FYMM/MMP IIIb Solutions to Exam Homework Dec 15 2020

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Exercise 1

1. $M_1 = \mathbb{R}^2 \backslash \{(0,0)\}$.
   For this S^1 is a retract of M_1. A topological group is simple connected if and only if X is path connected and the fundamental group is trivial, $\pi_1(X) = \{e\}$
   For M_1 we are on a 2D space. $\mathbb{R}^2$ is contractible and its fundamental group is $\{e\}$, i.e trivial. But for M_1 we have $\mathbb{R}^2$ minus a point at $(0,0)$. The $\mathbb{R}^2 \backslash \{(0,0)\}$ deformation retracts on S^1 via the homotopy $H : (\mathbb{R}^2 \backslash \{(0,0)\} \times [0,1]) \to S^1$. As a result
   $$\Pi_1(\mathbb{R}^2 \backslash \{(0,0)\}) = \Pi_1(S^1) = \mathbb{Z}$$

2. $M_2 = \mathbb{R}^2 \backslash \{(0,0),(1,0),(0,1)\}$
   If we consider some standard coordinates in $\mathbb{R}^2$ i.e $(x_1 \dots x_n) \in \mathbb{R}^3$ and that $(0,0) \in \{x_1 > 0\}$ and $(1,0) \in \{x_1 < 0\}$. And we have a variable $\varepsilon > 0$ such that
   $$H = \{(x_1 \dots x_3) \in \mathbb{R}^2 : |x_1| < \varepsilon\}$$
   Which is disjoint from the points. Consider the open subsets of $\mathbb{R}^2 \backslash \{(0,0),(1,0),(0,1)\}$
   $$A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > -\varepsilon\} \backslash \{(0,0),(1,0),(0,1)\}$$
   $$B = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < \varepsilon\} \backslash \{(0,0),(1,0),(0,1)\}$$
   Clearly the union of these two subsets is $\mathbb{R}^2 \backslash \{(0,0),(1,0),(0,1)\}$. If we denote the points in A and B as k_A, k_B since none of the points is contained in $A \cap B$ so $k_A + k_B = k$. We can now use Van Kampen's theorem for fundamental groups to say that
   $$\Pi_1(\mathbb{R}^2 \backslash \{(0,0),(1,0),(0,1)\}) = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$$

3. $M_3 = S^2$. On a sphere we can continuously deform a loop around it so
   $$\Pi_1(S^2) = \{e\}$$

4. $M_4 = S^2 \backslash \{(1,00)\}$. This is a sphere with a point removed at the equator. A sphere and a point are simply connected. Removing a point doesn't change the fact that you can go around the hole as there are two directions for the band to move across so it can be made into a single point thus
   $$\Pi_1(S^2 \backslash \{(1,0,0)\}) = \{e\}$$

5. $M_5 = S^2 \backslash \{(0,0,-1),(0,0,1)\}$
   $$\Pi_1(M_5) = \mathbb{Z}$$
   This manifold is simply connected.

Exercise 2

Assuming f is a smooth function. A tensor of type (0,1) is a vector field. Applying ∇_X to a tensor yields a tensor of the same type. The exterior derivative is a map $\Omega^r(M) \to \Omega^{r+1}(M)$ and has the property that $d^2 = 0 \to d_{r+1}d_r = 0$

$$T = \nabla_X(df) \to (1,1)q = 1, r = 1$$

$$U = d(\nabla_X f) \to (1,1)q = 1, r = 1$$

$$V = d(L_X(df)) = L_X(d^2 f) = L_X 0 = 0 \to (0,0), q = 0, r = 0$$

$$W = L_X(d(df)) = L_x(0f) = 0 \to (0,0), q = 0, r = 0$$

Exercise 3

1. X is a smooth vector field on M because we can assign a vector $v(p)$ to every point on the manifold i.e X is a linear map

$$X : C^\infty(M) \to C^\infty(M)$$

However, byy adding the (removal of) North and South pole points to the manifold, we cannot have a smooth vector field as we no longer have continuous derivatives i.e it is not differentiable everywhere now. M is of course a smooth manifold.

2. To compute the flow generated by the vector field we need to solve an ODE in coordinates. So, suppose we have some points $p = (\theta\sigma, \phi\sigma)$ and we want the integral curve y with $y(\sigma) = p$, i.e $y(t) = (\theta(t), \phi(t))$. The condition that y be an integral curve gives us

$$y'(t) = X_{y(t)}$$

Therefore

$$\theta'(t) = 0$$

And

$$\phi'(t) = \frac{1}{\sin(\theta(t))}$$

Now θ must be constant $\theta\sigma$ so

$$\phi'(t) = \frac{1}{\sin(\theta\sigma)}$$

is constant. And

$$\phi(t) = \frac{t}{\sin(\theta\sigma) + \phi\sigma}$$

This means that the integral curve at $p = (\theta\sigma, \phi\sigma)$ is

$$y(t) = (\theta\sigma, \frac{t}{\sin(\theta\sigma)} + \phi\sigma)$$

The flow is then

$$F(p,t) = (\theta, \frac{t}{\sin(\theta)} + \phi)$$

I belive this is the maximal flow if $y(t,p) = \phi_p(t)$ for (t,p) in an open subset of $\mathbb{R} \times M$, which we have. The vector field is well defined for $\sin(\theta\sigma) = \sigma$, therefore the maximal flow is not complete.

Exercise 4

1.
$$M = \{(u,v) | uv < 1\}$$
$$ds^2 = -\frac{1}{1-uv)^2}(du \otimes dv + dv \otimes dv)$$

   A pseudo-Riemannian manifold has a pseudo-Riemannian metric which is a metric which is symmetric and non-degenerate. As we can see this is a manifold with signature 0 and therefore is pseudo-riemannian. I think it may actually be Lorentizan as it is 2 dimensional.

2. The Levi-Civita connection ∇ associated with the metric has the property that $\nabla_g g = 0$ and the non-zero Christoffel symbols are

$$\Gamma^u_{uu} = -\frac{v}{1-uv}$$

$$\Gamma^u_{uv} = -\frac{u}{1-uv}$$

$$\Gamma^u_{vv} = \frac{v}{1-uv}$$

$$\Gamma^v_{uu} = \frac{u}{1-uv}$$

$$\Gamma^v_{uv} = -\frac{v}{1-uv}$$

$$\Gamma^v_{vv} = -\frac{u}{1-uv}$$

   I have skipped those that can be deduced by symmetry.

3. Which of the following curves are geodesics

$$c(t) = (0,t), c(t) = (t,-t), c(t) = (e^t, -e^{-t})$$

For this I calculated the associated tangent vector fields for each curve w.r.t the manifold and checked to see if

$$\nabla_V V = 0$$

$$\frac{d\nu^i}{dt} + \Gamma^i_{jk}\nu^i\nu^k = 0$$

The curve is a geodesic if $\gamma(t) = 0$.

$$c(t) = (0,t) \Rightarrow \gamma_1' = \frac{\partial}{\partial v}$$

$$c(t) = (t,-t) \Rightarrow \gamma_2' = \frac{\partial}{\partial u} - \frac{\partial}{\partial v}$$

$$c(t) = (e^t, -e^{-t}) \Rightarrow \gamma_3' = e^t\frac{\partial}{\partial u} + e^{-t}\frac{\partial}{\partial v}$$

So we have

$$\gamma_1(t) = -t\frac{\partial}{\partial u}$$

$$\gamma_2(t) = (-\frac{2t}{t^2+1})\frac{\partial}{\partial u} + (\frac{2t}{t^2+1})\frac{\partial}{\partial v}$$

$$\gamma_3(t) = (\frac{1}{2}e^{-3t} + \frac{3}{2}e^t)\frac{\partial}{\partial u} + (-\frac{1}{2}e^{3t} - \frac{3}{2}e^{-t})\frac{\partial}{\partial v}$$

From my calculations, none of these are geodesics. For this question I heavily used Mathematica referencing the Problem Set 5 solutions to calculate the coefficients and various derivatives.

Exercise 5

Describe the classifications of all irreducible unitary finite-dimensional representations of the simple Lie algebra $\mathfrak{su}(3)$ (or $\mathfrak{su}(2)$) of the simple connected compact simple Lie group $SU(3)$ or $SU(2)$. I am not entirely sure what the classifications are, however, I will attempt to answer as best I can, after reading appendix A.4 in the lecture notes. I am not sure but I believe the theorem of the highest weight plays a substantial part, however this may be for semi-simple Lie groups.

To describe the classifications of all the irreducible representations we first need the Cartan matrix. As the defining representation of $\mathfrak{su}(3)$ consists of 3x3 trace 0 unitary matrices we can easily find the cartan matrix. The basis of generators T^a is normalzied such that $Tr(T^aT^b) = \frac{1}{2}\delta^{ab}$ for $a = 1 \ldots 8$. The Cartan generators H_m are easily found

$$H_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\lambda_3$$

Where λ_a are the Gell-Mann matrices.

$$H_2 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \frac{1}{2\sqrt{3}}\lambda_8$$

In the definining representation we have basis vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

To find the weights μ we find the eigenvalues of the Cartan generators in the basis of the defining representation, giving

$$\mu^1 = (\frac{1}{2}, \frac{1}{2\sqrt{3}})$$

$$\mu^2 = (-\frac{1}{2}, \frac{1}{2\sqrt{3}})$$

$$\mu^3 = (0, -\frac{1}{\sqrt{3}})$$

The raising and lowering operators of SU(3)

$$E^1_\alpha = i(\lambda_1 + i\lambda_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E^2_\alpha = i(\lambda_4 + i\lambda_5) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E^3_\alpha = i(\lambda_6 + i\lambda_7) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E^1_{-\alpha} = i(\lambda_1 - i\lambda_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E^2_{-\alpha} = i(\lambda_4 - i\lambda_5) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$E^3_{-\alpha} = i(\lambda_6 - i\lambda_7) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

To find the roots (denoted α) of SU(3) we find the weight states of the adjoint representation

$$[H_1, E^1_\alpha] = E^1_\alpha$$

$$[H_2, E^1_\alpha] = 0$$

Therefore the roots for $\alpha^1_+ = (1, 0)$ and $\alpha^1_- = (-1, 0)$. For the second generator

$$[H_1, E^2_\alpha] = \frac{1}{2} E^2_\alpha$$

$$[H_2, E^2_\alpha] = \frac{\sqrt{3}}{2} E^2_\alpha$$

Therefore the roots are $\pm\alpha^2 = \pm(\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $\pm\alpha^3 = \pm(-\frac{1}{2}, \frac{\sqrt{3}}{2})$. From these we can find the simple roots (denoted $\vec{\alpha}$)

$$\vec{\alpha}^1 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$$

$$\vec{\alpha}^2 = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$$

The fundamental weights (denoted $\vec{q}$) are found from

$$2\frac{\alpha^i \vec{q}^j}{|\alpha|^2} = \delta^{ij}$$

On SU(3) these represent the 3and $\bar{3}$ irreducible representations so we have

$$(\frac{1}{2}, \frac{1}{2\sqrt{3}}), (\frac{1}{2}, -\frac{1}{2\sqrt{3}})$$

From these the cartan matrix can be found

$$2\vec{\alpha}_1\vec{\alpha}_1 = -2$$

$$2\vec{\alpha}_1\vec{\alpha}_2 = -1$$

$$2\vec{\alpha}_2\vec{\alpha}_2 = -2$$

The cartan matrix is then

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

We can build another unitary representation from this by choosing another H_m and $E_{\pm\alpha}$ such that

$$H_1 = \frac{1}{2}\begin{pmatrix} 1 & - & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$H_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$E_{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_{\alpha_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_{\alpha_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

WIth weights

$$\mu^1 = (\frac{1}{2}, 0), \mu^2 = (-\frac{1}{2}, \frac{1}{2}), \mu^3 = (0, -\frac{1}{2})$$

And roots

$$\alpha_1 = (1, -\frac{1}{2}), \alpha_2 = (\frac{1}{2}, \frac{1}{2}), \alpha_3 = (-\frac{1}{2}, 1)$$

The simple roots are

$$\alpha^{12} = \alpha_1 - \alpha_2 = (\frac{1}{2}, 1), \alpha^{23} = \alpha_2 - (-\alpha_3) = (0, -\frac{3}{2})$$

The fundamental highest weights q^j are given by

$$q^1 = (\frac{1}{2}, 0), q^2 = (0, -\frac{1}{2})$$

In terms of the classification theorem (Appendix A.4) I believe these are represented by the A_n and this corresponds to the same Dynkin diagram given in the notes.