Quantum Information A   Fall 2020   Problem Set 3

Solutions are due in 4 pm on Tuesday Sep 22.
All problems except ♯5 are taken from Nielsen-Chuang, look them up from the book.

1. Excercise 2.48 from the book.
2. Excercise 2.50 from the book.
3. Exercise 2.60 from the book.
4. Exercise 2.61 from the book.
5. If you have an orthonormal basis $e_1, \ldots, e_n$ of a vector space V chosen so that the first $1 \leq k < n$ vectors are a basis of a k-dimensional subspace W, a projection operator P that projects to W is simply
$$P = \sum_{i=1}^{k} e_i e_i^\dagger \ .$$
(In ket notation with $e_i = |i\rangle$, $P = \sum_{i=1}^{k} |i\rangle\langle i|$.) What if you have a basis which is not even orthogonal? Consider the vectors
$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \ ; \ u_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$
spanning a two-dimensional subspace W (the xy-plane) of $\mathbf{R}^3$. Note that u_1, u_2 are not orthogonal. In this case one can construct a projection operator P which projects to W as follows. Construct a 3×2 matrix
$$A = [u_1 u_2] \ ,$$
the notation means that the two vectors u_1, u_2 are the columns of A. Then
$$P = A(A^T A)^{-1} A^T$$
is a projection operator to W. Verify this: show that in general the above $P^2 = P$, and by using the given u_1, u_2 calculate the matrix P explicitly and verify that it projects to W by showing that the vector Pv with an arbitrary
$$v = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$
is in W. Next, show that the matrix $G \equiv A^T A$ is in fact a Gram matrix with $G_{ij} = u_i \cdot u_j$.
6. Exercise 2.64 from the book.