Sphericity

One of the requirements for a repeated measures ANOVA is known to be sphericity. This assumption is met when the variances of the differences between related combinations of all factor levels are equal. Sphericity is related to homogeneity of variances in an ANOVA without repeated measures. The violation of sphericity causes the test to become too liberal. Therefore, determining whether sphericity has been violated is necessary if the omnibus F-test of a repeated measures ANOVA shall be interpreted. Though violation of sphericity is problematic, it is still possible to conduct and interpret the Omnibus F-test if some adjustments for the violation of sphericity are incorporated. This is achieved by estimating the degree to which sphericity has been violated. The corresponding estimator is commonly denoted by epsilon ($\epsilon \in [1, 0]$) where a value of 1 means the data is absolute spherical. This measure epsilon can then be used as a correction factor to the degrees of freedom of the F-distribution, which leads (in case of $\epsilon < 1$) to a decrease in the degrees of freedom and therefore to a more conservative F-test. Of course it will be only very rarely the case that sphericity is met exactly by the empirical data, so at first is it is necessary to tests, whether it can be assumed that sphericity is theoretically met or not. A popular Test to accomplish this is called Mauchly's sphericity test (Mauchly, 1940). It tests the H_0 that sphericity is met. Therefore a violation of sphericity shall be assumed whenever the Mauchly's sphericity test yields a significant results whereupon the H_0 is rejected. So if a repeated measures ANOVA is indicated it must be tested if the assumption of sphericity is met. If it is not, an appropriate adjustment of the omnibus F-Test has to be performed. This is implemented by the following function.

rma_spheri(rma_data, id = 1, append = FALSE)

As input parameter it needs a repeated measures data file which meets the above described requirements. It also contains the **id = 1** argument. If **append = TRUE** the function chooses the appropriate correction (see Girden, 1992) for the omnibus F-test if sphericity is violated and appends the thereby adjusted p-value to the ANOVA-table. Obviously it is nonsensical to test sphericity, if there are only two factor levels (**k = 2**). So the function will abort in this case and provide a notification.

```
# Check whether a test for sphericity is needed
if (k = 2) {
        stop("Note that there can't be a violation of sphericity since the factor of the one-way
                anova has only two factor levels")
}
```

To conduct the Mauchly's sphericity test the first step is to construct a Helmert matrix. The construction of the first row is omitted since it is not required for further use in this procedure.

```
helmert = function(k, df) {
	H = matrix(0, k, k)
	diag(H) = (0:df) * (-((0:df) * ((0:df) + 1))^(-0.5))
	for (i in 2:k) {
		H[i, 1:(i - 1)] = ((i - 1) * (i))^(-0.5)
	}
	# H[1,] = 1/sqrt(k)
	return(H)
}
```

```
# The first row of the helmert matrix is not required for further use in this procedure
C = helmert(k, df)[-1, ]
```

Thereby **df** denotes the effect degrees of freedom which equal **k - 1**. Then the test statistic (Mauchly's W) for the Mauchly's sphericity test is computed by

$$W = \frac{|C\ Var(Y)\ C^T|}{(tr(C\ Var(Y) C^T)/k-1)^{k-1}}$$

where C is the Helmert matrix and $Var(Y)$ is the depended data covariance matrix.

```
# Empirical covariance matrix
covariance_matix = cov(dependent_variable)
```

```
w = det(C %*% covariance_matix %*% t(C))/
	((sum(diag(C %*% covariance_matix %*% t(C)))/df)^(df))
```

The transformation

$$-(n-1) \times 1 - \frac{2 \times (k-1)^2 + k + 1}{6 \times k - 1 \times n - 1} \times \ln(W)$$

of Mauchly's W is χ^2 distributed with

$$\frac{k \times (k-1)}{2} - 1$$

degrees of freedom. So a p-value for the test can be computed.

```
# Computing the degrees of freedom for the chi-square value
df_w = ((k * df)/2) – 1
```

```
# Computing the chi-square value
f = 1 - ((2 * (df^2) + df + 2)/(6 * df * (n - 1)))
chi_sq_w = -(n - 1) * f * log(w)
```

```
# Computing the corresponding p-value
p_w = 1 - pchisq(chi_sq_w, df_w)
```

The results of the computational steps described above are summarized in a table, which is later returned by the function.

```
mauchly_table = data.frame(check.names = FALSE, Source = "Factor", `Mauchly's W` = w,
        `Chi square` = chi_sq_w, df = df_w, p = p_w)
rownames(mauchly_table) = NULL
```

The next step is to compute epsilon.

```
# Lower-Bound correction (Greenhouse & Geisser, 1959)
epsilon_lb = 1/df

# Box correction (Geisser & Greenhouse, 1958)
epsilon_gg = (sum(diag(C %*% covariance_matix %*% t(C)))^2)/
        (df * sum(diag(t((C %*% covariance_matix %*% t(C))) %*%
        (C %*% covariance_matix %*% t(C)))))

# Huynh-Feldt correction (Huynh & Feldt, 1976)
epsilon_hf = min((n * df * epsilon_gg - 2)/(df * (n - 1) - (df^2 * epsilon_gg)), 1)
```

There are three commonly used estimators for epsilon (see Rutherford, 2011). The Huynh-Feldt epsilon (Huynh & Feldt, 1976) on the one hand is a fairly liberal estimator, i.e. it tends to underestimate the degree of violation. The Box epsilon (Box, 1954; Geisser & Greenhouse, 1958) on the other hand is a more conservative estimator and therefore tends to overestimates the degree of violation. The Greenhouse-Geisser or lower bound epsilon (Greenhouse & Geisser, 1959) lastly is the most conservative estimation possible under the given conditions, i.e. it assumes the worst-case scenario of violation, thus it is actually not sensitive for variation in the degree by which sphericity is violated. Each of these can be used as a correction factor to adjust the degrees of freedom and therefore the resulting p-value of the omnibus F-test (note that the empirical F value stays the same since the correction factors cancel out, but the theoretical F distribution of the tests statistic under H_0 changes).

```
anova_table = rma(rma_data)[[1]]

corrected_factor_df = anova_table[2, 3] * epsilon_lb
corrected_error_df = anova_table[2, 3] * epsilon_lb
p_factor_lb = 1 - pf(anova_table[2, 5], corrected_factor_df, corrected_error_df)

corrected_factor_df = anova_table[2, 3] * epsilon_gg
corrected_error_df = anova_table[2, 3] * epsilon_gg
p_factor_gg = 1 - pf(anova_table[2, 5], corrected_factor_df, corrected_error_df)

corrected_factor_df = anova_table[2, 3] * epsilon_hf
corrected_error_df = anova_table[2, 3] * epsilon_hf
p_factor_hf = 1 - pf(anova_table[2, 5], corrected_factor_df, corrected_error_df)
```

The results of the computational steps described above are summarized in a table, which is later returned by the function.

```
epsilon_table = data.frame(check.names = FALSE, Source = c("Epsilon", "Adjusted p-Value"),
        `Lower-Bound correction (Greenhouse & Geisser, 1959)` = c(epsilon_lb, p_factor_lb),
        `Box correction (Geisser & Greenhouse, 1958)` = c(epsilon_gg, p_factor_gg),
        `Huynh-Feldt correction (Huynh & Feldt, 1976)` = c(epsilon_hf, p_factor_hf))
rownames(epsilon_table) = NULL
```

Greenhouse and Geisser (1959) recommended in case of a violation of sphericity (indicated by a significant Mauchly's sphericity test) to use the lower bound correction first. If the results become insignificant, the Box epsilon is used to estimate the degree of violation. If the degree of violation is high (i.e. $\epsilon < .75$) the Box correction is preferable, if the degree of violation is low (i.e. $\epsilon > .75$) the Huynh-Feldt epsilon shall be used to correct the degrees of freedom (Girden, 1992).

```
if (p_w < 0.05) {
        if (p_factor_lb < 0.05) {
                anova_table[, "Recommended Lower-Bound corrected p-Value
                        (Greenhouse & Geisser, 1959))"] = c(NA, p_factor_lb, NA, NA, NA, NA)
        } else {
                if (epsilon_gg < 0.75) {
                        anova_table[, "Recommended Box corrected p-Value
                                (Geisser & Greenhouse, 1958)"] =
                                c(NA, p_factor_gg, NA, NA, NA, NA)
                } else {
                        anova_table[, "Recommended Huynh-Feldt corrected p-Value
                                (Huynh & Feldt, 1976)"] = c(NA, p_factor_hf, NA, NA, NA, NA)
                }
        }
}
```

The function will eventually return both tables with the results of the Mauchly's sphericity test as well as the correction factors with their associated adjusted p-values. If the function argument **append = TRUE** the ANOVA-table with the recommended p-value is provided as well.

```
if (append == TRUE) {
        return(list(mauchly_table = mauchly_table, correction_factors_epsilon_table =
                epsilon_table, corrected_anova_table = anova_table))
 } else {
        return(list(mauchly_test_table = mauchly_table, correction_factors_epsilon_table =
                epsilon_table))
}
```

The analysis of the Ringelmann-data reveals that sphericity is violated ($W = X$, $\chi^2 = X$, $p < X$). The recommended correction is adjusting the degreees of freedom with the Box-epsilon ($\epsilon_{Box} = X$). The thereby for the violation in sphericity corrected omnibus F-test is still significant ($F(X, X) = X$, $p < X$) so it can be assumed that there is indeed an effect of the group size on the individual work performace.

Literature

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