Totally geodesic submanifolds of the homogeneous nearly Kähler six-manifolds

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UNIVERSIDADE DE SANTIAGO DE COMPOSTELA

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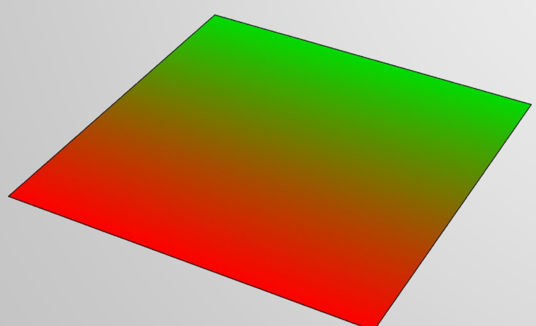
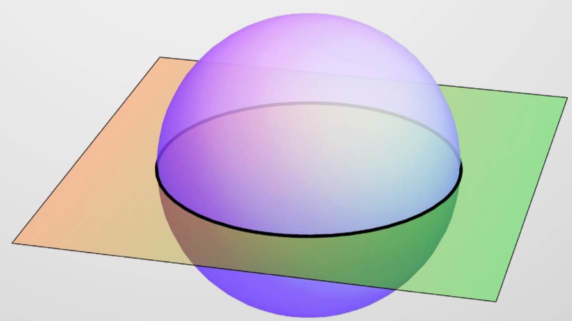
Totally geodesic submanifolds

- Σ, M complete Riemannian manifolds, $f\colon \Sigma \to M$ isometric immersion.
- f is *totally geodesic* if:
  - $\gamma(t)$ geodesic in $\Sigma \Rightarrow f(\gamma(t))$ geodesic in M.
  - $\mathbb{II} = 0$.



$\mathbb{R}^k \subseteq \mathbb{R}^n$



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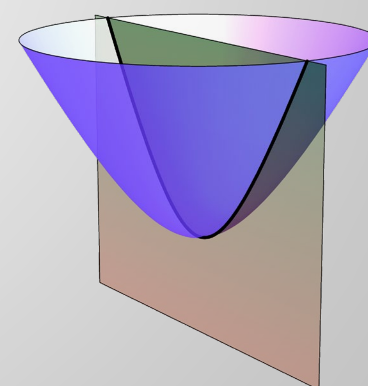

$\mathbb{R}\mathrm{H}^{\mathrm{k}} \subseteq \mathbb{R}\mathrm{H}^{\mathrm{n}}$

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General problem

Classify totally geodesic submanifolds of M up to congruence.

Nearly Kähler geometry

- (M, J) is nearly Kähler if $(\nabla_X J)X = 0$ for all $X \in \mathfrak{X}(M)$.
- Butruille (‘06): classification of homogeneous NK 6-manifolds:

$$\mathbb{S}^6 = \frac{\mathrm{G}_2}{\mathrm{SU}(3)} \qquad \mathbb{CP}^3 = \frac{\mathrm{Sp}(2)}{\mathrm{U}(1) \times \mathrm{Sp}(1)}$$

$$\mathrm{F}(\mathbb{C}^3) = \frac{\mathrm{SU}(3)}{\mathrm{T}^2} \qquad \mathbb{S}^3 \times \mathbb{S}^3 = \frac{\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)}{\Delta\mathrm{SU}(2)}$$

$$\begin{array}{c} \sigma \in \mathrm{Aut}(\mathrm{G}) \\ \sigma^3 = \mathrm{Id} \end{array} \longrightarrow \sigma_* = \frac{1}{2}\mathrm{Id} + \frac{\sqrt{3}}{2}J$$

Previously known results

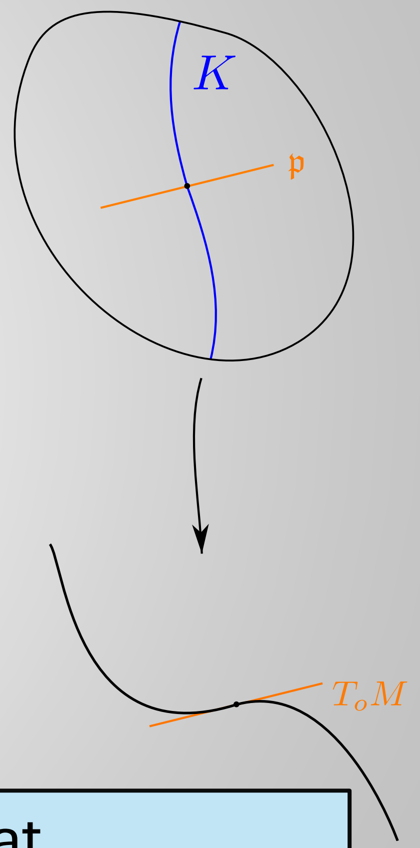
- $\mathrm{F}(\mathbb{C}^3)$:
  - Totally geodesic + Lagrangian (Storm '20).
  - Totally geodesic + almost complex + $2-\dim$ (Vrancken, Cwiklinski '22).
- $\mathbb{C}\mathrm{P}^3$:
  - Totally geodesic + Lagrangian (Aslan '23, Liefsoens '22).
- $\mathbb{S}^3 \times \mathbb{S}^3$:
  - Totally geodesic + Lagrangian (Zhang, Dioos, Hu, Vrancken, Wang '16).
  - Totally geodesic + almost complex + $2-\dim$ (Bolton, Dillen, Dioos, Vrancken '22).

Reductive homogeneous spaces

- $M = \mathrm{G}/\mathrm{K}$ homogeneous space, $o = e\mathrm{K}$.
- $\mathrm{G} \curvearrowright M$ gives a homomorphism $\mathrm{G} \to I(M)$.
- $X \in \mathfrak{g} \mapsto X^* \in \mathfrak{X}(M)$.

$$X_p^* = \frac{d}{dt}\Big|_{t=0} \mathrm{Exp}(tX) \cdot p$$

- $T_o M = \mathfrak{g}/\mathfrak{k}$ via $X + \mathfrak{k} \mapsto X_o^*$.



$M = \mathrm{G}/K$ is *reductive* if there is a $\mathfrak{p} \subseteq \mathfrak{g}$ such that
$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},\ \mathrm{Ad}(K)\mathfrak{p} = \mathfrak{p}.$$

- $\mathfrak{p} = \mathfrak{g}/\mathfrak{k} = T_o M$.
- Isotropy representation $\mathrm{K} \curvearrowright T_o M$ $\leftrightsquigarrow$ Adjoint representation $\mathrm{K} \curvearrowright \mathfrak{p}$.
- G-invariant tensor fields on M $\leftrightsquigarrow$ $\mathrm{Ad}(\mathrm{K})$-invariant tensors on $\mathfrak{p}$.

$\langle\cdot,\cdot\rangle$ G-invariant metric	$=$	$\langle\cdot,\cdot\rangle$ $\mathrm{Ad}(\mathrm{K})$-invariant metric on $\mathfrak{p}$

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Theorem (Nomizu '54)

Invariant connections $\nabla: \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$	=	Ad-equivariant maps $\alpha: \mathfrak{p} \otimes \mathfrak{p} \to \mathfrak{p}$

$$\alpha(X, Y) = \Lambda(X)Y = \nabla_{X^*} Y^* + [X, Y]_{\mathfrak{p}},$$

$$R(X, Y) = [\Lambda(X), \Lambda(Y)] - \Lambda\big([X, Y]_{\mathfrak{p}}\big) - \mathrm{ad}\big([X, Y]_{\mathfrak{p}}\big),$$

$$T(X, Y) = \alpha(X, Y) - \alpha(Y, X) - [X, Y]_{\mathfrak{p}}.$$

Natural reductivity

$$\alpha(X,Y) = \frac{1}{2}[X,Y]_{\mathfrak{p}} + U(X,Y),$$

$$2\langle U(X,Y), Z\rangle = \left\langle [Z,X]_{\mathfrak{p}}, Y\right\rangle + \left\langle X, [Z,Y]_{\mathfrak{p}}\right\rangle.$$

- $M = \mathrm{G}/\mathrm{K}$ is *naturally reductive* if $U = 0$.

  $\Leftrightarrow \exp_o(tX) = \mathrm{Exp}(tX) \cdot o$ for all $X \in \mathfrak{p}$.
- $M = \mathrm{G}/\mathrm{K}$ is *normal homogeneous* if:
  1. G has a bi-invariant metric.
  2. The complement $\mathfrak{p} = \mathfrak{g} \ominus \mathfrak{k}$.
  3. The metric on M is induced from the metric on G.

Totally geodesic submanifolds of G/K

- $\Sigma \subseteq M$ totally geodesic, $p \in \Sigma$. Then,

$$\Sigma = \exp_p(T_p\Sigma).$$

- If $V \subseteq T_pM$, when is $\exp_p(V)$ totally geodesic?

Theorem (Cartan '51, Hermann '59)

M analytic Riemannian manifold, $V \subseteq T_pM$. The following are equivalent:

- V generates a totally geodesic submanifold.
- V is $\nabla^k R$-invariant for all $k = 0,1,2, \dots$

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$$\Sigma = \exp_p(T_p\Sigma).$$
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Theorem (Cartan) for naturally reductive spaces

M naturally reductive, $\mathfrak{v} \subseteq \mathfrak{p}$. The following are equivalent:

- $\mathfrak{v}$ generates a totally geodesic submanifold.
- $\mathfrak{v}$ is $\nabla^k R$-invariant for all $k = 0,1,2,\ldots,d$.

Tojo's criterion ('96)

M naturally reductive, $\mathfrak{v} \subseteq \mathfrak{p}$. The following are equivalent:

- $\mathfrak{v}$ generates a totally geodesic submanifold.
- $R\left(e^{-\Lambda(X)}\mathfrak{v}, e^{-\Lambda(X)}\mathfrak{v}\right)e^{-\Lambda(X)}\mathfrak{v} \subseteq e^{-\Lambda(X)}\mathfrak{v}$ for all $X \in \mathfrak{v}$.

Sufficient criterion (LN—Rodríguez-Vázquez)

M naturally reductive, $\mathfrak{v} \subseteq \mathfrak{p}$ invariant under R and α:

1. The subalgebra $\mathfrak{s} = [\mathfrak{v}, \mathfrak{v}] + \mathfrak{v} = [\mathfrak{v}, \mathfrak{v}]_{\mathfrak{k}} \oplus \mathfrak{v}$ satisfies

$$\mathfrak{s} = (\mathfrak{s} \cap \mathfrak{k}) \oplus (\mathfrak{s} \cap \mathfrak{p}) = \mathfrak{s}_{\mathfrak{k}} \oplus \mathfrak{s}_{\mathfrak{p}}.$$

2. The set $\exp_o(\mathfrak{v}) = S \cdot o$ is totally geodesic.

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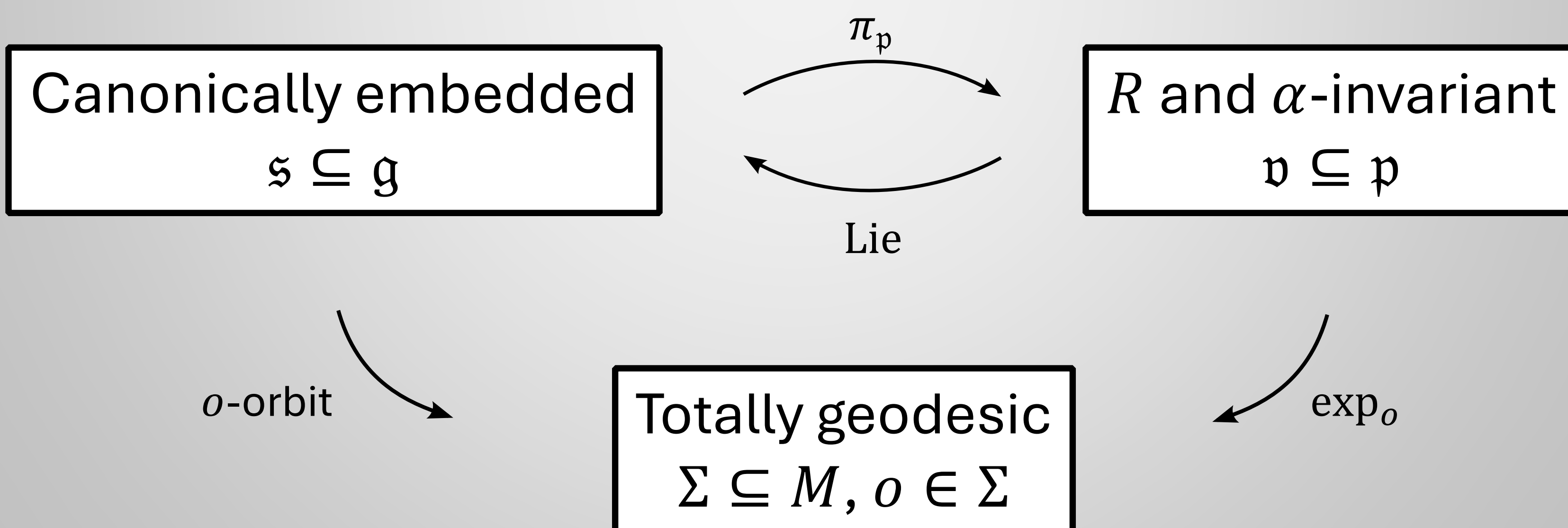

The flag manifold $\mathrm{F}(\mathbb{C}^3)$

- $\mathrm{F}(\mathbb{C}^3) = \frac{\mathrm{SU}(3)}{\mathrm{T}^2}$; $I^0\big(\mathrm{F}(\mathbb{C}^3)\big) = \mathrm{PSU}(3)$.

- $\mathfrak{p} = \left\{ \begin{pmatrix} 0 & x & z \\ -\bar{x} & 0 & y \\ -\bar{z} & -\bar{y} & 0 \end{pmatrix} : x, y, z \in \mathbb{C} \right\} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$.

$$\begin{tikzcd} & e^{i(\theta_1 - \theta_2)} \\ \mathrm{diag}\big(e^{i\theta_1}, e^{i\theta_2}, e^{-i(\theta_1+\theta_2)}\big) \arrow[ur] \arrow[r] \arrow[dr] & e^{i(\theta_1 + 2\theta_2)} \\ & e^{i(2\theta_1 + \theta_2)} \end{tikzcd}$$

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- $\langle X, Y \rangle = -\mathrm{tr}(XY)$

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General fact:
$\mathrm{K} \subseteq \mathrm{H} \subseteq \mathrm{G}$ gives a fibration
$\mathrm{H}/\mathrm{K} \to \mathrm{G}/\mathrm{K} \to \mathrm{G}/\mathrm{H}.$

- $\mathfrak{p} = \left\{ \begin{pmatrix} 0 & x & z \\ -\bar{x} & 0 & y \\ -\bar{z} & -\bar{y} & 0 \end{pmatrix} : x, y, z \in \mathbb{C} \right\} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3.$
- $T^2 \subseteq \mathrm{U}(2) \subseteq \mathrm{SU}(3)$ gives the fibration $\mathbb{C}\mathrm{P}^1 \to \mathrm{F}(\mathbb{C}^3) \to \mathbb{C}\mathrm{P}^2.$
- $\mathcal{V}_o = \mathfrak{p}_1, \mathcal{H}_o = \mathfrak{p}_2 \oplus \mathfrak{p}_3.$

The complex projective space $\mathbb{CP}^3$

- $\mathbb{CP}^3 = \frac{\mathrm{Sp}(2)}{\mathrm{U}(1)\times\mathrm{Sp}(1)}$; $I^0(\mathbb{CP}^3) = \frac{\mathrm{Sp}(2)}{\mathbb{Z}^2}$.

- $\mathfrak{p} = \left\{ \begin{pmatrix} zj & -\bar{q} \\ q & 0 \end{pmatrix} : z \in \mathbb{C}, q \in \mathbb{H} \right\} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$.

$$(\lambda, \mu) \cdot z = \lambda^2 z$$
$$(\lambda, \mu) \cdot q = \mu q \bar{\lambda}$$

The complex projective space $\mathbb{C}\mathrm{P}^3$

- $\mathbb{C}\mathrm{P}^3 = \frac{\mathrm{Sp}(2)}{\mathrm{U}(1)\times\mathrm{Sp}(1)}$; $I^0(\mathbb{C}\mathrm{P}^3) = \frac{\mathrm{Sp}(2)}{\mathbb{Z}^2}$.
- $\mathfrak{p} = \left\{ \begin{pmatrix} zj & -\bar{q} \\ q & 0 \end{pmatrix} : z \in \mathbb{C}, q \in \mathbb{H} \right\} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$.
- $\langle X, Y \rangle = -2\mathrm{Re}\,\mathrm{tr}_{\mathbb{H}}(XY)$.

The complex projective space $\mathbb{CP}^3$

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- $\mathfrak{p} = \left\{\begin{pmatrix} zj & -\bar{q} \\ q & 0 \end{pmatrix} : z \in \mathbb{C}, q \in \mathbb{H}\right\} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$.
- $\mathrm{U}(1) \times \mathrm{Sp}(1) \subseteq \mathrm{Sp}(1) \times \mathrm{Sp}(1) \subseteq \mathrm{Sp}(2)$ gives the fibration $\mathbb{CP}^1 \to \mathbb{CP}^3 \to \mathbb{HP}^1 = \mathbb{S}^4$.
- $\mathcal{V}_o = \mathfrak{p}_1, \mathcal{H}_o = \mathfrak{p}_2$.

The almost product $\mathbb{S}^3 \times \mathbb{S}^3$

- $\mathbb{S}^3 \times \mathbb{S}^3 = \frac{\mathrm{SU}(2)^3}{\Delta \mathrm{SU}(2)}$; $I^0(\mathbb{S}^3 \times \mathbb{S}^3) = \frac{\mathrm{SU}(2)^3}{\Delta \mathbb{Z}_2}$.
- $\mathfrak{p} = \{(X, Y, Z): X + Y + Z = 0\} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$.
  - $\mathfrak{p}_1 = \{(X, -2X, X): X \in \mathfrak{su}(2)\} \cong \mathrm{Ad}$.
  - $\mathfrak{p}_2 = \{(X, 0, -X): X \in \mathfrak{su}(2)\} \cong \mathrm{Ad}$.
- $\langle (X, Y, Z), (X', Y', Z') \rangle = -\mathrm{tr}(XX') - \mathrm{tr}(YY') - \mathrm{tr}(ZZ')$.

The almost product $\mathbb{S}^3 \times \mathbb{S}^3$

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- $\mathfrak{p} = \{(X, Y, Z) : X + Y + Z = 0\} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$.
- $\Delta SU(2) \subseteq \Delta_{13} SU(2) \times SU(2)_2 \subseteq SU(2)^3$ gives the fibration $\mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{S}^3$.
- $\mathcal{V}_o = \mathfrak{p}_1, \mathcal{H}_o = \mathfrak{p}_2$.

Low codimension

Theorem (Nikolayevsky ‘15)

M simply connected, compact and homogeneous.
If there exists $\Sigma \subseteq M$ totally geodesic hypersurface, then $M = \mathbb{S}^n$.

Corollary

$M \in \{\mathrm{F}(\mathbb{C}^3), \mathbb{C}\mathrm{P}^3, \mathbb{S}^3 \times \mathbb{S}^3\}$ has no totally geodesic hypersurfaces.

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If there exists $\Sigma \subseteq M$ totally geodesic hypersurface, then $M = \mathbb{S}^n$.

Theorem

$M \in \{\mathrm{F}(\mathbb{C}^3), \mathbb{C}\mathrm{P}^3, \mathbb{S}^3 \times \mathbb{S}^3\}$ has no totally geodesic submanifolds of codimension ≤ 2.

Dimension three

Ambient	Submanifold	Orbit of
$F(\mathbb{C}^3)$	$F(\mathbb{R}^3) = \mathbb{S}^3/Q_8$	$SO(3)$
	$\mathbb{S}^3_{\mathbb{C}}$	$SU(2)$
$\mathbb{CP}^3$	$\mathbb{RP}^3_{\mathbb{C}}$	$SU(2)^j$
$\mathbb{S}^3 \times \mathbb{S}^3$	$\mathbb{S}^3(2/\sqrt{3})$	$SU(2)_2$
	$\mathbb{S}^3_{\mathbb{C}}$	$\Delta_{13}SU(2) \times SU(2)_2$

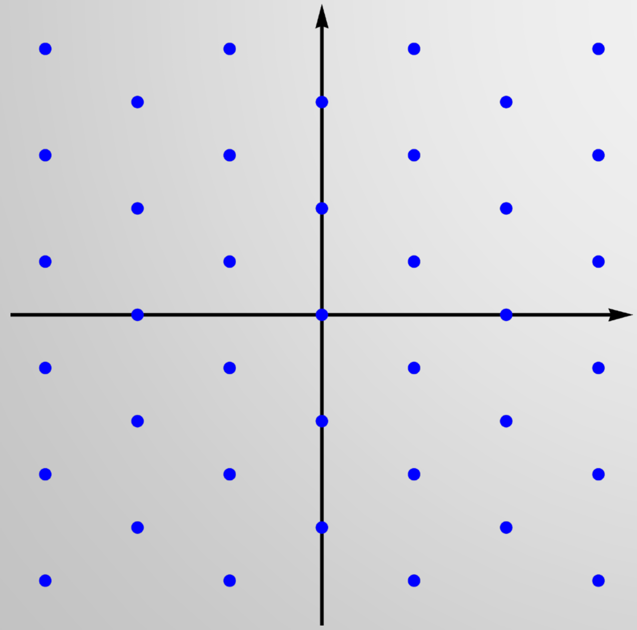
All of these are Lagrangian.

Dimension two in $\mathrm{F}(\mathbb{C}^3)$

Submanifold	Orbit of	Relationship with J
$\mathrm{T_A} = \mathbb{R}^2/\mathrm{A}$	T^2	Almost complex
$\mathbb{C}\mathrm{P}^1 = \mathbb{S}^2(1/\sqrt{2})$	$\mathrm{U}(2)$	Almost complex
$\mathbb{S}^2(\sqrt{2})$	$\mathrm{SO}(3)$	Almost complex
$\mathbb{R}\mathrm{P}^2 \to \mathrm{F}(\mathbb{R}^3)$	Inhomogeneous	Totally real

Dimension two in $F(\mathbb{C}^3)$

Submanifold	Orbit of	Relationship with J
$T_A = \mathbb{R}^2/A$	T^2	Almost complex



$$A = \mathbb{Z} - \mathrm{span}\left\{\left(0, \frac{2\sqrt{2}\pi}{\sqrt{3}}\right), \left(\sqrt{2}\pi, \frac{\sqrt{2}}{\sqrt{3}}\pi\right)\right\}$$

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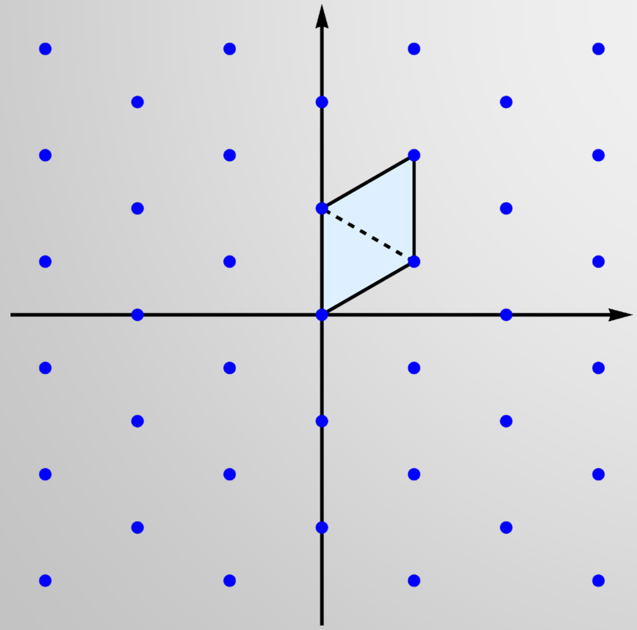

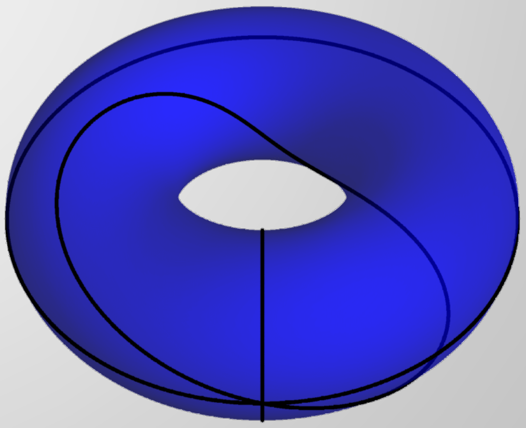

Dimension two in $\mathbb{C}\mathrm{P}^3$

Submanifold	Orbit of	Relationship with J
$\mathbb{S}^2(1/\sqrt{2})$	$\mathrm{Sp}(1)_f$	Almost complex
$\mathbb{S}^2(1)$	$\mathrm{SU}(2)$	Almost complex
$\mathbb{S}^2(\sqrt{5})$	$\mathrm{SU}(2)_{\Lambda_3}$	Almost complex

$\Lambda_3 = \mathrm{Sym}^3(\mathbb{C}^2)$: three-dimensional irrep. of $SU(2)$.

$$\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \longmapsto \begin{pmatrix} a(|a|^2 - 2|b|^2) & -\sqrt{3}a^2\bar{b} \\ \sqrt{3}a^2\bar{b} & a^3 \end{pmatrix} + j \begin{pmatrix} b(2|a|^2 - |b|^2) & -\sqrt{3}ab^2 \\ \sqrt{3}\bar{a}b^2 & -b^3 \end{pmatrix}$$

Dimension two in $\mathbb{S}^3 \times \mathbb{S}^3$

Submanifold	Orbit of	Relationship with J
$\mathrm{T_B} = \mathbb{R}^2/\mathrm{B}$	$\mathrm{T} \subseteq \mathrm{U}(1)^3$	Almost complex
$\mathbb{S}^2\left(\sqrt{3}/\sqrt{2}\right)$	$\Delta SU(2)$	Almost complex
$\mathbb{S}^2 \subseteq \mathbb{S}^3\left(2/\sqrt{3}\right)$	$\mathrm{H} \subseteq \Delta_{13}\mathrm{SU}(2) \times \mathrm{SU}(2)_2$	Totally real

Dimension two in $\mathbb{S}^3 \times \mathbb{S}^3$

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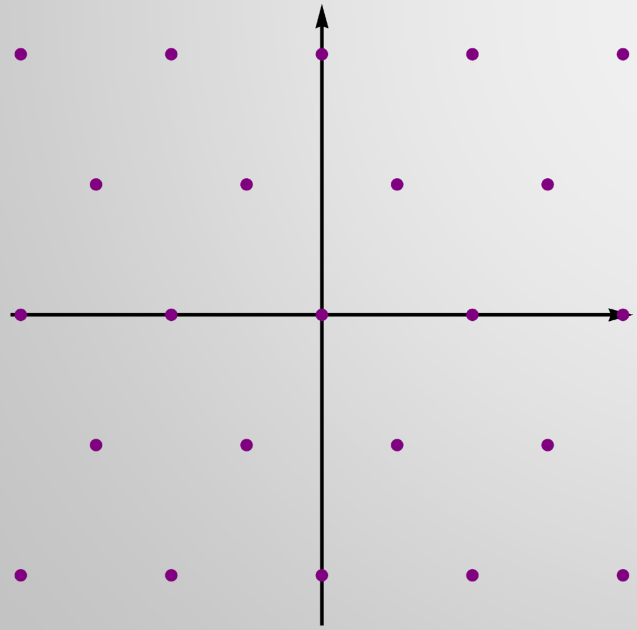

$$\mathrm{B} = \mathbb{Z} - \mathrm{span}\left\{\left(\frac{4\pi}{\sqrt{3}}, 0\right), \left(\frac{2\pi}{\sqrt{3}}, 2\pi\right)\right\}$$

Dimension two in $\mathbb{S}^3 \times \mathbb{S}^3$

Submanifold	Orbit of	Relationship with J
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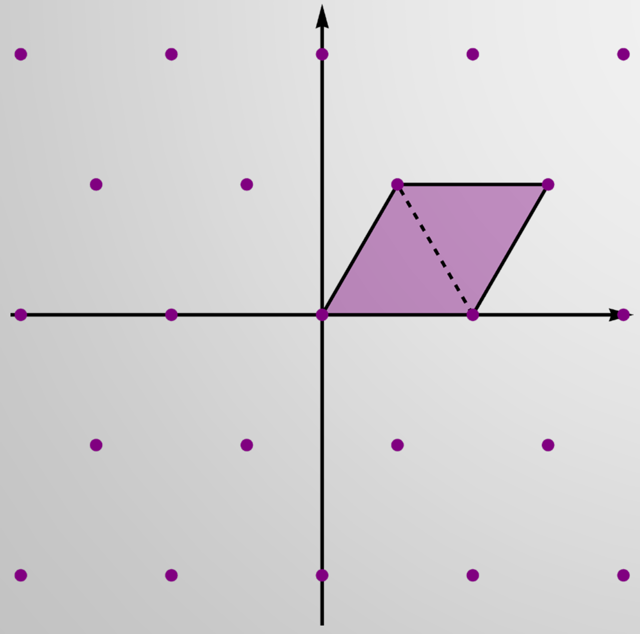

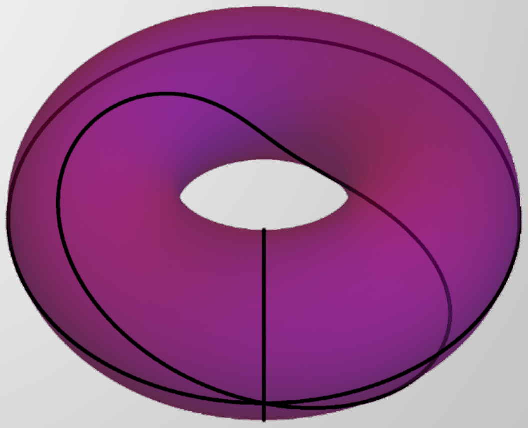

Ambient	Submanifold	Orbit of	Relationship with J
$F(\mathbb{C}^3)$	$F(\mathbb{R}^3)$	$SO(3)$	Lagrangian
	$\mathbb{S}^3_{\mathbb{C}}$	$SU(2)$	Lagrangian
	T_A	T^2	Almost complex
	$\mathbb{C}P^1 = \mathbb{S}^2(1/\sqrt{2})$	$U(2)$	Almost complex
	$\mathbb{S}^2(\sqrt{2})$	$SO(3)$	Almost complex
	$\mathbb{R}P^2 \to F(\mathbb{R}^3)$	Inhomogeneous	Totally real
$\mathbb{C}P^3$	$\mathbb{R}P^3_{\mathbb{C}}$	$SU(2)^j$	Lagrangian
	$\mathbb{S}^2(1/\sqrt{2})$	$Sp(1)_f$	Almost complex
	$\mathbb{S}^2(1)$	$SU(2)$	Almost complex
	$\mathbb{S}^2(\sqrt{5})$	$SU(2)_{\Lambda_3}$	Almost complex
$\mathbb{S}^3 \times \mathbb{S}^3$	$\mathbb{S}^3(2/\sqrt{3})$	$SU(2)_2$	Lagrangian
	$\mathbb{S}^3_{\mathbb{C}}$	$\Delta_{13}SU(2) \times SU(2)_2$	Lagrangian
	T_B	$T \subseteq U(1)^3$	Almost complex
	$\mathbb{S}^2(\sqrt{3}/\sqrt{2})$	$\Delta SU(2)$	Almost complex
	$\mathbb{S}^2 \subseteq \mathbb{S}^3(2/\sqrt{3})$	$H \subseteq \Delta_{13}SU(2) \times SU(2)_2$	Totally real


That's all Folks!

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	$\mathbb{S}^2 \subseteq \mathbb{S}^3(2/\sqrt{3})$	$H \subseteq \Delta_{13}SU(2) \times SU(2)_2$	Totally real