Quantum Information A Fall 2020 Answers to Problem Set 1

Jake Muff
Student number: 015361763
04/09/2020

1. Suppose V is a vector space with basis vectors $|0\rangle$ and $|1\rangle$, and A is a linear operator from V to V such that
$$A|0\rangle = |1\rangle \ ; \ A|1\rangle = |0\rangle \ .$$
Give a matrix representation for A, with respect to the input and output basis $|0\rangle, |1\rangle$. Find input and output bases which give rise to a different matrix representation of A. (For example, you can use the states $|\pm\rangle$ appearing in problem 5.)

   (a) ***Answer.*** From
$$A|v_j\rangle = \sum_i A_{ij}|w_i\rangle$$
   We get
$$A|0\rangle = A_{11}|0\rangle + A_{21}|1\rangle = |1\rangle$$
$$A|1\rangle = A_{12}|0\rangle + A_{22}|1\rangle = |0\rangle$$
$$A_{11} = 0, A_{21} = 1, A_{12} = 1, A_{22} = 0$$
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
   Input $(|0\rangle, |1\rangle)$ with output $(|1\rangle, |0\rangle)$

   (b) For a different matrix representation of A
$$A|0\rangle = A_{11}|1\rangle + A_{21}|0\rangle = |1\rangle$$
$$A|1\rangle = A_{12}|1\rangle + A_{22}|0\rangle = |0\rangle$$
$$A_{11} = 1, A_{21} = 0, A_{12} = 0, A_{22} = 1$$
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
   Input $(|0\rangle, |1\rangle)$ with output $(|0\rangle, |1\rangle)$

2. Let the basis of V be $\{|0\rangle, |1\rangle\} \equiv \{e_1, e_2\}$. The tensor product vector space $V \otimes V$ has the basis
$$\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\} = \{|0\rangle|0\rangle, |0\rangle|1\rangle, |1\rangle|0\rangle, |1\rangle|1\rangle\}$$
$$\equiv \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\} \ . \qquad (1)$$

Let us use the two-component notation

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ ; \ |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{2}$$

In the component notation, the tensor product of vectors becomes the Kronecker product,

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a \begin{pmatrix} c \\ d \end{pmatrix} \\ b \begin{pmatrix} c \\ d \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix} . \tag{3}$$

Show that

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \ ; \ |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \ ; \ |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \ ; \ |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} . \tag{4}$$

(a) ***Answer.***

$$|00\rangle = |0\rangle|0\rangle$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \times 1 \\ 1 \times 0 \\ 0 \times 0 \\ 0 \times 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} . \tag{5}$$

$$|01\rangle = |0\rangle|1\rangle$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \times 0 \\ 1 \times 1 \\ 0 \times 0 \\ 0 \times 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} . \tag{6}$$

$$|10\rangle = |1\rangle|0\rangle$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \times 1 \\ 0 \times 0 \\ 1 \times 1 \\ 1 \times 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} . \tag{7}$$

$$|11\rangle = |1\rangle|1\rangle$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \times 0 \\ 0 \times 1 \\ 1 \times 0 \\ 1 \times 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} . \tag{8}$$

3. Using the above 4-component notation, show that the CNOT operator

$$U_{CN} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{9}$$

acts on the (computational) basis states as

$$U_{CN}|00\rangle = |00\rangle \ ; \ U_{CN}|01\rangle = |01\rangle \ ; \ U_{CN}|10\rangle = |11\rangle \ ; \ U_{CN}|11\rangle = |10\rangle \ . \tag{10}$$

(a) ***<u>Answer.</u>*** Simply use matrix multiplication.

$$U_{CN}|00\rangle =$$

$$U_{CN} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |00\rangle. \tag{11}$$

$$U_{CN}|01\rangle =$$

$$U_{CN} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |01\rangle. \tag{12}$$

$$U_{CN}|10\rangle =$$

$$U_{CN} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |11\rangle. \tag{13}$$

$$U_{CN}|11\rangle =$$

$$U_{CN} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = |10\rangle. \tag{14}$$

4. Recall that (from Mathematical Methods of Physics IIIa or equivalent) the abelian group $\mathbf{Z}_2$ can be realized as the set $\{0,1\}$ with addition modulo two as the product. The latter is the same as the XOR operation of bits, and we denote it with the symbol $\oplus$:

$$0 \oplus 0 = 1 \oplus 1 = 0 \ ; \ 0 \oplus 1 = 1 \oplus 0 = 1 \ . \tag{15}$$

Note that $x \oplus x = 0$. Recall also that we can define a product group

$$\mathbf{Z}_2^n = \underbrace{\mathbf{Z}_2 \times \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2}_{n \ times} = \{0,1\}^n \tag{16}$$

with elements $x = (x_1, x_2, \ldots, x_n)$ with each $x_i \in \mathbf{Z}_2$, with the product

$$x \oplus y = (x_1 \oplus y_1, x_2 \oplus y_2, \ldots, x_n \oplus y_n) \ . \tag{17}$$

Next we shorten the notation and write

$$x = x_1 x_2 \cdots x_n = (x_1, x_2, \ldots, x_n) \ . \tag{18}$$

You notice that x is a binary number with n bits, e.g. $x = 011011$ for $n = 6$. The above rule then extends the XOR operation for n-bit binary numbers. Converting the binary numbers to decimal numbers and back gives fun results, e.g. $3 \oplus 5 = 011 \oplus 101 = 110 = 6$. Practise this by calculating the results (in binary form) of

(a) $01011 \oplus 10001$

(b) $10001 \oplus 01111$

(c) $(01101 \oplus 10101) \oplus 11100$

(a) ***Answer.*** Use the XOR operation on binary numbers and convert to decimal *Example:* $3 \oplus 5 = 011 \oplus 101 = 110 = 6$

$$\begin{array}{cccc} & 0 & 1 & 1 \\ \oplus & 1 & 0 & 1 \\ \hline & 1 & 1 & 0 \end{array} = 6_{10}$$

(a)$01011 \oplus 10001$

$$\begin{array}{cccccc} & 0 & 1 & 0 & 1 & 1 \\ \oplus & 1 & 0 & 0 & 0 & 1 \\ \hline & 1 & 1 & 0 & 1 & 0 \end{array} = 26_{10}$$

(b)$10001 \oplus 01111$

$$\begin{array}{cccccc} & 1 & 0 & 0 & 0 & 1 \\ \oplus & 0 & 1 & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & 1 & 0 \end{array} = 30_{10}$$

(b)$(01101 \oplus 10101) \oplus 11100$

$$\begin{array}{cccccc} & 0 & 1 & 1 & 0 & 1 \\ \oplus & 1 & 0 & 1 & 0 & 1 \\ \hline & 1 & 1 & 0 & 0 & 0 \end{array} \Longrightarrow \begin{array}{cccccc} & 1 & 1 & 0 & 0 & 0 \\ \oplus & 1 & 1 & 1 & 0 & 0 \\ \hline & 0 & 0 & 1 & 0 & 0 \end{array} = 4_{10}$$

5. The Hadamard gate is represented by

$$H = \frac{1}{\sqrt{2}} \left(\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right) . \tag{19}$$

Define the states

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = H|0\rangle \ ; \ |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = H|1\rangle \ . \tag{20}$$

Let H act on them, what are the resulting states $H|+\rangle, H|-\rangle$?

(a) ***Answer.*** $H|+\rangle$

$$\begin{aligned} H|+\rangle &= H(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)) \\ &= \frac{1}{\sqrt{2}} \left(\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right) (\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)) \\ &= \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle + \frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle \\ &= \frac{1}{2}(|0\rangle + |1\rangle) + \frac{1}{2}(|0\rangle - |1\rangle) = |0\rangle \\ H|+\rangle &= |0\rangle \end{aligned}$$

(b) $H|-\rangle$

$$\begin{aligned} H|-\rangle &= H(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)) \\ &= \frac{1}{\sqrt{2}} \left(\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right) (\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)) \\ &= \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle - \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle \\ &= \frac{1}{2}(|0\rangle + |1\rangle) + \frac{1}{2}(-|0\rangle + |1\rangle) = |1\rangle \\ H|-\rangle &= |1\rangle \end{aligned}$$