SOME USEFUL FORMULAE FOR HOMOGENEOUS SPACES

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Contents

Abstract. We gather here some general facts on homogeneous manifolds and their corresponding left-invariant connections. For the case of Riemannian naturally reductive homogeneous spaces, we also discuss their Jacobi fields.

1. Introduction and notation

Let $M^n = \mathsf{G}/\mathsf{K}$ be a homogeneous manifold of a Lie group G (which we will usually assume to be connected). We say that an affine connection ∇ on M is *left-invariant* if for every $X, Y \in \mathfrak{X}(M)$ and $g \in \mathsf{G}$, we have

$$\nabla_{g_* X} g_* Y = g_* \nabla_X Y. \tag{1}$$

On the other hand, consider a homogeneous S-structure $\pi\colon P \to M$, where $\mathsf{S} \leq \mathsf{GL}(n, \mathbb{R})$. This means that P is a reduction of the frame bundle $\mathsf{F}(M)$ of M with structure group S. A principal connection $\omega\colon TP \to \mathfrak{s}$ is *left-invariant* if for every $g \in \mathsf{G}$, we have

$$s^*\omega = \omega. \tag{2}$$

It is easy to see that the bijective correspondence between principal connections in P and affine conenctions in M restricts to a bijective correspondence between invariant connections in P and adapted invariant affine connections in M.

We aim to give a description of these connections in Lie algebraic terms, as well as explicit formulae for the curvature and torsion tensors, given by

$$\begin{aligned} R(X,Y) &= [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \\ T(X,Y) &= \nabla_X Y - \nabla_Y X - [X,Y]. \end{aligned}$$

The main reference for the first five sections is [2].

2. Wang's theorem for invariant connections

We start by stating the theorem of Wang, which allows us to characterize invariant connections on P. Let $o = eK$ and fix a frame $u_0 \in P$. Then we may define a Lie group homomorphism $\lambda\colon K \to S$ via the equation $k \cdot u_0 = u_0 \cdot \lambda(k)$ for $k \in K$. For geometric structures, we actually have $k \cdot u_0 = k_{*o} \circ u_0$ and $u_0 \cdot \lambda(k) = u_0 \circ \lambda(k)$, which means that

$$\lambda(k) = u_0^{-1} \circ k_{*o} \circ u_0. \tag{3}$$

This implies that λ is actually the isotropy representation of M.

We can now state the first version of Wang's theorem, copied verbatim from [2, Chapter II, Proposition 11.3].

Theorem 1. *There is a bijective correspondence between the set of invariant connections on P and the set of linear maps $\Lambda\colon \mathfrak{g} \to \mathfrak{s}$ that satisfy the following conditions:*

(1) $\Lambda(X) = \lambda_*(X)$ for all $X \in \mathfrak{k}$.
(2) $\Lambda(\mathrm{Ad}(k)X) = \mathrm{Ad}(\lambda(k))X$ for all $X \in \mathfrak{g}$ and $k \in K$, that is, Λ is K-equivariant.

The correspondence is explicitly given as follows: for an invariant connection ω, we define the associated Λ via

$$\Lambda(X) = \omega\left(\frac{d}{dt}\bigg|_{t=0} \mathrm{Exp}(tX) \cdot u_0\right), \quad X \in \mathfrak{g}. \tag{4}$$

For $S = GL(n, \mathbb{R})$ (that is, for arbitrary connections), we can rewrite the previous result without choosing a frame u_0. In order to do this, recall that there is a vector space isomorphism

$$\phi\colon X + \mathfrak{k} \in \mathfrak{g}/\mathfrak{k} \longmapsto X^*_o \in T_oM, \tag{5}$$

where X^* is the fundamental vector field corresponding to X. Choose an isomorphism $\nu_0\colon \mathbb{R}^n \to \mathfrak{g}/\mathfrak{k}$ such that $u_0 = \phi \circ \nu_0$ belongs to P. Then any map $\Lambda\colon \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{R})$ can be reinterpreted as a map $\Gamma\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{k})$ via

$$\Gamma(X) = \nu_0 \circ \Lambda(X) \circ \nu_0^{-1}.$$

On the other hand, the isotropy representation can be converted to a homomorphism $\mu\colon K \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{k})$ by taking

$$\mu(k) = \phi^{-1} \circ k_{*o} \circ \phi = \mathrm{Ad}(k).$$

Then Λ satisfies the conditions of the theorem if and only if $\Gamma(X) = \mu_*(X)$ for $X \in \mathfrak{k}$ (that is, $\Gamma|_{\mathfrak{k}}$ is the isotropy representation) and for $X \in \mathfrak{g}$ and $k \in \mathfrak{k}$ we have $\Gamma(\mathrm{Ad}(k)X) = \mathrm{Ad}(\mu(k))\Gamma(X)$.

In conclusion, we have:

Theorem 2. *There is a bijective correspondence between the set of invariant connections on $F(M)$ and the set of linear maps $\Gamma\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{k})$ that satisfy:*

(1) $\Gamma(X) = \mu_*(X)$ for all $X \in \mathfrak{k}$, where $\mu\colon K \to GL(\mathfrak{g}/\mathfrak{k})$ is the isotropy representation.
(2) $\Gamma(\mathrm{Ad}(k)X) = \mathrm{Ad}(\mu(k))\Gamma(X)$ for $X \in \mathfrak{g}$ and $k \in K$.

The correspondence is described as follows: for a fixed isomorphism $\nu_0\colon \mathbb{R}^n \to \mathfrak{g}/\mathfrak{k}$, let $u_0 = \phi \circ \nu_0$. Then to any connection $\omega \in \Omega^1(F(M), \mathfrak{gl}(n, \mathbb{R}))$ we associate the map Γ defined by

$$\Gamma(X) = \nu_0 \circ \omega\left(\frac{d}{dt}\bigg|_{t=0} \mathrm{Exp}(tX) \cdot u_0\right) \circ \nu_0^{-1}. \tag{6}$$

It is easy to check that the definition of Γ is independent of the frame chosen for $\mathfrak{g}/\mathfrak{k}$. From now on, we will also refer to Γ as a connection.

Remark 1. We can interpret Wang's theorem in the following way. Note that $\mathfrak{g}$ and $\mathfrak{k}$ are representations of K via the adjoint action, while $\mathfrak{gl}(\mathfrak{g}/\mathfrak{k})$ is also a K-module via the representation $k \in K \to \mathrm{Ad}(\mathrm{Ad}(k)) \in GL(\mathfrak{gl}(\mathfrak{g}/\mathfrak{k}))$. Then the theorem states that giving an invariant connection on M is the same as giving a K-equivariant extension of $\mathrm{ad}\colon \mathfrak{k} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{k})$ to $\mathfrak{g}$.

We now aim to derive a formula for the covariant derivative ∇ associated to a homogeneous connection Γ. Note that for $X \in \mathfrak{g}$ and $V \in \mathfrak{X}(M)$, we have

$$\nabla_{X^*} V = \nabla_V X^* + [X^*, V] + T(X^*, V),$$

so if we know how to take covariant derivatives of vector fields induced by $\mathfrak{g}$ and we know the torsion tensor at o, we have a complete description of ∇. The torsion tensor will be determined in Section 3.

Recall that, in general, if M is a smooth manifold and ω is a connection on its frame bundle, the covariant derivative ∇ associated with ω is given by

$$(\nabla_V W)_p = u(DF_W(\bar{V})),$$

where $p \in M$, u is a frame at p, $F_W \colon \mathsf{F}(M) \to \mathbb{R}^n$ is defined by $F_W(\nu) = \nu^{-1}(W_{\pi(\nu)})$, D denotes the exterior covariant derivative and $\bar{V}$ is any lift of V to p. Alternatively,

$$(\nabla_V W)_p = u(dF_W(\bar{V}) + \omega(\bar{V})F_W(u)). \tag{7}$$

In our case, we have $p = o$, $u = u_0$, and a lift of $Z^* \in \mathfrak{X}(M)$ is merely $Z^* \in \mathfrak{X}(P)$. A direct computation yields:

Theorem 3. *If $\Gamma \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{k})$ is any connection, then the corresponding covariant derivative ∇ satisfies*

$$\begin{aligned} \nabla_{X^*_o} Y^* &= -[X, Y]^*_o + \phi \circ \Gamma(X) \circ \phi^{-1}(Y^*_o) \\ &= \phi\left(-\overline{[X, Y]} + \Gamma(X)\overline{Y}\right) \end{aligned} \tag{8}$$

for all $X, Y \in \mathfrak{g}$, where $\overline{(\cdot)}$ denotes the projection to $\mathfrak{g}/\mathfrak{k}$.

3. Curvature and torsion

In this section we derive expressions for the curvature and torsion tensors associated to a connection Γ. To this end, we start by determining the curvature and torsion two-forms at vector fields of the form X^*, where $X \in \mathfrak{g}$. We keep the notation of the previous section.

Firstly, recall that the curvature form Ω can be computed by Cartan's first structure equation

$$\Omega(V, W) = D\omega(V, W) = d\omega(V, W) + [\omega(V), \omega(W)],$$

where $V, W \in \mathfrak{X}(P)$. If we let $V = X^*$ and $W = Y^*$ for $X, Y \in \mathfrak{g}$, we obtain

$$\Omega(X^*_{u_0}, Y^*_{u_0}) = [\Lambda(X), \Lambda(Y)] - \Lambda([X, Y]). \tag{9}$$

On the other hand, the curvature tensor R for an arbitrary connection ω is related to the curvature form via the equation

$$(R(U, V)W)_p = u(\Omega(\bar{U}, \bar{V})u^{-1}(W)),$$

where U, V, $W \in \mathfrak{X}(M)$, $p \in M$, u is a frame at p and $\bar{U}$, $\bar{V}$ are lifts of U and V to $\mathsf{F}(M)$. As a consequence, for X, Y, $Z \in \mathfrak{g}$, we get

$$(R(X^*, Y^*)Z^*)_o = \phi\left([\Gamma(X), \Gamma(Y)]\overline{Z} - \Gamma([X, Y])\overline{Z}\right). \tag{10}$$

Equivalently, we may think of the curvature tensor as a map $R \colon \mathfrak{g}/\mathfrak{k} \times \mathfrak{g}/\mathfrak{k} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{k})$, and it is given by

$$R\left(\overline{X}, \overline{Y}\right) = [\Gamma(X), \Gamma(Y)] - \Gamma([X, Y]). \tag{11}$$

We now compute the torsion: recall that $\mathsf{F}(M)$ is endowed with a *solder form* $\vartheta \colon T\mathsf{F}(M) \to \mathbb{R}^n$ defined by $\vartheta_u(\xi) = u^{-1}(\pi_{*u}(\xi))$. The torsion two-form Θ can be computed from the second structure equation

$$\Theta(V, W) = D\vartheta(V, W) = d\vartheta(V, W) + \omega(V)\vartheta(W) - \omega(W)\vartheta(V).$$

For $U = X^*$ and $V = Y^*$, a simple calculation gives us

$$\Theta(X^*_{u_0}, Y^*_{u_0}) = \Lambda(X)\nu_0^{-1}\left(\overline{Y}\right) - \Lambda(Y)\nu_0^{-1}\left(\overline{X}\right) - \nu_0^{-1}\left(\overline{[X, Y]}\right), \tag{12}$$

and since the torsion tensor is generally computed via

$$T(U,V)_p = u(\Theta(\bar{U},\bar{V})),$$

we finally obtain

$$T(X_o^*, Y_o^*) = \phi\left(\Gamma(X)\overline{Y} - \Gamma(Y)\overline{X} - \overline{[X,Y]}\right). \tag{13}$$

If we view the torsion tensor as a map $T\colon \mathfrak{g}/\mathfrak{k} \times \mathfrak{g}/\mathfrak{k} \to \mathfrak{g}/\mathfrak{k}$, then we have showed that

$$T\left(\overline{X},\overline{Y}\right) = \Gamma(X)\overline{Y} - \Gamma(Y)\overline{X} - \overline{[X,Y]}. \tag{14}$$

We summarize everything in this

Theorem 4. *Let $M = G/K$ be an arbitrary homogeneous space and $\Gamma\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{k})$ an invariant connection on M. Then the following formulae are valid for all $X, Y \in \mathfrak{g}$ and $V \in \mathfrak{X}(M)$:*

(1) *Formula for the covariant derivative for induced fields:*
$$\nabla_{X_o^*} Y^* = \phi\left(-\overline{[X,Y]} + \Gamma(X)\overline{Y}\right). \tag{15}$$
(2) *Formula for the covariant derivative of arbitrary vector fields:*
$$\nabla_{X_o^*} V = [X^*, V]_o + \phi\left(\Gamma(X)\overline{\xi}\right), \tag{16}$$
where $\xi \in \mathfrak{g}$ is any vector such that $\xi_o^* = V_o$.
(3) *Formula for the torsion tensor:*
$$T\left(\overline{X},\overline{Y}\right) = \Gamma(X)\overline{Y} - \Gamma(Y)\overline{X} - \overline{[X,Y]}. \tag{17}$$
(4) *Formula for the curvature tensor:*
$$R\left(\overline{X},\overline{Y}\right) = [\Gamma(X),\Gamma(Y)] - \Gamma([X,Y]). \tag{18}$$

4. Reductive homogeneous spaces

A homogeneous space $M = G/K$ is *reductive* if we can decompose $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as a direct sum of K-modules (this is known as a reductive decomposition). This is the same as saying that the short exact sequence of K-modules

$$0 \longrightarrow \mathfrak{k} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{k} \longrightarrow 0$$

splits. This allows us to give identifications $T_oM \cong \mathfrak{g}/\mathfrak{k} \cong \mathfrak{p}$, simply by restricting ϕ to $\mathfrak{p}$.

Remark 2. Every (effective) Riemannian homogeneous space is reductive, as is shown in [1].

The description of invariant connections in reductive homogeneous spaces becomes particularly nice, as giving an extension of $\mathrm{ad}\colon \mathfrak{k} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{k}) \cong \mathfrak{gl}(\mathfrak{p})$ in this case is the same as giving a K-equivariant map $\Gamma\colon \mathfrak{p} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{k}) \cong \mathfrak{gl}(\mathfrak{p})$, which in turn is the same as a K-invariant bilinear map $\alpha\colon \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$. This allows us to give (yet) another formulation of Wang's theorem, commonly known as Nomizu's theorem:

Theorem 5. *Let $M = G/K$ be a reductive homogeneous space with corresponding decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then there is a bijective correspondence between the set of invariant affine connections in M and the set of bilinear forms $\alpha\colon \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$ which are K-invariant, in the sense that*

$$\alpha\left(\mathrm{Ad}(k)X, \mathrm{Ad}(k)Y\right) = \mathrm{Ad}(k)\alpha(X,Y) \tag{19}$$

for all $X, Y \in \mathfrak{p}$ and $k \in K$.

Proof. The explicit isomorphism $\mathfrak{gl}(\mathfrak{g}/\mathfrak{k}) \cong \mathfrak{gl}(\mathfrak{p})$ takes $f\colon \mathfrak{g}/\mathfrak{k} \to \mathfrak{g}/\mathfrak{k}$ to $\psi|_{\mathfrak{p}}^{-1} \circ f \circ \psi|_{\mathfrak{p}}$, where ψ is the projection to $\mathfrak{g}/\mathfrak{k}$. Given a connection $\Gamma\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{k})$, we define $\alpha\colon \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$ by

$$\alpha(X,Y) = \psi|_{\mathfrak{p}}^{-1}(\Gamma(X)\overline{Y}), \tag{20}$$

and it is easily checked that α is invariant. Conversely, if α is an invariant bilinear form on $\mathfrak{p}$, define $\Gamma\colon \mathfrak{p} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{k})$ by

$$\Gamma(X)\overline{Y} = \psi(\alpha(X, \psi|_{\mathfrak{p}}^{-1}(\overline{Y}))) = \psi(\alpha(X, Y_{\mathfrak{p}})). \tag{21}$$

This map is equivariant, and we extend it to $\mathfrak{g}$ by simply declaring $\Gamma = \mathrm{ad}$ on $\mathfrak{k}$, giving us an invariant connection. It is straightforward to check that these constructions are mutually inverse to each other. □

Remark 3. Nomizu's theorem relates invariant connections on M to nonassociative algebra structures on $\mathfrak{p}$ such that K acts by algebra automorphisms.

As an application of Equation (20), we deduce a new (and simpler) set of formulae for invariant connections on the reductive case.

Theorem 6. *Let $M = G/K$ be a reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and consider an arbitrary invariant connection $\alpha\colon \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$. Then the following formulae hold for X, $Y \in \mathfrak{p}$ and $V \in \mathfrak{X}(M)$:*

(1) Formula for the covariant derivative of induced fields:

$$\nabla_{X^*_o} Y^* = -[X, Y]^*_o + \alpha(X, Y)^*_o = (-[X, Y]_{\mathfrak{p}} + \alpha(X, Y))^*_o. \tag{22}$$

(2) Formula for the covariant derivative of arbitrary vector fields:

$$\nabla_{X^*_o} V = [X^*, V]_o + \alpha(X, \xi)^*_o, \tag{23}$$

where $\xi \in \mathfrak{p}$ is the unique vector such that $\xi^*_o = V_o$.

(3) Formula for the torsion tensor:

$$T(X, Y) = \alpha(X, Y) - \alpha(Y, X) - [X, Y]_{\mathfrak{p}}. \tag{24}$$

(4) Formula for the curvature tensor:

$$R(X, Y)Z = \alpha(X, \alpha(Y, Z)) - \alpha(Y, \alpha(X, Z)) - [[X, Y]_{\mathfrak{k}}, Z] - \alpha([X, Y]_{\mathfrak{p}}, Z). \tag{25}$$

5. Examples of invariant connections

We now go with some examples:

5.1. The canonical connection.

This is the connection ∇^c associated with the map $\alpha = 0$. In this case we have for X, Y, $Z \in \mathfrak{p}$:

$$\begin{aligned} \nabla^c_X Y &= -[X, Y]_{\mathfrak{p}}, \\ T^c(X, Y) &= -[X, Y]_{\mathfrak{p}}, \\ R^c(X, Y)Z &= -[[X, Y]_{\mathfrak{k}}, Z]. \end{aligned} \tag{26}$$

Theorem 7. *A tensor field T on M is G-invariant if and only if $\nabla^c T = 0$.*

Remark 4. For any invariant connection ∇, the difference tensor $\nabla - \nabla^c$ is precisely α.

Observe that since $\nabla^c_X X = 0$ for all $X \in \mathfrak{p}$, the geodesics of M at o are given by

$$\exp_o(tX) = \mathrm{Exp}(tX) \cdot o. \tag{27}$$

For the same reason, any connection ∇ has the same geodesics as ∇^c if and only if α is skew-symmetric. We also note that parallel translation along $\mathrm{Exp}(tX) \cdot o$ is given by $(\mathrm{Exp}(tX))_{*o}$ (see for example [1, Corollary 1.4.13]).

5.2. **The natural torsion-free connection.** This is the connection $\widetilde{\nabla}$ associated with the map

$$\widetilde{\alpha}(X, Y) = \frac{1}{2}[X, Y]_{\mathfrak{p}}, \tag{28}$$

equivalently, it is the unique invariant connection on M which has zero torsion and whose geodesics are given by (27). We have for $X, Y, Z \in \mathfrak{p}$:

$$\begin{aligned} \widetilde{\nabla}_X Y &= -\frac{1}{2}[X, Y]_{\mathfrak{p}}, \\ \widetilde{R}(X, Y)Z &= \frac{1}{4}[X, [Y, Z]_{\mathfrak{p}}]_{\mathfrak{p}} - \frac{1}{4}[Y, [X, Z]_{\mathfrak{p}}]_{\mathfrak{p}} - [[X, Y]_{\mathfrak{k}}, Z] - \frac{1}{2}[[X, Y]_{\mathfrak{p}}, Z]_{\mathfrak{p}}. \end{aligned} \tag{29}$$

6. Riemannian homogeneous spaces

Let $M = G/K$ be a Riemannian homogeneous space. Then, if we assume that G acts effectively, we know that there exists a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We pull back the Riemannian metric on T_oM to an inner product on $\mathfrak{p}$. The Levi-Civita connection ∇ of M is an invariant affine connection, and its corresponding Nomizu map α is given by

$$\alpha(X, Y) = \frac{1}{2}[X, Y]_{\mathfrak{p}} + U(X, Y), \tag{30}$$

where $U\colon \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$ is the symmetric tensor defined implicitly by

$$2\langle U(X, Y), Z\rangle = \langle [Z, X]_{\mathfrak{p}}, Y\rangle + \langle X, [Z, Y]_{\mathfrak{p}}\rangle. \tag{31}$$

The operator ∇ satisfies

$$\nabla_X Y = -\frac{1}{2}[X, Y]_{\mathfrak{p}} + U(X, Y). \tag{32}$$

More generally, if $V \in \mathfrak{X}(M)$ and $\xi \in \mathfrak{p}$ is such that $\xi^*_o = V_o$, then

$$\nabla_{X^*_o} V = [X^*, V]_o + \frac{1}{2}[X, \xi]^*_o + U(X, \xi)^*_o. \tag{33}$$

We say that M is *naturally reductive* if $U = 0$, or equivalently, if ∇ is precisely the natural torsion-free connection. Symmetric spaces are naturally reductive and also satisfy $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$.

7. Olmos' magic formula

We now present a general formula that describes parallel transport along the flow lines of a Killing field for a torsion-free connection ∇ on a manifold M, which we will kindly refer to as *Olmos' magic formula*.

Remark 5. As far as I know, Olmos' magic formula was **not** introduced by Olmos.

Let (M, ∇) be an affine manifold. A complete vector field $X \in \mathfrak{X}(M)$ is an *affine Killing field* if the flow of X consists of affine maps, or equivalently, if the Lie derivative $\mathcal{L}_X\nabla$ is zero. Let $p \in M$ be arbitrary, and consider the flow line $\gamma(t) = \phi_t(p)$. We aim to compare the parallel transport $\mathcal{P}_t\colon T_pM \to T_{\gamma(t)}M$ and the differential of the flow $(\phi_t)_{*p}\colon T_pM \to T_{\gamma(t)}M$.

Let $v \in T_pM$ and consider any curve $\alpha(s)$ on M such that $\alpha'(0) = v$. Then we can construct a variation of α by letting $\Gamma(t, s) = \phi_t(\alpha(s))$. The mixed partial derivatives of Γ at $(0, 0)$ are

$$\begin{aligned} \frac{\nabla}{dt}\frac{\partial}{\partial s}\Gamma(0,0) &= \frac{\nabla}{dt}\bigg|_{t=0}(\phi_t)_{*p}v = \frac{d}{dt}\bigg|_{t=0}\mathcal{P}_t^{-1}(\phi_t)_{*p}(v), \\ \frac{\nabla}{ds}\frac{\partial}{\partial t}\Gamma(0,0) &= \frac{\nabla}{ds}\bigg|_{s=0}X_{\alpha(s)} = \nabla_v X. \end{aligned}$$

We conclude then that

$$\frac{d}{dt}\bigg|_{t=0}\mathcal{P}_t^{-1}(\phi_t)_{*p}(v) = \nabla_v X + T\left(\frac{\partial\Gamma}{\partial t}(0,0), \frac{\partial\Gamma}{\partial s}(0,0)\right) = \nabla_v X + T(X_p, v) =: \tau_X(v). \tag{34}$$

Now, since X is affine, the map $t \in \mathbb{R} \mapsto \mathcal{P}_t^{-1}(\phi_t)_{*p} \in \mathsf{GL}(T_o M)$ is a one-parameter subgroup. Combining this with (34), we deduce

Theorem 8 (Olmos' magic formula)**.**

$$\mathcal{P}_t^{-1} \circ (\phi_t)_{*p} = e^{t\tau_X}. \tag{35}$$

It will be especially interesting for us to consider the case of (Riemannian) Killing fields, where the formula becomes

$$\mathcal{P}_t^{-1} \circ (\phi_t)_{*p} = e^{t\nabla X}. \tag{36}$$

8. Naturally reductive homogeneous spaces

We now dive deeper into the geometry of a naturally reductive homogeneous space.

First of all, for each $X \in \mathfrak{p}$, we can apply Olmos' magic formula (36) to deduce that

$$\mathcal{P}_t^{-1} \circ \mathrm{Exp}(tX)_{*o} = e^{t\nabla X^*} \tag{37}$$

along the geodesic $\mathrm{Exp}(tX) \cdot o$. Note that ∇X^* corresponds to the map $\tau_X \colon \mathfrak{p} \to \mathfrak{p}$ defined by $\tau_X(Y) = \frac{1}{2}[X, Y]_{\mathfrak{p}}$. We will use this formula shortly to derive explicit expressions for Jacobi fields.

8.1. Jacobi fields.

Let $\gamma(t) = \mathrm{Exp}(t\xi) \cdot o$ for a nonzero vector $\xi \in \mathfrak{p}$. A vector field $J(t)$ along $\gamma(t)$ is called a *Jacobi field* if it satisfies the differential equation

$$\frac{\nabla^2}{dt^2} J(t) + R(J(t), \gamma'(t))\gamma'(t) = 0. \tag{38}$$

This differential equation arises when trying to compute the differential of the exponential map or the extrinsic geometry of certain submanifolds, such as geodesic spheres, tubes or equidistant hypersurfaces. The vector fields $\gamma'(t)$ and $t\gamma'(t)$ are trivial examples. We are interested in producing more examples and giving a computationally efficient description of those. We do this following Ziller [3].

Remark 6. Since ∇ and ∇^c have the same geodesics, it follows that $J(t)$ is a Jacobi field if and only if it satisfies the following differential equation involving ∇^c:

$$\frac{(\nabla^c)^2}{dt^2} J(t) + R^c(J(t), \gamma'(t))\gamma'(t) + T^c(\nabla^c_t J(t), \gamma'(t)) = 0. \tag{39}$$

If we write $J(t) = \mathrm{Exp}(t\xi)_{*o}(Z(t))$ for a curve $Z(t)$ in $\mathfrak{p}$, then using the fact that R^c and T^c are ∇^c-parallel we get that (39) becomes

$$Z''(t) + R^c(Z(t), \xi)\xi + T^c(Z'(t), \xi) = 0,$$

that is:

$$Z''(t) - [[Z(t), \xi]_{\mathfrak{k}}, \xi] - [Z'(t), \xi]_{\mathfrak{p}} = 0.$$

Define operators $R_\xi, T_\xi \colon \mathfrak{p} \to \mathfrak{p}$ via

$$R_\xi(v) = -[[v, \xi]_{\mathfrak{k}}, \xi], \quad T_\xi(v) = [\xi, v]_{\mathfrak{p}}. \tag{40}$$

We conclude that the Jacobi equation for $Z(t)$ becomes

$$Z''(t) + T_\xi Z'(t) + R_\xi Z(t) = 0. \tag{41}$$

8.1.1. *A reasonable approach.* We give a list of Jacobi fields that, in the compact case, actually generate all the Jacobi fields on M. Nevertheless, the fact that the fields presented satisfy the Jacobi equation is independent of the compactness of M.

(1) Let $X \in \mathfrak{g}$ be arbitrary. Then, since X^* is a Killing field, the restriction $J^k_X(t) = X^*(\gamma(t))$ is a Jacobi field with initial conditions

$$\begin{cases} J^k_X(0) = X_{\mathfrak{p}}, \\ \nabla_t J^k_X(0) = -[\xi, X]_{\mathfrak{p}} + \frac{1}{2}[\xi, X_{\mathfrak{p}}]_{\mathfrak{p}}. \end{cases} \tag{42}$$

It is clear that J^k_X is orthogonal to γ' if and only if $X \in \mathfrak{k} \oplus (\mathfrak{p} \ominus \mathbb{R}\xi)$. Applying Olmos' magic formula, we see that

$$J^k_X(t) = \mathcal{P}_t e^{t\tau_\xi}\left(e^{-t\,\mathrm{ad}(\xi)}X\right)_{\mathfrak{p}}. \tag{43}$$

(2) Let $X \in \mathfrak{p}$ and define $J^l_X(t) = \operatorname{Exp}(t\xi)_{*o}(X)$. Then $J^l_K(t)$ is a Jacobi field if and only if $R_\xi X = 0$. If this is the case, its initial conditions are

$$\begin{cases} J^l_X(0) = X, \\ \nabla_t J^l_X(0) = \left.\dfrac{d}{dt}\right|_{t=0} \mathcal{P}_t^{-1} \operatorname{Exp}(t\xi)_{*o}(X) = \nabla_X \xi^* = \frac{1}{2}[\xi, X]_{\mathfrak{p}}. \end{cases} \tag{44}$$

The field J^l_X is orthogonal to γ' if and only if $X \in \mathfrak{p} \ominus \mathbb{R}\xi$. From Olmos' Magic formula, we get

$$J^l_X(t) = \mathcal{P}_t\left(e^{t\tau_\xi}X\right). \tag{45}$$

(3) Let $X, Y \in \mathfrak{p}$, and define $J_{X,Y}(t) = \operatorname{Exp}(t\xi)_{*o}(X + tY)$. In this case, $J_{X,Y}$ is a Jacobi field if and only if $T_\xi Y + R_\xi X = 0$. The initial conditions of $J_{X,Y}$ are

$$\begin{cases} J_{X,Y}(0) = X, \\ \nabla_t J_{X,Y}(0) = \left.\dfrac{d}{dt}\right|_{t=0} e^{t\tau_\xi}(X + tY) = \frac{1}{2}[\xi, X]_{\mathfrak{p}} + Y. \end{cases} \tag{46}$$

We have that $J_{X,Y}$ is orthogonal to γ' if and only if $X, Y \in \mathfrak{p} \ominus \mathbb{R}\xi$. Using once again Olmos' magic formula:

$$J_{X,Y}(t) = \mathcal{P}_t\left(e^{t\tau_\xi}(X + tY)\right) \tag{47}$$

Then from [3] we obtain:

Theorem 9. *If $M = G/K$ is a compact naturally reductive homogeneous space, then every Jacobi field along $\operatorname{Exp}(t\xi) \cdot o$ is a linear combination of the fields given in 1, 2 and 3.*

8.1.2. *A brute force approach.* We define a system of linear differential equations on $\mathfrak{p} \oplus \mathfrak{p}$ whose solutions are Jacobi fields:

$$\begin{cases} A'(t) = B(t), \\ B'(t) = -R_\xi A(t) - T_\xi B(t). \end{cases} \tag{48}$$

From (41), we see that $(A(t), B(t))$ is a solution of the ODE system if and only if the vector field $J(t) = \operatorname{Exp}(tX)_{*o}(A(t))$ is a Jacobi field. Note that this is a system with constant coefficients, where the coefficient matrix is given by

$$C = \left(\begin{array}{c|c} 0 & \mathrm{Id}_{\mathfrak{p}} \\ \hline -R_\xi & -T_\xi \end{array}\right),$$

so the Jacobi fields along $\gamma(t)$ are obtained by projecting all curves of the form $e^{tC}(A_0, B_0)^t$. The initial conditions of $J(t)$ are

$$\begin{cases} J(0) = A_0, \\ \nabla_t J(0) = \left.\dfrac{d}{dt}\right|_{t=0} e^{t\tau_\xi}(A(t)) = \frac{1}{2}[\xi, A_0]_{\mathfrak{p}} + B_0, \end{cases} \tag{49}$$

and the Jacobi fields that are normal to γ' are obtained by choosing A_0, $B_0 \in \mathfrak{p} \ominus \mathbb{R}\xi$. We can rewrite $J(t)$ via Olmos' magic formula, obtaining

$$J(t) = \mathcal{P}_t\left(e^{t\tau_\xi} A(t)\right). \tag{50}$$

It is worth noting that the length of J coincides with $|A(t)|$, as $e^{t\tau_\xi}$ is an isometry.

8.2. **Tubes around a homogeneous submanifold** $N \subseteq M$. Let $N \subseteq M$ be an orbit of a subgroup of G at o, and let $\mathfrak{v} \subseteq \mathfrak{p}$ be its tangent space. Let $N^t \subseteq M$ be the tube of radius t around N, which we will assume that is a submanifold of M. We compute the extrinsic geometry of N^t.

Choose any $\xi \in \mathfrak{p} \ominus \mathfrak{v}$, and let $\gamma(t) = \mathrm{Exp}(t\xi) \cdot o$ be its corresponding geodesic. We recall that an N-*Jacobi field* along $\gamma(t)$ is a Jacobi field $J(t)$ such that

$$\begin{cases} J(0) \in T_o N, \\ \nabla_t J(0) + \mathcal{S}_\xi J(0) \in \nu_o N, \end{cases} \tag{51}$$

where $\mathcal{S}_\xi$ denotes the shape operator of ξ at o. The space of all N-Jacobi fields along γ has dimension $\dim M - 1$, and we actually have

$$T_{\gamma(t)} N^t = \{J(t) \colon J \text{ is N-Jacobi and } \nabla_t J(0) \perp \xi\} = T_{\gamma(t)} M \ominus \mathbb{R}\gamma'(t). \tag{52}$$

Furthermore, the shape operator $\mathcal{S}_{\gamma'(t)} \colon T_{\gamma(t)} N^t \to T_{\gamma(t)} N^t$ satisfies

$$\mathcal{S}_{\gamma'(t)} J(t) = \nabla_t J(t). \tag{53}$$

We now pull back everything to $\mathfrak{p}$ via Olmos' magic formula.

8.2.1. *Computations with left translates.* Let $J(t) = \mathrm{Exp}(t\xi)_{*o}(Z(t))$ for a curve $Z(t)$ in $\mathfrak{p}$. Then $J(t)$ is an N-Jacobi field if and only if (41) holds, $Z(0) \in \mathfrak{v}$ and $\frac{1}{2}[\xi, Z(0)]_{\mathfrak{p}} + Z'(0) + \mathcal{S}_\xi Z(0) \in \mathfrak{p} \ominus \mathfrak{v}$.

The tangent space of M^t is

$$T_{\gamma(t)} M^t = \mathcal{P}_t(\mathfrak{p} \ominus \mathbb{R}\xi). \tag{54}$$

Since $J(t) = \mathcal{P}_t\,(e^{t\tau_\xi} Z(t))$, the shape operator $\mathcal{S}_{\gamma'(t)}$ satisfies

$$\mathcal{S}_{\gamma'(t)} \mathcal{P}_t(e^{t\tau_\xi} Z(t)) = \mathcal{P}_t\left(\frac{1}{2}\left[\xi, e^{t\tau_\xi} Z(t)\right]_{\mathfrak{p}} + e^{t\tau_\xi} Z'(t)\right). \tag{55}$$

This means that we may identify the shape operator of N^t with the linear map $\mathcal{S}_t \colon \mathfrak{p} \ominus \mathbb{R}\xi \to \mathfrak{p} \ominus \mathbb{R}\xi$ defined by

$$\mathcal{S}_t e^{t\tau_\xi} Z(t) = \frac{d}{dt}\left(e^{t\tau_\xi} Z(t)\right) = \frac{1}{2}\left[\xi, e^{t\tau_\xi} Z(t)\right]_{\mathfrak{p}} + e^{t\tau_\xi} Z'(t). \tag{56}$$

Suppose we are given a basis $\{Z_1, ..., Z_{n-1}\}$ of N-Jacobi fields along γ. Let $(a_{ij}(t))$ be the matrix of $\mathcal{S}_t$ with respect to the basis $\{e^{t\tau_\xi} Z_1(t), \dots, e^{t\tau_\xi} Z_{n-1}(t)\}$. Then

$$\frac{d}{dt}\left(e^{t\tau_\xi} Z_j(t)\right) = \mathcal{S}_t e^{t\tau_\xi} Z_j(t) = \sum_{i=1}^{n-1} a_{ij}(t) e^{t\tau_\xi} Z_i(t),$$

and for each k,

$$\left\langle \frac{d}{dt}\left(e^{t\tau_\xi} Z_j(t)\right), e^{t\tau_\xi} Z_k(t) \right\rangle = \sum_{i=1}^{n-1} a_{ij}(t) \langle e^{t\tau_\xi} Z_i(t), e^{t\tau_\xi} Z_k(t)\rangle = \sum_{i=1}^{n-1} a_{ij}(t) \langle Z_i(t), Z_k(t)\rangle.$$

We may write this in terms of matrices as

$$\left(\left\langle e^{t\tau_\xi} Z_i(t), \frac{d}{dt}\left(e^{t\tau_\xi} Z_j(t)\right)\right\rangle\right) = \left(\langle Z_i(t), Z_j(t)\rangle\right)\left(a_{ij}(t)\right)$$

so

$$\left(a_{ij}(t)\right) = \left(\langle Z_i(t), Z_j(t)\rangle\right)^{-1}\left(\left\langle e^{t\tau_\xi} Z_i(t), \frac{d}{dt}\left(e^{t\tau_\xi} Z_j(t)\right)\right\rangle\right). \tag{57}$$

8.2.2. *Computations with parallel translates.* Keep the notation as before, so $J(t) = \mathrm{Exp}(t\xi)_{*o}(Z(t))$ for $Z\colon \mathbb{R} \to \mathfrak{p}$ a solution of the Jacobi equation (41). Define $Y(t) = e^{t\tau_\xi} Z(t)$. If $\{Z_1, \ldots, Z_{n-1}\}$ is a basis of N-Jacobi fields along γ, then the corresponding curves $\{Y_1, ..., Y_{n-1}\}$ are such that the set

$$\{\mathcal{P}_t Y_1(t), \ldots, \mathcal{P}_t Y_{n-1}(t)\}$$

is a basis of $T_{\gamma(t)} N^t$. Since

$$\mathcal{S}_{\gamma'(t)} \mathcal{P}_t Y(t) = \mathcal{P}_t Y'(t), \tag{58}$$

we can identify $\mathcal{S}_{\gamma'(t)}$ with the map $\mathcal{S}_t \colon \mathfrak{p} \ominus \mathbb{R}\xi \to \mathfrak{p} \ominus \mathbb{R}\xi$ defined by

$$\mathcal{S}_t Y(t) = Y'(t). \tag{59}$$

The matrix $(a_{ij}(t))$ of $\mathcal{S}_{\gamma'(t)}$ with respect to our basis is

$$(a_{ij}(t)) = (\langle Y_i(t), Y_j(t)\rangle)^{-1} \left(\left\langle Y_i(t), Y_j'(t)\right\rangle\right). \tag{60}$$

9. The second fundamental form of an orbit

Let $M = G/K$ be a Riemannian homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and S a subgroup of G. If $X, Y \in \mathfrak{s}_\mathfrak{p}$ are tangent vectors to $S \cdot o$, we compute their second fundamental form $\mathrm{I\!I}(X, Y)$.

Consider a vector $Z \in \mathfrak{g}$ such that $Z_\mathfrak{p} = Y$. Then we have

$$\begin{aligned} \nabla_{X^*} Z^* &= [X^*, Z^*]_o + \frac{1}{2}[X, Y]^*_o + U(X, Y)^*_o = \left(-[X, Z_\mathfrak{k}] - [X, Y] + \frac{1}{2}[X, Y] + U(X, Y)\right)^*_o \\ &= \left(-[X, Z_\mathfrak{k}] - \frac{1}{2}[X, Y] + U(X, Y)\right)^*_o = \left(\left[Z_\mathfrak{k} + \frac{1}{2}Y, X\right] + U(Y, X)\right)^*_o. \end{aligned}$$

By symmetry of the second fundamental form, we may rewrite this as

$$\mathrm{I\!I}(X, Y) = \left(\left[X^\mathfrak{k} + \frac{1}{2}X, Y\right] + U(X, Y)\right)^\perp_\mathfrak{p}, \tag{61}$$

where $X^\mathfrak{k}$ is any vector in $\mathfrak{k}$ such that $X^\mathfrak{k} + X \in \mathfrak{s}$, and $(\cdot)^\perp$ denotes the orthogonal projection from $\mathfrak{p}$ to $\mathfrak{p} \ominus \mathfrak{s}_\mathfrak{p}$.

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