FYMM/MMP IIIa 2020 Solutions to Problem Set 5

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1. Show that

$$P_n \equiv \{a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n | a_0, a_1, \ldots, a_n \in \mathbb{C}^n\}$$

is a vector space.
From the vector space axioms we can show that this is a vector space.

(a) Closure under addition

$$a_n Z^n + b_n Z^n = (a_n + b_n) Z^n$$

(b) Closure under multiplication

$$ca_n Z^n = (ca_n) Z^n$$

(c) Associativity

$$(a_n Z^n + b_n Z^n) + c_n Z^n = (a_n + b_n + c_n) Z^n$$
$$= a_n Z^n + cb_n Z^n + c_n Z^n$$

(d) Identity element of addition

$$a_n = 0$$

for all n

$$a_n Z^n + 0 = a_n Z^n$$

(e) Inverse elements of addition

$$(a_n^{-1} = -a_n) a_n Z^n + (-a_n) Z^n$$

$$= (a_n - a_n) Z^n = 0$$

(f) Commutivity of addition

$$a_n Z^n + b_n Z^n = (a_n + b_n) Z^n$$

$$= (b_n + a_n) Z^n = b_n Z^n + a_n Z^n$$

(g) Distributivity of scalar multiplication with respect to vector addition

$$c(a_n Z^n + b_n Z^n) = c(a_n + b_n) Z^n$$

$$= (ca_n) Z^n + (cb_n) Z^n$$

(h) Distributivity of scalar multiplication with respect to field addition.

$$(c+d)a_n Z^n = ca_n Z^n + Da_n Z^n$$

(i) Identity element of scalar multiplication

$$1 \cdot a_n Z^n = a_n Z^n$$

(j) Associativity of scalar multiplication

$$(cd)a_n Z^n = (cda_n)Z^n = c(da_n)Z^n$$

The dimension in the complex field is

$$dim P_n = n(n+1)$$

2. Find a faithful representation of $\mathbb{Z}_6$ in $\mathbb{R}^2$, thinking of group elements generated by anticlockwise 60 degree rotations.
60 degree $= \frac{\pi}{3}$. Rotation denoted R_θ. Representation

$$D : \mathbb{Z}_6 \to Aut(\mathbb{R}^2)$$

Where

$$\mathbb{Z}_6 = \{e, a, a^2, a^3, a^4, a^5\}$$
$$a^n \to R_{n\frac{\pi}{3}}$$

Because we have a rotation matrix

$$D(a^n \cdot a^m) = D(a^{(n+m)\ mod\ 6})$$

So that

$$R_{\frac{\pi}{3}[(n+m)\ mod\ 6]}$$
$$= R_{n\frac{\pi}{3}+m\frac{\pi}{3}\ mod\ 2\pi}$$
$$= R_{n\frac{\pi}{3}} R_{m\frac{\pi}{3}} = D(a^n)D(a^m)$$

So that D is a homomorphism.

$$ker D = \{a^n \in \mathbb{Z}_6 | R_{n\frac{\pi}{3}} = \mathbb{1}\}$$
$$= \{a^n \in \mathbb{Z}_6 | n = 0\ mod\ 6\}$$
$$= \{a^0\} = \{e\}$$

So D is a faithful representation

3. Show that $SL(n, \mathbb{R})$ is a normal subgroup of $GL(n, \mathbb{R})$, and identify the quotient group $GL(n, \mathbb{R})/SL(n, \mathbb{R})$.
   The first isomorphism theorem:

   Theorem 1 *Let G and H be two groups and $\phi : G \to H$ be a group homomorphism. Then $ker\phi$ is a normal subgroup of G and*

   $$G/ker\phi \cong Im\phi$$

   So we have

   $$\phi : GL(n, \mathbb{R}) \to \mathbb{R}\backslash\{\vec{0}\}$$

   Is a determinant map such that $A \to det(A)$. Determinants of $GL(, \mathbb{R})$ are non zero and determinats of $\mathbb{R}$ are real so ϕ works as a map here. So we have

   $$\phi(AB) = det(AB) = det(A)det(B) = \phi(A)\phi(B)$$

   So $\phi : GL(n, \mathbb{R}) \to \mathbb{R}\backslash\{\vec{0}\}$ is a homomorphism and

   $$ker\phi = \{A \in GL(n, \mathbb{R}) | det(A) = 1\}$$

   which is equivalent to the special linear transform group $SL(n, \mathbb{R})$. By the first isomorphism theorem then

   $$GL(, \mathbb{R})\backslash ker\phi = GL(n, \mathbb{R})\backslash SL(n, \mathbb{R}) \cong \mathbb{R}\backslash\{\vec{0}\}$$

4. Show that all group elements belonging to the same conjugacy class have the same order of element.
   In a group G, two elements h and g are conjugate when

   $$h = xgx^{-1}$$

   where $x \in G$. To show that they have the same order we need to show that g and xgx^{-1} have the same order.
   In a group where $(xgx^{-1})^n = xg^nx^{-1}$ for $n > 0$

   $$(xgx^{-1})^n = xg^nx^{-1} \forall n \in \mathbb{Z}^+$$

   If $g^n = 1$ then $(xgx^{-1})^n = xg^nx^{-1} = xx^{-1} = e$, and if $(xgx^{-1})^n = 1$ then

   $$xg^nx^{-1} = e$$

   so

   $$g^n = xx^{-1} = e$$

   so

   $$(xgx^{-1})^n = 1$$

   If and only if $g^n = 1$ so g and xgx^{-1} have the same order.

5. Show that a linear map $L : V \to V$ is an automorphism if and only if Ker $L = \{0\}$.

   Let $L \in Aut(V)$ and $x \in kerL$ such that $L(x) = 0$ and $L(x) = L(0)$. L is an automorphism meaning it is an injection and $x = 0$ and $kerL = \{0\}$.
   Now suppose that $L : V \to V$ is an automorphism, which means proving that it is a bijection, thus proving surjectivity and injectivity:
   For surjectivity suppose we have $\forall y \in Y \exists x \in X s.t f(x) = y$. Suppose we have a basis for V as $\{v_i\}$ then $L(v_i)$ is also a basis of V as

$$\sum_{i=1}^{n} a_i L(v_i) = L\Big(\sum_{i=1}^{n} a_i v_i\Big)$$

$$= \sum_{i=1}^{n} a_i v_i = 0$$

   a_i is such that $a_i = 0$. If we have a $u \in V$ such that u_i is in the basis $\{v_i\}$ and $L(v_i)$ then

$$U = \sum_{i=1}^{n} u_i L(v_i) = L\Big(\sum_{i=1}^{n} u_i v_i\Big)$$

   So that $u_i v_i \in V$
   For injectivity suppose we have a function $f(x) \neq f(x') \forall x \neq x'$ and we have $x, y \in V s.t L(x) = L(y)$ and $L(x-y) = 0$ due to $x - y \in kerL$ and $kerL = \{0\}$ so $x - y = 0$ and $x = y$.
   There is injection and surjection proved so there exists a bijection and $L : V \to V$ is a automorphism.