Maths pre-requisites for data analysis in neuroscience

1/ Linear Algebra

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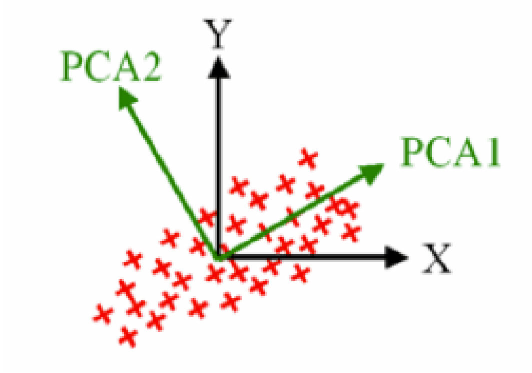
05/04/2019 - Workshop “computational neuroscience”, pre-requisite course

Ultimate goal: understand and develop tools for the analysis of neural data

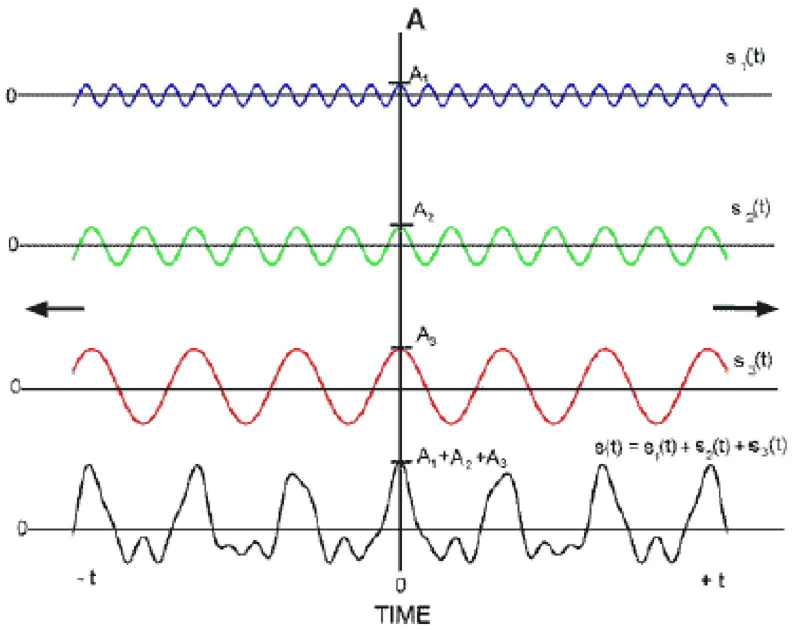
$$C_m \frac{dV}{dt} = -g_L(V - E_L) + g_L \Delta_T e^{\frac{V-V_T}{\Delta_T}} - u + I$$

$$\tau_w \frac{du}{dt} = a(V - E_L) - u$$

differential equations and modeling



dimensionality reduction



Fourier transforms, convolutions, and filtering out noise

“What is an eigenvector?”

“What exactly is PCA doing?”

“What *really* is a Fourier transform?”

Many slides from Lane McIntosh & Kiah Hardcastle

(NBIO course, Stanford Univ, https://web.stanford.edu/class/nbio228-01/info.html)

1/ Linear Algebra

- When and why is linear algebra useful?
- Vectors and their operations
- Matrices and their operations, special matrices
- Equation systems, matrices and determinants

Why linear algebra?

Why linear algebra?

1.63	5.20	7.66	8.12	3.22
4.98	5.90	8.21	9.29	20.10
10.10	8.57	5.73	8.17	2.22
0.02	0.21	0.14	0.93	1.40
9.27	10.27	13.12	8.90	9.01
7.44	6.98	5.62	8.20	7.21
100.10	8.22	7.54	60.10	1.69
40.20	29.21	12.45	10.41	8.90
32.33	21.59	10.21	4.99	2.62
2.99	1.67	1.01	0.80	0.07

Datasets are matrices

time →

neuron 1	1.63	5.20	7.66	8.12	3.22
neuron 2	4.98	5.90	8.21	9.29	20.10
neuron 3	10.10	8.57	5.73	8.17	2.22
neuron 4	0.02	0.21	0.14	0.93	1.40
neuron 5	9.27	10.27	13.12	8.90	9.01
neuron 6	7.44	6.98	5.62	8.20	7.21
neuron 7	100.10	8.22	7.54	60.10	1.69
neuron 8	40.20	29.21	12.45	10.41	8.90
neuron 9	32.33	21.59	10.21	4.99	2.62
neuron 10	2.99	1.67	1.01	0.80	0.07

Datasets are matrices

time →

voxel 1	1.63	5.20	7.66	8.12	3.22
voxel 2	4.98	5.90	8.21	9.29	20.10
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Datasets are matrices

patient →

gene 1	1.63	5.20	7.66	8.12	3.22
gene 2	4.98	5.90	8.21	9.29	20.10
gene 3	10.10	8.57	5.73	8.17	2.22
gene 4	0.02	0.21	0.14	0.93	1.40
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1/ Linear Algebra

A/ Vectors

- Definition and representation
- Basic operations
- Norm and angle
- Products

B/ Matrices

- Definition and basic operations
- Linear transformation M·V
- Matrix rank
- Multiplication
- Special matrices

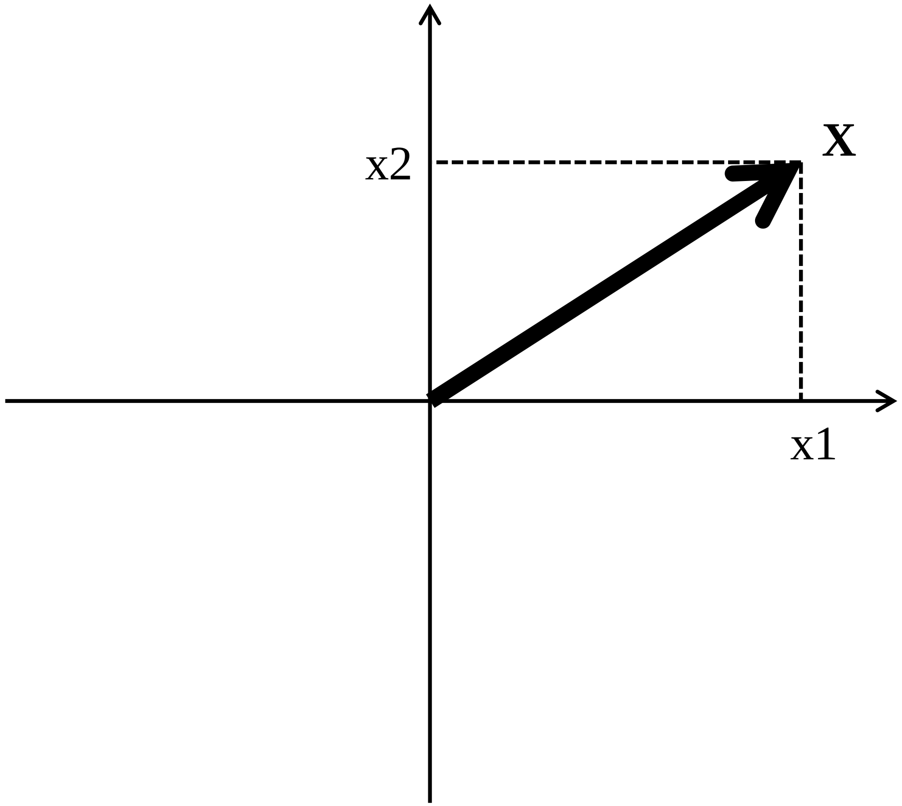
C/ Linear systems, matrices and determinant

- Linear equation systems
- Inverse matrix
- Determinant: geometry

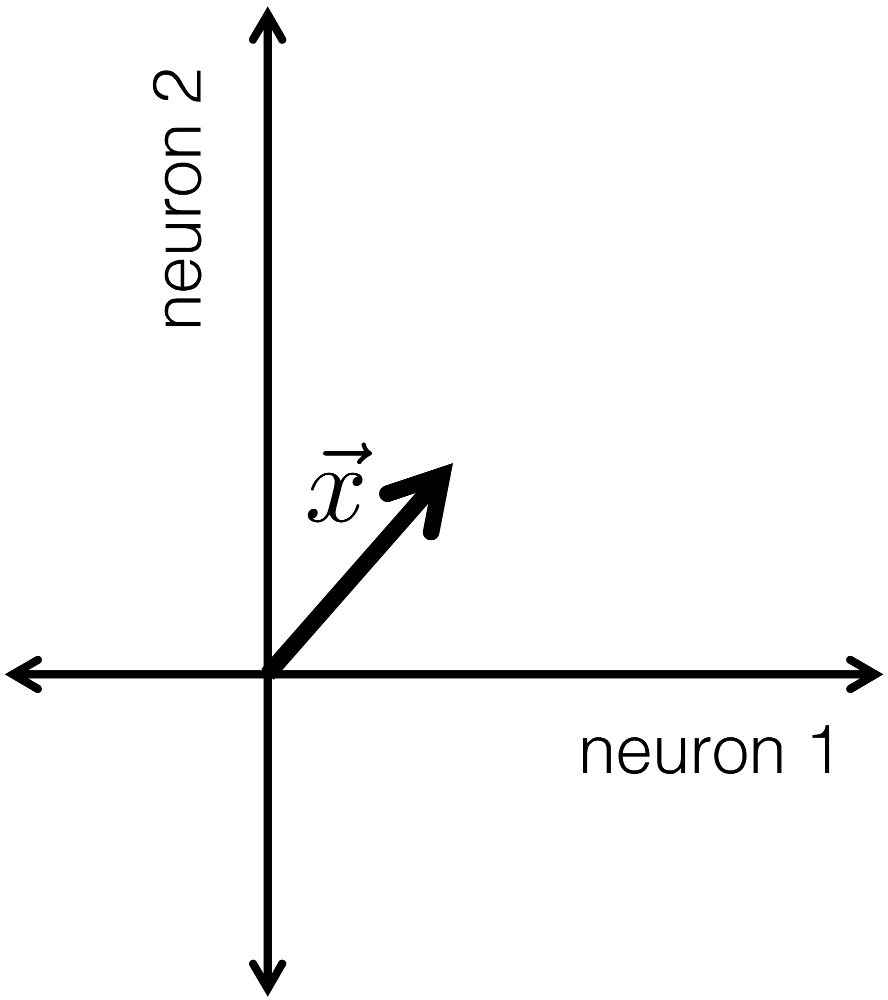
A/ Vectors

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

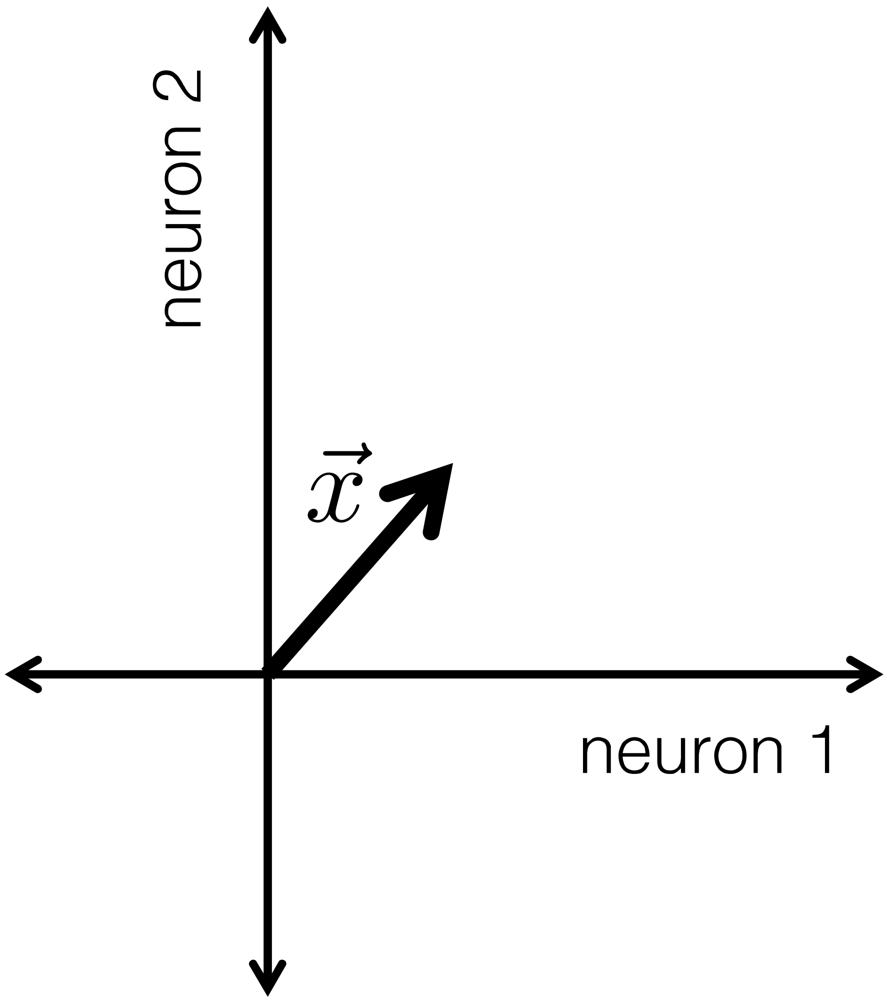


Scalar times vector



Measure 10 neurons at 100 HZ

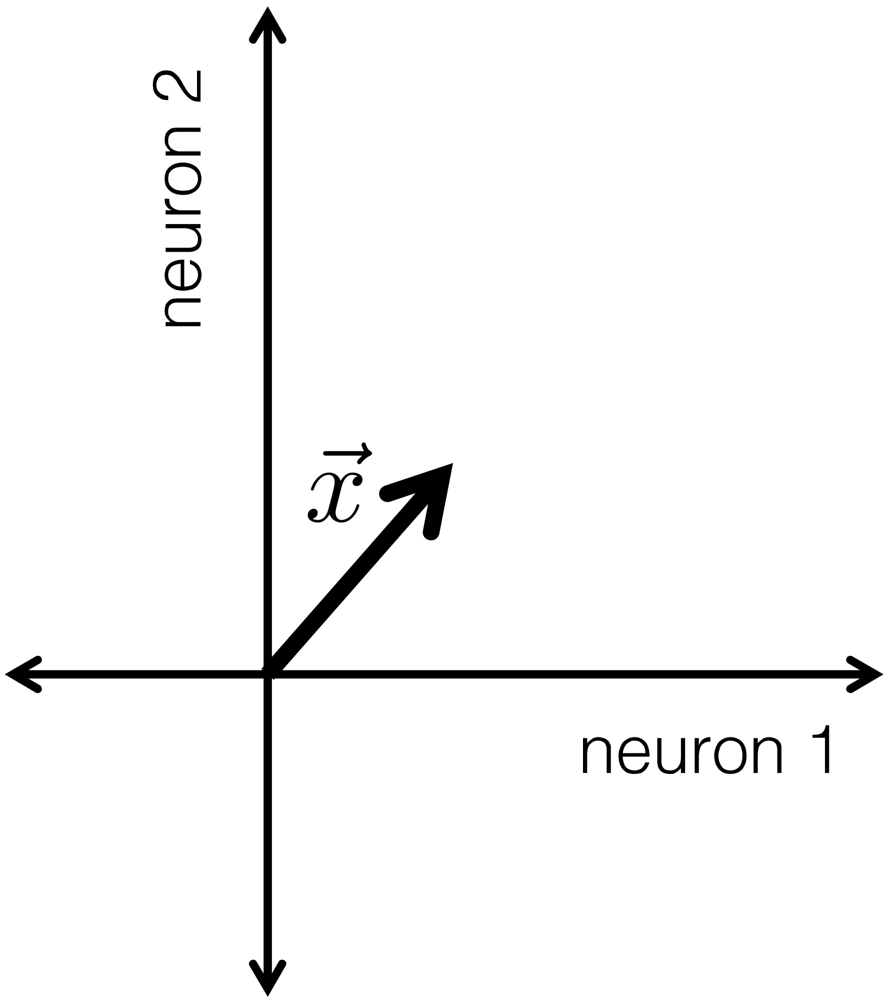
Scalar times vector



Measure 10 neurons at 100 HZ

Number of spikes per 10ms bin

Scalar times vector

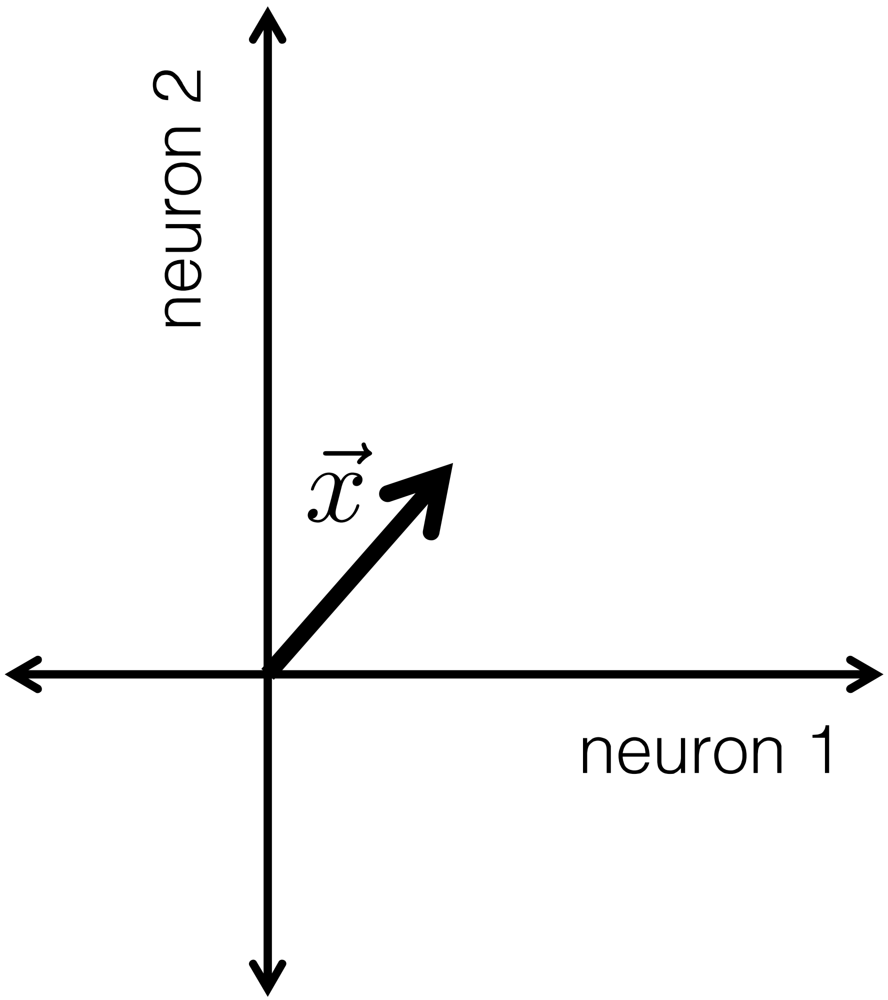


Measure 10 neurons at 100 HZ

Number of spikes per 10ms bin

How to convert to a firing rate in spikes/sec?

Scalar times vector



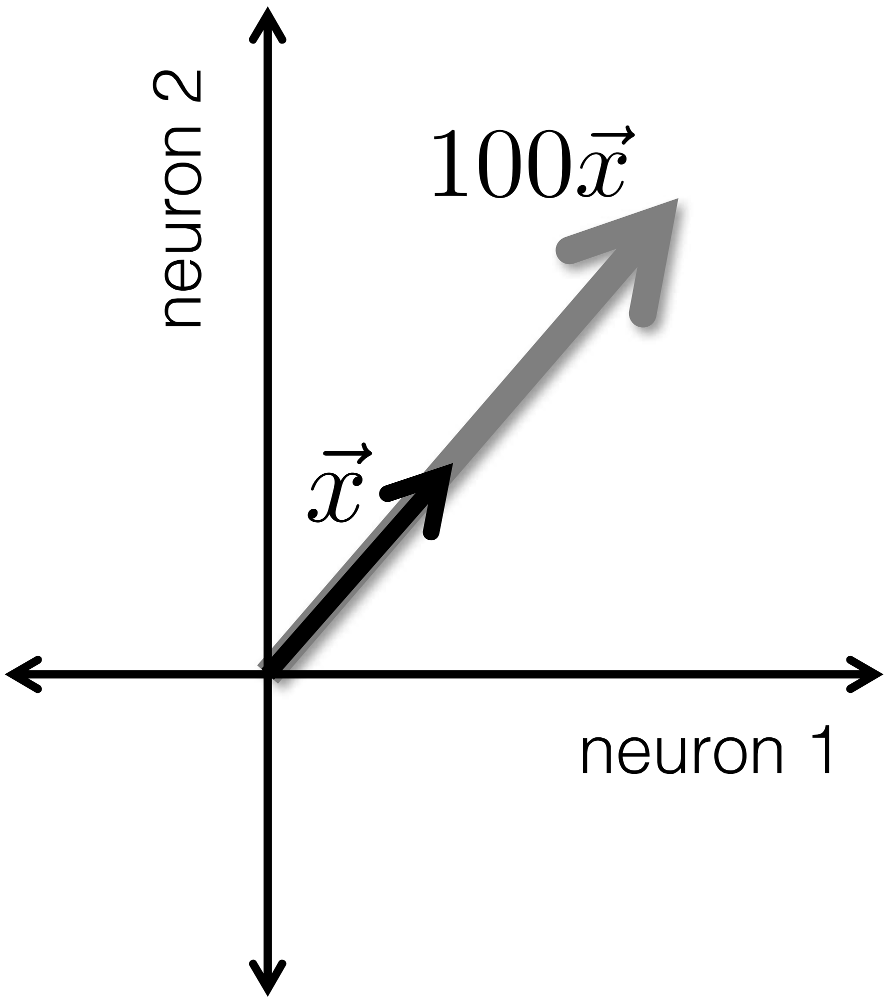
Measure 10 neurons at 100 HZ

Number of spikes per 10ms bin

How to convert to a firing rate in spikes/sec?

Multiply each measurement by 100

Scalar times vector



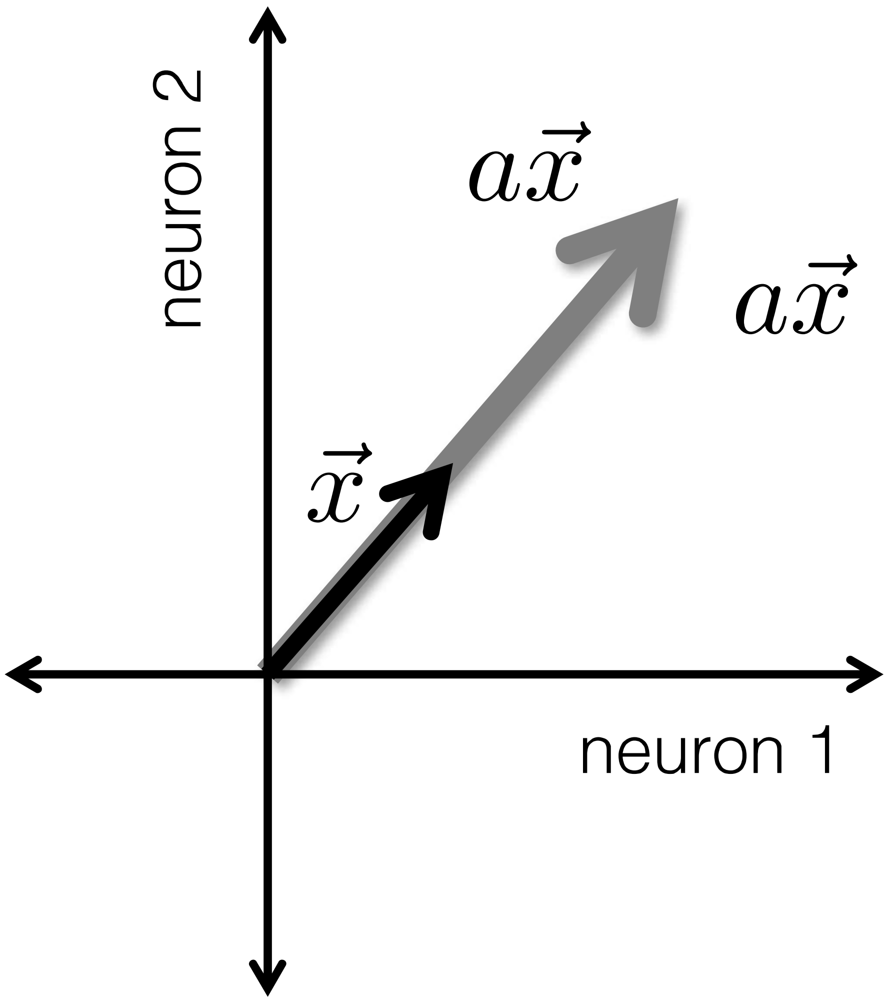
Measure 10 neurons at 100 HZ

Number of spikes per 10ms bin

How to convert to a firing rate in spikes/sec?

$\Rightarrow$ Multiply each measurement by 100

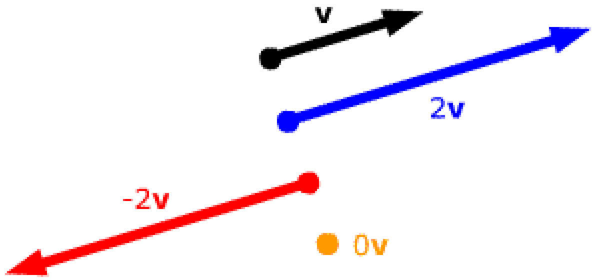
Scalar times vector



$$a\vec{x} = a \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_N \end{pmatrix}$$

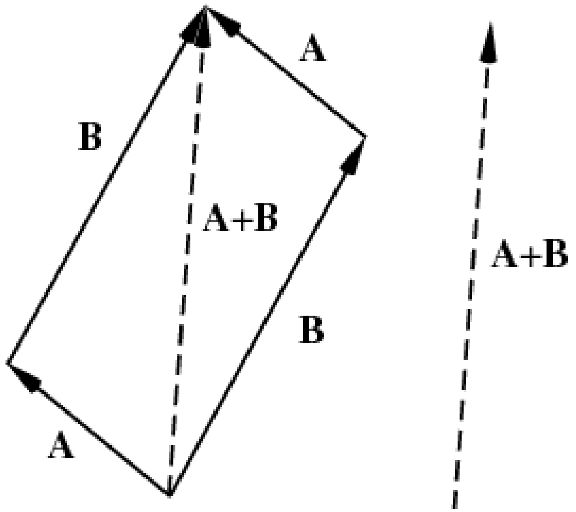
A/ Vectors

Scalar multiplication



$$\lambda \mathrm{x} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \equiv \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{pmatrix}$$

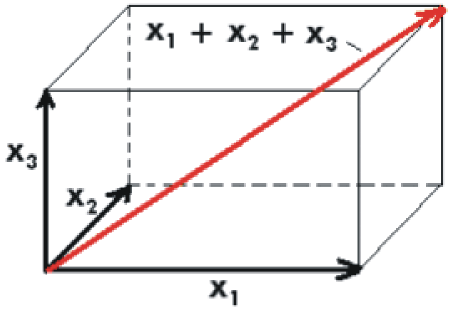
Addition



$$\mathrm{x} + \mathrm{y} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \equiv \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

Substraction

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}).$$

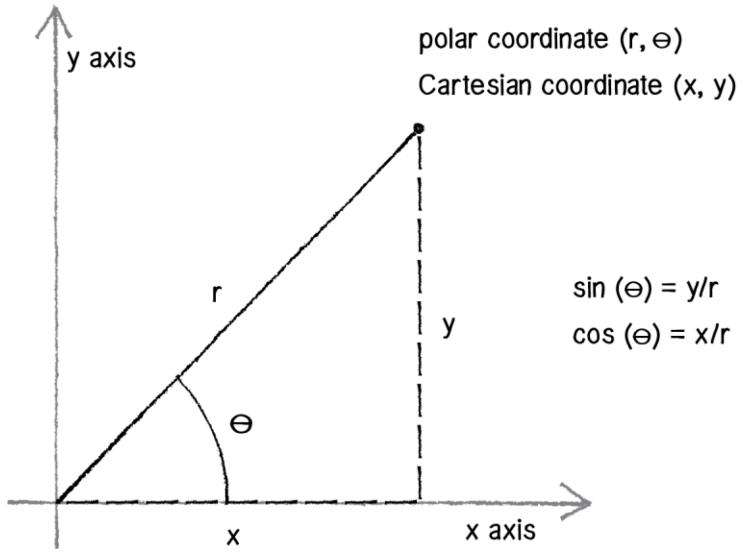


A/ Vectors

Norm (or magnitude): |x| or ||x||

$$|\mathbf{x}| = \sqrt{x_1^2 + x_2^2}$$

$$|\mathbf{x}| \equiv \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} = \sqrt{\sum x_j^2}$$



Dot product (inner product)

$$\vec{x} \cdot \vec{y} =$$

Multiplication:

Dot product (inner product)

$$\vec{x} \cdot \vec{y} =$$

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_N \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}$$

Multiplication:
Dot product (inner product)

$$\vec{x} \cdot \vec{y} =$$

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_N \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = x_1 y_1$$

Multiplication:
Dot product (inner product)

$$\vec{x} \cdot \vec{y} =$$

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_N \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = x_1 y_1 + x_2 y_2$$

Multiplication:
Dot product (inner product)

$$\vec{x} \cdot \vec{y} =$$

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_N \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_N y_N$$

Multiplication:
Dot product (inner product)

$$\vec{x} \cdot \vec{y} =$$

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_N \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_N y_N$$

$$= \sum_{i=1}^{N} x_i y_i$$

Multiplication:
Dot product (inner product)

$$\vec{x} \cdot \vec{y} =$$

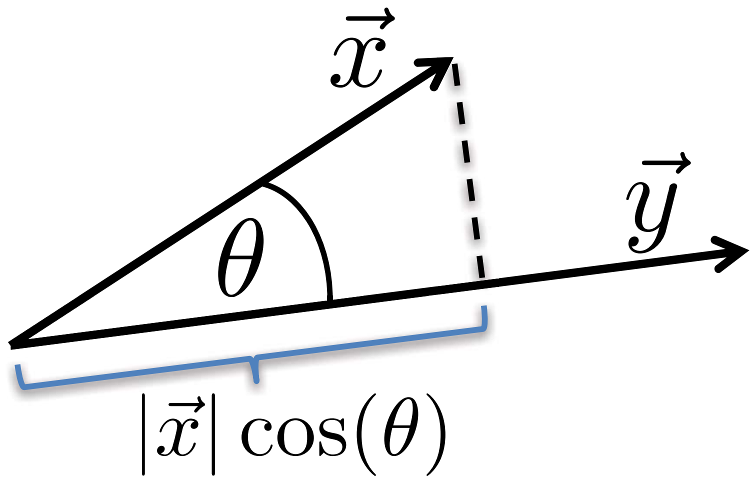
$$\begin{pmatrix} x_1 & x_2 & \cdots & x_N \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_N y_N$$

1 X N   N X 1   1 X 1

- MATLAB: 'inner matrix dimensions must agree'

Outer dimensions give size of resulting matrix

Dot product geometric intuition: "Overlap" of 2 vectors

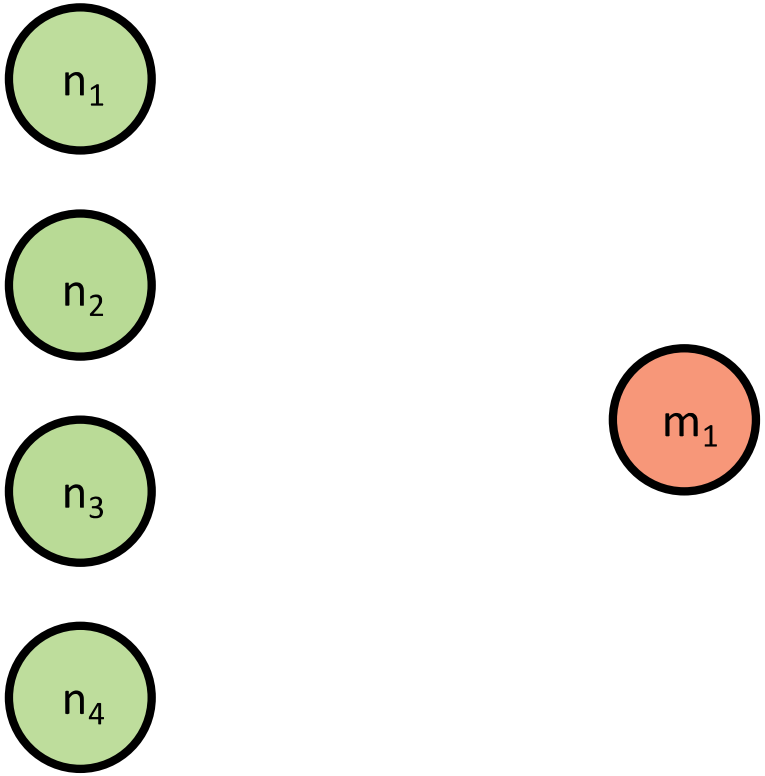


$$\vec{x} \cdot \vec{y} = |\vec{x}||\vec{y}|\cos(\theta)$$

Multiplication:

Dot product (inner product) Example 1

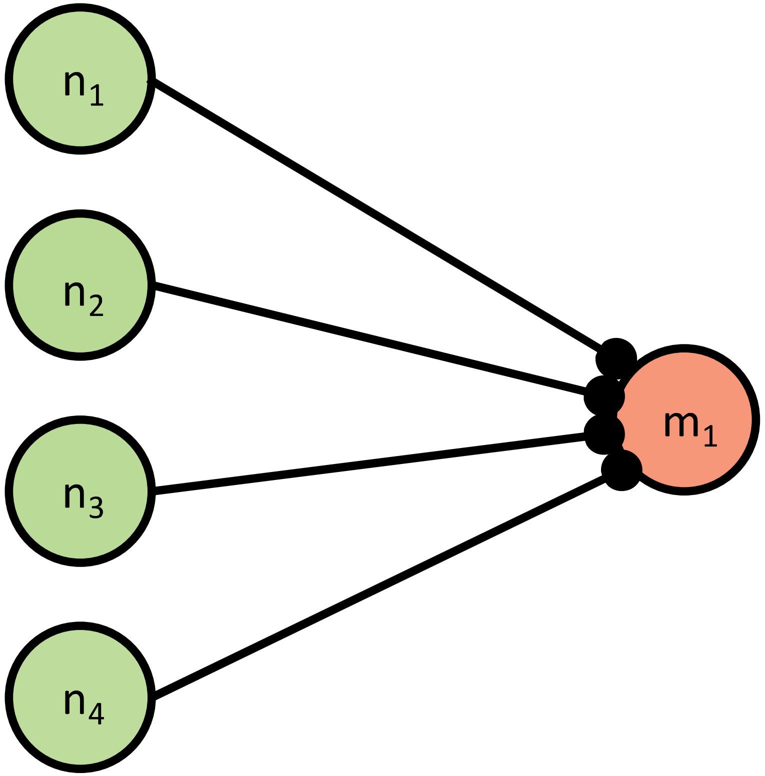
weighted average



Multiplication:

Dot product (inner product) Example 1

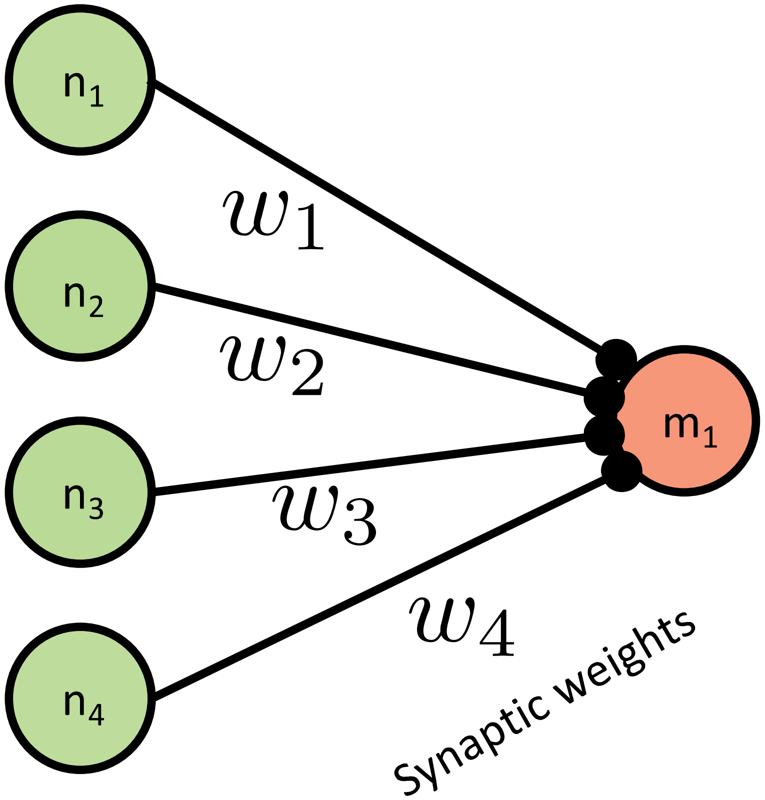
weighted average



Multiplication:

Dot product (inner product) Example 1

weighted average

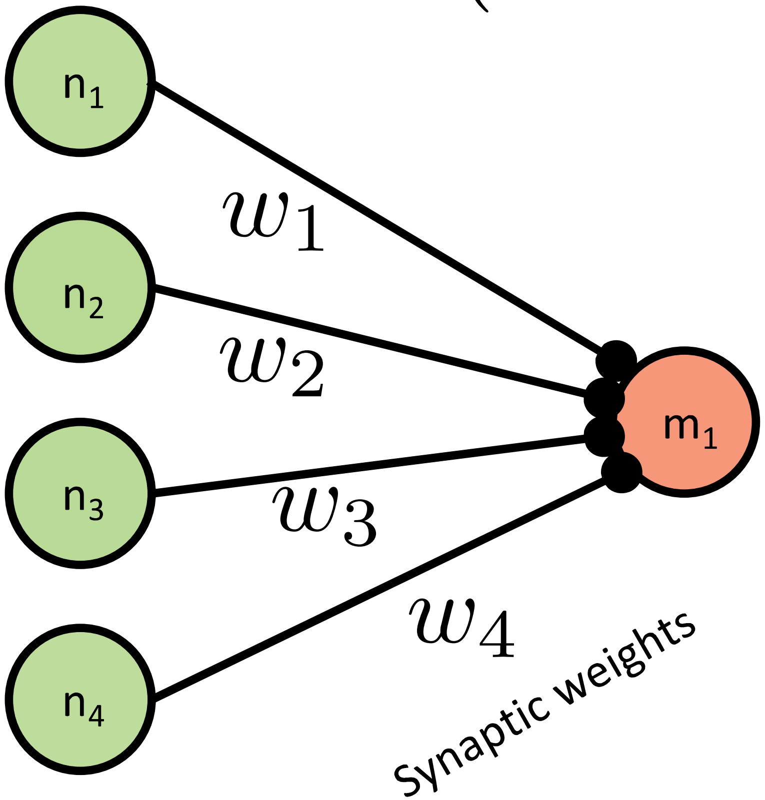


Multiplication:

Dot product (inner product) Example 1

weighted average

$$m_1 = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix}$$



Multiplication:

Dot product (inner product) Example 1

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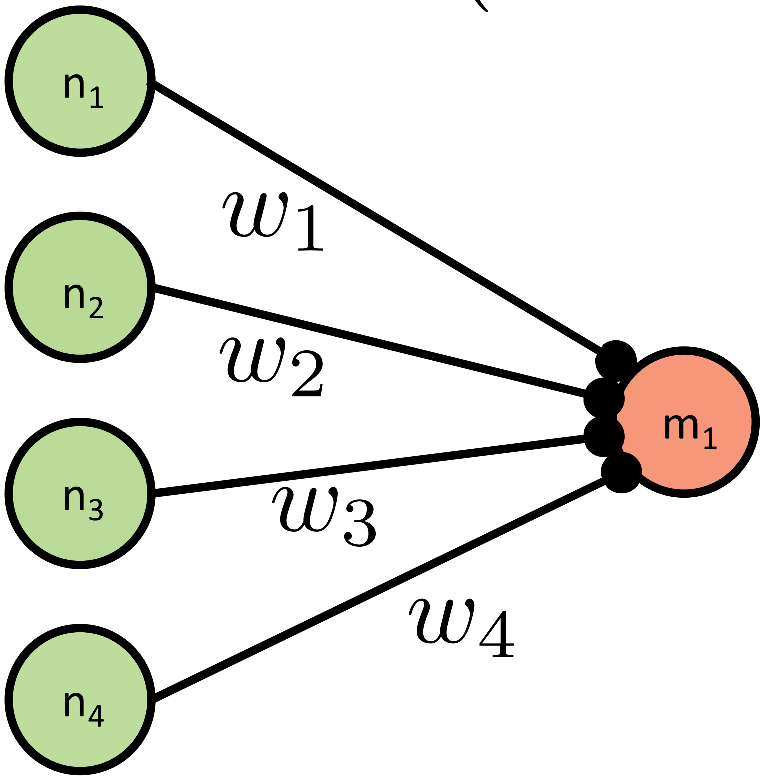

$$= w_1 n_1 + w_2 n_2 + w_3 n_3 + w_4 n_4$$

$$= w^T n$$ (`w'*n` in MATLAB)

Multiplication:

Dot product (inner product) Example 2

linear regression

patient →

gene 1	1.63	5.20	7.66	8.12	3.22
gene 2	4.98	5.90	8.21	9.29	20.10
gene 3	10.10	8.57	5.73	8.17	2.22
gene 4	0.02	0.21	0.14	0.93	1.40
gene 5	9.27	10.27	13.12	8.90	9.01
gene 6	7.44	6.98	5.62	8.20	7.21

Multiplication:
Dot product (inner product) Example 2

linear regression

patient →

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For each patient, you also measure their Asperger's disorder quotient

Multiplication:
Dot product (inner product) Example 2

linear regression

patient ⟶

gene 1	1.63	5.20	7.66	8.12	3.22
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gene 5	9.27	10.27	13.12	8.90	9.01
gene 6	7.44	6.98	5.62	8.20	7.21
score	2	0	9	44	48

Multiplication:
Dot product (inner product) Example 2

linear regression

patient $\longrightarrow$

gene 1	1.63	5.20	7.66	8.12	3.22
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gene 5	9.27	10.27	13.12	8.90	9.01
gene 6	7.44	6.98	5.62	8.20	7.21
score	2	0	9	44	48

$$\text{score} = w_1 \text{gene}_1 + w_2 \text{gene}_2 + \cdots + w_6 \text{gene}_6$$

Multiplication:

Dot product (inner product) Example 2

linear regression

$$
\begin{pmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \end{pmatrix}
\begin{array}{cc}
\begin{pmatrix} 8.12 \\ 9.29 \\ 8.17 \\ 0.93 \\ 8.90 \\ 8.20 \end{pmatrix} &
\begin{array}{l} \text{gene 1} \\ \text{gene 2} \\ \text{gene 3} \\ \text{gene 4} \\ \text{gene 5} \\ \text{gene 6} \end{array}
\end{array}
$$

score 44

$$44 = w_1 8.12 + w_2 9.29 + \cdots + w_6 8.20$$

Multiplication:
Dot product (inner product) Example 2

linear regression

$$
\begin{pmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \end{pmatrix}
\begin{pmatrix} 8.12 \\ 9.29 \\ 8.17 \\ 0.93 \\ 8.90 \\ 8.20 \end{pmatrix}
\begin{array}{l} \text{gene 1} \\ \text{gene 2} \\ \text{gene 3} \\ \text{gene 4} \\ \text{gene 5} \\ \text{gene 6} \end{array}
$$

$$\text{score} = w^T \vec{\text{genes}}$$

Multiplication:
Dot product (inner product) Example 2

linear regression

patient →

gene 1	1.63	5.20	7.66	8.12	3.22
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score	2	0	9	44	48

$$\text{score} = w^T \vec{\text{genes}}$$

(`regress(score',X')` in MATLAB to get w's, where `X` is the full genes by patients matrix)

A/ Vectors

Let's imagine a linear neuron receiving input from n presynaptic neurons which firing rates $(x_j)_{j=1:n}$ are scaled by synaptic weights $(w_j)_{j=1:n}$ and together determine the firing rate y of the output neuron:

$$y = w_1 x_1 + w_2 x_2 + \ldots + w_n x_n = \sum w_j x_j.$$

The output can be written as dot product of the weight vector and input vector:

$$y = \boldsymbol{W} \cdot \boldsymbol{X}$$

From vectors to matrices

Now let's suppose we have *m* output neurons with activities $(y_i)_{i=1:m}$, still given by a linear combination of the firing rates of their inputs neurons $(x_{ij})_{i=1:m,\,j=1:n}$ scaled by synaptic weights $(w_{ij})_{i=1:m,\,j=1:n}$

$$y_i = w_{i1}x_{i1} + w_{i2}x_{i2} + \ldots + w_{in}x_{in} = \sum w_{ij}x_j.$$

It is convenient to think of a doubly indexed array of $m \times n$ scalars such as the weights w_{ij} above as a single object W called an $m \times n$ matrix

$$W = (w_{ij}) = \begin{pmatrix} w_{11} & w_{12} & \ldots & w_{1n} \\ w_{21} & w_{22} & \ldots & w_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ w_{m1} & w_{m2} & \ldots & w_{mn} \end{pmatrix}$$

Conventionally the first index indicates row position and the second index column position.

From vectors to matrices

Product of a Matrix and a Vector. We start with the equation above for the output of a linear neuron

$$y_i = \sum_{j=1}^{m} w_{ij} x_j.$$

and use this to define the product $W\mathbf{x}$ of the $m \times n$ matrix $W = (w_{ij})$ and a *column* n-vector $\mathbf{x} = (x_j)$

$$\mathbf{y} = W\mathbf{x} \Leftrightarrow y_i = \sum_{k=1}^{m} w_{ij} x_j.$$

Now we can write the equation for the outputs of the neuronal assemby in the compact form

$$\mathbf{y} = W\mathbf{x}.$$

Notice that this defines a mapping (usually also called W)

$$W : \mathbb{R}^n \to \mathbb{R}^m \qquad W : \mathbf{x} \to \mathbf{y} = W\mathbf{x}.$$

This transformation "squashes and squeezes" vectors input space to give vectors in output space. Getting an feel for the nature of these *linear transformations* is a really useful mathematical tool.

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- Multiplication
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C/ Linear systems, matrices and determinant

- Linear equation systems
- Inverse matrix
- Determinant: geometry

Matrix times vector

$$\overrightarrow{y} = \overleftrightarrow{W}\,\overrightarrow{x}$$

Matrix times vector

$$\overrightarrow{y} = \overleftrightarrow{W}\,\overrightarrow{x}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{M1} & W_{M2} & \cdots & W_{MN} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

Matrix times vector

$$\overrightarrow{y} = \overleftrightarrow{W}\,\overrightarrow{x}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{M1} & W_{M2} & \cdots & W_{MN} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

M X 1 M X N N X 1

Matrix times vector:
inner product interpretation

$$
\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_M \end{pmatrix}
=
\begin{pmatrix}
W_{11} & W_{12} & \cdots & W_{1N} \\
W_{21} & W_{22} & \cdots & W_{2N} \\
\vdots & \vdots & & \vdots \\
W_{i1} & W_{i2} & \cdots & W_{iN} \\
\vdots & \vdots & \ddots & \vdots \\
W_{M1} & W_{M2} & \cdots & W_{MN}
\end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}
$$

- Rule: the i^{th} element of **y** is the dot product of the i^{th} row of W with **x**

Matrix times vector:

inner product interpretation

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_M \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{i1} & W_{i2} & \cdots & W_{iN} \\ \vdots & \vdots & \ddots & \vdots \\ W_{M1} & W_{M2} & \cdots & W_{MN} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

- Rule: the i^{th} element of **y** is the dot product of the i^{th} row of W with **x**

Matrix times vector: **inner product interpretation**

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_M \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{i1} & W_{i2} & \cdots & W_{iN} \\ \vdots & \vdots & \ddots & \vdots \\ W_{M1} & W_{M2} & \cdots & W_{MN} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

- Rule: the i^{th} element of **y** is the dot product of the i^{th} row of W with **x**

Matrix times vector: **inner product interpretation**

$$
\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_M \end{pmatrix}
=
\begin{pmatrix}
W_{11} & W_{12} & \cdots & W_{1N} \\
W_{21} & W_{22} & \cdots & W_{2N} \\
\vdots & \vdots & & \vdots \\
W_{i1} & W_{i2} & \cdots & W_{iN} \\
\vdots & \vdots & \ddots & \vdots \\
W_{M1} & W_{M2} & \cdots & W_{MN}
\end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}
$$

- Rule: the i^{th} element of **y** is the dot product of the i^{th} row of W with **x**

Matrix times vector: **inner product interpretation**

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_M \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{i1} & W_{i2} & \cdots & W_{iN} \\ \vdots & \vdots & \ddots & \vdots \\ W_{M1} & W_{M2} & \cdots & W_{MN} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

- Rule: the i^{th} element of **y** is the dot product of the i^{th} row of W with **x**

Matrix times vector:
outer product interpretation

$$
\begin{array}{c}
\begin{array}{l}
\overrightarrow{W}^{(1)} \\
\downarrow
\end{array} \\
\begin{pmatrix}
W_{11} & W_{12} & \cdots & W_{1N} \\
W_{21} & W_{22} & \cdots & W_{2N} \\
\vdots & \vdots & & \vdots \\
W_{M1} & W_{M2} & \cdots & W_{MN}
\end{pmatrix}
\end{array}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_N
\end{pmatrix}
=
$$

- The product is a weighted sum of the columns of W, weighted by the entries of x

Matrix times vector:
outer product interpretation

$$\begin{array}{l} \;\;\overrightarrow{W}^{(1)} \\ \;\;\;\downarrow \\ \begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{M1} & W_{M2} & \cdots & W_{MN} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = x_1 \overrightarrow{W}^{(1)} + \end{array}$$

- The product is a weighted sum of the columns of W, weighted by the entries of x

Matrix times vector: **outer product interpretation**

$$\begin{array}{c} \overrightarrow{W}^{(1)} \\ \downarrow \end{array}$$

$$\begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{M1} & W_{M2} & \cdots & W_{MN} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = x_1 \overrightarrow{W}^{(1)} + x_2 \overrightarrow{W}^{(2)}$$

- The product is a weighted sum of the columns of W, weighted by the entries of x

Matrix times vector:
outer product interpretation

$$\begin{array}{c} \overrightarrow{W}^{(1)} \\ \downarrow \\ \begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{M1} & W_{M2} & \cdots & W_{MN} \end{pmatrix} \end{array} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = x_1 \overrightarrow{W}^{(1)} + x_2 \overrightarrow{W}^{(2)} + \cdots + x_N \overrightarrow{W}^{(N)}$$

- The product is a weighted sum of the columns of W, weighted by the entries of x

B/ Matrices

Multiplication by a scalar:

$$\lambda W = \lambda(w_{ij}) = (\lambda w_{ij})$$

Addition of two matrices:

$$V + W = (v_{ij}) + (w_{ij}) = (v_{ij} + w_{ij})$$

(Matrix addition is both commutative and associative)

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}.$$

Transpose of a matrix:

$$V = W^T \Leftrightarrow V_{ij} = W_{ji}$$

$$W = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad \text{has transpose} \quad W^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Substraction of two matrices:

V – W = V + (-1)W

B/ Matrices

Product of two matrices:

$$\mathbf{y} = W\mathbf{x} \qquad \mathbf{z} = V\mathbf{y}.$$

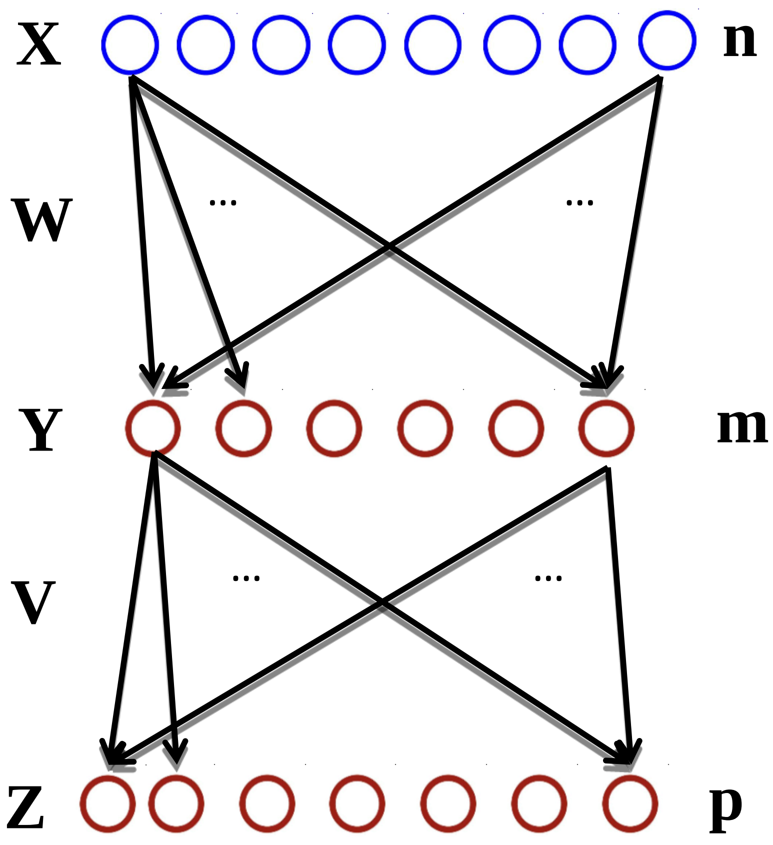
$$\mathbf{x} \xrightarrow{W} \mathbf{y} \xrightarrow{V} \mathbf{z}$$

$$\mathbb{R}^n \xrightarrow{W} \mathbb{R}^m \xrightarrow{V} \mathbb{R}^p.$$

$$z_i = \sum_j V_{ij} y_j = \sum_j V_{ij} \left(\sum_k W_{jk} x_k) \right) = \sum_k \left(\sum_j V_{ij} W_{jk} \right) x_k$$

$$z_i = \sum_k P_{ik} x_k \qquad P_{ik} = \sum_j V_{ij} W_{jk}.$$

$$\mathbf{z} = V\mathbf{y} = V(W\mathbf{x}) = (VW)\mathbf{x}.$$



Special matrices: **diagonal matrix**

$$\overleftrightarrow{D} = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}$$

$$\overleftrightarrow{D}\,\overrightarrow{x} = \begin{pmatrix} d_1 x_1 \\ d_2 x_2 \\ \vdots \\ d_n x_n \end{pmatrix}$$

- This acts like scalar multiplication

Special matrices: **identity matrix**

$$\overleftrightarrow{1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$\text{for all } \overleftrightarrow{A}, \quad \overleftrightarrow{1}\,\overleftrightarrow{A} = \overleftrightarrow{A}\,\overleftrightarrow{1} = \overleftrightarrow{A}$$

Special matrices: **inverse matrix**

$$\overleftrightarrow{A}\,\overleftrightarrow{A}^{-1} = \overleftrightarrow{A}^{-1}\overleftrightarrow{A} = \overleftrightarrow{1}$$

- Does the inverse always exist?

C/ Linear systems, matrices and determinants

Linear system

$$\begin{aligned} w_{11}x_1 + w_{12}x_2 + \ldots + w_{1n}x_n = &\quad y_1 \\ w_{21}x_1 + w_{22}x_2 + \ldots + w_{2n}x_n = &\quad y_2 \\ \ldots& \\ w_{n1}x_1 + w_{22}x_2 + \ldots + w_{nn}x_n = &\quad y_n \end{aligned}$$

$$\begin{aligned} 2x + 3y + z = &\quad 6 \\ x - y + z = &\quad 1 \\ x + y + z = &\quad 3 \end{aligned}$$

Matrix formulation:

Given: $W : \mathbf{x} \to \mathbf{y}$

Find $V : \mathbf{y} \to \mathbf{x}$ such that $\mathbf{x} \xrightarrow{W} \mathbf{y} \xrightarrow{V} \mathbf{x}$

i.e. VWx=x, or VW=I

$$W^{-1}(W\mathbf{x}) = \mathbf{x}$$

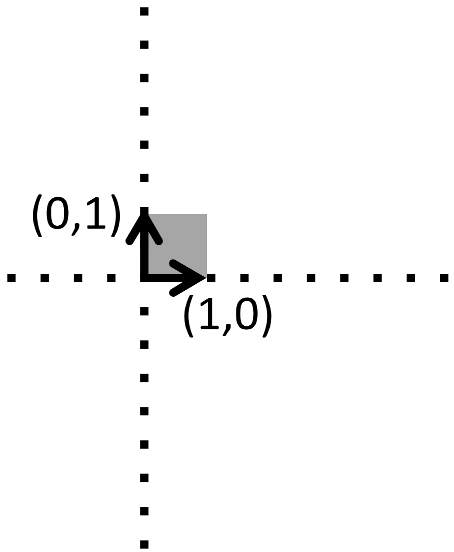
$$W^{-1}W = WW^{-1} = I$$

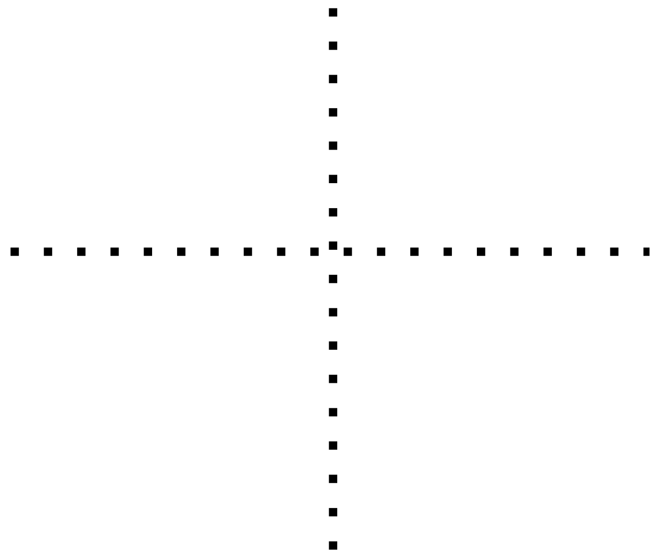
Some square matrices have no inverse and a non-invertible matrix is said to be *singular* (this is like the exception $k = 0$ for the scalar case). If an inverse does not exist it is for a very good reason. It happens when there is a non-zero vector $\mathbf{x}$ such that $W\mathbf{x} = 0$; $\mathbf{x}$ is then called a null-vector of W. If $\mathbf{x}$ is a null-vector then there are two different vectors, 0 and $\mathbf{x}$, which map to 0 under W, hence W cannot have an inverse (in general a function must be *one-to-one* to have an inverse).

Another test for singularity of a matrix W is to calculate its *determinant*, a scalar quantity denoted by $\det(W)$ (sometimes $|W|$) which we will discuss later. A matrix is singular if

$$\det(W) = 0$$

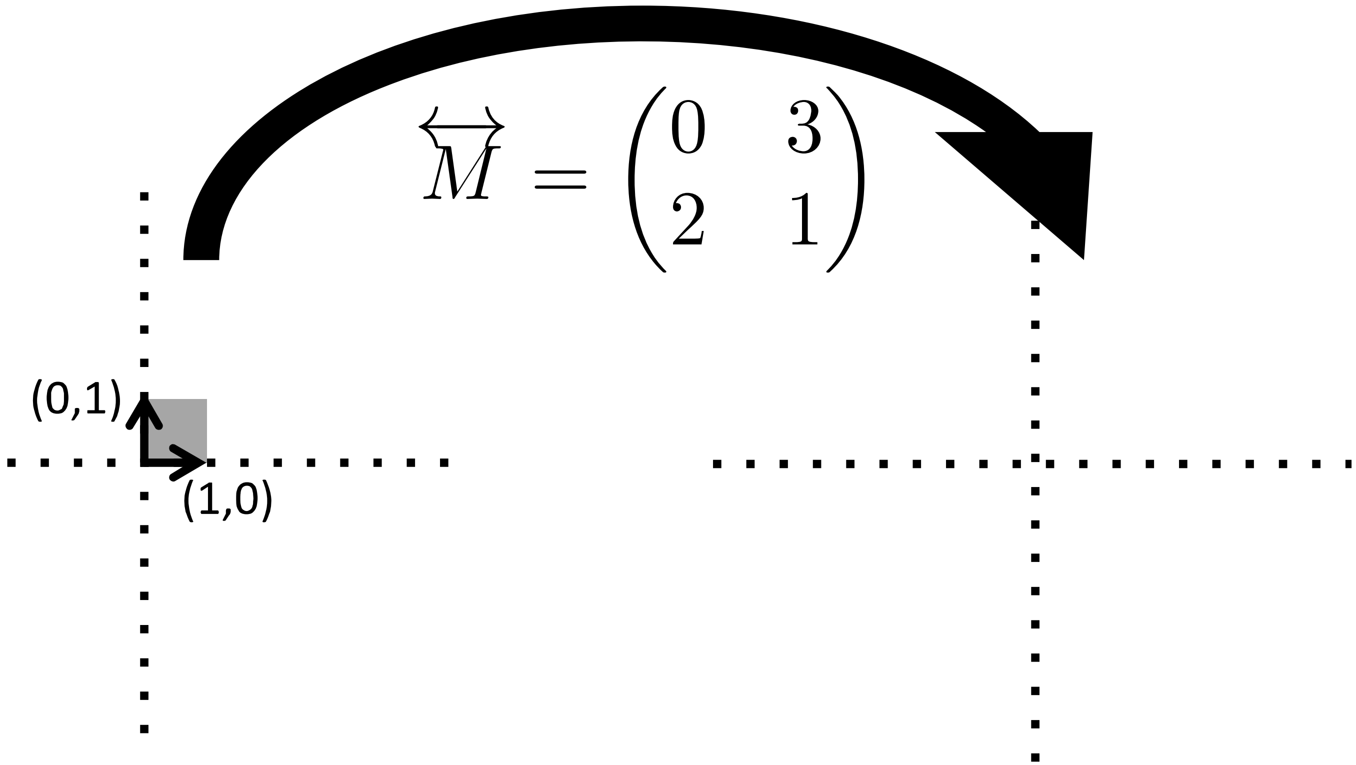
How does a matrix transform a square?





How does a matrix transform a square?

$$\overleftrightarrow{M} = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$$



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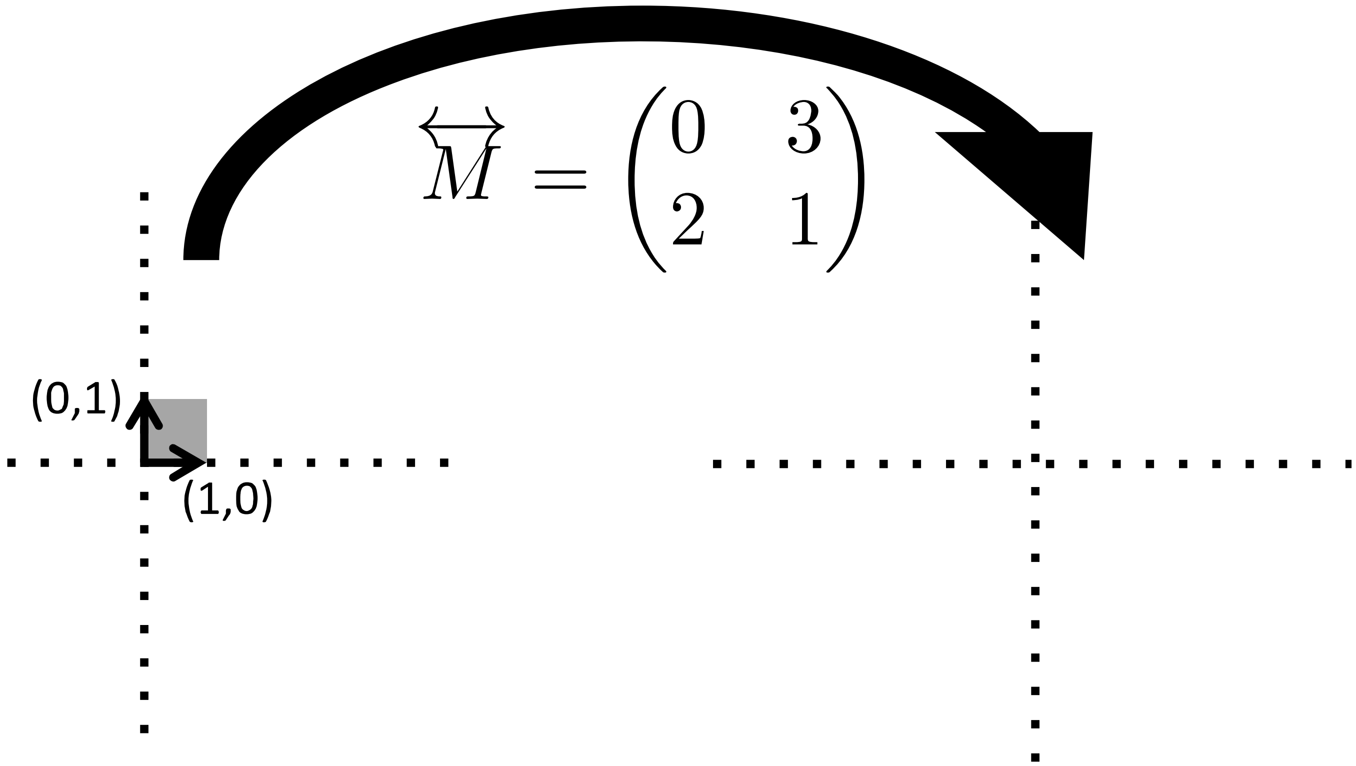
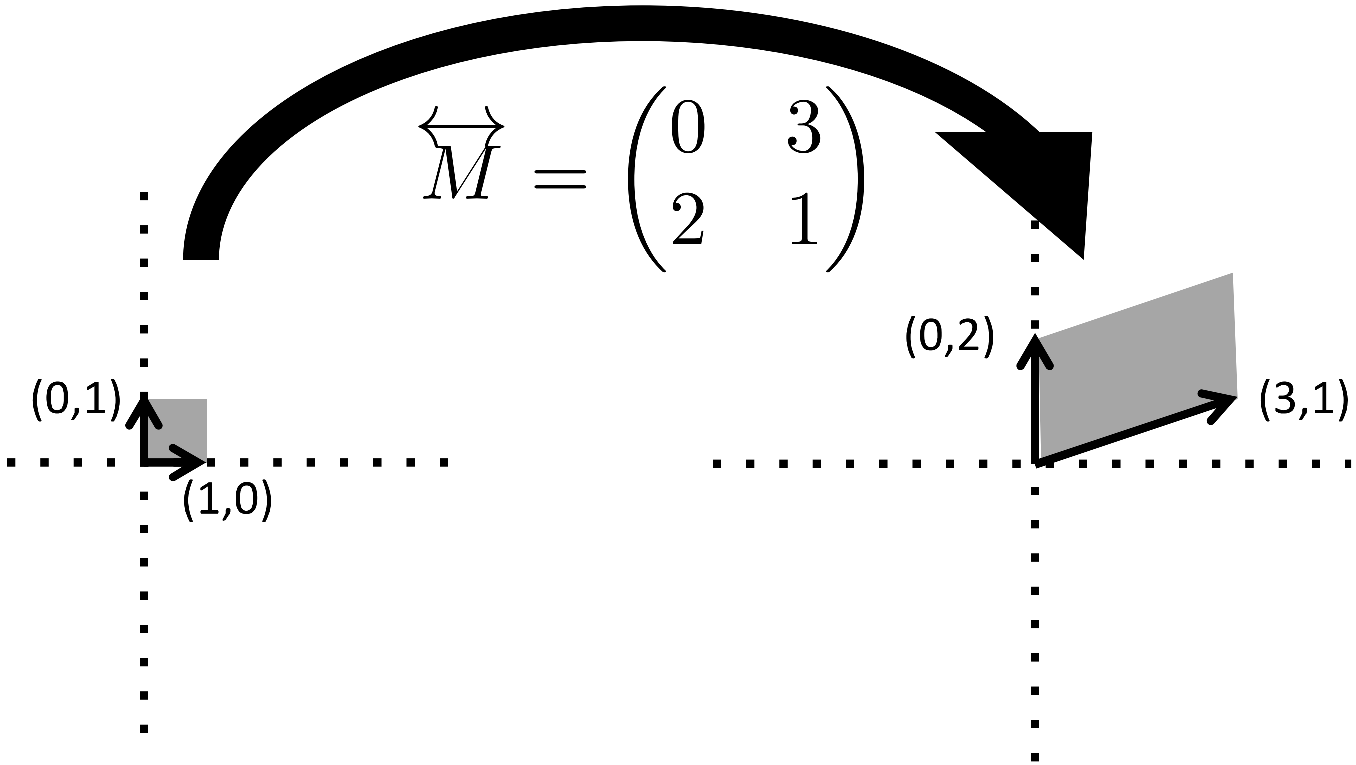
$$\overleftrightarrow{M} = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$$



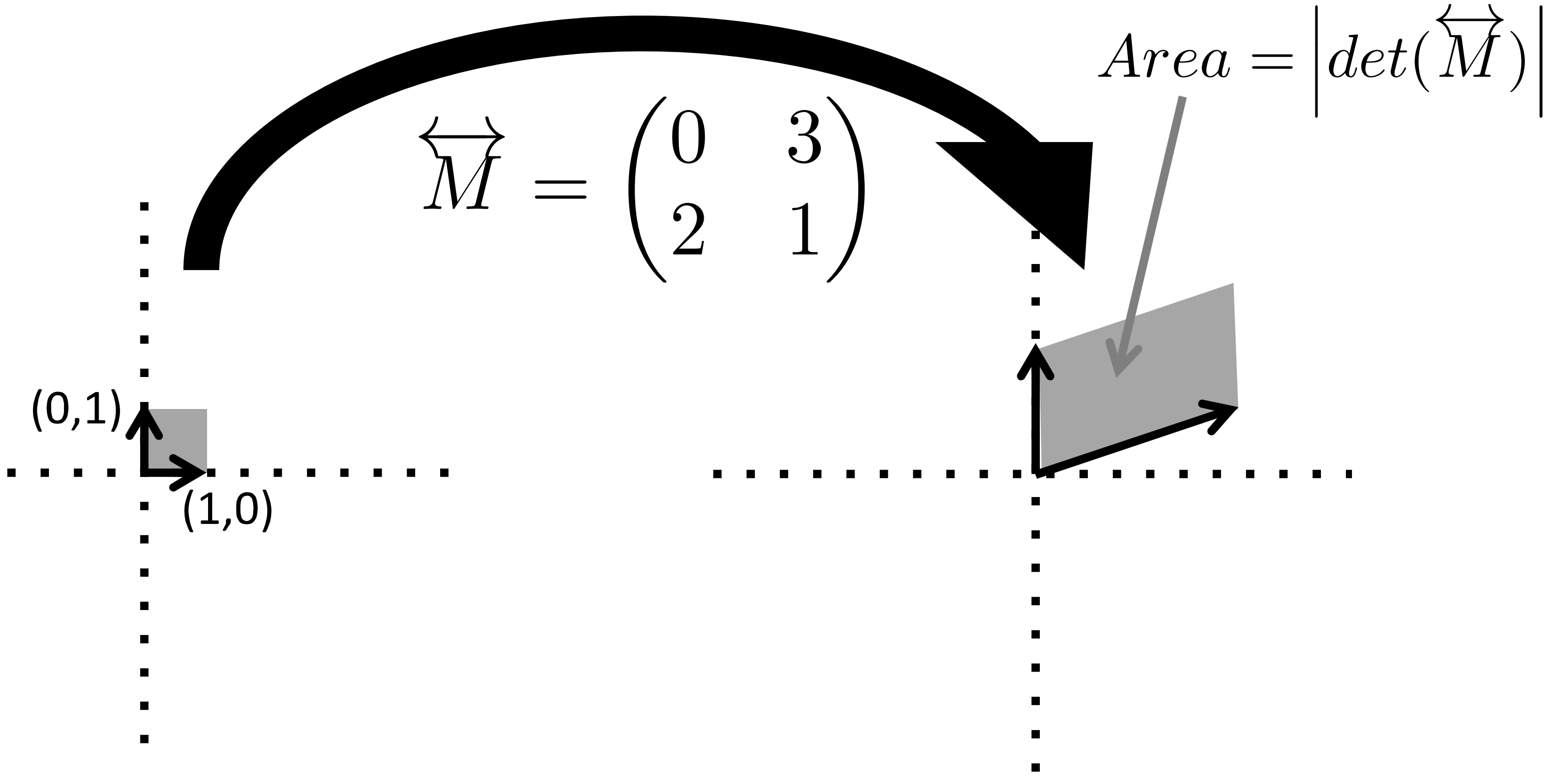
$$Mx?$$

How does a matrix transform a square?

$$\overleftrightarrow{M} = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$$



Geometric definition of the determinant: How does a matrix transform a square?



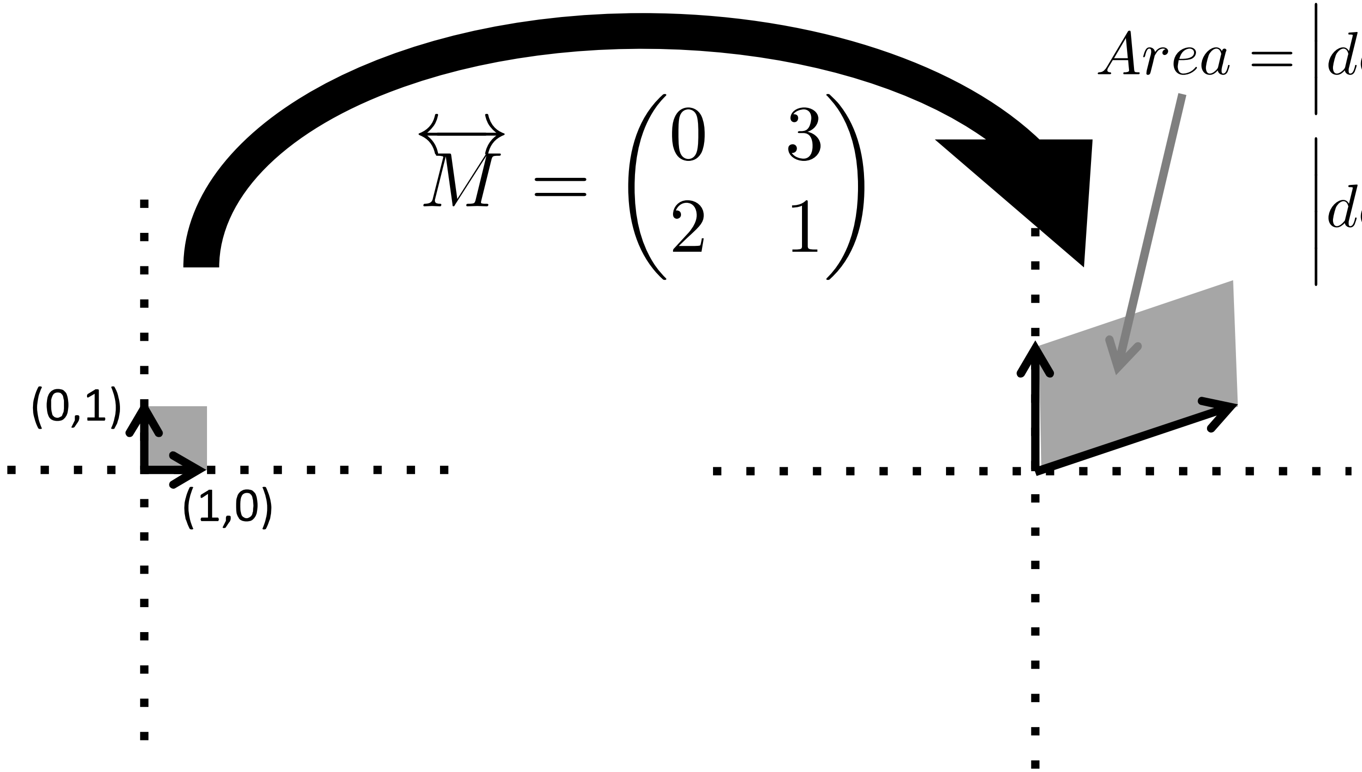
MATLAB: `det(M)`

Geometric definition of the determinant: How does a matrix transform a square?

$$\overleftrightarrow{M} = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$$

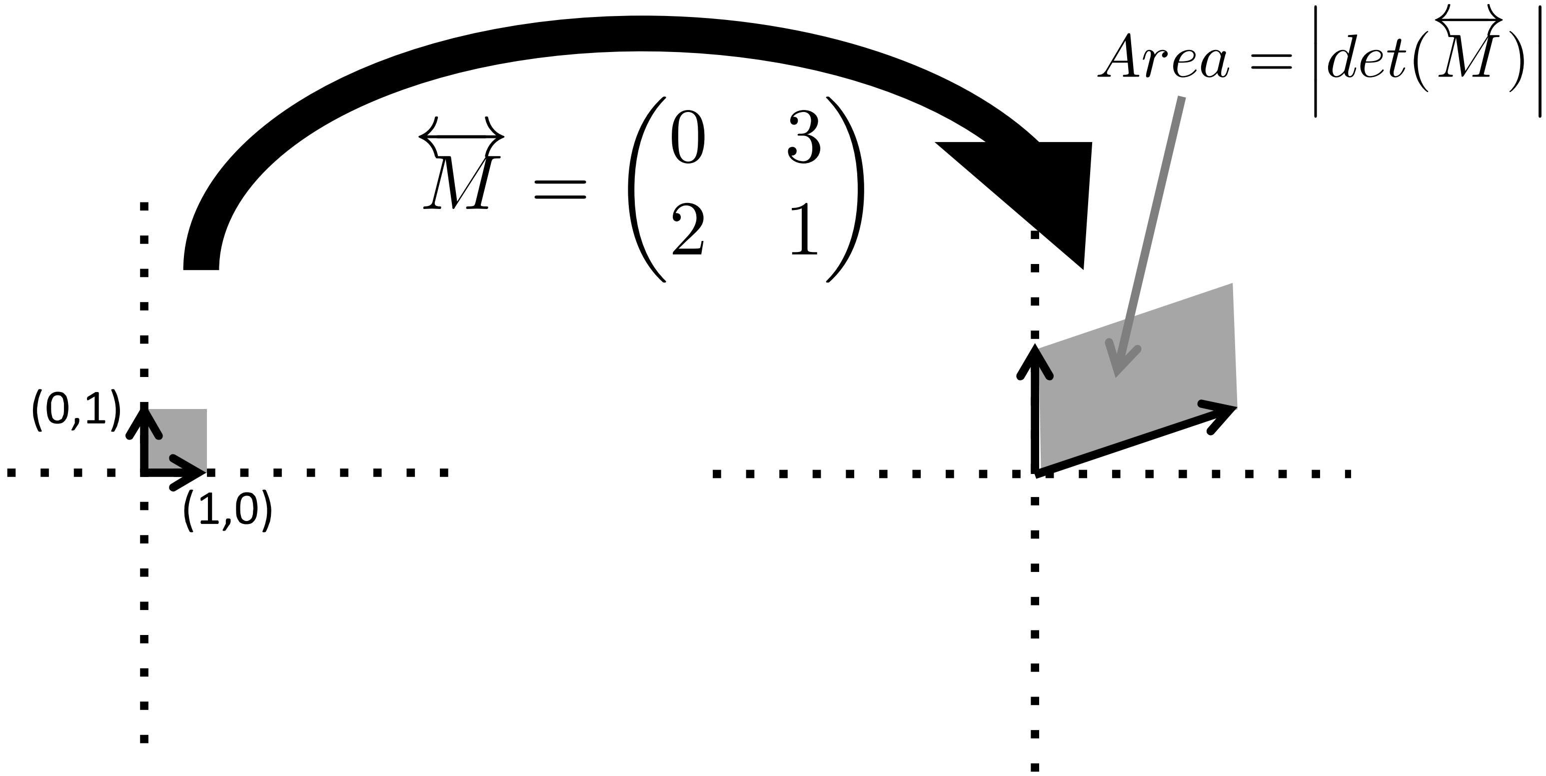
$$Area = \left| det(\overleftrightarrow{M}) \right|$$

$$\left| det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| = |ad - bc|$$



MATLAB: `det(M)`

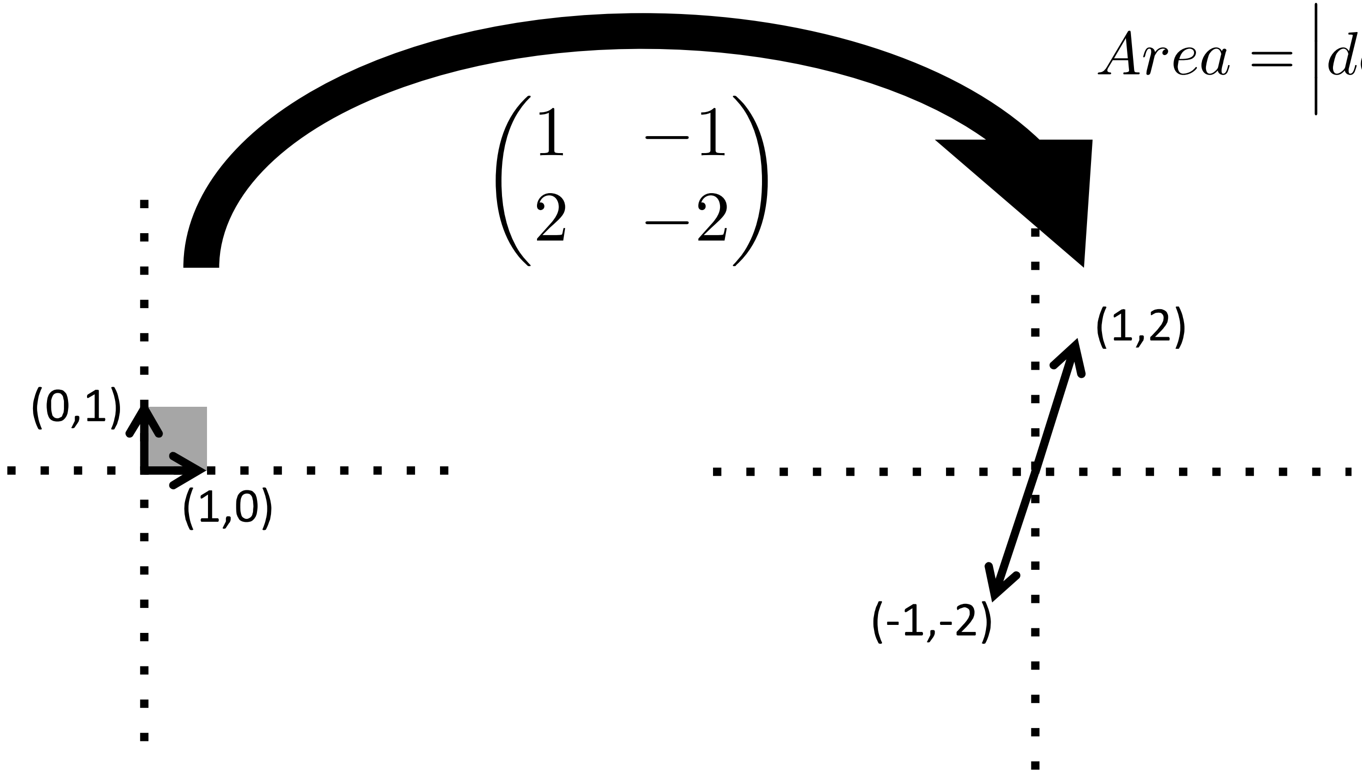
Geometric definition of the determinant: How does a matrix transform a square?



what about $\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$?

Geometric definition of the determinant: How does a matrix transform a square?

$$Area = \left| det(\overleftrightarrow{M}) \right|$$



no area!

this means the determinant is zero

what about $\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$?