FYMM/MMP IIIb 2020 Problem Set 3

Please submit your solutions for grading by **Monday 16.11.** in Moodle.

1. We define a map $f : S^1 \to S^2$ such that in local coordinates $\phi \in (0, 2\pi)$ on S^1 and $(\theta, \varphi) \in (0, \pi) \times (0, 2\pi)$ on S^2 it is

$$\phi \mapsto f(\phi) = (\theta(\phi), \varphi(\phi)) = (\frac{1}{2}\phi, \phi) \ .$$

Consider the tangent vector

$$V = \dot{\phi}(t) \frac{\partial}{\partial \phi}$$

of the curve $c(t) = \phi(t)$ on S^1, $t \in (0, 2\pi)$. Calculate V and its push $f_* V$ explicitly when the curve is

(a) $c(t) = \phi(t) = at$ where $a > 0$ is a constant

(b) $c(t) = \phi(t) = 2\pi \sin t$.

Can you visualize the vector field $f_* V$? (You don't have to draw it, but think about how it would look like on S^2.)

2. Let X, Y and Z be smooth vector fields on a differentiable manifold M and f a smooth function on M. Consider the Lie bracket $[X, Y]$ which is defined

$$[X, Y]f = X(Yf) - Y(Xf).$$

(Smooth) vector fields can be expressed in a coordinate basis as $X = X^\mu \partial_\mu$, where X^μ is a smooth function and $\partial_\mu = \frac{\partial}{\partial x^\mu}$. Show that XY is not a vector field (i.e. it cannot be expressed in the form given above). Show that $[X, Y]$ is a smooth vector field and write down its expression with derivatives, X^μ's and Y^μ's. Finally, show that the following identities are true

$$[X, fY] = (Xf)Y + f[X, Y] \text{ and } [X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0.$$

The latter is the Jacobi identity, which you might know from quantum mechanics. It can be shown that smooth vector fields and the Lie bracket form a Lie algebra. We will return to Lie algebras later on. (*Hint:* Despite the long explanations, the problem is a straightforward calculation.)

3. **Hamilton's Equations as an Example of a Flow Generated by Vector Field.** It should be familiar from Classical Mechanics that a system with N degrees of freedom can be described by generalized coordinates q_i and canonical momenta p_i, where $i = 1, \ldots, N$. The generalized coordinates and canonical momenta can be thought as coordinates of a $2N$-dimensional manifold M, called the *phase space.* The dynamics of the system is given by the Hamiltonian, $H : M \to \mathbb{R}$, $H = H(q_1, \ldots, q_N, p_1, \ldots, p_N)$; its equations of motion can be written as a group of first order differential equations

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}; \ \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}.$$

These are called the Hamilton's equations. We will next reformulate them in a different way. Let's define a vector field X_H in the phase space M,

$$X_H = \sum_{i=1}^{N} \left\{ \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right\}.$$

The vector field X_H gives rise to integral curves $x_H(t) = (q_1(t), \ldots, q_N(t), p_1(t), \ldots, p_N(t))$ on the manifold M.

   (a) Show that the equation defining the integral curves $x_H(t)$ is equivalent to Hamilton's equations.

   (b) Let $M = \mathbb{R}^2$, i.e. $N = 1$, and $H = \frac{1}{2}(p^2 + q^2)$. Find X_H and the generated flow $\sigma(t, x_0)$ where $x_0 = (q_0, p_0) = (1, 0)$. Illustrate it by a figure.

   (c) As before, but now with $H = \frac{1}{2}(p^2 - q^2)$ and $x_0 = (1, 1)$.

   (d) Now $M = T^2$, with coordinates $q, p \in [0, 2\pi]$, and $H = \cos(p)$. Find the equation of the integral curve. Draw a figure of the curves on T^2.

4. Let

$$X = X^\mu(x) \frac{\partial}{\partial x^\mu}$$

be a vector field and

$$g = g_{\mu\nu}(x) dx^\mu \otimes dx^\nu$$

be a (0,2)-tensor. Calculate the Lie derivative $\mathcal{L}_X g$ following the examples in the lecture notes.