FYMM/MMP III Answers to Problem Set 1

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1. Consider the following constructions; check each one whether it is a semigroup, monoid, group or none of them. Why?

   - The set of real numbers $\mathbb{R}$, with raising to power as multiplication: $x \cdot y \equiv x^y$, $x, y \in \mathbb{R}$.
   - The set of positive natural numbers $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$ with the greatest common divisor of $m, n \in \mathbb{N}_+$ as their product: $m, n \in \mathbb{N}_+$.
   - The set of nonzero rational numbers $\mathbb{Q} \setminus \{0\}$, with the usual product as multiplication: $(m/n) \cdot (p/q) = (mp/nq)$.

   (a) ***Answer.*** Set of Reals $\mathbb{R}$ with multiplication $\circ$

   $$x \cdot y \equiv x^y; x, y \in \mathbb{R}$$

   Meaning

   $$\circ : x \cdot y \equiv x^y$$

   This has no classification, not even a magma as to be a magma it requires that for all $a, b \in G$, $a \cdot b$ must also be in G. As such, there are some $x, y \in \mathbb{R}$ to which this does not apply, e.g,

   $$(-1) \circ \frac{1}{2} = i; (-1)^{\frac{1}{2}} = \sqrt{-1} = i \rightarrow i \notin \mathbb{R}$$

   $$0 \circ 5 = 0^5 = Undefined \rightarrow \notin \mathbb{R}$$

   (b) $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$;$m, n \in \mathbb{N}_+$; $m \cdot n \equiv \gcd(m, n)$ e.g

   $$\gcd(8, 12) = 4$$

   This is a magma because the GCD of any two numbers in $\mathbb{N}_+$ is in $\mathbb{N}_+$ since the set of of common divisors is always a subset of $\mathbb{N}_+$ for every $m, n \in \mathbb{N}_+$ and is always less than or equal to the lowest number of m, n.
   The set is a semigroup due to associativity holding:

   $$a, b, c \in G; a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

   In our case For any $m, n, p \in \mathbb{N}_+$

   $$(m \circ n) \circ p = \gcd(\gcd(m, n), p)$$

$$\gcd(\gcd(4,8),12) = \gcd(\gcd(8,12),4)$$

The set, however, is not a monoid as there is no existence of the unit element. For example, say there exists an element $e \in \mathbb{N}_+$ and there exists a natural positive number such that $e+1 \in \mathbb{N}_+$ then, assuming e is a unit element, the $\gcd(e, e+1) = e+1$ (with $e+1 > e$) which doesn't make any sense as the divisor of the number should at a minimum be the size of the number. Proof that the unit element doesn't exist by contradiction.

(c) $\mathbb{Q}\backslash 0$ with $\circ : (\frac{m}{n}) \cdot (\frac{p}{q}) = \frac{mp}{nq}$ This is closed and a magma, with $m, n, r, s \in \mathbb{Q}\backslash 0$:

$$(p,q) = (\frac{m}{n}, \frac{r}{s}) \rightarrow \frac{mr}{ns}$$

For associativity:

$$(p \circ q) \circ a = (\frac{mr}{ns})\frac{b}{c} = \frac{mrb}{nsc}$$

$$p \circ (q \circ a) = \frac{m}{n}(\frac{r}{s}\frac{b}{c}) = \frac{mrb}{nsc}$$

Existence of the unit element:

$$e \circ p = \frac{em}{en} = \frac{m}{n} = p$$

Existence of the inverse:

$$p^{-1} \circ p = \frac{n}{m} \circ p = \frac{nm}{mn} = e$$

and m must $\neq 0$
Communitivity:

$$p \circ q = \frac{mr}{ns} = \frac{r}{s} \circ \frac{m}{n} = q \circ p$$

Therefore it is an Abelian Group.

2. Show that $|S_N| = N!$.

(a) ***Answer.*** $|S_N|$ means the order of the group and is the smallest number n such that $g^n \equiv e$. In a symmetric group of N elements there are N ways to choose the position of the first element, $N-1$ ways to choose the position of the second element, $N-2$ for the third, and so on.

$$\therefore N \times (N-1) \times (N-2) \times \ldots 1 = N!$$

This is one-to-one mapping (injection) as well as being onto (surjection) meaning they are all bijections. The sum of all of these elements in $N!$ and thus the order

3. Consider the group $G = \{e, x_1, x_2, x_3, x_4, x_5\}$, where

$$e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; ; \; x_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$x_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \; ; \; x_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$x_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \; ; \; x_5 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \, ,$$

and the law of composition is the matrix multiplication. Show that G is isomorphic to a known group, give an explicit construction of the isomorphism.

(a) ***Answer.***
The Group

$$G = \{e, x_1, x_2, x_3, x_4, x_5\}$$

can be shown to be isomorphic to S_3 by defining a map $i : G \to S_3$

$$i(e) = (); i(x_1) = (12); i(x_2) = (13); i(x_3) = (23); i(x_4) = (132); i(x_5) = (123)$$

Using one line notation these are bijections.
This can be shown through the Cayley table to match S_3 using matrix multiplication and matching to the above matrices.

$$\begin{pmatrix} e & x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1 & e & x_4 & x_5 & x_2 & x_3 \\ x_2 & x_5 & e & x_4 & x_3 & x_1 \\ x_3 & x_4 & x_5 & e & x_1 & x_2 \\ x_4 & x_3 & x_1 & x_2 & x_5 & e \\ x_5 & x_2 & x_3 & x_1 & e & x_4 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 4 & 5 & 2 & 3 \\ 2 & 5 & 0 & 4 & 3 & 1 \\ 3 & 4 & 5 & 0 & 1 & 2 \\ 4 & 3 & 1 & 2 & 5 & e \\ 5 & 2 & 3 & 1 & 0 & 4 \end{pmatrix}$$

Which shows that the map i is a group homomorphism. As there is a bijection and group homomorphism it can be said that $G \cong S_3$ and i is an isomorphism.

4. An equilateral triangle is symmetric under reflections, with the line passing through the center and one of the vertices as the reflection axis; and symmetric under 120 degree counterclockwise rotations (with the center as the fixed point). Let e be the identity map (do nothing), a a rotation by 120 degrees, and b the above mentioned reflection. Consider the group generated by e, a ja b with composition of symmetry operations as the multiplication rule. What is the order of the group? (Hint: greater than three.) Construct the multiplication table (Cayley table) of the group.

   (a) ***Answer.***
   By looking at the operations on the equilateral triangle we can see that the order of the group is at least 6 as there are 6 operations which give a unique triangle. The order of the group is also at a maximum, 6 as $3! = 6$ through permutations of different vertices. The difference transformations are

$$e = \text{identity map}$$

$$a = \text{rotation of } 120^\circ$$

$$b = \text{Reflection down the symmetrical line}$$

   With operations e, a, a^2, b, ab, ba
   The original triangle is labelled

$$e = \overset{1}{\underset{3\ldots2}{\triangle}}$$

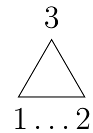
   The configurations of the triangle are then

$$ae = \overset{3}{\underset{2\ldots1}{\triangle}}$$

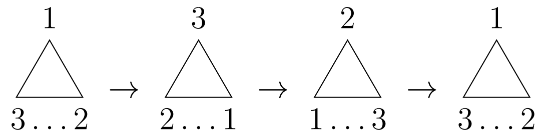
$$a^2e = \overset{2}{\underset{1\ldots3}{\triangle}}$$

$$be = \overset{1}{\underset{2\ldots3}{\triangle}}$$

$$abe = \overset{2}{\underset{3\ldots1}{\triangle}}$$

$bae =$ 

All other products simplify to these operations, shown in the Cayley table. For example

$a^3 =$  $= e$

Also $b^2 = e, (ab)^2 = e, (ba)^2 = e, a^2b = ba, ba^2 = ab, aba = b, ba^2 = ab, a^4 = a, a^3b = b, aba^2 = ba, ba^3 = b, ab^2 = a, bab = a^2, (ab)(ba) = a^2, (ba)(ab) = a, b^2a = a, a^2ba = ab.$
Note that $abe \neq bae$ and is not abelian, thus S_3.
The Cayley table is:

$$\begin{pmatrix} e & a & a^2 & b & ab & ba \\ a & a^2 & e & ab & ba & b \\ a^2 & e & a & ba & b & ab \\ b & ba & ab & e & a^2 & a \\ ab & b & ba & a & e & a^2 \\ ba & ab & b & a^2 & a & e \end{pmatrix}$$