Quantum Information B Fall 2020 Solutions to Problem Set 1

Jake Muff

1/11/20

1 Answers

1. Exercise 6.3 Nielsen and Chaung. Show that in the $|\alpha\rangle, |\beta\rangle$ basis we can write the Grover iteration as
$$G = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$
The grover iteration is written as
$$H^{\otimes n} P H^{\otimes n} = 2|\psi\rangle\langle\psi| - I$$
And the Oracle O
$$O = I - 2|\beta\rangle\langle\beta|$$
So that
$$G = O \cdot H^{\otimes n} P H^{\otimes n}$$
$$G = (I - 2|\beta\rangle\langle\beta|)(2|\psi\rangle\langle\psi| - I)$$
$$= 2|\psi\rangle\langle\psi| - I - 4|\beta\rangle\langle\beta||\psi\rangle\langle\psi| + 2|\beta\rangle\langle\beta|$$
We take the case where $M = 1$ so that
$$|\psi\rangle = \sqrt{1 - \frac{1}{N}}|\alpha\rangle + \sqrt{\frac{1}{N}}|\beta\rangle$$
Using the fact that $I = |\alpha\rangle\langle\alpha| + |\beta\rangle\langle\beta|$
$$G = 2\Big[\frac{1}{N}|\beta\rangle\langle\beta| + \frac{\sqrt{N-1}}{N}(|\beta\rangle\langle\alpha| + |\alpha\rangle\langle\beta|) + \frac{N-1}{N}|\alpha\rangle\langle\alpha|\Big]$$
$$-|\beta\rangle\langle\beta| - |\alpha\rangle\langle\alpha| - 4|\beta\rangle\Big(\sqrt{\frac{1}{N}}\Big)\Big(\sqrt{\frac{1}{N}}\langle\beta| + \sqrt{\frac{N-1}{N}}\langle\alpha|\Big) + 2|\beta\rangle\langle\beta|$$
$$G = (1 - \frac{2}{N})(|\beta\rangle\langle\beta| + |\alpha\rangle\langle\alpha|) + \frac{2\sqrt{N-1}}{N}(|\alpha\rangle\langle\beta| - |\beta\rangle\langle\alpha|)$$
If we have $\cos(\theta) = (1 - \frac{2}{N})$ then $\sin(\theta) = \sqrt{1 - \cos(\theta)} = \frac{2\sqrt{N-1}}{N}$, then G is
$$G = \cos(\theta)(|\beta\rangle\langle\beta| + |\beta\rangle\langle\beta|) + \sin(\theta)(|\alpha\rangle\langle\beta| - |\beta\rangle\langle\alpha|)$$
And we have
$$G = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

2. Exercise 6.5. Show that the augmented oracle O' may be constructed using one application of O and elementary quantum gates using the extra quibt $|q\rangle$. The augmented oracle only marks an item if it is a solution and the extra qubit is set to 0. Equation 6.1 from the book tells us that $|q\rangle$ is flipped if $f(x) = 1$ and unchanged otherwise. Much like in Fig 4.11 in the book where an X gate was used we can use the Pauli Z gate as we wanted to leave the qubit in state $|0\rangle$ unchanged and sign flip if the state is $|1\rangle$.
   The application of the normal oracle is

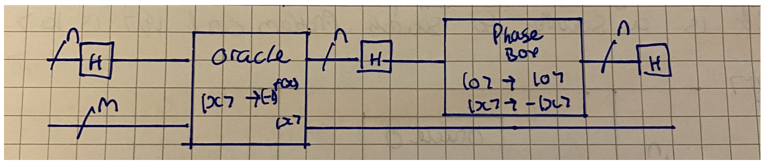

Figure 1: Circuit implementing the Oracle O

   The circuit for the augmented oracle O'. $\frac{|0\rangle - |1\rangle}{2}$ is the initial state. $\backslash n$ represents n wires.

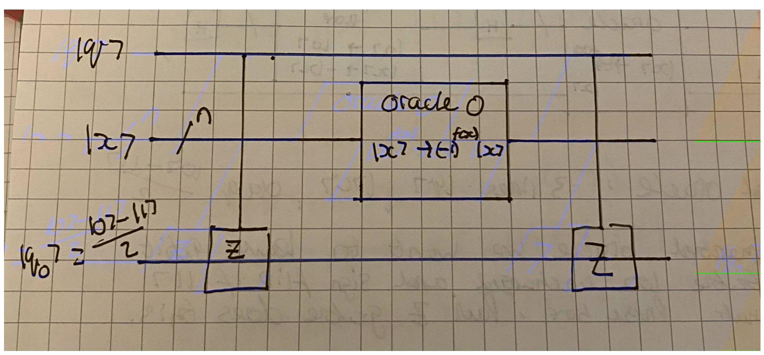

Figure 2: Circuit implementing the Augmented Oracle O'

3. Exercise 6.6. Verify that the gates in the dotted box in the second figure of box 6.1 perform the conditional phase shift operation $2|00\rangle\langle 00| - I$ up to an umimportant phase factor.

$$|x\rangle = |0\rangle^{\otimes n}|0\rangle^{\otimes n} \to a|0\rangle + b|1\rangle$$
$$|y\rangle = |0\rangle^{\otimes n}|1\rangle \to c|0\rangle + d|1\rangle$$

   The initial state is

$$|\psi_0\rangle = (|0\rangle \otimes |0\rangle)(|0\rangle \otimes |1\rangle)$$
$$= (a|0\rangle + b|1\rangle)(c|0\rangle + d|1\rangle)$$

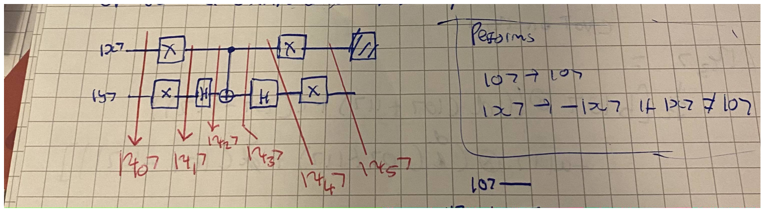

Figure 3: Circuit from dotted box in second figure of Box 6.1. The red lines are where we calculate the state

After the X gates

$$\begin{aligned}|\psi_1\rangle &= X(a|0\rangle + b|1\rangle)X(c|0\rangle + d|1\rangle)\\ &= (b|0\rangle + a|1\rangle)(d|0\rangle + c|1\rangle)\end{aligned}$$

After the H gate

$$\begin{aligned}|\psi_2\rangle &= (b|0\rangle + a|1\rangle)H(d|0\rangle + c|1\rangle)\\ &= (b|0\rangle + a|1\rangle)\Big(d\frac{|0\rangle + |1\rangle}{\sqrt{2}} + c\frac{|0\rangle - |1\rangle}{\sqrt{2}}\Big)\\ &= \frac{1}{\sqrt{2}}(b|0\rangle + a|1\rangle)(c(|0\rangle + |1\rangle) + d(|0\rangle - |1\rangle))\end{aligned}$$

Apply the CNOT to $|\psi_2\rangle$

$$\begin{aligned}|\psi_3\rangle &= \frac{1}{\sqrt{2}}\Big[b|0\rangle \otimes \Big(d(|0\rangle + |1\rangle) + c(|0\rangle - |1\rangle)\Big) + a|1\rangle \otimes \Big(d(|0\rangle + |1\rangle) + c(|0\rangle - |1\rangle)\Big)\Big]\\ &= \frac{1}{\sqrt{2}}\Big[b|0\rangle\big((d+c)|0\rangle + (d-c)|1\rangle\big) + a|1\rangle\big((d+c)|0\rangle + (d-c)|1\rangle\big)\Big]\end{aligned}$$

After the second H gate

$$|\psi_4\rangle = b|0\rangle(d|0\rangle + c|1\rangle) + a|1\rangle(d|0\rangle - c|1\rangle)$$

And the final state is

$$|\psi_5\rangle = a|0\rangle(-c|0\rangle + d|1\rangle) + b|1\rangle(c|0\rangle + d|1\rangle)$$

Expanding

$$= -ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle)$$

For this to perform the quantum search we have $|x\rangle \to -|x\rangle$ or in this case $|00\rangle \to -|00\rangle$. So $|\psi_5\rangle$ can be written as

$$-2|00\rangle\langle 00| + I$$

Or

$$\begin{aligned}&-(2|00\rangle\langle 00| - I)\\ &= -(2|00\rangle - I)|xy\rangle\end{aligned}$$

Where $-$ is the unimportant phase factor.

4. Exericse 6.9. Verify

$$U(\Delta t) = \Big(\cos^2(\frac{\Delta t}{2}) - \sin^2(\frac{\Delta t}{2})\vec{\psi}\cdot\hat{z}\Big)I$$

$$-2i\sin(\frac{\Delta t}{2})\Big(\cos(\frac{\Delta t}{2})\frac{\vec{\psi}+\hat{z}}{2} + \sin(\frac{\Delta t}{2})\frac{\vec{\psi}\times\hat{z}}{2}\Big)$$

From the book we know that

$$U(\Delta t) \equiv \exp(-i|\psi\rangle\langle\psi|\Delta t)\exp(-i|x\rangle\langle x|\Delta t)$$

From Chapter 4 (eq 4.8) we know that

$$\exp(iAx) = cos(x)I + isin(x)A$$

So we have

$$R_{\vec{\psi}} = \exp(-i|\psi\rangle\langle\psi|\Delta t)$$
$$= \cos(\frac{\Delta t}{2})I - i\sin(\frac{\Delta t}{2})(\vec{\psi}\cdot\vec{\sigma})$$
$$R_{\hat{z}} = \exp(-i|x\rangle\langle x|\Delta t)$$
$$= \cos(\frac{\Delta t}{2}) - i\sin(\frac{\Delta t}{2})(\hat{z}\cdot\vec{\sigma})$$

And

$$U(\Delta t) = R_{\vec{\psi}}R_{\hat{z}}$$
$$U(\Delta t) = \Big[\cos(\frac{\Delta t}{2}) - isin(\frac{\Delta t}{2})(\vec{\psi}\cdot\vec{\sigma})\Big]\cdot\Big[\cos(\frac{\Delta t}{2}) - isin(\frac{\Delta t}{2})(\hat{z}\cdot\vec{\sigma})\Big]$$
$$= \cos(\frac{\Delta t}{2})\cos(\frac{\Delta t}{2})I - \sin(\frac{\Delta t}{2})\sin(\frac{\Delta t}{2})(\vec{\psi}\cdot\vec{\sigma})(\hat{z}\cdot\vec{\sigma})$$
$$-i\sin(\frac{\Delta t}{2})\cos(\frac{\Delta t}{2})(\hat{z}\cdot\vec{\sigma}) - i\cos(\frac{\Delta t}{2})\sin(\frac{\Delta t}{2})(\vec{\psi}\cdot\vec{\sigma})$$

Introducing an identity (From Ex 4.15)

$$(\vec{\psi}\cdot\vec{\sigma})(\hat{z}\cdot\vec{\sigma}) = \hat{z}\cdot\vec{\psi}I + i(\vec{\psi}\times\hat{z})\cdot\vec{\sigma}$$

Implementing and simplifying

$$R_{\vec{\psi}}R_{\hat{z}} =$$
$$\cos^2(\frac{\Delta t}{2})I - \sin^2(\frac{\Delta t}{2})\Big[\hat{z}\cdot\vec{\psi} + i(\vec{\psi}\times\hat{z})\cdot\vec{\sigma}\Big]\dots$$
$$-i\sin(\frac{\Delta t}{2})\cos(\frac{\Delta t}{2})(\hat{z}\cdot\vec{\sigma}) - i\cos(\frac{\Delta t}{2})\sin(\frac{\Delta t}{2})(\vec{\psi}\cdot\vec{\sigma})$$

Take out i from the second part

$$= \cos^2(\frac{\Delta t}{2}) - \sin^2(\frac{\Delta t}{2})(\vec{\psi} \cdot \hat{z})I \dots$$

$$-i\Big(\sin(\frac{\Delta t}{2})\cos(\frac{\Delta t}{2})\hat{z} + \cos(\frac{\Delta t}{2})\sin(\frac{\Delta t}{2})\vec{\psi} + \sin(\frac{\Delta t}{2})\sin(\frac{\Delta t}{2})(\vec{\psi} \times \hat{z})\Big) \cdot \vec{\sigma}$$

Take out $\sin(\frac{\Delta t}{2})$ from the second part

$$= \cos^2(\frac{\Delta t}{2}) - \sin^2(\frac{\Delta t}{2})(\vec{\psi} \cdot \hat{z})I \dots$$

$$-i\sin(\frac{\Delta t}{2})\Big(\cos(\frac{\Delta t}{2})\hat{z} + \cos(\frac{\Delta t}{2})\vec{\psi} + \sin(\frac{\Delta t}{2})(\vec{\psi} \times \hat{z})\Big) \cdot \vec{\sigma}$$

Take out a factor of 2

$$= \cos^2(\frac{\Delta t}{2}) - \sin^2(\frac{\Delta t}{2})(\vec{\psi} \cdot \hat{z})I \dots$$

$$-2i\sin(\frac{\Delta t}{2})\Big(\frac{\cos(\frac{\Delta t}{2})}{2}\hat{z} + \frac{\cos(\frac{\Delta t}{2})}{2}\vec{\psi} + \frac{\sin(\frac{\Delta t}{2})}{2}(\vec{\psi} \times \hat{z})\Big) \cdot \vec{\sigma}$$

Which collating terms gives us

$$= \cos^2(\frac{\Delta t}{2}) - \sin^2(\frac{\Delta t}{2})(\vec{\psi} \cdot \hat{z})I - 2i\sin(\frac{\Delta t}{2})\Big(\cos(\frac{\Delta t}{2})(\frac{\hat{z} + \vec{\psi}}{2}) + \sin(\frac{\Delta t}{2})(\frac{\vec{\psi} \times \hat{z}}{2})\Big) \cdot \vec{\sigma}$$

And clearly this is equal to the equation in the book.

5. Problem 6.2 not attempted.