FYMM/MMP IIIa 2020 Problem Set 3

Please submit your solutions for grading by **Monday 21.9.** in Moodle.
The second part of problem four could prove to be more challenging than the rest: you could save it last when working on the problems.

1. Consider a subgroup of $2n \times 2n$ non-singular real matrices, the symplectic group $Sp(2n, \mathbb{R})$:
$$Sp(2n, \mathbb{R}) = \{M \in GL(2n, \mathbb{R}) | M^T \Omega M = \Omega\} ,$$
where Ω is the $2n \times 2n$ matrix
$$\Omega = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$$
where I_n denotes the $n \times n$ identity matrix and 0_n the $n \times n$ null matrix.
    (a) Show that it is a group.
    (b) Show that dim $Sp(2n, \mathbb{R}) = n(2n + 1)$.

2. Let the group $G \equiv \mathbb{Z} \times \mathbb{Z}$ act on $\mathbb{R}^2$ by
$$L_{(m,n)}(x, y) = (x + 2\pi m, y + 2\pi n) .$$
What is the quotient space $\mathbb{R}^2/G$?

3. Recall that the unitary group $U(5)$ leaves the complex scalar product of any pair of vectors of $\mathbb{C}^5$ invariant. Pick two vectors $x, y \in \mathbb{C}^5$. Consider then the subset
$$A = \{\alpha x + \beta y | \ \forall \alpha, \beta \in \mathbb{C}\} \subset \mathbb{C}^5 ,$$
the set of all complex linear combinations of x and y. Is there a subgroup H of $U(5)$ that leaves A invariant? (More precisely, we ask that for each matrix $M \in H$ we should have $Ma \in A$ for every $a \in A$.) What properties would H have?

4. In the lectures, it was shown that the SU(2) matrices can be written in the form
$$g = \begin{pmatrix} x_0 + ix_i & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix},$$
where the real parameters x_i, $i = 0, \ldots, 3$ satisfy the constraint $x_0^2 + \cdots + x_3^2 = 1$. Thus the parameters of SU(2) are coordinates of a unit sphere S^3. This time, we consider $\mathrm{SL}(2, \mathbb{R})$ and its conjugacy classes.

(a) Show that $\mathrm{SL}(2,\mathbb{R})$ can be parametrized by four real numbers $x_0, \ldots, x_3$ which satisfy a constraint of the form

$$c_0 x_0^2 + c_3 x_3^3 + x_1^2 + x_2^2 = c. \tag{1}$$

Find c_0, c_1, c.

Hint: The equation will describe one of the four so-called unit pseudospheres. In general, n-dimensional unit pseudospheres are n-dimensional hypersurfaces of constant curvature in $\mathbb{R}^{n+1}$, defined by an equation of type

$$c_0 x_0^2 + c_n x_n^2 + x_1^2 + \cdots + x_{n-1}^2 = c, \tag{2}$$

where c_0, c_n, $c = \pm 1$. There are four possible alternatives:

i. $(c_0, c_n, c) = (-1, -1, -1)$: n-dimensional anti-de Sitter space, AdS_n

ii. $(c_0, c_n, c) = (-1, 1, 1)$: n-dimensional de Sitter space, dS_n

iii. $(c_0, c_n, c) = (-1, 1, -1)$: n-dimensional hyperbolic space, H_n

iv. $(c_0, c_n, c) = (1, 1, 1)$: n-sphere S^n.

(b) Find the conjugacy classes of $\mathrm{SL}(2,\mathbb{R})$ and interpret them as 2-dimensional pseudospheres, if possible.

Hint: The trace and the eigenvalues of a matrix do not change under conjugation, i.e., they are shared by all elements in the conjugacy class (but they may also be shared with other conjugacy classes). Use diagonalization to classify the different conjugacy classes. There are three different types of conjugacy classes.

5. Given a group G, a left G-space X, and an element $x \in X$, prove that the isotropy group of x is a subgroup of G.