FYMM/MMP IIIb 2020 Problem Set 5

Please submit your solutions for grading by **Monday 30.11.** in Moodle.

1. Let θ, ϕ be the polar coordinates. Introduce the complex numbers $z, \bar{z}$, where

$$z = e^{i\phi}\tan(\theta/2) \equiv \xi + i\eta \quad , \tag{1}$$

and ξ, η are real numbers. Show that the metric of the two-sphere transforms as

$$\begin{aligned} ds^2 &= d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi \\ &= \frac{2}{(1+|z|^2)^2}(d\bar{z} \otimes dz + dz \otimes d\bar{z}) \\ &= \frac{2}{(1+\xi^2+\eta^2)^2}(d\xi \otimes d\xi + d\eta \otimes d\eta) \end{aligned}$$

and the area ("volume") 2-form ω transforms as

$$\begin{aligned} \omega &= \sin\theta d\theta \wedge d\phi \\ &= \frac{2i}{(1+|z|^2)^2}(dz \wedge d\bar{z}) \\ &= \frac{4}{(1+\xi^2+\eta^2)^2}(d\xi \wedge d\eta) \ . \end{aligned}$$

2. Let X, Y be vector fields and f a function on M. Calculate the "double covariant derivatives"

   i) $\nabla_X \nabla_Y f$,

   ii) $\nabla_\mu \nabla_\nu f$

   i.e, write them as sums of terms that involve partial derivatives and connection coefficients (if needed).

3. **Geodesics on a torus**. Let the metric of the torus T^2 with radii r and $R > r$ be

$$g = r^2 d\theta \otimes d\theta + (R + r\cos\theta)^2 d\phi \otimes d\phi \ , \tag{2}$$

where $\theta, \phi \in [0, 2\pi]$.

Find the geodesic equation(s) on a torus with the metric (2), by

i) using the variational principle and the action of a free massive point particle (the expression for the length of a curve, see section 5.9 of lecture notes). As a first step, substitute the torus metric into the (suitable) Lagrangian.

ii) calculating the Christoffel symbols using the formula on p. 72 of the notes and substituting them to the general geodesic equation.

Which method do you think would be easier? Next,

iii) Attempt to solve the equations (this is hard, you probably get stuck at $\dot{\theta} = f(\theta)$), and

iv) sketch some examples of geodesics on a torus.

4. Find all components of the matrix exponential $\exp(i\alpha A)$, $\alpha \in \mathbb{R}$, for the following two choices of the matrix A: do it both for $A = A_3$ and for $A = A_2$, where

$$A_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \quad A_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} .$$

(Recall that the matrix exponential can be defined by using matrix product in the series representation of the exponential: $\exp(M) = \sum_{n=0}^{\infty} \frac{1}{n!} M^n$. If M is diagonalizable, it coincides with exponentiation of the eigenvalues.)

This example will be relevant to both spin in quantum mechanics, and for examples of Lie groups in the last week of the course.