Nearly Kähler geometry and totally geodesic submanifolds

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CENTRO DE INVESTIGACIÓN
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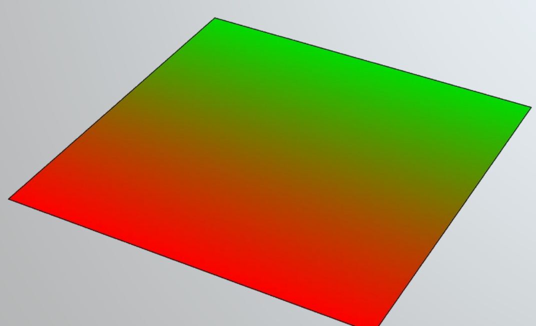
Preliminaries

- M complete Riemannian manifold.
- Metric $\langle \cdot,\cdot \rangle : T_pM \times T_pM \to \mathbb{R}$.
- Levi-Civita connection $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$.
- Curvature tensor $R : T_pM \times T_pM \times T_pM \to T_pM$.

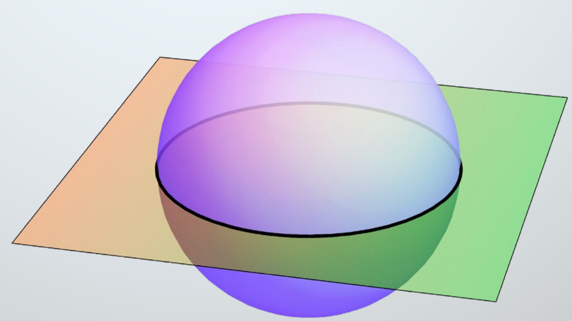
$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Totally geodesic submanifolds

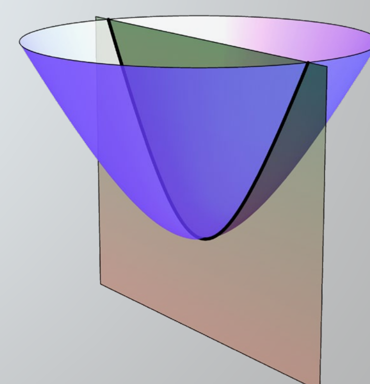
- M, Σ complete Riemannian manifolds, $f : \Sigma \to M$ isometric immersion.
- f is *totally geodesic* if:
  - $\gamma(t)$ geodesic in $\Sigma \Rightarrow f\big(\gamma(t)\big)$ geodesic in M.
  - The second fundamental form $\mathbb{II}$ vanishes.



$\mathbb{R}^k \subseteq \mathbb{R}^n$



$\mathbb{S}^k \subseteq \mathbb{S}^n$



$\mathbb{R}\mathrm{H}^{\mathrm{k}} \subseteq \mathbb{R}\mathrm{H}^{\mathrm{n}}$

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General problem

Given M, classify all totally geodesic submanifolds of M up to congruence.

Nearly Kähler spaces

- (M, J) almost Hermitian manifold.
  - $J: T_pM \to T_pM$ such that $J^2 = -\mathrm{Id}_{T_pM}$.
- M is *nearly Kähler* if ∇J is skew-symmetric.
- Butruille ('06): classification of homogeneous NK 6-manifolds:

$$\mathbb{S}^6 = \frac{\mathrm{G}_2}{\mathrm{SU}(3)} \qquad \mathbb{C}\mathrm{P}^3 = \frac{\mathrm{Sp}(2)}{\mathrm{U}(1) \times \mathrm{Sp}(1)}$$

$$\mathrm{F}(\mathbb{C}^3) = \frac{\mathrm{SU}(3)}{\mathrm{T}^2} \qquad \mathbb{S}^3 \times \mathbb{S}^3 = \frac{\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)}{\Delta\mathrm{SU}(2)}$$

Previously known results

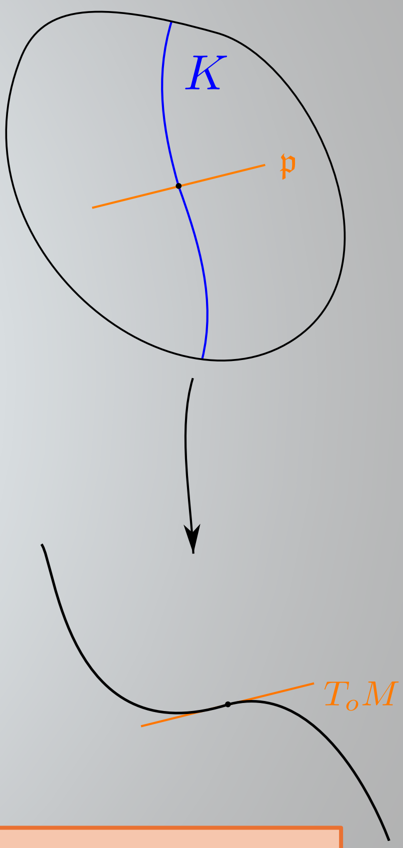
- $F(\mathbb{C}^3)$:
  - Totally geodesic + Lagrangian (Storm ‘20).
  - Totally geodesic + almost complex + $2-\dim$ (Vrancken, Cwiklinski ‘22).
- $\mathbb{C}P^3$:
  - Totally geodesic + Lagrangian (Aslan ‘23, Liefsoens ‘22).
- $\mathbb{S}^3 \times \mathbb{S}^3$:
  - Totally geodesic + Lagrangian (Zhang, Dioos, Hu, Vrancken, Wang ‘16).
  - Totally geodesic + almost complex + $2-\dim$ (Bolton, Dillen, Dioos, Vrancken ‘22).

Reductive homogeneous spaces

- $M = \mathrm{G}/\mathrm{K}$ homogeneous space, $o = e\mathrm{K}$.
- $\mathrm{G} \curvearrowright M$ gives a homomorphism $\mathrm{G} \to I(M)$.
- Anti-homomorphism $X \in \mathfrak{g} \to \mathfrak{X}(M)$.

$$X_p^* = \frac{d}{dt}\Big|_{t=0} \mathrm{Exp}(tX) \cdot p$$

- $T_oM = \mathfrak{g}/\mathfrak{k}$ via $X + \mathfrak{k} \mapsto X_o^*$.



M is *reductive* if there exists $\mathfrak{p} \subseteq \mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},\ \mathrm{Ad}(\mathrm{K})\mathfrak{p} = \mathfrak{p}.$$

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- $\mathfrak{p} = \mathfrak{g}/\mathfrak{k} = T_o M.$
- Isotropy representation $\mathrm{K} \curvearrowright T_o M \leftrightsquigarrow$ Adjoint representation $\mathrm{K} \curvearrowright \mathfrak{p}$.
- G-invariant tensor fields on $M \leftrightsquigarrow \mathrm{Ad}(\mathrm{K})$-invariant tensors on $\mathfrak{p}$.

Connection and curvature

- M has a *canonical connection* ∇^c.

$$(\nabla^c_{X^*} Y^*)_o = -[X, Y]_{\mathfrak{p}}, \quad X, Y \in \mathfrak{p}.$$

- $D = \nabla - \nabla^c$ *difference tensor*.

$$D_X Y = \frac{1}{2}[X, Y]_{\mathfrak{p}} + U(X, Y),$$

$$2\langle U(X, Y), Z\rangle = \big\langle [Z, X]_{\mathfrak{p}}, Y\big\rangle + \big\langle X, [Z, Y]_{\mathfrak{p}}\big\rangle,$$

$$(\nabla_{X^*} Y^*)_o = -[X, Y]_{\mathfrak{p}} + D_X Y,$$

$$R(X, Y) = [D_X, D_Y] - D_{[X,Y]_{\mathfrak{p}}} - \mathrm{ad}([X, Y]_{\mathfrak{k}}).$$

Natural reductivity

- $M = \mathrm{G}/\mathrm{K}$ is *naturally reductive* if $U = 0$.

  $\Leftrightarrow \exp_{\mathrm{o}}(tX) = \mathrm{Exp}(tX) \cdot o$ for all $X \in \mathfrak{p}$.

- M is normal homogeneous if:

  1. G has a bi-invariant metric.
  2. The complement $\mathfrak{p} = \mathfrak{g} \ominus \mathfrak{k}$.
  3. The metric on M is induced from the metric on G.

Totally geodesic submanifolds of G/K

- $\Sigma \subseteq M$ totally geodesic, $p \in \Sigma$. Then
$$\Sigma = \exp_p(T_p\Sigma).$$
- If $V \subseteq T_pM$, when is $\exp_p(V)$ totally geodesic?

Theorem (Cartan ’51, Hermann ‘59)

M analytic Riemannian manifold, $V \subseteq T_pM$. The following are equivalent:

- V generates a (complete) totally geodesic submanifold.
- V is $\nabla^k R$-invariant for all $k = 0,1,2, \ldots$

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$$\Sigma = \exp_p\left(T_p\Sigma\right).$$
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Theorem (Cartan) for naturally reductive spaces

M naturally reductive, $\mathfrak{v} \subseteq \mathfrak{p}$. The following are equivalent:

- $\mathfrak{v}$ generates a (complete) totally geodesic submanifold.
- $\mathfrak{v}$ is $\nabla^k R$-invariant for all $k = 0,1,2, \dots, d$.

Tojo's criterion ('96)

M naturally reductive, $\mathfrak{v} \subseteq \mathfrak{p}$. The following are equivalent:

- $\mathfrak{v}$ generates a (complete) totally geodesic submanifold.
- $e^{-D_X}\mathfrak{v}$ is R-invariant for all $X \in \mathfrak{v}$.

Sufficient criterion (Sagle '68, LN—Rodríguez-Vázquez)

M naturally reductive, $\mathfrak{v} \subseteq \mathfrak{p}$ invariant under R and D.

- The subalgebra $\mathfrak{s} = [\mathfrak{v}, \mathfrak{v}]_{\mathfrak{k}} \oplus \mathfrak{v}$ satisfies
$$\mathfrak{s} = (\mathfrak{s} \cap \mathfrak{k}) \oplus (\mathfrak{s} \cap \mathfrak{p}) = \mathfrak{s}_{\mathfrak{k}} \oplus \mathfrak{s}_{\mathfrak{p}}.$$
- The orbit $S \cdot o$ is totally geodesic with $T_o(S \cdot o) = \mathfrak{v}$.

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- The orbit $S \cdot o$ is totally geodesic.

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{Canonically embedded}\\ \mathfrak{s} \subseteq \mathfrak{g}\end{array}} \arrow[rr, bend left, "\pi_{\mathfrak{p}}"] \arrow[dr, "o\text{-orbit}"'] & & \boxed{\begin{array}{c}R \text{ and } D\text{-invariant}\\ \mathfrak{v} \subseteq \mathfrak{p}\end{array}} \arrow[ll, bend left, "\mathrm{Lie}"] \arrow[dl, "\exp_o"] \\
& \boxed{\begin{array}{c}\text{Totally geodesic}\\ \Sigma \subseteq M,\ o \in \Sigma\end{array}} &
\end{tikzcd}$$

The flag manifold $\mathrm{F}(\mathbb{C}^3)$

- $\mathrm{F}(\mathbb{C}^3) = \dfrac{\mathrm{SU}(3)}{\mathrm{T}^2}$; $I\big(\mathrm{F}(\mathbb{C}^3)\big) = \mathrm{PSU}(3) \rtimes (\mathfrak{S}_3 \times \mathbb{Z}_2)$.

- $\mathfrak{p} = \left\{ \begin{pmatrix} 0 & x & z \\ -\bar{x} & 0 & y \\ -\bar{z} & -\bar{y} & 0 \end{pmatrix} : x, y, z \in \mathbb{C} \right\} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3.$

$$
\begin{tikzcd}
 & e^{i(\theta_1-\theta_2)} \\
\mathrm{diag}\big(e^{i\theta_1}, e^{i\theta_2}, e^{-i(\theta_1+\theta_2)}\big) \arrow[ur] \arrow[r] \arrow[dr] & e^{i(\theta_1+2\theta_2)} \\
 & e^{i(2\theta_1+\theta_2)}
\end{tikzcd}
$$

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- $\langle X, Y \rangle = -\mathrm{Tr}(XY).$

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- $T^2 \subseteq \mathrm{U}(2) \subseteq \mathrm{SU}(3)$ gives the fibration $\mathbb{C}\mathrm{P}^1 \to \mathrm{F}(\mathbb{C}^3) \to \mathbb{C}\mathrm{P}^2$.

- $\mathcal{V}_o = \mathfrak{p}_1, \mathcal{H}_o = \mathfrak{p}_2 \oplus \mathfrak{p}_3$.

General fact:
$\mathrm{K} \subseteq \mathrm{H} \subseteq \mathrm{G}$ gives a fibration
$\mathrm{H}/\mathrm{K} \to \mathrm{G}/\mathrm{K} \to \mathrm{G}/\mathrm{H}$.

The complex projective space $\mathbb{CP}^3$

- $\mathbb{CP}^3 = \frac{\mathrm{Sp}(2)}{\mathrm{U}(1)\times\mathrm{Sp}(1)}$; $I(\mathbb{CP}^3) = \frac{\mathrm{Sp}(2)}{\mathbb{Z}^2} \rtimes \mathbb{Z}_2$.
- $\mathfrak{p} = \left\{\begin{pmatrix} zj & -\bar{q} \\ q & 0 \end{pmatrix} : z \in \mathbb{C}, q \in \mathbb{H}\right\} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$.

$$(\lambda,\mu)\cdot z = \lambda^2 z$$
$$(\lambda,\mu)\cdot q = \mu q \bar{\lambda}$$

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- $\langle X, Y \rangle = -2\mathrm{Re}\,\mathrm{tr}_{\mathbb{H}}(XY)$.

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- $\mathrm{U}(1) \times \mathrm{Sp}(1) \subseteq \mathrm{Sp}(1) \times \mathrm{Sp}(1) \subseteq \mathrm{Sp}(2)$ gives the fibration $\mathbb{CP}^1 \to \mathbb{CP}^3 \to \mathbb{HP}^1 = \mathbb{S}^4$.

- $\mathcal{V}_o = \mathfrak{p}_1, \mathcal{H}_o = \mathfrak{p}_2$.

The almost product $\mathbb{S}^3 \times \mathbb{S}^3$

- $\mathbb{S}^3 \times \mathbb{S}^3 = \frac{\mathrm{SU}(2)^3}{\Delta\mathrm{SU}(2)}$; $I(\mathbb{S}^3 \times \mathbb{S}^3) = \frac{\mathrm{SU}(2)^3}{\Delta\mathbb{Z}_2} \rtimes \mathfrak{S}_3$.
- $\mathfrak{p} = \{(X, Y, Z) : X + Y + Z = 0\} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$.
  - $\mathfrak{p}_1 = \{(X, -2X, X) : X \in \mathfrak{su}(2)\} \cong \mathrm{Ad}$.
  - $\mathfrak{p}_2 = \{(X, 0, -X) : X \in \mathfrak{su}(2)\} \cong \mathrm{Ad}$.
- $\langle (X, Y, Z), (X', Y', Z') \rangle = -\mathrm{tr}(XX') - \mathrm{tr}(YY') - \mathrm{tr}(ZZ')$.

The almost product $\mathbb{S}^3 \times \mathbb{S}^3$

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- $\mathfrak{p} = \{(X, Y, Z) : X + Y + Z = 0\} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$.
- $\Delta SU(2) \subseteq \Delta_{13}SU(2) \times SU(2)_2 \subseteq SU(2)^3$ gives the fibration $\mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{S}^3$.
- $\mathcal{V}_o = \mathfrak{p}_1, \mathcal{H}_o = \mathfrak{p}_2$.

Low codimension

Theorem (Nikolayevsky '15)

M simply connected, irreducible, compact and homogeneous.

If there exists a totally geodesic hypersurface $\Sigma \subseteq M$, then $M = \mathbb{S}^n$.

Corollary

$\mathrm{F}(\mathbb{C}^3)$, $\mathbb{C}\mathrm{P}^3$ and $\mathbb{S}^3 \times \mathbb{S}^3$ have no totally geodesic hypersurfaces.

Low codimension

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Theorem

$\mathrm{F}(\mathbb{C}^3)$, $\mathbb{C}\mathrm{P}^3$ and $\mathbb{S}^3 \times \mathbb{S}^3$ have no totally geodesic submanifolds of codimension ≤ 2.

Dimension three

Ambient	Submanifold	Orbit of
$F(\mathbb{C}^3)$	$F(\mathbb{R}^3) = \mathbb{S}^3(2\sqrt{2})/Q_8$	$SO(3)$
	$\mathbb{S}^3_{\mathbb{C},1/4}(2)$	$SU(2)$
$\mathbb{C}P^3$	$\mathbb{R}P^3_{\mathbb{C},1/2}(\sqrt{2})$	$SU(2)^j$
$\mathbb{S}^3 \times \mathbb{S}^3$	$\mathbb{S}^3(2/\sqrt{3})$	$SU(2)_2$
	$\mathbb{S}^3_{\mathbb{C},1/3}(2)$	$\Delta_{13}SU(2) \times SU(2)_2$

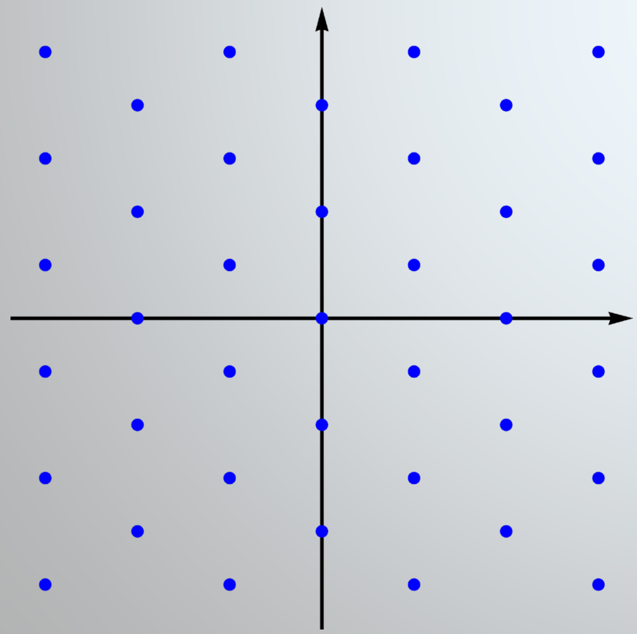
All of them are Lagrangian.

Dimension two in $\mathrm{F}(\mathbb{C}^3)$

Submanifold	Orbit of	Relationship with J
$\mathrm{T_A} = \mathbb{R}^2/\mathrm{A}$	T^2	Almost complex
$\mathbb{C}\mathrm{P}^1 = \mathbb{S}^2(1/\sqrt{2})$	$\mathrm{U}(2)$	Almost complex
$\mathbb{S}^2(\sqrt{2})$	$\mathrm{SO}(3)$	Almost complex
$\mathbb{R}\mathrm{P}^2(2\sqrt{2}) \to \mathrm{F}(\mathbb{R}^3)$	Inhomogeneous	Totally real

Dimension two in $F(\mathbb{C}^3)$

Submanifold	Orbit of	Relationship with J
$T_A = \mathbb{R}^2/A$	T^2	Almost complex



$$A = \mathbb{Z} - \operatorname{span}\left\{\left(0, \frac{2\sqrt{2}\pi}{\sqrt{3}}\right), \left(\sqrt{2}\pi, \frac{\sqrt{2}}{\sqrt{3}}\pi\right)\right\}$$

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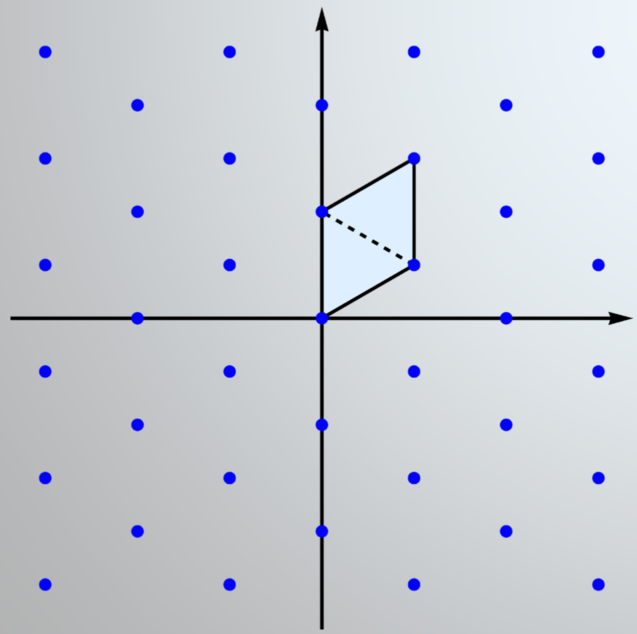
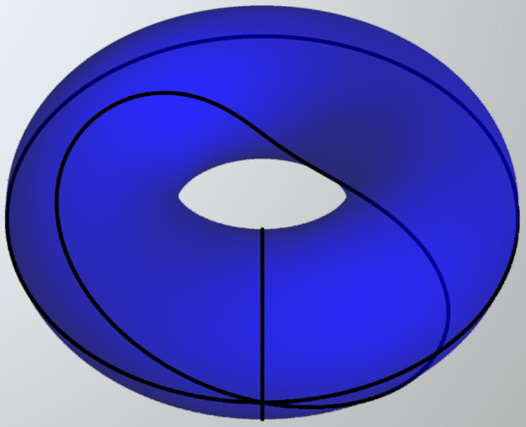




Dimension two in $\mathbb{C}\mathrm{P}^3$

Submanifold	Orbit of	Relationship with J
$\mathbb{S}^2(1/\sqrt{2})$	$\mathrm{Sp}(1)_f$	Almost complex
$\mathbb{S}^2(1)$	$\mathrm{SU}(2)$	Almost complex
$\mathbb{S}^2\left(\sqrt{5}\right)$	$\mathrm{SU}(2)_{\Lambda_3}$	Almost complex

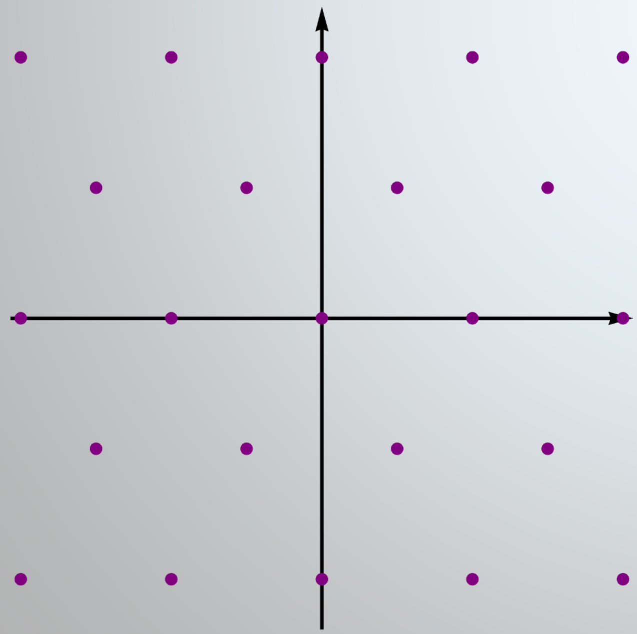
$\Lambda_3 = \mathrm{Sym}^3(\mathbb{C}^2)$: four-dimensional irrep. of $SU(2)$.

Dimension two in $\mathbb{S}^3 \times \mathbb{S}^3$

Submanifold	Orbit of	Relationship with J
$\mathrm{T_B} = \mathbb{R}^2/\mathrm{B}$	$\mathrm{T} \subseteq \mathrm{U}(1)^3$	Almost complex
$\mathbb{S}^2\left(\sqrt{3}/\sqrt{2}\right)$	$\Delta\mathrm{SU}(2)$	Almost complex
$\mathbb{S}^2 \subseteq \mathbb{S}^3\left(2/\sqrt{3}\right)$	$\mathrm{H} \subseteq \Delta_{13}\mathrm{SU}(2) \times \mathrm{SU}(2)_2$	Totally real

Dimension two in $\mathbb{S}^3 \times \mathbb{S}^3$

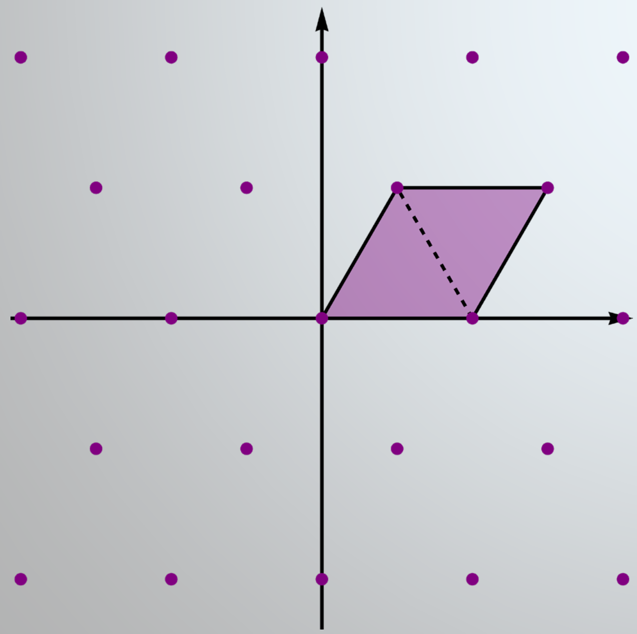
Submanifold	Orbit of	Relationship with J
$\mathrm{T}_{\mathrm{B}} = \mathbb{R}^2/\mathrm{B}$	$\mathrm{T} \subseteq \mathrm{U}(1)^3$	Almost complex

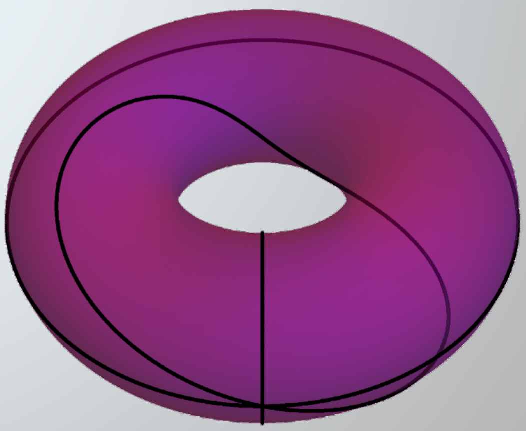


$$\mathrm{B} = \mathbb{Z} - \mathrm{span}\left\{\left(\frac{4\pi}{\sqrt{3}}, 0\right), \left(\frac{2\pi}{\sqrt{3}}, 2\pi\right)\right\}$$

Dimension two in $\mathbb{S}^3 \times \mathbb{S}^3$

Submanifold	Orbit of	Relationship with J
$\mathrm{T_B} = \mathbb{R}^2/\mathrm{B}$	$\mathrm{T} \subseteq \mathrm{U}(1)^3$	Almost complex





Ambient	Submanifold	Orbit of	Relationship with J
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	$\mathbb{S}^3_{\mathbb{C},1/4}(2)$	$SU(2)$	Lagrangian
	T_A	T^2	Almost complex
	$\mathbb{C}P^1 = \mathbb{S}^2(1/\sqrt{2})$	$U(2)$	Almost complex
	$\mathbb{S}^2(\sqrt{2})$	$SO(3)$	Almost complex
	$\mathbb{R}P^2(2\sqrt{2}) \to F(\mathbb{R}^3)$	Inhomogeneous	Totally real
$\mathbb{C}P^3$	$\mathbb{R}P^3_{\mathbb{C},1/2}(\sqrt{2})$	$SU(2)^j$	Lagrangian
	$\mathbb{S}^2(1/\sqrt{2})$	$Sp(1)_f$	Almost complex
	$\mathbb{S}^2(1)$	$SU(2)$	Almost complex
	$\mathbb{S}^2(\sqrt{5})$	$SU(2)_{\Lambda_3}$	Almost complex
$\mathbb{S}^3 \times \mathbb{S}^3$	$\mathbb{S}^3(2/\sqrt{3})$	$SU(2)_2$	Lagrangian
	$\mathbb{S}^3_{\mathbb{C},1/3}(2)$	$\Delta_{13}SU(2) \times SU(2)_2$	Lagrangian
	T_B	$T \subseteq U(1)^3$	Almost complex
	$\mathbb{S}^2(\sqrt{3}/\sqrt{2})$	$\Delta SU(2)$	Almost complex
	$\mathbb{S}^2 \subseteq \mathbb{S}^3(2/\sqrt{3})$	$H \subseteq \Delta_{13}SU(2) \times SU(2)_2$	Totally real