) + e^{-i\omega_0 t}\Big(\frac{1-e^{i(\omega-\omega_0)t}}{\omega-\omega_0}\Big)\Big)$$

$$= \frac{1}{\hbar}\frac{\hbar}{2m\omega_0}F_0\Big(\Big(\frac{e^{-i\omega_0 t}-e^{i\omega t}}{\omega+\omega_0}\Big) + \Big(\frac{e^{-i\omega_0 t}-e^{-i\omega t}}{\omega-\omega_0}\Big)\Big)$$

$$= \frac{1}{\hbar}\frac{\hbar}{2m\omega_0}F_0\frac{\cos(\omega_0 t)-\cos(\omega t)}{\omega_0^2-\omega^2}\Big((\omega-\omega_0)+(\omega+\omega_0)\Big)$$

$$= \frac{1}{\hbar}\frac{\hbar}{2m\omega_0}F_0 2\omega_0\Big(\frac{\cos(\omega_0 t)-\cos(\omega t)}{\omega_0^2-\omega^2}\Big)$$

$$= \frac{F_0}{m}\frac{\cos(\omega_0 t)-\cos(\omega t)}{\omega_0^2-\omega^2}$$

Is this valid for $\omega = \omega_0$? No. The equation becomes invalid when at resonance because as ω_0 increases toward ω the soltion tends towards infinty, thus perturbation theory breaks down.

Problem 2

Simple Harmonic Oscillator with

$$V(x,t) = Ax^2 e^{\frac{-t}{\tau}}$$

Probability that after $t >> \tau$ system transitions to a higher excited state. Transistion probability for $|i\rangle \to |n\rangle$ with $n \neq i$ is

$$P(i \to n) = |c_n^{(1)}(t) + c_n^{(2)}(t) + \dots|^2$$

For this we need

$$\langle n'|x^2|n\rangle = \sqrt{\frac{\hbar}{2m\omega_0}}\Big(\sqrt{n}\langle n'|x|n-1\rangle + \sqrt{n-1}\langle n'|x|n+1\rangle\Big)$$

$$= \frac{\hbar}{2m\omega_0}\Big(\sqrt{n(n-1)}\delta_{n-2,n'} + (2n+1)\delta_{nn'} + \sqrt{(n+1)(n+2)}\delta_{n+2,n'}\Big)$$

So

$$\langle n'|x^2|0\rangle = \frac{\hbar}{2m\omega_0}(\delta_{0n'} + \sqrt{2}\delta_{2n'})$$

Ignoring $\delta_{0n'}$.

$$c_n^{(0)} = \delta_{n0}$$

$$c_n^{(1)} = \frac{-i}{\hbar}\int_0^t e^{i(E_n - E_0)t'/\hbar}\langle n'|Ax^2 e^{-t/\tau}|0\rangle dt'$$

$$= \frac{-i}{\hbar}A\int_0^t e^{i\omega_0 t'}e^{-t/\tau}\langle n'|x^2|0\rangle dt'$$

$$= \frac{-i}{\hbar}A\frac{\hbar}{2m\omega_0}\sqrt{2}\delta_{n2}\int_0^t e^{i\omega_0 t'}e^{-t/\tau}dt'$$

$$= \frac{-i}{\hbar}A\frac{\hbar}{2m\omega_0}\sqrt{2}\delta_{n2}\Big[\frac{e^{i\omega_0 t'} - \frac{t'}{\tau}}{i\omega_0 - \frac{1}{\tau}}\Big]_0^t$$

$$= \frac{-i}{\hbar}A\frac{\hbar}{2m\omega_0}\sqrt{2}\delta_{n2}\Big[\frac{e^{i\omega_0 t - \frac{t}{\tau}}}{i\omega_0 - \frac{1}{\tau}} - \frac{1}{i\omega_0 - \frac{1}{\tau}}\Big]$$

$$= \frac{-iA}{m\omega_0}\frac{1}{\sqrt{2}}\delta_{n2}\Big[\frac{e^{i\omega_0 t - \frac{t}{\tau}}}{i\omega_0 - \frac{1}{\tau}}\Big]$$

Trying for $c_0^{(1)}$, $c_1^{(1)}$ and $c_2^{(1)}$ we find that

$$c_0^{(1)} \neq 0 \to \delta_0 \neq 0$$

$$c_2^{(1)} \neq 0 \rightarrow \delta_{22} \neq 0$$

So we can say $c_n^{(1)} = 0$ for $n \neq 0, 2$. This may be extended to $n \neq 0, 2n$ (aka even n) but I didn't test this. Therefore we have,

$$\begin{aligned} c_0^{(1)} &= \frac{-i}{\hbar} \int_0^t \frac{\hbar}{2m\omega_0} A e^{\frac{-t'}{\tau}} dt' \\ &= \frac{iA}{2m\omega_0} \tau (e^{-\frac{t}{\tau}} - 1) \end{aligned}$$

With $t >> \tau$ we have

$$c_0^{(1)} = -\frac{iA}{2m\omega_0} \tau$$

For $c_2^{(1)}$ we have

$$c_2^{(1)} = \frac{-iA}{m\omega_0} \frac{1}{\sqrt{2}} \frac{e^{i\omega_0 t - \frac{t}{\tau}}}{i\omega_0 - \frac{1}{\tau}}$$

The transistion to the $|2\rangle$ state is

$$|c_2^{(1)}|^2 = \frac{A^2 \tau^2 |e^{\frac{i\omega_0 t\tau - t}{\tau}}|^2}{2m^2 \omega_0^2 (\omega_0^2 \tau^2 + 1)}$$

With $t >> \tau$ we have

$$|c_2^{(1)}|^2 = \frac{A^2 \tau^2}{2m^2 \omega_0^2 (\omega_0^2 \tau^2 + 1)}$$

Problem 3

Hydrogen atom in ground state $(nl, m) = (1, 0, 0)$ with

$$\vec{E} = \begin{cases} 0 & t < 0, \\ \vec{E_0} e^{-t/\tau} & t > 0. \end{cases}$$

We want to calculate the probability for atom to found at at $t >> \tau$ in

$$(n, l, m) = (2, 1, \pm 1)$$

$$(n, l, m) = (2, 1, 0)$$

$$(n, l, m) = (2, 0, 0)$$

The potential is

$$V = -eE_0 \hat{z} e^{-t/\tau}$$

So we have

$$\begin{aligned} c_n^{(1)} &= \frac{-i}{\hbar} \int_0^t e^{i\omega_{ni} t'} \langle n | eE_0 \hat{z} e^{-t'/\tau} | i \rangle dt' \\ &= \frac{i}{\hbar} \int_0^t e^{i\omega_{ni} t'} \langle n | eE_0 \hat{z} | i \rangle e^{-t'/\tau} dt' \end{aligned}$$

1. For $(n,l,m) = (2,1,\pm 1)$ we have

$$\langle 2,1,\pm 1|\hat{z}|1,0,0\rangle = 0$$

And because of the $\Delta m = 0$ selection rule we have

$$c^{(1)}_{2,1,\pm 1} = 0$$

2. For $(n,l,m) = (2,1,0)$ we have

$$\langle 2,1,0|\hat{z}|1,0,0\rangle$$

We have a radial integral which we don't have to evaluate, which is

$$\langle 210|\hat{z}|100\rangle = \int_0^\infty drr^3 R^*_{21}R_{10}\int_{-1}^1 d(\cos\theta)\cos\theta Y^0_1 Y^0_0$$

$$= \int_0^\infty R^*_{21}R_{10}r^3 dr$$

Normalizing this gives

$$= \frac{1}{\sqrt{3}}\int_0^\infty R_{21}R_{10}r^3 dr$$

Let's call this I_r for ease. So that

$$c^{(1)}_{210} = I_r \cdot \frac{-i}{\hbar}\int_0^t e^{i\omega_{ni}t'}eE_0 e^{-t'/\tau}dt'$$

$$= \frac{i}{\hbar}eE_0\tau \cdot I_r \cdot \frac{e^(i\omega_{ni}t - \frac{1}{\tau})\tau - 1}{1 - i\omega_{ni}\tau}$$

With $t >> \tau$ the probability transistion becomes

$$|c^{(1)}_{210}|^2 = \frac{e^2E_0^2\tau^2}{\hbar^2}\cdot |I_r|^2 \cdot \frac{1}{1+\omega^2_{ni}\tau^2}$$

$$\frac{e^2E_0^2\tau^2}{\hbar^2}|\int_0^\infty R_{21}R_{10}r^3 dr|^2 \cdot \frac{1}{1+\omega^2_{ni}\tau^2}$$

3. Replacing the final state with 2s gives

$$\langle 200|\hat{z}|100\rangle = 0$$

Due to selection rules $\Delta l = +1$ and the transistion probability is thus 0.

Problem 4

For $t < 0$, $H = 0$. For $t > 0$

$$H = \left(\frac{4\Delta}{\hbar^2}\right)\vec{S_1} \cdot \vec{S_2}$$

The state is initially in $|+-\rangle$ for $t \leq 0$. We need to find the probability of being in $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$ as a function of time.

1. Solving exactly. Firstly:

$$\vec{S_1} \cdot \vec{S_2} = S_{12}S_{22} + \frac{1}{2}S_{1+}S_{2-} + \frac{1}{2}S_1 - S_{2+}$$

So H can be expanded using Spin Operator identities in a coupled basis, i.e

$$S_+|-\rangle = \hbar|+\rangle$$

$$S_-|+\rangle = \hbar|-\rangle$$

$$S_2|+\rangle = \hbar 2|+\rangle$$

and we denote $|1\rangle = |++\rangle, |2\rangle = |+-\rangle, |3\rangle = |-+\rangle, |4\rangle = |--\rangle$. So the Hamiltonian can now be expanded as a matrix

$$H = 4\Delta \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}$$

This is worked from 11.230 of the QM 1 notes. The eigenvalues are

$$E_1 = \Delta, E_0 = -3\Delta$$

With E_1 for spin 1, E_0 for spin 0. From the Clebsch-Gordan coefficients we see that $|++\rangle, |--\rangle, \frac{1}{\sqrt{2}}(|+-\rangle, +|-+\rangle)$ are all spin 1 and therefore have energy Δ. $\frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$ is spin 0 and has energy -3Δ.
If we denote a new basis with

$$|1,0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle)$$

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$$

Then the initial state is

$$|+-\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle + |0,0\rangle)$$

Solving exactly for $t > 0$ we have

$$U(t, t_0) = e^{\frac{-i}{\hbar}H(t-t_0)}$$

$$|{+}{-}\rangle = \frac{1}{\sqrt{2}}(e^{-i\Delta t/\hbar}|1,0\rangle + e^{i\Delta t3/\hbar}|0,0\rangle)$$

$$= \frac{1}{\sqrt{2}}e^{-i\Delta t/\hbar}(|{+}{-}\rangle + |{-}{+}\rangle) + \frac{1}{\sqrt{2}}e^{3i\Delta t/\hbar}(|{+}{-}\rangle - |{-}{+}\rangle)$$

$$= \frac{1}{2}\Big[(e^{it\Delta/\hbar} + e^{3it\Delta/\hbar})|{+}{-}\rangle + (e^{-it\Delta/\hbar} + e^{3i\Delta t/\hbar})|{-}{+}\rangle\Big]$$

The probability to find the system in state $|i\rangle$ is $|\langle i| + -\rangle|^2$, thus, clearly

$$|\langle + + | + -\rangle|^2 = 0$$

$$|\langle - - | + -\rangle|^2 = 0$$

The other two are

$$|\langle + - | + -\rangle|^2 = \frac{1}{4}(2 + e^{4i\Delta t/\hbar} + e^{-4it\Delta/\hbar})$$

$$= \frac{1 + \cos(\frac{4\Delta t}{\hbar})}{2}$$

$$|\langle - + | + -\rangle|^2 = \frac{1}{4}(2 - e^{4i\Delta t/\hbar} - e^{-4i\Delta t/\hbar})$$

$$= \frac{1 - \cos(\frac{4\Delta t}{\hbar})}{2}$$

2. Now solved using perturbation theory.

$$c_n^{(0)} = \delta_{ni}$$

$$c_n^{(1)} = \frac{-i}{\hbar}\int_0^t \langle n|H|i\rangle dt'$$

$$= \frac{-it}{\hbar}\langle n|H|i\rangle$$

Using the matrix form of H from before we have

$$\langle + - |H| + -\rangle = 0$$

$$\langle - + |H| + -\rangle = 2\Delta$$

$$\langle + + |H| + -\rangle = 0$$

$$\langle - - |H| + -\rangle = 0$$

The bottom two agree with the exactly solved version. So we have

$$c^{(0)}_{|+-\rangle} = 1$$

$$c^{(1)}_{|+-\rangle} = 1 + \frac{it\Delta}{\hbar}$$

$$c^{(1)}_{|-+\rangle} = \frac{-2i\Delta t}{\hbar}$$

$$P^{(1)}_{|+-\rangle} = |c^{(1)}_{|+-\rangle}|^2 = 1 + \frac{t^2\Delta^2}{\hbar^2}$$

$$P^{(1)}_{|-+\rangle} = |c^{(1)}_{|-+\rangle}|^2 = \frac{4\Delta^2 t^2}{\hbar^2}$$

Using the small angle approxmation for the exact solution we can then compare answers as

$$= \frac{1 - \cos(\frac{4\Delta t}{\hbar})}{2} \approx \frac{4\Delta^2 t^2}{\hbar^2}$$

Thus, for small times, first order perturbation theory gives the exact solution. The answer for the state $|+-\rangle$ does not match the exact solution, even when applying the same approxmation

$$= \frac{1 + \cos(\frac{4\Delta t}{\hbar})}{2} \approx 1 - \frac{4\Delta^2 t^2}{\hbar^2}$$

However they do match in the sense that the probability must add up to equal 1

$$P^{(1)}_{|+-\rangle} + P^{(1)}_{|-+\rangle} = 1$$

Thus,

$$P^{(1)}_{|+-\rangle} = 1 - \frac{4\Delta^2 t^2}{\hbar^2}$$

I am not sure if I made a mistake somewhere, but it is interesting.