FYMM/MMP IIIa 2020 Solutions to Problem Set 3

Jake Muff
20/9/20

1. Question 1

   (a) Show that it is a group.
   For associativity and closed conditions (G0 & G1) we can say:
   For $M, N \in Sp$

   $$(MN)^T\Omega(MN) = N^T M^T \Omega M N = N^T \Omega N$$
   $$= \Omega$$

   This $M \cdot N \in Sp$. And associativity follows from this.
   Proving the existence of the unit element (G2):

   $$I^T \Omega I = \Omega \to I \in Sp$$

   And existence of the inverse(G3): Suppose that $M \in Sp$ and $M^T\Omega M = \Omega$ such that

   $$M^T\Omega M = (M^{-1})^T M^t \Omega M M^{-1}$$
   $$= (MM^{-1})^T \Omega M M^{-1}$$
   $$= I^T \Omega I = \Omega$$

   And $M^{-1} \in Sp$, thus, it is a group!

   (b) Show that dim $Sp(2n, \mathbb{R}) = n(2n+1)$.
   $M \in Sp$ can be thought of as a block matrix or "partitioned"matrix where each element is a matrix in itself or submatrix. Representing this with the symplectic condition

   $$\begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} \Omega \begin{pmatrix} A & B \\ C & D \end{pmatrix} = M^T \Omega M$$

   If we think of the $2n$ columns of $M^T\Omega M$ such that $M \in Sp$ with columns $a_1 \dots a_n$ and $b_1 \dots b_n$ then there are $(2n)^2$ constraints which are:

   $$a_i^T \Omega a_j = 0$$
   $$b_i^T \Omega b_j = 0$$
   $$a_i^T \Omega b_i = \delta_{ij} \text{ and } a_i^T \Omega b_i = \delta ij$$

   For the first two there are only $\frac{n^2-n}{2}$ independent equations for each one of them. For the two delta function equations there are only n^2 independent constraints. So we have:

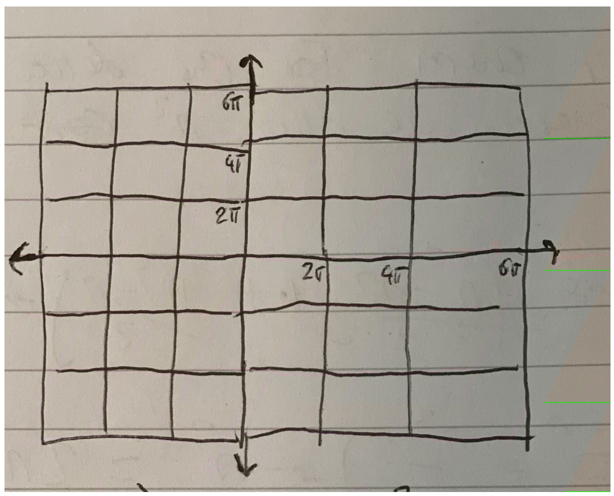
   $$(2n)^2 - (\frac{n^2-n}{2} + \frac{n^2-n}{2}) - n^2 = 2n^2 + n$$

2. Question 2
   (x, y) and (x', y') are in $\mathbb{R}^2$ and equivalent. If there exists a $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ such that:

$$x' = x + 2\pi n$$

$$y' = y + 2\pi m$$

   Note that the 2π is not included in the area and is part of the next square i.e $(0, 0)$ $(2\pi, 2\pi)$.
   Therefore $\mathbb{R}^2/G = [0, 2\pi]/\{0, 2\pi\}$



Kuva 1: Plot for question two showing how the action represents squares on an xy plane

3. Question 3
   To answer whether there is a subgroup H of $U(5)$ we need to explore the different dimensions of A, as different dimensions of A will give different dimensions of H and thus different properties of H:

   Consider the case where A has $dimA = 0$, meaning that x and y are 0. This is a simple case as H is all of $U(5)$ such that H has dimensions $dimH = 5^2 = 25$. It is also a subgroup.

   Consider the case where A only have 1 dimension, where $x \neq 0$ and $y = 0$ or $x = 0$ and $y \neq 0$ or importantly showing linear dependence $y = cx \neq 0 \leftrightarrow x = cy \neq 0$. A then has unit vector $v = \frac{x}{|x|}$ if x is 0, (if y is 0 then swap for ys) with an orthonormal basis in $\mathbb{C}^5$. In an orthonormal basis there exists a vector $u \in A$ such that

$$u = \begin{pmatrix} a & 0 & 0 & 0 & 0 \end{pmatrix}$$

Of which $a \in \mathbb{C}$. A matrix $B \in H$ in which A is invariant and of course unitary $B \in H \in U(5)$ can be represented by a block matrix with diagonal entries

$$B = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & U_4 \end{pmatrix}$$

Where $e^{i\theta} \in U(1)$ and $U_4 \in U(4)$ and thus H is

$$H = \left\{ B | e^{i\theta} \in U(1) \& U_4 \in U(4) \right\}$$

$$= \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & U_4 \end{pmatrix} | e^{i\theta} \in U(1) \& U_4 \in U(4) \right\}$$

Which shows that H is a proper subgroup with $dimH = dimU(1) + dimU(4) = 17$
Finally, the case where if both x and y don't equal 0 and are linearly independent. Again, if we have a orthonormal basis for A such that a vector $u \in A$ is

$$u = \begin{pmatrix} a & b & 0 & 0 & 0 \end{pmatrix}$$

Then $B \in H$ has the block matrix form with diagonal entries:

$$B = \begin{pmatrix} U_2 & 0 \\ 0 & U_3 \end{pmatrix}$$

So we have

$$H = \left\{ \begin{pmatrix} U_2 & 0 \\ 0 & U_3 \end{pmatrix} | U_2 \in U(2) \& U_3 \in U(3) \right\}$$

Which has dimension $dimH = dimU(2) + dimU(3) = 13$ and is a proper subgroup.

4. Question 4

   (a) In $SL(2, \mathbb{R})$ we have a matrix A such that

   $$A = \begin{pmatrix} x_0 - x_1 & x_2 + x_3 \\ x_2 - x_3 & x_0 + x_1 \end{pmatrix}$$

   Taking the determinant with the constraint $det(A) = 1$

   $$det(A) = (x_0 - x_1)(x_0 + x_1) - (x_2 + x_3)(x_2 - x_3)$$
   $$= x_0^2 - x_1^2 - x_2^2 + x_3^2 = 1$$

   Now we have $-x_1^2$ and $-x_2^2$ which is not in the forn needed as they have negative constants in front of them. Multiplying by -1 to get into the form required gives us

   $$-x_0^2 - x_3^2 + x_1^2 + x_2^2 = -1$$

   Giving $(c_0, c_n, c) = (-1, -1, -1)$ and $SL(2, \mathbb{R})$ is a 3 dimensional anti-de sitter space AdS_n.

(b) Part b wasn't answered fully as I don't really understand Conjugacy Classes
If $A \in SL(2,\mathbb{R})$ we say that the conjugacy class of A is A_{CL}

$$A_{CL} = \Big\{B \in SL(2,\mathbb{R}) | S \in SL(2,\mathbb{R}) : B = S^{-1}AS\Big\}$$

Where S is a symmetric matrix and $B \in A_{CL}$
B can be considered equivalent to A such that $A\ B$ and in terms of traces $Tr(A) = Tr(B)$

5. Given a group G, a left G-space X, and an element $x \in X$, prove that the isotropy group of x is a subgroup of G.
Let G_x by the isotropy group of G. To be a subgroup we have that

$$\forall g_1 g_2 \in G\ g_1 g_2 \in G$$

$$\forall g \in G\ g^{-1} \in G$$

Let $g_1 g_2 \in G_x$ such that the left action $L_{g_1}(x) = x\ L_{g_2}(x) = x$ where L is a homomorphism

$$L_{g_1 g_2}(x) = L_{g_1}(L_{g_2}(x)) = L_{g_1}(x) = x$$

And this $g_1 g_2 \in G_x$. Note that this also proves associativity. For the second property of a subgroup lets state $g \in G_x$ such that

$$L_{g^{-1}g}(x) = L_{g^{-1}}(L_g(x)) = L_{g^{-1}}(x)$$

Thus $g^{-1} \in G_x$.
Hence G_x is a subgroup of G.
Not entirely sure if this is true but if I say that the indentity element of L such that $L_e = id_X$ then we can prove the existence of the identity element in G_x

$$L_e(x) = id_X(x) = x$$

Hence $e \in G_x$ and G_x is a group and thus a subset of G, solidying evidence that G_x is a subgroup of G